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Mixed categories, formality for the nilpotent cone, and a derived Springer correspondence

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MIXED CATEGORIES, FORMALITY FOR
THE NILPOTENT CONE,
AND A DERIVED SPRINGER CORRESPONDENCE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in

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by

Laura Rider

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Abstract

Recall that the Springer correspondence relates representations of the Weyl group to perverse sheaves on the nilpotent cone. We explain how to extend this to an equivalence between the triangulated category generated by the Springer perverse sheaf and the derived category of differential graded modules over a dg-ring related to the Weyl group.

Chapter 1

Introduction

An important problem in geometric representation theory is describing the (equivariant) derived category of sheaves on a variety attached to an algebraic group. For instance, this has been done by Bernstein and Lunts in [13] for a point, by Lunts in [22] for projective and affine toric varieties, by Arkhipov, Bezrukavnikov, and Ginzburg in [7] for the affine Grassmannian, and by Schnürer in [34] for flag varieties.

We consider this problem for the nilpotent cone \mathcal{N} of a connected reductive algebraic group G . In particular, we focus on $D_{G,\text{Spr}}^b(\mathcal{N})$ — the triangulated subcategory of $D_{G,c}^b(\mathcal{N})$ generated by the simple summands of the Springer perverse sheaf \mathbf{A} . It is in this setting that we prove Theorem 4.2.9: there is an equivalence of triangulated categories

$$D_{G,\text{Spr}}^b(\mathcal{N}) \cong \mathcal{D}_f^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B})). \quad (1.0.1)$$

Here $\mathcal{D}_f^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B}))$ is the derived category of finitely generated differential graded (dg) modules over the smash product algebra $\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B})$ with W , the Weyl group of G and $H_G^\bullet(\mathcal{B})$, the G -equivariant cohomology of the flag variety. This theorem can be viewed as a derived category version of the Springer correspondence. Along the way, we prove the following *mixed* version of the above (see Theorem 3.2.3): an equivalence of triangulated categories

$$K^b(\mathcal{P}\text{ure}_G \mathcal{N}_0) \cong D^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B}))$$

relating a category built out of mixed sheaves $K^b(\mathcal{P}\text{ure}_G \mathcal{N}_0)$ and the derived category of graded modules over $\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B})$. We also prove the obvious non-equivariant analogue, i.e. an equivalence

$$K^b(\mathcal{P}\text{ure } \mathcal{N}_0) \cong D_{\text{per}}^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# H^\bullet(\mathcal{B})).$$

There are two key components to the proof of (1.0.1): formality and Koszulity. A dg-ring \mathcal{R} is called formal if it is quasi-isomorphic to its cohomology $H^\bullet(\mathcal{R})$. The role of formality in derived equivalences such as (1.0.1) is well established, but in our case, the construction of the dg-ring and functor is less straightforward than the analogue for the flag variety. Roughly, the machinery of quasi-hereditary categories does not apply, so new techniques are required. Instead, we exploit a non-standard t -structure on the triangulated category $K^b(\mathcal{P}\text{ure}_G \mathcal{N}_0)$ that arises via Koszul duality.

One might also expect a non-equivariant version of (1.0.1). Unfortunately, the ring $\bar{\mathbb{Q}}_\ell[W] \# H^\bullet(\mathcal{B})$ is not Koszul, so the methods of the present paper do not yield that result.

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Organization of the Dissertation

Let X_0 be a variety over \mathbb{F}_q . In Chapter 2, we introduce a category $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0) \subset \mathbb{D}_m^b(X_0)$ of weight zero objects (see Section 2.1) and construct a realization functor $\mathbb{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \rightarrow \mathbb{D}_m^b(X_0)$ in Section 2.2. In Section 2.3, we prove that $\mathbb{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is a *mixed version* of its analogue over $\overline{\mathbb{F}}_q$. In Chapter 3, we introduce notation related to \mathcal{N} and prove Frobenius invariance of certain Ext groups for objects related to the Springer sheaf \mathbf{A} (see Corollary 3.1.3 and Proposition 3.1.5). We prove a *mixed* version of the Springer correspondence (see Theorem 3.2.3) in Section 3.2, and in Chapter 4, we prove a *derived* version of the Springer correspondence (Theorem 4.2.9). In Section 4.3, we prove that our functor for the equivalence in Theorem 4.2.9 is triangulated. Finally in Chapter 5, we mention joint work with Russell proving an orthogonal decomposition of the category $\mathbb{D}_{G,c}^b(\mathcal{N})$ and pose the conjectural analogue of Theorem 4.2.9 for the other blocks in $\mathbb{D}_{G,c}^b(\mathcal{N})$ (see Conjecture 5.3.1).

Chapter 2

Setting Up the Mixed Framework

Mixed Geometry

In Section 2.1, we recall the definition of pointwise purity for constructible sheaves on an \mathbb{F}_q -variety. This notion is extended to mixed sheaves and the triangulated category of mixed ℓ -adic sheaves $D_m^b(X_0)$. This category is well-known by the experts to be too big: there are some unwanted naturally occurring extensions. In [10], this is fixed by considering only perverse sheaves with semisimple Frobenius action. This works fine at the derived level in the case of flag varieties. However, it is not sufficient for our purposes. Thus, we define a triangulated category built out of pure perverse sheaves denoted $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$. The inspiration for considering $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ as a replacement is [5]. We define the property of *Frobenius invariance* for $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ and prove a couple of easy Lemmas in this case that will facilitate our construction in Section 2.2.

Realization Functor

The goal of Section 2.2 is to connect our two categories: $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ and $D_m^b(X_0)$. The connection is a functor of triangulated categories called a realization functor. Historically, realization functors were introduced in the context of t -structures. Let \mathcal{T} be a triangulated category with a t -structure. Denote the heart of this t -structure by P . A *realization functor* is a triangulated functor $D^b(P) \rightarrow \mathcal{T}$ that restricts to the identity on P . In our setting, the defining structure is not the t -structure, but the mixed structure (related to something referred to as a co- t -structure in the literature). Most of our proof closely follows those in the literature (see [8, 9], for instance). However, an essential ingredient is Frobenius invariance (which may be replaced by a slightly weaker condition for this section). We use Frobenius invariance to prove Hom-vanishing which occurs for trivial reasons in the setting of a t -structure.

Suitable replacement

In Section 2.3, we finish our study of $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$. We recall the definitions of *mixed categories* and *Orlov categories*. Using the theory of Orlov categories as developed in [5], we prove that $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ has a mixed structure, a t -structure, and that it is a *mixed version* (see Definition 2.3.2) of its analogue over $\bar{\mathbb{F}}_q$ when $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ is Frobenius invariant. All together, this proves that $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ is a *suitable replacement* for $D_m^b(X_0)$. We also define a second t -structure on $K^b(\mathcal{P}ure_{\mathcal{S}}X_0)$ which is Koszul dual to the first in special situations (see Theorem 4.2.1).

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2.1 Basics on Mixed Sheaves and Purity

We fix a finite field \mathbb{F}_q and a prime number ℓ different from the characteristic of \mathbb{F}_q . Let X_0 be a variety defined over \mathbb{F}_q . Let \mathcal{F}_0 be an ℓ -adic sheaf on X_0 .

Definition 2.1.1. We call \mathcal{F}_0 *pointwise pure of weight* $w \in \mathbb{Z}$ if for all $n \geq 1$ and for all fixed points x of Frobenius $Fr^n := Fr_q^n (= Fr_{q^n})$ the eigenvalues of the automorphism on the stalk $\mathcal{F}_x \cong \mathcal{F}_{Fr^n(x)} \cong \mathcal{F}_x$ are algebraic numbers all of whose complex conjugates have absolute value $(q^n)^{w/2}$.

The sheaf \mathcal{F}_0 is called *mixed* if it has a finite filtration whose subquotients are pointwise pure. The weights appearing in the (non-zero) quotients are the weights of \mathcal{F}_0 . Let $D_m^b(X_0)$ be the full subcategory of $D_m^b(X_0)$ consisting of objects \mathcal{F}_0 whose cohomology $H^i(\mathcal{F}_0)$ is a mixed sheaf for each $i \in \mathbb{Z}$. $D_m^b(X_0)$ is referred to as the category of mixed complexes of sheaves on X_0 . This category studied extensively in [9, 16, 10, 5] and others. Note that in Section 2.3, we will use a different meaning for the word *mixed* ([10, Section 4]). While $D_m^b(X_0)$ shares some characteristics of that definition (i.e. the notion of weights and purity for objects), it is not mixed in the sense of [10].

If we have a sheaf, a complex of sheaves, or a perverse sheaf $\mathcal{F}_0 \in D_m^b(X_0)$, it will often be useful to extend scalars to get an object \mathcal{F} in $D_c^b(X)$, where $X := X_0 \times_{\text{Spec} \mathbb{F}_q} \text{Spec}(\bar{\mathbb{F}}_q)$. Let $Fr : X \rightarrow X$ be the Frobenius map. After extending scalars, \mathcal{F} is endowed with additional structure: an isomorphism $Fr^*(\mathcal{F}) \cong \mathcal{F}$. Let $a : X_0 \rightarrow \text{Spec}(\mathbb{F}_q)$ be the structure map. For \mathcal{F}_0 and \mathcal{G}_0 in $D_m^b(X_0)$, we let $\underline{\text{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0) = R^i a_* R\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$. This is a vector space with an action of Frobenius. Forgetting that action yields $\text{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}[i])$.

Recall, from [9, 5.1.2.5], that for \mathcal{F}_0 and \mathcal{G}_0 in $D_m^b(X_0)$, we have a short exact sequence relating morphisms in $D_m^b(X_0)$ to the Frobenius coinvariants and invariants (i.e. the cokernel and kernel of the map $Fr - Id$) of the morphisms in $D_c^b(X)$.

$$0 \rightarrow \underline{\text{Hom}}^{i-1}(\mathcal{F}, \mathcal{G})_{\text{Frob}} \rightarrow \text{Hom}^i(\mathcal{F}_0, \mathcal{G}_0) \rightarrow \underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})^{\text{Frob}} \rightarrow 0 \quad (2.1.1)$$

Remark 2.1.2. The Frobenius invariants $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})^{\text{Frob}}$ inject into the weight 0 part of $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})$ and the Frobenius coinvariants $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})_{\text{Frob}}$ are a quotient of the weight 0 part. Thus, $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})^{\text{Frob}}$ and $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})_{\text{Frob}}$ vanish when $\underline{\text{Hom}}^i(\mathcal{F}, \mathcal{G})$ is pure of non-zero weight.

Fix a square-root of the Tate sheaf. Note that Tate twist affects the weights of an object in the following way: for $\mathcal{F}_0 \in D_m^b(X_0)$, the weights of $\mathcal{F}_0(-\frac{i}{2})$ equals the weights of \mathcal{F}_0 plus i .

Pure of Weight Zero

Let \mathcal{S} be a (finite up to isomorphism) collection of simple perverse sheaves that have weight 0. Then for any $i \in \mathbb{Z}$ and $S \in \mathcal{S}$, $S[2i](i)$ is also pure of weight 0. Define $\mathcal{P}ure_{\mathcal{S}}(X_0)$ as a full subcategory of $D_m^b(X_0)$ containing objects that are finite direct sums of such objects, i.e. if $M \in \mathcal{P}ure_{\mathcal{S}}(X_0)$, then there exist $S_1, \dots, S_N \in \mathcal{S}$ (possibly repeating) and integers i_1, \dots, i_N such that $M = S_1[2i_1](i_1) \oplus \dots \oplus S_N[2i_N](i_N)$. We define the *length* of such an object to be the number of terms in the direct sum.

Remark 2.1.3. We could have also defined $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ to be closed under integral shift-twist. The results in this section (appropriately modified), the construction of the realization functor in Section 2.2, and the mixed results from Section 2.3 still hold with this modification. However, the second t -structure as discussed in Section 2.3.1 need not exist.

Definition 2.1.4. If $\underline{\text{Hom}}(S, S'[2n](n))^{\text{Frob}} \cong \underline{\text{Hom}}(S, S'[2n](n))$ and $\underline{\text{Hom}}^{2n+1}(S, S')$ vanishes for all $S, S' \in \mathcal{S}$ and $n \in \mathbb{Z}$, we call $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ *Frobenius invariant*. If $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant, then it is easy to see that for all simple perverse sheaves $S, S' \in \mathcal{S}$, we have that $\underline{\text{Hom}}^{2n}(S, S')$ is pure of weight $2n$.

Lemma 2.1.5. *If $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant, then for all objects M, N in $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$, we have that $\text{Hom}(M, N[n]) = 0$ for all integers n with $n \neq 0, 1$.*

Proof. Note that M is pure of weight 0 and $N[n]$ is pure of weight n . For $n > 1$, the result follows from properties of mixed perverse sheaves [9, Proposition 5.1.15]. Assume that M and N have length 1 and $n < 0$. Then $M = S[2i](i)$ and $N = S'[2j](j)$ for integers i and j with $S, S' \in \mathcal{S}$. Of course,

$$\text{Hom}(S[2i](i), S'[2j](j)[n]) = \text{Hom}(S[2i - 2j](i - j), S'[n]).$$

Thus, it suffices to consider the case with $M = S[2i](i)$ and $N = S'$.

Note that Definition 2.1.4 implies that $\underline{\text{Hom}}^j(M, N[n])$ is pure of weight $n + j$ since

$$\underline{\text{Hom}}^j(M, N[n]) = \underline{\text{Hom}}^j(S[2i](i), S'[n]) = \underline{\text{Hom}}^{j-2i+n}(S, S')(-i).$$

In particular, for $j = 0, -1$, $\underline{\text{Hom}}^j(M, N[n])$ is pure of non-zero weight. This implies that $\underline{\text{Hom}}(M, N[n])^{\text{Frob}}$ and $\underline{\text{Hom}}^{-1}(M, N[n])^{\text{Frob}}$ are both zero (see Remark 2.1.2). Thus, by the short exact sequence (2.1.1), we have that $\text{Hom}(M, N[n]) = 0$.

For objects M and N in $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ of lengths greater than 1, the claim follows since Hom commutes with finite direct sums. \square

The following lemmas are preparation for Section 2.3 when we prove that a certain category $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}}\mathcal{N}_0)$ (or more generally $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}}X_0)$) is a mixed version of $\text{D}_{\text{Spr}}^b(\mathcal{N})$ (or $\text{D}_{\mathcal{S}}^b(X)$).

Lemma 2.1.6. *If $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant, then $\underline{\text{Hom}}^i(M, N)$ is pure of weight i for i even and vanishes for i odd for all $M, N \in \mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$.*

Proof. By Definition 2.1.4, the result follows for M and N in $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ of length 1. For arbitrary objects, the claim follows since $\underline{\text{Hom}}$ commutes with finite sums. \square

Corollary 2.1.7. *Suppose that $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant. Then for all M, N in $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$, we have $\text{Hom}(M, N) \cong \underline{\text{Hom}}(M, N)^{\text{Frob}}$.*

Proof. Let $M, N \in \mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$. Then $\underline{\text{Hom}}^{-1}(M, N)$ vanishes by Lemma 2.1.6. Thus, the Frobenius coinvariants $\underline{\text{Hom}}^{-1}(M, N)_{\text{Frob}}$ are also trivial. Hence, by the short exact sequence (2.1.1), we see that $\text{Hom}(M, N) \cong \underline{\text{Hom}}(M, N)^{\text{Frob}}$. \square

Lemma 2.1.8. *Suppose that $\mathcal{P}\text{ure}_{\mathcal{G}}(X_0)$ is Frobenius invariant. Then $\text{Hom}(M, N) \cong \underline{\text{Hom}}(M, N)^{\text{Frob}} \cong \underline{\text{Hom}}(M, N)$ for all $M, N \in \mathcal{P}\text{ure}_{\mathcal{G}}(X_0)$.*

Proof. The claim holds for objects of length 1 by the definition of Frobenius invariant and Corollary 2.1.7. The general case holds since Hom and $\underline{\text{Hom}}$ commute with finite direct sums. \square

2.2 A Realization Functor

In this section, we construct a triangulated functor from $K^b(\mathcal{P}\text{ure}_{\mathcal{G}}X_0)$ to $D_m^b(X_0)$ assuming that $\mathcal{P}\text{ure}_{\mathcal{G}}(X_0)$ is Frobenius invariant. Our method is based on Beilinson's construction of a *realization* functor in [8]. We briefly review the definition of a filtered triangulated category and some of its important properties. Note that we consider increasing filtrations, while Beilinson considers decreasing filtrations.

Definition 2.2.1. We say that a triangulated category \mathcal{D} is *filtered* if it has a collection of pairs of strictly full triangulated subcategories $(F^{\leq n}\mathcal{D}, F^{\geq n}\mathcal{D})_{n \in \mathbb{Z}}$ satisfying the following properties:

1. If $M \in F^{\leq n}\mathcal{D}$ and $N \in F^{\geq n+1}\mathcal{D}$, then $\text{Hom}(M, N) = 0$.
2. We have $F^{\leq n}\mathcal{D} \subset F^{\leq n+1}\mathcal{D}$ and $F^{\geq n}\mathcal{D} \supset F^{\geq n+1}\mathcal{D}$.
3. For any $Z \in \mathcal{D}$ and $n \in \mathbb{Z}$, there is a distinguished triangle $A \rightarrow Z \rightarrow B \rightarrow$ with $A \in F^{\leq n-1}\mathcal{D}$ and $B \in F^{\geq n}\mathcal{D}$.
4. The filtration is *bounded*, i.e. $\bigcup_{n \in \mathbb{Z}} F^{\leq n}\mathcal{D} = \bigcup_{n \in \mathbb{Z}} F^{\geq n}\mathcal{D} = \mathcal{D}$.
5. We have a *shift of filtration* (s, α) . Here $s : \mathcal{D} \rightarrow \mathcal{D}$ is an autoequivalence so that $s(F^{\leq n}\mathcal{D}) = F^{\leq n+1}\mathcal{D}$ and $s(F^{\geq n}\mathcal{D}) = F^{\geq n+1}\mathcal{D}$ and α is a natural transformation $s \rightarrow \text{id}_{\mathcal{D}}$ with $\alpha_M = s(\alpha_{s^{-1}M})$.
6. For all $M \in F^{\geq 1}\mathcal{D}$ and $N \in F^{\leq 0}\mathcal{D}$, the natural transformation α induces isomorphisms

$$\text{Hom}(M, N) \cong \text{Hom}(M, sN) \cong \text{Hom}(s^{-1}M, N). \quad (2.2.1)$$

The inclusion functor $F^{\leq n}\mathcal{D} \rightarrow \mathcal{D}$ admits a right adjoint denoted $w_{\leq n} : \mathcal{D} \rightarrow F^{\leq n}\mathcal{D}$. Similarly, $F^{\geq n}\mathcal{D} \rightarrow \mathcal{D}$ admits a left adjoint denoted $w_{\geq n} : \mathcal{D} \rightarrow F^{\geq n}\mathcal{D}$. It is shown in [8, Proposition A.3] that for each $n \in \mathbb{Z}$ the distinguished triangle in (3) is canonically isomorphic to

$$w_{\leq n-1}Z \rightarrow Z \rightarrow w_{\geq n}Z \rightarrow . \quad (2.2.2)$$

Let $\mathcal{D}^n = F^{\leq n}\mathcal{D} \cap F^{\geq n}\mathcal{D}$. The compositions $w_{\leq n}w_{\geq n}$ and $w_{\geq n}w_{\leq n}$ are naturally equivalent, and we denote them by $\text{gr}_n : \mathcal{D} \rightarrow \mathcal{D}^n$. For an object M in \mathcal{D} , we define the *filtered support* of M to be the smallest interval $[m, n]$ satisfying $M \in F^{\leq n}\mathcal{D} \cap F^{\geq m}\mathcal{D}$; in other words, $m = \max\{i \mid M \in F^{\geq i}\mathcal{D}\}$ and $n = \min\{i \mid M \in F^{\leq i}\mathcal{D}\}$.

