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Subgradient formulas for optimal control problems with constant dynamics

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**SUBGRADIENT FORMULAS FOR OPTIMAL CONTROL
PROBLEMS WITH CONSTANT DYNAMICS**

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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by

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Abstract

In this thesis our first concern is the study of the minimal time function corresponding to control problems with constant convex dynamics and closed target sets. Unlike previous work in this area, we do not make any nonempty interior or calmness assumptions and the minimal time functions is generally non-Lipschitzian. We show that the Proximal and Fréchet subgradients of the minimal time function are computed in terms of normal vectors to level sets. And we also computed the subgradients of the minimal time function in terms of the F -projection.

Secondly, we consider the value function for Bolza Problem in optimal control and the calculus of variations. The main results present refined formulas for calculating the Fréchet subgradient of the value function under minimal requirements, and are similar to those obtained for the minimal time function.

Chapter 1

Introduction

This thesis includes two main distinct and classical but related optimal control problems: (1) Minimal Time Problem and (2) Bolza Problem. In the first half of this thesis we focus on the subgradient formulas for the minimal time function in the case of constant dynamics. For the second part of this thesis, we study the Fréchet subgradient of the value function for the Bolza problem whose dynamics are also constant.

We are given a closed convex set $F \subseteq \mathbb{R}^n$. Let's consider an (autonomous) control system with constant dynamics. The state equation associated with the system is

$$\begin{cases} \dot{y}(s) \in F, & s \in [0, T_0] \text{ a.e.} \\ y(0) = x, \end{cases} \quad (1.0.1)$$

where $x \in \mathbb{R}^n$. We denote the solution of the above control system by $y(\cdot; 0, x)$ and it is called the trajectory of the control system corresponding to the initial condition $y(0) = x$. The set of endpoints of all such trajectories is called the reachable set from x and at time T_0 .

Given a nonempty closed set $S \subseteq \mathbb{R}^n$, the minimal time function $T(\cdot)$ defined by

$$T(x) := \inf \{t \geq 0 : S \cap \{x + tF\} \neq \emptyset\} \quad (1.0.2)$$

and can be viewed as

$$T(x) := \inf \{T_0 : \text{there exists } y(\cdot) \text{ such that } \dot{y}(t) \in F \text{ a.e. } t \in [0, T_0] \quad (1.0.3)$$

with $y(0) = x$ and $y(T_0) \in S\}$.

In other words,

$$T(x) = \inf \{T_0 : I_S(y(T_0; 0, x)) = 0\},$$

where I_S is the indicator function of the set S . Mayer problem is a special case of Bolza problem. In fact, the minimal time function at x , denoted as $T(x)$, equals the smallest value T_0 such that the infimum in the Mayer problem with final cost $g = I_S$ and initial point $(0, x)$ is attained.

Let $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a proper and lsc function and let $T_0 \in \mathbb{R}$. For any $(t, x) \in [0, T_0] \times \mathbb{R}^n$, we consider the following problem:

$$(MP) \quad \text{minimize } g(y(T_0; t, x)). \quad (1.0.4)$$

When the infimum in (MP) is attained, the corresponding solution $y(\cdot) = y(\cdot; 0, x)$ of the state equation (1.0.4) is called an optimal trajectory. The function g is called the final cost of the Mayer problem. Observe that the Mayer problem is a problem with finite time horizon.

Next, let's consider another kind of optimal control problem with finite time horizon, the Bolza problem. As in the Mayer problem, we are given the function g and a time $T_0 > 0$. In addition, let's assign a convex, proper and lsc function L , which is called the running cost. For any $(t, x) \in [0, T_0] \times \mathbb{R}^n$, let's consider the functional

$$J_t(y(\cdot)) := g(y(T_0)) + \int_t^{T_0} L(\dot{y}(t))dt, \quad (1.0.5)$$

where the Lagrangian L only depends on \dot{y} , and the control problem

$$(BP) \quad \text{minimize } J_t(y(\cdot)), \quad (1.0.6)$$

where the minimization takes place over all the absolutely continuous $y(\cdot) : [t, T_0] \rightarrow \mathbb{R}^n$ with derivative $\dot{y}(\cdot) \in L_n^p[t, T_0]$. Such type of control problem is called a Bolza problem. For $(t, x) \in [0, T_0] \times \mathbb{R}^n$, we define the function

$$V(t, x) = \inf\{J_t(y(\cdot))\} \quad (1.0.7)$$

as the value function of the control problem (BP) with initial point (t, x) . The value function (1.0.7) propagates the final cost function g backward from time T_0 , which is the usual setting in optimal control.

The minimal time control problem consists of a given nonempty closed set S (we call it the target set) and a control system in which the dynamic equation is $\dot{x}(t) \in F$, where $F \subseteq \mathbb{R}^n$ is nonempty, closed, convex, bounded, and with 0 as the relative interior point of F . The goal is to steer an initial value point x to the target set S along a trajectory of the system in minimal time. The minimal time value function $T : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is given by

$$T(x) := \inf_{t \geq 0} \{t : S \cap \{x + tF\} \neq \emptyset\}. \quad (1.0.8)$$

If no trajectory of F originating from x can reach S in finite time, then the above infimum is taken over an empty set. So $T(x)$ can be $+\infty$ in this case.

The classic distance function defined as

$$d_S(x) = \inf_{s \in S} \|x - s\| \quad (1.0.9)$$

is a minimal time function, where $F = \overline{\mathbb{B}}$, a closed unit ball. Under the interiority condition $0 \in \text{int } F$, the minimal time function $T(\cdot)$ is globally Lipschitz on \mathbb{R}^n . A lot of recent work on minimal time problems with interiority condition $0 \in \text{int } F$ are analogous to the results concerning classic distance function $d_S(\cdot)$.

Let's review some of the recent work about minimal time problems here. The minimal time function for general nonlinear control problems was first studied in the case of finite dimensional systems by Bardi in [3] using viscosity solutions. Wolenski and Yu in [11] extended these results to allow for noncontrollability and more general boundary conditions using an invariance-based approach. Soravia in [17] proved the existence, uniqueness and representation formulas for the solution

in the sense of viscosity solution. Cannarsa and Sinestrari in [18] studied some convexity properties of the minimal time function for a nonlinear control system with a general target. For linear control systems with convex targets, a semiconvexity result holds and there is an analogy between the distance function and the minimal time function. A necessary and sufficient condition for local Lipschitz continuity is obtained by Veliov in [19].

The general formula for the proximal and the Fréchet subgradients of $T(\cdot)$ in terms of normal vectors to its level sets in infinite dimension are given by Colombo and Wolenski in [2] and [10]. He and Ng in [5] studied the subdifferentials of $T(\cdot)$ in Banach spaces.

However, in the absence of the interiority condition $0 \in \text{int } F$, the minimal time function (1.0.8) could be quite different from the classic distance function (1.0.9). For example, for $F = [-1, 1] \times \{0\} \subset \mathbb{R}^2$ and $S = \mathbb{R} \times \{0\}$, the minimal time function

$$T(x) := \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{otherwise} \end{cases}$$

is not globally Lipschitz on \mathbb{R}^2 .

Jiang and He [9] provided an analysis of the minimal time function without the interiority condition $0 \in \text{int } F$. They gave formulas for calculating the proximal and Fréchet subdifferentials through corresponding normal cones of level sets of the minimal time function. However, the main results in the case $x \notin S$ are under the calmness condition at x . For example, $T(x)$ is calm at x if there exists $k > 0$ and a neighborhood V of x such that

$$|T(y) - T(x)| \leq k\|y - x\|, \quad \forall y \in V. \quad (1.0.10)$$

Mordukhovich and Nam in [13] proved the ϵ -subgradients of minimal time functions at out-of-set points via ϵ -normals to target enlargements without imposing the interiority condition $0 \in \text{int } F$. But the calmness condition is still required in the proof. They also found the horizon subgradient of the minimal time function with convex target set.

The main goal of the first half of this thesis is to extend the results of Colombo and Wolenski [2] and [10] to the case with relative interiority condition $0 \in \text{ri } F$. We pay primary attention to developing the proximal and Fréchet subgradient properties of the minimal time function (1.0.8) without imposing the calmness condition (1.0.10).

In the second half of this dissertation we consider the Bolza problem, another kind of optimal control problem. We are interested in the value functions $V : [0, \infty) \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ of the type

$$\begin{aligned} V(\tau, \xi) &:= \inf \left\{ g(x(0)) + \int_0^\tau L(\dot{x}(t)) dt \mid x(\tau) = \xi \right\}, & (1.0.11) \\ V(0, \xi) &= g(\xi), \end{aligned}$$

with an initial cost function $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ forward from time 0 and a Lagrangian function $L : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, where L only depends on $\dot{x}(t)$. The value function (1.0.7), the usual setting in the optimal control, is covered by (1.0.11) through time reversal.

Rockafellar and Wolenski in [15] showed that the value function, which can be $+\infty$, satisfies the generalized Hamilton-Jacobi equation. They gave an analysis of value functions and the Hamilton-Jacobi equation in a convex Lagrangian case.

The primary goal of the second part of this thesis is to show that the positive hull of the Fréchet subgradient of the value function at the interior point of its domain with Lagrangian $L = I_F$, where $F \subseteq \mathbb{R}^n$ is nonempty, closed, convex, bounded and with $0 \in \text{int } F$, is representable by virtue of Fréchet normal cones

of the lower level sets of the value function under minimal requirements. In this particular case when $L = I_F$, the Hamiltonian function $H : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ defined through the Legendre-Fenchel transform is actually the gauge function associated to the polar of F .

The rest of this dissertation is organized as follows. Chapter 2 contains a review of the required background plus some preliminary results that will be used in the thesis. In Section 2.1, we review some of the concepts and tools in variational analysis. Section 2.2 contains definitions and general properties of gauge functions and polars. Finally, Section 2.3 provides the differential properties of the gauge functions in the $0 \in F$ case, as well as the duality relationships among them.

In Chapter 3, we concern the study of a class of minimal time functions with constant dynamics under the nonempty relative interior condition for the nonconvex target case or without imposing the nonempty relative interior condition for convex target set case. Section 3.1 defines and introduces some properties of the minimal time function, which are widely used in deriving subdifferential results.

In Section 3.2-3.4 we present the main result of the thesis related to evaluating proximal and Fréchet subgradients of the minimal time function under minimal requirements in both the convex and nonconvex cases. We extend the results in Colombo and Wolenski [2] from the nonempty interior case to the nonempty relative interior case or without the nonempty relative interior condition. Most of the results obtained in those sections are new and are inspired by the proof in Colombo and Wolenski [2]. Section 3.2 contains upper estimates and equalities under nonempty relative interior condition as follows:

- (a) A general formula for the proximal subgradient of a minimal time function in terms of proximal normal vectors to its level sets that are scaled in a manner to satisfy the Hamilton-Jacobi equation.

- (b) A general formula for the Fréchet subgradient of a minimal time function in terms of Fréchet normal vectors to its level sets that are scaled in a manner to satisfy the Hamilton-Jacobi equation.
- (c) Upper inclusions for the proximal subgradient at some out-of-set point $x \notin S$, which does not involve the level sets of the minimal time function.
- (d) Upper inclusions for the Fréchet subgradient at some out-of-set point $x \notin S$, which does not involve the level sets of the minimal time function.

Section 3.3 provides the subgradient formulas at the out-of-set point $x \notin S$ and the horizon subgradient formula at in-set point $x \in S$ of the minimal time function without imposing any nonempty relative interior condition when the target set S is a convex set. Mordukhovich and Nam in [12] got a similar result using a different approach.

In Section 3.4, we use the strict convexity of the dynamics to balance and control the nonconvexity of the target set. Under the sort of one-sided Lipschitz condition of the F -projection map and minimal time function, we get the proximal subgradient formula at some out-of-set point $x \notin S$, which does not involve the level sets of the minimal time function.

In Chapter 4, we turn to the Bolza problem in optimal control and the calculus of variations. In Section 4.1, we describe the Bolza problem and collect some properties of the value function of the Bolza problem that relate to this thesis. And in Section 4.2 we derive some new results based on Theorem 2.5 by Rockafellar and Wolenski in [15]. We prove an upper inclusion for both the proximal and Fréchet subgradients of the value function, which involve the level sets of the value function and the Hamilton-Jacobi equation. Next, we give a formula for the positive hull of the Fréchet subgradient of the value function of the Bolza Problem

with a finite convex initial cost function g and a Lagrangian $L = I_F$ under minimal requirements, where F is a closed, bounded, convex set with $0 \in \text{int } F$.

We will discuss some future work in Chapter 5.

Chapter 2

Preliminaries

In this chapter we will give some definitions and important results in convex and nonsmooth analysis that will be used throughout the thesis. See [1] for a detailed development of nonsmooth analysis in finite dimensions. Section 2.1 reviews some of the concepts and basic tools in variational analysis. In Section 2.2, we introduce gauge functions and polars, and their properties that will be used in this thesis. Finally, in Section 2.3, the subgradient properties of the gauge functions are shown.

2.1 Background in variational analysis

Let $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be extended real-valued. In order to single out those points where f is not $+\infty$, we define the (effective) *domain* as the set

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < \infty\}.$$

If $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$, and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, then we call f a *proper* function. The *indicator* function I_C of a set $C \subset \mathbb{R}^n$ is a very useful type of function. It is defined as

$$I_C(x) := \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C. \end{cases}$$

We also define the *epigraph* of f to be the set

$$\text{epi } f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

Thus the epigraph of f consists of all the points in \mathbb{R}^{n+1} lying on or above the graph of f . We also define the lower level sets of f as

$$\text{lev } f := \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}.$$

For $\alpha = \inf f$, one has $\text{lev } f = \text{argmin } f$.

Definition 2.1.1. *The function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is lower semicontinuous (lsc) at \bar{x} if*

$$\liminf_{x \rightarrow \bar{x}} f(x) = f(\bar{x}).$$

In other words, for all $\epsilon > 0$, there exists $\delta > 0$ such that when $y \in B(\bar{x}, \delta)$, we have $f(x) \geq f(\bar{x}) - \epsilon$. And f is lower semicontinuous on \mathbb{R}^n if this holds for every $\bar{x} \in \mathbb{R}^n$.

The following theorem gives characterizations of lower semicontinuity.

Theorem 2.1.2. *(Theorem 1.6 in [1]). For a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, the following properties are equivalent:*

- (a) *f is lower semicontinuous on \mathbb{R}^n ;*
- (b) *the epigraph set $\text{epi } f$ is a closed set in $\mathbb{R}^n \times \mathbb{R}$;*
- (c) *for each α , the level sets of type $\text{lev } f$ are all closed in \mathbb{R}^n .*

A set $C \subseteq \mathbb{R}^n$ is said to be *convex* if for any elements $x_0, x_1 \in C$ the line segment that joins them is contained in C , or in other words, one has

$$(1 - \tau)x_0 + \tau x_1 \in C \quad \text{for all } \tau \in (0, 1). \quad (2.1.1)$$

It's clear that any arbitrary intersection of convex sets is a convex set as well.

Let f be an extended-real valued function defined on a convex set $C \subseteq \mathbb{R}^n$. The function f is said to be *convex* relative to C if for every choice of $x_0 \in C$ and $x_1 \in C$ one has

$$f((1 - \tau)x_0 + \tau x_1) \leq (1 - \tau)f(x_0) + \tau f(x_1) \quad \text{for all } \tau \in (0, 1). \quad (2.1.2)$$

As a consequence, the effective domain $\text{dom } f$ of a convex function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is itself convex because it is the image set of convex set $\text{epi } f$ under a linear transformation (the projection map on \mathbb{R}^n). The indicator function I_C of a set $C \subseteq \mathbb{R}^n$ is convex if and only if C is convex.

Proposition 2.1.3. *For a convex function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, all the level sets $\text{lev } f$ are convex.*

A set $C \subseteq \mathbb{R}^n$ is said to be affine if for any $x_0, x_1 \in C$, the line that goes through them is contained in C , or in other words, one has $(1 - \tau)x_0 + \tau x_1 \in C$ for all $\tau \in (-\infty, \infty)$. For any convex set C in \mathbb{R}^n , the affine hull of C is the smallest affine set that includes C . The interior of C relative to its affine hull is called the relative interior of C , which is denoted by $\text{ri } C$. The relative interior coincides with the true interior if the affine hull is all of \mathbb{R}^n . But it can be used as a substitute for $\text{int } C$ when $\text{int } C$ is an empty set. For arbitrary convex sets C_1 and C_2 and any scalar $\lambda \in \mathbb{R}$, we have the following identities:

$$(a) \text{ ri } (C_1 + C_2) = \text{ri } C_1 + \text{ri } C_2.$$

$$(b) \text{ ri } (\lambda C_1) = \lambda(\text{ri } C_1).$$

Theorem 2.1.4. *(Theorem 6.2 in [16]) Let $C \subseteq \mathbb{R}^n$ be a convex set. Then $\text{ri } C$ is a convex set in \mathbb{R}^n . In particular, if $C \neq \emptyset$, then $\text{ri } C \neq \emptyset$.*

A set $K \subseteq \mathbb{R}^n$ is called a cone if $0 \in K$ and $\lambda x \in K$ for all $x \in K$ and $\lambda > 0$. If a set is a cone and is convex as well, then we call it a convex cone.

We are also interested in forming the smallest cone containing a set $C \subseteq \mathbb{R}^n$. This cone has the formula

$$\text{pos } C = \{0\} \cup \{\lambda x \mid x \in C, \lambda > 0\} \tag{2.1.3}$$

and is called the positive hull of C . If $C = \emptyset$, we have $\text{pos } C = \{0\}$, but if $C \neq \emptyset$, we have $\text{pos } C = \{\lambda x \mid x \in C, \lambda \geq 0\}$.

For a subset C in \mathbb{R}^n , the *horizon cone* is defined as the closed cone C^∞ such that

$$C^\infty = \begin{cases} \{x \mid \exists x_n \in C, \lambda_n \searrow 0, \text{ with } \lambda_n x_n \rightarrow x\} & \text{if } C \neq \emptyset \\ \{0\} & \text{if } C = \emptyset. \end{cases}$$

A function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is said to be *positively homogeneous* if $0 \in \text{dom } f$ and $f(\lambda x) = \lambda f(x)$ for all x and all $\lambda > 0$. The function f is called *sublinear* if in addition we have

$$f(x + x') \leq f(x) + f(x') \quad \text{for all } x \text{ and } x'. \quad (2.1.4)$$

For a set $C \subseteq \mathbb{R}^n$, the *proximal normal cone* to C at $x \in C$, denoted by $N_C^p(x)$, is the set of all $\zeta \in \mathbb{R}^n$ for which there exist $\sigma = \sigma(\zeta, x) \geq 0$ such that

$$\langle \zeta, x' - x \rangle \leq \sigma \|x' - x\|^2 \quad \forall x' \in C. \quad (2.1.5)$$

If C is convex, then the proximal normal cone coincides with the normal cone $N_C(x)$ of convex analysis, and in this case, without loss in generality we can use $\sigma = 0$.

Suppose $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is lower semicontinuous and proper. For $x \in \text{dom } f$, the proximal subgradient $\partial_p f(x)$ is (the possibly empty) set in \mathbb{R}^n defined as those ζ satisfying $(\zeta, -1) \in N_{\text{epi } f}^p(x, f(x))$. Then $\zeta \in \partial_p f(x)$ if and only if there exist positive constants σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in x + \eta \mathbb{B}. \quad (2.1.6)$$

If f is a convex function, then the proximal subgradient $\partial_p f(x)$ coincides with the subgradient of convex analysis, and in this case, it can be simply denoted as ∂f .

