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Subrepresentation semirings and an analogue of 6j-symbols

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SUBREPRESENTATION SEMIRINGS AND AN ANALOGUE OF 6j-SYMBOLS

A Dissertation
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in partial fulfillment of the
requirements for the degree of
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in
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Abstract

Let $G$ be a quasi simply reducible group, and let $V$ be a representation of $G$ over the complex numbers $\mathbb{C}$. In this thesis, we introduce the twisted $6j$-symbols over $G$ which have their origin to Wigner’s $6j$-symbols over the group $SU(2)$ to study the structure constants of the subrepresentation semiring $S_G(\text{End}(V))$, and we study the representation theory of a quasi simply reducible group $G$ laying emphasis on our new $G$-module objects. We also investigate properties of our twisted $6j$-symbols by establishing the link between the twisted $6j$-symbols and Wigner’s $3j$-symbols over the group $G$. 
Introduction

In this thesis, we examine the structure of the semiring of subrepresentations of certain matrix algebra on which a group acts by algebra automorphism. The study of this semiring, while very natural from a representation-theoretic perspective, was first motivated by a problem in materials science. We begin by describing how this semiring arises in the theory of composite materials.

In materials science, it is an important problem to create a composite material with desired properties. However, it is not easy to predict effective properties of composites because their physical properties are usually strongly dependent on the microstructure. For these reasons, it is natural to consider the set of all possible values of a given physical properties of a composite material that is made with given materials of fixed proportions as one changes the microstructure of the composite. We call this set a $G$-closure. It is a subset of an appropriate vector space tensors. Even though most $G$-closure sets have a non-empty interior, in exceptional cases they degenerate to a surface which is called an exact relation. Because exact relations give the information about a composite material regardless of its microstructure, it has been an important problem in materials to find such exact relations. Unfortunately, the classical approach to find exact relations through analytical computations was limited by heavily dependence on the details of the physical context. Moreover, these techniques could not be used to determine whether all exact relations in a specific contest had been found.

Recently, in [GMS] the authors developed an abstract theory of exact relations, which not only led to the discovery of many new exact relations, but also gave complete lists in many physical situations. The success of this approach was due
to its reduction of the problem of finding exact relations to algebraic questions concerned with the representation theory of the group $SO(3)$. More specifically, it was shown that finding an exact relation was equivalent to solving an equation involving the multiplication of subrepresentations in a certain matrix $SO(3)$.

We now define subrepresentation semiring following [S2]. Let $A$ be an associative algebra with identity over a field $k$. Assume that the algebra $A$ has a $G$-module structure with the additional property $\alpha \cdot (xy) = (\alpha \cdot x)(\alpha \cdot y)$ for $\alpha \in G$ and $x, y \in A$. In other words, $G$ acts on $A$ by algebra automorphism. We call $A$ a $G$-algebra. For a given $G$-algebra $A$, let $S_G(A)$ be the set of all subrepresentations (i.e., $G$-submodules) of $A$. Then we can give a semiring structure on $S_G(A)$ with the usual addition of subspaces and multiplication given by $XY = \text{span}\{xy \mid x \in X, y \in Y\}$. We call the semiring $S_G(A)$ the subrepresentation semiring of the $G$-algebra $A$.

A fundamental example is given by $A = \text{End}(V)$, where $V$ is a representation of a group $G$. This was the case that arose in the study of exact relation. In particular, it was shown in [GMS] how the search for exact relations reduce to the algebraic problem of computing the structure constants of certain subrepresentation semiring $S_G(\text{End}(V))$ for $G = SO(3)$. Moreover, it was observed by Etingof and Sage independently that the structure constants of $S_{SO(3)}(\text{End}(V))$ are in fact related to the vanishing of Wigner’s $6j$-symbols which arise in the quantum theory of angular momentum. It is well known that there is a double covering homomorphism $\pi : SU(2) \longrightarrow SO(3)$, and $\pi$ yields a canonical isomorphism between $S_{SO(3)}(\text{End}(V))$ and $S_{SU(2)}(\text{End}(V))$. Here $V$ denotes a representation of $SO(3)$ over $\mathbb{C}$. Thus for a given representation $V$ over the complex numbers $\mathbb{C}$ the structure constants of $S_{SU(2)}(\text{End}(V))$ are also related to the vanishing of Wigner’s $6j$-symbols. This is an unexpected link because Wigner initially developed his $6j$-symbols for $SU(2)$ in the quite different context of the quantum theory of angular
momentum. Wigner himself generalized his construction of 6j-symbols to a more general class of groups called simply reducible groups, and Sharp further generalized them to quasi simply reducible groups. Quasi simply reducible groups were introduced by Mackey in [M3]; their representation theory has broad similarities to the representation theory of the group \(SU(2)\). Recall that every irreducible representation of the group \(SU(2)\) can be parameterized by the set of nonnegative half integers \(\frac{1}{2}\mathbb{Z}_{\geq 0}\), and each irreducible representation is self-dual (i.e., \(V_j \simeq V_j^*\)). Moreover, the tensor product of two irreducible representations of \(SU(2)\) is multiplicity-free, which can be easily checked by the Clebsch-Gordan formula. Keeping these representation theoretic properties in mind, we define quasi simply reducible groups as follows. A finite or compact group \(G\) is called a quasi simply reducible group if there exists an involutory anti-automorphism on \(G\) that leaves the conjugacy classes invariant, and irreducible representations of \(G\) satisfy the multiplicity-free property. If we take the involutory anti-automorphism on \(G\) to be the multiplication inverse, then in this case we call the group \(G\) a simply reducible group. \(SU(2)\) is the fundamental example of a simply reducible group.