Lemma 2.2.2. *Let M be an object in \mathcal{D} . The morphism $\alpha_{sM} : sM \rightarrow M$ induced by the natural transformation α defined above has the property that $\mathrm{gr}_i(\alpha_{sM}) = 0$ for all $i \in \mathbb{Z}$.*

Proof. We prove the lemma by induction on the length of filtered support of M . If $M = \mathrm{gr}_n M$ for some n , the claim follows immediately since $\mathrm{gr}_n sM = s \mathrm{gr}_{n-1} M = 0$. To prove the general case, let M have filtered support $[m, n]$ and consider the morphism of distinguished triangles induced by α :

$$\begin{array}{ccccccc} w_{\leq n-1} sM & \longrightarrow & sM & \longrightarrow & w_{\geq n} M & \longrightarrow & w_{\leq n-1} sM[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ w_{\leq n-1} M & \longrightarrow & M & \longrightarrow & w_{\geq n} M & \longrightarrow & w_{\leq n-1} M[1] \end{array}$$

Note that the filtered support of $w_{\leq n-1} M$ and $w_{\geq n} M$ has strictly shorter length than that of M . The functor gr_i is triangulated ([6, Proposition 2.3]); thus, we have an induced morphism of distinguished triangles:

$$\begin{array}{ccccccc} \mathrm{gr}_i w_{\leq n-1} sM & \longrightarrow & \mathrm{gr}_i sM & \longrightarrow & \mathrm{gr}_i w_{\geq n} sM & \longrightarrow & \mathrm{gr}_i w_{\leq n-1} sM[1] \\ \downarrow 0 & & \downarrow \mathrm{gr}_i \alpha_{sM} & & \downarrow 0 & & \downarrow \\ \mathrm{gr}_i w_{\leq n-1} M & \longrightarrow & \mathrm{gr}_i M & \longrightarrow & \mathrm{gr}_i w_{\geq n} M & \longrightarrow & \mathrm{gr}_i w_{\leq n-1} M[1] \end{array}$$

If $i \leq n-1$, then $\mathrm{gr}_i w_{\geq n} sM = \mathrm{gr}_i w_{\geq n} M = 0$, so the maps $\mathrm{gr}_i w_{\leq n-1} sM \rightarrow \mathrm{gr}_i sM$ and $\mathrm{gr}_i w_{\leq n-1} M \rightarrow \mathrm{gr}_i M$ are isomorphisms. Commutativity of the above squares implies that $\mathrm{gr}_i \alpha_{sM} = 0$. Similar arguments prove the cases $i \geq n+1$ and $i = n$. \square

We say that a filtered category $\widetilde{\mathcal{D}}$ is a *filtered version* of a triangulated category \mathcal{D} if there is an equivalence $\mathcal{D} \rightarrow \mathcal{D}^0$ of triangulated categories. Beilinson proves in [8] the existence of a unique functor (up to unique isomorphism) $\omega : \widetilde{\mathcal{D}} \rightarrow \mathcal{D}$ satisfying the following conditions:

1. $\omega|_{F^{\geq 0} \widetilde{\mathcal{D}}}$ is left adjoint to the inclusion functor $\mathcal{D} \rightarrow F^{\geq 0} \widetilde{\mathcal{D}}$,
2. $\omega|_{F^{\leq 0} \widetilde{\mathcal{D}}}$ is right adjoint to $\mathcal{D} \rightarrow F^{\leq 0} \widetilde{\mathcal{D}}$,
3. and $\omega(\alpha_M) : \omega(sM) \rightarrow \omega(M)$ is an isomorphism.

We may think of ω as the functor that forgets the filtration. For $M \in F^{\geq 0} \widetilde{\mathcal{D}}$ and $N \in F^{\leq 0} \widetilde{\mathcal{D}}$, ω induces an isomorphism

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}}(M, N) \simeq \mathrm{Hom}_{\mathcal{D}}(\omega(M), \omega(N)). \quad (2.2.3)$$

From now on, denote by $\widetilde{\mathcal{D}}$ a *filtered version* of $D_m^b(X_0)$. Let $\widetilde{\mathcal{A}}$ be the full subcategory of $\widetilde{\mathcal{D}}$ consisting of objects M with the property that $\mathrm{gr}_i M \in s^i \mathcal{P}ure_{\mathcal{S}}(X_0)[i]$ for all $i \in \mathbb{Z}$.

Remark 2.2.3. If $w_{\leq n}M$ and $w_{\geq n+1}M$ are both in $\tilde{\mathcal{A}}$ for some $n \in \mathbb{Z}$, then $M \in \tilde{\mathcal{A}}$. Also $M \in \tilde{\mathcal{A}}$ implies that $sM[1] \in \tilde{\mathcal{A}}$. Thus, if $f : M \rightarrow N$ is a morphism in $\tilde{\mathcal{A}}$, then the cone of the composition

$$sM \xrightarrow{\alpha} M \xrightarrow{f} N$$

is also in $\tilde{\mathcal{A}}$ since Lemma 2.2.2 implies the graded pieces are given by $\text{gr}_n \text{cone}(f \circ \alpha) = \text{gr}_n sM[1] \oplus \text{gr}_n N$.

Here is an outline of the construction of the realization functor: first, we show $\tilde{\mathcal{A}}$ is equivalent to $\text{C}^b(\mathcal{P}\text{ure}_{\mathcal{G}} X_0)$ via the functor β (to be defined in (2.2.4)). The composition $\omega \circ \beta^{-1}$ gives a functor from $\text{C}^b(\mathcal{P}\text{ure}_{\mathcal{G}} X_0)$ to $\text{D}_m^b(X_0)$. Next, we show that this functor takes null-homotopic maps to 0. Thus, $\omega \circ \beta^{-1}$ factors through $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{G}} X_0)$.

$$\begin{array}{ccccc} \text{C}^b(\mathcal{P}\text{ure}_{\mathcal{G}} X_0) & \xrightarrow[\sim]{\beta^{-1}} & \tilde{\mathcal{A}} & \hookrightarrow & \tilde{\mathcal{D}} & \xrightarrow{\omega} & \text{D}_m^b(X_0) \\ \downarrow \text{dashed} & & & & & \nearrow & \\ \text{K}^b(\mathcal{P}\text{ure}_{\mathcal{G}} X_0) & & & & & \tilde{\beta} & \end{array}$$

Lemma 2.2.4. *Let M and N be objects in $\tilde{\mathcal{A}}$. Then $\text{Hom}(\text{gr}_n M, w_{\leq n-1}N) = 0$ for all $n \in \mathbb{Z}$*

Proof. This is a direct consequence of Lemma 2.1.5. □

Our situation differs from Beilinson's in that neither $\mathcal{P}\text{ure}_{\mathcal{G}}(X_0)$ nor $\tilde{\mathcal{A}}$ is the heart of a t -structure. In particular, we need the following lemma.

Lemma 2.2.5. *Let $f : M \rightarrow N$ be a morphism in $\tilde{\mathcal{A}}$. Then $f = 0$ if and only if $\text{gr}_i f = 0$ for all $i \in \mathbb{Z}$.*

Proof. First we show that $w_{\leq n-1}f = 0$ implies that $w_{\leq n}f = 0$. Consider the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} w_{\leq n-1}M & \xrightarrow{u} & w_{\leq n}M & \xrightarrow{v} & \text{gr}_n M & \xrightarrow{w} & w_{\leq n-1}M[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ w_{\leq n-1}f = 0 & & w_{\leq n}f & & \text{gr}_n f = 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ w_{\leq n-1}N & \xrightarrow{u'} & w_{\leq n}N & \xrightarrow{v'} & \text{gr}_n N & \xrightarrow{w'} & w_{\leq n-1}N[1] \end{array}$$

Since the squares commute, we see that $v'w_{\leq n}f = 0$. Thus, there is a morphism s in $\text{Hom}(w_{\leq n}M, w_{\leq n-1}N)$ so that $u's = w_{\leq n}f$. Similarly, since $w_{\leq n}fu = 0$, there exists t in $\text{Hom}(\text{gr}_n M, w_{\leq n}N)$ so that $tv = w_{\leq n}f$. This gives a morphism of distinguished triangles.

$$\begin{array}{ccccccc}
w_{\leq n-1}M & \xrightarrow{u} & w_{\leq n}M & \xrightarrow{v} & \mathrm{gr}_n M & \xrightarrow{w} & w_{\leq n-1}M[1] \\
\downarrow h & & \downarrow s & & \downarrow t & & \downarrow h[1] \\
\mathrm{gr}_n N[-1] & \xrightarrow{-w'} & w_{\leq n-1}N & \xrightarrow{u'} & w_{\leq n}N & \xrightarrow{v'} & \mathrm{gr}_n N
\end{array}$$

Now, we have that $h = 0$ and $h[1] = 0$ since $\mathrm{Hom}(w_{\leq n-1}M, \mathrm{gr}_n N[-1]) = 0$ by property (1) of the filtered derived category. Thus, we see that $v't = 0$. Next, we apply the functor $\mathrm{Hom}(\mathrm{gr}_n M, -)$ to the bottom distinguished triangle to get the exact sequence

$$\mathrm{Hom}(\mathrm{gr}_n M, w_{\leq n-1}N) \xrightarrow{u'} \mathrm{Hom}(\mathrm{gr}_n M, w_{\leq n}N) \xrightarrow{v'} \mathrm{Hom}(\mathrm{gr}_n M, \mathrm{gr}_n N).$$

We have that $v't = 0$; thus, $t \in \mathrm{Ker} v' = \mathrm{Im} u'$. However, Lemma 2.2.4 implies that $\mathrm{Hom}(\mathrm{gr}_n M, w_{\leq n-1}N) = 0$. Thus, $t = 0$ and hence, $w_{\leq n}f = 0$.

Let $m = \max\{i \mid M \in F^{\geq i}\tilde{\mathcal{D}}\}$ and $n = \min\{i \mid M \in F^{\leq i}\tilde{\mathcal{D}}\}$. We proceed by induction on the length of the interval $[m, n]$. If $m = n$, then $M = \mathrm{gr}_n M$. In this case, $f = \mathrm{gr}_n f = 0$ by hypothesis. If $m < n$, then $w_{\leq m}f = \mathrm{gr}_m f = 0$. The above argument and induction implies that $w_{\leq n}f = 0$, but $w_{\leq n}f = f$. \square

Now we define the functor

$$\beta : \tilde{\mathcal{A}} \rightarrow \mathrm{C}^b(\mathrm{Pure}_{\mathcal{F}} X_0). \quad (2.2.4)$$

For an object M in $\tilde{\mathcal{A}}$, let $\beta(M)$ be the chain complex M^\bullet with $M^i = \omega(\mathrm{gr}_{-i} M)[i] = \mathrm{gr}_0(s^i M)[i]$ and differential $\delta^i : M^i \rightarrow M^{i+1}$ given by the third morphism in the functorial distinguished triangle

$$\omega(\mathrm{gr}_{-i-1} M)[i] \rightarrow \omega(w_{\geq -i-1}w_{\leq -i}M)[i] \rightarrow \omega(\mathrm{gr}_{-i} M)[i] \xrightarrow{\delta^i} \omega(\mathrm{gr}_{-i-1} M)[i+1].$$

Lemma 2.2.6. *The functor β takes M to a chain complex M^\bullet with differential δ .*

Proof. It is sufficient to show that the composition $\delta^{i+1} \circ \delta^i = 0$. Consider the following commutative diagrams.

$$\begin{array}{ccc}
& \mathrm{gr}_{-i-1} M & \\
& \nearrow & \searrow \\
w_{\geq -i-2}w_{\leq -i-1}M & & w_{\geq -i-1}w_{\leq -i}M
\end{array}
\qquad
\begin{array}{ccc}
& w_{\geq -i-2}w_{\leq -i}M & \\
& \nearrow & \searrow \\
w_{\geq -i-2}w_{\leq -i-1}M & & w_{\geq -i-1}w_{\leq -i}M
\end{array}$$

Let N be the cone of the morphism $w_{\geq -i-2}w_{\leq -i-1}M \rightarrow w_{\geq -i-1}w_{\leq -i}M$. The octahedral axiom applied to each diagram yields two distinguished triangles

$$\begin{array}{c}
\mathrm{gr}_{-i-2} M[1] \rightarrow N \rightarrow \mathrm{gr}_{-i} M \xrightarrow{q} \mathrm{gr}_{-i-2} M[2], \\
\mathrm{gr}_{-i} M \rightarrow N \rightarrow \mathrm{gr}_{-i-2} M[1] \rightarrow \mathrm{gr}_{-i} M[1].
\end{array}$$

Note that the second triangle splits since $\mathrm{Hom}(\mathrm{gr}_{-i-2} M, \mathrm{gr}_{-i} M) = 0$. This implies that the first triangle must split as well, so $q = 0$. Thus, our composition $\delta^{i+1} \circ \delta^i = 0$ since $\delta^{i+1} \circ \delta^i = \omega(q)[i]$. \square

Proposition 2.2.7. *Let $\tilde{\mathcal{A}}$ be defined as above, and assume that $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant.*

1. *The functor $\beta : \tilde{\mathcal{A}} \rightarrow \text{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is an equivalence of additive categories.*
2. *The composition $\omega \circ \beta^{-1} : \text{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \rightarrow \text{D}_m^b(X_0)$ factors through the category $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ and induces a functor $\tilde{\beta} : \text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \rightarrow \text{D}_m^b(X_0)$ such that the restriction*

$$\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \big|_{\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)} : \mathcal{P}\text{ure}_{\mathcal{S}}(X_0) \rightarrow \text{D}_m^b(X_0)$$

is isomorphic to the inclusion functor.

Proof. To show the equivalence, we must show that β is full, faithful, and essentially surjective. Lemma 2.2.5 implies that β is faithful. We prove fullness by induction on filtered support. Let M and N be objects in $\tilde{\mathcal{A}}$. First, we assume that $M, N \in \tilde{\mathcal{D}}^n$ for some $n \in \mathbb{Z}$. Then $\beta(M)$ and $\beta(N)$ are chain complexes concentrated in degree $-n$ and it follows from the isomorphism (2.2.3) that we have an isomorphism

$$\text{Hom}_{\tilde{\mathcal{D}}}(M, N) \simeq \text{Hom}_{\text{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)}(\beta(M), \beta(N)).$$

Now suppose that $M, N \in F^{\geq m} \tilde{\mathcal{D}} \cap F^{\leq n} \tilde{\mathcal{D}}$. We consider the truncations given by

$$M' = w_{\leq n-1} M, \quad M'' = w_{\geq n} M, \quad N' = w_{\leq n-1} N, \quad N'' = w_{\geq n} N.$$

Let $q : \beta(M'')[-1] \rightarrow \beta(M')$ be the chain map induced by the differential $\delta^{-n} : \beta(M)^{-n} \rightarrow \beta(M)^{-n+1}$. We will need to make use of a lift \tilde{q} of q to $\tilde{\mathcal{D}}$. Note that $\beta(M'')[-1] \cong \beta(s^{-1}M''[-1])$. Let $\tilde{q} : s^{-1}M''[-1] \rightarrow M'$ be the natural map obtained by applying the isomorphism (2.2.1) to the first map of the distinguished triangle $M''[-1] \rightarrow M' \rightarrow M \rightarrow M''$. It is easy to see that $\beta(\tilde{q}) = q$.

Let $f \in \text{Hom}_{\text{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)}(\beta(M), \beta(N))$, and let $f' : \beta(M') \rightarrow \beta(N')$ and $f'' : \beta(M'') \rightarrow \beta(N'')$ be the induced chain maps. The diagram

$$\begin{array}{ccc} \beta(M'')[-1] & \xrightarrow{q} & \beta(M') \\ \downarrow f''[-1] & & \downarrow f' \\ \beta(N'')[-1] & \xrightarrow{q} & \beta(N') \end{array}$$

commutes since f is a chain map. By induction, there are morphisms $\tilde{f}' : M' \rightarrow N'$ and $\tilde{f}'' : s^{-1}M''[-1] \rightarrow s^{-1}N''[-1]$ such that $\beta(\tilde{f}') = f'$ and $\beta(\tilde{f}'') = f''[-1]$. Consider the diagram

$$\begin{array}{ccccc} M''[-1] & \xrightarrow{\alpha} & s^{-1}M''[-1] & \xrightarrow{\tilde{q}} & M' \\ \downarrow s\tilde{f}'' & & \downarrow \tilde{f}'' & & \downarrow \tilde{f}' \\ N''[-1] & \xrightarrow{\alpha} & s^{-1}N''[-1] & \xrightarrow{\tilde{q}} & N' \end{array}$$

Since α is a natural transformation, we see that the left-hand square commutes. The right-hand square commutes because β is faithful. Thus, the outer square commutes as well, so we may complete it to a morphism of distinguished triangles

$$\begin{array}{ccccccc}
M''[-1] & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \\
s\tilde{f}'' \downarrow & & \tilde{f}' \downarrow & & \tilde{f} \downarrow & & \downarrow \\
N''[-1] & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N''
\end{array}$$

We have $\beta(\tilde{f}) = f$, so β is full.