The above description is equivalent with $\sigma = 0$ and $\eta = \infty$, if f is convex.

Proposition 2.1.5. (Proposition 1.10 in [14]) Let S be closed and convex. Then

(a) $\zeta \in N_S^p(s)$ iff

$$\langle \zeta, s' - s \rangle \leq 0 \quad \forall s' \in S. \quad (2.1.7)$$

(b) If $s \in \text{bdry}(S)$, then $N_S^p(s) \neq \{0\}$.

Let $C \subseteq \mathbb{R}^n$ and $x \in C$, the Fréchet normal cone $N_C^f(x)$ to C at x is the set of all $\zeta \in \mathbb{R}^n$ such that

$$\limsup_{x' \rightarrow x, x' \in C} \left\langle \zeta, \frac{x' - x}{\|x' - x\|} \right\rangle \leq 0. \quad (2.1.8)$$

Consider a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and a point x with $f(x)$ finite. The corresponding function concept, the Fréchet subgradient $\partial_f f(x)$, can be defined via the epigraph as in the proximal case. Or equivalently by

$$\zeta \in \partial_f f(x) \quad \text{if and only if} \quad \liminf_{x' \rightarrow x} \frac{f(x') - f(x) - \langle \zeta, x' - x \rangle}{\|x' - x\|} \geq 0. \quad (2.1.9)$$

For example, the Fréchet subgradient $\partial_f f(x)$ is the subset of \mathbb{R}^n defined as those ζ satisfying $(\zeta, -1) \in N_{\text{epi } f}^f(x, f(x))$.

The *limiting normal cone* to C at x is the set $N_C^l(x)$ such that

$$N_C^l(x) = \left\{ \zeta : \zeta = \lim_{n \rightarrow \infty} \zeta_n, \zeta_n \in N_C^p(x_n), x_n \rightarrow x \right\}. \quad (2.1.10)$$

A vector ζ is the limiting subgradient of a lower semicontinuous proper function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ at $x \in \text{dom } f$, written as $\zeta \in \partial_l f(x)$, if there are sequences $x_n \rightarrow x$ and $\zeta_n \in \partial_f f(x_n)$ with $\zeta_n \rightarrow \zeta$. Next, let's define the horizon subgradient for a function f . For a vector $\zeta \in \mathbb{R}^n$ one says that ζ is a horizon subgradient of f at x , written as $\zeta \in \partial^\infty f(x)$, if there are sequences $x_n \rightarrow x$, $\lambda_n \downarrow 0$, and $\zeta_n \in \partial_f f(x_n)$ with $\lambda_n \zeta_n \rightarrow \zeta$. Observe that $N_C^p(x) \subseteq N_C^f(x)$ and $N_C^p(x) \subseteq N_C^l(x)$. Similar chains of inclusions hold also for subgradients.

Corollary 2.1.6. (Corollary 8.11 in [1]) Consider a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and a point \bar{x} with $f(\bar{x})$ finite and $\partial_l f(\bar{x}) \neq \emptyset$, one has f regular at \bar{x} if and only if f is locally lsc at \bar{x} with

$$\partial_l f(\bar{x}) = \partial_f f(\bar{x}), \quad \partial^\infty f(\bar{x}) = \partial_f f(\bar{x})^\infty. \quad (2.1.11)$$

Proposition 2.1.7. (Proposition 10.3 in [1]) Suppose C is the lower level set $\{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ for a proper, lsc function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, and let \bar{x} be a point with $f(\bar{x}) = \alpha$. Then

$$N_C^f(\bar{x}) \supset \text{pos } \partial_f f(\bar{x}). \quad (2.1.12)$$

If $0 \notin \partial_l f(\bar{x})$, then we also have

$$N_C^l(\bar{x}) \subset \text{pos } \partial_l f(\bar{x}) \cup \partial^\infty f(\bar{x}). \quad (2.1.13)$$

If f is regular at \bar{x} with $0 \notin \partial_l f(\bar{x})$, then C is regular at \bar{x} and

$$N_C^f(\bar{x}) = \text{pos } \partial_l f(\bar{x}) \cup \partial^\infty f(\bar{x}). \quad (2.1.14)$$

Proposition 2.1.8. (Proposition 8.12 in [1]) Consider a proper, convex function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ with $\bar{x} \in \text{dom } f$, one has

$$\partial_l f(\bar{x}) = \{v \mid f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \text{ for all } x\} = \partial_f f(\bar{x}).$$

and

$$\partial^\infty f(\bar{x}) \subset \{v \mid 0 \geq \langle v, x - \bar{x} \rangle \text{ for all } x \in \text{dom } f\} = N_{\text{dom } f}(\bar{x}).$$

The horizon subgradient inclusion is an equality when f is locally lsc at \bar{x} or when $\partial_l f(\bar{x}) \neq \emptyset$, and in the latter case one also has $\partial^\infty f(\bar{x}) = \partial_f f(\bar{x})^\infty$.

2.2 Gauge functions and polars

We are interested in convex sets $F \subset \mathbb{R}^n$ that are closed, bounded, and with $0 \in \text{ri } F$. However, let's first consider more generally that F is assumed only to be closed, bounded, convex and with $0 \in F$.

The (Minkowski) *gauge* function $\rho_F : \mathbb{R}^n \rightarrow [0, \infty]$ associated to F is defined as

$$\rho_F(\zeta) = \inf \{t \geq 0 : \zeta \in tF\}, \quad (2.2.1)$$

The gauge function $\rho_F(\cdot)$ is nonnegative, lsc, positively homogenous and sublinear with level sets

1. $F = \{\zeta \mid \rho_F(\zeta) \leq 1\}$.
2. $F^\circ = \{\zeta \mid \rho_F(\zeta) = 0\}$.
3. $\text{pos } F = \{\zeta \mid \rho_F(\zeta) < \infty\}$.

Observe that the gauge function $\rho_F(\cdot)$ is convex. The polar F° of F is the set defined as

$$F^\circ := \{\zeta \mid \langle \zeta, v \rangle \leq 1 \text{ for all } v \in F\}, \quad (2.2.2)$$

which is closed and convex with $0 \in F^\circ$. The *bipolar* of F , the set

$$F^{\circ\circ} := (F^\circ)^\circ = \{v \mid \langle v, \zeta \rangle \leq 1 \text{ for all } \zeta \in F^\circ\}, \quad (2.2.3)$$

always agrees with F , because of the closedness and convexity of F . The set F is bounded if and only if $0 \in \text{int } F^\circ$. In fact, for a positive constant M , the following holds

$$F \subseteq M\overline{\mathbb{B}} \quad \text{if and only if} \quad \frac{1}{M}\overline{\mathbb{B}} \subseteq F^\circ. \quad (2.2.4)$$

Define $\|F\| := \sup\{\|f\| \mid f \in F\}$. Let's look at the following proposition for some further consequences.

Proposition 2.2.1. *Suppose F is closed, convex, bounded, and with $0 \in F$. Then*

(a) *For all $\zeta \notin (F^\circ)^\infty$, we have*

$$0 < \rho_{F^\circ}(\zeta) = \max_{v \in F} \langle \zeta, v \rangle < \infty. \quad (2.2.5)$$

(b) *For all nonzero $v \in \text{pos } F$ we have*

$$0 < \rho_F(v) = \sup_{\zeta \in F^\circ} \langle \zeta, v \rangle < \infty. \quad (2.2.6)$$

Proof. (a) Observe that $\zeta \notin (F^\circ)^\infty$ implies that $\rho_{F^\circ}(\zeta)$ is positive and $0 \in \text{int } F^\circ$ implies that there exists $\epsilon > 0$ such that $\overline{B(0, \epsilon)} \subset F^\circ$. Hence, we have $\epsilon \frac{\zeta}{\|\zeta\|} \in F^\circ$ and then

$$\rho_{F^\circ}(\zeta) = \inf\{t : \zeta \in tF^\circ\} \leq \frac{\|\zeta\|}{\epsilon} < \infty. \quad (2.2.7)$$

A calculation shows that

$$\rho_{F^\circ}(\zeta) = \inf\{t : \zeta \in tF^\circ\} = \inf\{t : \langle \zeta, v \rangle \leq t \text{ for all } v \in F\} = \sup_{v \in F} \langle \zeta, v \rangle.$$

Let $\{v_n\}_{n=1}^\infty$ be a sequence in F such that $\lim_{n \rightarrow \infty} \langle \zeta, v_n \rangle = \rho_{F^\circ}(\zeta)$. Because F is bounded, there exists a subsequence $\{v_{n_k}\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} v_{n_k} = \bar{v} \in F$ and therefore $\rho_{F^\circ}(\zeta) = \max_{v \in F} \langle \zeta, v \rangle$.

(b) Observe that $v \in \text{pos } F$ and $v \neq 0$ imply that $0 < \rho_F(v) < \infty$. The following calculation

$$\rho_F(v) = \inf\{t \geq 0 \mid v \in tF\} = \inf\{t \mid \langle \zeta, v \rangle \leq t \text{ for all } \zeta \in F^\circ\} = \sup_{\zeta \in F^\circ} \langle \zeta, v \rangle,$$

completes the proof. □

If we further assume that $0 \in \text{ri } F$, the next proposition shows some further consequences.

Proposition 2.2.2. *Let F be closed, bounded and convex with $0 \in \text{ri } F$. The subspace spanned by F is denoted by $\text{ssp } F$ and the orthogonal complement of $\text{ssp } F$ is denoted by $(\text{ssp } F)^\perp$. Then the following holds.*

- (a) *There exists $m > 0$ such that $F^\circ \cap \text{ssp } F \subseteq \overline{B(0, \frac{1}{m})}$.*
- (b) *Suppose that $w = w_1 + w_2 \in F^\circ$, where $w_1 \in \text{ssp } F$ and $w_2 \in (\text{ssp } F)^\perp$. Then w_1 is bounded by $\frac{1}{m}$.*
- (c) *Let $z \in \text{ssp } F$ and $\zeta \in F^\circ$, then there exists $m > 0$ such that $\langle \zeta, z \rangle \leq \frac{1}{m} \|z\|$.*
- (d) *For all $z \in \text{ssp } F$, there exists $m > 0$ so that $\rho_F(z) \leq \frac{1}{m} \|z\|$.*
- (e) *The gauge function $\rho_F(\cdot)$ is Lipschitz relative to $\text{ssp } F$ with the modulus $\frac{1}{m}$.*

Proof. (a) Observe that $0 \in \text{ri } F$ implies that there exists $m > 0$ such that $\overline{B(0, m)} \cap \text{ssp } F \subseteq F$ because of the definition of the relative interior. For all $w \in F^\circ \cap \text{ssp } F$. The case of $w = 0$ is trivial. Now assume that $w \neq 0$. We have $m \frac{w}{\|w\|} \in \overline{B(0, m)} \cap \text{ssp } F$ and hence $m \frac{w}{\|w\|} \in F$. It implies that $m \|w\| = \langle w, m \frac{w}{\|w\|} \rangle \leq 1$, which completes the proof.

(b) Again $0 \in \text{ri } F$ implies that there exists $m > 0$ so that $\overline{B(0, m)} \cap \text{ssp } F \subseteq F$ by the definition of the relative interior. Hence, $m \frac{w_1}{\|w_1\|} \in F$ and the following calculation

$$m \|w_1\| = \langle m \frac{w_1}{\|w_1\|}, w_1 \rangle + \langle m \frac{w_1}{\|w_1\|}, w_2 \rangle = \langle m \frac{w_1}{\|w_1\|}, w \rangle \leq 1. \quad (2.2.8)$$

completes the proof.

(c) Rewrite ζ as $\zeta = \zeta_1 + \zeta_2$, where $\zeta_1 \in \text{ssp } F$ and $\zeta_2 \in (\text{ssp } F)^\perp$, which along with Part (b) implies that $\|\zeta_1\| \leq \frac{1}{m}$ for some positive m . The latter directly implies the following inequality

$$\langle \zeta, z \rangle = \langle \zeta_1, z \rangle + \langle \zeta_2, z \rangle = \langle \zeta_1, z \rangle \leq \|\zeta_1\| \|z\| \leq \frac{1}{m} \|z\|. \quad (2.2.9)$$

and finish the proof.

(d) For all $z \in \text{ssp } F$, Part (c) implies that there exists $m > 0$ such that $\langle z, \zeta \rangle \leq \frac{1}{m} \|z\|$ for all $\zeta \in F^\circ$. Hence,

$$\rho_F(z) = \sup_{\zeta \in F^\circ} \langle z, \zeta \rangle = \frac{1}{m} \|z\| \quad (2.2.10)$$

and finish the proof.

(e) For all $x, y \in \text{ssp } F$, the subadditivity of ρ_F implies that

$$\rho_F(x) \leq \rho_F(x - y) + \rho_F(y), \quad (2.2.11)$$

together with Part (d) gives us

$$\rho_F(x) - \rho_F(y) \leq \rho_F(x - y) \leq \frac{1}{m} \|x - y\|, \quad \forall x, y \in \text{ssp } F. \quad (2.2.12)$$

Exchange the roles of x and y we get the following inequality:

$$\rho_F(y) - \rho_F(x) \leq \frac{1}{m} \|x - y\|. \quad (2.2.13)$$

Hence, $\rho_F(\cdot)$ is Lipschitz relative to $\text{ssp } F$ with modulus $\frac{1}{m}$. □

2.3 Subgradient properties of the gauge functions

We will review the differentiable properties of the gauge functions in this section and the duality relationships between them as well.

Definition 2.3.1. *Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an extended-valued function. The convex conjugate or the Legendre transform of f is the function f^* defined by*

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle y, x \rangle - f(x)\}. \quad (2.3.1)$$

When f itself is proper, lsc and convex, we have $f^{**} = f$. Moreover, the following proposition gives us a relationship between subgradients of f and those of f^* .

Proposition 2.3.2. (Proposition 11.3 in [1]) For a proper, lsc, convex function f , one has the following equivalence:

$$v \in \partial f(x) \iff x \in \partial f^*(v) \iff f(x) + f^*(v) = \langle v, x \rangle. \quad (2.3.2)$$

For any set $C \subset \mathbb{R}^n$, the conjugate of the indicator function δ_C is the support function of C , defined as

$$\sigma_C(\cdot) := \sup_{v \in C} \langle \cdot, v \rangle. \quad (2.3.3)$$

On the other hand, for any positively homogeneous function h on \mathbb{R}^n the conjugate function h^* is the indicator function δ_C of the set $C = \{x \mid \langle v, x \rangle \leq h(v) \text{ for all } v\}$. One has (See Example 11.4(a) in [1])

$$\bar{v} \in N_C(\bar{x}) \iff \bar{x} \in \partial \sigma_C(\bar{v}) \iff \bar{x} \in C, \langle \bar{v}, \bar{x} \rangle = \sigma_C(\bar{v}). \quad (2.3.4)$$

Proposition 2.3.3. Assume that F is closed, convex, bounded and with $0 \in F$. For given nonzero vectors $v \in \text{pos } F$, $\zeta \notin (F^\circ)^\infty$, the following statements are equivalent.

- (a) $\langle \zeta, v \rangle = \rho_F(v) \rho_{F^\circ}(\zeta)$;
- (b) $\frac{v}{\rho_F(v)}$ attains the max over $u \in F$ of the map $u \mapsto \langle \zeta, u \rangle$;
- (c) $\zeta \in N_F\left(\frac{v}{\rho_F(v)}\right)$;
- (d) $\frac{\zeta}{\rho_{F^\circ}(\zeta)}$ attains the max over $\xi \in F^\circ$ of the map $\xi \mapsto \langle \xi, v \rangle$;
- (e) $v \in N_{F^\circ}\left(\frac{\zeta}{\rho_{F^\circ}(\zeta)}\right)$;
- (f) $\frac{\zeta}{\rho_{F^\circ}(\zeta)} \in \partial \rho_F(v)$;
- (g) $\frac{v}{\rho_F(v)} \in \partial \rho_{F^\circ}(\zeta)$.

Proof. Observe that $\rho_{F^\circ}(\cdot)$ is the support function of F by Proposition 2.2.1 (b).

The conjugate of the indicator function I_F is the support function ρ_{F° . Since I_F is a proper, lsc and convex function, we have $I_F^{**} = I_F$ and conclude that the Legendre-fenchel conjugate

$$\rho_{F^\circ}^*(v) := \sup_{\zeta \in \mathbb{R}^n} \{\langle \zeta, v \rangle - \rho_{F^\circ}(\zeta)\} \quad (2.3.5)$$

is the indicator function

$$I_F(v) := \begin{cases} 0 & \text{if } v \in F \\ \infty & \text{if } v \notin F \end{cases} \quad (2.3.6)$$

of F . Recall that the convex subgradient $\partial I_F(v)$ is the normal cone to F at v . Notice that ρ_F and ρ_{F° are both proper, lsc, convex functions. Because F° is also a closed, convex set as F , the same holds for F° . For example, $\rho_F^*(\cdot) = I_{F^\circ}(\cdot)$ and $\partial I_{F^\circ}(\cdot) = N_{F^\circ}(\cdot)$.

((a) \iff (c)). Part (a) holds if and only if

$$\rho_F(v)\rho_{F^\circ}(\zeta) + \rho_F(v)I_F\left(\frac{v}{\rho_F(v)}\right) = \langle \zeta, v \rangle, \quad (2.3.7)$$

because $\frac{v}{\rho_F(v)} \in F$ and therefore $I_F\left(\frac{v}{\rho_F(v)}\right) = 0$. It's also equivalent to the following equation

$$\rho_F(v)\rho_{F^\circ}(\zeta) + \rho_F(v)\rho_{F^\circ}^*\left(\frac{v}{\rho_F(v)}\right) = \langle \zeta, v \rangle, \quad (2.3.8)$$

because $I_F(\cdot) = \rho_{F^\circ}^*(\cdot)$. Divide both sides by $\rho_F(v)$, we get

$$\rho_{F^\circ}(\zeta) + \rho_{F^\circ}^*\left(\frac{v}{\rho_F(v)}\right) = \left\langle \zeta, \frac{v}{\rho_F(v)} \right\rangle, \quad (2.3.9)$$

together with Proposition 2.3.2 we get

$$\zeta \in \partial \rho_{F^\circ}^*\left(\frac{v}{\rho_F(v)}\right) = \partial I_F\left(\frac{v}{\rho_F(v)}\right) = N_F\left(\frac{v}{\rho_F(v)}\right),$$

which shows Part (c).

((a) \iff (d)). Part (a) means that

$$\sup_{\xi \in F^\circ} \langle \xi, v \rangle = \rho_F(v) = \left\langle \frac{\zeta}{\rho_{F^\circ}(\zeta)}, v \right\rangle, \quad (2.3.10)$$

which is equivalent to (d), because $\frac{\zeta}{\rho_{F^\circ}(\zeta)} \in F^\circ$.

((a) \iff (b)). Part (a) means that

$$\left\langle \zeta, \frac{v}{\rho_F(v)} \right\rangle = \rho_{F^\circ}(\zeta) = \sup_{w \in F} \langle \zeta, w \rangle, \quad (2.3.11)$$

which is equivalent to (b), because $\frac{v}{\rho_F(v)} \in F$.