Now the following natural question arises:

How are the structure constants of \(S_G(\text{End}(V))\) related to the 6j-symbols over \(G\) when we replace the group \(SU(2)\) by an arbitrary quasi simply reducible group \(G\)?

To answer this question, in this thesis we explicitly calculate the structure constants of the subrepresentation semiring \(S_G(\text{End}(V))\) by introducing a new class of 6j-symbols over \(G\) which we will call the twisted 6j-symbols in this thesis.

Before giving a more detailed description of the content of this thesis, we briefly digress to mention that the subrepresentation semiring \(S_G(\text{End}(V))\) can also be used to describe the \(G\)-invariant ideals and subalgebras of \(\text{End}(V)\).
Let $A$ be a $G$-algebra, and let $I$ be a $G$-invariant left ideal of $A$. Then we define the saturation of $I$ by $\mathcal{I} = \{ J \in S_G(A) \mid J \subseteq I \}$. Clearly $\mathcal{I}$ is a saturated left ideal containing the maximum element $I$, where we consider the inclusion as a partial order on $S_G(A)$ (Saturated means that if $J \in \mathcal{I}$ and $J' \subset J$, then $J' \in \mathcal{I}$). Thus we can assign each $G$-invariant left ideal $I$ of $A$ to the saturated left ideal $\mathcal{I}$ of $S_G(A)$ containing a maximum element, and this mapping is a bijective correspondence. Furthermore, when $A = \text{End}(V)$ it is known explicitly about the types of saturated left ideals of $S_G(\text{End}(V))$. More precisely, let $W$ be a subrepresentation of $V$. Then we define $G$-invariant left ideal $\text{Ann}(V)$ called annihilator of $W$ by $\text{Ann}(W) = \{ f \in \text{End}(V) \mid f(W) = 0 \}$. It is known that every saturated left ideal of $S_G(\text{End}(V))$ is of the form $\text{Ann}(W)$ (see [S1]). Similarly, for a given $G$-algebra $A$ there is a bijection between $G$-invariant subalgebras and saturated subhemirings of $S_G(A)$ containing their maximum elements. Recall that we call a set $R$ a hemiring if $R$ is an additive monoid under multiplication, but not containing the unity. In the $G$-algebra $\text{End}(V)$, it is also known that every nonzero saturated subhemiring is given by the saturation of a certain induced $G$-module. A complete description of the invariant subalgebras is given in [S2] for the case of $V$ irreducible.

Now we give an in depth description of the contents of this thesis. From Chapter 1 to 3, we cover basic material and motivations for this thesis. The main results are exhibited in Chapter 4 through Chapter 6.

In Chapter 1, we review the classical $3j$ and $6j$ symbols through the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Our approach is different from that of Wigner. It does not generalized to arbitrary groups, but it is quickened and more elegant for $SU(2)$. We also introduce the definition of quasi simply reducible groups and give some examples of quasi simply reducible groups.
In Chapter 2 and 3, we give some background on subrepresentation semirings. In Chapter 2, we define subrepresentation semiring and recall some basic concepts. We then focus on the important class of subrepresentation semirings coming from central simple algebras of the form $\text{End}(V)$, where $V$ is a representation. In Chapter 3 we review how the structure constants of $S_{\text{SU}(2)}(\text{End}(V))$ are related to the vanishing of $6j$-symbols.

In Chapter 4, we consider a group $G$ endowed with an involutory anti-automorphism. We first introduce new $G$-modules which are called twisted dual $G$-modules and twisted homomorphism $G$-modules respectively. As vector spaces, these will coincide the usual notion of dual spaces $V^*$ and homomorphism spaces $\text{Hom}(V, W)$, but they will have new $G$-module structures. Using these new $G$-modules, we define Clebsch-Gordan coefficients and twisted $6j$-symbols for a quasi simply reducible group $G$. Then in Theorem (4.13) we use the twisted $6j$-symbols to describe the structure constants of the subrepresentation semiring $S_G(\text{End}(V))$ for a given irreducible representation $V$ of the quasi simply reducible group $G$.