A similar argument proves that β is essentially surjective. It is easy to see that any chain complex concentrated in a single degree is in the image of β . Now, if $M^\bullet \in \mathbf{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ such that $M^i = 0$ except when $-n \leq i \leq -m$, then the differential δ^{-n} induces a chain map $q : M''[-1] \rightarrow M'$, where M'' is concentrated in degree $-n$ and M' vanishes except in degrees $-n+1, \dots, -m$. By induction, we have objects $\tilde{M}'', \tilde{M}' \in \tilde{\mathcal{A}}$ so that $\beta(\tilde{M}'') = M''$ and $\beta(\tilde{M}') = M'$. Since β is fully faithful, we have a morphism $\tilde{q} : \tilde{M}'' \rightarrow \tilde{M}'$ with $\beta(\tilde{q}) = q$. Let \tilde{M} be the cone of the morphism $\tilde{q} \circ \alpha : s\tilde{M}'' \rightarrow \tilde{M}'$. Then $\beta(\tilde{M}) \cong M$.

Now, we consider part (2). Let $f : M^\bullet \rightarrow N^\bullet$ be a morphism in $\mathbf{C}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$, corresponding via β to $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$. Let Z^\bullet denote the cone of f , and let \tilde{Z} denote the cone of the composition

$$s\tilde{M} \xrightarrow{\alpha} \tilde{M} \xrightarrow{\tilde{f}} \tilde{N}.$$

Note that $\tilde{Z} \in \tilde{\mathcal{A}}$ by Remark 2.2.3. Since $\omega(sM) \cong \omega(M)$, we see that $\omega \circ \beta^{-1}$ takes the diagram $M^\bullet \rightarrow N^\bullet \rightarrow Z^\bullet \rightarrow M^\bullet[1]$ to a distinguished triangle in $\mathbf{D}_m^b(X_0)$. If f is null-homotopic, then the homotopy induces a chain map $Z^\bullet \rightarrow N^\bullet$ which induces a splitting of the triangle

$$\tilde{N} \rightarrow \tilde{Z} \rightarrow s\tilde{M}[1] \xrightarrow{\tilde{f} \circ \alpha[1]} \tilde{N}[1]$$

in $\tilde{\mathcal{D}}$. Thus, $\omega \circ \beta^{-1}(f) = 0$. □

Remark 2.2.8. We assumed that $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ was Frobenius invariant. However, the above construction still holds if we replace *Frobenius invariance* with the weaker condition that $\underline{\text{Hom}}^i(S, S')$ is pure of weight i for all $i \in \mathbb{Z}$ and $S, S' \in \mathcal{S}$.

2.3 Mixed and Orlov Categories

Let \mathcal{M} be a finite-length abelian category. As in [10, Definition 4.1.1], a *mixed structure* on \mathcal{M} is a function $\text{wt} : \text{Irr}(\mathcal{M}) \rightarrow \mathbb{Z}$ such that

$$\text{Ext}^1(S, S') = 0 \text{ if } S, S' \text{ are simple objects with } \text{wt}(S) \leq \text{wt}(S'). \quad (2.3.1)$$

As in [5, Section 2.2], we can extend the notion of a mixed structure to a triangulated category in the following way. Let \mathcal{D} be a triangulated category with a bounded t -structure whose heart is \mathcal{M} . A *mixed structure* on \mathcal{D} is a mixed structure on \mathcal{M} satisfying

$$\text{Hom}_{\mathcal{D}}^i(S, S') = 0 \text{ if } S, S' \in \mathcal{M} \text{ are simple objects with } \text{wt}(S) < \text{wt}(S') + i. \quad (2.3.2)$$

Let \mathcal{A} be an additive category and $\text{Ind}(\mathcal{A})$ be the set of isomorphism classes of indecomposable objects in \mathcal{A} . The category \mathcal{A} , equipped with a function $\text{deg} : \text{Ind}(\mathcal{A}) \rightarrow \mathbb{Z}$, is called an *Orlov category* (see [5, Definition 4.1]) if the following conditions hold:

1. All Hom-spaces in \mathcal{A} are finite-dimensional.
2. For any $S \in \text{Ind}(\mathcal{A})$, we have $\text{End}(S) \cong \bar{\mathbb{Q}}_\ell$.
3. If $S, S' \in \text{Ind}(\mathcal{A})$ with $\text{deg}(S) \leq \text{deg}(S')$ and $S \not\cong S'$, then $\text{Hom}(S, S') = 0$.

According to [5, Proposition 5.4], the homotopy category of an Orlov category $\text{K}^b(\mathcal{A})$ has a natural t -structure whose heart is a finite-length abelian category containing irreducibles given by $A[\text{deg}(A)]$ for $A \in \text{Ind}(\mathcal{A})$. Also, the function $\text{wt}(A[\text{deg}(A)]) = \text{deg}(A)$ makes $\text{K}^b(\mathcal{A})$ into a mixed category.

Remark 2.3.1. The category $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Orlov. An indecomposable object is given by $S[2m](m)$ where S is a simple perverse sheaf in \mathcal{S} . We define the degree function by $\text{deg}(S[2m](m)) = -2m$. To see that this degree function makes $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ into an Orlov category, we simply note that for $S, S' \in \mathcal{S}$ with $S[2m](m) \not\cong S'[2n](n)$

$$\text{Hom}_{\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)}(S[2m](m), S'[2n](n)) = \text{Hom}_{\text{D}_{\text{m}}^b(X_0)}^{2n-2m}(S(m), S'(n)).$$

When $-2m < -2n$, $2n - 2m$ is negative, and this vanishes since $S(m)$ and $S'(n)$ are objects in the heart of a t -structure on $\text{D}_{\text{m}}^b(X_0)$. If $-2m = -2n$, this vanishes since we assume that $S[2m](m) \not\cong S'[2n](n)$ implying that S and S' are nonisomorphic simple objects.

We will denote the heart of $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ by $\mathcal{P}\text{erv}^{\text{mix}}(X_0)$. The simple objects in $\mathcal{P}\text{erv}^{\text{mix}}(X_0)$ are given by $(S[2i](i))[-2i]$ for any $S \in \mathcal{S}$, $i \in \mathbb{Z}$. Note that the two shifts do not cancel since they occur in different triangulated categories. By [5], the category $\mathcal{P}\text{erv}^{\text{mix}}(X_0)$ is a mixed category with weight function $\text{wt}(S[2i](i))[-2i] = -2i$ and a degree 2 Tate twist $\langle 2 \rangle := [-2](-1)[2]$. In the remainder of this section, we will show that $\text{K}^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is a mixed version of its analogue over $\bar{\mathbb{F}}_q$. This is defined as follows:

Definition 2.3.2. Let \mathcal{D} and \mathcal{D}' be t -categories such that

- \mathcal{D} is a mixed triangulated category with a t -exact autoequivalence (*a degree d Tate twist*) $\langle d \rangle : \mathcal{D} \rightarrow \mathcal{D}$ satisfying $\text{wt}(S\langle d \rangle) = \text{wt}(S) + d$;
- there is a t -exact functor $F : \mathcal{D} \rightarrow \mathcal{D}'$ such that the essential image generates \mathcal{D}' as a triangulated category;
- and there is an isomorphism $\varepsilon : F \circ \langle d \rangle \xrightarrow{\sim} F$.

Then \mathcal{D} is called a *mixed version* of \mathcal{D}' if ε induces an isomorphism for all objects $M, N \in \mathcal{D}$

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(M, N\langle nd \rangle) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}'}(FM, FN). \quad (2.3.3)$$

Let $D_c^b(X)$ be the ℓ -adic derived category of complexes of sheaves on $X := X_0 \times_{\text{Spec} \mathbb{F}_q} \text{Spec}(\overline{\mathbb{F}}_q)$. Define $\mathcal{F} : K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \rightarrow D_c^b(X)$ to be the functor given by the composition

$$K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \xrightarrow{\tilde{\beta}} D_m^b(X_0) \xrightarrow{\xi} D_c^b(X).$$

Recall that $K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0) \xrightarrow{\tilde{\beta}} D_m^b(X_0)$ is the realization functor defined in Section 2.2. The functor $D_m^b(X_0) \xrightarrow{\xi} D_c^b(X)$ is given by extending scalars to $\overline{\mathbb{F}}_q$. Let $D_{\mathcal{S}}^b(X)$ be the triangulated category generated in $D_c^b(X)$ by the objects $\mathcal{F}(S)$ for $S \in \mathcal{S}$, and let $\mathcal{P}\text{erv}_{\mathcal{S}}(X)$ be the Serre subcategory of $\mathcal{P}\text{erv}(X)$ generated by the perverse sheaves $\mathcal{F}(S)$ for $S \in \mathcal{S}$. Note that $D_{\mathcal{S}}^b(X)$ contains the image of \mathcal{F} in $D_c^b(X)$. Thus, we may think of \mathcal{F} as a functor with target $D_{\mathcal{S}}^b(X)$. Note that shifts in $K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ and $D_m^b(X_0)$ combine under \mathcal{F} . Thus, for $M \in K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$, we have that $\mathcal{F}(M[2i](i)[-2i]) \cong \mathcal{F}(M)$. Also, \mathcal{F} commutes with shifts so $\mathcal{F}(M[i]) \cong \mathcal{F}(M)[i]$. We now show that $K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is a mixed version of $D_{\mathcal{S}}^b(X)$ assuming $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant on morphisms.

Theorem 2.3.3. *Assume that $\mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$ is Frobenius invariant. Then $K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is a mixed version of $D_{\mathcal{S}}^b(X)$, where $D_{\mathcal{S}}^b(X)$ is the triangulated category generated by the image of \mathcal{S} in $D_c^b(X)$.*

Proof. Let M and $N \in K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$. We proceed by double induction on the lengths of the chain complexes M and N . First, assume that M and N are concentrated in one degree. Without loss of generality, assume M is concentrated in degree 0, i.e. that $M \in \mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$. Let $j \in \mathbb{Z}$ be such that $N[-j] \in \mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$. Then $\text{Hom}(M, N\langle 2n \rangle) = \text{Hom}(M, N[-2n](-n)[2n]) \neq 0$ implies that $2n = -j$ because otherwise, M and $N\langle 2n \rangle$ would be chain complexes concentrated in different degrees. Now, if j is odd, then $\bigoplus_{n \in \mathbb{Z}} \text{Hom}(M, N\langle 2n \rangle) = 0$. In this case, we must show that $\text{Hom}(\mathcal{F}M, \mathcal{F}N) = 0$. Recall that \mathcal{F} commutes with shift and that $N[-j] \in \mathcal{P}\text{ure}_{\mathcal{S}}(X_0)$. Thus, we see that

$$\text{Hom}(\mathcal{F}M, \mathcal{F}N) = \text{Hom}(\mathcal{F}M, \mathcal{F}(N[-j])[j]) = \text{Hom}^j(\mathcal{F}M, \mathcal{F}(N[-j])).$$

This vanishes by Lemma 2.1.6.

Now assume that j is even.

$$\begin{aligned} \bigoplus_{n \in \mathbb{Z}} \text{Hom}(M, N\langle 2n \rangle) &= \text{Hom}(M, N[j](\tfrac{j}{2})[-j]) \\ &\cong \text{Hom}_{D_m^b(X_0)}(M, (N[-j])(\tfrac{j}{2})[j]) \\ &\cong \text{Hom}_{D_{\mathcal{S}}^b(X)}(\mathcal{F}M, \mathcal{F}N), \quad \text{by Lemma 2.1.8} \end{aligned}$$

Suppose that the theorem holds for M a chain complex of length less than $n + 1$ and N concentrated in a single degree. Now, assume that $M^\bullet \in K^b(\mathcal{P}\text{ure}_{\mathcal{S}} X_0)$ is a chain complex of length $n + 1$ and that N is a chain complex concentrated in one degree. Let $i \in \mathbb{Z}$ be such that M^\bullet vanishes in degrees less than i and more than $i + n$. Note that the differential δ^i induces a chain map $M''[-1] \rightarrow M'$ where M'' and M' are the obvious truncations of M . This gives a distinguished triangle $M' \rightarrow M^\bullet \rightarrow M'' \rightarrow$. This triangle gives us the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^{-1}(M', N\langle 2n \rangle) & \longrightarrow & \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(M'', N\langle 2n \rangle) & \longrightarrow & \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(M^\bullet, N\langle 2n \rangle) \xrightarrow{u} \\
\alpha_1 \downarrow \wr & & \alpha_2 \downarrow \wr & & \downarrow f \\
\mathrm{Hom}^{-1}(\mathcal{F}(M'), \mathcal{F}(N)) & \longrightarrow & \mathrm{Hom}(\mathcal{F}(M''), \mathcal{F}(N)) & \longrightarrow & \mathrm{Hom}(\mathcal{F}(M^\bullet), \mathcal{F}(N)) \xrightarrow{u'} \\
\\
& & \xrightarrow{u} \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(M', N\langle 2n \rangle) & \longrightarrow & \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^1(M'', N\langle 2n \rangle) \\
& & \wr \downarrow \alpha_3 & & \wr \downarrow \alpha_4 \\
& & \xrightarrow{u'} \mathrm{Hom}(\mathcal{F}(M'), \mathcal{F}(N)) & \longrightarrow & \mathrm{Hom}^1(\mathcal{F}(M''), \mathcal{F}(N))
\end{array}$$

Note that $\alpha_1, \alpha_2, \alpha_3$, and α_4 are isomorphisms by the induction hypothesis. Thus, the five lemma implies that f is also an isomorphism.

A similar argument proves the claim for general M and N in $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_{\mathcal{S}} X_0)$. \square

Corollary 2.3.4. *The heart $\mathcal{P}\mathrm{erv}^{\mathrm{mix}}(X_0)$ of the t -structure on $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_{\mathcal{S}} X_0)$ is a mixed version of the category of perverse sheaves $\mathcal{P}\mathrm{erv}_{\mathcal{S}}(X)$.*

2.3.1 A Second t -Structure

We now define a new t -structure on $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_{\mathcal{S}} X_0)$, which we will refer to as the non-standard t -structure. To do so, we regard $\mathcal{P}\mathrm{ure}_{\mathcal{S}}(X_0)$ as an Orlov category with a different degree function. Recall that an indecomposable in $\mathcal{P}\mathrm{ure}_{\mathcal{S}}(X_0)$ is given by $S[2i](i)$ for a simple perverse sheaf $S \in \mathcal{S}$ and $i \in \mathbb{Z}$. We define the new degree function on indecomposables by $\mathrm{deg}(S[2i](i)) = -i$. The same argument used earlier verifies that this degree function also makes $\mathcal{P}\mathrm{ure}_{\mathcal{S}}(X_0)$ into an Orlov category. Thus, [5, Proposition 5.4] implies that we have a t -structure on $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_{\mathcal{S}} X_0)$. We will denote the heart of this second t -structure by $\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(X_0)$. It is also mixed with a degree 1 Tate twist $((1)) := [-2](-1)[1]$ and contains irreducible objects $S((i))$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$.

Chapter 3

Mixed Sheaves on the Nilpotent Cone

In the following Chapter, we apply the theory developed in Chapter 2 to the Springer correspondence and constructible sheaves on the nilpotent cone. In Section 3.1, we briefly recall the Springer correspondence, the Springer sheaf \mathbf{A} , and Borel–Moore homology.

The interested reader may find Springer’s original discovery of Weyl group representations in the cohomology of Springer fibers [36] in 1976. In 1981, Lusztig conjectured that Springer’s Weyl group representations can be interpreted in terms of intersection cohomology sheaves on the unipotent variety [23], which was proven by Borho–MacPherson [14] in 1981. More recently, the modular case has been investigated by Juteau [19] in 2007 and Mautner [28] in 2012. Other examples include Kato’s definition of an *exotic* Springer correspondence [20] in 2009, Yun’s study of *global* Springer theory [37] in 2011, Nadler’s interpretation of Springer theory via the Hitchin Fibration [30] in 2011, and Russell’s study of Graham fibers [33] in 2011. Connections between the Springer correspondence and geometric Satake have been studied by Achar–Henderson [2] in 2011 and by Achar–Henderson–Riche [3] in 2012. In 2013, Ben-Zvi and Nadler have defined *elliptic* Springer theory in [11]. This is only a small sampling of the wealth of interest and approaches to Springer theory.

Our main result in this section is pinning down the Frobenius action on $\mathcal{P}\text{ure}_{\mathcal{N}_0}$. In particular, using the Borel–Moore homology of the Steinberg variety, we are able to prove that the Frobenius action on $\text{End}(\mathbf{A})$ is trivial. We also prove that $\mathbf{K}^b(\mathcal{P}\text{ure}_G \mathcal{N}_0)$ is a mixed version of the Springer block $\mathbf{D}_{G, \text{Spr}}^b(\mathcal{N})$ (see Theorem 3.1.6).

In Section 3.2, we complete our study of mixed Springer sheaves on the nilpotent cone. Our main result is an equivalence of two (mixed) categories: $\mathbf{K}^b(\mathcal{P}\text{ure}_G \mathcal{N}_0)$, related to the geometry of nilpotent cone, and $\mathbf{D}^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*)$, related to representations of the Weyl group W (see Theorem 3.2.3). We also prove the non-equivariant analogue.

3.1 Background on the Springer Correspondence

Let G be a connected, reductive algebraic group defined over $\bar{\mathbb{F}}_q$. Let $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements in the Lie algebra of G . Our goal is to understand the representation theory of the Weyl group W for G by studying the Springer sheaf \mathbf{A} .

