On the Witt Group of Finitely Generated Torsion Modules and Its Applications to Algebraic Topology.

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ON THE WITT GROUP OF FINITELY GENERATED TORSION MODULES

AND ITS APPLICATIONS TO ALGEBRAIC TOPOLOGY

A Dissertation

Submitted to the Graduate Faculty of the
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Doctor of Philosophy

in

The Department of Mathematics

by

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B.S., National Taiwan University, 1967
M.S., National Taiwan University, 1970
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The main idea in this dissertation is the development of a new approach to the well known exact sequence by M. Knebusch and W. Scharlau; see [14]. This exact sequence of Witt rings is

\[ 0 \to W(D) \to W(F) \to \sum_P W(D/P) \to \frac{C}{C^2} \to 0 \]

where \( F \) is the quotient field of a Dedekind domain \( D \), \( P \) is a prime ideal of \( D \) and \( C \) is the ideal class group of \( D \).

We will develop this new approach by proving two main theorems. The first of these theorems is the following isomorphism, see Chapter III,

\[ W(T(F/D)) \cong \sum_P W(D/P) \]

where \( W(T(F/D)) \) is the Witt group of finitely generated torsion modules with the values of bilinear form in \( F/D \).

En route to proving this isomorphism, we generalize the fundamental theorem of finitely generated torsion modules over a Dedekind domain, (see Chapter II). The second of the aforementioned theorems establishes that the sequence
is exact. Combining these two theorems, we show that the sequences

\[ 0 \to W(D) \to W(F) \to \sum_{P} W(D/P) \]

\[ 0 \to W(D) \to W(F) \to W(T(F/D)) \to \frac{\mathbb{C}}{\mathbb{C}^2} \to 0 \]

are exact.

We will equip every module with a group action of \( G \) as a group of \( D \)-linear automorphisms to obtain a new exact sequence, (see Chapter IV),

\[ 0 \to W(G;D) \to W(G;F) \to W(G;T(F/D)) \]

For this purpose, we must generalize the definition of Witt equivalence so that the module may be acted on by \( G \). Therefore, in Chapter I, we must prove that the relation is indeed an equivalence relation and prove the anisotropic representation theorem under the generalized definition of Witt equivalence.

Finally, we apply the exact sequence in Chapter IV to the bordism theory and cohomology theory of manifolds. We obtain an important result (Chapter V, section 5) that the image of inner product torsion module derived from the manifold under the boundary operator coincides with that derived from the boundary of a manifold.
INTRODUCTION

The quadratic forms over integers, rational numbers and Dedekind domains are discussed in [20], part III and Part IV. If the quadratic form on a finitely generated projective A-module \( V \) is defined as a map

\[
q: V \to A
\]

such that

\[
q(\alpha x) = \alpha^2 q(x), \quad \alpha \in A
\]

and

\[
b_q(x,y) = q(x+y) - q(x) - q(y)
\]

is bilinear, where \( A \) is an integral domain, and if 2 is a unit in \( A \), then a quadratic form \( q \) determines a symmetric bilinear form \( b_q \) and vice versa by

\[
2b(x,y) = b_q(x,y) = q(x+y) - q(x) - q(y)
\]

\[
q(x) = b(x,x).
\]

However, if 2 is not a unit in \( A \), then the theory of quadratic forms and the theory of symmetric bilinear forms...
divide. We are mainly interested in the symmetric bilinear forms in this dissertation.

In appendices I and II of [9], Hirzebruch and Scharlau discuss the relation between the Grothendieck group of nonsingular symmetric bilinear forms and Witt group.

Some of the Witt rings $W(D)$ of finitely generated projective $D$-modules over a Dedekind domain $D$ have been computed. For example, let $D = Z$, then $W(Z) \cong Z$; let $D = R$, then $W(R) \cong Z$; let $D = C$, then $W(C) \cong Z_2$, the field of two elements; let $D = Z_p$, then $W(Z_p) \cong Z_4$ if $p \equiv 3 \pmod{4}$; $W(Z_p) \cong Z_2 \oplus Z_2$ if $p \equiv 1 \pmod{4}$; let $D = Z_2$, then $W(Z_2) \cong Z_2$, since every element in $Z_2^*$ (the set of units) is a square, see [19], p. 66; let $D = Q$, then $W(Q) \cong \bigoplus_{p \text{ primes}} W(Z_p) \cong Z \oplus Z_2 \oplus Z \oplus Z_4$. All of these results can be found in [19]. The last isomorphism is from an exact sequence in the paper by Knebusch and Scharlau, see [14], which states that the sequence

$$O \rightarrow W(D) \rightarrow W(F) \rightarrow \Sigma_P (D/F) \rightarrow \frac{C}{C^2} \rightarrow O$$

is exact, where $C$ is the ideal class group of the Dedekind domain $D$. Let $D = Z$, the sequence

$$O \rightarrow W(Z) \rightarrow W(Q) \rightarrow \Sigma_P W(Z_p) \rightarrow 0$$

is then split exact.
The main goal in this dissertation is to provide a different approach to the above exact sequence so that we can equip every module with a D-linear group action $G$ to get a new exact sequence

$$0 \rightarrow W(G; D) \rightarrow W(G; F) \rightarrow W(G; T(F/D)).$$

In order to reach this goal, we will define the Witt group $W(T(F/D))$ of finitely generated torsion modules over a Dedekind domain and describe the isomorphism between $W(T(F/D))$ and $\bigoplus_{P} W(D/P)$, where $F$ is the quotient field of $D$ and $P$ is a prime ideal of $D$.

Chapter I will define the Witt group or Witt ring following the generalized definition by Conner [3], instead of the definition in [19] and discuss the split torsion modules and anisotropic torsion modules. Furthermore, we will prove the Witt equivalent relation in the sense of the generalized definition of Conner.

In Chapter II, we generalize the fundamental theorem of finitely generated torsion modules over a Dedekind domain and prove some theorems about the case when $D$ is a local ring as a digression. In Chapter III, we apply the generalized theorem of Chapter II to prove the isomorphism

$$W(T(F/D)) \cong \bigoplus_{P} W(D/P)$$

and the exact sequence
Up to this, we have proved the well known exact sequence

$$0 \to W(D) \to W(F) \to W(T(F/D)) \to \frac{C}{C^2} \to 0$$

by a new approach putting the isomorphism and exact sequence together.

Since we can put group action of $G$ on a torsion module, in Chapter IV, we will equip every kind of module, such as finitely generated projective modules, vector spaces, finite modules or torsion modules with group action of $G$. We denote the module with group action by $(G;M)$. We will then obtain the exact sequence with action

$$0 \to W(G;D) \to W(G;F) \to W(G;T(F/D)).$$

As we know the concept of a split inner product space is closely related to cobordism theory. According to Thom [23], if the closed manifold $M^{2n}$ bounds a compact manifold $N^{2n+1}$, then the kernel $K$ of the inclusion homomorphism

$$H_n(M^{2n}; F) \to H_n(N^{2n+1}; F)$$

satisfies $\theta(K, K) = 0$ and \text{rank} $(K) = \frac{1}{2} \text{rank} (H_n(M^{2n}; F))$, where $F$ is a coefficient field. Therefore $K = K^1$, so that the inner product space $H_n(M^{2n}; F)$ is split. (Compare [10], p. 84, §8.) We will have some beautiful results in Chapter V.
as the application of Witt rings (groups) to cohomology theory.

Before we end the introduction, we would like to compute some concrete examples of Witt ring of quadratic forms by the classical method rather than the complicated method of bilinear forms. If the quadratic form is over \( \mathbb{Z}_3 \), then
\[
  x^2 + y^2 \equiv -x^2 - y^2 ,
\]
indicates quadratic form equivalence. The ideal \((x^2 - y^2)\) generated by \(x^2 - y^2\) is equivalent to 0. And
\[
  x^2 + y^2 \sim -x^2 - y^2 .
\]
By the following table under the operation \(\oplus\):

<table>
<thead>
<tr>
<th>(\oplus)</th>
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<th>(x^2)</th>
<th>(-x^2)</th>
<th>(x^2 + y^2)</th>
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<tr>
<td>0</td>
<td>0</td>
<td>(x^2)</td>
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<td>(-x^2)</td>
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<td>(x^2 + y^2)</td>
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<td>(-x^2)</td>
<td>(x^2)</td>
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and the multiplication \(\otimes\) with this rule:

\[
(x_1^2 + 2x_2^2) \otimes (-y_1^2 + y_2^2 + 3y_3^2)
\]

\[
= -t_1^2 + t_2^2 + 3t_3^2 - 2t_4^2 + 2t_5^2 + 6t_6^2 \quad (\text{mod } \mathbb{Z}_3),
\]

we know \(W(\mathbb{Z}_3) \cong \{ x^2, -x^2, x^2 + y^2, 0 \} \cong \mathbb{Z}_4 \).

Similarly, if the quadratic form is over \( \mathbb{Z}_5 \), then
\(W(\mathbb{Z}_5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\), a four group.

If the quadratic form is over \( \mathbb{R} \), the real numbers, then \(W(\mathbb{R}) \cong \mathbb{Z}\) since \(x^2 + y^2 \not\equiv -x^2 - y^2\) over \(\mathbb{R}\) and by the following:
dim. 1: \( x^2 \), \(-x^2\)

dim. 2: \( x^2 + y^2 \), \(-x^2 - y^2\)

dim. 3: \( x^2 + y^2 + z^2 \), \(-x^2 - y^2 - z^2\)

\[ \vdots \]

dim. \( n \): \( \sum_{i=1}^{n} x_i^2 \), \(-\sum_{i=1}^{n} x_i^2\).

Therefore, \( W(R) \cong Z \) after comparing the dimensions and \( Z \).

If the quadratic form is over \( C \), then \( W(C) \cong Z_2 \) since \( x^2 + ay^2 = 0 \) is solvable over \( C \) and \( x^2 \cong -y^2 \), \( W(C) = \{0, x^2\} \).

By the way, we should mention here that the work about Witt ring (group) in [19] is the case of finitely generated projective modules; and, the works in [1] and [24] are the case of finite modules. This dissertation will deal mainly with finitely generated torsion modules.

In this dissertation, if we quote a lemma or a theorem without indicating the chapter, we will mean that lemma or that theorem in the same chapter. For example, if in Chapter III, we quote Lemma 2.1, then we mean that lemma of Chapter III. If in Chapter III, we quote a lemma of Chapter I, we will say Lemma 2.1, Chapter I.
CHAPTER I

THE WITT GROUP OF FINITELY GENERATED TORSION MODULES
OVER A DEDEKIND DOMAIN

§1. Some Preliminary Definitions.

In this dissertation, unless otherwise specified, D will denote a Dedekind domain with quotient field F. Recall that a Dedekind domain is an integral domain in which any non-zero ideal can be expressed uniquely as a product of maximal ideals.

An inner product module \( (M, \beta) \) over D is a D-module M together with a bilinear form \( \beta : M \times M \to K \) and with the non-singular condition that \( M \cong \text{Hom}_D(M, K) \), where K is a D-module and K is suitably chosen according to the condition of M. For example, if M is a finitely generated projective module, then \( K = D \) is chosen; if M is a vector space, then \( K = F \) and \( D = F \) are chosen; if M is finitely generated torsion module, \( K = F/D \) is chosen; etc.

We should observe the following remark. If \( (M, \beta) \) is a finitely generated torsion D-module with the bilinear form
then we choose the values of \( \beta \) in \( \mathbb{F}/D \) by the following reason. Suppose \( d \neq 0 \) and \( dx = 0 \). If \( \beta(x,y) \in D \), then 
\[
\beta(dx,y) = d\beta(x,y) = 0. 
\]
It follows that \( \beta(x,y) = 0 \) for all \( y \in M \); and similarly, \( \beta(x,y) = 0 \) for all \( x \in M \). Therefore, \( \beta(x,y) = 0 \) for all \( x \) and \( y \) in \( M \), the bilinear form \( \beta \) is trivial, and this is not the case we want.

**Definition 1.1.** A bilinear form or an inner product is **symmetric** if \( \beta(x,y) = \beta(y,x) \) for all \( x \) and \( y \). Similarly, it is **skew-symmetric** if \( \beta(x,y) = -\beta(y,x) \) for all \( x \) and \( y \) and **symplectic** (or **alternating**) if \( \beta(x,x) = 0 \) for all \( x \).

**Definition 1.2.** Two elements \( x \) and \( y \) of an inner product module are called **orthogonal** if \( \beta(x,y) = 0 \in K \).

**Lemma 1.3.** Let \((M_1, \beta_1), \ldots, (M_n, \beta_n)\) be inner product modules over \( D \), with bilinear forms \( \beta_1, \ldots, \beta_n \) respectively. If \( K \) is a \( D \)-module and also a ring, then the **tensor product** \( M_1 \otimes \cdots \otimes M_n \) over \( D \) can be made into an inner product module with unique bilinear form \( \beta \) on \( M_1 \otimes \cdots \otimes M_n \) which satisfies the identity
\[
\beta(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \prod_{i=1}^{n} \beta_1(x_i, y_i)
\]
for all \( x_i \) and \( y_i \) in \( X_i \), \( 1 \leq i \leq n \), see [19], p. 10. Similarly, if \( K \) is a \( D \)-module, then the **orthogonal sum** \( M_1 \oplus \cdots \oplus M_n \) is defined to be the direct sum of modules \( M_i \) with bilinear form \( \beta \) defined by the equation
\[ \beta(x_1 \oplus \cdots \oplus x_n, y_1 \oplus \cdots \oplus y_n) = \sum_{i=1}^{n} \beta_i(x_i, y_i). \]

We should clarify here that if the value of \( \beta_1, \beta_2, \ldots, \beta_n \) are in \( D \) or in \( F \), then the tensor product can be made into an inner product \( \beta = \prod_{i=1}^{n} \beta_i \). However, if the values of \( \beta_1, \ldots, \beta_1, \ldots, \beta_n \) are in \( F/D \), then the pair \((M_1 \oplus \cdots \oplus M_n, \prod_{i=1}^{n} \beta_i)\) is no longer an inner product module since \( \prod_{i=1}^{n} \beta_i = \beta \) is not well defined.

**Definition 1.4.** Let \( N \) be a submodule of \( M \), \( N^\perp \) denotes the orthogonal complement of \( N \), which consists of all \( x \in M \) such that \( \beta(x, N) = 0 \).

**Definition 1.5.** A symmetric inner product module \( M \) over \( D \) is **split** if there exists a submodule \( N \subset M \), such that \( N \) is precisely equal to its own orthogonal complement \( N^\perp \).

This definition is somewhat more general than the definition in Milnor and Husemoller [19]. This generalized definition is in Conner [3], p. 17.

**Definition 1.6.** Two symmetric inner product modules \((M, \beta)\) and \((M', \beta')\) over \( D \) **belong to the same Witt class**, written \( M \sim M' \) or simply \( M \sim M' \), if \((M, \beta) \oplus (M', -\beta')\) is split. We will use \( M \oplus (-M') \) to denote \((M, \beta) \oplus (M', -\beta')\).

Clearly, if \( M \) is a **split inner product module** then
$M \sim 0$, since $M \otimes \{-0\}$ is split. So if $M \sim M'$, then $M \otimes (-M') \sim 0$.

We will prove this is an equivalence relation and that the collection $W(K)$ of all Witt classes of symmetric inner product modules over $D$ with values in $K$ forms a commutative ring with 1 if $K$ is a $D$-module and also a ring; and that $W(K)$ forms an abelian group if $K$ is only a $D$-module, using the orthogonal sum as addition and the tensor product as multiplication. $W(K)$ is called the Witt ring of $K$ if the inner product modules are finitely generated projective $D$-modules with $\beta$-values in $D$ or if they are finite dimensional vector spaces with values in a field $F$. The Witt ring $W(D)$ or $W(F)$ is treated in [19]. However, we will mainly deal with finitely generated torsion inner product modules in Chapters I, II and III. In order to distinguish the Witt group of finitely generated torsion modules from the Witt ring of finitely generated projective modules, we denote the Witt group of the torsion $D$-modules as $W(T(F/D))$ with $\beta$-values in $F/D$.