((c) \iff (g)). Part (c) means that $\zeta \in \partial \rho_{F^\circ}^* \left(\frac{v}{\rho_F(v)} \right)$, which is the same as Part (g) by Proposition 2.3.2.

((a) \iff (e)). Part (a) can be written as

$$\rho_F(v) \rho_{F^\circ}(\zeta) + \rho_{F^\circ}(\zeta) I_{F^\circ} \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right) = \langle \zeta, v \rangle, \quad (2.3.12)$$

because $\frac{\zeta}{\rho_{F^\circ}(\zeta)} \in F^\circ$. Because $I_{F^\circ}(\cdot) = \rho_F^*(\cdot)$ is true, Part (a) also can be written as

$$\rho_F(v) \rho_{F^\circ}(\zeta) + \rho_{F^\circ}(\zeta) \rho_F^* \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right) = \langle \zeta, v \rangle. \quad (2.3.13)$$

Divide both sides by $\rho_{F^\circ}(\zeta)$, we get the following equality

$$\rho_F(v) + \rho_F^* \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right) = \left\langle \frac{\zeta}{\rho_{F^\circ}(\zeta)}, v \right\rangle, \quad (2.3.14)$$

which is equivalent to the following by Proposition 2.3.2

$$v \in \partial \rho_F^* \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right) = \partial I_{F^\circ} \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right) = N_{F^\circ} \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right), \quad (2.3.15)$$

which is Part (e).

((e) \iff (f)). Part (e) holds if and only if $v \in \partial \rho_F^* \left(\frac{\zeta}{\rho_{F^\circ}(\zeta)} \right)$, which is Part (f) by Proposition 2.3.2, and therefore completes the proof. \square

Corollary 2.3.4. *Suppose v is a nonzero vector in $\text{pos } F$. Then*

$$\partial\rho_F(v) = \{\zeta : \rho_{F^\circ}(\zeta) = 1\} \cap N_F\left(\frac{v}{\rho_F(v)}\right). \quad (2.3.16)$$

Proof. (\supseteq) Suppose that $\zeta \in N_F\left(\frac{v}{\rho_F(v)}\right)$ with $\rho_{F^\circ}(\zeta) = 1$, which imply that $\zeta \neq 0$ and $\zeta = \frac{\zeta}{\rho_{F^\circ}(\zeta)} \in \partial\rho_F(v)$ by Proposition 2.3.3.

(\subseteq) For all $\zeta \in \partial\rho_F(v)$, the convexity of ρ_F implies that

$$\rho_F(y) - \rho_F(v) \geq \langle \zeta, y - v \rangle, \quad \text{for all } y \in \mathbb{R}^n. \quad (2.3.17)$$

Let $y = v + w$, where $w \in F$, then we can get the following

$$\begin{aligned} \rho_F(w) &= \rho_F(v) + \rho_F(w) - \rho_F(v) \\ &\geq \rho_F(v + w) - \rho_F(v), \quad \text{since } \rho_F(\cdot) \text{ is subadditive.} \\ &= \rho_F(y) - \rho_F(v) \\ &\geq \langle \zeta, w \rangle. \end{aligned} \quad (2.3.18)$$

Then we have

$$1 \geq \rho_F(w) \geq \sup_{v \in F} \langle \zeta, w \rangle = \rho_{F^\circ}(\zeta). \quad (2.3.19)$$

Let $y = 0$, then $0 - \rho_F(v) \geq \langle \zeta, -v \rangle$, which yields $\langle \zeta, \frac{v}{\rho_F(v)} \rangle \geq 1$. Hence,

$$\rho_{F^\circ}(\zeta) = \max_{v \in F} \langle \zeta, v \rangle \geq \left\langle \zeta, \frac{v}{\rho_F(v)} \right\rangle \geq 1. \quad (2.3.20)$$

It follows that $\rho_{F^\circ}(\zeta) = 1$ and $\zeta \neq 0$ and moreover, $\frac{\zeta}{\rho_{F^\circ}(\zeta)} = \zeta \in \partial\rho_F(v)$. Using Proposition 2.3.3 again, we get $\zeta \in N_F\left(\frac{v}{\rho_F(v)}\right)$ and finish the proof of this equality. \square

Chapter 3

Subgradients of Minimal Time Functions

3.1 The minimal time function

In this section we necessarily define and collect some properties of the minimal time function, which are not related to generalized differentiation. For this section, we assume that F is closed, bounded, convex with $0 \in \text{ri } F$ and S is a nonempty closed set. Note that, the minimal time function is merely the extended-real-valued $T : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ under the assumption that F is closed, convex, bounded and with $0 \in \text{ri } F$, and does not share many common properties with the distance function (1.0.9) as in the $0 \in \text{int } F$ case.

Given a nonempty closed set S , the minimal time function $T(\cdot) : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ defined as

$$T(x) := \inf_{t \geq 0} \{t \mid S \cap \{x + tF\} \neq \emptyset\} \quad (3.1.1)$$

can also be expressed as (see Proposition 3.3 in [12])

$$T(x) = \inf_{s \in S} \rho_F(s - x) \quad (3.1.2)$$

with $\text{dom } T = S + \text{sps } F$. Observe that $T(x) = \min_{s \in S} \rho_F(s - x)$ for $x \in \text{dom } T$.

The lower level sets $S(r)$ of $T(\cdot)$ are defined by

$$S(r) = \{y \in \mathbb{R}^n : T(y) \leq r\} \quad (3.1.3)$$

and will play an important role in our analysis. Given $x \in \mathbb{R}^n$ with $T(x) < \infty$, the F -projection of the point is the set

$$\Pi_S^F(x) = \{s \in S : \rho_F(s - x) = T(x)\}. \quad (3.1.4)$$

It is clear that the nonemptiness of $\Pi_S^F(x)$ means that the infimum in (3.1.2) is attained. Observe that $x \in S$ if and only if $T(x) = 0$. Clearly, $\Pi_S^F(x) \neq \emptyset$ for $x \in \text{dom } T$.

Proposition 3.1.1. *Let F be closed, convex, bounded and with $0 \in \text{ri } F$. Let $x, y \in \text{dom } T$ such that $x - y \in \text{ssp } F$. Then there exists $m > 0$ such that*

$$|T(x) - T(y)| \leq \frac{1}{m} \|x - y\|. \quad (3.1.5)$$

Proof. For all $x, y \in \text{dom } T$ with $x - y \in \text{ssp } F$ and fix $\epsilon > 0$, there exists $s_0 \in S$ such that $\rho_F(s_0 - y) - \epsilon \leq T(y)$. By Proposition 2.2.2 (d) and the subadditivity of $\rho_F(\cdot)$, we have

$$\begin{aligned} T(x) - T(y) &\leq \rho_F(s_0 - x) - \rho_F(s_0 - y) + \epsilon \\ &\leq \rho_F(y - x) + \epsilon \leq \frac{1}{m} \|y - x\| + \epsilon, \end{aligned} \quad (3.1.6)$$

for some $m > 0$. Letting $\epsilon \downarrow 0$ and switching the roles of x and y complete the proof of the proposition. \square

Proposition 3.1.2. *Let $F \subseteq \mathbb{R}^n$ be a closed, convex, bounded set with $0 \in \text{ri } F$. Suppose that $y \in \text{dom } T$ and $v \in F$. Define $g(t) := T(y + tv)$, then the function g is Lipschitz on \mathbb{R} .*

Proof. For all $t_1, t_2 \in \mathbb{R}$, we have $y + t_1 v \in \text{dom } T$ and $y + t_2 v \in \text{dom } T$ and moreover,

$$(y + t_1 v) - (y + t_2 v) = (t_1 - t_2)v \in \text{ssp } F. \quad (3.1.7)$$

By Proposition 3.1.1, there exists $m > 0$ such that

$$\begin{aligned}
|g(t_1) - g(t_2)| &= |T(y + t_1v) - T(y + t_2v)| & (3.1.8) \\
&\leq \frac{1}{m} \|(y + t_1v) - (y + t_2v)\| \\
&\leq \frac{\|F\|}{m} |t_1 - t_2|.
\end{aligned}$$

Hence, g is Lipschitz on \mathbb{R} with constant $\|F\|/m$. \square

Proposition 3.1.3. *Let $F \subseteq \mathbb{R}^n$ be a closed, convex, bounded set with $0 \in \text{ri } F$. Suppose that $y \in \text{dom } T$ and $v \in F$. Define $g(t) := T(y - tv)$, then the function g is Lipschitz on \mathbb{R} .*

Proof. For all $t_1, t_2 \in \mathbb{R}$, we have $y - t_1v \in \text{dom } T$ and $y - t_2v \in \text{dom } T$ and moreover,

$$(y - t_1v) - (y - t_2v) = (-t_1 + t_2)v \in \text{ssp } F. \quad (3.1.9)$$

By Proposition 3.1.1, there exists $m > 0$ such that

$$\begin{aligned}
|g(t_1) - g(t_2)| &= |T(y - t_1v) - T(y - t_2v)| & (3.1.10) \\
&\leq \frac{1}{m} \|(y - t_1v) - (y - t_2v)\| \\
&\leq \frac{\|F\|}{m} |t_1 - t_2|.
\end{aligned}$$

Hence, g is Lipschitz on \mathbb{R} with constant $\|F\|/m$. \square

Proposition 3.1.4. *Let $F \subseteq \mathbb{R}^n$ be a closed, convex, bounded set with $0 \in \text{ri } F$ and $x \in \text{dom } T$. Suppose that there exist constants $k' > 0$ and $\delta > 0$ such that*

$$T(y) - T(x) \leq k' \|y - x\| \quad \forall y \in (x + (\text{ssp } F)^\perp) \cap B(x, \delta). \quad (3.1.11)$$

Then there exists $k > 0$ such that

$$T(z) - T(x) \leq k \|z - x\| \quad \forall z \in \text{dom } T \cap B(x, \delta). \quad (3.1.12)$$

Proof. For all $z \in \text{dom } T \cap B(x, \delta)$. Let's first rewrite $z - x$ as $z - x = w_1 + w_2$, where $w_1 \in \text{ssp } F$ and $w_2 \in (\text{ssp } F)^\perp$. Since $z = w_1 + (x + w_2) \in \text{dom } T = S + \text{ssp } F$, we have $x + w_2 \in S + \text{ssp } F = \text{dom } T$ and $x + w_2 \in B(x, \delta) \cap (x + (\text{ssp } F)^\perp)$. By Proposition 3.1.1 and (3.1.14), we have

$$\begin{aligned}
T(z) - T(x) &= (T(x + w_1 + w_2) - T(x + w_2)) + (T(x + w_2) - T(x)) \\
&\leq \frac{1}{m} \|w_1\| + k' \|w_2\| \\
&\leq \frac{1}{m} \|z - x\| + k' \|z - x\| \\
&= k \|z - x\|,
\end{aligned}$$

where $k := k' + 1/m$ and then complete the proof. \square

Proposition 3.1.5. *Let $F \subseteq \mathbb{R}^n$ be a closed, convex, bounded set with $0 \in \text{ri } F$ and $x \in \text{dom } T$. Suppose that there exist constants $k' > 0$ and $\delta > 0$ such that*

$$T(x) - T(y) \leq k' \|y - x\| \quad \forall y \in (x + (\text{ssp } F)^\perp) \cap B(x, \delta). \quad (3.1.13)$$

Then there exists $k > 0$ such that

$$T(x) - T(z) \leq k \|z - x\| \quad \forall z \in \text{dom } T \cap B(x, \delta). \quad (3.1.14)$$

Proof. For all $z \in \text{dom } T \cap B(x, \delta)$. Let's first rewrite $z - x$ as $z - x = w_1 + w_2$, where $w_1 \in \text{ssp } F$ and $w_2 \in (\text{ssp } F)^\perp$. Since $z = w_1 + (x + w_2) \in \text{dom } T = S + \text{ssp } F$, we have $x + w_2 \in S + \text{ssp } F = \text{dom } T$ and $x + w_2 \in B(x, \delta) \cap (x + (\text{ssp } F)^\perp)$. By Proposition 3.1.1 and (3.1.14), we have

$$\begin{aligned}
T(x) - T(z) &= (-T(x + w_1 + w_2) + T(x + w_2)) + (-T(x + w_2) + T(x)) \\
&\leq \frac{1}{m} \|w_1\| + k' \|w_2\| \\
&\leq \frac{1}{m} \|z - x\| + k' \|z - x\| \\
&= k \|z - x\|,
\end{aligned}$$

where $k := k' + 1/m$ and then complete the proof. \square

Proposition 3.1.6. *(Proposition 3.1 in [13]) Let F be closed, convex and bounded with $0 \in F$. Let $r > 0$ and $x \notin S$ be such that $T(x) < \infty$. Then*

$$T(x) \leq r + \inf_{s \in S(r)} \rho_F(s - x). \quad (3.1.15)$$

The next property is elementary while useful in what follows.

Proposition 3.1.7. *Let F be closed, convex, and bounded with $0 \in \text{ri } F$. Suppose that $x \in \text{dom } T$, for all $v \in F$ and $t \geq 0$, we have*

$$T(x - tv) \leq T(x) + t. \quad (3.1.16)$$

Proof. For all $\epsilon > 0$. There exists $s \in S$ such that

$$\rho_F(s - x) < T(x) + \epsilon. \quad (3.1.17)$$

By subadditivity and positive homogeneity, we have

$$T(x - tv) \leq \rho_F(s - x + tv) \leq \rho_F(s - x) + t\rho_F(v) < T(x) + \epsilon. \quad (3.1.18)$$

Letting $\epsilon \downarrow 0$ proves the proposition. \square

Proposition 3.1.8. *(Proposition 3.5 in [13]) Let F be closed, convex and bounded with $0 \in F$. Then the minimal time function T is lower semicontinuous on its domain.*

Proposition 3.1.9. *(Proposition 3.6 in [13]) The minimal time function T is convex if and only if its target set S is convex.*

Corollary 3.1.10. *Let F be closed, bounded, convex and with $0 \in \text{ri } F$. Suppose that $x \notin S$, $\epsilon > 0$ and $0 < r \leq T(x) < \infty$. Let $s \in S$ satisfy*

$$\rho_F(s - x) \leq T(x) + \epsilon. \quad (3.1.19)$$

Let $v := \frac{s-x}{\rho_F(s-x)} \in F$, and define $z_t := x + tv$ for $t \geq 0$. Now suppose \bar{t} satisfies $T(z_{\bar{t}}) = r$. Then

$$\bar{t} \leq \inf_{z \in S(r)} \rho_F(z - x) + \epsilon. \quad (3.1.20)$$

is true.

Proof. We have

$$\begin{aligned} r &= T(z_{\bar{t}}) = \inf_{s' \in S} \rho_F(s' - z_{\bar{t}}) \leq \rho_F(s - z_{\bar{t}}) \\ &= \rho_F\left(s - x - \bar{t} \frac{s-x}{\rho_F(s-x)}\right) = \rho_F\left([\rho_F(s-x) - \bar{t}] \frac{s-x}{\rho_F(s-x)}\right) \\ &= \inf_{t' \geq 0} \left\{ t' \left| \frac{\rho_F(s-x) - \bar{t}}{t'} \frac{s-x}{\rho_F(s-x)} \in F \right. \right\} \leq \rho_F(s-x) - \bar{t}, \end{aligned} \quad (3.1.21)$$

because $\frac{\rho_F(s-x) - \bar{t}}{\rho_F(s-x) - \bar{t}} \frac{s-x}{\rho_F(s-x)} = \frac{s-x}{\rho_F(s-x)} \in F$. By (3.1.19) and Proposition 3.1.6 we get

$$\bar{t} \leq -r + \rho_F(s-x) \leq -r + T(x) + \epsilon \leq \inf_{z \in S(r)} \rho_F(z-x) + \epsilon. \quad (3.1.22)$$

Let $\epsilon \downarrow 0$, we get the conclusion (3.1.20). \square

3.2 General formulas for $\partial_p T$ and $\partial_f T$

In this section, we first characterize the proximal and the Fréchet subgradient of $T(\cdot)$ in general terms. Next, we prove an upper inclusion for both proximal and Fréchet subgradients at some point $x \notin S$.

Theorem 3.2.1. *(Theorem 3.2 and Theorem 4.2 in [9]) Suppose that S is closed and $F \subseteq \mathbb{R}^n$ is closed, convex, bounded and with $0 \in \text{ri } F$. Let $x \notin S$ and $r := T(x) < \infty$, we have*

$$\partial_p T(x) \subseteq N_{S(r)}^p(x) \cap \{\zeta : \rho_{F^\circ}(-\zeta) = 1\} \quad (3.2.1)$$

and

$$\partial_f T(x) \subseteq N_{S(r)}^f(x) \cap \{\zeta : \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.2.2)$$

Theorem 3.2.2. *Suppose that S is closed and $F \subseteq \mathbb{R}^n$ is closed, convex, bounded and with $0 \in \text{ri } F$. Let $x \notin S$ and $r := T(x) < \infty$. Suppose that there exist constants $k' > 0$ and $\delta' > 0$ such that*

$$T(z) - T(x) \leq k' \|z - x\|, \quad \forall z \in (x + (\text{ssp } F)^\perp) \cap B(x, \delta'). \quad (3.2.3)$$

Then

$$\partial_p T(x) = N_{S(r)}^p(x) \cap \{\zeta : \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.2.4)$$

Proof. (\subseteq) The upper inclusion is proved by Theorem 3.2.1.

(\supseteq) For all $\zeta \in N_{S(r)}^p(x)$ with $\rho_{F^\circ}(-\zeta) = 1$. There exists $\sigma' > 0$ such that

$$\langle \zeta, z - x \rangle \leq \sigma' \|z - x\|^2 \quad \forall z \in S(r). \quad (3.2.5)$$

Let $\eta < \min\{1, \delta', \frac{1}{4\sigma'\|F\|}, \frac{1}{16\sigma'\|F\|^2(\sigma'+\|\zeta\|)}\}$. We need to show that there exists $\sigma > 0$ such that

$$T(y) \geq r + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2, \quad \forall y \in x + \eta \overline{\mathbb{B}}. \quad (3.2.6)$$

There are four possibilities for a point y , which we shall consider separately: (1) $T(y) = \infty$, (2) $T(y) = r$, (3) $r < T(y) < \infty$ and (4) $T(y) < r$.

(1) The case $T(y) = \infty$ is trivial.

(2) The case $T(y) = r$ is also trivial, since (3.2.6) follows automatically from (3.2.5)

for $\sigma = \sigma'$.