3.1.1 The Springer Sheaf

Let $\mu : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the Springer resolution. Then the Springer sheaf $\mathbf{A} \in \mathbf{D}_c^b(\mathcal{N})$ is defined by

$$\mathbf{A} := \mu_*(\mathcal{C}_{\tilde{\mathcal{N}}})[d]\left(\frac{d}{2}\right)$$

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where $\mathcal{C}_{\tilde{\mathcal{N}}}$ is the constant sheaf on $\tilde{\mathcal{N}}$ and $d = \dim \tilde{\mathcal{N}}$. Let $D_{G,c}^b(\mathcal{N})$ be the G -equivariant derived category. For background on the equivariant derived category, see [13]. The Springer sheaf \mathbf{A} is also a G -equivariant perverse sheaf. For most of what follows, we will consider both non-equivariant and G -equivariant versions of statements. The proofs in both cases are essentially the same. We will not distinguish by notation objects that belong to both $D_c^b(\mathcal{N})$ and $D_{G,c}^b(\mathcal{N})$. The following fact is well-known and a consequence of [9, Proposition 4.2.5]: for G -equivariant perverse sheaves \mathcal{F} and \mathcal{G} on a G variety X with G connected, we have that $\mathrm{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D_{G,c}^b(X)}(\mathcal{F}, \mathcal{G})$. A proof can be found in [27, Section 1.16].

The Springer sheaf \mathbf{A} has a natural action of the Weyl group $\sigma : W \rightarrow \mathrm{Aut}(\mathbf{A})$ defined by Lusztig in [23]. According to [14, Theorem 3], we have an isomorphism

$$\bar{\mathbb{Q}}_\ell[W] \cong \mathrm{End}(\mathbf{A}).$$

Let \mathcal{B} be the variety of Borel subgroups of G . Then we also have an action $\kappa : \bar{\mathbb{Q}}_\ell[W] \rightarrow \mathrm{End}(\mathbf{H}^\bullet(\mathcal{B}))$ induced by the W -action on G/T where T is a maximal torus. We have G -equivariant analogues of the above. The structure of the cohomology ring of the flag variety is well understood. There is a degree doubling isomorphism of graded rings between (1) the algebra of W -coinvariants $\mathrm{Coinv}(W)$ and $\mathbf{H}^\bullet(\mathcal{B})$, and (2) the symmetric algebra on the dual of the Cartan subalgebra Sh^* and $\mathbf{H}_G^\bullet(\mathcal{B})$. We will often make use of the categories $D_c^b(\mathcal{B})$ and $D_{G,c}^b(\mathcal{B})$ of ℓ -adic sheaves on \mathcal{B} constructible with respect to the trivial stratification.

We will usually focus our attention on the categories $D_{\mathrm{Spr}}^b(\mathcal{N})$ and $D_{G,\mathrm{Spr}}^b(\mathcal{N})$ defined as the triangulated categories generated by the simple summands of \mathbf{A} in $D_c^b(\mathcal{N})$ and $D_{G,c}^b(\mathcal{N})$, respectively.

3.1.2 Borel–Moore Homology of the Steinberg Variety

Another approach to studying the relationship between W -representations and the Springer sheaf \mathbf{A} involves the Borel–Moore homology of the Steinberg variety $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. For our purposes, we use the definition of the Borel–Moore homology in terms of the hypercohomology of the dualizing complex ω_Z as developed in [15, 8.3.7].

$$\mathbf{H}_i^{BM}(Z) := \mathbf{H}^{-i}(Z, \omega_Z).$$

Many details of the relationship between $\mathbf{H}_\bullet^{BM}(Z)$ and the Springer sheaf \mathbf{A} are developed in [15]. In particular, we have an isomorphism of graded rings

$$\mathbf{H}_{2d-\bullet}^{BM}(Z) \cong \mathrm{Hom}^\bullet(\mathbf{A}, \mathbf{A}). \quad (3.1.1)$$

Let X_0 be a variety defined over \mathbb{F}_q . Then we may define Borel–Moore homology with an n -twist (with ℓ -adic coefficients) by

$$\mathbf{H}_i^{BM}(X_0, n) = \mathrm{Hom}_{D_{\mathrm{m}}^b(X_0)}^{-i}(\mathcal{C}_{X_0}, \omega_{X_0}(n)).$$

Extending scalars to $\bar{\mathbb{F}}_q$, we get the usual Borel–Moore homology (with an n -twist), and it has inherited an action of Frobenius. To denote this, we write $\underline{\mathbf{H}}_i^{BM}(X_0, n) =$

$\underline{\mathrm{Hom}}^{-i}(\mathcal{C}_{X_0}, \omega_{X_0}(n))$. Applying the short exact sequence (2.1.1), we get the following short exact sequence relating the Borel–Moore homology groups of X_0 over \mathbb{F}_q and X over $\overline{\mathbb{F}}_q$:

$$0 \rightarrow \underline{H}_{i+1}^{BM}(X_0, n)_{\mathrm{Frob}} \rightarrow H_i^{BM}(X_0, n) \rightarrow \underline{H}_i^{BM}(X_0, n)^{\mathrm{Frob}} \rightarrow 0$$

In particular, we have the following theorem:

Theorem 3.1.1. *Let X_0 be a variety defined over \mathbb{F}_q . Then the Borel–Moore homology of X_0 vanishes for $i < -1$ and $i > 2 \dim(X_0)$. In particular, $H_{2 \dim(X_0)}^{BM}(X_0, n) \cong \underline{H}_{2 \dim(X_0)}^{BM}(X_0, n)^{\mathrm{Frob}}$.*

Proof. This follows from the fact that $\underline{H}_i^{BM}(X_0, n)$ is only non-zero for $0 \leq i \leq 2 \dim(X_0)$ and the above short exact sequence. \square

3.1.3 Mixed Springer Sheaves

Now, in order to apply the mixedness machinery of [9], we need analogues of \mathcal{N} , \mathfrak{g} , G , \mathcal{B} and other related varieties defined over a finite field \mathbb{F}_q of characteristic p . We make the following assumptions before proceeding.

1. These varieties are equipped with an \mathbb{F}_q -rational structure.
2. The field \mathbb{F}_q is *big enough* and has good characteristic.
3. The reductive group G is \mathbb{F}_q -split.

The reason for our first assumption is clear. Let \mathcal{N}_0 , G_0 , et cetera denote \mathbb{F}_q -schemes whose extension of scalars are \mathcal{N} , G , et cetera. Now, G_0 acts on \mathcal{N}_0 by the adjoint action; however, it may be the case that not all G -orbits of \mathcal{N} appear. Since there are only finitely many nilpotent orbits, we can ensure that all of them are defined by taking a finite field extension of \mathbb{F}_q if necessary. This is the reasoning behind our second assumption. We assume that \mathbb{F}_q is *big enough* so that the Frobenius fixed point set of each nilpotent G -orbit is non-empty. Our final requirement is that we must assume that G is \mathbb{F}_q -split. In other words, G_0 has a split maximal torus of the same dimension as a maximal torus in G . We now show that in this setting, we do not lose information by passing to the \mathbb{F}_q -setting.

Proposition 3.1.2. *The top degree Borel–Moore homology $\underline{H}_{2d}^{BM}(Z, -d)$ has a basis that is Frobenius invariant.*

Proof. Recall that the irreducible components of the Steinberg variety are given by closures of conormal bundles $T_{Y_w}^*(\mathcal{B} \times \mathcal{B})$, where Y_w is the G -orbit of $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in W$. Each of these is defined over \mathbb{F}_q since for split G , the Bruhat decomposition is defined over \mathbb{F}_q .

The top-degree Borel–Moore homology of an algebraic variety has a basis given by fundamental classes associated to the top-dimensional irreducible components. In order to show that $\underline{H}_{2d}^{BM}(Z, -d)$ is Frobenius invariant, it suffices to show that our basis can be defined over \mathbb{F}_q . Let X be an irreducible component of Z such that

X_0 is the corresponding irreducible component of Z_0 over \mathbb{F}_q . Then the fundamental class associated to X is defined over \mathbb{F}_q . To see this, we need a canonical element $f \in \text{Hom}(\mathcal{C}_{X_0}, \omega_{X_0}[-2d](-d)) = \mathbb{H}^{-2d}(X_0, \omega_{X_0}(-d)) = \mathbb{H}_{2d}^{BM}(X_0, -d)$. Let $U_0 \subset X_0$ be a smooth dense open set and set $F_0 = X_0 \setminus U_0$. Then we have inclusions

$$U_0 \xrightarrow{j} X_0 \xleftarrow{i} F_0.$$

This gives a distinguished triangle in $\mathbb{D}_m^b(X_0)$

$$i_* i^! \omega_{X_0}[-2d](-d) \rightarrow \omega_{X_0}[-2d](-d) \rightarrow j_* j^* \omega_{X_0}[-2d](-d) \rightarrow i_* i^! \omega_{X_0}[-2d+1](-d).$$

Now we apply $\text{Hom}(\mathcal{C}_{X_0}, -)$ to get the exact sequence in Borel–Moore homology

$$\mathbb{H}_{2d}^{BM}(F_0, -d) \rightarrow \mathbb{H}_{2d}^{BM}(X_0, -d) \rightarrow \mathbb{H}_{2d}^{BM}(U_0, -d) \rightarrow \mathbb{H}_{2d+1}^{BM}(F_0, -d)$$

Note that $\dim F_0 < \dim X_0$. Thus, Theorem 3.1.1 implies that

$$\mathbb{H}_{2d}^{BM}(F_0, -d) = \mathbb{H}_{2d+1}^{BM}(F_0, -d) = 0$$

since both are Borel–Moore homology groups in degree greater than $2 \dim F_0$. Therefore we have an isomorphism $\mathbb{H}_{2d}^{BM}(X_0, -d) \cong \mathbb{H}_{2d}^{BM}(U_0, -d)$, i.e. an isomorphism

$$\text{Hom}(\mathcal{C}_{X_0}, \omega_{X_0}[-2d](-d)) \cong \text{Hom}(\mathcal{C}_{U_0}, \mathcal{C}_{U_0}).$$

Let f be the morphism corresponding to $id : \mathcal{C}_{U_0} \rightarrow \mathcal{C}_{U_0}$ under this isomorphism. \square

We use the Borel–Moore homology as a stepping stone to get the following result.

Corollary 3.1.3. *We have an isomorphism $\text{Hom}_{\mathbb{D}_m^b(\mathcal{N}_0)}(\mathbf{A}, \mathbf{A}) \cong \underline{\text{Hom}}(\mathbf{A}, \mathbf{A}) \cong \bar{\mathbb{Q}}_\ell[W]$. Thus, we also have $\text{Hom}_{\mathbb{D}_{G,m}^b(\mathcal{N}_0)}(\mathbf{A}, \mathbf{A}) \cong \bar{\mathbb{Q}}_\ell[W]$.*

Proof. Recall the short exact sequence

$$0 \rightarrow \underline{\text{Hom}}^{-1}(\mathbf{A}, \mathbf{A})_{\text{Frob}} \rightarrow \text{Hom}(\mathbf{A}, \mathbf{A}) \rightarrow \underline{\text{Hom}}(\mathbf{A}, \mathbf{A})^{\text{Frob}} \rightarrow 0.$$

Since \mathbf{A} is a perverse sheaf, $\underline{\text{Hom}}^{-1}(\mathbf{A}, \mathbf{A}) = 0$ implying the map

$$\text{Hom}(\mathbf{A}, \mathbf{A}) \rightarrow \underline{\text{Hom}}(\mathbf{A}, \mathbf{A})^{\text{Frob}}$$

is an isomorphism. Thus, we have an injection $\text{Hom}(\mathbf{A}, \mathbf{A}) \hookrightarrow \underline{\text{Hom}}(\mathbf{A}, \mathbf{A})$, so it suffices to show that $\text{Hom}(\mathbf{A}, \mathbf{A})$ and $\underline{\text{Hom}}(\mathbf{A}, \mathbf{A})$ have the same dimension. Recall the isomorphism (3.1.1) from [15]. This restricts to an isomorphism of the degree 0 piece: $\underline{\text{Hom}}(\mathbf{A}, \mathbf{A}) \cong \underline{\mathbb{H}}_{2d}^{BM}(Z_0, -d)$. The following string of isomorphisms gives us the conclusion:

$$\begin{aligned} \underline{\text{Hom}}(\mathbf{A}, \mathbf{A})^{\text{Frob}} &\cong \underline{\mathbb{H}}_{2d}^{BM}(Z_0, -d)^{\text{Frob}} \\ &\cong \underline{\mathbb{H}}_{2d}^{BM}(Z_0, -d), && \text{by Proposition 3.1.2} \\ &\cong \underline{\text{Hom}}(\mathbf{A}, \mathbf{A}). && \square \end{aligned}$$

Thus, we have a natural action of the Weyl group W on $\mathbf{A} \in D_m^b(\mathcal{N}_0)$. Let $\sigma : W \rightarrow \text{Aut}(\mathbf{A})$ denote this action. By [9, Proposition 5.3.9], we may decompose \mathbf{A} as follows:

$$\mathbf{A} \simeq \bigoplus_{\chi \in \text{Irr}(W)} \text{IC}_\chi \otimes V_\chi,$$

where the IC_χ are various distinct simple perverse sheaves and V_χ is a vector space with an action of Frobenius, which is not a priori semisimple.

Remark 3.1.4. The above decomposition occurs for \mathcal{N}_0 defined over \mathbb{F}_q and differs from what is usually found in the literature when \mathcal{N} is defined over an algebraically closed field \mathbb{C} or $\overline{\mathbb{F}}_q$.

Note that both the Frobenius action and the Weyl group action on V_χ arise via the identification: $V_\chi = \underline{\text{Hom}}(\text{IC}_\chi, \mathbf{A})$. However this identification requires a choice of $\text{IC}_\chi \in D_m^b(\mathcal{N}_0)$. Regardless of the choice of IC_χ , it is easy to see that Frobenius is a W -equivariant endomorphism of V_χ by Corollary 3.1.3. Of course V_χ is an irreducible W -representation. Hence, Frobenius must act by a scalar (with absolute value 1). Since a scalar may travel across the tensor product $\text{IC}_\chi \otimes V_\chi$, we declare that Frobenius acts trivially on V_χ , and this gives a unique choice up to isomorphism for the simple perverse sheaf $\text{IC}_\chi \in D_m^b(\mathcal{N}_0)$. We fix this choice for all $\chi \in \text{Irr}(W)$.

Define $\mathcal{S}\text{pr}$ as the full subcategory of $D_m^b(\mathcal{N}_0)$ (or $D_{G,m}^b(\mathcal{N}_0)$) consisting of objects that are finite direct sums of the simple perverse sheaves IC_χ as above. Let $\mathcal{P}\text{ure}\mathcal{N}_0$, respectively $\mathcal{P}\text{ure}_G\mathcal{N}_0$, be the full subcategory of $D_m^b(\mathcal{N}_0)$, respectively $D_{G,m}^b(\mathcal{N}_0)$, consisting of semisimple objects that are pure of weight 0 and whose length 1 indecomposable summands are in $\mathcal{S}\text{pr}[2n](n)$ for some n . In other words,

$$\mathcal{P}\text{ure}\mathcal{N}_0 = \bigoplus_{n \in \mathbb{Z}} \mathcal{S}\text{pr}[2n](n).$$

In the following section, we will show that the categories $\mathcal{P}\text{ure}\mathcal{N}_0$ and $\mathcal{P}\text{ure}_G\mathcal{N}_0$ are Frobenius invariant by relating morphisms between simple perverse sheaves with the cohomology (or G -equivariant cohomology) of the flag variety \mathcal{B} . To do so, we must first introduce a pair of functors.

3.1.4 The Functors Ψ and Φ

We now introduce an adjoint pair of functors studied thoroughly in [1]. Consider the maps

$$\mathcal{N} \xleftarrow{\mu} \tilde{\mathcal{N}} \xrightarrow{\pi} \mathcal{B}.$$

We define the functors

$$\Psi : D_c^b(\mathcal{B}) \rightarrow D_c^b(\mathcal{N}) \quad \text{and} \quad \Phi : D_c^b(\mathcal{N}) \rightarrow D_c^b(\mathcal{B})$$

by $\Psi = \mu_* \pi^!$ and $\Phi = (\pi_* \mu^!)[-d](-\frac{d}{2})$. Note that $\Psi \simeq \mu_! \pi^*[d](\frac{d}{2})$ since μ is proper and π is smooth of relative dimension $\frac{d}{2}$. It is easy to see that (Ψ, Φ) forms an adjoint pair. We denote the isomorphism induced by adjunction by

$$\theta : \text{Hom}^i(\mathcal{C}_{\mathcal{B}}, \Phi(\mathbf{A})) \xrightarrow{\sim} \text{Hom}^i(\Psi(\mathcal{C}_{\mathcal{B}}), \mathbf{A}) = \text{Hom}^i(\mathbf{A}, \mathbf{A}).$$

The following is a refinement [1, Theorem 4.6]:

Proposition 3.1.5. *The category $\mathcal{P}\text{ure}\mathcal{N}_0$ is Frobenius invariant. That is, for all $\text{IC}_\chi, \text{IC}_\psi$ in \mathcal{S} and $i \in \mathbb{Z}$*

$$\underline{\text{Hom}}(\text{IC}_\chi, \text{IC}_\psi[i](\frac{i}{2}))^{\text{Frob}} \cong \underline{\text{Hom}}(\text{IC}_\chi, \text{IC}_\psi[i](\frac{i}{2})).$$

In particular, $\underline{\text{Hom}}(\text{IC}_\chi, \text{IC}_\psi[i](\frac{i}{2})) \cong \text{Hom}_W(V_\chi^, \text{H}^i(\mathcal{B}) \otimes V_\psi^*)$ and $\underline{\text{Hom}}^i(\text{IC}_\chi, \text{IC}_\psi)$ vanishes for i odd.*

Proof. Note that $\underline{\text{Hom}}^i(\mathbf{A}, \text{IC}_\psi(\frac{i}{2})) \cong \bigoplus_\chi \underline{\text{Hom}}^i(\text{IC}_\chi, \text{IC}_\psi(\frac{i}{2})) \otimes V_\chi^*$. Thus, we may

compute $\underline{\text{Hom}}^i(\text{IC}_\chi, \text{IC}_\psi(\frac{i}{2}))$ by the χ^* -isotypic component of $\underline{\text{Hom}}^i(\mathbf{A}, \text{IC}_\psi(\frac{i}{2}))$ under its W action. Therefore, we have that

$$\begin{aligned} \underline{\text{Hom}}^i(\text{IC}_\chi, \text{IC}_\psi(\frac{i}{2})) &\cong \text{Hom}_W(V_\chi^*, \underline{\text{Hom}}^i(\mathbf{A}, \text{IC}_\psi(\frac{i}{2}))) \\ &\cong \text{Hom}_W(V_\chi^*, \text{H}^i(\mathcal{B})(\frac{i}{2}) \otimes \underline{\text{Hom}}(\mathcal{C}_\mathcal{B}, \Phi(\text{IC}_\psi))) \\ &\cong \text{Hom}_W(V_\chi^*, \text{H}^i(\mathcal{B})(\frac{i}{2}) \otimes \underline{\text{Hom}}(\mathbf{A}, \text{IC}_\psi)) \\ &\cong \text{Hom}_W(V_\chi^*, \text{H}^i(\mathcal{B})(\frac{i}{2}) \otimes \underline{\text{Hom}}(\text{IC}_\psi \otimes V_\psi, \text{IC}_\psi)) \\ &\cong \text{Hom}_W(V_\chi^*, \text{H}^i(\mathcal{B})(\frac{i}{2}) \otimes V_\psi^*) \\ &\cong (V_\chi \otimes \text{H}^i(\mathcal{B})(\frac{i}{2}) \otimes V_\psi^*)^W \end{aligned}$$