§2. The Witt Equivalence Relation

In this section, unless otherwise specified, all modules will be finitely generated torsion modules. The first lemma proves that the submodule $A \subset M$ has the property that $A = (A^\perp)^\perp$ for a finitely generated torsion module $M$. If $M$ is a finite dimensional vector space, then it is trivial
that \( A = (A^\perp)^\perp \); if \( M \) is a finitely generated projective module, \((A^\perp)^\perp\) does not generally equal \( A \). However, we will discuss this case later in (2.10) remarks.

Lemma 2.1. If \( M \) is a finitely generated torsion \( D \)-module, and \((M,\beta)\) is an inner product module with values in \( F/D = K \), where \( F \) is the quotient field of a Dedekind domain \( D \), then \( A = (A^\perp)^\perp \).

Proof. Since \( M \) is a finitely generated torsion \( D \)-module, by Cartan and Eilenberg [2], p. 137, we have

\[
M \cong \text{Hom}_D(M,K) \cong \text{Tor}_1(M,K) \cong \text{Hom}_D(\text{Hom}_D(M,K),K)
\]

and

\[
(A) \cong \text{Hom}_D(\text{Hom}_D(\text{Hom}_D(A,K),K),K)
\]

where \( M \cong \text{Hom}_D(M,K) \) since \( M \) is an inner product module with \( K \)-values. It follows that the following sequences are exact,

\[
0 \rightarrow A^\perp \rightarrow M \rightarrow \text{Hom}_D(A,K) \rightarrow 0
\]

\[
0 \rightarrow A^\perp \rightarrow \text{Hom}_D(M,K) \rightarrow \text{Hom}_D(A,K) \rightarrow 0
\]

and

\[
0 \rightarrow (A^\perp)^\perp \rightarrow M \rightarrow \text{Hom}_D(A^\perp,K) \rightarrow 0 .
\]

\( K = F/D \) is divisible, see [2], p. 129. Since \( D \) is a Dedekind domain and \( F \) is the quotient field of \( D \), we have \( K = F/D \) is an injective \( D \)-module, see [2], p. 134. Taking
the Hom functor in (S.2), we obtain the following exact sequence

$$0 \to \text{Hom}_D(\text{Hom}_D(A,K),K) \to \text{Hom}_D(M,K) \to \text{Hom}_D(A^\perp,K) \to \text{Ext}(\text{Hom}_D(A,K),K) \to \cdots.$$  

By the theorems in [11], pp. 145-149, the above sequence and (S.1), it follows that the sequences

$$0 \to \text{Hom}_D(\text{Hom}_D(A,K),K) \to \text{Hom}_D(M,K) \to \text{Hom}_D(A^\perp,K) \to 0$$

(S.4)

$0 \to A \to M \to \text{Hom}_D(A^\perp,K) \to 0$

are exact.

Combining (S.3) and (S.4), the following sequences are exact:

$$0 \to A \to M \to \text{Hom}_D(A^\perp,K) \to 0$$

where $i$ is an inclusion map since $A \subseteq (A^\perp)^\perp$.

From the 3x3 lemma (see [16], p. 49), and the commutativity of the following diagram
we have \( A \to (A^\perp)^\perp \) is onto and, hence, \( A = (A^\perp)^\perp \).

**Lemma 2.2.** If \((S,\beta)\) is a split torsion module and \(L \subset S\) is a maximal self-annihilating submodule (i.e., maximal with respect to \(L \subset L^\perp\)), then \(L = L^\perp\), where \(L^\perp\) is the orthogonal complement of \(L\) in \(S\).

We will call \(L\) a **splitter** in the split module \(S\) if \(L = L^\perp\).

**Proof.** Let \(N \subset S\) be the splitter such that \(N = N^\perp\). Consider \(L + (NL^\perp)\). Let \(e, e_1 \in L\), \(e', e_1' \in L^\perp \cap N\).

\[
\beta(e+e', e_1+e_1') = \beta(e,e_1) + \beta(e,e_1') + \beta(e',e_1) + \beta(e',e_1') = 0.
\]

So \(L + (NL^\perp) \subset (L + (NL^\perp))^\perp\). \(L = (L + (NL^\perp))^\perp\) since \(L\) is a maximal self-annihilating submodule. We obtain \(N \subset L^\perp\). Since \(N = N^\perp\), we will have

\[
(N+L)^\perp = (N^\perp N L^\perp) = N \cap L^\perp \subset L.
\]

Taking the orthogonal complement of above inclusion, we get
\[(N+L) = L^\perp.\] Let \(l' \in L^\perp\), we have \(l' = l+n\), \(n \in N\) and \(l \in L\). Hence, \(n = l' - l\), and \(n \in L^\perp \cap N\) for \(l \in L \subset L^\perp\). So \(l' = n+l \in L + (N\cap L^\perp)\); consequently, \(L^\perp \subset L + (N\cap L^\perp)\). Therefore, \(L = L + (N\cap L^\perp) = L\), and \(L = L^\perp\).

**Lemma 2.3.** If \(K \subset K^\perp \subset S\) and \(S\) is a split module, then \(K\) is contained in a maximal self-annihilating submodule.

**Proof.** Let \(K_i \subset K_i^\perp\) for all \(i\) in some index. Take the maximal chain of \(\{K_i\}\) and let \(L = \bigcup K_i\), we have \(L \subset L^\perp\). \(L\) does exist as a result of the ascending chain condition of finitely generated modules. By (2.2), \(L = L^\perp\) and \(K = K_j \subset K_j^\perp \subset L^\perp = L\).

**Lemma 2.4.** If \(S\) is a split module and \(K \subset S\) is self-annihilating, then \(K\) is contained in a splitter.

**Proof.** This is trivial from (2.3).

We can use (2.2), (2.3) and (2.4) to prove (2.6). However, we will use the lattice lemma (2.5) as a short-cut to prove (2.6).

**Lattice Lemma 2.5.** Suppose \(M\) is split, and suppose \(L \subset M\) and \(L \subset L^\perp\), then \(L\) is contained in a splitter \(L + (N\cap L^\perp)\), where \(N \subset M\) and \(N = N^\perp\).
Proof. \([L + (N\cap L)]^\perp = L^\perp \cap (N\cap L)^\perp = L^\perp \cap [N^\perp + (L^\perp)^\perp]\). Since (2.1) and the remark before (2.1), \((L^\perp)^\perp = L\) and since \(N = N^\perp\),

\[(L + (N\cap L^\perp)]^\perp = L^\perp \cap (N+L) = (L^\perp \cap N) + L^\perp \cap L = L + (L^\perp \cap N)\).

Therefore, \(L + (N\cap L^\perp) = [L + (N\cap L^\perp)]^\perp\), a splitter.

Lemma 2.6. If \((M,\beta_1)\) is a torsion module, and \((S,\beta_2)\) is a split torsion module for which \((M\otimes S,\beta)\) is split, then \(M\) is split (i.e., stably split implies split).

Proof. Since \(M \otimes S\) is split, we can choose \(N \subset M \otimes S\) such that \(N = N^\perp\). And since \(S\) is split, we can choose \(N_1 \subset S\) such that \(N_1 = N_1^\perp\). By (2.4) or (2.5) lattice lemma, we can assume \(N_1 = 0\otimes N_1 \subset N\).

If \((a,s) \in N\), then \(s \in N_1\), for if \((0,s_1) \in N_1 \subset N = N^\perp\), \(0 = \beta((0,s),(0,s_1)) = \beta_2(s,s_1)\), then \(s \in N_1 = N_1^\perp\) and \((0,s) \in N_1\). Since \((a,0) = (a,s) - (0,s) \in N\), we have \((a,0) \in N\). Thus \((a,s) = (a,0) + (0,s) \in N_2 + N_1\), where

\[N_2 = \{(a,0) \in M \otimes S | (a,0) \in N\} \subset M \otimes \{0\} \cong M; \] we have \(N = N_2 \otimes N_1\). We claim \(N_2 \subset M\) is a splitter, that is, \(N_2 = N_2^\perp\), for \(N_2 \subset N_2^\perp\) and if we let \(b \in N_2^\perp\), then \((b,0) \in N_1 = N\). Hence, it follows that \(b \in N_2\) and \(N_2 = N_2^\perp\). Therefore \(N_2\) is a splitter. \(M\) is split with splitter \(N_2\).

Lemma 2.7. The following are equivalent:
(1) If \( M \) is torsion module, and \( S \) is a split torsion module for which \( M \oplus S \) is split, then \( M \) is split.

(2) If \( M_1 \oplus (-M_2) \sim 0 \), and \( M_2 \oplus (-M_3) \sim 0 \), then \( M_1 \oplus (-M_3) \sim 0 \).

**Proof.** (2) ⇒ (1): Since \( S \sim 0 \), we have \( (-S) \oplus [0] \sim 0 \), and \( M \oplus S \sim 0 \). By (2), \( M \oplus [0] \sim 0 \), therefore \( M \sim 0 \).

(1) ⇒ (2): Let \( M_1 \oplus (-M_2) = S_1 \) and \( M_2 \oplus (-M_3) = S_2 \). Adding \( M_3 \) to the first equality, we obtain \( M_1 \oplus (-M_2) \oplus M_3 = S_1 \oplus M_3 \). It follows that \( M_1 \oplus (-M_2) \oplus M_3 = S_1 \oplus M_3 \). So \( M_1 \oplus (-S_2) = S_1 \oplus M_3 \). Adding \( (-M_3) \) to both sides, we obtain \( M_1 \oplus (-M_3) \oplus (-S_2) = S_1 \oplus M_3 \oplus (-M_3) \). Thus, \( M_1 \oplus (-M_3) \oplus (-S_2) = S_1 \oplus S_3 \) implies that \( M_1 \oplus (-M_3) \sim 0 \) by (1).

**Theorem 2.8.** Witt equivalence is an equivalence relation for a finitely generated torsion inner product module over a Dedekind domain.

**Proof.** From (2.6), the first statement of Lemma (2.7) is true; as a result, the second statement of (2.7) is true, and the transitive law is proven. The symmetric law and the reflexivity are trivial. Therefore, the Witt equivalence defined in (1.6) is an equivalence relation.

**Remark 2.9.** The Witt equivalence relation defined on (1.6) is more general than that defined in [19], p. 14. For if there exists split inner product modules, \( S \) and \( S' \), so that
\[ \text{Remark 2.10. The Witt equivalence defined in (1.6) for finite} \\
\text{dimensional vector space over a field } F \text{ is also an equivalence} \\
\text{relation, for if } A \text{ is a subspace of } M, \text{ then } (A^\perp)^\perp = A \\
\text{implies the conclusion in (2.6). It is also true that stably} \\
\text{split implies split.}
\]

\text{It is also an equivalence relation for finitely generated} \\
\text{projective modules. Suppose } (M,\beta) \text{ is a finitely generated} \\
\text{projective module. Consider } (M\otimes F,\beta') \text{ which is a finite} \\
\text{dimensional vector space with inner product } \beta' \text{ satisfying} \\
\beta'((x,s),(y,t)) = \beta(x,y) \cdot st
\]

\text{where } (x,s),(y,t) \in M\otimes F. \text{ We claim that } M \text{ is split if,} \\
\text{and only if, } M\otimes F \text{ is split: Since } M \text{ is embedded in } M\otimes F \\
as } M\otimes 1, \text{ if } M \text{ is split, then } M\otimes F \text{ is split following} \\
immediately by the equality } \beta'(x,1),(y,1)) = \beta(x,y). \text{ In the} \\
other way, suppose } M\otimes F \text{ is split with splitter } N = N^\perp \text{ in} \\
M\otimes F. \text{ Let } N_1 = N \cap (M\otimes 1) \subseteq M\otimes 1. \text{ Then } N_1 = N_1^\perp \text{ in } M\otimes 1 \\
\text{for the following reason. If } (x,1) \in N_1^\perp, \text{ then} \\
\beta'((x,1),(y,1)) = 0 \text{ for all } (y,1) \in N_1. \text{ For any } (y,s) \in N, \\
\text{we have } (y,1) \in N_1 \text{ and}
\[ \beta'(x, l)(y, s) = \beta(x, y) s = s \beta'(x, l)(y, 1) = 0. \]

Hence, \((x, l) \in N^{-1} = N\). Therefore \((x, l) \in N_1\). This shows that \(N_1^{-1} \subset N_1\). So \(N_1 = N_1^{-1}\). Now we have proven the claim.

We will show that if \(M \otimes S\) is split with split module \(S\), then \(M\) is split. Consider

\[ (M \otimes S) \otimes F = M \otimes F \otimes S \otimes F. \]

If \(M \otimes S\) is split with split module \(S\), then both \((M \otimes S) \otimes F\)`
and \(S \otimes F\) are split vector spaces over \(F\). Thus \(M \otimes F \otimes S \otimes F\)
\(D\)
is a split vector space with split vector space \(S \otimes F\). It follows that \(M \otimes F\) is split; and, by the above claim, \(M\) is split.

Since the statement that stably split implies split
and the statement that Witt equivalence is transitive are equi-
valent. Hence Witt equivalence is transitive for finitely
generated projective \(D\)-modules also.

§3. Anisotropic Representation

**Lemma 3.1.** Let \(D\) be a Dedekind domain, and let \(M\) be a
finitely generated torsion \(D\)-module. Then \(M\) has the ascending
chain condition and the descending chain condition.

**Proof.** \(M\) has A.C.C. since \(M\) is finitely generated, see
[15], p. 144. Suppose \(M = \langle x_1, \ldots, x_n \rangle\). Let \(a_i x_i = 0\),
\(a_i \in D\) for all \(i = 1, 2, \ldots, n\). Let \(a = a_1 a_2 \cdots a_n\) and
I = (a). M is a D-module implies M is D/I-module, since 
(d+I)m = dm and Im = 0, for all m ∈ M. D has A.C.C.
implies D/I has A.C.C., and every prime ideal of D/I dif-
ferent from D/I is a maximal ideal. As a result of the
theorem in [25], p. 203, D/I has D.C.C. And M has D.C.C.,
because of the result of [25], p. 158.

Lemma 3.2. Let (M,β) be an inner product module over D with
β-values in F/D = K. Suppose A is a submodule of M, and
A ⊆ A^⊥. Then A^⊥/A has a non-singular inner product structure.

Proof. The exact sequence

\[ 0 \rightarrow A \rightarrow A^\perp \rightarrow A^\perp/A \rightarrow 0 \]

implies another exact sequence

\[ 0 \rightarrow \text{Hom}(A^\perp/A, K) \rightarrow \text{Hom}(A^\perp, K) \rightarrow \text{Hom}(A, K) \rightarrow 0. \]

Let \( \overline{\varphi} \in \text{Hom}(A^\perp/A, K) \). From the above exact sequence, we have
\( \varphi \in \text{Hom}(A^\perp, K) \). There exists unique \( a \in M \ni \varphi(a') = \beta(a', a) \)
for all \( a' \in A^\perp \) and \( \varphi = \varphi_a \). Since \( A \subseteq \text{Ker} \varphi \), \( \varphi_a(a') = 0 \)
implies \( a_1 \in A \); the unique \( a \), corresponding to \( \varphi_a \), is
in \( A^\perp/A \). So, for any \( \overline{\varphi} \in \text{Hom}(A^\perp/A, K) \), we associate
\( \overline{a} \in A^\perp/A \) such that \( \overline{\varphi} = \overline{\varphi_{\overline{a}}} \).

Let \( \overline{a} \in A^\perp/A \). We have \( a \in A^\perp \), \( a \) is a representative
of \( \overline{a} \). Therefore \( \varphi_a \in \text{Hom}(M, K) \subseteq \text{Hom}(A^\perp, K) \) with \( \varphi_a(a') = \beta(a, a') \)
for all \( a' \in A^\perp \). If \( \varphi_a = 0 \), then \( a \in A \). Hence
is well defined, for if $a \in A$, then we have $\overline{\varphi}_a = 0$ since $\overline{\varphi}_a(a') = \beta(a,a') = 0$ for all $a' \in A^\perp$. Therefore, for any $\overline{a} \in A^\perp/A$, we have well defined $\overline{\varphi}_\overline{a}$ associated with $\overline{a}$. Because the two sides composition is an identity, we have the following homomorphism

$$A^\perp/A \rightarrow \text{Hom}(A^\perp/A, K)$$

is bijective.