In Chapter 5, we introduce an analogue of the classical Frobenius-Schur invariants. These Frobenius-Schur invariants actually coincide with Mackay’s invariants appeared in [M2]. Sharp also used the same invariants in his book [SH]. In particular, Sharp used the invariants to generalize the concepts of even and odd representations. However, his argument has some errors. Actually, it turns out that there is a counterexample of a quasi simply reducible group having an irreducible representation which is both even and odd. This counterexample indicates that all of Sharp’s results in [SH] on $3j$ and $6j$ symbols over a quasi simply reducible group that are based on his extended even and odd definitions are also false as stated. We will present our counterexample in Chapter 5.
In Chapter 6, we first review the relationship between Clebsch-Gordan coefficients and Wigner’s 3\(j\)-symbols. We then show that there is an expression for our twisted 6\(j\)-symbols in terms of 3\(j\)-symbols similar to that for classical 6\(j\)-symbols. Finally, in Theorem (6.7), we use properties of the 3\(j\)-symbols to derive some properties of the twisted 6\(j\)-symbols.

In Chapter 7, we treat calculation examples of Clebsch-Gordan coefficients and the twisted 6\(j\)-symbols for the symmetric group \(S_3\).
1. Preliminary

1.1 Two Basic Lemmas

We review the following two basic facts concerned with the representation theory of finite groups (or compact groups).

**Lemma 1.1.** Let \( \rho : G \to GL(V) \) be a representation of a finite group (or a compact group) \( G \) over a complex vector space \( V \). Then there exists a \( G \)-invariant, positive-definite hermitian form on \( V \). In other words, every representation of a finite group (or a compact group) over \( \mathbb{C} \) is a unitary representation.

**Proof.** Let \( (\ , \ ) \) be an arbitrary positive-definite hermitian form on \( V \). Then we define the form \( (\ , \ ) \) on \( V \) by the rule (if \( G \) is a compact group, then we replace a summation by an integration)

\[
(v, w) := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle.
\]

It is easy to check that the form \( (\ , \ ) \) is a \( G \)-invariant, positive-definite hermitian form on \( V \).

**Lemma 1.2.** Let \( V \) be an irreducible representation of a finite group (or compact group) \( G \) over \( \mathbb{C} \). Then any two \( G \)-invariant, positive-definite hermitian inner products on \( V \) differ by a constant factor.

**Proof.** Let \( (\ , \ )_1 \) and \( (\ , \ )_2 \) be two \( G \)-invariant, positive-definite hermitian inner products on \( V \). Then the inner products \( (\ , \ )_1 \) and \( (\ , \ )_2 \) yield two bijections \( \phi_1 : V \to V^* \) defined by \( \phi_1(v) = (v, \ )_1 \) and \( \phi_2 : V \to V^* \) defined by \( \phi_2(v) = (v, \ )_2 \) respectively. Now the lemma is immediate if we apply Schur’s lemma to a \( G \)-module isomorphism \( \phi_1^{-1} \circ \phi_2 : V \to V \).
1.2 Representation Theory of $SU(2)$

In this section, we review the representation theory of $SU(2)$ because some part of this thesis has its motivation in extending a certain result of $SU(2)$ to groups whose representation theory is similar to that of $SU(2)$.

Recall that the special unitary group is defined

$$SU(2) = \{ A \in GL(2, \mathbb{C}) \mid ^t \overline{A} A = I \text{ and } \det A = 1 \}.$$  

Let $V_j$ ($j \in \frac{1}{2} \mathbb{Z}_{\geq 0}$) be the set of homogeneous polynomials of degree $2j$ in two variables $z_1$ and $z_2$. The dimension of $V_j$ is $2j + 1$. Viewing the polynomials as functions on $\mathbb{C}^2$, we obtain a left action of $SU(2)$ on the polynomials defined by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot f(z_1, z_2) = f( (z_1, z_2) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) ) .$$  \hfill (1.1)

Then, the subspaces $V_j$ are $SU(2)$-invariant, and each $V_j$ possesses a $G$-invariant inner product because $SU(2)$ is a compact group. In other words, the subspaces $V_j$ are unitary representations of $SU(2)$.

Now we present the following interesting representation theoretic properties of $SU(2)$.

**Theorem 1.3.**

1. The representations $V_j$ are irreducible.

2. Every irreducible unitary representation of $SU(2)$ is isomorphic to one of the $V_j$ (hence each $V_j$ is self-dual).