Recall that Frobenius acts on $\text{H}^i(\mathcal{B})$ by $q^{\frac{i}{2}}$. Hence, the Frobenius action on $\text{H}^i(\mathcal{B})(\frac{i}{2})$ is trivial. Since $\text{H}^i(\mathcal{B})$ vanishes for i odd, so does $\underline{\text{Hom}}^i(\text{IC}_\chi, \text{IC}_\psi)$. \square

Theorem 3.1.6. *We have that $\text{K}^b(\mathcal{P}\text{ure}\mathcal{N}_0)$ is a mixed version of $\text{D}_{\text{Spr}}^b(\mathcal{N})$, where $\text{D}_{\text{Spr}}^b(\mathcal{N})$ is the triangulated category generated by the image of $\mathcal{S}\text{pr}$ in $\text{D}_c^b(\mathcal{N})$. Similarly, in the G equivariant case, we have that $\text{K}^b(\mathcal{P}\text{ure}_G\mathcal{N}_0)$ is a mixed version of $\text{D}_{G, \text{Spr}}^b(\mathcal{N})$, where $\text{D}_{G, \text{Spr}}^b(\mathcal{N})$ is the triangulated category generated by the image of $\mathcal{S}\text{pr}$ in $\text{D}_{G, c}^b(\mathcal{N})$.*

Proof. By Proposition 3.1.5, we see that $\mathcal{P}\text{ure}\mathcal{N}_0$ has Frobenius invariant morphisms. Now, we apply Theorem 2.3.3. \square

Corollary 3.1.7. *The category $\mathcal{P}\text{erv}^{\text{mix}}(\mathcal{N})$ is a mixed version of the category of perverse sheaves $\mathcal{S}\text{pr} \subset \text{D}_{\text{Spr}}^b(\mathcal{N})$. Similarly, $\mathcal{P}\text{erv}_G^{\text{mix}}(\mathcal{N})$ is a mixed version of the category of perverse sheaves $\mathcal{S}\text{pr} \subset \text{D}_{G, \text{Spr}}^b(\mathcal{N})$.*

3.2 Mixed Springer Correspondence

We now prove a *mixed version* of the Springer correspondence, i.e. an equivalence of two mixed triangulated categories: $\text{K}^b(\mathcal{P}\text{ure}_G\mathcal{N}_0)$, related to the geometry of \mathcal{N} , and $\text{D}^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*)$, related to the representation theory of W . (We also consider the obvious non-equivariant analogue.)

Since we have a W -action on $\text{H}^\bullet(\mathcal{B})$, we may define the smash product algebra $\bar{\mathbb{Q}}_\ell[W] \# \text{H}^\bullet(\mathcal{B})$. As a vector space, $\bar{\mathbb{Q}}_\ell[W] \# \text{H}^\bullet(\mathcal{B}) = \bar{\mathbb{Q}}_\ell[W] \otimes \text{H}^\bullet(\mathcal{B})$. The multiplication is given by

$$(w_1 \otimes f_1)(w_2 \otimes f_2) = w_1 w_2 \otimes \kappa(w_2^{-1})(f_1) f_2$$

for $w_1, w_2 \in W$ and $f_1, f_2 \in H^\bullet(\mathcal{B})$. This algebra is discussed in [17] where they show that $\bar{\mathbb{Q}}_\ell[W] \# H^\bullet(\mathcal{B})$ and $H_{2d-\bullet}^{BM}(Z)$ are isomorphic as graded algebras. Of course, this combined with the isomorphism from (3.1.1) implies that

$$\mathrm{Hom}^\bullet(\mathbf{A}, \mathbf{A}) \cong \bar{\mathbb{Q}}_\ell[W] \# H^\bullet(\mathcal{B}) .$$

Now we consider the following G -equivariant version of the above proposition. A proof can be found in [21].

Proposition 3.2.1. *The rings $\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B})$ and $\mathrm{Hom}_G^\bullet(\mathbf{A}, \mathbf{A})$ are isomorphic.*

The above isomorphisms are isomorphisms of graded rings. In particular, the vanishing of the cohomology of the flag variety in odd degrees implies $\mathrm{Hom}^i(\mathbf{A}, \mathbf{A})$ and $\mathrm{Hom}_G^i(\mathbf{A}, \mathbf{A})$ vanish when i is odd. Thus, we may consider the graded algebra \mathcal{A} defined by $\mathcal{A}^i := \mathrm{Hom}^{2i}(\mathbf{A}, \mathbf{A})$. Similarly, we define \mathcal{A}_G in the G -equivariant case. Then, it is easy to see that

$$\mathcal{A} \cong \bar{\mathbb{Q}}_\ell[W] \# \mathrm{Coinv}(W) \text{ and } \mathcal{A}_G \cong \bar{\mathbb{Q}}_\ell[W] \# \mathrm{Sh}^* .$$

Graded \mathcal{A} and \mathcal{A}_G Modules

Let R be a positively graded ring, $R = \bigoplus_{i \in \mathbb{N}} R^i$. Let $\mathrm{gMod}(R)$ denote the category of finitely generated right graded modules over R . For a graded module $M \in \mathrm{gMod}(R)$ with $M = \bigoplus_{i \in \mathbb{Z}} M^i$, we denote by $\{1\}M$ the object obtained by shifting the grading $(\{1\}M)^i = M^{i+1}$. Let $\mathrm{gProj}(R)$ denote the full subcategory consisting of finitely generated projectives. Define $R^+ = \bigoplus_{i > 0} R^i$, and suppose that the quotient $R_0 = R/R^+$ is a semisimple ring.

Remark 3.2.2. In this case, the projective modules for R are easy to describe. Let L be a simple (right) R_0 -module. Then $L \otimes_{R_0} R$ is a projective R -module. In fact, any projective R -module is a direct sum of shifts of these. A proof of this can be found in [35, Lemma 6].

In our case, we consider the graded rings $\mathcal{A} = \bar{\mathbb{Q}}_\ell[W] \# \mathrm{Coinv}(W)$ and $\mathcal{A}_G = \bar{\mathbb{Q}}_\ell[W] \# \mathrm{Sh}^*$. The degree 0 piece is isomorphic to $\bar{\mathbb{Q}}_\ell[W]$ in both cases. Thus, for any irreducible W representation V_χ , a shift of $V_\chi \otimes \mathrm{Coinv}(W)$ (or $V_\chi \otimes \mathrm{Sh}^*$ in the G -equivariant case) is an indecomposable projective module.

Let $D^b(\mathcal{A})$ and $D^b(\mathcal{A}_G)$ be the bounded derived category of finitely generated graded modules over $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{Coinv}(W)$ and $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{Sh}^*$. We also consider the perfect derived category: $D_{\mathrm{per}}^b(\mathcal{A})$. It is the full triangulated subcategory of $D^b(\mathcal{A})$ generated by modules with a finite projective resolution. In other words, $D_{\mathrm{per}}^b(\mathcal{A})$ is equivalent to $K^b(\mathrm{gProj} \bar{\mathbb{Q}}_\ell[W] \# \mathrm{Coinv}(W))$, the bounded homotopy category of (finitely generated) graded $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{Coinv}(W)$ modules that are projective. We note that

$$D^b(\mathcal{A}_G) \cong D_{\mathrm{per}}^b(\mathcal{A}_G)$$

since $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{Sh}^*$ has finite homological dimension.

Theorem 3.2.3 (Mixed Springer Correspondence). *We have the following equivalences of triangulated categories:*

- $\mathrm{K}^b(\mathcal{P}\mathrm{ure}\mathcal{N}_0)$ is equivalent to $\mathrm{D}_{\mathrm{per}}^b(\mathcal{A})$;
- $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G\mathcal{N}_0)$ is equivalent to $\mathrm{D}^b(\mathcal{A}_G)$.

Proof. It suffices to show that the additive categories $\mathrm{gProj}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Coinv}(W))$ and $\mathcal{P}\mathrm{ure}(\mathcal{N}_0)$ (or $\mathrm{gProj}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Sh}^*)$ and $\mathcal{P}\mathrm{ure}_G\mathcal{N}_0$ in the G -equivariant case) are equivalent since we have

$$\mathrm{D}_{\mathrm{per}}^b(\mathcal{A}) \cong \mathrm{K}^b(\mathrm{gProj}\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Coinv}(W)) \quad \text{and} \quad \mathrm{D}^b(\mathcal{A}_G) \cong \mathrm{K}^b(\mathrm{gProj}\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Sh}^*).$$

This is the content of the following proposition. □

Remark 3.2.4. The analogue of the above in the $G \times \mathbb{G}_m$ -equivariant setting should also hold by the same methods. In this case, the category $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_{G \times \mathbb{G}_m}\mathcal{N}_0)$ should be equivalent to $\mathrm{D}^b(\mathbb{H})$, the derived category of graded modules over the graded Hecke algebra considered by Lusztig in [26]. We also note the discrepancy between the above statement and that in the introduction. The statement in the introduction holds with a slight modification of the definition of $\mathcal{P}\mathrm{ure}(\mathcal{N}_0)$ and $\mathcal{P}\mathrm{ure}_G(\mathcal{N}_0)$ to allow *integral* shift-twist $[1]_{(\frac{1}{2})}$.

Proposition 3.2.5. *The categories $\mathcal{P}\mathrm{ure}(\mathcal{N}_0)$ and $\mathrm{gProj}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Coinv}(W))$ are equivalent as additive categories.*

Proof. We apply the functor $\varphi := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}_m^b(\mathcal{N}_0)}(\mathbf{A}[-2m](-m), -)$. For an indecomposable $\mathrm{IC}_\chi[2i](i)$, we get

$$\underline{\mathrm{Hom}}^{2m+2i}(\mathbf{A}, \mathrm{IC}_\chi(i+m)) \cong V_\chi^* \otimes \mathrm{H}^{2m+2i}(\mathcal{B})(m+i)$$

in degree m . Summing gives $\varphi(\mathrm{IC}_\chi[2i](i)) = \{i\}V_\chi^* \otimes \mathrm{Coinv}(W)$. It is easy to see that we get all objects in $\mathrm{gProj}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Coinv}(W))$ in this way based on the Remark 3.2.2 above.

Now, we need to show that $\mathrm{Hom}(\mathrm{IC}_\chi, \mathrm{IC}_\psi[2i](i)) \cong \mathrm{Hom}(\varphi(\mathrm{IC}_\chi), \varphi(\mathrm{IC}_\psi[2i](i)))$. Recall from the proof of Proposition 3.1.5, we have that

$$\mathrm{Hom}(\mathrm{IC}_\chi, \mathrm{IC}_\psi[2i](i)) \cong \mathrm{Hom}_W(V_\chi^*, \mathrm{H}^{2i}(\mathcal{B})(i) \otimes V_\psi^*).$$

It is easy to see that a W -equivariant map $V_\chi^* \rightarrow \mathrm{H}^{2i}(\mathcal{B})(i) \otimes V_\psi^*$ uniquely determines a map of graded $\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Coinv}(W)$ modules from $V_\chi^* \otimes \mathrm{Coinv}(W) \rightarrow \{i\}V_\psi^* \otimes \mathrm{Coinv}(W)$. □

Proposition 3.2.6. *The categories $\mathcal{P}\mathrm{ure}_G(\mathcal{N}_0)$ and $\mathrm{gProj}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Sh}^*)$ are equivalent as additive categories.*

Proof. Same as in Proposition 3.2.5. □

Note that the equivalence in Theorem 3.2.3 does not preserve the usual t -structures, i.e. $\varphi(\mathcal{P}\mathrm{erv}_G^{\mathrm{mix}}(\mathcal{N}_0)) \not\subset \mathrm{gMod}(\bar{\mathbb{Q}}_\ell[W]\#\mathrm{Sh}^*)$.

Chapter 4

Formality for the Nilpotent Cone

In this chapter, we prove our main result: an equivalence of triangulated categories between $D_{G, \text{Spr}}^b(\mathcal{N})$ and $\mathcal{D}_f^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# H_G^\bullet(\mathcal{B}))$ (see Theorem 4.2.9). That is, we relate the triangulated category of constructible sheaves on the nilpotent cone with differential graded (dg) modules over the corresponding Ext-algebra which we regard as a differential graded algebra with trivial differential. A necessary ingredient for the proof is formality. In order to preserve the triangulated structure of the category, we must pass through some intermediate dg category whose governing dg-algebra is complicated. It is well-known that a formal dg-ring has the same (derived) module category as its cohomology. The usual proof of formality (due to an idea of Deligne) involves endowing this dg-ring with an extra grading so that the cohomology is concentrated on the diagonal. This extra grading is inherited from the mixed category, which we thoroughly studied in Chapters 2 and 3.

The next obstacle is defining such an intermediate dg-ring and a triangulated functor into its category of dg-modules. The definition of such a functor is not obvious. The reason for this is that the category $D_{G, \text{Spr}}^b(\mathcal{N})$ is not actually the derived category of anything. In particular, its objects are not represented by chain complexes of sheaves. In order to define a dg-ring, we need a chain complex with nice homological properties representing the Springer sheaf \mathbf{A} . The solution comes from considering the Koszul dual category in the mixed setting.

Recall that the G -equivariant version of our Springer correspondence involves modules over the graded ring $\bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*$. Thus, it is natural to consider the additional structure of the Koszulity of this ring. Koszul duality between the symmetric and exterior algebras was first described in [12]. This was extended to a more general class of rings in [10] where they describe a derived equivalence between the categories of graded (finitely generated, right) modules over a Koszul ring and its Koszul dual. In our setting, we have an equivalence

$$D^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \bigwedge \mathfrak{h}) \cong D^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*).$$

This equivalence transports the standard t -structure on $D^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \bigwedge \mathfrak{h})$ to a non-standard t -structure on $D^b(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*)$. A description of this is given in [10, 2.13]. Also recall the non-standard t -structure on $K^b(\mathcal{P}\text{ure}_G \mathcal{N}_0)$ defined in Section 2.3.1 with heart $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{N}_0)$. We will show that our mixed Springer equivalence 3.2.3 is exact with respect to this non-standard t -structure. Therefore, it restricts to an equivalence of the hearts $\text{gMod}(\bar{\mathbb{Q}}_\ell[W] \# \bigwedge \mathfrak{h})$ and $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{N}_0)$.

It is well known that the category $\text{gMod}(\bar{\mathbb{Q}}_\ell[W] \# \bigwedge \mathfrak{h})$ has enough injective and projective objects. In particular, we see that $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{N}_0)$ also has enough injective

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and projective objects. Using this, we prove Theorem 4.2.9: we have an equivalence of triangulated categories

$$D_{G,\text{Spr}}^b(\mathcal{N}) \cong \mathcal{D}_f^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})),$$

where $\mathcal{D}_f^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$ is the derived category of finitely generated dg-modules over $\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})$.

The outline for the approach somewhat follows that of [34].

1. Take a projective resolution $\tilde{P}^\bullet \rightarrow \mathbf{A}$ in $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{N}_0)$.
2. Let P^\bullet be the image of \tilde{P}^\bullet in $D_{G,c}^b(\mathcal{N})$.
3. Define the dgg-algebra $\mathcal{H}\text{om}(P^\bullet, P^\bullet)$ and the functor $\mathcal{H}\text{om}(P^\bullet, -)$.
4. Show that $\mathcal{H}\text{om}(P^\bullet, P^\bullet)$ is formal and that its cohomology is $\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})$.

In contrast with [34], we note that step (3) is more involved, and that we should include another step:

5. Show the functor $\mathcal{H}\text{om}(P^\bullet, -)$ is triangulated.

4.1 Differential Graded Algebras and Modules

We briefly review definitions related to differential graded (dg) algebras and modules. For a more thorough treatment, see [13] for example.

Definition 4.1.1. A *differential graded (dg) algebra* is a graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ together with a differential $d : A \rightarrow A[1]$ (an additive map with $d \circ d = 0$) that satisfies the graded Leibniz rule. That is, for homogeneous elements a and b , we have that

$$d(ab) = d(a)b + (-1)^{\deg(a)}d(b).$$

We will also assume that A has a unit 1_A and that $d(1_A) = 0$. A *right dg-module* is a graded unitary module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ together with a differential $d_M : M \rightarrow M[1]$ (an additive map with $d \circ d = 0$) such that

$$d_M(ma) = d_M(m)a + (-1)^{\deg(m)}md(a)$$

for homogeneous $m \in M$ and $a \in A$.

A *morphism* of dg-modules is a degree preserving A -module homomorphism that commutes with differentials. The cohomology $\mathbf{H}(-)$ of a dg-algebra or module is simply the cohomology of the chain complex. A dg-module homomorphism $f : M \rightarrow N$ is called a *quasi-isomorphism* if it induces an isomorphism on cohomology $\mathbf{H}(f) : \mathbf{H}(M) \xrightarrow{\sim} \mathbf{H}(N)$. Similarly, we have quasi-isomorphisms for dg-algebras. Two dg-modules or algebras are *quasi-isomorphic* if they are linked by a sequence of quasi-isomorphisms.

Definition 4.1.2. Let A be a dg-algebra. We denote the (abelian) category of right dg A -modules by $\mathcal{M}^{\text{dg}}(A)$. The derived category of (right) dg A -modules, denoted $\mathcal{D}^{\text{dg}}(A)$ is obtained by localizing $\mathcal{M}^{\text{dg}}(A)$ at quasi-isomorphisms.