(3) Now suppose that $r < T(y) < \infty$. Since

$$T(y) = \min_{s \in S} \rho_F(s - y). \quad (3.2.7)$$

there exists $s \in S$ such that

$$\rho_F(s - y) = T(y). \quad (3.2.8)$$

Let $v := \frac{s-y}{\rho_F(s-y)}$, which is a point in F and define $z_t := y + tv$ and $g(t) := T(z_t)$. By Proposition 3.1.2, g is Lipschitz on \mathbb{R} . Let $t_0 = 0$, then $g(t_0) = T(y) > r$. Now let $t_1 = \rho_F(s - y)$, which implies that $g(t_1) = T(y + \rho_F(s - y)v) = T(s) = 0$. By Intermediate value Theorem, there exists $\bar{t} \in (0, \rho_F(s - y))$ such that $T(z_{\bar{t}}) = g(\bar{t}) = r$. Hence, $z_{\bar{t}} \in S(r)$. We claim that

$$r + \bar{t} \leq \rho_F(s - y). \quad (3.2.9)$$

Let $t' = \rho_F(s - y) - \bar{t}$ and we have

$$\begin{aligned} \frac{s - z_{\bar{t}}}{t'} &= \frac{1}{t'} \left[s - y - \bar{t} \frac{s - y}{\rho_F(s - y)} \right] \\ &= \frac{\rho_F(s - y) - \bar{t}}{t'} \cdot \frac{s - y}{\rho_F(s - y)} = \frac{s - y}{\rho_F(s - y)}. \end{aligned} \quad (3.2.10)$$

Hence we get $\rho_F\left(\frac{s - z_{\bar{t}}}{t'}\right) = 1$ and therefore $\frac{s - z_{\bar{t}}}{t'} \in F$. Notice that

$$\begin{aligned} r &= T(z_{\bar{t}}) = \inf_{s' \in S} \rho_F(s' - z_{\bar{t}}) \\ &\leq \rho_F(s - z_{\bar{t}}) = \inf \left\{ \lambda \geq 0 : \frac{1}{\lambda} (s - z_{\bar{t}}) \in F \right\} \\ &\leq t' = \rho_F(s - y) - \bar{t}, \end{aligned} \quad (3.2.11)$$

which implies (3.2.9). By Proposition 3.1.4, there exists $k_0 > 0$ such that

$$\bar{t} \leq T(y) - T(x) \leq k_0 \|y - x\|. \quad (3.2.12)$$

Since $v := \frac{s-y}{\rho_F(s-y)} \in F$, we have:

$$1 = \rho_{F^\circ}(-\zeta) = \max_{v' \in F} \langle -\zeta, v' \rangle \geq \langle v, -\zeta \rangle. \quad (3.2.13)$$

It follows that $\langle \zeta, v \rangle \geq -1$ and therefore $\bar{t}(1 + \langle \zeta, v \rangle) \geq 0$. Combining (3.2.9) with (3.2.8) yields

$$\begin{aligned} T(y) &= \rho_F(s-y) \geq r + \bar{t} \\ &\geq r + \bar{t} + \langle \zeta, z_{\bar{t}} - x \rangle - \sigma' \|z_{\bar{t}} - x\|^2 \\ &= r + \bar{t} + \langle \zeta, z_{\bar{t}} - y \rangle + \langle \zeta, y - x \rangle - \sigma' \|z_{\bar{t}} - x\|^2 \\ &= r + (\bar{t} + \bar{t} \langle \zeta, v \rangle) + \langle \zeta, y - x \rangle - \sigma' \|z_{\bar{t}} - x\|^2 \\ &\geq r + \langle \zeta, y - x \rangle - \sigma' \|z_{\bar{t}} - x\|^2. \end{aligned} \quad (3.2.14)$$

We claim that there exists a constant $k > 0$ independent of y such that

$$\|z_{\bar{t}} - x\| \leq k \|y - x\|. \quad (3.2.15)$$

In fact,

$$\begin{aligned} \|z_{\bar{t}} - x\| &\leq \|z_{\bar{t}} - y\| + \|y - x\| = \bar{t} \|v\| + \|y - x\| \\ &\leq k_0 \|y - x\| \|v\| + \|y - x\| \quad \text{by (3.2.12)} \\ &= [k_0 \|F\| + 1] \|y - x\| =: k \|y - x\| \end{aligned} \quad (3.2.16)$$

Hence, we can get

$$T(y) \geq r + \langle \zeta, y - x \rangle - \sigma' k^2 \|y - x\|^2. \quad (3.2.17)$$

Letting $\sigma = \sigma' k^2$ directly implies that

$$T(y) \geq r + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2. \quad (3.2.18)$$

(4) Now assume that $T(y) < r$. Observe that $1 = \rho_{F^\circ}(-\zeta) = \max_{v \in F} \langle -\zeta, v \rangle$ and

there exists $\bar{v} \in F$ such that $\langle -\zeta, \bar{v} \rangle = 1$. Define $z_t := y - t\bar{v}$ and $g(t) := T(z_t)$. Observe that z_t is in $S + \text{ssp } F$ for all t . It means that $T(z_t) < \infty$ for all t . By Proposition 3.1.3, g is Lipschitz on \mathbb{R} . We now claim that there exists $\bar{t} \geq 0$ such that $T(z_{\bar{t}}) = r$ with $\bar{t} \leq k\|y - x\|$ for some constant k . The case $t_0 = 0$ implies that $g(t_0) = T(y) < r$. Next, let's show that there exists $z_t \notin S(r)$. In fact, we have

$$\begin{aligned} \|z_t - x\|^2 &= \langle y - x - t\bar{v}, y - x - t\bar{v} \rangle \\ &= \|y - x\|^2 - 2t\langle y - x, \bar{v} \rangle + t^2\|\bar{v}\|^2, \end{aligned} \quad (3.2.19)$$

and

$$\langle \zeta, z_t - x \rangle = \langle \zeta, y - x - t\bar{v} \rangle = \langle \zeta, y - x \rangle - t\langle \zeta, \bar{v} \rangle, \quad (3.2.20)$$

and therefore

$$\begin{aligned} &\sigma'\|z_t - x\|^2 - \langle \zeta, z_t - x \rangle \\ &= \sigma't^2\|\bar{v}\|^2 - 2\sigma't\langle y - x, \bar{v} \rangle + \sigma'\|y - x\|^2 - \langle \zeta, y - x \rangle + t\langle \zeta, \bar{v} \rangle \\ &= \sigma'\|\bar{v}\|^2 t^2 + (-2\sigma'\langle y - x, \bar{v} \rangle - 1)t + (\sigma'\|y - x\|^2 - \langle \zeta, y - x \rangle) \\ &=: at^2 + bt + c. \end{aligned} \quad (3.2.21)$$

Since we have

$$2\sigma'\langle y - x, \bar{v} \rangle \geq -2\sigma'\|F\|\|y - x\| \geq -2\sigma'\|F\|\frac{1}{4\sigma'\|F\|} = -\frac{1}{2}, \quad (3.2.22)$$

and

$$b = -2\sigma'\langle y - x, \bar{v} \rangle - 1 \leq -\frac{1}{2} \quad (3.2.23)$$

with

$$\begin{aligned}
4ac &= 4(\sigma'\|\bar{v}\|^2)(\sigma'\|y-x\|^2 - \langle \zeta, y-x \rangle) \\
&\leq 4\sigma'\|F\|^2(\sigma'\|y-x\| + \|\zeta\|\|y-x\|), \quad \text{since } \|y-x\| \leq 1 \\
&= 4\sigma'\|F\|^2(\sigma' + \|\zeta\|)\|y-x\| \\
&\leq 4\sigma'\|F\|^2(\sigma' + \|\zeta\|)\frac{1}{16\sigma'\|F\|^2(\sigma' + \|\zeta\|)} \\
&= \frac{1}{4} < b^2,
\end{aligned}$$

the above quadratic function has real roots. Then smallest root is given by

$$\hat{t} := \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b + \sqrt{b^2 - 4ac}}. \quad (3.2.24)$$

Let t_1 be slightly larger than \hat{t} , then

$$\sigma'\|z_{t_1} - x\|^2 - \langle \zeta, z_{t_1} - x \rangle < 0. \quad (3.2.25)$$

But

$$\langle \zeta, z - x \rangle \leq \sigma'\|z - x\|^2 \quad \forall z \in S(r). \quad (3.2.26)$$

It follows that $z_{t_1} \notin S(r)$ for all t_1 such that t_1 is slightly larger than \hat{t} , in other words, $g(t_1) = T(z_{t_1}) > r$. By Intermediate value Theorem, there exists $\bar{t} \in (0, \hat{t}]$ such that $T(z_{\bar{t}}) = g(\bar{t}) = r$. Hence,

$$-b = 1 + 2\sigma'\langle y-x, \bar{v} \rangle \geq 1 - 2\sigma'\|y-x\|\|\bar{v}\|. \quad (3.2.27)$$

and $-b + \sqrt{b^2 - 4ac} \geq -b \geq \frac{1}{2}$, which imply the following inequality:

$$\begin{aligned}
\bar{t} &\leq \hat{t} = \frac{2c}{-b + \sqrt{b^2 - 4ac}} \\
&\leq 4\{\sigma'\|y-x\|^2 - \langle \zeta, y-x \rangle\} \\
&= 4(\sigma' + \|\zeta\|)\|y-x\| =: k\|y-x\|.
\end{aligned} \quad (3.2.28)$$

Therefore $T(z_{\bar{t}}) = r$ and $\bar{t} \leq k\|y - x\|$. Since Proposition 3.1.7 implies that

$$T(z_{\bar{t}}) = T(y - \bar{t}\bar{v}) \leq T(y) + \bar{t}, \quad (3.2.29)$$

we have

$$\begin{aligned} T(y) &\geq T(z_{\bar{t}}) - \bar{t} = r + \bar{t}\langle \zeta, \bar{v} \rangle, \quad \text{since } \langle \zeta, \bar{v} \rangle = -1 \\ &= r + \langle \zeta, y - x \rangle - \langle \zeta, y - \bar{t}\bar{v} - x \rangle \\ &= r + \langle \zeta, y - x \rangle - \langle \zeta, z_{\bar{t}} - x \rangle. \end{aligned} \quad (3.2.30)$$

Since $z_{\bar{t}} \in S(r)$, we have

$$\langle \zeta, z_{\bar{t}} - x \rangle \leq \sigma' \|z_{\bar{t}} - x\|^2, \quad (3.2.31)$$

and therefore

$$\begin{aligned} \|z_{\bar{t}} - x\| &= \|y - x - \bar{t}\bar{v}\| \leq \|y - x\| + \bar{t}\|\bar{v}\| \\ &\leq \|y - x\| + k\|y - x\|\|F\| \\ &= [1 + k\|F\|]\|y - x\|. \end{aligned} \quad (3.2.32)$$

Hence,

$$\begin{aligned} T(y) &\geq r + \langle \zeta, y - x \rangle - \sigma' \|z_{\bar{t}} - x\|^2 \\ &\geq r + \langle \zeta, y - x \rangle - \sigma'(1 + k\|F\|)^2 \|y - x\|^2 \end{aligned} \quad (3.2.33)$$

Let $\sigma = \sigma'(1 + k\|F\|)^2$ and we arrive at

$$T(y) \geq r + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2. \quad (3.2.34)$$

and complete the proof of the theorem. \square

Theorem 3.2.3. *Suppose that S is closed and $F \subseteq \mathbb{R}^n$ is closed, convex, bounded and with $0 \in \text{ri } F$. Let $x \notin S$ with $r := T(x) < \infty$. Suppose that there exist constants $k' > 0$ and $\delta' > 0$ such that*

$$T(z) - T(x) \leq k' \|z - x\|, \quad \forall z \in (x + (\text{spp } F)^\perp) \cap B(x, \delta'). \quad (3.2.35)$$

Then we have

$$\partial_f T(x) = N_{S(r)}^f(x) \cap \{\zeta : \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.2.36)$$

Proof. (\subseteq) The upper inclusion is proved by Theorem 3.2.1.

(\supseteq) For all $\zeta \in N_{S(r)}^f(x)$ with $\rho_{F^\circ}(-\zeta) = 1$. Or

$$\limsup_{z \rightarrow x, z \in S(r)} \frac{\langle \zeta, z - x \rangle}{\|z - x\|} \leq 0. \quad (3.2.37)$$

Let $\epsilon > 0$ be fixed. In order to prove that $\zeta \in \partial_f T(x)$ we must find $\delta > 0$ such that if $\|y - x\| \leq \delta$ then

$$T(y) \geq r + \langle \zeta, y - x \rangle - \epsilon \|y - x\|. \quad (3.2.38)$$

There are four possibilities for a point y , which we shall consider separately: (1) $T(y) = \infty$, (2) $T(y) = r$, (3) $r < T(y) < \infty$ and (4) $T(y) < r$.

(1) The case $T(y) = \infty$ is trivial.

(2) The case $T(y) = r$ is also trivial, since (3.2.38) follows automatically from (3.2.37).

(3) Now suppose that $r < T(y) < \infty$. Since

$$T(y) = \min_{s \in S} \rho_F(s - y), \quad (3.2.39)$$

there exists $s \in S$ such that

$$\rho_F(s - y) = T(y). \quad (3.2.40)$$

Let $v := \frac{s-y}{\rho_F(s-y)}$, which is a point in F and define $z_t := y + tv$ and $g(t) := T(z_t)$. By Proposition 3.1.2, g is Lipschitz on \mathbb{R} . Let $t_0 = 0$, then $g(t_0) = T(y) > r$. Now let $t_1 = \rho_F(s-y)$, which implies that $g(t_1) = T(y + \rho_F(s-y)v) = T(s) = 0$. By Intermediate value Theorem, there exists $\bar{t} \in (0, \rho_F(s-y))$ such that $T(z_{\bar{t}}) = g(\bar{t}) = r$. Hence, $z_{\bar{t}} \in S(r)$. We claim that

$$r + \bar{t} \leq \rho_F(s-y). \quad (3.2.41)$$

Let $t' = \rho_F(s-y) - \bar{t}$ and we have

$$\begin{aligned} \frac{s - z_{\bar{t}}}{t'} &= \frac{1}{t'} \left[s - y - \bar{t} \frac{s-y}{\rho_F(s-y)} \right] \\ &= \frac{\rho_F(s-y) - \bar{t}}{t'} \cdot \frac{s-y}{\rho_F(s-y)} = \frac{s-y}{\rho_F(s-y)}. \end{aligned} \quad (3.2.42)$$

Hence we get $\rho_F\left(\frac{s-z_{\bar{t}}}{t'}\right) = 1$ and therefore $\frac{s-z_{\bar{t}}}{t'} \in F$. Notice that

$$\begin{aligned} r &= T(z_{\bar{t}}) = \inf_{s' \in S} \rho_F(s' - z_{\bar{t}}) \\ &\leq \rho_F(s - z_{\bar{t}}) = \inf \left\{ \lambda \geq 0 : \frac{1}{\lambda}(s - z_{\bar{t}}) \in F \right\} \\ &\leq t' = \rho_F(s-y) - \bar{t}. \end{aligned} \quad (3.2.43)$$

If y is close enough to x , by (3.2.35) and Proposition 3.1.4, there exists $k_0 > 0$ such that

$$\bar{t} \leq T(y) - T(x) \leq k_0 \|y - x\|. \quad (3.2.44)$$

Since $v = \frac{s-y}{\rho_F(s-y)} \in F$, we have:

$$1 = \rho_{F^\circ}(-\zeta) = \max_{v' \in F} \langle -\zeta, v' \rangle \geq \langle v, -\zeta \rangle. \quad (3.2.45)$$

It follows that $\langle \zeta, v \rangle \geq -1$ and therefore $\bar{t}(1 + \langle \zeta, v \rangle) \geq 0$. Let $\epsilon' = \epsilon/(k_0\|F\| + 1)$.

If y is close enough to x , by (3.2.37) we get

$$\begin{aligned}
T(y) &= \rho_F(s - y) \geq r + \bar{t} & (3.2.46) \\
&\geq r + \bar{t} + \langle \zeta, z_{\bar{t}} - x \rangle - \epsilon' \|z_{\bar{t}} - x\| \\
&= r + \bar{t} + \langle \zeta, z_{\bar{t}} - y \rangle + \langle \zeta, y - x \rangle - \epsilon' \|z_{\bar{t}} - x\| \\
&= r + (\bar{t} + \bar{t}\langle \zeta, v \rangle) + \langle \zeta, y - x \rangle - \epsilon' \|z_{\bar{t}} - x\| \\
&\geq r + \langle \zeta, y - x \rangle - \epsilon' \|z_{\bar{t}} - x\|.
\end{aligned}$$

We claim that there exists a constant $k > 0$ independent of y such that

$$\|z_{\bar{t}} - x\| \leq k\|y - x\|. \quad (3.2.47)$$

In fact,

$$\begin{aligned}
\|z_{\bar{t}} - x\| &\leq \|z_{\bar{t}} - y\| + \|y - x\| = \bar{t}\|v\| + \|y - x\| & (3.2.48) \\
&\leq k_0\|y - x\|\|v\| + \|y - x\| \\
&\leq [k_0\|F\| + 1]\|y - x\| =: k\|y - x\|
\end{aligned}$$

and

$$T(y) \geq r + \langle \zeta, y - x \rangle - \epsilon'k\|y - x\|, \quad (3.2.49)$$

Then we have

$$T(y) \geq r + \langle \zeta, y - x \rangle - \epsilon\|y - x\|. \quad (3.2.50)$$

(4) Now assume that $T(y) < r$. Observe that $1 = \rho_{F^\circ}(-\zeta) = \max_{v \in F} \langle -\zeta, v \rangle$ and there exists $\bar{v} \in F$ such that $\langle -\zeta, \bar{v} \rangle = 1$. Assume that $\epsilon < 1/\|F\|$. Take $\delta > 0$ so that $\|z - x\| < \delta$ and $z \in S(r)$ imply that

$$\langle \zeta, z - x \rangle < \epsilon\|z - x\|. \quad (3.2.51)$$

Define $z_t := y - t\bar{v}$, which is in $S + \text{ssp } F$ and $g := T(z_t)$. By Proposition 3.1.2, g is Lipschitz on \mathbb{R} . Hence, $T(z_t) < \infty$, for all t . We now claim that there exists \bar{t} and k so that $T(z_{\bar{t}}) \notin S(r)$ with $0 \leq \bar{t} \leq k\|y - x\|$. If t is small enough, then by Proposition 3.1.7 we have

$$T(z_t) = T(y - t\bar{v}) \leq T(y) + t \leq r. \quad (3.2.52)$$

Hence, $z_t \in S(r)$ for small t . Let $\epsilon' < \min(\frac{1}{\|F\|}, \epsilon/(1 + \|F\|\|\zeta\| + \epsilon\|F\|))$. Indeed, if y is close enough to x and t is small, then

$$\langle \zeta, z_t - x \rangle < \epsilon'\|z_t - x\|. \quad (3.2.53)$$

Moreover,

$$\begin{aligned} \frac{\langle \zeta, z_t - x \rangle}{t} &= \frac{\langle \zeta, y - x \rangle - t\langle \zeta, \bar{v} \rangle}{t} \\ &= 1 + \frac{\langle \zeta, y - x \rangle}{t} \rightarrow 1 \quad \text{for } t \rightarrow +\infty. \end{aligned} \quad (3.2.54)$$

On the other hand,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{(\epsilon')^2 \|z_t - x\|^2}{t^2} &= \lim_{t \rightarrow +\infty} \frac{(\epsilon')^2 \langle y - x - t\bar{v}, y - x - t\bar{v} \rangle}{t^2} \\ &= \lim_{t \rightarrow +\infty} (\epsilon')^2 \left(\frac{\|y - x\|^2}{t^2} - \frac{2t\langle y - x, \bar{v} \rangle}{t^2} + \|\bar{v}\|^2 \right) \\ &= (\epsilon')^2 \|\bar{v}\|^2 \leq (\epsilon')^2 \|F\|^2 < 1. \end{aligned} \quad (3.2.55)$$

Hence,

$$\lim_{t \rightarrow +\infty} \frac{\epsilon'\|z_t - x\|}{t} < 1. \quad (3.2.56)$$