3. The tensor product of any two irreducible representation of $SU(2)$ satisfies the multiplicity-free property. More precisely, the Clebsch-Gordan formula states that

$$V_j \otimes V_k = \sum_{i=|j-k|}^{j+k} V_i.$$  

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Remark 1.4. 1. The complexified Lie algebra $\mathfrak{su}(2) \otimes \mathbb{C}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ also satisfies Theorem (1.3).

2. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ acts on $V_j$ as follows:

\begin{align*}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \cdot z_1^r z_2^s &= s z_1^{r+1} z_2^{s-1}, \\
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \cdot z_1^r z_2^s &= r z_1^{r-1} z_2^{s+1}, \\
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \cdot z_1^r z_2^s &= (r - s) z_1^r z_2^s,
\end{align*}

where $r + s = j$.

1.3 The Classical 3j and 6j Symbols

Wigner initially developed his 3j and 6j symbols over $SU(2)$ which have a connection with the quantum theory of angular momentum. However, in this section we will define 3j and 6j symbols over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ to give more concrete mathematical approach.

Roughly speaking, 3j-symbols are obtained from the matrix coefficients of imbeddings $V_a \hookrightarrow V_b \otimes V_c$ for irreducible representations $V_a$, $V_b$ and $V_c$ of $\mathfrak{sl}(2, \mathbb{C})$. On the other hand, 6j-symbols arise from the base change of the space $\text{Hom}_{\mathfrak{sl}(2, \mathbb{C})}(V_k, V_a \otimes V_b \otimes V_c)$, where iterating the Clebsch-Gordan formula yields two bases, one from $(V_a \otimes V_b) \otimes V_c \simeq (\oplus_k V_k) \otimes V_c$ and the other from $V_a \otimes (V_b \otimes V_c) \simeq V_a \otimes (\oplus_j V_j)$.

Definition 1.5. Let $\delta$ be an element of a commutative ring $R$, and let $j \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. The *Temperley-Lieb* algebra $TL_{2j}(\delta)$ is a $R$-algebra generated by the symbols $\{I, h_1, h_2, \ldots, h_{2j-1}\}$ that are subject to the following relations:

1. $I^2 = I$,
2. \( Ih_k = h_k I = h_k \) for \( k = 1, \cdots, 2j - 1 \),

3. \( h_k h_l = h_l h_k \) for \( |k - l| > 1 \),

4. \( h_k h_k = \delta h_k \) for \( k = 1, \cdots, 2j - 1 \),

5. \( h_k h_{k+1} h_k = h_k \).

In our case, we are interested in the Temperley-Lieb algebra \( TL_{2j}(-2) \) over the complex number \( \mathbb{C} \).

**Lemma 1.6.** Let \( V_\frac{1}{2} \) be the fundamental representation of \( \mathfrak{sl}(2, \mathbb{C}) \). Then, there is a representation

\[
\theta : TL_{2j}(-2) \longrightarrow \text{End}\left( \left(V_{\frac{1}{2}}\right)^{\otimes 2j} \right).
\]

Here \( \theta(I) \) is the identity, and for \( k = 1, 2, \cdots, 2j - 1 \), \( \theta(h_k) \) is an endomorphism of \( \left(V_{\frac{1}{2}}\right)^{\otimes 2j} \) which acts on the \((k, k+1)\) factors of the tensor product as

\[
\theta(h_k)(z_1 \otimes z_1) = \theta(h_k)(z_2 \otimes z_2) = 0,
\]

\[
\theta(h_k)(z_1 \otimes z_2) = z_2 \otimes z_1 - z_1 \otimes z_2,
\]

and

\[
\theta(h_k)(z_2 \otimes z_1) = z_1 \otimes z_2 - z_2 \otimes z_1.
\]

**Proof.** See [CFS].

**Lemma 1.7.** There is a homomorphism \( \rho \) of the permutation group \( S_{2j} \) on \( 2j \) letters into \( TL_{2j}(-2) \) given by \( \rho(\sigma_k) = I_k + h_k \) where \( \sigma_k \) is the transposition \((k, k+1)\).

**Proof.** See [CFS].
Now we note that $V_j$ can be imbedded into $(V_\frac{1}{2})^{\otimes 2j}$ via the map

$$\psi_j : x_1 \cdots x_{2j} \mapsto \frac{1}{(2j)!} \sum_{\sigma \in S_{2j}} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(2j)},$$

(1.2)

where $x_i \in \{z_1, z_2\}$.

On the other hand, we also can consider the following projection $(V_\frac{1}{2})^{\otimes 2j}$ onto the image $\psi_j(V_j)$ through the Temperley-Lieb algebra:

$$\pi_{2j} : (V_\frac{1}{2})^{\otimes 2j} \rightarrow \psi_j(V_j)$$

given by

$$\pi_{2j}(v) = \left( \frac{1}{(2j)!} \sum_{\sigma \in S_{2j}} \rho(\sigma) \right) \cdot v,$$

where $v \in (V_\frac{1}{2})^{\otimes 2j}$ and $\rho$ is a homomorphism defined in Lemma (1.7).