As in the case with usual derived categories, the first step in the construction is to form the homotopy category $\mathcal{K}(A)$ of dg-modules. Then, $\mathcal{D}^{\text{dg}}(A)$ inherits the triangulated structure from $\mathcal{K}(A)$. See [13], for instance. In particular, a distinguished triangle (the replacement for short exact sequences in the triangulated setting) of dg-modules is one that is isomorphic (in $\mathcal{D}^{\text{dg}}(A)$) to a sequence of the form

$$M \xrightarrow{f} N \rightarrow \text{cone}(f) \rightarrow M[1].$$

Here, $\text{cone}(f)$ denotes the *cone* of the morphism f , which is the dg-module $M[1] \oplus N$ with differential $d_{\text{cone}(f)} = (-d_M, f + d_N)$.

Definition 4.1.3. Let $\phi : A \rightarrow B$ be a dg-algebra homomorphism. Then ϕ induces functors on the categories of dg-modules. The *extension of scalars* functor, denoted

$$\phi^* : \mathcal{M}^{\text{dg}}(A) \rightarrow \mathcal{M}^{\text{dg}}(B)$$

is defined by $M \mapsto M \otimes_A B$. The *restriction* functor, denoted

$$\phi_* : \mathcal{M}^{\text{dg}}(B) \rightarrow \mathcal{M}^{\text{dg}}(A)$$

is defined by letting A act on a B -module by its image under ϕ . The functors ϕ^* and ϕ_* have derived versions (see [13, Section 10.12] for the definition) and form an adjoint pair

$$\text{Hom}_{\mathcal{D}^{\text{dg}}(B)}(\phi^* M, N) \cong \text{Hom}_{\mathcal{D}^{\text{dg}}(A)}(M, \phi_* N).$$

A dg-algebra A is called *formal* if it is quasi-isomorphic to its cohomology $\text{H}(A)$. A proof of the following theorem can be found in [13, Theorem 10.12.5.1]:

Theorem 4.1.4. *Suppose that $\phi : A \rightarrow B$ is a homomorphism of dg-algebras that induces an isomorphism on cohomology $\text{H}(\phi) : \text{H}(A) \xrightarrow{\sim} \text{H}(B)$. Then the extension and restriction functors*

$$\phi^* : \mathcal{D}^{\text{dg}}(A) \rightarrow \mathcal{D}^{\text{dg}}(B) \text{ and } \phi_* : \mathcal{D}^{\text{dg}}(B) \rightarrow \mathcal{D}^{\text{dg}}(A)$$

are equivalences and inverse to each other. In particular, if a dg-algebra A is formal, then

$$\mathcal{D}^{\text{dg}}(A) \cong \mathcal{D}^{\text{dg}}(\text{H}(A)).$$

4.2 Application to the Nilpotent Cone

4.2.1 The Koszul Dual t -Structure

We begin this section by showing our equivalence (Theorem 3.2.3) is exact with respect to the non-standard t -structure defined in [10].

Theorem 4.2.1. *The non-standard t -structure defined in Section 2.3.1 coincides with the geometric t -structure defined in [10] on $\text{D}^{\text{b}}(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*)$, thus the equivalence*

$$\text{K}^{\text{b}}(\mathcal{P}_{\text{ure}_G} \mathcal{N}_0) \cong \text{D}^{\text{b}}(\text{gMod } \bar{\mathbb{Q}}_\ell[W] \# \text{Sh}^*)$$

is exact with respect to that t -structure. In particular, it restricts to an equivalence of the hearts

$$\text{gMod}(\bar{\mathbb{Q}}_\ell[W] \# \bigwedge \mathfrak{h}) \cong \mathcal{P}_{\text{erv}_{\text{KD}}}(\mathcal{N}_0).$$

Proof. As before, we let $\mathcal{A}_G = \bar{\mathbb{Q}}_\ell[W] \# \mathbf{Sh}^*$. First, we recall the geometric t -structure, denoted $(D^{\leq 0, g}, D^{\geq 0, g})$, on $D^b(\mathcal{A}_G)$ defined in [10, Section 2.13] obtained from the standard t -structure on $D^b(\mathfrak{gMod} \bar{\mathbb{Q}}_\ell[W] \# \wedge \mathfrak{h})$ under the Koszul duality equivalence. The subcategory $D^{\leq 0, g} \subseteq D^b(\mathcal{A}_G)$ (respectively $D^{\geq 0, g} \subseteq D^b(\mathcal{A}_G)$) consists of objects isomorphic to complexes of graded projective modules

$$\dots \rightarrow P^i \rightarrow P^{i+1} \rightarrow \dots$$

such that P^i is generated by its components of degree $\leq -i$ (respectively $\geq -i$) for all i . To show that our functor is exact with respect to this t -structure, it suffices to show that $\varphi(\mathcal{Perv}_{\text{KD}}(\mathcal{N}_0))$ is contained within the heart $D^{\leq 0, g} \cap D^{\geq 0, g}$. Recall that an irreducible in $\mathcal{Perv}_{\text{KD}}(\mathcal{N}_0)$ has the form $(\text{IC}_\chi[2i](i))[-i]$, i.e. a chain complex with $\text{IC}_\chi[2i](i)$ in degree i and 0 elsewhere. Thus, the chain complex $\varphi((\text{IC}_\chi[2i](i))[-i])$ is concentrated in degree i with $\varphi((\text{IC}_\chi[2i](i))[-i])^i = \{-i\} V_\chi \otimes \mathbf{Sh}^*$. It is easy to see that this is an object in the heart $D^{\leq 0, g} \cap D^{\geq 0, g}$. \square

The equivalence $D^b(\mathfrak{gMod} \bar{\mathbb{Q}}_\ell[W] \# \wedge \mathfrak{h}) \cong D^b(\mathfrak{gMod} \bar{\mathbb{Q}}_\ell[W] \# \mathbf{Sh}^*)$ proves that we have an equivalence

$$D^b(\mathcal{Perv}_{\text{KD}}(\mathcal{N}_0)) \cong K^b(\mathcal{Pure}_G \mathcal{N}_0).$$

4.2.2 Formality

Recall that $\mathcal{Perv}_{\text{KD}}(\mathcal{N}_0)$ is the heart of the non-standard t -structure on $K^b(\mathcal{Pure}_G \mathcal{N}_0)$ discussed in Section 2.3.1. By Theorem 4.2.1, $\mathcal{Perv}_{\text{KD}}(\mathcal{N}_0)$ has enough projectives. Let \tilde{P}^\bullet be a projective resolution of \mathbf{A}

$$(\dots \rightarrow \tilde{P}^{-2} \rightarrow \tilde{P}^{-1} \rightarrow \tilde{P}^0) \simeq \mathbf{A}$$

so each $\tilde{P}^i \in \mathcal{Perv}_{\text{KD}}(\mathcal{N}_0) \subseteq K^b(\mathcal{Pure}_G \mathcal{N}_0)$. By Theorem 3.1.6, $K^b(\mathcal{Pure}_G \mathcal{N}_0)$ is a mixed version of $D_{G, \text{Spr}}^b(\mathcal{N})$. In particular, we have a triangulated functor

$$\nu : K^b(\mathcal{Pure}_G \mathcal{N}_0) \rightarrow D_{G, \text{Spr}}^b(\mathcal{N})$$

such that

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{K^b(\mathcal{Pure}_G \mathcal{N}_0)}(M, N\langle 2n \rangle) \xrightarrow{\sim} \text{Hom}_{D_{G, \text{Spr}}^b(\mathcal{N})}(\nu M, \nu N)$$

for all $M, N \in K^b(\mathcal{Pure}_G \mathcal{N}_0)$.

Now define the chain complex $P^\bullet \in C^- D_{G, \text{Spr}}^b(\mathcal{N})$ by the following: $P^i = \nu(\tilde{P}^i)$ with differential d_P given by the image of the differential of \tilde{P}^\bullet . Let \mathcal{R} be the dg-ring given by

$$\mathcal{R}^n = \prod_{\substack{n=i+j, \\ k \in \mathbb{Z}}} \text{Hom}_{D_{G, \text{Spr}}^b(\mathcal{N})}(P^{-i+k}, P^j[k])$$

in degree n and differential $d_{\mathcal{R}} f = d_P f - (-1)^n f d_P$ for f homogeneous of degree n . We will refer to this grading as the vertical grading. Note that \mathcal{R} has an extra

grading (called the internal or horizontal grading) arising from the mixed structure of $\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)$. For $m \in \mathbb{Z}$, we have

$$\mathcal{R}^{n,2m} = \prod_{\substack{n=i+j, \\ k \in \mathbb{Z}}} \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^{-i+k}\langle 2m \rangle, \tilde{P}^j[k])$$

and $\mathcal{R}^{n,2m+1} = 0$. Also, note that the differential $d_{\mathcal{R}}$ respects the internal grading, i.e. $d_{\mathcal{R}}$ is a degree $(1,0)$ map. The cohomology $\mathrm{H}(\mathcal{R})$ is a bigraded ring. We will regard it as a dg-ring with trivial differential in the vertical direction.

Lemma 4.2.2. *The dg-ring \mathcal{R} vanishes below the diagonal.*

Proof. First, we show that $k = m$ since the \tilde{P} 's are projective in $\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)$.

$$\begin{aligned} \mathcal{R}^{n,2m} &= \prod_{\substack{n=i+j, \\ k \in \mathbb{Z}}} \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^{-i+k}\langle 2m \rangle, \tilde{P}^j[k]) \\ &= \prod_{\substack{n=i+j, \\ k \in \mathbb{Z}}} \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^{-i+k}((m)), \tilde{P}^j[k-m]) \\ &= \prod_{\substack{n=i+j, \\ k \in \mathbb{Z}}} \mathrm{Ext}_{\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)}^{k-m}(\tilde{P}^{-i+k}((m)), \tilde{P}^j) \\ &= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)}(\tilde{P}^{m-i}((m)), \tilde{P}^{n-i}). \end{aligned}$$

Now, in the mixed category $\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)$, $\tilde{P}^{m-i}((m))$ has weights less than or equal to $2m - i$ with head pure of weight $2m - i$ and \tilde{P}^{n-i} has weights less than or equal to $n - i$. Any morphism is strictly compatible with the weight filtration. Thus, for $f : \tilde{P}^{m-i}((m)) \rightarrow \tilde{P}^{n-i}$, we have that the head of the image of f is pure of weight $2m - i$. This vanishes when $2m > n$. \square

Theorem 4.2.3. *The differential graded ring \mathcal{R} is formal. In other words, \mathcal{R} is quasi-isomorphic to its cohomology. Its cohomology ring is*

$$\mathrm{H}(\mathcal{R}) = \mathrm{Hom}_{\mathrm{D}_{G,\mathrm{Spr}}^b(\mathcal{N})}(\mathbf{A}, \mathbf{A}) \cong \bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}).$$

Proof. As a consequence of an idea of Deligne, [16, 5.3.1, Corollary 5.3.7], purity of the cohomology $\mathrm{H}(\mathcal{R})$ with respect to the internal grading implies formality of \mathcal{R} . A proof of this can be found in [34, Proposition 4]. Purity means the cohomology in vertical degree i should be concentrated in horizontal degree i . In other words, $\mathrm{H}^i(\mathcal{R}) = \mathrm{H}^i(\mathcal{R}^{\bullet,i})$. Since $\mathrm{D}^b(\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)) \cong \mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)$ and \tilde{P}^\bullet is a projective resolution of \mathbf{A} , we have that

$$\begin{aligned} \mathrm{H}^i(\mathcal{R}^{\bullet,2m}) &= \mathrm{H}^i(\mathcal{H}\mathrm{om}(\tilde{P}^{\bullet+m}\langle 2m \rangle, \tilde{P}^\bullet[m])) \\ &= \mathrm{H}^i(\mathcal{H}\mathrm{om}(\tilde{P}^{\bullet+m}((m)), \tilde{P}^\bullet)) \\ &= \mathrm{Ext}_{\mathrm{D}^b(\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0))}^i(\mathbf{A}((m))[m], \mathbf{A}) \\ &= \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}^i(\mathbf{A}\langle 2m \rangle, \mathbf{A}). \end{aligned}$$

Then we have

$$\mathrm{Hom}_{\mathbf{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}^i(\mathbf{A}\langle 2m \rangle, \mathbf{A}) = \mathrm{Hom}_{\mathbf{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}^{i-2m}(\mathbf{A}[-2m](-m), \mathbf{A}).$$

In $\mathbf{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)$, $\mathbf{A}[-2m](-m)$ and \mathbf{A} are chain complexes concentrated in degree 0. Clearly, if this is nonzero, then $i = 2m$. When $i = 2m$ since our realization functor $\mathbf{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0) \rightarrow \mathbf{D}_{G,m}^b(\mathcal{N}_0)$ restricts to inclusion on $\mathcal{P}\mathrm{ure}_G \mathcal{N}_0$, we have that

$$\mathrm{Hom}_{\mathbf{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\mathbf{A}[-2m](-m), \mathbf{A}) = \mathrm{Hom}_{\mathbf{D}_{G,m}^b(\mathcal{N}_0)}(\mathbf{A}[-2m](-m), \mathbf{A}).$$

By Proposition 3.2.1,

$$\mathrm{Hom}_{\mathbf{D}_{G,m}^b(\mathcal{N}_0)}(\mathbf{A}[-2m](-m), \mathbf{A}) = \underline{\mathrm{Hom}}_G^{2m}(\mathbf{A}, \mathbf{A})(m) \cong \bar{\mathbb{Q}}_\ell[W] \otimes \mathrm{H}_G^{2m}(\mathcal{B}). \quad \square$$

Remark 4.2.4. In fact, more is true. By Lemma 4.2.2 and Schnürer's argument [34, Proposition 4], we have a dg-ring homomorphism $\mathcal{R} \rightarrow \bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})$ which is a quasi-isomorphism.

4.2.3 The Main Theorem: A Derived Springer Correspondence

For any $M \in \mathbf{D}_{G,\mathrm{Spr}}^b(\mathcal{N})$, we define the dg-module $\mathcal{H}\mathrm{om}(P^\bullet, M)$ over \mathcal{R} . In degree i , we have

$$\mathcal{H}\mathrm{om}(P^\bullet, M)^i = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{G,\mathrm{Spr}}^b(\mathcal{N})}(P^{-i+j}, M[j])$$

and has differential induced by that of P^\bullet : if $f \in \mathcal{H}\mathrm{om}(P^\bullet, M)^i$, then $d_{\tilde{M}}f = (-1)^{i+1}d_P f$.

Remark 4.2.5. Note that in each degree $i \in \mathbb{Z}$, the module $\mathcal{H}\mathrm{om}(P^\bullet, M)$ has only finitely many non-zero terms. One can show directly that

$$\mathcal{H}\mathrm{om}(P^\bullet, \mathrm{IC}_X)^i = \mathrm{Hom}_{\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)}(\tilde{P}^{-i/2}((i/2)), \mathrm{IC}_X)$$

for i even and vanishes for i odd using properties of the t -structure and mixedness of $\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)$. Dévissage proves the general case.

Let $\mathcal{D}_{\mathrm{per}}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}))$ be the perfect derived category, i.e. the smallest full triangulated category generated by the free $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})$ module and closed under direct summands. We note that $\mathcal{D}_{\mathrm{per}}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})) \cong \mathcal{D}_{\mathrm{f}}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}))$. Since \mathcal{R} is quasi-isomorphic to $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})$, we have an equivalence between the derived categories of dg-modules over these dg-rings. Let $\tilde{\mathcal{L}}$ denote the composition

$$\mathbf{D}_{G,\mathrm{Spr}}^b(\mathcal{N}) \xrightarrow{\mathcal{H}\mathrm{om}(P^\bullet, -)} \mathcal{D}^{\mathrm{dg}}(\mathcal{R}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})).$$

Because P^\bullet is not an object of a triangulated category, the definition of our functor $\mathcal{H}\mathrm{om}(P^\bullet, -)$ is somewhat non-standard. We provide the following lemma to prove that $\mathcal{H}\mathrm{om}(P^\bullet, -)$ commutes with shift [1], and we prove the functor is triangulated in the appendix.

Lemma 4.2.6. *The functor $\mathcal{H}\mathrm{om}(P^\bullet, -) : \mathbf{D}_{G,c}^b(\mathcal{N}) \rightarrow \mathcal{D}^{\mathrm{dg}}(\mathcal{R})$ commutes with shift.*

Proof. Recall that $\mathcal{H}\mathrm{om}(P^\bullet, M)[1]^i = \mathcal{H}\mathrm{om}(P^\bullet, M)^{i+1}$ with differential $d_{\tilde{M}[1]} = -d_{\tilde{M}}$.

$$\begin{aligned}\mathcal{H}\mathrm{om}(P^\bullet, M[1])^i &= \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}_{G, \mathrm{Spr}}^b(\mathcal{N})}^{\mathrm{b}}(P^{-i+j}, M[j+1]) \\ &= \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}_{G, \mathrm{Spr}}^b(\mathcal{N})}^{\mathrm{b}}(P^{-i-1+j}, M[j]) \\ &= \mathcal{H}\mathrm{om}(P^\bullet, M)[1]^i\end{aligned}$$

The differential of $\mathcal{H}\mathrm{om}(P^\bullet, M[1])$ is given by $d_{\tilde{M}[1]} = (-1)^{i+1}d_P$. Hence, we have that $d_{\tilde{M}[1]} = (-1)^{i+1}d_P = -1(-1)^i d_P = -d_{\tilde{M}}$. \square

Lemma 4.2.7. *The functor $\mathcal{H}\mathrm{om}(P^\bullet, -)$ is triangulated.*

Proof. Given in the appendix. \square

Lemma 4.2.8. *The dg-module $\mathcal{H}\mathrm{om}(P^\bullet, \mathbf{A})$ is quasi-isomorphic to the free module $\mathcal{H}\mathrm{om}(P^\bullet, P^\bullet)$.*

Proof. Let $\tilde{P}^\bullet \xrightarrow{q} \mathbf{A}$ be the quasi-isomorphism in $\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)$. The image of q in $\mathrm{D}_{G, \mathrm{Spr}}^b(\mathcal{N})$ induces a morphism of dg-modules

$$\mathcal{H}\mathrm{om}(P^\bullet, P^\bullet) \xrightarrow{q^*} \mathcal{H}\mathrm{om}(P^\bullet, \mathbf{A}).$$

The fact that q^* induces an isomorphism in cohomology is almost by definition since

$$\mathcal{H}\mathrm{om}(P^\bullet, P^\bullet) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}\mathrm{om}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^\bullet, \tilde{P}^\bullet \langle 2m \rangle)$$

and

$$\mathcal{H}\mathrm{om}(P^\bullet, \mathbf{A}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}\mathrm{om}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^\bullet, \mathbf{A} \langle 2m \rangle),$$

and since cohomology commutes with the direct sum. We provide the following computation anyway.

$$\begin{aligned}\mathrm{H}^i(\mathcal{H}\mathrm{om}(P^\bullet, \mathbf{A})) &= \mathrm{H}^i\left(\bigoplus_{m \in \mathbb{Z}} \mathcal{H}\mathrm{om}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^\bullet, \mathbf{A} \langle 2m \rangle)\right) \\ &= \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^i(\mathcal{H}\mathrm{om}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\tilde{P}^\bullet, \mathbf{A} \langle 2m \rangle)) \\ &= \bigoplus_{m \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{P}\mathrm{erv}_{\mathrm{KD}}(\mathcal{N}_0)}^i(\mathbf{A}, \mathbf{A} \langle 2m \rangle) \\ &= \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\mathbf{A}, \mathbf{A}[-2m](-m)[2m+i]).\end{aligned}$$

Of course, $\mathrm{Hom}_{\mathrm{K}^b(\mathcal{P}\mathrm{ure}_G \mathcal{N}_0)}(\mathbf{A}, \mathbf{A}[-2m](-m)[2m+i]) \neq 0$ implies that $2m+i=0$ because otherwise we would have chain complexes concentrated in different degrees.