Lemma 3.3. Let $(M, \beta)$ be an inner product torsion D-module with $\beta$-values in $K = F/D$. Suppose $A \subset A^\perp$ and $A$ is a submodule of $M$. Then $M \cong A^\perp/A$.

Proof. Since $(A^\perp/A, \delta)$ is a non-singular inner product module by Lemma 3.2, $(M \otimes A^\perp/A, \beta-\delta)$ is an inner product module. Let $\nu : A^\perp \rightarrow A^\perp/A$ be a quotient homomorphism. Clearly, $\nu$ is isometric (i.e., $\beta(x,y) = \delta(\nu(x),\nu(y))$) since $(A^\perp/A, \delta)$ is non-singular. Consider

$$N = \{(a, \nu(a)) \in M \otimes (-A^\perp/A) | a \in A^\perp, \nu(a) \in A^\perp/A\}.$$ 

If we can prove $N = N^\perp$, then $M \otimes (-A^\perp/A)$ is split. Thus $M \otimes (-A^\perp/A) \sim 0$. This shows that $M \cong A^\perp/A$.

Claim $N \subset N^\perp$: Let $a_1', a_2' \in A^\perp$, $\nu(a_1'), \nu(a_2') \in A^\perp/A$.

$(\beta-\delta)[(a_1', \nu(a_1')), (a_2', \nu(a_2'))] = \beta(a_1', a_2') - \delta(\nu(a_1'), \nu(a_2')) = \beta(a_1', a_2') - \beta(a_1', a_2') = 0$. The next to the last equality is by
isometry. This shows that \( N \subset N^\perp \).

Claim \( N^\perp \subset N \): Let \((b,\nu(c)) \in N^\perp \subset M \oplus (-A^\perp/A)\), where \( c \in A^\perp \), \( b \in M \).

\[(\delta-\delta)[(b,\nu(c)),(a,\nu(a))] = 0 \quad \text{for all} \quad (a,\nu(a)) \in N\]

with \( a \in A^\perp \). It follows that

\[0 = \delta(b,a) - \delta(\nu(c),\nu(a)) = \delta(b,a) - \delta(c,a) = \delta(b-c,a)\]

for all \( a \in A^\perp \). Hence, \( b-c \in (A^\perp)^\perp = A \), the last equality is a result of Lemma (2.1). Since \( c \in A^\perp \) and \( A \subset A^\perp \), \( b = c + (b-c) \in A^\perp \). And \( b-c \in A \Rightarrow \nu(b-c) = 0 \Rightarrow \nu(b) = \nu(c) \). Therefore, \((b,\nu(c)) = (b,\nu(b)) \in N\). This shows \( N^\perp \subset N \).

Definition 3.4. An inner product module \( M \) is \textbf{anisotropic} if \( x \in M \) and \( \delta(x,x) = 0 \) imply \( x = 0 \).

In the next two theorems we will show that every element in the Witt group \( W(T(F/D)) \) is represented by an anisotropic inner product module which is unique up to isomorphism.

Theorem 3.5. Let \( M \) be a finitely generated torsion D-module, and let \((M,\delta)\) be an inner product module with values of \( \delta \) in \( F/D \). For any submodule \( A \) of \( M \) if \( A \subset A^\perp \), then \( M \wedge W \) anisotropic part of \( M \).

Proof. By Lemma (3.3), \( A \subset M \) and \( A \subset A^\perp \), it follows that \( M \wedge W \cong A^\perp/A \). Let \( A^\perp/A = M_1 \), \( A_1 \subset M_1 \) and \( A_1 \subset A_1^\perp \). Again,
by Lemma (3.3), \( M_1 \sim A_1^+/A_1 \). Let \( A_1^+/A_1 = M_2 \). After repeating the same procedure, we have an interpretation of the above induction:

\[
M \supset A_1^+ \supset A_2^+ \supset \cdots \supset A_2^+ \supset A_1^+ \supset A
\]

From Lemma (3.1), we obtain two chains

\[
A_1^+ \supset A_1^+ \supset A_2^+ \supset \cdots \supset A_n^+ = A_{n+1}^+ = \cdots
\]

\[
A \subset A_1 \subset A_2 \subset \cdots \subset A_m = A_{m+1} = \cdots
\]

Choose \( \ell = \max (n,m) \). Finally, we have

\[
A_1^+ \supset A_1^+ \supset A_2^+ \supset \cdots \supset A_\ell^+ \supset A_\ell \supset \cdots \supset A_2 \supset A_1 \supset A
\]

Therefore, \( M \overset{W}{\sim} \) anisotropic part of \( M \), that is,

\[
M \overset{W}{\sim} A_\ell^+/A_\ell
\]

Otherwise, the chains would not be stationary.

**Theorem 3.6.** Every element of the Witt group \( W(T(F/D)) \) is represented by one, and up to isomorphism, only one, anisotropic inner product module.

**Proof.** Theorem (3.5) proves the existence of anisotropic representation. If \( A \sim A' \), two anisotropic inner product modules belonging to the same Witt class, then we must prove that \((A,\beta) \sim (A',\beta')\).

\[A \otimes (-A') \sim 0\] implies that there is a submodule
\[ N \subset A \oplus (-A') \] such that \[ N = N^\perp. \] We have \( N \cap A' = \{0\} \) and \( N \cap A' = \{0\} \) since \( A \) and \( A' \) are anisotropic. Let \( B = \{b \in A | \exists a' \in A' \text{ such that } (b, a') \in N \} \). For fixed \( b, a' \) is unique for if \( (b, a'), (b, a'') \in N \), then \( (b, a') - (b, a'') = (0, a' - a'') \in N \cap A' = \{0\} \). It follows that \( a'' = a' \). Therefore \( B \cong N \). Similarly, let \( B' = \{b' \in A' | \exists a \in A \text{ such that } (a, b') \in N \} \). We obtain \( B' \cong N \). Now, we know \( B \subset A \), if we can prove \( B^\perp = \{0\} \) for \( B^\perp \) in \( A \), then \( B = A \). So we claim \( B^\perp = \{0\} \) in \( A \). Let \( b_\perp \in B^\perp \subset A \), \( (b_\perp, 0) \in A \oplus A' \). And let \( (b, a') \in N \). \( (\beta - \beta')((b, a'), (b_\perp, 0)) = \beta(b, b_\perp) = 0 \) for \( b \in B \), \( b_\perp \in B^\perp \). It follows that \( (b_\perp, 0) = N^\perp \), but \( N^\perp = N \) and therefore, \( (b_\perp, 0) \in N \). This shows that \( b_\perp \in B \) and that \( b_\perp \in B \cap B^\perp \). But \( A \cap A^\perp = \{0\} \) (since \( \beta(x, x) = 0 \) implies \( x = 0 \)) and \( B \subset A \) implies that \( B \cap B^\perp = \{0\} \). This proves the claim that \( b_\perp = 0 \), for any \( b_\perp \in B^\perp \).

Hence, \( B = A \); and \( B' = A' \) similarly. Therefore, \( N \) is the graph of an isomorphism \( f : A \rightarrow A' \), which satisfies \( \beta'(f(a), f(a_\perp)) = \beta(a, a_\perp) \) for \( a, a_\perp \in A \), and \( f(a), f(a_\perp) \in A' \).

Remark 3.7. We can also see that the split inner product torsion D-module is Witt equivalent to \( O \) from Lemma 3.3 since \( A^\perp = A \).
CHAPTER II
GENERALIZATION OF THE FUNDAMENTAL THEOREM
OF FINITELY GENERATED TORSION MODULES

In order to compute the Witt group \( W(T(F/D)) \) of \( F/D \),
where \( F \) is the quotient field of a Dedekind domain \( D \),
Professor Conner made the following conjecture:

Let \( M \) be a finitely generated torsion module over \( D \)
and \( D \) a Dedekind domain with quotient field \( F \). Suppose
\((M,\beta)\) is an anisotropic inner product module with the values
of \( \beta \) in \( F/D \). Then \( M \) can be split as an orthogonal direct
sum of submodules in a "nice" manner.

This chapter proves this conjecture, see Theorem (2.1),
even without the condition that \( M \) is an anisotropic inner
product module if it does not require orthogonal sum splitting
(see Theorem (2.2)).

The fundamental theorem of finitely generated torsion
modules is true over a principal ideal domain [21]. This
chapter is an analog of this theorem for a Dedekind domain.
§1. Some Lemmas

Let $D$ be a ring and $M$ a $D$-module, and let $P$ be an ideal in $D$ that is maximal among the family $\mathcal{F}$ of all annihilators of nonzero elements of $M$. Then by [12], $P$ is prime. We will call $P$ a prime of $D$ associated with $M$ as in the definition of [15]. To insure that the associated prime exists, we have the following statement [15]: If $D$ is a Noetherian ring and $M$ is a nontrivial $D$-module, there exists a prime ideal associated with $M$.

Lemma 1.1. [12] Let $D$ be a Noetherian ring, $M$ a finitely generated nonzero $D$-module, then there is only a finite number of maximal primes of $M$ in $\mathcal{F}$ and each is the annihilator of a non-zero element of $M$. Denote these associated primes by $P_1, \ldots, P_n$.

Lemma 1.2. [12] Let $N$ be a non-zero $D$-module, $I$ the annihilator of $N$, and $P$ a prime ideal in $D$, minimal over $I$. Then $P \subseteq Z(N) = \{r \in D | rm = 0 \text{ for some } 0 \neq m \in N\}$.

Theorem 1.3. Let $0 \neq m \in M$ and $M$ be a module over a Dedekind domain $D$, and $A = \text{Ann}(m) \in \mathcal{F} = \{\text{Ann}(m) | 0 \neq m \in M\}$, where $\text{Ann}$ denotes annihilator. Then $A = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_s^{\alpha_s}, \alpha_i$ a positive integer and $P_i$ an associated prime as in Lemma 1.1.

Proof. By Lemma 1.1, $A \subseteq P_i$ for some $i$. Now suppose
A \subseteq P_1, \ldots, \text{and } P_s. \text{ Let } Z(M) \text{ be the zero divisor of } M, \text{ where } Z(M) = \{ y \in D \mid ym = 0, 0 \neq m \in M \} = \cup \mathfrak{J} = P_1 \cup \cdots \cup P_n.

The last equality follows by Lemma 1.1. Since \( A \) is an ideal of \( D \) and \( D \) is a Dedekind domain, \( A \) is the product of prime ideals.

Let \( A = Q_1 \cdot Q_2 \cdots Q_k \). Each \( Q_j \) must be a prime ideal in \( \mathfrak{J} \) for the following reason: Let \( N = \langle m \rangle \), \( I = \text{Ann}(N) = \text{Ann}(m) = A \subseteq Q_j \). It is clear that \( Q_j \) is a prime ideal minimal over \( A \). Then by Lemma 1.2,

\[ Q_j \subseteq Z(N) = Z(\langle m \rangle) \subseteq Z(M) = P_1 \cup P_2 \cdots \cup P_n. \]

Therefore by [17], \( Q_j \subseteq P_i \) for some \( i = 1, 2, \ldots, n \).

\( P_1, \ldots, P_n \) are distinct maximal prime ideals, so we have \( Q_j = P_i \).

Lemma 1.4. Let \( M(P) = \{ x \in M \mid Px = 0 \} \), \( P \) an ideal of \( D \).

Let \( M \) be an anisotropic inner product module with the values of \( \beta \) in \( F/D \). Then \( M(P) = M(P^\alpha) \) for any positive integer \( \alpha \).

Proof. \( Px = 0 \) implies \( P^\alpha x = 0 \). Therefore \( M(P) \subseteq M(P^\alpha) \).

Now let \( x \in M(P^\alpha) \). Take \( k = 1 \) if \( \alpha \) is odd, \( k = 0 \) if \( \alpha \) is even, and we obtain

\[ \beta(P^{\frac{\alpha+k}{2}} x, P^{\frac{\alpha+k}{2}} x) = \beta(P^\alpha x, P^k x) = \alpha(0, P^k x) = 0. \]

Since \( (M, \beta) \) is anisotropic, \( P^{\frac{\alpha+k}{2}} x = 0 \). Since \( \frac{\alpha+k}{2} < \alpha \),
by induction we obtain $Px = 0$. Therefore $x \in M(P)$, proving $M(P^\alpha) \subseteq M(P)$. Hence $M(P) = M(P^\alpha)$ for any positive integer $\alpha$.

§2. Fundamental Theorems

**Theorem 2.1.** (Fundamental theorem of anisotropic finitely generated torsion modules) Let $M$ be a finitely generated torsion $D$-module, where $D$ is a Dedekind domain and $F$ is the quotient field of $D$. Suppose $M$ is an anisotropic inner product module with the values of $\phi$ in $F/D$. Then

$$M \cong M(P_1) \oplus M(P_2) \oplus \cdots \oplus M(P_n),$$

an orthogonal sum, where the $P_j$'s are the primes of $D$ associated with $M$.

**Proof.** Let $\mathfrak{J}$ be the family of all annihilators of nonzero elements of $M$. By Lemma 1.1, there is only a finite number of associated primes of $M$ in $\mathfrak{J}$, say, $P_1, \ldots, P_n$.

**Claim 1.** $M(P_i) \cap [M(P_1) + \cdots + M(P_{i-1}) + M(P_{i+1}) + \cdots + M(P_n)] = \{0\}$.

**Proof of Claim 1.** Let $0 \neq x \in M$, $P_1x = 0$ and

$$x = x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n,$$

where $P_jx_j = 0$ for all $j \neq i$, $j = 1, 2, \cdots, i-1, i+1, \cdots, n$. In particular,

$$P_1P_2 \cdots P_{i-1}P_{i+1} \cdots P_nx = 0.$$ Since the $P_i$'s are prime ideals in a Dedekind domain, the $P_i$'s are maximal and pairwise
comaximal. By [25], $P_i + P_1P_2 \cdots P_{i-1}P_{i+1} \cdots P_n = D$ for $i = 1, 2, \ldots, n$. Now

$$(P_i + P_1 \cdots P_{i-1}P_{i+1} \cdots P_n)x = P_i x + P_1P_2 \cdots P_{i-1}P_{i+1} \cdots P_n x = 0$$

implies $lx = 0$, and therefore $x = 0$. This proves claim 1.

Now let $0 \neq x \in M$ and $A = \text{Ann}(x) \in \mathfrak{A}$. By Theorem 1.3, $A = p^1 \cdots p_s$, $a_i$ positive integer. Let

$$J_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_s, i = 1, 2, \ldots, s.$$

**Claim 2.** $J_1 + J_2^+ \cdots + J_s = D$.

**Proof of Claim 2.** If $J_i \subset Q$, $Q$ a prime ideal of $D$, then

$$J_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_s \subset Q$$

implies $P_j \subset Q$ for some $j \neq i$. It follows that $P_j = Q$ for $P_j$ and $Q$ are maximal primes. Therefore

$$J_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_s \subset P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_s,$$

and $J_i$ is not contained in any other maximal prime ideals $\not\subset D$. From this, we have $J_i \notin P_1$, and is not contained in any other maximal prime ideals $\not\subset D$. So

$$J_1 + J_2 + \cdots + J_s \notin P_1, \ldots, P_s$$

or any other maximal prime ideals $\not\subset D$. But $J_1 + J_2^+ \cdots + J_s \not\subset \{0\}$, the above statements imply $J_1 + J_2^+ \cdots + J_s = D$. This proves Claim 2.
Claim 3. \( M \cong M(P_1) + M(P_2) + \cdots + M(P_n) \), an orthogonal sum:

Proof of Claim 3. By Claim 2, \( 1 = d_1 + d_2 + \cdots + d_s \) for some elements \( d_1 \in P_1 \cdots P_{i-1} P_{i+1} \cdots P_s \). Therefore \( x = d_1 x + \cdots + d_s x \), with \( P_i(d_i x) = P_1 \cdots P_s x = Ax = 0 \), so that \( x \in M(P_1) + M(P_2) + \cdots + M(P_s) = M(P_1) + \cdots + M(P_s) \).