Define

$$f(t) := \frac{\langle \zeta, z_t - x \rangle - \epsilon'\|z_t - x\|}{t}, \quad (3.2.57)$$

which is continuous on $(0, \infty)$. We know that $f(t) < 0$ for small t and $f(t) > 0$ for large t . By Intermediate value theorem, there exists \hat{t} such that $f(\hat{t}) = 0$, or in

other words,

$$\langle \zeta, z_{\hat{t}} - x \rangle = \epsilon' \|z_{\hat{t}} - x\|. \quad (3.2.58)$$

Then

$$\begin{aligned} \langle \zeta, y - x \rangle + \hat{t} &= \langle \zeta, y - x \rangle - \hat{t} \langle \zeta, \bar{v} \rangle \\ &= \epsilon' \|z_{\hat{t}} - x\| \\ &\leq \epsilon' \|y - x\| + \epsilon' \hat{t} \|\bar{v}\| \\ &\leq \epsilon' \|y - x\| + \epsilon' \hat{t} \|F\|. \end{aligned} \quad (3.2.59)$$

Hence,

$$\begin{aligned} \hat{t} - \epsilon' \hat{t} \|F\| &\leq -\langle \zeta, y - x \rangle + \epsilon' \|y - x\| \\ &\leq (\|\zeta\| + \epsilon') \|y - x\|. \end{aligned} \quad (3.2.60)$$

Define $k := (\|\zeta\| + \epsilon') / (1 - \epsilon' \|F\|)$ and we have $\hat{t} \leq k \|y - x\|$. If y is close enough to x , then $z_{\hat{t}} \notin S(r)$ because (3.2.51) is violated. Hence, $g(\hat{t}) = T(z_{\hat{t}}) > r$. For small t , we have $g(t) = T(z_t) < r$. By Intermediate value theorem, for y is close enough to x , there exists $0 < \bar{t} < \hat{t}$ such that $T(z_{\bar{t}}) = r$ and $\bar{t} \leq k \|y - x\|$. By Proposition 3.1.7, we have $T(z_{\bar{t}}) \leq T(y) + \bar{t}$. Hence,

$$\begin{aligned} T(y) &\geq T(z_{\bar{t}}) - \bar{t} = r + \langle \zeta, \bar{t} \bar{v} \rangle \\ &= r + \langle \zeta, y - z_{\bar{t}} \rangle \\ &= r + \langle \zeta, y - x \rangle - \langle \zeta, z_{\bar{t}} - x \rangle \end{aligned} \quad (3.2.61)$$

Since

$$\begin{aligned} \langle \zeta, z_{\bar{t}} - x \rangle &< \epsilon' \|z_{\bar{t}} - x\| \\ &\leq \epsilon' (\|y - x\| + \bar{t} \|\bar{v}\|) \\ &\leq \epsilon' (1 + k \|F\|) \|y - x\|, \end{aligned} \quad (3.2.62)$$

We have

$$\begin{aligned} T(y) &\geq r + \langle \zeta, y - x \rangle - \epsilon'(1 + k\|F\|)\|y - x\| \\ &\geq r + \langle \zeta, y - x \rangle - \epsilon\|y - x\|, \end{aligned} \quad (3.2.63)$$

and then complete the proof. \square

Corollary 3.2.4. *Let F be closed, convex, bounded and with $0 \in \text{ri } F$. Let $x \in S^c \cap \text{dom } T$, $s \in \Pi_S^F(x)$ and $\zeta \in \partial\rho_F(s - x)$. Then*

$$\left\langle \zeta, \frac{s - x}{\rho_F(s - x)} \right\rangle = 1. \quad (3.2.64)$$

Proof. Because $s \in \Pi_S^F(x)$ and $x \in S^c \cap \text{dom } T$, we have $0 < T(x) = \rho_F(s - x) < \infty$ and $s - x \neq 0$. $\zeta \in \partial\rho_F(s - x)$ implies that $\rho_{F^\circ}(\zeta) = 1$ and $\zeta \in N_F\left(\frac{s - x}{\rho_F(s - x)}\right)$ by Corollary 2.3.4. We have

$$\left\langle \zeta, f - \frac{s - x}{\rho_F(s - x)} \right\rangle \leq 0 \quad \text{for all } f \in F \quad (3.2.65)$$

and then

$$1 = \rho_{F^\circ}(\zeta) = \sup_{f \in F} \langle \zeta, f \rangle \leq \left\langle \zeta, \frac{s - x}{\rho_F(s - x)} \right\rangle \leq 1, \quad (3.2.66)$$

because $\zeta \in F^\circ$ and $\frac{s - x}{\rho_F(s - x)} \in F$. This justify the equality in (3.2.64) and completes the proof. \square

Theorem 3.2.5. *Let S be closed and F be closed, convex, bounded and with $0 \in \text{ri } F$. Suppose that $x \in S^c \cap \text{dom } T$, then we have*

$$\partial_p T(x) \subseteq N_S^p(s) \cap (-\partial\rho_F(s - x)), \quad \forall s \in \Pi_S^F(x). \quad (3.2.67)$$

Proof. For all $\epsilon > 0$. The case when $\partial_p T(x) = \emptyset$ is trivial. Now suppose that $\partial_p T(x) \neq \emptyset$. $x \in \text{dom } T$ implies that $\Pi_S^F(x) \neq \emptyset$. For all $\zeta \in \partial_p T(x)$ and for all

$s \in \Pi_S^F(x)$, let $r = T(x) = \rho_F(s - x)$. Then $\rho_{F^\circ}(-\zeta) = 1$ and $\zeta \in N_{S(r)}^p(x)$. Since $x \in S^c \cap \text{dom } T$ implies that $\infty > T(x) = \rho_F(s - x) > 0$, we have $0 \neq s - x \in \text{ssp } F$.

First, let's show that $\zeta \in (-\partial\rho_F(s - x))$. Let $\bar{v} := \frac{s-x}{\rho_F(s-x)}$, which is in F and let $y_i = x + t_i\bar{v}$, where $t_i > 0$.

$$\begin{aligned} \frac{1}{\rho_F(s-x) - t_i}(s - y_i) &= \frac{s - x - t_i \frac{s-x}{\rho_F(s-x)}}{\rho_F(s-x) - t_i} \\ &= \frac{\frac{s-x}{\rho_F(s-x)}(\rho_F(s-x) - t_i)}{\rho_F(s-x) - t_i} \\ &= \frac{s-x}{\rho_F(s-x)} \in F. \end{aligned} \tag{3.2.68}$$

Then

$$T(y_i) \leq \rho_F(s - y_i) \leq \rho_F(s - x) - t_i = r - t_i, \tag{3.2.69}$$

and therefore for y_i close enough to x , we have

$$\begin{aligned} r - t_i &\geq T(y_i) \\ &\geq r + \langle \zeta, y_i - x \rangle - \epsilon \|y_i - x\|, \text{ since } \zeta \in \partial_p T(x) \subseteq \partial_f T(x) \\ &= r + t_i \langle \zeta, \bar{v} \rangle - \epsilon t_i \|\bar{v}\|. \end{aligned} \tag{3.2.70}$$

Now subtract r on both sides and divide by t_i we get: $-1 \geq \langle \zeta, \bar{v} \rangle - \epsilon \|\bar{v}\|$. Let $\epsilon \downarrow 0$.

We have $\langle \zeta, \bar{v} \rangle \leq -1$. And $\sup_{v \in F} \langle -\zeta, v \rangle = \rho_{F^\circ}(-\zeta) = 1$ implies that $\langle -\zeta, v \rangle \leq 1$,

for all $v \in F$. Therefore we have $\langle -\zeta, v - \bar{v} \rangle \leq 0$, for all $v \in F$ and then

$$-\zeta \in N_F(\bar{v}) = N_F\left(\frac{s-x}{\rho_F(s-x)}\right). \tag{3.2.71}$$

Combining with $\rho_{F^\circ}(-\zeta) = 1$ implies that $-\zeta \in \partial\rho_F(s - x)$ by Corollary 2.3.4.

Second, let's show that $\zeta \in N_S^p(s)$. Since $\zeta \in N_{S(r)}^p(x)$, there exists $\sigma = \sigma(\zeta, x) \geq 0$ such that

$$\langle \zeta, y - x \rangle \leq \sigma \|y - x\|^2, \quad \forall y \in S(r). \tag{3.2.72}$$

For all $s' \in S$, let $y := s' + x - s$. Then $T(y) \leq \rho_F(s' - y) = \rho_F(s - x) = r$ and thus $y \in S(r)$. Then (3.2.72) implies that

$$\langle \zeta, s' - s \rangle = \langle \zeta, y - x \rangle \leq \sigma \|y - x\|^2 \leq \sigma \|s' - s\|^2, \quad \forall s' \in S. \quad (3.2.73)$$

Thus $\zeta \in N_S^p(s)$ and finish the proof. \square

Theorem 3.2.6. *Let S be closed and let F be closed, convex, bounded and with $0 \in \text{ri } F$. Suppose that $x \in S^c \cap \text{Dom}(T)$, then we have*

$$\partial_f T(x) \subseteq N_S^f(s) \cap (-\partial \rho_F(s - x)), \quad \forall s \in \Pi_S^F(x). \quad (3.2.74)$$

Proof. For all $\epsilon > 0$. The case when $\partial_f T(x) = \emptyset$ is trivial. Now suppose that $\partial_f T(x) \neq \emptyset$. $x \in \text{dom } T$ implies that $\Pi_S^F(x) \neq \emptyset$. For all $\zeta \in \partial_f T(x)$ and for all $s \in \Pi_S^F(x)$, let $r = T(x) = \rho_F(s - x)$. Then $\rho_{F^\circ}(-\zeta) = 1$ and $\zeta \in N_{S(r)}^f(x)$. Since $x \in S^c \cap \text{dom } T$ implies that $\infty > T(x) = \rho_F(s - x) > 0$, we have $0 \neq s - x \in \text{ssp } F$. First, let's show that $\zeta \in (-\partial \rho_F(s - x))$.

Let $\bar{v} := \frac{s-x}{\rho_F(s-x)}$, which is in F and let $y_i = x + t_i \bar{v}$, where $t_i > 0$.

$$\begin{aligned} \frac{1}{\rho_F(s-x) - t_i} (s - y_i) &= \frac{s - x - t_i \frac{s-x}{\rho_F(s-x)}}{\rho_F(s-x) - t_i} \\ &= \frac{\frac{s-x}{\rho_F(s-x)} (\rho_F(s-x) - t_i)}{\rho_F(s-x) - t_i} \\ &= \frac{s-x}{\rho_F(s-x)} \in F. \end{aligned} \quad (3.2.75)$$

Then

$$T(y_i) \leq \rho_F(s - y_i) \leq \rho_F(s - x) - t_i = r - t_i, \quad (3.2.76)$$

and therefore for y_i close enough to x , we have

$$\begin{aligned} r - t_i &\geq T(y_i) \\ &\geq r + \langle \zeta, y_i - x \rangle - \epsilon \|y_i - x\|, \text{ since } \zeta \in \partial_f T(x) \\ &= r + t_i \langle \zeta, \bar{v} \rangle - \epsilon t_i \|\bar{v}\|. \end{aligned} \quad (3.2.77)$$

Now subtract r on both sides and divide by t_i we get: $-1 \geq \langle \zeta, \bar{v} \rangle - \epsilon \|\bar{v}\|$. Let $\epsilon \downarrow 0$. We have $\langle \zeta, \bar{v} \rangle \leq -1$. And $\sup_{v \in F} \langle -\zeta, v \rangle = \rho_{F^\circ}(-\zeta) = 1$ implies that $\langle -\zeta, v \rangle \leq 1$, for all $v \in F$. Therefore we have $\langle -\zeta, v - \bar{v} \rangle \leq 0$, for all $v \in F$. And then $-\zeta \in N_F(\bar{v}) = N_F\left(\frac{s-x}{\rho_F(s-x)}\right)$. Combining with $\rho_{F^\circ}(-\zeta) = 1$ implies that $-\zeta \in \partial \rho_F(s-x)$ by Corollary 2.3.4.

Next, let's show that $\zeta \in N_S^f(s)$. Take $s' \in S$ and set $y := s' + x - s$. Since $T(y) \leq \rho_F(s' - y) = \rho_F(s - x) = T(x) = r$, we know that $y \in S(r)$ for all $s' \in S$.

Thus

$$\limsup_{s' \rightarrow s, s' \in S} \frac{\langle \zeta, s' - s \rangle}{\|s' - s\|} = \limsup_{y \rightarrow x, y \in S(r)} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \leq 0, \quad (3.2.78)$$

where the inequality follows from $\zeta \in N_{S(r)}^f(x)$. Hence, $\zeta \in N_S^f(s)$. \square

3.3 The case where S is convex

In this section, we are assuming that F is closed, convex, bounded with $0 \in F$.

Observe that $T(x) = 0$ if and only if $x \in S$. Clearly, $T(x) = \min_{s \in S} \rho_F(s - x)$ and $\Pi_S^F(x) \neq \emptyset$ for $x \in \text{dom } T$.

Proposition 3.3.1. *Let $\bar{x} \in S^c \cap \text{dom } T$ and $\bar{s} \in \Pi_S^F(\bar{x})$. Then*

$$N_{\bar{x}+T(\bar{x})F}(\bar{s}) = N_F\left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right). \quad (3.3.1)$$

Proof. Observe that $\bar{s} \in \Pi_S^F(\bar{x})$ and $\bar{x} \in S^c$ imply that $T(\bar{x}) = \rho_F(\bar{s} - \bar{x}) > 0$. Then

$$\begin{aligned} N_{\bar{x}+T(\bar{x})F}(\bar{s}) &= \{\zeta \mid \langle \zeta, s' - \bar{s} \rangle \leq 0, \quad \forall s' \in \bar{x} + T(\bar{x})F\} \\ &= \{\zeta \mid \langle \zeta, \bar{x} + T(\bar{x})f - \bar{s} \rangle \leq 0, \quad \forall f \in F\} \\ &= \{\zeta \mid \left\langle \zeta, T(\bar{x})\left(f - \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right)\right\rangle \leq 0, \quad \forall f \in F\} \\ &= N_F\left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right). \end{aligned} \quad (3.3.2)$$

and thus complete the proof of the proposition. \square

For the convex set S case, it is convenient to have the following concept.

Definition 3.3.2. (Definition 4.1 in [2]) Let F be closed, bounded, convex and with $0 \in F$ and suppose S is convex, $\bar{x} \in S^c \cap \text{dom } T$, and $\bar{s} \in \Pi_S^F(\bar{x})$. Then S/F separating normal cone $SEP(S/F, \bar{s}, \bar{x})$ for (\bar{s}, \bar{x}) is defined by

$$SEP(S/F, \bar{s}, \bar{x}) := N_S(\bar{s}) \cap \left\{ -N_F\left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right) \right\}. \quad (3.3.3)$$

Proposition 3.3.3. Let F be closed, bounded, convex with $0 \in F$. For all $v \in \text{ri } F$, we have $\rho_F(v) < 1$.

Proof. Since $v \in \text{ri } F \subseteq F$, we have $\rho_F(v) \leq 1$ and there exists $\delta > 0$ such that $\text{aff } F \cap B(v, \delta) \subseteq F$. For ϵ small enough, we have

$$\frac{v}{1 - \epsilon} = \left(1 - \frac{1}{1 - \epsilon}\right)0 + \frac{1}{1 - \epsilon}v \in \text{aff } F \quad (3.3.4)$$

and

$$\frac{v}{1 - \epsilon} \in B(v, \delta). \quad (3.3.5)$$

Hence, $\frac{v}{1 - \epsilon} \in F$ implies that $\rho_F\left(\frac{v}{1 - \epsilon}\right) \leq 1$. Hence, $\rho_F(v) \leq 1 - \epsilon < 1$. \square

Theorem 3.3.4. Let S_1 and S_2 be closed and convex sets with S_2 bounded and $S_1 \cap (\text{ri } S_2) = \emptyset$ and $S_1 \cap S_2 \neq \emptyset$. Then for all $\bar{s} \in S_1 \cap S_2$, we have

$$N_{S_1}(\bar{s}) \cap \left\{ -N_{S_2}(\bar{s}) \right\} \neq \{0\} \quad (3.3.6)$$

and the cone $N_{S_1}(\bar{s}) \cap \{-N_{S_2}(\bar{s})\}$ is independent of the choice of $\bar{s} \in S_1 \cap S_2$.

Proof. Let $\tilde{S} = S_1 - S_2$. Since S_2 is bounded, \tilde{S} is closed and convex. $S \cap (\text{ri } S_2) = \emptyset$ implies that $0 \notin S_1 - (\text{ri } S_2)$. Since

$$\text{int } (S_1 - S_2) \subseteq \text{ri } (S_1 - S_2) = \text{ri } S_1 - \text{ri } S_2 \subseteq S_1 - \text{ri } S_2, \quad (3.3.7)$$

we have $0 \notin \text{int}(S_1 - S_2)$, but $S_1 \cap S_2 \neq \emptyset$ and hence $0 \in (S_1 - S_2)$, which implies that $0 \in \text{bdry}(S_1 - S_2) = \text{bdry} \tilde{S}$. By Proposition 2.1.5 Part (b) implies that $N_{\tilde{S}}(0) \neq \{0\}$, which also implies that there exists a nonzero vector $\zeta \in N_{\tilde{S}}(0)$. Since \tilde{S} is closed and convex and by Proposition 2.1.5 Part (a) we have

$$\langle \zeta, s_1 - s_2 \rangle \leq 0, \quad \forall s_1 \in S_1 \quad \text{and} \quad \forall s_2 \in S_2. \quad (3.3.8)$$

Let $\bar{s} \in S_1 \cap S_2$. Then $\langle \zeta, s_1 - \bar{s} \rangle \leq 0, \forall s_1 \in S_1$, and

$$\langle -\zeta, s_2 - \bar{s} \rangle = \langle \zeta, \bar{s} - s_2 \rangle \leq 0, \quad \forall s_2 \in S_2. \quad (3.3.9)$$

Then $\zeta \in N_{S_1}(\bar{s})$ and $-\zeta \in N_{S_2}(\bar{s})$. Then nonzero $\zeta \in N_{S_1}(\bar{s}) \cap \{-N_{S_2}(\bar{s})\}$ and ζ does not depend on the choice of $\bar{s} \in S_1 \cap S_2$. \square

Theorem 3.3.5. *Let F be closed, bounded, convex and with $0 \in F$ and suppose that S is convex and $\bar{x} \in S^c \cap \text{Dom}(T)$, then $\text{SEP}(S/F, \bar{s}, \bar{x}) \neq \{0\}$ and the cone $\text{SEP}(S/F, \bar{s}, \bar{x})$ is independent of the choice of $\bar{s} \in \Pi_S^F(\bar{x})$.*

Proof. Let $S_1 = S$ and $S_2 = \bar{x} + T(\bar{x})F$. Then S_1 is closed and convex and S_2 is closed, convex and bounded. Then

$$\text{ri } S_2 = \text{ri}(\bar{x} + T(\bar{x})F) = \bar{x} + T(\bar{x})(\text{ri } F). \quad (3.3.10)$$

Suppose that $S_1 \cap (\text{ri } S_2) \neq \emptyset$. There exists $s \in S$ with

$$s \in \text{ri}(\bar{x} + T(\bar{x})F) = \bar{x} + T(\bar{x})(\text{ri } F), \quad (3.3.11)$$

which implies that $\frac{s - \bar{x}}{T(\bar{x})} \in \text{ri } F$. Then $\rho_F\left(\frac{s - \bar{x}}{T(\bar{x})}\right) < 1$ by Proposition 3.3.3, but

$$\rho_F\left(\frac{s - \bar{x}}{T(\bar{x})}\right) = \frac{1}{T(\bar{x})} \rho_F(s - \bar{x}) \geq 1. \quad (3.3.12)$$

It's a contradiction. Hence, $S_1 \cap (\text{ri } S_2) = \emptyset$. For all $\bar{s} \in \Pi_S^F(\bar{x})$, we have $\bar{s} \in S \cap (\bar{x} + T(\bar{x})F) \neq \emptyset$. By Theorem 3.3.4, we have

$$N_S(\bar{s}) \cap \{-N_{(\bar{x} + T(\bar{x})F)}(\bar{s})\} \neq \{0\}, \quad (3.3.13)$$

and the cone $N_S(\bar{s}) \cap \{-N_{\bar{x}+T(\bar{x})F}(\bar{s})\}$ is independent of the choice of $\bar{s} \in S \cap (\bar{x} + T(\bar{x})F) = \Pi_S^F(x)$. By Proposition 3.3.1, we have

$$N_S(\bar{s}) \cap \left\{ -N_F \left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} \right) \right\} \neq \{0\}, \quad (3.3.14)$$

and the cone $SEP(S/F, \bar{s}, \bar{x})$ is independent of the choice of $\bar{s} \in \Pi_S^F(\bar{x})$. \square

Theorem 3.3.6. (Theorem 7.1 in [13]) *Let S be closed and convex. Let F be closed, convex, bounded and with $0 \in F$. Suppose that $x \in S^c \cap \text{dom } T$ and $r := T(x)$, then*

$$\partial T(x) = N_{S(r)}(x) \cap \{\zeta \mid \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.3.15)$$

Theorem 3.3.7. *Let F be closed, bounded, convex and with $0 \in F$ and suppose that S is convex. Then*

$$\begin{aligned} \partial T(\bar{x}) &= \{\zeta \mid \zeta \in SEP(S/F, \bar{s}, \bar{x}) \text{ for some } \bar{s} \in \Pi_S^F(\bar{x}) \text{ and } \rho_{F^\circ}(-\zeta) = 1\} \\ &= \{\zeta \mid \zeta \in SEP(S/F, \bar{s}, \bar{x}) \text{ for all } \bar{s} \in \Pi_S^F(\bar{x}) \text{ and } \rho_{F^\circ}(-\zeta) = 1\}, \end{aligned}$$

where $\bar{x} \in \text{dom}(T) \cap S^c$.