Thus, we see that

$$\begin{aligned}
\mathrm{H}^i(\mathcal{H}\mathrm{om}(P^\bullet, \mathbf{A})) &= \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathcal{P}_{\mathrm{ure}_G \mathcal{N}_0})}(\mathbf{A}, \mathbf{A}[i](\frac{i}{2})) \\
&= \mathrm{Hom}_{\mathrm{D}_{G, \mathrm{m}}^{\mathrm{b}}(\mathcal{N}_0)}(\mathbf{A}, \mathbf{A}[i](\frac{i}{2})) \\
&\cong \underline{\mathrm{Hom}}_{\mathrm{D}_{G, \mathrm{Spr}}^{\mathrm{b}}(\mathcal{N})}(\mathbf{A}, \mathbf{A}[i](\frac{i}{2})) \\
&\cong \bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}^i(\mathcal{B}). \quad \square
\end{aligned}$$

Theorem 4.2.9 (A derived Springer correspondence). *The category $\mathrm{D}_{G, \mathrm{Spr}}^{\mathrm{b}}(\mathcal{N})$ is equivalent to $\mathcal{D}_{\mathrm{f}}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}))$ as a triangulated category.*

Proof. Recall that $\mathrm{D}_{G, \mathrm{Spr}}^{\mathrm{b}}(\mathcal{N})$ is the triangulated category generated by Spr in $\mathrm{D}_G^{\mathrm{b}}(\mathcal{N})$ and that $\mathcal{D}_{\mathrm{f}}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}))$ is the triangulated category generated by the summands of the free module $\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})$ in $\mathcal{D}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B}))$. Thus, by [34, Lemma 6] a refinement of Beilinson's Lemma [8, Lemma 1.4], it suffices to prove that

$$\mathrm{Hom}(\mathrm{IC}_\chi, \mathrm{IC}_\psi[i]) \cong \mathrm{Hom}(\tilde{\mathcal{L}}\mathrm{IC}_\chi, \tilde{\mathcal{L}}\mathrm{IC}_\psi[i])$$

for all irreducible perverse sheaves $\mathrm{IC}_\chi, \mathrm{IC}_\psi \in \mathrm{Spr}$, any $i \in \mathbb{Z}$. Note that $\tilde{\mathcal{L}}\mathrm{IC}_\chi \cong V_\chi^* \otimes \mathrm{H}_G^\bullet(\mathcal{B})$ and $\tilde{\mathcal{L}}\mathrm{IC}_\psi[i] \cong V_\psi^* \otimes \mathrm{H}_G^{\bullet+i}(\mathcal{B})$. A morphism of dg-modules in this case is simply a morphism of graded modules since the differentials vanish. In fact, it is easy to see that such a morphism is determined by a W -equivariant map $V_\chi^* \rightarrow V_\psi^* \otimes \mathrm{H}_G^i(\mathcal{B})$. Hence,

$$\mathrm{Hom}(\tilde{\mathcal{L}}\mathrm{IC}_\chi, \tilde{\mathcal{L}}\mathrm{IC}_\psi[i]) \cong \mathrm{Hom}_W(V_\chi^*, V_\psi^* \otimes \mathrm{H}_G^i(\mathcal{B})).$$

A slight modification (remove Tate twists) of Lemma 3.1.5 proves that we also have

$$\mathrm{Hom}(\mathrm{IC}_\chi, \mathrm{IC}_\psi[i]) \cong \mathrm{Hom}_W(V_\chi^*, V_\psi^* \otimes \mathrm{H}_G^i(\mathcal{B})). \quad \square$$

Remark 4.2.10. The usual Springer correspondence can be recovered by composing with H^0 .

4.3 Proof of Lemma 4.2.7

Proving our functor is triangulated is not at all straightforward. The problem of course lies at the heart of the problem with triangulated categories: non-functoriality of cones. Thus, our approach is very roundabout. It is similar to successfully killing a mosquito: the mosquito must not see you coming.

First, we prove the restriction functor $i_* : \mathcal{D}^{\mathrm{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})) \rightarrow \mathcal{D}^{\mathrm{dg}}(\mathrm{H}_G^\bullet(\mathcal{B}))$ induced by the injection $i : \mathrm{H}_G^\bullet(\mathcal{B}) \rightarrow \bar{\mathbb{Q}}_\ell[W] \# \mathrm{H}_G^\bullet(\mathcal{B})$ reflects distinguished triangles.

Then we prove some straightforward facts about the flag variety. It is well known by work of [13] that the G -equivariant derived category of the flag variety is equivalent to a category of dg-modules $\mathcal{D}^{\mathrm{dg}}(\mathrm{H}_G^\bullet(\mathcal{B}))$. We treat this as a sort of enhancement to the category $\mathrm{D}_{G, c}^{\mathrm{b}}(\mathcal{B})$ and then use our adjoint pair (Ψ, Φ) to say something about the nilpotent cone.

4.3.1 Averaging

Let W be a finite group. Suppose that $f : V_1 \rightarrow V_2$ is a linear map between W -representations that is not necessarily W -equivariant. Then we can easily produce a W -equivariant map f^a by averaging:

$$f^a(x) = \frac{1}{|W|} \sum_{w \in W} w^{-1} f(wx).$$

It is straightforward to check that if f is W -equivariant, then $f = f^a$. We call f *locally W -equivariant at $m \in V_1$* if $f(wm) = wf(m)$ for all $w \in W$. The following lemma says that f matches f^a where f is *locally W -equivariant*.

Lemma 4.3.1. *Let $m \in V_1$ have the property that $f(wm) = wf(m)$ for all $w \in W$. Then, $f^a(m) = f(m)$.*

Proof. This is an easy computation. □

Lemma 4.3.2. *Suppose we have a commutative diagram*

$$\begin{array}{ccccc} M_1 & \xrightarrow{g} & M_2 & \xrightarrow{h} & M_3 \\ r \downarrow & & f \downarrow & & s \downarrow \\ N_1 & \xrightarrow{g'} & N_2 & \xrightarrow{h'} & N_3 \end{array}$$

so that M_i, N_i are W -representations, the maps g, h, g', h', r, s are W -equivariant, and f is linear (not necessarily W -equivariant). Then the diagram where we replace f with its average f^a commutes.

Proof. We begin with the left square. Note that $fg(wm) = g'r(wm) = wg'r(m) = wfg(m)$. Thus, f is *locally W -equivariant* on the image of g . By Lemma 4.3.1, $f^a g = fg = g'r$.

Now, we consider the right square. Note that the composition $h'f$ is W -equivariant since it equals the W -equivariant map sh .

$$h'f^a(m) = \frac{1}{|W|} \sum_{w \in W} h'(w^{-1}f(wm)) = \frac{1}{|W|} \sum_{w \in W} w^{-1}h'f(wm) = h'f(m).$$

Since $h'f = sh$, the result follows. □

For a dg-ring \mathcal{R} , we denote by $\mathcal{KP}(\mathcal{R})$ the homotopy category of homotopically projective dg-modules (also referred to as K -projectives). It is well known that we have an equivalence $\mathcal{D}^{\text{dg}}(\mathcal{R}) \cong \mathcal{KP}(\mathcal{R})$ [13, Corollary 10.12.2.9].

We regard $\mathbf{H}_G^\bullet(\mathcal{B})$ as a dg-ring with trivial differential. There is a natural functor $i_* : \mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})) \rightarrow \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))$ forgetting the W action. This functor, although clearly not an equivalence, has a very special property: it reflects triangles. The meaning of this is the following:

Proposition 4.3.3. *Let $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} L[1]$ be a sequence in $\mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$ ($\cong \mathcal{K}\mathcal{P}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$) such that its image under i_* is a distinguished triangle in $\mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))$ ($\cong \mathcal{K}\mathcal{P}(\mathbf{H}_G^\bullet(\mathcal{B}))$). Then it is also distinguished in $\mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$.*

Proof. Let $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} L[1]$ be a candidate triangle in $\mathcal{K}\mathcal{P}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$ that becomes distinguished in $\mathcal{K}\mathcal{P}(\mathbf{H}_G^\bullet(\mathcal{B}))$. By the hypothesis, we have an isomorphism between our candidate triangle and a standard triangle in the category $\mathcal{K}\mathcal{P}(\mathbf{H}_G^\bullet(\mathcal{B}))$

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N & \xrightarrow{h} & L[1] \\ \downarrow \parallel & & \downarrow \parallel & & q \downarrow & & \downarrow \parallel \\ \tilde{L} & \xrightarrow{f} & \tilde{M} & \xrightarrow{\pi_1} & \text{cone}(f) & \xrightarrow{\pi_2} & \tilde{L}[1] \end{array}$$

where the map q is not necessarily W -equivariant. Note that the standard triangle $\tilde{L} \xrightarrow{f} \tilde{M} \xrightarrow{\pi_1} \text{cone}(f) \xrightarrow{\pi_2} \tilde{L}[1]$ is distinguished in $\mathcal{K}\mathcal{P}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$. To get a morphism of triangles in $\mathcal{K}\mathcal{P}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$, we replace q with q^a to get the following:

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N & \xrightarrow{h} & L[1] \\ \downarrow \parallel & & \downarrow \parallel & & q^a \downarrow & & \downarrow \parallel \\ \tilde{L} & \xrightarrow{f} & \tilde{M} & \xrightarrow{\pi_1} & \text{cone}(f) & \xrightarrow{\pi_2} & \tilde{L}[1] \end{array}$$

The fact that each square commutes is proven in Lemma 4.3.8. We take cohomology in $\mathcal{K}\mathcal{P}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$ to get a long exact sequence and then apply the five lemma. This proves that $\mathbf{H}^i(q^a)$ is an isomorphism for all $i \in \mathbb{Z}$. Thus, q^a is a quasi-isomorphism. \square

4.3.2 Statements for the Flag Variety

Definition 4.3.4. Define $\mathcal{P}\text{ure}_G(\mathcal{B}_0)$ as the full subcategory of $\mathbf{D}_{G,m}^b(\mathcal{B}_0)$ with objects in $\bigoplus_{i \in \mathbb{Z}} \mathcal{C}_{\mathcal{B}_0}[2i](i)$.

Lemma 4.3.5. *Let $\mathcal{P}\text{ure}_G(\mathcal{B}_0)$ be defined as above. Then we have*

1. $\mathbf{K}^b(\mathcal{P}\text{ure}_G \mathcal{B}_0)$ is a mixed version of $\mathbf{D}_{G,c}^b(\mathcal{B})$.
2. $\mathbf{K}^b(\mathcal{P}\text{ure}_G \mathcal{B}_0) \cong \mathbf{D}^b(\text{gMod Sh}^*)$ as triangulated categories.
3. $\mathbf{K}^b(\mathcal{P}\text{ure}_G \mathcal{B}_0)$ has a non-standard t -structure with heart $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{B}_0)$ containing enough projectives.
4. Choose a projective resolution \tilde{Q}^\bullet of $\mathcal{C}_{\mathcal{B}_0}$ in $\mathcal{P}\text{erv}_{\text{KD}}(\mathcal{B}_0)$ such that the image Q^\bullet in $\mathbf{C}^- \mathbf{D}_{G,c}^b(\mathcal{B})$ satisfies $\Psi(Q^\bullet) = P^\bullet$. Define $\mathcal{H}\text{om}(Q^\bullet, Q^\bullet)$ as above. Then $\mathcal{H}\text{om}(Q^\bullet, Q^\bullet)$ is formal.

Proof. The proof of each statement is exactly the same as the analogous statement above for \mathcal{N} . \square

Let $\gamma : D_{G,c}^b(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}_f^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))$ denote the equivalence described in [13, Theorem 2.6.3, Theorem 12.7.2 ii]. Note that it is given by composition

$$D_{G,c}^b(\mathcal{B}) \xrightarrow{\sim} D_{B,c}^b(pt) \cong \mathcal{D}_f^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))$$

where the first functor is induction equivalence. Define the chain complex of dg-modules \mathcal{Q}^\bullet by $\mathcal{Q}^i = \gamma(Q^i)$ and differential $d_{\mathcal{Q}} = \gamma(Q^i \rightarrow Q^{i+1})$. Now we define a functor

$$\mathcal{L}_{\mathcal{B}} : \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B})) \rightarrow \mathcal{D}^{\text{dg}}(\mathcal{H}\text{om}(Q^\bullet, Q^\bullet))$$

by the following formula $\mathcal{L}_{\mathcal{B}}(M)^n = \bigoplus_{n=j-i} \text{Hom}_{\mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))}(\mathcal{Q}^i, M[j])$ with differential

given by $d_{\mathcal{L}_{\mathcal{B}}(M)}f = d_M f - (-1)^n f d_{\mathcal{Q}}$ for f homogeneous of degree n . Note that the dg-rings $\mathcal{H}\text{om}(\mathcal{Q}^\bullet, \mathcal{Q}^\bullet)$ and $\mathcal{H}\text{om}(Q^\bullet, Q^\bullet)$ are isomorphic as dg-rings (not just quasi-isomorphic).

Lemma 4.3.6. *The functor $\mathcal{L}_{\mathcal{B}} : \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B})) \rightarrow \mathcal{D}^{\text{dg}}(\mathcal{H}\text{om}(Q^\bullet, Q^\bullet))$ is triangulated.*

Proof. To see that the functor is triangulated, we will prove that for a morphism $M \xrightarrow{f} N$ of $\mathbf{H}_G^\bullet(\mathcal{B})$ dg-modules, $\mathcal{L}_{\mathcal{B}}(\text{cone}(f)) \cong \text{cone}(\mathcal{L}_{\mathcal{B}}(f))$. Let

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \text{cone}(\mathcal{L}_{\mathcal{B}}(f))$$

be homogeneous of degree i . Note that this implies g_1 is homogeneous of degree $i+1$ for $\mathcal{L}_{\mathcal{B}}(M)$. Of course, $\text{cone}(\mathcal{L}_{\mathcal{B}}(f)) = \mathcal{L}_{\mathcal{B}}(M)[1] \oplus \mathcal{L}_{\mathcal{B}}(N)$ with differential given by

$$d_1(g) = \begin{pmatrix} -d_{\mathcal{L}(M)} & 0 \\ \mathcal{L}(f) & d_{\mathcal{L}(N)} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -d_M g_1 - (-1)^i g_1 d_Q \\ f g_1 + d_N g_2 - (-1)^i g_2 d_Q \end{pmatrix}.$$

On the other hand, the induced differential applied to $g \in \mathcal{L}_{\mathcal{B}}(\text{cone}(f))$ is given by

$$d_2(g) = \begin{pmatrix} -d_M & 0 \\ f & d_N \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} - (-1)^i \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d_Q.$$

It is clear that these two are the same. \square

Remark 4.3.7. There is an isomorphism of functors between

$$\mathcal{L}_{\mathcal{B}} \circ \gamma : D_{G,c}^b(\mathcal{B}) \rightarrow \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(Q^\bullet)) \text{ and } \mathcal{H}\text{om}(Q^\bullet, -) : D_{G,c}^b(\mathcal{B}) \rightarrow \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(Q^\bullet)).$$

To see this, it suffices to compare the differentials of the dg-modules $\mathcal{L}_{\mathcal{B}} \circ \gamma(M)$ and $\mathcal{H}\text{om}(Q^\bullet, M)$ for $M \in D_{G,c}^b(\mathcal{B})$. Note that $\mathcal{H}\text{om}(Q^\bullet, -)$ is defined as before: for $M \in D_{G,c}^b(\mathcal{B})$, we define the dg-module $\mathcal{H}\text{om}(Q^\bullet, M)$ over $\mathcal{E}\text{nd}(Q^\bullet)$. In degree i , we have

$$\mathcal{H}\text{om}(Q^\bullet, M)^i = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{D_{G,c}^b(\mathcal{B})}(Q^{-i+j}, M[j])$$

and has differential induced by that of Q^\bullet : if $f \in \mathcal{H}\text{om}(Q^\bullet, M)^i$, then $d_{\tilde{M}} f = (-1)^{i+1} d_Q f$. The differential for $\mathcal{L}_{\mathcal{B}}(M)$ is given by $d_{\mathcal{L}_{\mathcal{B}}(M)} f = d_{\gamma(M)} f - (-1)^i f d_{\mathcal{Q}}$. These two differ by $d_{\gamma(M)} f$, but $d_{\gamma(M)}$ is a null-homotopic map of dg-modules. Since $\mathcal{L}_{\mathcal{B}} \circ \gamma(M)$ is a direct product of Hom groups in $\mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))$, composition with $d_{\gamma(M)}$ takes maps to zero.