The last equality follows from Lemma 1.4. Further,
\[
(M(P_1), \sigma | M(P_1)) = (M(P_1), \sigma_1)
\]

is an inner product since
\[
\sum_{i=1}^{n} \sigma_i M(P_i) \cong \text{Hom}_D\left( \sum_{i=1}^{n} M(P_i), \sigma \right) \cong \prod_{i=1}^{n} \text{Hom}_D(M(P_i), \sigma_1) \cong \\
\sum_{i=1}^{n} \text{Hom}_D(M(P_i), \sigma_1).
\]

This completes the proof of Theorem 2.1.

Theorem 2.2. (Fundamental theorem of finitely generated torsion modules over a Dedekind domain) Let \( M \) be a finitely generated torsion \( D \)-module, where \( D \) is a Dedekind domain and \( \tilde{M}(P) = \{ x \in M | P^m x = 0 \text{ for some positive integer } m \} \), \( P \) a prime ideal in \( D \). Then

\[
M \cong \tilde{M}(P_1) \oplus \tilde{M}(P_2) \oplus \cdots \oplus \tilde{M}(P_n).
\]

Proof. This proof is similar to that of Theorem 2.1. \( P_1, P_2, \ldots, P_n \) are pairwise comaximal and we get that \( \gamma_1, \ldots, \gamma_n \) are pairwise comaximal by [25]. Hence \( \gamma_1 P_1 + \gamma_1 P_1 \cdots P_{i-1} P_{i+1} \cdots P_n = D \), \( \gamma_j \) positive integer.
So \( \tilde{M}(P_i) \cap [\tilde{M}(P_i) + \cdots + \tilde{M}(P_{i-1}) + \tilde{M}(P_{i+1}) + \cdots + \tilde{M}(P_n)] = \{0\} \).

If \( A = \text{Ann}(x) \), then \( A = P_1^\alpha_1 \cdots P_s^\alpha_s \) implies
\( J_1 + J_2 + \cdots + J_n = D \), and hence \( M \simeq \tilde{M}(P_1) \oplus \cdots \oplus \tilde{M}(P_n) \).

Remarks 2.3. (1) If we take \( A = \text{Ann}(M) \subset \text{Ann}(x) \), then
\[ A = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_s^{\alpha_s} \]
and we still have the same result:
\[ M \simeq \tilde{M}(P_1) \oplus \cdots \oplus \tilde{M}(P_n) \, . \]

(2) Note that the decomposition of Theorems 2.1 and 2.2 is unique up to order.

(3) If we add the condition that \((M, \beta)\) is an anisotropic inner product module to Theorem 2.2, then we have
Theorem 2.1 for \( \tilde{M}(P_1) = \bigcup_{Y_1=1}^{\infty} M(P_{Y_1}) = \bigcup_{Y_1=1}^{\infty} M(P_1) = M(P_1) \).

§3. Localization

We will have a short digression here about local rings.

Lemma 3.1. Let \( R \) be a local ring, \( M \) a finitely generated torsion \( R \)-module, and \((M, \beta)\) an anisotropic module. If the maximal ideal \( I \) of \( R \) is principal, say \( I = (q) \), then \( qM = 0 \).

Proof. Since \( RM = 0 \), and \( M = \langle m_1, \ldots, m_n \rangle \), we can assume \( d_i m_i = 0 \) for all \( i=1, \ldots, n \), where \( d_i \in R \), and is not a unit. Otherwise, \( d_i^{-1} d_i m_i = d_i^{-1} 0 = 0 \) implies
$m_i = 0$, a contradiction. Therefore, $d_i \in I$ for all $i$ and $d_i = q^{k_i}$ for some integer $k_i > 0$. Hence there exists $k > 0$, $k = k_1 k_2 \cdots k_n$ such that $q^{k}M = 0$. This shows that $q^{k}m = 0$ for all $m \in M$. Since

$$\beta(q^{k-1}m, q^{k-1}m) = \beta(q^{k-1}m, q^{k-2}m)$$

$$= \beta(q^{k}m, q^{k-2}m) = \beta(0, q^{k}m) = 0$$

and $M$ is anisotropic, we obtain $q^{k-1}m = 0$. Similarly, $q^{k-2}m = 0, \ldots, q^{2}m = 0, q^{m} = 0$ for all $m \in M$. Hence $q^{m}M = 0$.

**Remark 3.2.** Let $N$ be a submodule of $M$, where $M$ is a $D$-module. Suppose $S$ is a multiplicative set for $D$. Then $S^{-1}(M/N) \cong S^{-1}/S^{-1}N$, where $S^{-1}M = \{m/s|m \in M, s \in S\}$. If $P$ is a prime ideal of $D$, and $S = D-P$, we denote $S^{-1}D$ by $D_P$ as a local ring of $D$ at $P$.

Let $(M, \beta)$ be an inner product torsion module with values in $K = F/D$. Let $P$ be a prime ideal of $D$. Consider $M \otimes D_P$, where $D_P$ is the local ring of $D$ at $P$. Let $m \otimes a, n \otimes r \in M \otimes D_P$. Define $\beta'(m \otimes a, n \otimes r) = \beta(m, n) \otimes ar \in K \otimes D_P$, where $K \otimes D_P = (F/D)_P = F/D_P$. $D_P$ is torsion free since we have the short exact sequence

$$0 \to D \to F \to F/D \to 0$$

and since $D_P$ is a submodule of $F$, which is a torsion free module. By [2], p. 130, we have the sequence

$$0 \to D \otimes D_P \to F \otimes D_P \to K \otimes D_P \to 0$$
is exact. Hence, the sequence
\[ 0 \rightarrow D_P \rightarrow F \otimes D_P \rightarrow F/D_P \rightarrow 0 \]
is exact.

**Lemma 3.3.** If \( \varphi ' \in \text{Hom}(M \otimes D_P, F/D_P) \) is defined by \( \varphi'(m \otimes \alpha) = \varphi(m) \otimes \alpha \), for \( \varphi \in \text{Hom}(M, F/D) \), then \( (M \otimes D_P, \beta ') \) is an inner product module with values of \( \beta ' \) in \( F/D_P \).

**Proof.** What we need to prove is the isomorphism
\[ M \otimes D_P \cong \text{Hom}(M \otimes D_P, F/D_P), \]
where \( F/D_P = F/D \otimes D_P \). Define \( f \)
\[ f : m \otimes \alpha \mapsto \varphi'_m \otimes \alpha \in \text{Hom}(M \otimes D_P, F/D_P) \]
where \( \varphi'_m (n \otimes r) = \beta'(m \otimes \alpha, n \otimes r) \). We claim \( f \) is well defined for if \( m \otimes \alpha = m_1 \otimes \alpha_1 \) and \( \varphi'_m \neq \varphi'_{m_1} \otimes \alpha_1 \), then there exists \( n \otimes r \in M \otimes D_P \) such that \( \varphi'_m (n \otimes r) \neq \varphi'_{m_1} \otimes \alpha_1 (n \otimes r) \). Thus, \( \beta'(m \otimes \alpha, n \otimes r) \neq \beta'(m_1 \otimes \alpha, n \otimes r) \), but this is impossible. We also claim \( f \) is one to one for if \( \varphi'_m (n \otimes r) = 0 \) for all \( n \otimes r \in M \otimes D_P \), then \( \beta'(m \otimes \alpha, n \otimes r) = 0 \). It follows that \( \beta(m, n) \otimes \alpha r = 0 \) for all \( r \in D_P \) and that \( \beta(m, n) = 0 \) for all \( n \in M \). Hence, \( m = 0 \) and \( m \otimes \alpha = 0 \). Finally, we claim \( f \) is onto for let \( \varphi ' \in \text{Hom}(M \otimes D_P, F/D_P) \), we have \( \varphi'(n \otimes r) \in F/D \otimes D_P \). Since
\[ \varphi'(n \otimes r) = \varphi_m(n) \otimes r = \varphi(m, n) \otimes r = \varphi'(m \otimes l, n \otimes r). \]

It shows that \( \varphi' = \varphi'_{m \otimes l} \). Therefore, \( f \) is onto.

**Lemma 3.4.** If \( M \) is a finitely generated torsion \( D \)-module, then \( M \otimes_{D} D' \) is a finitely generated torsion \( D' \)-module.

**Proof.** Let \( M = \langle m_1, \ldots, m_n \rangle \) be a \( D \)-module. We have
\[
M \otimes_{D} D' = \langle m_1 \otimes 1, m_2 \otimes 1, \ldots, m_n \otimes 1 \rangle \quad \text{as a} \quad D' \text{-module. Also,} \quad d(m) = 0 \quad \text{implies} \quad d(m \otimes \alpha) = 0 \quad \text{for any} \quad \alpha \in D'.
\]

**Theorem 3.5.** Let \( M \) be a finitely generated torsion \( D \)-module, and let \( (M \otimes_{D} D', \beta') \) be an anisotropic inner product module, where \( D' \) is a local ring of \( D \) at \( P \). If \( I \) is the unique maximal ideal of \( D' \), then \( M \otimes_{D} D' \) is a finite dimensional vector space over \( D'/I \).

**Proof.** \( D' \) is a discrete valuation ring by [25], p. 278.

So \( I = (q) \) for some uniformizing parameter \( q \). By Lemma 3.1 and Lemma 3.4, we have \( q(M \otimes_{D} D') = 0 \). It follows that
\[
I(M \otimes_{D} D') = 0, \quad \text{where} \quad M \otimes_{D} D' \quad \text{is a} \quad D' \text{-module. Therefore,}
\]
\[
M \otimes_{D} D' \quad \text{is a vector space over the field} \quad D'/I. \quad \text{Now,}
\]
\[
M \otimes_{D} D' = \langle m_1 \otimes 1, m_2 \otimes 1, \ldots, m_n \otimes 1 \rangle, \quad \text{so} \quad M \otimes_{D} D' \quad \text{is a finite dimensional vector space over} \quad D'/I.
\]
CHAPTER III
AN ISOMORPHISM AND AN EXACT SEQUENCE

§1. The Isomorphism \( W(T(F/D)) \cong \sum_P W(D/P) \).

Lemma 1.1. Let \( D \) be a Dedekind domain with quotient field \( F \), and let \( P \) be a prime ideal of \( D \). Then \( H(P) = \{ x \in F/D | P x = 0 \} \) is isomorphic to \( D/P \), which is a field.

Proof. Let \( P^{-1} = \{ x \in F | x P \subseteq D \} \) be the fractional ideal of \( D \). \( PP^{-1} = D \) since every proper prime ideal in a Dedekind domain is invertible, see [25], p. 273 and since \( P^{-1} \) is a \( D \)-submodule of \( F \). It is clear that \( H(P) = P^{-1}/D \). We will prove that \( P^{-1}/D \cong D/P \). First, claim there is no submodule of \( F \) strictly between \( D \) and \( P^{-1} \). Suppose there exists a submodule, say \( Q \), such that \( D \subseteq Q \not\subseteq P^{-1} \subseteq F \). Let \( Q^{-1} = \{ x \in F | x Q \subseteq D \} \). Clearly, \( D^{-1} \supset Q^{-1} \supset (P^{-1})^{-1} \) implies that \( D \not\supset Q^{-1} \not\subseteq P \), the inequalities are strict. Otherwise, if we take fractional ideal of \( D \supseteq Q^{-1} \supseteq P \), we get a contradiction. Now there is no \( Q^{-1} \) strictly between \( D \) and \( P \) since \( P \) is a maximal ideal. So \( D = Q^{-1} \) or \( P = Q^{-1} \). It follows
that $D^{-1} = (Q^{-1})^{-1}$ or $(Q^{-1})^{-1} = P^{-1}$. Hence, $D = Q$ or $Q = P^{-1}$. This shows the first claim.

Before we continue to the second claim, we will define a homomorphism. Let $g \in P^{-1} - D$. We obtain $D \subset <D, g> \subset P^{-1}$; but since there is no submodule between $D$ and $P^{-1}$, $<D, g> = P^{-1}$. Define a mapping $\varphi$ as follows:

\[
\begin{array}{ccc}
D & \longrightarrow & P^{-1} \\
\varphi & \longrightarrow & P^{-1}/D \\
d & \longmapsto & dg + D
\end{array}
\]

$\varphi$ is well defined from the definition. $\varphi$ is a $D$-homomorphism since $\varphi(rd) = rdg + D = rdg + r(r^{-1}D) = r(dg + r^{-1}D) = r\varphi(D)$, and $\varphi(d_1 + d_2) = (d_1 + d_2)g + D = \varphi(d_1) + \varphi(d_2)$.

$\varphi$ is onto for if we suppose $d_1 + dg \in <D, g> = P^{-1}$, then $\varphi(d) = dg + D$ followed by the equality $dg + d_1 + D = dg + D$.

Now we claim $\ker \varphi = P$. $\varphi(P) = Pg + D \subset PP^{-1} + D = D$. It follows that $P \subseteq \ker \varphi$. Since $g \notin D$ and $\varphi(1) = g + D \notin D$, $1 \notin \ker \varphi$, we get $P \subseteq \ker \varphi \neq D$. But, $P$ is a maximal ideal of $D$, therefore, $P = \ker \varphi$. This ends the proof of Lemma 1.1 and we have

$D/P \cong P^{-1}/D \cong H(P)$.

Remarks 1.2. (1) If $D$ is a PID, then $H(P) \cong P^{-1}/D$ is trivial because the homomorphism $\psi$
has kernel $P = \langle p \rangle$. Hence $H(P) \cong D/P$.

(2) If $D = \mathbb{Q}$, the rational numbers, then

$$H(P) = \left\{ \frac{0}{p}, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p} \right\} \cong \mathbb{Z}/\langle p \rangle = \mathbb{Z}_p.$$

(3) For the case of local ring $D_p$,

$$\{ x \in F/D_p \mid \pi x = 0 \} = H(\pi) \subset F/D_p.$$

$D_p$ is a discrete valuation ring and a local ring with the unique maximal ideal $\pi D_p$, which is a PID. Hence,

$$D_p/\pi D_p = D_p/\langle \pi \rangle \cong H(\pi) \subset F/D_p,$$

for

$$D_p \twoheadrightarrow F \twoheadrightarrow H(\pi) \subset F/D_p$$

and $\ker \phi = \langle \pi \rangle = \pi D_p$.

Theorem 1.3. The following is an isomorphism

$$W(T(F/D)) \cong \sum_{P} W(D/P)$$

where $P$ ranges over all the prime ideals of $D$.

Proof. Let $(M, \beta)$ be a finitely generated torsion inner product $D$-module with $\beta$-values in $F/D$. Suppose $(M, \beta)$ is anisotropic.
By Chapter II, Theorem 2.1, we have

$$M \cong M(P_1) \oplus M(P_2) \oplus \cdots \oplus M(P_n).$$

Since $P_1 M(P_1) = 0$, $M(P_1)$ is a finite dimensional vector space over $D/P_1$ with $\beta$-values in $H(P_1)$, which is the set of all those elements in $F/D$ killed by $P$. By Lemma 1.1, we have $H(P_1) \cong D/P_1$. It follows that $(M(P_1), \beta_1)$ is a finite dimensional inner product space over $D/P_1$ with $\beta_1$-values in $D/P_1$, where $\beta_1 = \beta|_{M(P_1)}$. Finally, since $<M> \in W(T(F/D))$ and $<M(P_1)> \in W(D/P_1)$, we obtain the isomorphism

$$W(T(F/D)) \cong \bigoplus_{P} W(D/P).$$

§2. The Exact Sequence $0 \to W(D) \to W(F) \to W(T(F/D)) \to \frac{C}{C^2} \to 0$.

In this section, we will define $\alpha$,

$$\alpha : W(F) \to W(T(F/D)),$$

where $F$ is the quotient field of $D$.