Proof. Theorem 3.3.5 implies that

$$\begin{aligned} &\{ \zeta : \zeta \in SEP(S/F, \bar{s}, \bar{x}) \text{ for some } \bar{s} \in \Pi_S^F(\bar{x}) \text{ and } \rho_{F^\circ}(-\zeta) = 1 \} \\ &= \{ \zeta : \zeta \in SEP(S/F, \bar{s}, \bar{x}) \text{ for all } \bar{s} \in \Pi_S^F(\bar{x}) \text{ and } \rho_{F^\circ}(-\zeta) = 1 \}. \end{aligned}$$

Fix $\bar{s} \in \Pi_S^F(\bar{x})$. We need to show that

$$\partial T(\bar{x}) = \{\zeta : \zeta \in SEP(S/F, \bar{s}, \bar{x}) \text{ and } \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.3.16)$$

Let $r := T(\bar{x}) > 0$. Since $T(\cdot)$ is convex, the lower level set $S(r)$ is convex as well.

By Theorem 3.3.6 we have

$$\partial T(\bar{x}) = N_{S(r)}(\bar{x}) \cap \{\zeta : \rho_{F^\circ}(-\zeta) = 1\}. \quad (3.3.17)$$

We need to show that $N_{S(r)}(\bar{x}) = \text{SEP}(S/F, \bar{s}, \bar{x})$.

(\supseteq) For all $\zeta \in \text{SEP}(S/F, \bar{s}, \bar{x})$ and for all $x \in S(r)$. Then $\zeta \in N_S(\bar{s})$ implies that

$$\langle \zeta, s - \bar{s} \rangle \leq 0 \quad \forall s \in S \quad (3.3.18)$$

and $-\zeta \in N_F\left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right)$ implies that

$$\left\langle -\zeta, v - \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} \right\rangle \leq 0 \quad \forall v \in F. \quad (3.3.19)$$

For all $s \in \Pi_S^F(x)$ we have

$$\rho_F(s - x) = T(x) \leq r = \rho_F(\bar{s} - \bar{x}), \quad (3.3.20)$$

which implies that $0 \leq \frac{\rho_F(s-x)}{r} \leq 1$. Then together with (3.3.18) we have

$$\begin{aligned} \frac{1}{r} \langle \zeta, x - \bar{x} \rangle &= \frac{1}{r} [\langle \zeta, x - s \rangle + \langle \zeta, s - \bar{s} \rangle + \langle \zeta, \bar{s} - \bar{x} \rangle] \\ &\leq \left\langle \zeta, \frac{\bar{s} - \bar{x}}{r} - \frac{s - x}{r} \right\rangle. \end{aligned} \quad (3.3.21)$$

Since $\frac{s-x}{\rho_F(s-x)} \in F$ and F is convex, we have

$$\frac{s-x}{r} = \frac{\rho_F(s-x)}{r} \frac{s-x}{\rho_F(s-x)} + \left(1 - \frac{\rho_F(s-x)}{r}\right) 0 \in F. \quad (3.3.22)$$

Then (3.3.22) and (3.3.19) gives us

$$\frac{1}{r} \langle \zeta, x - \bar{x} \rangle \leq \left\langle -\zeta, \frac{s-x}{r} - \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} \right\rangle \leq 0, \quad (3.3.23)$$

and therefore $\langle \zeta, x - \bar{x} \rangle \leq 0$ for all $x \in S(r)$, which means that $\zeta \in N_{S(r)}(\bar{x})$.

(\subseteq) For all $\zeta \in N_{S(r)}(\bar{x})$. Then $\langle \zeta, x_1 - \bar{x} \rangle \leq 0$, for all $x_1 \in S(r)$. For all $s \in S$, let $x := s + \bar{x} - \bar{s}$. Then

$$T(x) \leq \rho_F(s - x) = \rho_F(\bar{s} - \bar{x}) = T(\bar{x}) = r. \quad (3.3.24)$$

Hence, $x \in S(r)$ and

$$\langle \zeta, s - \bar{s} \rangle = \langle \zeta, x - \bar{x} \rangle \leq 0, \quad (3.3.25)$$

which implies that $\zeta \in N_S(\bar{s})$. For all $v \in F$, let $x' := \bar{s} - \rho_F(\bar{s} - \bar{x})v$. Then

$$T(x') \leq \rho_F(\bar{s} - x') = \rho_F(\rho_F(\bar{s} - \bar{x})v) = \rho_F(\bar{s} - \bar{x})\rho_F(v) \leq \rho_F(\bar{s} - \bar{x}) = r$$

Then $x' \in S(r)$, we have

$$\begin{aligned} \left\langle -\zeta, v - \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} \right\rangle &= \left\langle \zeta, \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} - v \right\rangle \\ &= \left\langle \zeta, \frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})} - \frac{\bar{s} - x'}{\rho_F(\bar{s} - \bar{x})} \right\rangle \\ &= \frac{\langle \zeta, x' - \bar{x} \rangle}{\rho_F(\bar{s} - \bar{x})} \leq 0. \end{aligned} \quad (3.3.26)$$

and then $-\zeta \in N_F\left(\frac{\bar{s} - \bar{x}}{\rho_F(\bar{s} - \bar{x})}\right)$. Therefore we conclude that $\zeta \in \text{SEP}(S/F, \bar{s}, \bar{x})$. \square

Theorem 3.3.8. (Theorem 3.1 in [9]) *Let S be closed and convex. Let F be closed, convex, bounded and with $0 \in F$. Let $x \in S$. Then*

$$\partial T(x) = N_S(x) \cap \{\zeta \mid \rho_{F^\circ}(-\zeta) \leq 1\} \quad (3.3.27)$$

and $0 \in \partial T(x)$.

Proposition 3.3.9. (Proposition 3.9 in [1]) *For any collection of sets $C_i \subset \mathbb{R}^n$ for $i \in I$, where I is an arbitrary index set, we have*

$$[\bigcap_{i \in I} C_i]^\infty \subset \bigcap_{i \in I} C_i^\infty. \quad (3.3.28)$$

If C_i are closed, convex sets with nonempty intersection, then the above inclusion holds as an equation.

Theorem 3.3.10. *Let S be closed and convex. Let F be closed, bounded and convex with $0 \in F$. Let $x \in S$, then*

$$\partial^\infty T(x) = N_S(x) \cap (-F^\circ)^\infty. \quad (3.3.29)$$

Proof. By Theorem 3.3.8, we have

$$\begin{aligned}\partial T(x) &= N_S(x) \cap \{\zeta \mid \rho_{F^\circ}(-\zeta) \leq 1\} \\ &= N_S(x) \cap (-F^\circ).\end{aligned}\tag{3.3.30}$$

Since $0 \in N_S(x) \cap (-F^\circ)$, we have $N_S(x) \cap (-F^\circ) \neq \emptyset$. Since $N_S(x)$ is a closed convex cone and $(-F^\circ)$ is closed and convex, By Proposition 3.3.9, we have

$$\begin{aligned}\left(N_S(s) \cap (-F^\circ)\right)^\infty &= N_S(s)^\infty \cap (-F^\circ)^\infty \\ &= N_S(s) \cap (-F^\circ)^\infty.\end{aligned}\tag{3.3.31}$$

By Proposition 2.1.8, we have

$$\partial^\infty T(x) = \partial T(x)^\infty = N_S(x) \cap (-F^\circ)^\infty.\tag{3.3.32}$$

□

3.4 Results for nonconvex S

In this section, let's consider condition on F and a possibly nonconvex target S . The projection map Π_S^F is a very useful tool for the analysis here. We use the strict convexity of the dynamics F to balance and control the nonconvexity of the target set S .

Definition 3.4.1. (*Definition 5.1 in [2]*) A closed set $S \subseteq \mathbb{R}^n$ is *proximally smooth* if there exists $\varphi \geq 0$ so that for all $s_1, s_2 \in S$ and $\zeta_1 \in N_S^p(s_1)$, $\zeta_2 \in N_S^p(s_2)$ such that $\|\zeta_1\| \leq 1$ and $\|\zeta_2\| \leq 1$. We have

$$\langle \zeta_2 - \zeta_1, s_2 - s_1 \rangle \geq -\varphi \|s_2 - s_1\|^2.\tag{3.4.1}$$

Or S is called φ -proximally smooth if (3.4.1) holds.

Definition 3.4.2. Let $\gamma > 0$ be given. A closed bounded convex set $F \subseteq \mathbb{R}^n$ with $0 \in \text{ri } F$ is called γ -strictly uniformly convex if for all $v_1, v_2 \in F$ and $\zeta_i \in$

$\{0\} \cup N_F(v_i) \cap [(ssp F)^\perp]^c$, such that $\|\zeta_i\| \leq 1$, $i = 1, 2$, and ζ_1, ζ_2 are not both 0, we have

$$\langle \hat{\zeta}_2 - \hat{\zeta}_1, v_2 - v_1 \rangle \geq \gamma \|v_2 - v_1\|^2. \quad (3.4.2)$$

Proposition 3.4.3. *Let S be closed, let F be closed, bounded and convex with $0 \in F$, let $x \in (\text{dom } T) \cap S^c$. Then $\Pi_S^F(x) \subseteq \text{bdry } S$.*

Proof. For all $s \in \Pi_S^F(x)$, we have $\rho_F(s - x) = T(x) > 0$ and $s \in S$. Suppose that $s \in \text{int } S$, then there exists $r > 0$ such that $\overline{B(s, r)} \subseteq S$. Let $r' = \min(\frac{r}{2}, \frac{\|s-x\|}{2})$, then $s - r' \frac{s-x}{\|s-x\|} \in S$ and

$$\rho_F(s - x) - \frac{r' \rho_F(s - x)}{\|s - x\|} = \rho_F(s - x) \left(1 - \frac{r'}{\|s - x\|}\right) \geq 0. \quad (3.4.3)$$

Then

$$\begin{aligned} T(x) &\leq \rho_F\left(s - r' \frac{s-x}{\|s-x\|} - x\right) \\ &= \rho_F\left((s-x) - \frac{r' \rho_F(s-x)}{\|s-x\|} \frac{s-x}{\rho_F(s-x)}\right) \\ &= \rho_F\left(\frac{s-x}{\rho_F(s-x)} \left(\rho_F(s-x) - \frac{r' \rho_F(s-x)}{\|s-x\|}\right)\right) \\ &= \rho_F(s-x) - \frac{r' \rho_F(s-x)}{\|s-x\|} \\ &< \rho_F(s-x) = T(x). \end{aligned} \quad (3.4.4)$$

It's a contradiction. Hence, $s \in \text{bdry } S$ and then $\Pi_S^F(x) \subseteq \text{bdry } S$. \square

Proposition 3.4.4. *Let S be φ -proximally smooth and F be γ -strictly uniformly convex, and let $x \in \text{dom } T \cap S^c$. Suppose that there exists $s_1 \in \Pi_S^F(x)$ such that*

$$N_S^p(s_1) \cap (-\partial \rho_F(s_1 - x)) \neq \emptyset. \quad (3.4.5)$$

If $\varphi T(x) < \gamma$, then $\Pi_S^F(x)$ is a singleton.

Proof. By (3.4.5), there exists $\zeta \in N_F\left(\frac{s_1-x}{\rho_F(s_1-x)}\right) \cap (-N_S^p(s_1))$ such that $\rho_{F^\circ}(\zeta) = 1$. Hence, $\zeta \neq 0$ and $\zeta \notin (F^\circ)^\infty \supseteq (\text{ssp } F)^\perp$. Or in other words, $\frac{\zeta}{\|\zeta\|} \notin (\text{ssp } F)^\perp$. Hence,

$$\frac{\zeta}{\|\zeta\|} \in N_F\left(\frac{s_1-x}{\rho_F(s_1-x)}\right) \cap (-N_S^p(s_1)). \quad (3.4.6)$$

Then $\frac{-\zeta}{\|\zeta\|} \in N_S^p(s_1)$. For all $s_2 \in \Pi_S^F(x)$. Set $\rho := \rho_F(s_1-x) = \rho_F(s_2-x)$. Hence, $\frac{s_1-x}{\rho} \in F$ and $\frac{s_2-x}{\rho} \in F$. Since $0 \in N_S^p(s_2)$ and S is φ -proximal smooth, we have

$$\left\langle \frac{-\zeta}{\|\zeta\|}, s_2 - s_1 \right\rangle \leq \varphi \|s_2 - s_1\|^2. \quad (3.4.7)$$

Then

$$\begin{aligned} \left\langle \frac{-\zeta}{\|\zeta\|}, \frac{s_2 - s_1}{\rho} \right\rangle &= \left\langle 0 - \frac{\zeta}{\|\zeta\|}, \frac{s_2 - x}{\rho} - \frac{s_1 - x}{\rho} \right\rangle \\ &\geq \frac{\gamma}{\rho^2} \|s_2 - s_1\|^2, \end{aligned} \quad (3.4.8)$$

where the inequality is given by strict convexity because $\frac{\zeta}{\|\zeta\|} \in N_F\left(\frac{s_1-x}{\rho_F(s_1-x)}\right) \cap [(\text{ssp } F)^\perp]^c$ and $0 \in N_F\left(\frac{s_2-x}{\rho_F(s_2-x)}\right)$. Then

$$\frac{\gamma}{\rho} \|s_2 - s_1\|^2 \leq \left\langle -\frac{\zeta}{\|\zeta\|}, s_2 - s_1 \right\rangle \leq \varphi \|s_2 - s_1\|^2. \quad (3.4.9)$$

Then $(\frac{\gamma}{\rho} - \varphi) \|s_2 - s_1\|^2 \leq 0$. Since $\frac{\gamma}{\rho} - \varphi > 0$, we get $\|s_2 - s_1\|^2 \leq 0$. Hence, $s_2 = s_1$.

Then $\Pi_S^F(x)$ is a singleton. \square

Proposition 3.4.5. *Let S be φ -proximally smooth and F be γ -strictly uniformly convex. Let $x \in \text{dom } T \cap S^c$, set $r := T(x)$. Assume that $\varphi r < \frac{\gamma m}{\|F\|} \wedge \frac{1}{\|F\|}$ and there exists $s' \in \Pi_S^F(x)$ such that*

$$N_S^p(s') \cap (-\partial\rho_F(s' - x)) \neq \emptyset. \quad (3.4.10)$$

Suppose that there exist constant $k' > 0$ and $\delta > 0$ such that

$$T(x) - T(y) \leq k' \|y - x\|, \quad \forall y \in (x + (\text{ssp } F)^\perp) \cap B(x, \delta). \quad (3.4.11)$$

Let D be any compact neighborhood of x in $S(r) \cap S^c$. Then there exists constant $C > 0$ such that

$$\|s' - \Pi_S^F(x)\| \leq C\|x' - x\|^{\frac{1}{2}}, \quad \forall x' \in D \cap B(x, \delta). \quad (3.4.12)$$

Proof. By Proposition 3.4.4, $\Pi_S^F(x)$ is a singleton. Set now $\{s\} = \Pi_S^F(x)$. For all $x' \in D$, let $s' \in \Pi_S^F(x')$ and $r' = \rho_F(s' - x')$. (3.4.10) implies that there exists nonzero vector $\zeta_0 \in N_F\left(\frac{s-x}{\rho_F(s-x)}\right) \cap (-N_S^p(s))$ with $\rho_{F^\circ}(\zeta_0) = 1$. Then $\zeta_0 \neq 0$ and $\zeta_0 \notin (F^\circ)^\infty \supseteq (\text{sps } F)^\perp$ and hence, $\frac{\zeta_0}{\|\zeta_0\|} \notin (\text{sps } F)^\perp$. Let $\zeta := \frac{\zeta_0}{\|\zeta_0\|}$, then $\zeta \in N_F\left(\frac{s-x}{\rho_F(s-x)}\right) \cap (-N_S^p(s))$ and $\zeta \notin (\text{sps } F)^\perp$.

Case 1: Suppose that $r' = r$.