Recall the adjoint pair (Ψ, Φ) defined in Section 3.1.4. By construction, we have an isomorphism of chain complexes $P^\bullet = \Psi(Q^\bullet)$. Hence Ψ induces a map of dg-rings $\Psi : \mathcal{E}\text{nd}(Q^\bullet) \rightarrow \mathcal{E}\text{nd}(P^\bullet)$. Similarly we have homomorphisms of dg-rings $\pi_1 : \mathcal{E}\text{nd}(P^\bullet) \rightarrow \bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})$, $i : \mathbf{H}_G^\bullet(\mathcal{B}) \rightarrow \bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})$, and $\pi_2 : \mathcal{E}\text{nd}(Q^\bullet) \rightarrow \mathbf{H}_G^\bullet(\mathcal{B})$. (We note the existence of π_1 and π_2 is implied by Remark 4.2.4.) These are related by the following lemma.

Lemma 4.3.8. *Let i , π_1 , π_2 , and Ψ be defined as above. Then $i \circ \pi_2 = \pi_1 \circ \Psi$.*

Proof. We need to show the following diagram commutes.

$$\begin{array}{ccc} \mathcal{E}\text{nd}(Q^\bullet) & \xrightarrow{\pi_2} & \text{Hom}^\bullet(\mathcal{C}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}}) \\ \Psi \downarrow & & \downarrow i \\ \mathcal{E}\text{nd}(P^\bullet) & \xrightarrow{\pi_1} & \text{Hom}^\bullet(\mathbf{A}, \mathbf{A}) \end{array}$$

Note that the inclusion i is induced by the functor Ψ . We may rewrite the above diagram in the following way:

$$\begin{array}{ccc} \mathcal{E}\text{nd}(Q^\bullet) & \xrightarrow{\pi_2} & \mathbf{H}(\mathcal{E}\text{nd}(Q^\bullet)) \\ \Psi \downarrow & & \downarrow i \\ \mathcal{E}\text{nd}(\Psi(Q)^\bullet) & \xrightarrow{\pi_1} & \mathbf{H}(\mathcal{E}\text{nd}(\Psi(Q)^\bullet)) \end{array}$$

Note that Ψ is a chain map $\mathcal{E}\text{nd}(Q^\bullet) \rightarrow \mathcal{E}\text{nd}(P^\bullet)$. The differential on $\mathcal{E}\text{nd}(Q^\bullet)$ is given by $d_Q f - (-1)^n f d_Q$. After we apply Ψ , we get $\Psi(d_Q)\Psi(f) - (-1)^n \Psi(f)\Psi(d_Q)$. Since $\Psi(d_Q) = d_P$, we have that

$$\Psi(d_Q)\Psi(f) - (-1)^n \Psi(f)\Psi(d_Q) = d_P \Psi(f) - (-1)^n \Psi(f) d_P,$$

which is exactly what we get when we apply the differential of $\mathcal{E}\text{nd}(P^\bullet)$ to $\Psi(f)$.

Let $f \in \mathcal{E}\text{nd}(Q^\bullet)^n$. Let \bar{f} denote the cohomology class determined by f , i.e. $i \circ \pi_2(f) = \bar{\Psi(f)}$. We want to compare this to $\bar{\Psi(f)}$.

$$\begin{aligned} \bar{\Psi(f)} - \Psi(\bar{f}) &= \Psi(f) + \text{Im}(d_P) - \Psi(f + \text{Im}(d_Q)) \\ &= \Psi(f) + \text{Im}(d_P) - \Psi(f) - \Psi(\text{Im}(d_Q)) \\ &= \text{Im}(d_P) \end{aligned}$$

since $\Psi(\text{Im}(d_Q)) \subset \text{Im}(d_P)$ since $\Psi d_Q = d_P \Psi$. Of course, $\text{Im}(d_P) = 0$ in $\mathbf{H}(\mathcal{E}\text{nd}(P^\bullet))$. \square

Lemma 4.3.9. *The functor $\tilde{\mathcal{L}} : \mathbf{D}_{G, \text{Spr}}^b(\mathcal{N}) \rightarrow \mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B}))$ (and hence $\mathcal{L} : \mathbf{D}_{G, \text{Spr}}^b(\mathcal{N}) \rightarrow \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(P^\bullet))$) is triangulated.*

Proof. We denote by Ψ_* , π_{1*} , π_{2*} , and i_* the restriction functors defined in [13, 10.12.5] induced by the corresponding dg-ring homomorphisms. Lemma 4.3.8 implies that the following diagram commutes:

$$\begin{array}{ccccc}
D_{G,\text{Spr}}^b(\mathcal{N}) & \xrightarrow{\mathcal{L}} & \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(P^\bullet)) & \xleftarrow{\pi_{1*}} & \mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})) \\
\downarrow \Phi & & \downarrow \Psi_* & & \downarrow i_* \\
D_{G,c}^b(\mathcal{B}) & \xrightarrow{\mathcal{L}_{\mathcal{B}}} & \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(Q^\bullet)) & \xleftarrow{\pi_{2*}} & \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))
\end{array}$$

Now, since π_{1*} and π_{2*} are equivalences, the following diagram where we replace them with their inverses also commutes.

$$\begin{array}{ccccc}
D_{G,\text{Spr}}^b(\mathcal{N}) & \xrightarrow{\mathcal{L}} & \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(P^\bullet)) & \xrightarrow{\pi_1^*} & \mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})) \\
\downarrow \Phi & & \downarrow \Psi_* & & \downarrow i_* \\
D_{G,c}^b(\mathcal{B}) & \xrightarrow{\mathcal{L}_{\mathcal{B}}} & \mathcal{D}^{\text{dg}}(\mathcal{E}\text{nd}(Q^\bullet)) & \xrightarrow{\pi_2^*} & \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))
\end{array}$$

Thus, we have that the outer diagram commutes, where $\tilde{\mathcal{L}} = \pi_1^* \circ \mathcal{L}$ and $\tilde{\mathcal{L}}_{\mathcal{B}} = \pi_2^* \circ \mathcal{L}_{\mathcal{B}}$.

$$\begin{array}{ccc}
D_{G,\text{Spr}}^b(\mathcal{N}) & \xrightarrow{\tilde{\mathcal{L}}} & \mathcal{D}^{\text{dg}}(\bar{\mathbb{Q}}_\ell[W] \# \mathbf{H}_G^\bullet(\mathcal{B})) \\
\downarrow \Phi & & \downarrow i_* \\
D_{G,c}^b(\mathcal{B}) & \xrightarrow{\tilde{\mathcal{L}}_{\mathcal{B}}} & \mathcal{D}^{\text{dg}}(\mathbf{H}_G^\bullet(\mathcal{B}))
\end{array}$$

Suppose $L \rightarrow M \rightarrow N \rightarrow L[1]$ is distinguished in $D_{G,\text{Spr}}^b(\mathcal{N})$. We need to show that the triangle $\tilde{\mathcal{L}}L \rightarrow \tilde{\mathcal{L}}M \rightarrow \tilde{\mathcal{L}}N \rightarrow \tilde{\mathcal{L}}L[1]$ is distinguished. Note that the triangle $\tilde{\mathcal{L}}_{\mathcal{B}}\Phi L \rightarrow \tilde{\mathcal{L}}_{\mathcal{B}}\Phi M \rightarrow \tilde{\mathcal{L}}_{\mathcal{B}}\Phi N \rightarrow \tilde{\mathcal{L}}_{\mathcal{B}}\Phi L[1]$ is distinguished since $\tilde{\mathcal{L}}_{\mathcal{B}}\Phi$ is a composition of triangulated functors. By commutativity of the above square, this implies the triangle $i_*\tilde{\mathcal{L}}L \rightarrow i_*\tilde{\mathcal{L}}M \rightarrow i_*\tilde{\mathcal{L}}N \rightarrow i_*\tilde{\mathcal{L}}L[1]$ is distinguished. By Lemma 4.3.3, the functor i_* reflects triangles. Hence, $\tilde{\mathcal{L}}L \rightarrow \tilde{\mathcal{L}}M \rightarrow \tilde{\mathcal{L}}N \rightarrow \tilde{\mathcal{L}}L[1]$ is distinguished. \square

Chapter 5

Open Questions

5.1 Hyperbolic Localization Functors

Let G be a connected, reductive group over a field k , and let \mathcal{N} be the nilpotent cone for G . In this section, we define induction and restriction functors for the nilpotent cone. Let P be a parabolic subgroup of G with Levi decomposition $P = LU$. We denote by \mathcal{N}_L the nilpotent cone for L and $\mathfrak{u} = \text{Lie}(U)$. We consider the following G -varieties and G -maps

$$\begin{aligned} \tilde{\mathcal{N}}_P &:= G \times^P (\mathfrak{u} + \mathcal{N}_L) \quad \text{and} \quad \mathcal{C}_P := G \times^P \mathcal{N}_L, \\ \mathcal{N} &\xleftarrow{\mu} \tilde{\mathcal{N}}_P \xrightarrow{\pi} \mathcal{C}_P. \end{aligned}$$

Note that μ is proper and π is smooth, so we have $\mu_! = \mu_*$ and $\pi^! = \pi^*[2d]$, where d is the dimension of G/P . We will consider the following functors

$$\begin{aligned} \mathcal{I}_P^G &= \mu_! \pi^*[d] \cong \mu_* \pi^![-d], \\ \mathcal{R}_P^G &= \pi_* \mu^![-d] \quad \text{and} \quad \tilde{\mathcal{R}}_P^G = \pi_! \mu^*[d], \end{aligned}$$

which we will refer to as induction and restriction functors. We have adjoint pairs $(\mathcal{I}_P^G, \mathcal{R}_P^G)$ and $(\tilde{\mathcal{R}}_P^G, \mathcal{I}_P^G)$. In the case when our parabolic is a Borel B , we get the usual Springer resolution diagram

$$\mathcal{N} \longleftarrow \mathcal{N} \longrightarrow G/B$$

from Section 3.1.4, where we used the notation Ψ , Φ , and Φ' for the induction and restriction functors. By induction ([13]), we have an equivalence $D_G^b(\mathcal{C}_P) \cong D_L^b(\mathcal{N}_L)$.

Theorem 5.1.1 ([25, 29, 3, 4]). *The functors \mathcal{I}_P^G , \mathcal{R}_P^G , and $\tilde{\mathcal{R}}_P^G$ are exact with respect to the perverse t -structure.*

5.2 Cuspidal Data

Definition 5.2.1. A simple perverse sheaf $\mathcal{F} \in D_G^b(\mathcal{N})$ is called *cuspidal* if $\mathcal{R}_P^G(\mathcal{F}) = \tilde{\mathcal{R}}_P^G(\mathcal{F}) = 0$ for all parabolics P . Let \mathcal{L} be a local system on a nilpotent orbit $\mathcal{O} \subset \mathcal{N}$. Then \mathcal{L} is called a *cuspidal local system* if $\text{IC}(\mathcal{O}, \mathcal{L})$ is cuspidal. A *cuspidal datum* (for \mathcal{N}) is a tuple $(L, \mathcal{O}_L, \mathcal{L})$ where L is a Levi subgroup of G , \mathcal{O}_L is an L orbit in \mathcal{N}_L , and \mathcal{L} is a cuspidal local system.

We will consider two cuspidal data (for G) equivalent if they are conjugate (in G). In [24], Lusztig classifies all such data up to conjugacy.

Remark 5.2.2. The only cuspidal datum when the Levi is a torus T is the datum (T, pt, triv) .

Definition 5.2.3. Let $(L, \mathcal{O}_L, \mathcal{L})$ be a cuspidal datum for G . We define the perverse sheaf $\mathbb{A}_{\mathcal{L}} = \mathcal{I}_P^G(\text{IC}(\mathcal{O}_L, \mathcal{L}))$, and call it a *Lusztig sheaf*. Let $D_G^b(\mathcal{N}, \mathbb{A}_{\mathcal{L}})$ be the triangulated subcategory of $D_G^b(\mathcal{N})$ generated by the simple summands of $\mathbb{A}_{\mathcal{L}}$.

Remark 5.2.4. For the cuspidal datum $(T, pt, triv)$, the Lusztig sheaf \mathbb{A}_{triv} is the Springer sheaf \mathbb{A} as defined in Section 3.1, and $D_G^b(\mathcal{N}, \mathbb{A})$ is the Springer block which we denoted $D_{G, \text{Spr}}^b(\mathcal{N})$.

Let X be the set of Lusztig sheaves up to isomorphism. By Lusztig's classification, X is finite. The main result of [32] is an orthogonal decomposition of the category $D_G^b(\mathcal{N})$ into blocks, each corresponding to a cuspidal datum for G . In other words, we have that

$$D_G^b(\mathcal{N}) = \bigoplus_{\mathbb{A}_{\mathcal{L}} \in X} D_G^b(\mathcal{N}, \mathbb{A}_{\mathcal{L}}).$$

The proof of this decomposition mostly follows from computations that Lusztig carried out before 1984 in [24].

5.3 Conjecture for the Generalized Springer Correspondence

Now, in order to completely describe the category $D_G^b(\mathcal{N})$ in the spirit of Theorem 4.2.9, it suffices to consider each block $D_G^b(\mathcal{N}, \mathbb{A}_{\mathcal{L}})$ independently. Consider the algebra

$$\mathcal{R}_{\mathcal{L}} := \text{Hom}^{\bullet}(\mathbb{A}_{\mathcal{L}}, \mathbb{A}_{\mathcal{L}}).$$

We regard $\mathcal{R}_{\mathcal{L}}$ as a dg-algebra with trivial differential. Recall that $\mathcal{D}_{\mathfrak{f}}^{\text{dg}}(\mathcal{R}_{\mathcal{L}})$ denoted the derived category of finitely generated differential graded modules over $\mathcal{R}_{\mathcal{L}}$.

Conjecture 5.3.1. *We have an equivalence of triangulated categories*

$$D_G^b(\mathcal{N}, \mathbb{A}_{\mathcal{L}}) \cong \mathcal{D}_{\mathfrak{f}}^{\text{dg}}(\mathcal{R}_{\mathcal{L}}).$$

References

- [1] P. N. Achar, *Green functions via hyperbolic localization*, Doc. Math. **16** (2011), 869–884.
- [2] P. N. Achar and A. Henderson, *Geometric Satake, Springer correspondence, and small representations*, arXiv:1108.4999, Selecta Math., to appear.
- [3] P. N. Achar, A. Henderson, and S. Riche, *Geometric Satake, Springer correspondence, and small representations II*, arXiv:1205.5089, 2012.
- [4] P. N. Achar and C. Mautner, *Sheaves on nilpotent cones, Fourier transform, and a geometric Ringel duality*, arXiv:1207.7044, 2012.
- [5] P. N. Achar and S. Riche, *Koszul duality and semisimplicity of Frobenius*, Ann. Inst. Fourier, to appear.
- [6] P. N. Achar and D. Treumann, *Baric structures on triangulated categories and coherent sheaves*, Int. Math. Res. Not. IMRN **2011** (2011), no. 16, 3688–3743.
- [7] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), no. 3, 595–678.
- [8] A. Beilinson, *On the derived category of perverse sheaves*, Lecture Notes in Mathematics, vol. 1289, Springer, Berlin, Berlin, 1987, pp. 27–41.
- [9] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, 1982, pp. 5–171.
- [10] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.
- [11] D. Ben-Zvi and D. Nadler, *Elliptic Springer Theory*, arXiv:1302.7053, 2013.
- [12] J. Bernstein, I. Gelfand, and S. Gelfand, *Algebraic vector bundles on \mathbb{P}^n and problems of linear algebra*, Funktsional. Anal. i Prilozhen **12** (1978), no. 3, 66–67.
- [13] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture notes in Mathematics, Springer-Verlag, Berlin, 1994.
- [14] W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d’intersection pour les variétés nilpotentes*, C. R. Acad. Sci. Sér. I Math. **292** (1981), no. 15, 707–710.
- [15] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., 1997.

- [16] P. Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252.
- [17] J. M. Douglass and G. Röhrle, *Homology of the Steinberg Variety and Weyl Group Coinvariants*, Doc. Math. **14** (2009), 339–357.
- [18] M. Grinberg, *A generalization of Springer theory using nearby cycles.*, Represent. Theory **2** (1998), 410431.
- [19] D. Juteau, *Modular Springer Correspondence and Decomposition Matrices*, PhD thesis, arXiv:0901.3671.
- [20] S. Kato, *An exotic Deligne-Langlands correspondence for symplectic groups*, Duke Math. J. **148** (2009), no. 2, 305371.
- [21] S. Kato, *A homological study of Green polynomials*, arXiv:1111.4640, 2011.
- [22] V. Lunts, *Equivariant sheaves on toric varieties*, Compositio Math. **96** (1995), no. 1, 63–83.
- [23] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), no. 2, 169–178.
- [24] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. math. **75** (1984), no. 2, 205–272.
- [25] G. Lusztig, *Fourier transforms on a semisimple Lie algebra over F_q* , Algebraic groups Utrecht 1986, 177188, Lecture Notes in Math., 1271, Springer, Berlin, 1987.
- [26] G. Lusztig, *Cuspidal local systems and graded Hecke algebras. I*, Inst. Hautes Études Sci. Publ. Math. **75** (1988), no. 67, 145–202.
- [27] G. Lusztig, *Cuspidal local systems and graded Hecke algebras. II. With errata for Part I*, CMS Conf. Proc., 16, Representations of groups (Banff, AB, 1994), vol. 16, Amer. Math. Soc., 1995, pp. 217–275.
- [28] C. Mautner, *A geometric Schur functor*, arXiv:1210.3635, 2012.
- [29] I. Mirković, *Character sheaves on reductive Lie algebras*, Mosc. Math. J. **4** (2004), no. 4, 897910, 981.
- [30] D. Nadler, *Springer theory via the Hitchin fibration*, Compos. Math. **147** (2011), no. 5, 16351670.
- [31] L. Rider, *Formality for the nilpotent cone and a derived Springer correspondence*, Adv. Math. **235** (2013), 208-236.
- [32] L. Rider and A. Russell, *An Orthogonal Decomposition for the Nilpotent Cone via Lusztig’s Generalized Springer Correspondence*, in preparation.

- [33] A. Russell, *Graham's Variety and Perverse Sheaves on the Nilpotent Cone*, arXiv:1111.2343, 2011.
- [34] O. Schnürer, *Equivariant sheaves on flag varieties*, Math. Z. **267** (2011), no. 1-2, 27–80.
- [35] O. Schnürer, *Perfect derived categories of positively graded DG algebras*, Appl. Categ. Structures **19** (2011), no. 5, 757–782, arXiv:0809.4782v2.
- [36] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173207.
- [37] Z. Yun, *Global Springer theory*, Adv. Math. **228** (2011), no. 1, 266328.

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