Let $(M, \beta)$ be an inner product space over $F$. We can form a $D$-lattice $L$ in $M$. A $D$-lattice $L$ is a finitely generated $D$-submodule of $M$ such that

$$L = Dx_1 + Dx_2 + \cdots + Dx_k \subseteq M$$

spanned by $x_1, \cdots, x_k$, where the set \{x_1, \cdots, x_k\} contains a basis for $M$ over $F$, see [19], p. 91.
Let \( L^\# \) be the dual lattice, a \( D \)-lattice,

\[
L^\# = \{ r \mid r \in M, \beta (r, L) \subseteq D \}.
\]

As we know, \( M \supset L^\# \supset L \), and \( L^\# \cong \text{Hom}_D(L, D) \). Define \( \alpha : \langle M \rangle \mapsto \langle L^\#/L \rangle \) by the following manner. The exact sequence

\[
0 \to L \to \text{Hom}_D(L, D) \to \frac{\text{Hom}_D(L, D)}{L} \to 0
\]

implies another exact sequence

\[
0 \to L \to L^\# \to L^\#/L \to 0,
\]

where \( L^\# \cong \text{Hom}_D(L, D) \) because every \( D \)-linear mapping \( L \to D \) extends linearly and uniquely to a \( F \)-linear mapping \( M \to F \) having the form \( x \mapsto \beta(x, x_0) \) for a unique \( x_0 \in L^\# \). Let \( \beta' \) be the linear symmetric form induced by \( \beta \) on \( L^\#/L \). We claim that \( (L^\#/L, \beta') \) is an inner product torsion \( D \)-module with \( \beta' \)-values in \( F/D \). To see the values of \( \beta' \) are in \( F/D \), let \( w^#, w_0^# \in L^\# \), and let \( w, w_0 \in L \). Then

\[
\beta(w^# + w, w_0^# + w_0) = \beta(w^#, w_0^#) + \beta(w^#, w_0) + \beta(w, w_0^#) + \beta(w, w_0),
\]

the first term is in \( F \); however, the other terms are each in \( D \). Thus \( \beta'(w^#, w_0^#) \in F/D \). That \( L^\#/L \) is a torsion \( D \)-module is clear by the definition of \( L^\# \). To see that \( \beta' \) is an inner product, we need to prove \( L^\#/L \cong \text{Hom}_D(L^\#/L, F/D) \). For any \( \varphi \in \text{Hom}(L^\#, F) \) with \( \varphi(L) \subseteq D \), there exists
$w \in M \Rightarrow \varphi = \varphi_w$. \quad \varphi_w(w^*) = \beta(w,w^*) \in F$ and $\beta(w,L) \subset D$.

It follows that $w \in L^\#$. If we let $w = w^*_0$, then, for any $\varphi \in \text{Hom}(L^\#,F)$, we have $\varphi = \varphi_{w^*_0}$ and this induces $\varphi_{w^*_0} \in \text{Hom}(L^\# / L,F/D)$. Now, we can define $f: L^\# \to \text{Hom}(L^\# / L,F/D)$ by $f(w^*) = \varphi_{w^*}$. If $w^* \in L$, then it is clear that $\varphi_{w^*} = 0$, hence, $w^* \in \ker f$. If $w^* \in \ker f$, then $\varphi_{w^*} = 0$. This implies that $\varphi_{w^*_0}(w^*_0) = 0$ for all $w^*_0 \in L^\# / L$. So $\beta(w^*_0,w^*_0) \subset D$ for all $w^*_0 \in L^\#$ and this implies that $w \in L$. Therefore $L^\# / L \cong \text{Hom}(L^\# / L,F/D)$.

**Lemma 2.1.** The mapping $\varphi$ defined above is a homomorphism.

**Proof.** $\varphi$ is well defined: $<M> \sim 0$ implies $<L> \sim 0$. Then there exists a splitter $L_1 \subset L$ with $L_1 = L_1^\perp$ in $L$. Consider $L_1 \subset L \subset L^\#$. Define $\tilde{N} \supset L_1$ as $\tilde{N} = \{n \in L^\# | \beta(n,L_1) = 0\}$.

Now $\tilde{N}$ is a $D$-submodule of $L^\#$ and $\tilde{N} = \tilde{N}^\perp$ in $L^\#$. If $N \subset L^\# / L$ denotes the quotient image of $\tilde{N}$, where $\tilde{N} \subset L^\#$ and $\tilde{N} \to \tilde{N}/L = N$, then it is clear that $N \subset N^\perp$ for $\beta(n,\tilde{N}) = 0 \in D$ where $n \in \tilde{N} = \tilde{N}^\perp$. We need to show $N^\perp \subset N$.

For this purpose, suppose $v \in L^\#$ such that $\beta(v,x) \in D$ for all $x \in \tilde{N} = \tilde{N}^\perp$. Since $\tilde{N}$ is a submodule of $L^\#$, there exists $\varphi_v: L^\# \to D$ such that $\varphi_v(x) = \beta(v,x)$ for all $x \in \tilde{N}^\perp$. However, $(L^\#)^\# = L$ implies that there exists $w \in L$ with $\varphi_w(x) = \varphi_v(x)$ and $\beta(v-w,x) = 0$ for all $x \in \tilde{N}^\perp$. And it
follows that \( v \cdot w \in \widetilde{N} \) and \( v \rightarrow v + L \in N \). Therefore, \( \langle L^\# / L \rangle = 0 \). Thus \( a \) is well defined.

Finally, \( a(<M_1> \otimes <M_2>) = a(<M_1 \otimes M_2>) \)

\[
\frac{(L_1 \oplus L_2)^\#}{L_1 \oplus L_2} = \frac{L_1^\# \times L_2^\#}{L_1 \oplus L_2},
\]

the last equality is by [15], p. 82. Furthermore

\[
\frac{L_1^\# \times L_2^\#}{L_1 \oplus L_2} = \langle \frac{L_1^\# \times 0 \oplus 0 \times L_2^\#}{L_1 \times 0 \oplus 0 \times L_2} \rangle
\]

\[
= \langle \frac{L_1^\# / L_1 \oplus L_2^\# / L_2} \rangle = \langle \frac{L_1^\# / L} \rangle + \langle \frac{L_2^\# / L} \rangle.
\]

**Theorem 2.2**. The sequence \( 0 \rightarrow W(D) \xrightarrow{i} W(F) \xrightarrow{\delta} W(T(F/D)) \) is exact.

**Proof**. The sequence \( 0 \rightarrow W(D) \rightarrow W(F) \) is exact by [19], p. 93. Let \( L \) be a D-module with \( L = L^\# \). Therefore, in the following composition

\[
L \xrightarrow{i} L \otimes F \xrightarrow{\delta} L^\# / L,
\]

\( \delta \cdot i = 0 \), since \( L^\# / L = 0 \), where \( i \) is an inclusion. Suppose \( <M> \in \ker(\delta) \), then \( L \subseteq L^\# \subseteq M \), that is, \( <L^\# / L> = 0 \), where \( L \) is a D-lattice, it follows that there exists a D-submodule \( N \subseteq L^\# / L \) with \( N^\perp = N \). Let \( \widetilde{N} = \delta^{-1}(N) \) and we know \( L \subseteq \widetilde{N} \subseteq L^\# \). We claim that \( \widetilde{N} = \widetilde{N}^\# \). That \( N \subseteq \widetilde{N} \subseteq L^\# \) is clear by the definition of \( L^\# \). If \( v \in \widetilde{N}^\# \), then \( v \in L^\# \) and \( v \rightarrow v + L = \overline{v} \in \widetilde{N}^\# / L = N^\perp = N \). Hence, \( \overline{v} \in N \). So there
is a $w \in \tilde{N}$ with $v-w \in L$ and it implies that $v \in \tilde{N}$.

Finally, since $\tilde{N}$ is a D-module, $<M> = i(<\tilde{N}>)$. This shows $\ker (a) = \text{im} (i)$.

**Theorem 2.3.** The sequence

$$0 \to W(D) \to W(F) \to \sum_{P} W(D/P)$$

is exact.

**Proof.** Theorem 1.3 says that

$$W(T(F/D)) \cong \sum_{P} W(D/P).$$

Theorem 2.2 states that the following sequence is exact,

$$0 \to W(D) \to W(F) \to W(T(F/D)).$$

Therefore, $0 \to W(D) \to W(F) \to \sum_{P} W(D/P)$ is exact. This is an exact sequence by Knebusch and Scharlau [14], (compare with [9]).

Up to this point, we have shown a new approach to the above well known exact sequence.

**Corollary 2.4.** The sequence

$$0 \to W(D) \to W(F) \to W(T(F/D)) \to \frac{C}{C^{2}} \to 0$$

is exact, where $C$ is the ideal class group of a Dedekind domain $D$ with quotient field $F$, and $C^{2}$ is the subgroup of
Proof. From [19], p. 94, we have the sequence

\[ 0 \to W(D) \to W(F) \to \sum_{P} W(D/P) \to \frac{C}{C^2} \to 0 \]

exact. From Theorem 1.3, we have the isomorphism

\[ \sum_{P} W(D/P) \cong W(T(F/D)) \]

Hence, the sequence \[ 0 \to W(D) \to W(F) \to W(T(F/D)) \to \frac{C}{C^2} \to 0 \] is exact.

Remark 2.5. As [18], p. 14 indicates that \( C(D) \), the ideal class group, is isomorphic naturally to the Picard group \( Pic(D) \), the set of isomorphism classes of invertible modules under tensor product.
CHAPTER IV
APPLICATIONS AND GROUP ACTION

In this chapter, we equip every module with a group action $G$ as a $D$-linear automorphism and modify definitions, lemmas and theorems from the previous chapters.


As before, $D$ indicates a Dedekind domain. We consider finitely generated inner product $D$-modules $(G;M)$, where $G$ is a $D$-linear group action on $M$ (i.e., $g(dx) = d(gx)$ for all $g \in G$, $d \in D$ and $x \in M$) acting as a group of $D$-module isometries, that is, $\beta(gx,gy) = \beta(x,y)$ for all $g \in G$ and all $x,y \in M$. Furthermore, the values of inner product are in $F/D$, where $F$ is as usual the quotient field of a Dedekind domain $D$. Now $(G;M) \sim 0$ if, and only if, there is a $G$-invariant submodule $N \subset M$ for which $N = N^\perp$, the orthogonal complement. We will call $(G;M)$ split if $(G;M) \sim 0$. Furthermore, $(G;M) \sim (G;M')$ if, and only if, $(G;M) \oplus (G;-M') \sim 0$. The Witt class of $(G;M)$ is denoted by $\langle G;M \rangle$ and the collection of all such classes by $\text{W}(G;\text{T}(F/D))$ for torsion modules.
Lemma 1.1. Suppose $N$ is an invariant submodule of finitely generated inner product torsion $D$-module $(G;M)$ whose values are in $F/D$, then $N^\perp$ is also invariant and $(N^\perp)^\perp = N$.

Proof. Let $x \in N^\perp$. Let $g$ be any element in $G$. If $\beta(x,y) = 0$ for all $y \in N$, then $\beta(gx,y) = \beta(gx,g^{-1}y) = \beta(x,g^{-1}y) = 0$ since $g^{-1}y \in N$. Hence, $gx \in N^\perp$, $N^\perp$ is invariant. The proof of $(N^\perp)^\perp = N$ is similar to that of Lemma 2.1 in Chapter I, because $N$ and $N^\perp$ are invariant submodules.

Lemma 1.2. Let $(G;M)$ be a torsion module with group action $G$, and let $(G;S)$ be a split torsion module for which $(G;M\otimes S)$ is split, where the action $G$ on $M\otimes S$ is a diagonal action, i.e., $g(m,s) = (gm,gs)$ for $g \in G$, $(m,s) \in M\otimes S$, then $(G;M)$ is split.

Proof. The proof is similar to that of Lemma 2.6, Chapter I, using last lemma and Lemma 2.5 of Chapter I. Therefore, we have an analog of Lemma 2.7 of Chapter I.

This will prove the transitivity of Witt equivalence with the group action. Hence, the Witt equivalence with the group action is an equivalence relation.

Lemma 1.3. The following are equivalent:

1. If $(G;M)$ is torsion module, and $(G;S)$ is a split torsion module for which $(G;M\otimes S)$ is split, then $(G;M)$ is
split.

(2) If \((G;M_1) \oplus (G;-M_2) \sim 0\) and \((G;M_2) \oplus (G;-M_3) \sim 0\), then \((G;M_1) \oplus (G;-M_3) \sim 0\).

For the case of finitely generated projective modules with group action, stably split implies split is also true as remarked in 2.9 and 2.10 of Chapter I.

**Lemma 1.4.** If \(N \subset M\) is an invariant submodule with \(N \subset N^1\), then \(<G;M> = <G;N^1/N>\).

**Proof.** Lemma 3.2 of Chapter I shows that \(N^1/N\) has a non-singular inner product structure. We know the action \(G\) on \(N^1/N\) is the induced action, i.e., \((g,x+N) = gx + N\) since \(N\) is invariant. Let \(v:N^1 \rightarrow N^1/N\) be a quotient homomorphism. This is an equivariant isometry. The rest of the proof is similar to that of Lemma 3.5, Chapter I.

**Definition 1.5.** The inner product module \((G;M)\) is \(G\)-anisotropic if, and only if, for every invariant submodule, \(N \subset M\) with

\[
\text{rad (N) = N\cap N^1 = \{0\} .}
\]

**Theorem 1.6.** Every element in \(W(G;T(F/D))\) has a \(G\)-anisotropic representative which is unique up to equivariant isometry.

**Proof.** The repeated application of Lemma 1.4 proves the existence of the required representative.
The proof of the uniqueness is similar to that of Theorem 3.6, Chapter I. We only need to observe that the G-anisotropic means that for any G-invariant submodule $A$, $\text{rad}(A) = A\cap A^1 = \{0\}$. Finally, $N$ is the graph of an equivariant isomorphism $f:A \to A'$, which satisfies $\beta'(f(a), f(a')) = \beta(a, a')$.

§2. Localization and Group Action.

**Lemma 2.1.** Let $(G; M)$ be a finitely generated $G$-anisotropic torsion $R$-module, where $R$ is a local ring. If the maximal ideal $I$ of $R$ is principal, say $I = (q)$, then $qM = 0$.

**Proof.** As in the proof of 3.1, Chapter II, we have $q_iM = 0$. Suppose there is a $m \in M$ such that $q^{-1}m \neq 0$. Let

$A = \langle q^{-1}m \rangle = Rq^{-1}m$ be $G$-invariant. $\beta(dq^{-1}m, q^{-1}m) = \beta(dq^{-2}m, q^{-1}m) = 0$, where $d \in R$. It follows that $dq^{-1}m \in A^1$.

In particular, $q^{-1}m \in A\cap A^1 = \{0\}$, because $(G; M)$ is anisotropic. Therefore, $q^{-1}m = 0$. Similarly, $q^{-2}m = \cdots = qm = 0$ for all $m \in M$.

Let $D_P$ be the local ring of $D$ at a prime ideal $P$.

Let $G$ be the group action on $M$. If we define $G$ acting on $M \otimes D_P$ by $g(m, a) = (gm, a)$, where $g \in G$, $m \in M$ and $a \in D_P$, then we have a result similar to Theorem 3.5 of Chapter II.

**Theorem 2.2.** Let $(G; M)$ be a finitely generated torsion
D-module with inner product $\iota$. Suppose $(M \otimes_{D_P} \beta')$ is a $G$-anisotropic D-module, where $D_P$ is a local ring of a Dedekind domain $D$ at $P$. If $I$ is a unique maximal ideal of $D_P$, then $(G; M \otimes_{D_P} D)$ is a finite dimensional vector space over $D_P/I$ with group action $G$.

§3. The Isomorphism and the Exact Sequence with Group Action.

Recall Theorem 2.1, Chapter II, if $M$ is a finitely generated torsion D-module and if $(M, \beta)$ is an anisotropic inner product module with values of $\beta$ in $F/D$, then

$$M \cong M(P_1) \oplus M(P_2) \oplus \cdots \oplus M(P_n),$$

where $M(P_i) = \{ x \in M | P_i x = 0 \}$.

Now if we put a group action $G$ on $M$, then $M(P_1)$ is an invariant submodule since $P_i(gx) = g(P_i x) = 0$ for all $g \in G$, $x \in M(P_1)$. Hence, $(G; M) \cong \bigoplus_{P_i} (G; M(P_i))$, where $P_i$ runs a certain finite number of $P_1$'s.