Since $-\zeta \in N_S^p(s)$ and $0 \in N_S^p(0)$, by φ -proximal smoothness of S , we have

$$\langle -\zeta - 0, s' - s \rangle \leq \varphi\|s' - s\|^2. \quad (3.4.13)$$

Since $\rho_F\left(\frac{s'-x'}{r}\right) \leq \frac{1}{r}T(x') \leq 1$, we have $\frac{s'-x'}{r} \in F$. Since $0 \in N_F\left(\frac{s'-x'}{r}\right)$ and $\zeta \in N_F\left(\frac{s-x}{r}\right)$, by strictly convexity of F and Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left\langle 0 - \zeta, \frac{s' - x'}{r} - \frac{s - x}{r} \right\rangle &\geq \frac{\gamma}{r^2} \left\| s' - x' - (s - x) \right\|^2 \\ &\geq \frac{\gamma}{r^2} (\|s' - s\|^2 + \|x' - x\|^2 - 2\|s' - s\|\|x' - x\|). \end{aligned}$$

Then

$$\begin{aligned} \varphi\|s' - s\|^2 &\geq \langle -\zeta, s' - s \rangle \\ &= \langle -\zeta, x' - x \rangle + \langle -\zeta, (s' - x') - (s - x) \rangle \\ &\geq \langle -\zeta, x' - x \rangle + \frac{\gamma}{r} (\|s' - s\|^2 + \|x' - x\|^2 - 2\|s' - s\|\|x' - x\|). \end{aligned}$$

Since

$$\varphi\|s' - s\|^2 + \|\zeta\|\|x' - x\| \geq \varphi\|s' - s\|^2 + \langle \zeta, x' - x \rangle,$$

we have

$$\begin{aligned}\varphi r \|s' - s\|^2 + r \|\zeta\| \|x' - x\| &\geq \gamma (\|s' - s\|^2 + \|x' - x\|^2 - 2\|s' - s\| \|x' - x\|) \\ &\geq \gamma \|s' - s\|^2 - 2\|s' - s\| \|x' - x\|.\end{aligned}$$

Since $\|\zeta\| = 1$, we have

$$(\gamma - \varphi r) \|s' - s\|^2 - 2\gamma \|x' - x\| \|s' - s\| - r \|x' - x\| \leq 0.$$

Since $\gamma - \varphi r > 0$, we have

$$\begin{aligned}\|s' - s\| &\leq \frac{\gamma \|x' - x\| + \sqrt{\gamma^2 \|x' - x\|^2 + (\gamma - \varphi r)r \|x' - x\|}}{\gamma - \varphi r} \\ &= \frac{\gamma \sqrt{\|x' - x\|} + \sqrt{\gamma^2 \|x' - x\| + (\gamma - \varphi r)r}}{\gamma - \varphi r} \|x' - x\|^{\frac{1}{2}}.\end{aligned}$$

The compactness of D implies that $\|D\| \leq M$ for some positive M . Hence

$$\|x' - x\| \leq 2\|D\| = 2M. \quad (3.4.14)$$

Hence,

$$\|s' - s\| \leq \frac{\gamma \sqrt{2M} + \sqrt{2\gamma^2 M + (\gamma - \varphi r)r}}{\gamma - \varphi r} \|x' - x\|^{\frac{1}{2}}. \quad (3.4.15)$$

Let

$$C = \frac{\gamma \sqrt{2M} + \sqrt{2\gamma^2 M + (\gamma - \varphi r)r}}{\gamma - \varphi r}. \quad (3.4.16)$$

Then $\|s' - s\| \leq C \|x' - x\|^{\frac{1}{2}}$.

Case 2 Suppose that $0 < r' < r$.

(3.4.11) and Proposition 3.1.5 implies that

$$r - r' = T(x) - T(x') \leq k \|x - x'\| \quad \text{for all } x' \in \text{dom } T \cap B(x, \delta).$$

Let $v := \frac{s-x}{\rho_F(s-x)}$, then $s = x + rv$ and $\rho_F(v) = 1$. Let $x'' = x + (r - r')v$. By

Proposition 3.1.7, we have

$$r = T(x) = T(x'' - (r - r')v) \leq T(x'') + r - r'. \quad (3.4.17)$$

Then $r' \leq T(x'')$. Then

$$\begin{aligned}
r' &\leq T(x'') \leq \rho_F(s - x'') \\
&= \rho_F\left(s - x - (r - r') \frac{s - x}{\rho_F(s - x)}\right) \\
&= \rho_F\left(\frac{s - x}{\rho_F(s - x)} r'\right) = r'.
\end{aligned} \tag{3.4.18}$$

Then $r' = T(x'') = \rho_F(s - x'')$ and hence $s \in \Pi_S^F(x'')$. Observe that

$$\zeta \in N_F\left(\frac{s - x}{\rho_F(s - x)}\right) \cap (-N_S^p(s)) \cap [(\text{ssp } F)^\perp]^c. \tag{3.4.19}$$

We have

$$\frac{s - x''}{\rho_F(s - x'')} = \frac{s - x - (r - r') \frac{s - x}{r}}{r'} = \frac{1}{r'} \frac{s - x}{r} r' = \frac{s - x}{r} = \frac{s - x}{\rho_F(s - x)}.$$

For all $f \in F$, since $\zeta \in N_F\left(\frac{s - x}{\rho_F(s - x)}\right)$, we have

$$\left\langle \zeta, f - \frac{s - x''}{\rho_F(s - x'')} \right\rangle = \left\langle \zeta, f - \frac{s - x}{\rho_F(s - x)} \right\rangle \leq 0, \tag{3.4.20}$$

which implies that $\zeta \in N_F\left(\frac{s - x''}{\rho_F(s - x'')} \right)$. Then

$$\zeta \in N_F\left(\frac{s - x''}{\rho_F(s - x'')} \right) \cap (-N_S^p(s)) \cap [(\text{ssp } F)^\perp]^c. \tag{3.4.21}$$

Since $s \in \Pi_S^F(x'')$ and $s' \in \Pi_S^F(x')$ with $r' = \rho_F(s - x'') = \rho_F(s - x')$, by the proof of Step 1, we get

$$\|s' - s\| \leq \frac{\gamma\sqrt{2M} + \sqrt{2\gamma^2 M + (\gamma - \varphi r')r'}}{\gamma - \varphi r'} \|x'' - x'\|^{\frac{1}{2}}. \tag{3.4.22}$$

Define

$$C' := \frac{\gamma\sqrt{2M} + \sqrt{2\gamma^2 M + (\gamma - \varphi r')r'}}{\gamma - \varphi r'}, \tag{3.4.23}$$

and we have the following

$$\begin{aligned}
\|s' - s\| &\leq C' \|x'' - x'\|^{\frac{1}{2}} & (3.4.24) \\
&= C' \sqrt{\|(x - x') + (T(x) - T(x'))v\|} \\
&\leq C' \sqrt{\|x - x'\| + k\|x - x'\|\|F\|} \\
&= C' \sqrt{1 + k\|F\|} \|x - x'\|^{\frac{1}{2}}.
\end{aligned}$$

□

Proposition 3.4.6. *Let S be φ -proximally smooth and F be γ -strictly convex. Let $x \in \text{Dom } T \cap S^c$, set $r := T(x)$. Assume that $\varphi r < \frac{\gamma m}{\|F\|} \wedge \frac{1}{\|F\|}$. Suppose that there exist $k > 0$ and $\delta > 0$ such that*

$$T(x) - T(y) \leq k\|y - x\|, \quad \forall y \in (x + (\text{ssp } F)^\perp) \cap B(x, \delta). \quad (3.4.25)$$

and there exists $s' \in \Pi_S^F(x)$ such that $N_S^p(s) \cap (-\partial\rho_F(s' - x)) \neq \emptyset$. Then

$$N_S^p(\Pi_S^F(x)) \cap (-\partial\rho_F(\Pi_S^F(x) - x)) \subseteq N_{S(r)}^f(x). \quad (3.4.26)$$

Proof. By Proposition 3.4.4, $\Pi_S^F(x)$ is a singleton. Set now $\{s\} = \Pi_S^F(x)$. The case when $N_S^p(\Pi_S^F(x)) \cap (-\partial\rho_F(\Pi_S^F(x) - x)) = \emptyset$ is trivial. Now let $\zeta \in -\partial\rho_F(s - x) \cap N_S^p(s)$ and let $x_n \rightarrow x$, $x_n \in S(r)$. Need to show that

$$\limsup_{n \rightarrow \infty} \left\langle \zeta, \frac{x_n - x}{\|x_n - x\|} \right\rangle \leq 0 \quad (3.4.27)$$

Let $s_n \in \Pi_S^F(x_n)$. Then

$$\begin{aligned}
\langle \zeta, x_n - x \rangle &= \langle \zeta, x_n - s_n \rangle + \langle \zeta, s_n - s \rangle + \langle \zeta, s - x \rangle & (3.4.28) \\
&= r \left\langle \zeta, \frac{s - x}{r} - \frac{s_n - x_n}{r} \right\rangle + \langle \zeta, s_n - s \rangle.
\end{aligned}$$

Since $\rho_F(\frac{s_n - x_n}{r}) = \frac{1}{r}T(x_n) \leq 1$, we have $\frac{s_n - x_n}{r} \in F$. And $\rho_{F^\circ}(-\zeta) = 1$ implies that $-\zeta \notin (F^\circ)^\infty \supseteq (\text{ssp } F)^\perp$. Since $\frac{-\zeta}{\|\zeta\|} \in N_F(\frac{s-x}{r})$ and $0 \in N_F(\frac{s_n - x_n}{r})$, by γ strictly

uniform convexity of F , we have

$$\left\langle 0 + \frac{\zeta}{\|\zeta\|}, \frac{s_n - x_n}{r} - \frac{s - x}{r} \right\rangle \geq \gamma \left\| \frac{s_n - x_n}{r} - \frac{s - x}{r} \right\|^2.$$

Then

$$\left\langle \zeta, \frac{s - x}{r} - \frac{s_n - x_n}{r} \right\rangle \leq \frac{-\gamma\|\zeta\|}{r^2} \|s_n - x_n - (s - x)\|^2.$$

Since $\zeta/\|\zeta\| \in N_S^p(s)$, $0 \in N_S^p(s_n)$ and S is φ -proximally smooth, we have the following

$$\left\langle \frac{\zeta}{\|\zeta\|} - 0, s_n - s \right\rangle \leq \varphi \|s_n - s\|^2. \quad (3.4.29)$$

Thus we have

$$\begin{aligned} \langle \zeta, x_n - x \rangle &\leq \frac{-\gamma\|\zeta\|}{r} \|s_n - s - (x_n - x)\|^2 + \langle \zeta, s_n - s \rangle \\ &\leq \frac{-\gamma\|\zeta\|}{r} \|s_n - s - (x_n - x)\|^2 + \varphi\|\zeta\| \|s_n - s\|^2 \\ &= (\varphi - \frac{\gamma}{r})\|\zeta\| \|s_n - s\|^2 + \frac{2\gamma\|\zeta\|}{r} \langle s_n - s, x_n - x \rangle - \frac{\gamma\|\zeta\|}{r} \|x_n - x\|^2. \end{aligned}$$

Let n be large enough such that $x_n \in S^c \cap B(x, \delta)$. By Proposition 3.4.5, we have

$$\|s_n - s\| = O(\sqrt{\|x_n - x\|}). \quad (3.4.30)$$

Observe that $m \leq \|F\|$ because $m\overline{\mathbb{B}} \cap \text{spp } F \subseteq F$. Hence, $\varphi - \frac{\gamma}{r} < 0$. Then

$$\left\langle \zeta, \frac{x_n - x}{\|x_n - x\|} \right\rangle \leq (\varphi - \frac{\gamma}{r})\|\zeta\| \frac{\|s_n - s\|^2}{\|x_n - x\|} + \frac{2\gamma\|\zeta\|}{r} \|s_n - s\| - \frac{\gamma\|\zeta\|}{r} \|x_n - x\|.$$

Then

$$\limsup_{n \rightarrow \infty} \left\langle \zeta, \frac{x_n - x}{\|x_n - x\|} \right\rangle \leq 0. \quad (3.4.31)$$

Hence, $\zeta \in N_{S(r)}^f(x)$. □

Proposition 3.4.7. *Let S be φ -proximally smooth, and let F be closed convex bounded and with $0 \in \text{ri } F$. Suppose that $x \in S^c \cap \text{Dom } (T)$, and there exist constants $\eta = \eta(x) > 0$ and $k = k(x) > 0$ so that*

$$\Pi_S^F(y) \subseteq \Pi_S^F(x) + k\|y - x\|\overline{\mathbb{B}}, \quad \forall y \in x + \eta\overline{\mathbb{B}}. \quad (3.4.32)$$

Suppose that there exist constants $\delta = \delta(x) > 0$, $M = M(x) > 0$ such that

$$T(y) - T(x) \leq M\|y - x\|^2, \quad \forall y \in (x + (\text{sps } F)^\perp) \cap B(x, \delta), \quad (3.4.33)$$

and that the set $N_S^p(s) \cap (-\partial\rho_F(s - x))$ is independent of the choice of $s \in \Pi_S^F(x)$.

Then one has

$$\partial_p T(x) = N_S^p(s) \cap (-\partial\rho_F(s - x)) \quad \forall s \in \Pi_S^F(x). \quad (3.4.34)$$

Proof. Let $x \in S^c \cap \text{Dom } (T)$ and $s \in \Pi_S^F(x)$, so we have $0 < r := T(x) = \rho_F(s - x) < \infty$. So $0 \neq s - x \in \text{sps } F$.

(\subseteq) The upper inclusion is the result of Theorem 3.2.5.

(\supseteq) If $N_S^p(s) \cap (-\partial\rho_F(s - x)) = \emptyset$, then it's trivial. Let $y \in (x + \eta\overline{\mathbb{B}}) \cap S(r)$. Clearly, $y \in \text{dom } T$. For all $s' \in \Pi_S^F(y)$, By (3.4.32), there exists $s \in \Pi_S^F(x)$ such that

$$\|s' - s\| \leq k\|y - x\|. \quad (3.4.35)$$

Let $\zeta \in N_S^p(s) \cap (-\partial\rho_F(s - x))$. Then $-\zeta \in \partial\rho_F(s - x)$ implies that $\rho_{F^\circ}(-\zeta) = 1$, $-\zeta \in N_F\left(\frac{s-x}{\rho_F(s-x)}\right)$ and $\zeta \neq 0$. then

$$\langle \zeta, y - x \rangle = r \left\langle -\zeta, \frac{s' - y}{r} - \frac{s - x}{r} \right\rangle + \langle \zeta, s' - s \rangle. \quad (3.4.36)$$

Since $y \in S(r)$ and $s' \in \Pi_S^F(y)$, we have $\rho_F(s' - y) = T(y) \leq r$, which implies that $\rho_F\left(\frac{s' - y}{r}\right) \leq 1$ and then $\frac{s' - y}{r} \in F$. Since $-\zeta \in N_F\left(\frac{s-x}{\rho_F(s-x)}\right)$, we have

$$\left\langle -\zeta, \frac{s' - y}{r} - \frac{s - x}{r} \right\rangle \leq 0. \quad (3.4.37)$$

$\zeta \in N_S^p(s)$ implies that $\frac{\zeta}{\|\zeta\|} \in N_S^p(s)$ and $0 \in N_S^p(s')$. The φ -proximally smoothness of S implies that

$$\left\langle \frac{\zeta}{\|\zeta\|} - 0, s - s' \right\rangle \geq -\varphi \|s - s'\|^2. \quad (3.4.38)$$

Hence,

$$\langle \zeta, s' - s \rangle \leq \varphi \|\zeta\| \|s' - s\|^2 \leq \varphi \|\zeta\| k^2 \|y - x\|^2, \quad (3.4.39)$$

where the second inequality is implied by (3.4.35). Then

$$\begin{aligned} \langle \zeta, y - x \rangle &\leq \langle \zeta, s' - s \rangle && \text{by (3.4.36) and (3.4.37)} \\ &\leq \varphi k^2 \|\zeta\| \|y - x\|^2, \quad \forall y \in (x + \eta \bar{\mathbb{B}}) \cap S(r). && \text{by (3.4.39)} \end{aligned}$$

Since $\zeta \in N_{S(r)}^p(x)$ and $\rho_{F^\circ}(-\zeta) = 1$, together with Theorem 3.2.2, we have $\zeta \in \partial_p T(x)$. Since $N_S^p(s) \cap (-\rho_F(s - x))$ is independent of the choice of $s' \in \Pi_S^F(x)$, we have

$$N_S^p(s') \cap (-\partial \rho_F(s' - x)) \subseteq \partial_p T(x) \quad \text{for all } s' \in \Pi_S^F(x).$$

□

Chapter 4

Subgradients of Value Function for Bolza Problem

In this chapter, we turn to the Bolza problem in optimal control and the calculus of variations, and prove some properties of the value function, which is similar to those obtained earlier for the minimal time function.

4.1 Value function for Bolza problem

Now let's consider the Bolza problem, an optimal control problem with finite time horizon. We are going to adopt the notations in Rockafellar and Wolenski [15]. For any $\tau \in [0, \infty)$, we consider the functional

$$J_\tau(x(\cdot)) := g(x(0)) + \int_0^\tau L(\dot{x}(t))dt, \quad (4.1.1)$$

where $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is called the initial cost function and $L : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is a Lagrangian function.

We now introduce the so-called Bolza problem. For any $(\tau, \xi) \in [0, \infty) \times \mathbb{R}^n$, we consider the following problem:

$$(BP) \quad \inf\{J_\tau(x(\cdot)) \mid x(\tau) = \xi\}, \quad (4.1.2)$$

where the minimization takes place over all the absolutely continuous $x(\cdot) : [0, \tau] \rightarrow \mathbb{R}^n$ with derivative $\dot{x}(\cdot) \in L_n^p[0, \tau]$.

We are interested in the value function of the control problem (BP), defined as

$$V(\tau, \xi) := \inf\{J_\tau(x(\cdot)) \mid x(\tau) = \xi\}, \quad V(0, \xi) = g(\xi), \quad (4.1.3)$$

which propagate an initial cost function g forward from time 0. The value function (1.0.7), which propagates a final cost function backward from time T_0 , is the usual

setting in optimal control and covered by (4.1.3) through time reversal. If $\tau > 0$, the value function in (4.1.3) can also be written as

$$V(\tau, \xi) := \inf_{y \in \mathbb{R}^n} \left\{ \tau L\left(\frac{\xi - y}{\tau}\right) + g(y) \right\}. \quad (4.1.4)$$

The right hand side of (4.1.4) is called the Hopf-Lax formula.

The hamiltonian function $H : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is a key function in characterizing the value function V in Hamiltonian-Jacobi theory. It is defined associated with L by

$$H(y) := \sup_v \{ \langle v, y \rangle - L(v) \}. \quad (4.1.5)$$

The definition (4.1.5) can be viewed as a generalization of the Legendre-Fenchel transformation, which defines the hamiltonian in the calculus of variations. Observe that $H(\cdot)$ is a convex function.

Basic assumptions (A).

(A1) The initial function g is convex, proper, and lsc on \mathbb{R}^n .

(A2) The Lagrangian function L is convex, proper, and lsc on $\mathbb{R}^n \times \mathbb{R}^n$.

(A3) The set $F(x) := \text{dom } L(x, \cdot)$ is nonempty for all x , and there is a constant k such that $\text{dist}(0, F(x)) \leq k(1 + |x|)$ for all x .

(A4) There are constants α and β and a coercive, proper, nondecreasing function θ on $[0, \infty)$ such that $L(x, v) \geq \theta(\max\{0, |v| - \alpha|x|\}) - \beta|x|$ for all x and v .

Under assumption (A), the convexity of $L(\cdot)$ and $g(\cdot)$ guarantees that the function $J_\tau(x(\cdot))$ in (4.1.1) is well-defined and convex. Observe that $J_\tau(x(\cdot)) = \infty$ unless the arc $x(\cdot)$ satisfies the constraints

$$\dot{x}(t) \in F \text{ a.e. } t, \text{ with } x(0) \in D := \text{dom } g. \quad (4.1.6)$$

L and H are dual to each other, for example,

$$L(v) = \sup_v \{ \langle v, y \rangle - H(y) \}. \quad (4.1.7)$$

Basic assumptions (B):

(B1) The initial function g is convex, proper and lsc on \mathbb{R}^n .

(B2) $L = I_F$, where $F \subseteq \mathbb{R}^n$ is a closed bounded convex set with $0 \in \text{int } F$.