**Definition 1.8.** Suppose that $a, b \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. A triple of half-integers $(a, b, c)$ is said to be admissible if $c$ is appeared in the set \{|a - b|, |a - b| + 1, \ldots, a + b - 1, a + b\}.

**Example 1.9.** Let $V_j$ and $V_k$ be irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. Then by Clebsch-Gordan formula we have $V_j \otimes V_k = \bigoplus_{i=|j-k|}^{j+k} V_i$. Thus, the triple $(j, k, i)$ is admissible.

In what follows, we will define several homomorphisms which will play an important role in defining the $6j$-symbols.

**Definition 1.10.** The $\mathfrak{sl}(2, \mathbb{C})$-module homomorphisms $\omega_n : \mathbb{C} \rightarrow (V_\frac{1}{2})^{\otimes 2n}$ are defined inductively as follows.

- $\omega_1 : \mathbb{C} \rightarrow V_\frac{1}{2} \otimes V_\frac{1}{2}$ is defined $\omega_1(1) = \sqrt{-1} (z_1 \otimes z_2 - z_2 \otimes z_1)$. 
• Having defined \(\omega_{n-1} : \mathbb{C} \rightarrow \left(V_\frac{1}{2}\right)^{\otimes 2(n-1)}\), define \(\omega_n\) to be the composition

\[
\mathbb{C} \xrightarrow{\omega_{n-1}} \left(V_\frac{1}{2}\right)^{\otimes 2(n-1)} \xrightarrow{\sim} \left(V_\frac{1}{2}\right)^{\otimes (n-1)} \otimes \mathbb{C} \otimes \left(V_\frac{1}{2}\right)^{\otimes (n-1)} \xrightarrow{id \otimes \omega_1 \otimes id} \left(V_\frac{1}{2}\right)^{\otimes 2n}.
\]

Now we let

\[
Y_{ab}^j = (\pi_{2a} \otimes \pi_{2b}) \circ (id_{a+j-b} \otimes \omega_{a+b-j} \otimes id_{b+j-a}) \circ \psi_j : V_j \rightarrow \left(V_{\frac{1}{2}}\right)^{\otimes 2a} \otimes \left(V_{\frac{1}{2}}\right)^{\otimes 2b}
\]

and

\[
\mu_l : \left(V_{\frac{1}{2}}\right)^{\otimes 2l} \rightarrow V_l \text{ given by } \mu_l(x_1 \otimes \cdots \otimes x_{2l}) = x_1 \cdots x_{2l}. \tag{1.4}
\]

Then we denote by \(\psi_{ab}^j\) an imbedding \(V_j \rightarrow V_a \otimes V_b\) defined by the composition

\[
\psi_{ab}^j = (\mu_a \otimes \mu_b) \circ Y_{ab}^j. \tag{1.5}
\]

We recall from Remark (1.4) that the weight vector \(z_{1}^{j+t}z_{2}^{j-t}\) in \(V_j\) has a weight \(2t\) relative to the action of \(H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). If we denote by \(e_i^j\) the weight vector \(z_{1}^{j+t}z_{2}^{j-t}\), then we obtain the standard basis \(\{e_{i}^j, \cdots, e_{i}^j\}\) for \(V_j\).

From now on, we fix the standard basis for each irreducible representation of \(\mathfrak{sl}(2, \mathbb{C})\).

**Definition 1.11.** Suppose that \((a, b, j)\) is an admissible triple, and consider an imbedding \(\psi_{ab}^j : V_j \rightarrow V_a \otimes V_b\). Then for the fixed basis \(\{e_{i}^j, \cdots, e_{i}^j\}\) of \(V_j\) we have

\[
\psi_{ab}^j(e_{i}^j) = \sum_{p_1 + p_2 = u} \begin{pmatrix} a & b & j \\ p_1 & p_2 & u \end{pmatrix} e_{p_1}^a \otimes e_{p_2}^b. \tag{1.6}
\]
In this case, we call the coefficients \[
\begin{pmatrix}
  a & b & j \\
  p_1 & p_2 & u
\end{pmatrix}
\] the 3\(j\)-symbols.

Let us now consider the space \(\text{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V_k, V_a \otimes V_b \otimes V_c)\) for irreducible representations \(V_a, V_b, V_c\) and \(V_k\) of \(\mathfrak{sl}(2,\mathbb{C})\).

We will construct two bases of \(\text{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V_k, V_a \otimes V_b \otimes V_c)\) through iterating the tensor product \(V_a \otimes V_b \otimes V_c\).