Theorem 3.1. We have the following isomorphism

$$W(G; T(F/D)) \cong \bigoplus_P W(G; D/P),$$

where $D$, $P$ and $F$ are as usual, and $G$ is a group action.

Proof. This is analogous to Theorem 1.3, Chapter III.

To define the boundary homomorphism $\partial$, 

\[ \alpha : W(G; F) \to W(G; T(F/D)) , \]

we have the same procedure as §2, Chapter III. We only need observe that if \((G; L) \sim 0\), then we require the submodule \(L_1 \subseteq L\) with \(L_1^T = L_1\) be invariant under the group action. Therefore, we have an analog of Theorems 2.2 and 2.3, Chapter III, as following theorem.

**Theorem 3.2.** The following two sequences are exact:

\[
0 \to W(G; D) \to W(G; F) \to W(G; T(F/D)) \\
0 \to W(G; D) \to W(G; F) \to \sum_P W(G; D/P) .
\]

We should note that if \(G\) is a group action on \(M\), then the action \(G\) on \(M \otimes F\) will be defined as

\[ g(m, f) = (gm, f) \text{ for } g \in G, m \in M \text{ and } f \in F . \]
CHAPTER V
APPLICATIONS TO SMOOTH FINITE GROUP ACTIONS

§1. The Bordism Algebra and the Witt Ring.

Let us recall that a Lie group $G$ has a unique $C^\infty$ structure for which the map $G \times G \to G$ taking $(g,h) \mapsto gh^{-1}$ is smooth. By a smooth action of a Lie group $G$ on a smooth manifold $M$, we mean an action $\theta: G \times M \to M$ which is a smooth map. We will simply call this a smooth $G$-action on $M$.

Throughout this chapter, we will assume that $G = \pi$ is a finite group acting on a compact oriented manifold as a group of orientation preserving diffeomorphisms. We define the bordism equivalence in the following definition.

**Definition 1.1.** Let $(\pi;M^n)$ be a finite group of orientation preserving diffeomorphism acting on a closed oriented manifold $M^n$. Two closed $n$-manifolds $(\pi;M^n)$ and $(\pi;N^n)$ are in the same bordism class if there exists a compact $(n+1)$-manifold $(\pi';W^{n+1})$ such that $\partial W^{n+1} = M^n \sqcup -N^n$, $\pi' | M^n = \pi$ and $\pi' | N^n = \pi$, where $\sqcup$ indicates the disjoint union, and $\pi'$ is a finite group of orientation preserving diffeomorphisms.
Let $\mathcal{O}_n(\pi)$ be the collection of equivalent bordism classes.

**Definition 1.2.** Let $[\pi;M^n]$ denote the oriented bordism class of $(\pi;M^n)$. And, let $[\pi;M^n], [\pi;N^n] \in \mathcal{O}_n(\pi)$. We define the addition as $[\pi;M^n] + [\pi;N^n] = [\pi;M^n \sqcup N^n] \in \mathcal{O}_n(\pi)$, where $M^n \sqcup N^n$ is the disjoint union of $M^n$ and $N^n$.

Since $M^n \sqcup N^n = N^n \sqcup M^n$, then $\mathcal{O}_n(\pi)$, the oriented bordism group of orientation preserving finite group acting on closed $n$-manifolds, is an Abelian group; compare [7, l-10]. If we assign to each closed oriented manifold the trivial group action $I$, then $\mathcal{O}_n(I) = \Omega_n$; see [5; p. 99] and [7; p. 89].

**Definition 1.3.** Let $\mathcal{O}_*(\pi) = \bigoplus_{n=0}^{\infty} \mathcal{O}_n(\pi)$. We define the multiplication in $\mathcal{O}_*(\pi)$ as $[\pi;M^m][\pi';N^n] = [\pi \pi';M^m \times N^n]$.

$\mathcal{O}_*(\pi)$ is a ring by the above definition. Furthermore, we can embed $\Omega = \bigoplus_{n=0}^{\infty} \Omega_n$ into $\mathcal{O}_*(\pi)$, then we give to $\mathcal{O}_*(\pi) = \bigoplus_{n=0}^{\infty} \mathcal{O}_n(\pi)$ the structure of graded commutative algebra over $\Omega$.

Before we can define a mapping from $\mathcal{O}_*(\pi)$ to the Witt ring over $\mathbb{Z}$, we will introduce the symmetric, bilinear and non-singular inner product $\beta$ on cohomology module with rational coefficients. Let $(\pi;M^H_n)$ be a finite group of orientation preserving diffeomorphisms acting on a closed oriented manifold $M^H_n$. And, let $(\pi^*;H^{2n}(M;\mathbb{Q}))$ be the cohomology module with
the group action $\pi^*$ induced by $\pi$. We will write
$(\pi; H^{2n}(M;Q))$ instead of $(\pi^*; H^{2n}(M;Q))$ for convenience.
Define the bilinear form $\beta : H^{2n}(M;Q) \times H^{2n}(M;Q) \to Q$ by
$\beta(x,y) = \epsilon_*( (xu y) \cap \sigma) \in Q$, where $\epsilon_*$ is the augmentation
homomorphism and $\sigma$ is the orientation class in $H_{4n}(M;Z)$.
$\beta$ is preserved by $\pi$ since $\pi$ preserves the orientation. The
non-singularity follows immediately from Poincare duality, since
for any $x \in H^{2n}(M;Q)$, there exists $y \in H^{2n}(M;Q) \cong H_{2n}(M;Q)$
such that $\beta(x,y) \in Q$. Therefore, the Witt class $<\pi; H^{2n}(M;Q)>$
belong$s to $W(\pi;Q)$. Since $H^{2n}(M;Z)/\text{Tor}$ can be embedded in
$H^{2n}(M;Q)$, $<\pi; H^{2n}(M;Z)/\text{Tor}>$ belongs to $W(\pi;Z)$ in a similar
manner. In order to define a mapping from $O_*(\pi)$ to $W(\pi;Z)$,
we will define a mapping $\phi$ from $O_*(\pi)$ to $W(\pi;Q)$ first.
Then, by $0 \to W(\pi;Z) \to W(\pi;Q)$ and by taking restriction of
$H^{2n}(M;Q)$ to $H^{2n}(M;Z)$, we will have $\psi : O_*(\pi) \to W(\pi;Z)$.

**Lemma 1.4.** Let $[\pi, M^{2n}]$ be an oriented bordism class in $O_*(\pi)$
of orientation preserving finite group action on closed mani-
 folds. Define $\phi([\pi, M^{2n}]) = <\pi ; H^{2n}(M;Q)> \in W(\pi;Q)$. Then $\phi$
is a bordism invariant.

**Proof.** Let $(\pi'; B^{4n+1})$ be an action on a compact oriented mani-
fold such that $(\pi'; \tau B) = (\pi; M^{2n})$. If
$\sigma' \in H_{4n+1}(B^{4n+1}, M^{2n}; Z)$ is the orientation class, then
$\phi_*(\sigma') = \sigma$ is an orientation class in $H_{4n}(M^{2n}; Z)$. Consider
the exact sequence
Let $W \subset H^{2n}(M^{4n}; Q)$ be the image of $i^*$. $W$ is a $\pi$-invariant subspace of $H^{2n}(M; Q)$. Let $v, w \in H^{2n}(B; Q)$, we have

$$i^*(i^*(vw) \cap \sigma) = (vww) \cap i_*(\sigma) = (vww) \cap 0 = 0.$$ 

Therefore, by $0 \to H_0(M; Q) \to H_0(B; Q)$, we obtain $i^*(vww) \cap \sigma = 0$ and $\varepsilon_*(i^*(v) \cup i^*(w)) \cap \sigma = 0$. That is $\beta(i^*(v), i^*(w)) = 0$, and $W$ is self-annihilating. We will prove that $2 \dim W = \dim H^{2n}(M; Q)$ so that $W$ is a splitter in $H^{2n}(M; Q)$. Let $U$ be a direct summand of $W$. $U$ does exist since $H^{2n}(M; Q)$ is a vector space and since $W = \text{im}(i^*) = \ker(\delta^*)$. If $v \in U$, then $\delta^*(v) \neq 0 \in H^{2n+1}(B^{4n+1}, M^{4n}; Q) \cong H^{2n}(B; Q)$, so there exists $u \in H^{2n}(B^{4n+1}; Q)$ with

$$\varepsilon_*(\left(\delta^*(v)uu\right) \cap \sigma^{'}) = \varepsilon_*(\left(vw^*i^*(u)\right) \cap \sigma) \neq 0.$$ 

It follows that $U$ is dually paired with $W$ into $Q$. So $H^{2n}(M; Q) \cong W \oplus U$ and $W \cong U$. Thus $\dim W = \dim U = \frac{1}{2} \dim H^{2n}(M^{4n}; Q)$. Therefore $(\tau; H^{2n}(M; Q))$ is split, and $\varphi([\tau]; M^{4n}) = <\varphi; H^{2n}(M; Q)> = 0 \in W(\tau; Q)$. Hence $<\varphi; H^{2n}(M; Q)>$ depends on the bordism class only. This proves that $\varphi$ is bordism invariant.

**Lemma 1.5.** Let $\sigma_*(\pi)$ and $\varphi$ be as in the previous lemma. Then
\( \varphi \) is a ring homomorphism.

**Proof.** Let \([\pi;M^m],[\pi';N^r] \in \mathcal{O}_*(\pi)\). Then
\[
\varphi([\pi;M^m],[\pi';N^r]) = \varphi([\pi \times \pi';M^m \times N^r]) = \\
<\pi \times \pi';H^{2(m+n)}(M^m \times N^r;\mathbb{Q})> = <\pi \times \pi';\bigoplus_{p+q=2(m+n)} H^p(M;\mathbb{Q}) \otimes H^q(N;\mathbb{Q})>
\]
\[
= <\pi \times \pi';H^{2m}(M;\mathbb{Q}) \otimes H^{2n}(N;\mathbb{Q}) \otimes \bigoplus_{p+q=2(m+n)} H^p(M;\mathbb{Q}) \otimes H^q(N;\mathbb{Q})> .
\]

The second to the last equality is due to Künneth formula for cohomology over \(\mathbb{Q}\).

\[
<\pi \times \pi';\bigoplus_{p+q=2(m+n)} H^p(M;\mathbb{Q}) \otimes H^q(N;\mathbb{Q})> = 0
\]
\[
p+q=2(m+n)
\]
\[
p \neq 2m, q \neq 2n
\]

since \(\bigoplus_{p+q=2(m+n)} H^p(M;\mathbb{Q}) \otimes H^q(N;\mathbb{Q})\) has a splitter
\[
\bigoplus_{p+q=2(m+n)} H^i(M;\mathbb{Q}) \otimes H^j(N;\mathbb{Q})\)

with half of the dimensions.

It follows that \(\varphi([\pi;M],[\pi';N]) = <\pi \times \pi';H^{2m}(M;\mathbb{Q}) \otimes H^{2n}(N;\mathbb{Q})> = <\pi;H^{2m}(M;\mathbb{Q})><\pi';H^{2n}(N;\mathbb{Q})>\). Furthermore, let \(\bigcup\) indicate the disjoint union. We have

\[
\varphi([\pi;M^m] + [\pi;N^r]) = \varphi([\pi;M^m \bigcup N^r])
\]
\[
= <\pi;H^{2m}(M \bigcup N;\mathbb{Q})> = <\pi;H^{2m}(M;\mathbb{Q}) \otimes H^{2n}(N;\mathbb{Q})>
\]
\[
= <\pi;H^{2m}(M;\mathbb{Q})> + <\pi;H^{2n}(N;\mathbb{Q})> = \varphi([\pi;M^m])
\]
\[+ \varphi([\pi;N^r]) .\]
Let \([\tau; M]^{spin}\) be a bordism class in \(\Omega_*^{spin}(\tau)\) of orientation preserving finite group action on closed manifolds. Define 

\[\psi([\tau; M]^{spin}) = <\tau; H^{2n}(M; Z)/\text{Tor}> \in W(\tau; Z)\]

by taking the image of \(\varphi\) restricted to the cohomology module over \(Z\).

**Theorem 1.6.** Let \(\psi\) be defined as above, then \(\psi\) is a bordism invariant and a ring homomorphism.

**Proof.** Since \(W(\tau; Z)\) is embedded in \(W(\tau; Q)\) and 

\[(\tau; H^{2n}(M; Z)/\text{Tor})\]

can be embedded in \((\tau; H^{2n}(M; Q))\), if 

\((\tau; H^{2n}(M; Q))\) splits in \(W(\tau; Q)\), then \((\tau; H^{2n}(M; Z)/\text{Tor})\) splits in \(W(\tau; Z)\). Thus \(\psi\) is also a bordism invariant as a result of Lemmas 1.4 and 1.5. The Künneth formula for cohomology over \(Z\) and 

\[<\tau; H^{2m}(M; Q) \otimes H^{2n}(N; Q)> = 0\]

imply 

\[<\tau; H^{2m}(M; Z)/\text{Tor} \otimes H^{2n}(N; Q)/\text{Tor}> = 0\].

It yields the same result as in Lemmas 1.4 and 1.5. So \(\psi\) is a ring homomorphism.

**Remark 1.7.** We know if \(\tau\) is a \(p\)-group, \(p = \text{odd prime}\), then 

\(W_*(\tau; Z) = W(\tau; Z) \oplus W_2(\tau; Z)\) has no torsion; see [3; p. 43],

where \(W_2(\tau; Z)\) is the Witt ring of skew-symmetric finitely generated, projective inner product modules over \(Z\) with isometric group action. For \(p = 2\), then \(W(Z_2; Z) \cong Z @_2 Z_2\), [4, p. 1]. The torsion element in \(W(Z_2; Z)\) is really in the image of \(O(Z_2) \to W(Z_2; Z)\), [5, p. 7-9].

§2. **The Torsion Module** \(\text{Tor}^{2n}(M)\) from the Boundary of a Manifold

In this section and the following sections, we will define
three inner product torsion modules from the boundary of a manifold, from the boundary operator and from the manifold respectively. We will prove the Witt classes of three torsion modules are equal in section 5. And, therefore, we will reach the main theorem of this chapter, (Theorem 5.1).

We are going to consider a smooth orientation preserving action $(\pi; M^{4n-1})$ on a closed oriented $(4n-1)$-manifold and then define and study an invariant $w(\pi; M) \in W(\pi; T(Q/Z))$.

Let $\text{Tor}^q(M)$ be the torsion subgroup of $H^q(M; Z)$. Define the inner product $\beta$ on $\text{Tor}^{2n}(M)$ with values in $Q/Z$ (see [13]) as follows. From the coefficient sequence

$$0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0,$$

we have the Bockstein homomorphism $\beta^*: H^{2n-1}(M; Q/Z) \rightarrow H^{2n}(M; Z)$. The image of $\beta^*$ is exactly $\text{Tor}^{2n}(M) = \text{Tor} H^{2n}(M; Z)$ for if we consider the following exact sequence

$$0 \rightarrow H^{2n-1}(M; Q/Z) \rightarrow H^{2n}(M; Z) \rightarrow H^{2n}(M; Q) \rightarrow 0,$$

then $\alpha^*(\text{Tor} H^{2n}(M; Z)) = 0 \in H^{2n}(M; Q)$ which is torsion free and $\text{ker}(\alpha^*) = \text{Tor} H^{2n}(M; Z)$. Let $x, y \in \text{Tor}^{2n}(M)$, choose $z \in H^{2n-1}(M; Q/Z)$ with $\beta(z) = x$. The cup product $z \cup y \in H^{4n-1}(M; Q/Z \otimes Z)$ is $H^{4n-1}(M; Q/Z)$ for $Q/Z$ is a $Z$-module. Now, let $\sigma \in H^{4n-1}(M; Z)$ be the orientation class. The cap product $(z \cup y) \wedge \sigma \in H_0(M; Q/Z \otimes Z) = H_0(M; Q/Z)$.