Under assumption (B), the Lagrangian L defines a differential inclusion in terms of the F set and $L = I_F$ is a convex, proper and lsc function. Hence,

$$H(y) = \sup_{v \in F} \langle v, y \rangle = \rho_{F^\circ}(y) \quad (4.1.8)$$

Clearly, if the initial function g and the Lagrangian function L satisfy the assumptions (B), then they satisfy the assumptions (A), too.

Theorem 4.1.1. *(Theorem 2.1 in [15]) Under (A), the function $V_\tau := V(\tau, \cdot)$ is proper, lsc, and convex on \mathbb{R}^n for each $\tau \in [0, \infty)$. In particular, V is proper and lsc as a function on $[0, \infty) \times \mathbb{R}^n$.*

The value function (4.1.3) of a Bolza problem under assumption (A) satisfies the generalized Hamilton Jacobi equation, which is given by the following theorem.

Theorem 4.1.2. *(Theorem 2.5 in [15]) Under assumption (A), the subgradients of V on $(0, \infty) \times \mathbb{R}^n$ have the property that*

$$\begin{aligned} (\sigma, \eta) \in \partial_t V(\tau, \xi) &\iff (\sigma, \eta) \in \partial_f V(\tau, \xi) \\ &\iff \eta \in \partial V_\tau(\xi), \quad \sigma + H(\eta) = 0. \end{aligned} \quad (4.1.9)$$

In particular, the value function $V(\cdot, \cdot)$ satisfies the generalized Hamilton-Jacobi equation

$$\sigma + H(\eta) = 0 \quad \text{for all } (\sigma, \eta) \in \partial_t V(\tau, \xi) \quad \text{when } \tau > 0. \quad (4.1.10)$$

The level sets $S(r)$ of $V(\cdot, \cdot)$ will be an important role in our analysis, and are defined by

$$S(r) := \{(\tau, \xi) \mid V(\tau, \xi) \leq r\}. \quad (4.1.11)$$

Observe that $S(r)$ is a closed set in \mathbb{R}^n because $V(\cdot, \cdot)$ is a lsc function. We are also interested in the r level sets $R_\tau(r)$ of V_τ , which are defined as

$$R_\tau(r) := \{\xi \mid V_\tau(\xi) \leq r\}. \quad (4.1.12)$$

Since $V_\tau(\cdot)$ is a lsc convex function by Theorem 4.1.1, we get that $R_\tau(r)$ is a closed convex set for all r . It's easy to see that the upper inclusion $(\tau, R_\tau(r)) \subseteq S(r)$ holds.

Example 4.1.3. *Suppose that the initial cost function $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is the indicator function $I_S(\cdot)$. Under (B), we have*

$$V(\tau, \xi) = \begin{cases} 0 & \text{if } \tau \geq \min_{\xi \in S} \rho_F(\xi - \xi'), \\ \infty & \text{otherwise.} \end{cases} \quad (4.1.13)$$

4.2 Subgradient formulas

Theorem 4.2.1. *Let $\tau \in (0, \infty)$ and assume that $V(\tau, \xi) = r$. Under (A), we have*

$$\partial_p V(\tau, \xi) \subseteq N_{S(r)}^p(\tau, \xi) \cap \{(\sigma, \eta) \mid \sigma + H(\eta) = 0\}. \quad (4.2.1)$$

Proof. If $\partial_p V(\tau, \xi) = \emptyset$, then it's trivial. Now let's assume that $\partial_p V(\tau, \xi) \neq \emptyset$. Let $(\sigma, \eta) \in \partial_p V(\tau, \xi)$, there exist constants $k > 0$, $\delta > 0$ such that

$$V(s, y) \geq r + \langle (\sigma, \eta), (s, y) - (\tau, \xi) \rangle - k \|(s, y) - (\tau, \xi)\|^2, \quad (4.2.2)$$

$$\forall (s, y) \in (t, x) + \delta \overline{\mathbb{B}}.$$

If $(s, y) \in S(r) \cap [(t, x) + \delta \overline{\mathbb{B}}]$, then $V(s, y) \leq r$. Then

$$\langle (\sigma, \eta), (s, y) - (\tau, \xi) \rangle \leq k \|(s, y) - (\tau, \xi)\|^2 \quad (4.2.3)$$

$$\forall (s, y) \in S(r) \cap [(t, x) + \delta \overline{\mathbb{B}}].$$

Hence, $(\sigma, \eta) \in N_{S(r)}^p(\tau, \xi)$. We have $(\sigma, \eta) \in \partial_l V(\tau, \xi)$ because $\partial_p V(\tau, \xi) \subseteq \partial_l V(\tau, \xi)$. By Theorem 4.1.2, we have $\sigma + H(\eta) = 0$. \square

Theorem 4.2.2. *Let $\tau \in (0, \infty)$. Assume that $V(\tau, \xi) = r$. Under (A), we have*

$$\partial_f V(\tau, \xi) \subseteq N_{S(r)}^f(\tau, \xi) \cap \{(\sigma, \eta) | \sigma + H(\eta) = 0\}. \quad (4.2.4)$$

Proof. If $\partial_f V(\tau, \xi) = \emptyset$, then it's trivial. Now let's assume that $\partial_f V(\tau, \xi) \neq \emptyset$. Let $(\sigma, \eta) \in \partial_f V(\tau, \xi)$. Since $\partial_f V(\tau, \xi) \subseteq \partial_l V(\tau, \xi)$, we have $(\sigma, \eta) \in \partial_l V(\tau, \xi)$. By Theorem 4.1.2, we have $\sigma + H(\eta) = 0$. Since $(\sigma, \eta) \in \partial_f V(\tau, \xi)$, we have

$$\liminf_{(s,y) \rightarrow (\tau,\xi)} \frac{V(s, y) - V(\tau, \xi) - \langle (\sigma, \eta), (s, y) - (\tau, \xi) \rangle}{\|(s, y) - (\tau, \xi)\|} \geq 0. \quad (4.2.5)$$

If we restrict the limit to $(s, y) \in S(r)$, the above formula becomes

$$\limsup_{(s,y) \in (\tau,\xi), (s,y) \in S(r)} \frac{\langle (\sigma, \eta), (s, y) - (\tau, \xi) \rangle}{\|(s, y) - (\tau, \xi)\|} \leq 0, \quad (4.2.6)$$

which says that $(\sigma, \eta) \in N_{S(r)}^f(\tau, \xi)$. \square

Theorem 4.2.3. *Let $\tau_0 \in (0, \infty)$. Suppose that $V(\tau_0, \xi_0) = r$. Under (A), we have*

$$\partial_f V(\tau_0, \xi_0) = N_{S(r)}^f(\tau_0, \xi_0) \cap \{(\sigma, \eta) | \sigma + H(\eta) = 0, \eta \in \partial V_{\tau_0}(\xi_0)\}. \quad (4.2.7)$$

Proof. (\subseteq) If $\partial_f V(\tau_0, \xi_0) = \emptyset$, then it's trivial. Now let's suppose that $\partial_f V(\tau_0, \xi_0) \neq \emptyset$. For all $(\sigma, \eta) \in \partial_f V(\tau_0, \xi_0)$. By Theorem 4.2.2, we have

$$(\sigma, \eta) \in N_{S(r)}^f(\tau_0, \xi_0) \cap \{(\sigma, \eta) | \sigma + H(\eta) = 0\}. \quad (4.2.8)$$

$(\sigma, \eta) \in \partial_f V(\tau_0, \xi_0)$ implies that

$$\liminf_{(\tau,\xi) \rightarrow (\tau_0,\xi_0)} \frac{V(\tau, \xi) - V(\tau_0, \xi_0) - \langle (\sigma, \eta), (\tau, \xi) - (\tau_0, \xi_0) \rangle}{\|(\tau, \xi) - (\tau_0, \xi_0)\|} \geq 0. \quad (4.2.9)$$

Now fix $\tau = \tau_0$, we have

$$\liminf_{\xi \rightarrow \xi_0} \frac{V_{\tau_0}(\xi) - V_{\tau_0}(\xi_0) - \langle \eta, \xi - \xi_0 \rangle}{\|\xi - \xi_0\|} \geq 0, \quad (4.2.10)$$

which implies that $\eta \in \partial V_{\tau_0}(\xi_0)$.

(\supseteq) By Theorem 4.1.2, we have

$$\{(\sigma, \eta) | \sigma + H(\eta) = 0, \eta \in \partial V_{\tau_0}(\xi_0)\} \subseteq \partial_f V(\tau_0, \xi_0). \quad (4.2.11)$$

Therefore,

$$N_{S(r)}^f(\tau_0, \xi_0) \cap \{(\sigma, \eta) | \sigma + H(\eta) = 0, \eta \in \partial V_{\tau_0}(\xi_0)\} \subseteq \partial_f V(\tau_0, \xi_0).$$

□

Theorem 4.2.4. *Let $\tau_0 \in (0, \infty)$. Assume that $V(\tau_0, \xi_0) = r$. Under (A), we have*

$$(\sigma, \eta) \in \partial_f V(\tau_0, \xi_0) \Rightarrow \begin{cases} \sigma + H(\eta) = 0 \\ \eta \in N_{R_{\tau_0}(r)}(\xi_0). \end{cases} \quad (4.2.12)$$

Proof. For all $(\sigma, \eta) \in \partial_f V(\tau_0, \xi_0)$. By Theorem 4.2.2, we have

$$(\sigma, \eta) \in N_{S(r)}^f(\tau_0, \xi_0) \quad \text{and} \quad \sigma + H(\eta) = 0. \quad (4.2.13)$$

So we have

$$\limsup_{(\tau, \xi) \rightarrow (\tau_0, \xi_0), (\tau, \xi) \in S(r)} \frac{\langle (\sigma, \eta), (\tau, \xi) - (\tau_0, \xi_0) \rangle}{\|(\tau, \xi) - (\tau_0, \xi_0)\|} \leq 0. \quad (4.2.14)$$

For all $\epsilon > 0$, there exists $\delta' > 0$ such that

$$\begin{aligned} \langle (\sigma, \eta), (\tau, \xi) - (\tau_0, \xi_0) \rangle &\leq \epsilon \|(\tau, \xi) - (\tau_0, \xi_0)\|, \\ \forall (\tau, \xi) &\in S(r) \cap B((\tau_0, \xi_0), \delta'). \end{aligned}$$

For all $\xi' \in B(\xi_0, \delta') \cap R_{\tau_0}(r)$, then $V(\tau_0, \xi') = V_{\tau_0}(\xi') \leq r$ and

$$\|(\tau_0, \xi') - (\tau_0, \xi_0)\| = \|\xi - \xi_0\| \leq \delta', \quad (4.2.15)$$

which imply that $(\tau_0, \xi') \in B((\tau_0, \xi_0), \delta') \cap S(r)$. Therefore,

$$\begin{aligned} \langle (\sigma, \eta), (\tau_0, \xi') - (\tau_0, \xi_0) \rangle &\leq \epsilon \|(\tau_0, \xi') - (\tau_0, \xi_0)\|, \\ \forall \xi' &\in B(\xi_0, \delta') \cap R_{\tau_0}(r). \end{aligned}$$

The above inequality also can be written as

$$\langle \eta, \xi' - \xi_0 \rangle \leq \epsilon \|\xi' - \xi_0\|, \quad \forall \xi' \in B(\xi_0, \delta) \cap R_{\tau_0}(r).$$

Therefore, $\eta \in N_{R_{\tau_0}(r)}^f(\xi_0)$. Since $V_{\tau_0}(\cdot)$ is convex, the level set $R_{\tau_0}(r)$ of $V_{\tau_0}(\cdot)$ is a convex set. Hence, $\eta \in N_{R_{\tau_0}(r)}(\xi_0)$. \square

Theorem 4.2.5. *Suppose $0 \notin \partial V_{\tau_0}(\xi_0)$ and $\partial V_{\tau_0}(\xi_0) \neq \emptyset$. Suppose that V_{τ_0} is finite around ξ_0 . Let $\tau_0 \in (0, \infty)$. Suppose that $V(\tau_0, \xi_0) = r$. Under (B), we have*

$$(\sigma, \eta) \in \text{pos } \partial_f V(\tau_0, \xi_0) \Leftrightarrow \begin{cases} \sigma + H(\eta) = 0 \\ (\sigma, \eta) \in N_{S(r)}^f(\tau_0, \xi_0). \end{cases} \quad (4.2.16)$$

Proof. Theorem 4.1.2 and $\partial V_{\tau_0}(\xi_0) \neq \emptyset$ imply that $\partial_f V(\tau_0, \xi_0) \neq \emptyset$.

(\Rightarrow) For all $(\sigma, \eta) \in \text{pos } \partial_f V(\tau_0, \xi_0)$. If $\sigma = 0$, then $H(\eta) = 0$. Hence, $\eta = 0$ and we have

$$\begin{cases} 0 + H(0) = 0 \\ (0, 0) \in N_{S(r)}^f(\tau_0, \xi_0). \end{cases} \quad (4.2.17)$$

Now let's assume that $(\sigma, \eta) \neq (0, 0)$. Then there exists $\lambda > 0$ such that

$$\left(\frac{\sigma}{\lambda}, \frac{\eta}{\lambda}\right) \in \partial_f V(\tau_0, \xi_0). \quad (4.2.18)$$

By Theorem 4.2.2, we have

$$\left(\frac{\sigma}{\lambda}, \frac{\eta}{\lambda}\right) \in N_{S(r)}^f(\tau_0, \xi_0) \text{ and } \sigma + H(\eta) = 0. \quad (4.2.19)$$

Then $(\sigma, \eta) \in N_{S(r)}^f(\tau_0, \xi_0)$ because $N_{S(r)}^f(\tau_0, \xi_0)$ is a cone.

(\Leftarrow) If $\eta = 0$, then $H(\eta) = \rho_{F^\circ}(\eta) = 0$, which implies that $\sigma = -H(\eta) = 0$, and then $(\sigma, \eta) \in \text{pos } \partial_f V(\tau_0, \xi_0)$. Now let's assume that $(\sigma, \eta) \neq (0, 0)$. Let $(\sigma, \eta) \in N_{S(r)}^f(\tau_0, \xi_0)$ such that $\sigma + H(\eta) = 0$. Then for any $\epsilon > 0$, there exists $\delta' > 0$ such that

$$\begin{aligned} \langle (\sigma, \eta), (\tau, \xi) - (\tau_0, \xi_0) \rangle &\leq \epsilon \|(\tau, \xi) - (\tau_0, \xi_0)\|, \\ \forall (\tau, \xi) &\in S(r) \cap B((\tau_0, \xi_0), \delta'). \end{aligned}$$

Let $\xi' \in B(\xi_0, \delta') \cap R_{\tau_0}(r)$, then $V(\tau_0, \xi') = V_{\tau_0}(\xi') \leq r$ and

$$\|(\tau_0, \xi') - (\tau_0, \xi_0)\| = \|\xi' - \xi_0\| \leq \delta', \quad (4.2.20)$$

which imply that $(\tau_0, \xi') \in B((\tau_0, \xi_0), \delta') \cap S(r)$. Therefore,

$$\begin{aligned} \langle (\sigma, \eta), (\tau_0, \xi') - (\tau_0, \xi_0) \rangle &\leq \epsilon \|(\tau_0, \xi') - (\tau_0, \xi_0)\|, \\ \forall \xi' &\in B(\xi_0, \delta') \cap R_{\tau_0}(r). \end{aligned}$$

The above inequality also can be written as

$$\langle \eta, \xi' - \xi_0 \rangle \leq \epsilon \|\xi' - \xi_0\|, \quad \forall \xi' \in B(\xi_0, \delta) \cap R_{\tau_0}(r). \quad (4.2.21)$$

Therefore, $\eta \in N_{R_{\tau_0}(r)}^f(\xi_0)$. Since $V_{\tau_0}(\cdot)$ is convex, the level set $R_{\tau_0}(r)$ of $V_{\tau_0}(\cdot)$ is a convex set. Hence, $\eta \in N_{R_{\tau_0}(r)}(\xi_0)$, or

$$\langle \eta, \xi' - \xi_0 \rangle \leq 0, \quad \forall \xi' \in R_{\tau_0}(r). \quad (4.2.22)$$

By Theorem 4.1.1, $V_{\tau_0}(\cdot)$ is a proper, lsc and convex function on \mathbb{R}^n . Since $\partial V_{\tau_0}(\xi_0) \neq \emptyset$, by Proposition 2.1.8, we have

$$\partial^\infty V_{\tau_0}(\xi_0) = \partial V_{\tau_0}(\xi_0)^\infty \text{ and } \partial_l V_{\tau_0}(\xi_0) = \partial_f V_{\tau_0}(\xi_0).$$

Then by Corollary 2.1.6, $V_{\tau_0}(\cdot)$ is regular at ξ_0 . Since $0 \notin \partial V_{\tau_0}(\xi_0)$, by Proposition 2.1.7, we have

$$N_{R_{\tau_0}(r)}(\xi_0) = \text{pos } \partial V_{\tau_0}(\xi_0) \cup \partial^\infty V_{\tau_0}(\xi_0). \quad (4.2.23)$$

Suppose that $\eta \in \partial^\infty V_{\tau_0}(\xi_0)$, then

$$\langle \eta, \xi' - \xi_0 \rangle \leq 0, \quad \forall \xi' \in \text{dom } V_{\tau_0}. \quad (4.2.24)$$

But $\xi_0 \in \text{int dom } V_{\tau_0}$, which implies that $\partial^\infty V_{\tau_0}(\xi_0) = \{0\}$. Hence, $\eta \in \text{pos } \partial V_{\tau_0}(\xi_0)$.

Then there exists $\lambda > 0$ such that $\frac{\eta}{\lambda} \in \partial V_{\tau_0}(\xi_0)$ and $\frac{\sigma}{\lambda} + H(\frac{\eta}{\lambda}) = 0$. By Theorem 4.1.2, we have $(\frac{\sigma}{\lambda}, \frac{\eta}{\lambda}) \in \partial_f V(\tau_0, \xi_0)$. Therefore, $(\sigma, \eta) \in \text{pos } \partial_f V(\tau_0, \xi_0)$. \square

Chapter 5

Future Work

For the value function of the Bolza problem with constant dynamics defined as

$$\begin{aligned} V(\tau, \xi) &:= \inf \left\{ g(0) + \int_0^\tau I_F(\dot{x}(t)) dt \mid x(\tau) = \xi \right\}, \\ V(0, \xi) &= g(\xi), \end{aligned} \tag{5.0.1}$$

where $F \subseteq \mathbb{R}^n$ is closed, convex, bounded with $0 \in \text{int } F$, we tried to find a general formula for Fréchet subgradient of $V(\cdot, \cdot)$ in terms of normal vectors to its lower level set. We thought that if a vector in the normal cone to its lower level set satisfies the Hamiltonian-Jacobi equation, then it's a Fréchet subgradient of the value function. For example,

$$\partial_f V(\tau, \xi) = N_{S(\tau)}^f(\tau, \xi) \cap \{(\sigma, \eta) \mid \sigma + H(\eta) = 0\}. \tag{5.0.2}$$

Then we realized that the right hand side of (5.0.2) is a cone, while the left hand side of (5.0.2) is generally not a cone. So in the future, we need an equation rather than the Hamiltonian-Jacobi equation to regulate a vector in the normal cone to the level set to be a Fréchet subgradient of the value function of the Bolza problem.

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