First, let us consider the following homomorphisms which are already defined in Equations (1.2), (1.3) and (1.4).

- \(\psi_k : V_k \hookrightarrow \left( V_\frac{1}{2} \right)^{\otimes 2k}\).
- \(Y_{aj}^k : \left( V_\frac{1}{2} \right)^{\otimes 2j} \rightarrow \left( V_\frac{1}{2} \right)^{\otimes 2a} \otimes \left( V_\frac{1}{2} \right)^{\otimes 2j}\).
- \(\text{id} \otimes Y_{bc}^j : \left( V_\frac{1}{2} \right)^{\otimes 2a} \otimes \left( V_\frac{1}{2} \right)^{\otimes 2j} \rightarrow \left( V_\frac{1}{2} \right)^{\otimes 2a} \otimes \left( \left( V_\frac{1}{2} \right)^{\otimes 2b} \otimes \left( V_\frac{1}{2} \right)^{\otimes 2c} \right)\).
- \(\mu_a \otimes \mu_b \otimes \mu_c : \left( V_\frac{1}{2} \right)^{\otimes 2a} \otimes \left( V_\frac{1}{2} \right)^{\otimes 2b} \otimes \left( V_\frac{1}{2} \right)^{\otimes 2c} \rightarrow V_a \otimes V_b \otimes V_c\).

Then we define \(S_{j}^{abc}\) as a composition of homomorphisms

\[S_{j}^{abc} = (\mu_a \otimes \mu_b \otimes \mu_c) \circ (\text{id} \otimes Y_{j}^{bc}) \circ (Y_{j}^{aj} \circ \psi_k).\]

Similarly, we define

\[T_n^{abc} = (\mu_a \otimes \mu_b \otimes \mu_c) \circ (Y_n^{ab} \otimes \text{id}) \circ Y_k^{nc} \circ \psi_k.\]

Here we assume that all of triples \((b, c, j),\ (a, j, k),\ (a, b, n)\) and \((n, c, k)\) are admissible.

The following lemma is the crucial step in defining 6\(j\)-symbols.

**Lemma 1.12.** Two sets

\[\{T_n^{abc} \mid \text{the index } n \text{ ranges such that } (a, b, n) \text{ and } (n, c, k) \text{ are admissible}\}\]
and

\[ \{ S_{j}^{abc} \mid \text{the index } j \text{ ranges such that } (b, c, j) \text{ and } (a, j, k) \text{ are admissible} \} \]

form bases for the vector space \( \text{Hom}_{sl(2, \mathbb{C})}(V_k, V_a \otimes V_b \otimes V_c) \).

**Proof.** See [CFS]. \( \square \)

Finally the following definition of \( 6j \)-symbols follows from Lemma (1.12).

**Definition 1.13.** We define the \( 6j \)-symbols to be the coefficients

\[
\begin{cases}
  a & b & n \\
  c & k & j
\end{cases}
\]

in the following base change equation

\[
S_{j}^{abc} = \sum_{n} \begin{cases}
  a & b & n \\
  c & k & j
\end{cases} T_{n}^{abc}.
\] (1.7)

By convention, we define

\[
\begin{cases}
  a & b & n \\
  c & k & j
\end{cases} = 0
\]

if any of the triples \((b, c, j), (a, j, k), (a, b, n)\) and \((n, c, k)\) is not admissible.

### 1.4 Quasi Simply Reducible Groups

In this section, we introduce a certain group which has its origin in the representation theory of the group \( SU(2) \).

**Definition 1.14.** A finite or compact group \( G \) is called a **quasi simply reducible group** if

1. there exists an involutory anti-automorphism \( i : G \rightarrow G \) such that \( g \) is conjugate to \( i(g) \) for all \( g \in G \).

2. the tensor product of any two irreducible representations of \( G \) satisfies the multiplicity-free property (i.e., for given two irreducible representation of \( G \), their tensor product can be decomposed into the direct sum of distinct irreducible representations of \( G \)).
Remark 1.15. The concept of a quasi simply reducible group is a generalized concept of a simply reducible group. Recall that a group is called a simply reducible group if every element in $G$ is conjugate to its inverse, and irreducible representations of $G$ satisfy the multiplicity-free property.

Example 1.16. 1. The group $SU(2)$ and simply reducible groups are quasi simply reducible groups if we consider the multiplication inverse map on $G$ as an involutory anti-automorphism of $G$.

2. The direct product of an abelian group and a simply reducible group is a quasi simply reducible group under an involutory anti-automorphism $i((a, b)) = (a, b^{-1})$. This example implies that there are quasi simply reducible groups that are not simply reducible groups. For example, the group $\mathbb{Z} \times S_3$ is a quasi simply reducible group, but not a simply reducible group, where the group $S_3$ means the symmetric group on 3 letters (see [M3]).