Suppose $M$ is a connected, closed, oriented manifold. The augmentation $\epsilon_\sigma : H_0(M; Q/Z) \rightarrow H_0(\text{pt}; Q/Z) = Q/Z$ is induced by the map $\epsilon: \text{pt} \rightarrow M$. Now, define $\beta: \text{Tor}^{2n}(M) \times \text{Tor}^{2n}(M) \rightarrow Q/Z$.
by \( \delta(x,y) = e_\ast((z \cup y) \cap c) \in Q/\mathbb{Z} \). Clearly, \( \delta \) is bilinear.

We will prove that \( \delta \) is a nonsingular symmetric bilinear form on \( \text{Tor}^{2n}(M) \) with isometric group action. Denote the Witt class of \( (\pi;\text{Tor}^{2n}(M)) \) by \( w(\pi;M) = \langle \pi;\text{Tor}^{2n}(M) \rangle \).

**Lemma 2.1.** Let \( \delta \) be the bilinear form defined as above. Then \( \delta \) is well defined.

**Proof.** Suppose \( z, z' \in H^{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \) such that \( b^\ast(z) = b^\ast(z') = x \). Hence \( b^\ast(z-z') = 0 \). There exists \( s \in H^{2n-1}(M;\mathbb{Q}) \) such that \( r^\ast(s) = z-z' \). \( s \cup y \in H^{4n-1}(M;\mathbb{Q}) \) implies \( r^\ast(s \cup y) \in H^{4n-1}(M;\mathbb{Q}/\mathbb{Z}) \). Furthermore, we have \( r^\ast(s \cup y) = r^\ast(s) \cup y = (z-z') \cup y = (z\cup y) - (z' \cup y) \). Since \( b^\ast[(z\cup y) - (z' \cup y)] = b^r^\ast(s \cup y) = 0 \), we obtain \( (z\cup y) - (z' \cup y) = r^\ast(t) \) for some \( t \in H^{4n-1}(M;\mathbb{Q}) \). \( H^{4n-1}(M;\mathbb{Q}) \) is torsion free implies \( r^\ast \) is a trivial homomorphism. Thus \( r^\ast(t) = 0 \). It follows that \( z \cup y = z' \cup y \). This proves \( \delta \) is well defined.

**Lemma 2.2.** Let \( \delta \) be as in the previous lemma. Then \( \delta \) is a symmetric bilinear form.

**Proof.** For any \( \psi^{2n}_x \in C^{2n}(M;\mathbb{Z}) \) representing \( x \in H^{2n}(M;\mathbb{Z}) \), there exists \( \phi^{2n}_x \in C^{2n}(M;\mathbb{Q}) \) as an extension of \( \psi^{2n}_x \) to the rational coefficient, such that \( \delta \phi^{2n-1}_x = \psi^{2n}_x \), where

\[
\begin{array}{ccc}
C^{2n-1}(M,\mathbb{Q}) & \to & C^{2n}(M,\mathbb{Q}) \\
\phi^{2n-1}_x & \mapsto & \phi^{2n}_x \\
\end{array}
\]
If we pass \( \varphi_{2n-1}^x \) to \( \mathbb{Q}/\mathbb{Z} \), then the corresponding 
\( \xi_{2n-1}^x \in C_{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \). Since \( \delta \xi_{2n-1}^x = 0 \) over \( \mathbb{Q}/\mathbb{Z} \), there 
exists \( z \in H_{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \) representing \( \xi_{2n-1}^x \) and therefore 
b\( (z) = x \). Similarly, there exists \( z' \in H_{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \) such 
that \( b^*(z') = y \). Let \( a_{2n}^x \in Z_{2n}(M;\mathbb{Z}) \) represent \( x \in H_{2n}(M;\mathbb{Z}) \)
and \( \beta_{2n}^y \in Z_{2n}(M;\mathbb{Z}) \) represent \( y \in H_{2n}(M;\mathbb{Z}) \). Let 
\( \varphi_{2n-1}^x \in C_{2n-1}(M;\mathbb{Q}) \mapsto \xi_{2n-1}^x \in Z_{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \) and 
\( \varphi_{2n-1}^y \in C_{2n-1}(M;\mathbb{Q}) \mapsto \xi_{2n-1}^y \in Z_{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \), where \( \delta \varphi_{2n-1}^x = a_{2n}^x \)
and \( \delta \varphi_{2n-1}^y = \beta_{2n}^y \). We have \( z \cup y = \xi_{2n-1}^x \cup \beta_{2n}^y \) and 
z\( ' \cup x = \xi_{2n-1}^x \cup a_{2n}^x \).

Since \( \varphi_{2n-1}^x \cup \varphi_{2n-1}^y \in C^4_{2n-2}(M;\mathbb{Q}) \), we obtain 
\( \delta(\varphi_{2n-1}^x \cup \varphi_{2n-1}^y) = \delta \varphi_{2n-1}^x \cup \varphi_{2n-1}^y + (-1)^{2n-1} \varphi_{2n-1}^x \cup \delta \varphi_{2n-1}^y \).
Thus \( \delta(\varphi_{2n-1}^x \cup \varphi_{2n-1}^y) = a_{2n}^x \cup \varphi_{2n-1}^y - \varphi_{2n-1}^x \cup \beta_{2n}^y \). Passing to
\( \mathbb{Q}/\mathbb{Z} \), we get \( a_{2n}^x \xi_{2n}^y - \xi_{2n}^y \beta_{2n}^y = 0 \). Therefore, \( x \cup z' = z \cup y \). But 
x \cup z' = (-1)^{2n} z' \cup x = z' \cup x \). So \( z \cup y = z' \cup x \). Hence, \( \beta(y,x) = \epsilon_*[(z' \cup x) \wedge \sigma] = \epsilon_*[(z \cup y) \wedge \sigma] = \beta(x,y) \).

**Lemma 2.3.** Let \( \beta \) be the symmetric bilinear form on \( \text{Tor}^{2n}(M) \)
as in the previous lemmas. Then \( \beta \) is a non-singular, symmetric
bilinear form. And, therefore, \( (\text{Tor}^{2n}(M),\beta) \) is an inner
product torsion module. And, \( <\pi;\text{Tor}^{2n}(M)> = w(\pi;\mathbb{M}) \in W(\pi;T(\mathbb{Q}/\mathbb{Z})) \).

**Proof.** We have proved that the bilinear form \( \beta \) on \( \text{Tor}^{2n}(M) \)
is well defined and symmetric in the previous lemmas. Now we
only need to prove that for each \( \varphi \in \text{Hom}(\text{Tor}^{2n}(M);\mathbb{Q}/\mathbb{Z}) \), there
exists one and only one element \( x \in \text{Tor}^{2n}(M) \) so that
$\beta(x,y) = \varphi(y)$ for all $y \in \text{Tor}^{2n}(M)$. If there are $x$ and $x'$ such that $\beta(x',y) = \beta(x,y) = \varphi(y)$ for all $y \in \text{Tor}^{2n}(M)$, then $\beta(x-x',y) = 0$ for all $y \in \text{Tor}^{2n}(M)$. Now, suppose $x_0 = x-x'$, then $\beta(x_0,y) = \beta(y,x_0) = \epsilon_*[(z \cup x_0) \cap \sigma] = 0$ for all $b^*(z) = y \in \text{Tor}^{2n}(M)$, where $\sigma$ is the orientation class in $H_{4n-1}(M;\mathbb{Z})$. Since $H^{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \to \text{Tor}^{2n}(M)$ is onto, we can say that $\epsilon_*[(z \cup x_0) \cap \sigma] = 0$ for all $z \in H^{2n-1}(M;\mathbb{Q}/\mathbb{Z})$. Since $M$ is connected, $z \cup x_0 = 0$ for all $z \in H^{2n-1}(M;\mathbb{Q}/\mathbb{Z})$. It follows that $x_0 = 0$. Therefore $x = x'$. This proves the uniqueness.

To prove the existence: let $\varphi \in \text{Hom}(\text{Tor}^{2n}(M);\mathbb{Q}/\mathbb{Z})$ with $\varphi(y) \in \mathbb{Q}/\mathbb{Z} = H_0(M;\mathbb{Q}/\mathbb{Z})$, then there exists $c_y \in H_0(M;\mathbb{Q}/\mathbb{Z})$ such that $\varphi(y) = \epsilon_*(c_y)$. Since $H_0(M;\mathbb{Q}/\mathbb{Z}) \cong H_{4n-1}(M;\mathbb{Q}/\mathbb{Z})$, for $\sigma \in H_{4n-1}(M;\mathbb{Z})$, the orientation class, we have $d_y \in H^{4n-1}(M;\mathbb{Q}/\mathbb{Z})$ such that $d_y \cap \sigma = c_y$. So far, we have $\varphi(y) = \epsilon_*(d_y \cap \sigma)$ for all $y \in \text{Tor}^{2n}(M) \subset H^{2n}(M;\mathbb{Z})$. Let $z_y \in H^{2n-1}(M;\mathbb{Q}/\mathbb{Z})$ correspond to $y$ with $b^*(z_y) = y$. Since for any $y \in \text{Tor}^{2n}(M)$, we associate $d_y \in H^{4n-1}(M;\mathbb{Q}/\mathbb{Z})$ and $z_y \in H^{2n-1}(M;\mathbb{Q}/\mathbb{Z})$ and since $H^{2n-1}(M;\mathbb{Q}/\mathbb{Z}) \to \text{Tor}^{2n}(M)$ is onto, there exists $x \in H^{2n}(M;\mathbb{Z})$ such that $z_y \cup x = d_y$. And, hence $\varphi(y) = \epsilon_*[(z_y \cup x) \cap \sigma] = \beta(y,x)$. $x \in \text{Tor}^{2n}(M;\mathbb{Z})$ follows immediately from the symmetry of $\beta$. This proves the non-singularity of $\beta$. Combining Lemma 2.1 and Lemma 2.2, we have $(\text{Tor}^{2n}(M),\beta)$ is an inner product torsion module.

Now $(\pi;M)$ induces $(\pi;\text{Tor}^{2n}(M))$ and $\pi$ acts as a group
of isometries with respect to $\beta$. Thus we have $\langle\pi;\text{Tor}^{2n}(M)\rangle = w(\pi;M) \in W(\pi;T(Q/Z))$. We should note that $w(\pi,M)$ is an invariant of the oriented equivariant homotopy type of $(\pi;M)$. It is not a bordism invariant.

§3. The Torsion Module $L^#_L$ from the Boundary Operator of Witt Groups

Suppose $(\pi;M^{4n-1})$ bounds as an action; that is, there is a $(\pi;B^{4n})$ on a compact oriented $4n$-manifold with $(\pi;\partial B) = (\pi;M^{4n-1})$. We can also define $w(\pi;B) \in W(\pi;Q)$. Look at the following exact sequence

$$
\begin{array}{ccccccccc}
H^{2n}(B,\partial B;\mathbb{Z}) & \xrightarrow{j^*} & H^{2n}(B;\mathbb{Z}) & \xrightarrow{i^*} & H^{2n}(\partial B;\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
H^{2n}(B,\partial B;\mathbb{Q}) & \xrightarrow{j^*} & H^{2n}(B;\mathbb{Q}) & \xrightarrow{i^*} & H^{2n}((\partial B;\mathbb{Q})
\end{array}
$$

Define $\beta: \text{im}(j^*) \times \text{im}(j^*) \to \mathbb{Q}$ by

$$
\beta(j^*x, j^*y) = \epsilon_*[(x \cup y) \cap \sigma_{4n}] \in \mathbb{Q},
$$

where $j_*\sigma = \sigma_{4n} \in H_{4n}(B,\partial B;\mathbb{Z})$ is the orientation class.

**Lemma 3.1.** Let the bilinear form $\beta$ be defined as above. Then $(\text{im}(j^*), \beta)$ is a nonsingular, symmetric bilinear $\mathbb{Q}$-valued inner product on $\text{im}(j^*)$ with an isometric group action.

**Proof.** We claim that $\beta$ is well defined since if $j^*x = j^*x'$, then $j^*(x-x') = 0$ and there is a $z \in H^{2n-1}(\partial B;\mathbb{Q})$ such that $\delta z = x-x'$. We have $(x-x')\cup y = \delta z \cup y$ with $\delta z \in B^{2n}(B,\partial B;\mathbb{Q})$. 

and \( y \in H^{2n}(B,\alpha B;\mathbb{Q}) \). It is clear that \( \delta z \cup y \) is zero since \( \delta z \) is in the coboundary class but \( y \) is a relative class. The proof of non-singularity is similar to that of Lemma 2.3. Therefore, there is a symmetric \( Q \)-valued inner product \( \beta \) on \( \text{im}(J^*) \) and \( \pi \) acts on \( \text{im}(J^*) \) as a group of isometries to yield \( w(\pi; B) = \langle \pi; \text{im}(J^*) \rangle \in W(\pi; \mathbb{Q}) \).

From Chapter IV, section 3, we have an exact sequence

\[
0 \rightarrow W(\pi; \mathbb{Z}) \rightarrow W(\pi; \mathbb{Q}) \rightarrow W(\pi; \mathbb{Q}/\mathbb{Z}) \rightarrow 0.
\]

Now, we know \( w(\pi; B) \in W(\pi; \mathbb{Q}) \). The question is that "what does the image of \( w(\pi; B) \) look like under the boundary operator \( \partial \)"? As we know from Chapter III, section 2, \( \partial(w(\pi; B)) = \langle \pi; L^#/L \rangle \), for some \( Z \)-lattice \( L \) of \( \text{im}(J^*) \) and its dual lattice \( L^# \). In the rest of this section, we will figure out what \( L^#/L \) looks like.

Consider the diagram (c.1). Now let \( \text{Tor}^{2n}(B) = \text{Tor} H^{2n}(B;\mathbb{Z}) \), the torsion subgroup of \( H^{2n}(B;\mathbb{Z}) \). Put

\[
L = \text{im}(j^*)/[\text{Tor}^{2n}(B) \cap \text{im}(j^*)] \subset H^{2n}(B;\mathbb{Z})/\text{Tor}^{2n}(B).
\]

Clearly, by \( \alpha_2^* \), we can see that \( L \) is embedded in \( \text{im}(J^*) \) and \( L \) is a \( \pi \)-invariant \( Z \)-lattice for \( \text{im}(J^*) \) with \( \beta(j^* x, j^* y) = \epsilon_4^* [(x \cup y) \wedge \sigma_{4n}] \), where \( \sigma_{4n} \in H_{4n}(B,\alpha B;\mathbb{Z}) \), and \( j^* x, j^* y \notin \text{Tor}^{2n}(B) \).

Let \( K = \{ z \in H^{2n}(B;\mathbb{Z})| i^*(z) \in \text{Tor}^{2n}(\alpha B) \} \). Then we have the following inclusions, \( \text{Tor}^{2n}(B) \subset K \), \( \text{im}(j^*) \subset K \subset \text{im}(j^*) \)
for \( a_3^* \) is \( 0 \) and \( L = \text{im}(j^*)/[\text{Tor}^{2n}(B) \cap \text{im}(j^*)] \subset K/\text{Tor}^{2n}(B) \subset \text{im}(j^*) \).

The dual lattice \( L^# \subset \text{im}(j^*) \) will be characterized in the following lemma.

**Lemma 3.2.** Let \( L = \text{im}(j^*)/[\text{Tor}^{2n}(B) \cap \text{im}(j^*)] \) as described above. Then \( L^# \cong K/\text{Tor}^{2n}(B) \).

**Proof.** Suppose \( z \in K \) and \( z \) is not a torsion element, there is \( z' \in H^{2n}(\partial B; \mathbb{Q}) \) with \( j^*(z') = a_2^*(z) \in H^{2n}(B; \mathbb{Q}) \), where \( a_2^* \) is an embedding from \( H^{2n}(B; \mathbb{Z})/\text{Tor} \) to \( H^{2n}(B; \mathbb{Q}) \). If \( x \in \text{im}(j^*) \) and is not a torsion element, then there exists \( x' \in H^{2n}(\partial B; \mathbb{Z}) \) with \( j^*(x') = x \). We get \( \beta(x, y) = \beta(j^*(x'), j^*(z)) = \epsilon_x[(x' \cup z') \cap \sigma_{4n}] = \epsilon_x[(x' \cup j^*(z')) \cap \sigma_{4n}] = \epsilon_x[(x' \cup z) \cap \sigma_{4n}] \). The value of \( \epsilon_x[(x' \cup z) \cap \sigma_{4n}] \) belongs to \( Z \) because \( x' \) and \( z \) are both integral classes. So \( z \in L^# \).