3. As a nontrivial example of a quasi simply reducible group that is not a simply reducible group, we can find an example of the dicyclic group $Q_3 = \langle R, S : R^3 = S^2 = (RS)^2 \rangle$ of order 12.

The conjugacy classes of $Q_3$ are $\{e\}$, $\{R, R^5\}$, $\{R^2, R^4\}$, $\{R^3\}$, $\{S, R^2S, R^4S\}$ and $\{RS, R^3S, R^5S\}$. In this case, we take an involutory anti-automorphism $i$ defined as follows:

$$i(e) = e, i(R^i) = R^i \quad (1 \leq i \leq 5), \quad i(S) = S, \quad \text{and} \quad i(R^iS) = R^{6-i}S \quad (1 \leq i \leq 5).$$

Remark 1.17. For a finite group $G$ with an involutory anti-automorphism $i$, an interesting characterization of the quasi simply reducibility for $G$ was given by Mackey in [M1]. Actually he had shown that a finite group $G$ is quasi simply reducible if and only if $\sum_{x \in G} \zeta(x)^3 = \sum_{x \in G} v(x)^2$, where $v(x)$ and $\zeta(x)$ are the
number of elements in the set \( \{ g \in G : gx = xg \} \) and \( \{ g \in G : gi(g^{-1}) = x \} \) respectively.
2. Subrepresentation Semirings

2.1 Definitions

Let $A$ be an associative algebra with identity over a field $k$. Assume that the algebra $A$ has a $G$-module structure with the additional property $\alpha \cdot (xy) = (\alpha \cdot x)(\alpha \cdot y)$ for $\alpha \in G$ and $x, y \in A$. In this case, we call $A$ a $G$-algebra. For a given $G$-algebra $A$, let $S_G(A)$ be the set of all subrepresentations (i.e., $G$-submodules) of $A$. Then we can give a semiring structure on $S_G(A)$ with the usual addition of subspaces and multiplication given by $XY = \text{span}\{xy \mid x \in X, y \in Y\}$. We call a semiring $S_G(A)$ of the $G$-algebra $A$ a subrepresentation semiring.

2.2 A Specific Case of $\text{End}(V)$

In this section, let us introduce an important class of subrepresentation semirings.

Let $V$ be a finite dimensional representation of $G$ over a field $k$, and consider the central simple algebra $\text{End}(V)$. We can make the algebra $\text{End}(V)$ into a $G$-algebra via $(\alpha \cdot f)(v) = \alpha \cdot f(\alpha^{-1} \cdot v)$ for $\alpha \in G$ and $v \in V$, and we have a subrepresentation semiring $S_G(\text{End}(V))$. In this case, the question on the structure constants of $S_G(\text{End}(V))$ was motivated by work on material science. In particular, understanding the structure constants of $S_{SU(2)}(\text{End}(V))$ was a key ingredient in the recent work of [GMS] on physical properties of composite materials. It also has been known that the structure constants of $S_{SU(2)}(\text{End}(V))$ are closely related with the vanishing of Wigner’s 6j-symbols which are familiar from the quantum theory of angular momentum (see [S2, W2]).

Example 2.1. 1. If $V$ is one-dimensional, then $\text{End}(V)$ is the trivial $G$-module $k$. Thus $S_G(\text{End}(V)) = S_G(k) = \{\{0\}, k\}$ which is isomorphic to Boolean semiring $\{0, 1\}$ with $1 + 1 = 1$. 

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2. Let $V_\frac{1}{2}$ be the fundamental representation of $SU(2)$. Then

$$\text{End}(V_\frac{1}{2}) \simeq V_\frac{1}{2} \otimes V_\frac{1}{2} \simeq V_0 \oplus V_1.$$ 

Thus

$$S_{SU(2)}(\text{End}(V_\frac{1}{2})) = \{\{0\}, V_0, V_1, V_0 \oplus V_1\}$$

with $V_1 V_1 = V_0 \oplus V_1$.

### 2.3 $G$-invariant Ideals and Subalgebras

We now return to an arbitrary $G$-algebra $A$.

Let $I$ be a $G$-invariant left ideal of $A$. Then we define the saturation of $I$ by $\bar{I} = \{J \in S_G(A) \mid J \subset I\}$. Clearly $\bar{I}$ is a saturated (i.e., there exits the maximum element $I \in \bar{I}$ such that every $J \in S_G(A)$ satisfying $J \subset I$ is an element of $\bar{I}$) left ideal containing the maximum element $I$, where we consider the inclusion as a partial order on $S_G(A)$. Thus we can assign each $G$-invariant left ideal $I$ of $A$ to the saturated left ideal $\bar{I}$ of $S_G(A)$ containing a maximum element. Conversely, for a given any left ideals $P$ of $S_G(A)$, $\text{sup}(P) = \sum_{V \in P} V$ is a $G$-invariant left ideal of $A$. These mappings give a bijections between $G$-invariant left ideals and saturated left ideals with a maximum element. We also have similar bijections for $G$-invariant right ideals, $G$- invariant subalgebras, etc.