Conversely, suppose \( J^*(z') = z \in L^# \). Then for all \( x' \in H^{2n}(\partial B; \mathbb{Z}) \) with \( j^*(x') = x \), we have \( \beta(J^*(z'), j^*(x')) = \beta(J^*(z'), x) \in Z \). But \( \beta(J^*(z'), j^*(x')) = \epsilon_x[(z' \cup x') \cap \sigma_{4n}] = \epsilon_x[(z \cup x') \cap \sigma_{4n}] \). For any \( x' \in H^{2n}(\partial B; \mathbb{Z}) \), Lefschetz duality, there is \( z'' \in H^{2n}(B; \mathbb{Z}) \) with \( \beta(z'', x') = \beta(J^*(z'), j^*(x')) \). Because \( \beta(z'', x') = \epsilon_x[(z'' \cup x') \cap \sigma_{4n}] \) and because \( \beta(J^*(z'), j^*(x')) = \epsilon_x[(z' \cup x') \cap \sigma_{4n}] \), we obtain \( z = J^*(z') = a_2^*(z'') \in H^{2n}(B; \mathbb{Q}) \). Now, \( I^*(z) = I^*J^*(z') = 0 = I^*a_2^*(z'') \), thus \( i^*(z'') \in \text{Tor}^{2n}(\partial B) \). Therefore, \( z'' \in K \) and \( J^*(z') \in K/\text{Tor}^{2n}(B) \). This proves \( L^# \cong K/\text{Tor}^{2n}(B) \).
Theorem 3.3. Let $\text{Tor}^{2n}(B) = \text{Tor} H^{2n}(B; Z)$. Let
$L = \text{im}(j^*)/[\text{Tor}^{2n}(B) \cap \text{im}(j^*)]$ be a $Z$-lattice of $\text{im}(j^*)$ as in the diagram (c.1). If $L^#$ is the dual lattice of $L$, then
$L^#/L \cong [K/\text{Tor}^{2n}(B)]/[\text{im}(j^*)/\text{Tor}^{2n}(B) \cap \text{im}(j^*)]$, which is the image of $w(\pi; B) = \langle \pi; \text{im}(j^*) \rangle$ under the boundary operator
$\partial: W(\pi; Q) \rightarrow W(\pi; T(Q/Z))$.

Proof. By Chapter III, section 2, the image of $w(\pi; B)$ under the boundary homomorphism is $\langle \pi; L^#/L \rangle$, i.e.,
$\partial(w(\pi; B)) = \langle \pi; L^#/L \rangle \in W(\pi; T(Q/Z))$.

§4. The Torsion Module $A^4/A$ from the Manifold Only

From Theorem 3.3, we have $\partial(w(\pi; B)) = \langle \pi; L^#/L \rangle \in W(\pi; T(Q/Z))$, where $L^#$ is the dual lattice of $L$ as described in Theorem 3.3.

On the other hand, if $M^{4n-1}$ is the boundary of $4n$-manifold $B^{4n}$, then, from Theorem 2.3, we have another inner product torsion module with $\pi$-action $(\pi; \text{Tor}^{2n}(M))$, whose Witt class is denoted by $w(\pi; M)$. Our main result is that
$\partial(w(\pi; B)) = w(\pi; \partial B) = w(\pi; M)$.

In order to reach this goal we will introduce an inner product torsion module in this section to connect these two Witt classes.

From the diagram (c.1), we have $i^*(K) \subset \text{Tor}^{2n}(\partial B)$, and $i^*(\text{Tor}^{2n}(B)) \subset i^*(K) \subset \text{Tor}^{2n}(\partial B)$. The inner product
Lemma 4.1. Let \( A = i^*(\text{Tor}^{2n}(B)) \) and so \( A^\perp = (i^*(\text{Tor}^{2n}(B)))^\perp \). Then \( A^\perp = i^*(K) \).

Proof. Let \( x \in \text{Tor}^{2n}(\partial B) \), but \( x \not\in i^*(K) \). We will prove \( x \not\in A^\perp \). Under \( \delta: H^{2n}(\partial B;Z) \rightarrow H^{2n+1}(B,\partial B;Z) \), we see that \( \delta x \neq 0 \) is a nonzero torsion class. By Lefschetz duality, there exists \( z \in H^{2n-1}(B;Q/Z) \) for which \( \epsilon_*(z \cup \delta x) \cdot \sigma_{4n} \neq 0 \), where \( \sigma_{4n} \in H_{4n}(B,\partial B;Z) \) is the orientation class. But there is \( i^*: H^{2n-1}(B;Q/Z) \rightarrow H^{2n-1}(\partial B;Q/Z) \) and \( \beta(b^i^*(z),x) = \epsilon_*(i^*(z) \cup x) \cdot \sigma_{4n} \). Since \( i^*(z) \cup x \cap \sigma_{4n} = 0 \) and \( \epsilon_*(z \cup \delta x) \cdot \sigma_{4n} \neq 0 \), we have \( \epsilon_*(i^*(z) \cup x) \cap \sigma_{4n} = 0 \). Therefore \( \beta(b^i^*(z),x) = \epsilon_*(i^*(z) \cup x) \cap \sigma_{4n} \neq 0 \). Because \( b^i^*(z) \in i^*(\text{Tor}^{2n}(B)) \) and \( \beta(b^i^*(z),x) \neq 0 \), we have that if \( x \not\in i^*(K) \), then \( x \not\in A^\perp \).

This proves \( A^\perp \subset i^*(K) \).

Now suppose \( u \in K \) and \( i^*(u) \in i^*(K) \). Let \( v \in \text{Tor}^{2n}(B) \), there is \( z \in H^{2n-1}(B;Q/Z) \) with \( b^i^*(z) = i^*(v) \). So \( \beta(i^*(v),i^*(u)) = \epsilon_*(i^*(z) \cup i^*(u)) \cap \sigma_{4n} \). Since \( i^*(z \cup u) = i^*(z) \cup i^*(u) \) and \( i^* \sigma_{4n} = 0 \in H_{4n-1}(B;Z) \), we obtain \( i^*(z \cup u) \cap i^* \sigma_{4n} = 0 \). Hence \( \beta(i^*(v),i^*(u)) = 0 \) for all \( i^*(v) \in i^*(\text{Tor}^{2n}(B)) = A \). Therefore, \( i^*(u) \in A^\perp \).

This proves the lemma that \( A^\perp = i^*(K) \).
As we know, $\text{Tor}^{2n}(B) \subset K$. $i^*(\text{Tor}^{2n}(B)) \subset i^*(K)$. It follows that $A = i^*(\text{Tor}^{2n}(B)) \subset i^*(K) = A^\perp$. From Lemma 3.2 of Chapter I, we know $A^\perp/A$ has a non-singular inner product structure. So $(\pi;A^\perp/A)$ forms a nonsingular inner product torsion module with finite group action $\pi$, whose Witt class is $<\pi;A^\perp/A> \in W(\pi; T(Q/Z))$. Furthermore, from Lemma 3.3 of Chapter I and Lemma 1.4 of Chapter IV, we know $w(\pi;\mathfrak{a}B) = <\pi;\text{Tor}^{2n}(\mathfrak{a}B)> = <\pi;A^\perp>$.

§5. Coincidence of Witt classes of Torsion Modules

The Witt class $<\pi;A^\perp/A>$ in section 4 and the Witt class $<\pi;\text{Tor}^{2n}(\mathfrak{a}B)>$ in section 2 are equal which follows immediately by Lemma 1.4 of Chapter IV and Lemma 3.3 of Chapter I since $A$ is $\pi$-invariant. If we can prove the following Lemma 5.1 that $(\pi;L^\#_3/L)$ in section 3 is equivariant isometric with $(\pi;A^\perp/A)$, then Theorem 5.2 will follow immediately.

Lemma 5.1. Let $A$, $L$, $K$ and $\text{Tor}^{2n}(B)$ be defined as in the previous sections. Then there is an equivariant isometry:

$L^\#_3/L \cong i^*(K)/i^*(\text{Tor}^{2n}(B)) = A^\perp/A$.

Once we prove Lemma 5.1, we will have the beautiful result that the three modules described in section 2, 3 and 4, respectively, belong to the same Witt class in $W(\pi; T(Q/Z))$, i.e., we have the following main theorem:

Theorem 5.2. Let $(\pi; M^{'(n-1)})$ bound $(\mathfrak{w}; \mathfrak{B}^{'n})$ as a group action.
Suppose $\text{Tor}^{2n}(M) = \text{Tor} H^{2n}(M;\mathbb{Z})$, the torsion subgroup of $H^{2n}(M;\mathbb{Z})$, with inner product defined by $\beta(x,y) = \epsilon_*(z \cup y) \cap \sigma$, where $b^*$ is the Bockstein homomorphism with $b^*(z) = x$, $\epsilon_*$ is the augmentation homomorphism, and $\sigma \in H_{4n-1}(M;\mathbb{Z})$ the orientation class. Denote $\langle \pi;\text{Tor}^{2n}(M) \rangle$ by $w(\pi;M) \in W(\pi;T(\mathbb{Q}/\mathbb{Z}))$. And, let $J^*(H^{2n}(B;\mathbb{Q};\mathbb{Q})) \subset H^{2n}(B;\mathbb{Q})$ be the inner product vector space over $\mathbb{Q}$ defined by $\beta(J^*(x),J^*(y)) = \epsilon_*(x \cup y) \cap \sigma_{4n} \in \mathbb{Q}$, where $\sigma_{4n} \in H_{4n}(B,\mathbb{Z};\mathbb{Q})$, the orientation class. Denote $\langle \pi;\text{im}(J^*) \rangle = w(\pi;B) \in W(\pi;\mathbb{Q})$.

Then, under the boundary homomorphism $\partial:W(\pi;\mathbb{Q}) \rightarrow W(\pi;T(\mathbb{Q}/\mathbb{Z}))$, we have

$$\partial(w(\pi;B)) = w(\pi;M) = w(\pi;\partial B).$$

**Proof.** Theorem 3.3 says that

$$\partial(w(\pi;B)) = \partial(\langle \pi;\text{im}(J^*) \rangle) = \langle \pi;L^#/I \rangle.$$

Lemma 5.1 says that $\langle \pi;L^#/I \rangle = \langle \pi;A^#/A \rangle$. Therefore, since $\langle \pi;A^#/A \rangle = \langle \pi;\text{Tor}^{2n}(M) \rangle = w(\pi;M)$, we have

$$\partial(w(\pi;B)) = \langle \pi;L^#/I \rangle = \langle \pi;A^#/A \rangle = \langle \pi;\text{Tor}^{2n}(M) \rangle = w(\pi;M).$$

**Proof of Lemma 5.1.**

$$A^#/A \cong i^*(K)/i^*(\text{Tor}^{2n}(B)) \cong [K/\ker(i^*|K)]/i^*(\text{Tor}^{2n}(B))$$

$$\cong [K/\ker(i^*|K)]/[\text{Tor}^{2n}(B)/\ker(i^*|\text{Tor}^{2n}(B))]$$

$$\cong [K/(\text{im}(J^*) \cap K)]/[\text{Tor}^{2n}(B)/\text{im}(J^*) \cap \text{Tor}^{2n}(B)]$$

$$\cong [K/\text{im}(J^*)]/[\text{Tor}^{2n}(B)/\text{im}(J^*) \cap \text{Tor}^{2n}(B)]$$
\[\cong [K/Tor^{2n}(B)]/[\text{im}(j^*)/Tor^{2n}(B) \cap \text{im}(j^*)] \cong L^#/L.\]

So \(A^1/A\) is isomorphic to \(L^#/L\). Now, to prove the equivariant isometry. Let \((L^#/L, \beta_1)\) and \((i^*(K)/i^*(Tor^{2n}(B), \beta_2))\) be two inner product torsion modules as described in section 3 and section 4. Consider the following diagram

\[
\begin{array}{cccccccc}
H^{2n-1}(B, \partial B; \mathbb{Q}/\mathbb{Z}) & \rightarrow & H^{2n-1}(B; \mathbb{Q}/\mathbb{Z}) & \rightarrow & H^{2n-1}(\partial B; \mathbb{Q}/\mathbb{Z}) & \rightarrow & H^{2n}(B, \partial B; \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{2n}(B, \partial B; \mathbb{Z}) & \rightarrow & H^{2n}(B; \mathbb{Z}) & \rightarrow & H^{2n}(\partial B; \mathbb{Z}) & \rightarrow & H^{2n+1}(B, \partial B; \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{2n}(B, \partial B; \mathbb{Q}) & \rightarrow & H^{2n}(B; \mathbb{Q}) & \rightarrow & H^{2n}(\partial B; \mathbb{Q}) & \rightarrow & H^{2n+1}(B, \partial B; \mathbb{Q}). \\
\end{array}
\]

Suppose \(u, v \in K\), but \(u, v \notin Tor^{2n}(B)\), i.e., \(\bar{u}, \bar{v} \in L^#\). And, the corresponding \(\bar{u}, \bar{v}\) are in \(L^#/L\). We have the following diagram

\[
\begin{array}{ccccc}
z & \rightarrow & \delta(z) \\
\downarrow & & \downarrow b^* \\
j^* \quad & i^* \quad & b^* \\
x & \rightarrow & u & \rightarrow & i^*(u) \\
y & \rightarrow & v & \rightarrow & i^*(v). \\
\end{array}
\]

In \((L^#, \beta_1)\), we have \(\beta_1(\bar{u}, \bar{v}) = \beta_1(j^*(x), j^*(y)) = e_*[(x \cup y) \cap \sigma_{4n}]\), where \(\sigma_{4n} \in H_{4n}(B; \partial B; \mathbb{Z})\) the orientation class. Since \((L^#/L, \beta_1)\) is with values in \(\mathbb{Q}/\mathbb{Z}\), if we pass the cohomology classes to \(\mathbb{Q}/\mathbb{Z}\), then \(\beta_1(\bar{u}, \bar{v}) = e_*[(\delta z u v) \cap \sigma_{4n}] = e_*[(\delta z u v) \cap \sigma_{4n}]\). On the other hand, if \(\beta_2\) is the inner product on \(i^*(K)/i^*(Tor^{2n}(B))\), then \(\beta_2(i^*(u), i^*(v)) = e_*[(z u i^*(v)) \cap \sigma_{4n}]\) with \(b^*(z) = i^*(u)\). Thus \(\beta_2(i^*(u), i^*(v)) = e_*[(\delta z u v) \cap \sigma_{4n}]\). So
$\beta_2(i^*(u),i^*(v)) = \beta_1(\bar{u},\bar{v})$. This proves $L^\# / L$ and $i^*(K)/i^*(\text{Tor}^{2n}(B)) \equiv A^+/A$ are equivariant isometric.

**Corollary 5.3.** Let $(\pi;M^{4n-1})$ and $(\pi;B^{4n})$ as in Theorem 5.2. Suppose $w(\pi;M^{4n-1}) \neq 0$, then if $(\pi;M^{4n-1})$ bounds on $(\pi;B^{4n})$, then the image of $j^* : H^{2n}(B;\mathbb{Z}) \rightarrow H^{2n}(B;\mathbb{Z})$ cannot be a direct summand.

**Proof.** If $\text{im}(j^*)$ is a direct summand, then $K \cong \text{Tor}^{2n}(B) \oplus \text{im}(j^*)$. So $K/\text{im}(j^*) \cong \text{Tor}^{2n}(B)$ and $[K/\text{Tor}^{2n}(B)]/[\text{im}(j^*)/\text{Tor}^{2n}(B) \cap \text{im}(j^*)] = 0$. Hence $L^\# / L = 0$, and therefore $\langle \pi;L^\# / L \rangle = 0$. It says $w(\pi;M) = \langle \pi;L^\# / L \rangle = 0$, a contradiction to $w(\pi;M) \neq 0$. 

BIBLIOGRAPHY


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