Recall that hemiring is an additive monoid closed under multiplication, but not containing 1.

**Theorem 2.2.** Let $A$ be a $G$-algebra. Then there is a bijection between $G$-invariant ideals (resp. left or right) of $A$ and saturated ideals (resp. left or right) of $S_G(A)$ containing their suprema. There is a similar bijection between $G$-invariant subalgebras and saturated subhemirings containing their suprema.

**Proof.** See [S2].
Let us now discuss about the saturated ideals of subrepresentation semirings \( S_G(End(V)) \).

Let \( V \) be a finite dimensional representation of \( G \), and let \( W \) be any subrepresentation of \( V \). Then we define \( G \)-invariant left and right ideals of \( \text{End}(V) \) called the annihilator and coannihilator of \( W \) through the formulas \( \text{Ann}(W) = \{ f \in \text{End}(V) \mid f(W) = 0 \} \) and \( \text{Coann}(W) = \{ f \in \text{End}(V) \mid f(V) \subseteq W \} \).

Concerned with the saturated ideals of \( S_G(End(V)) \), the following theorem is known [S1].

**Theorem 2.3.** Let \( V \) be a finite dimensional representation of a group \( G \). Then the saturated left (resp. right) ideals of \( S_G(End(V)) \) are of the form \( \text{Ann}(W) \) (resp. \( \text{Coann}(W) \)) for any subrepresentation \( W \) of \( V \). There are no nontrivial saturated two-sided ideals of \( S_G(End(V)) \)

**Proof.** See [S1].

As a quick application of Theorem (2.3), the semiring \( S_G(End(V)) \) has no nontrivial saturated one-sided ideals if and only if \( V \) is irreducible.

**Remark 2.4.** There is a version of Theorem (2.3) for the saturated subhemirings of \( S_G(End(V)) \). More precisely, let \( H \) be a subgroup of \( G \) of index \( n \) and \( B \) an \( H \)-algebra. Choose left coset representatives \( \{ g_1 = e, g_2, \ldots, g_n \} \). Then we define the induced \( G \)-module \( \text{Ind}_H^G(B) \) as a \( G \)-module \( \oplus_{i=1}^n g_i B \). The induced \( G \)-module \( \oplus_{i=1}^n g_i B \) becomes a \( G \)-algebra via \( (g_i b)(g_j b') = \delta_{ij} g_i b b' \). Now we consider a quadruple \((H, W, U, U')\), where \( H \) is a finite index subgroup of \( G \), \( W \) is a representation of \( H \) such that \( \text{Ind}_H^G(W) = V \), and \( U \) and \( U' \) are projective representations of \( H \) such that \( W \cong U \otimes U' \). It is known that every nonzero invariant subalgebra of \( \text{End}(V) \) is of the form \( \text{Ind}_H^G(End(U) \otimes k) \) for some quadruple \((H, W, U, U')\) as
above. So, every nonzero saturated subhemiring of $S_G(\text{End}(V))$ is given by the form $\text{Ind}_H^G (\text{End}(U) \otimes k)$. For details, see [S1].
3. Structure Constants of $S_{SU(2)}(\text{End}(V))$ and The Vanishing of $6j$-Symbols

3.1 Structure Constants

Let $G$ be a finite (or a compact) group. Let $\mathcal{X} = \{V_j : j \in J\}$ be the set of all irreducible $G$-modules over $\mathbb{C}$. Then we can express a tensor product $V_i \otimes V_k$ of two elements in terms of elements in $\mathcal{X}$, say

$$V_i \otimes V_k = \sum_{l} C_{ik}^{l} V_l,$$  \hspace{1cm} (3.8)

where the coefficients $C_{ik}^{l}$ are positive integers.

In this case, we call the numbers $C_{ik}^{l}$ the structure constants of $\mathcal{X}$.

Let us now present the following definition which yields a convenient notation for the dual $G$-module $V_i^*$ of an irreducible $G$-module $V_i$.

**Definition 3.1.** Suppose that $V$ is a $G$-module over $\mathbb{C}$.

1. By a conjugate space of $V$ we mean a vector space $\overline{V}$ which has the same additive structure as $V$ but scalar multiplication defined by

   $$\mathbb{C} \times V \longrightarrow V$$

   $$(z, v) \longmapsto \overline{z}v.$$

2. By a conjugate $G$-module of $V$ we mean a $G$-module $\overline{V}$ which has the same $G$-action structure as the $G$-module $V$.

The conjugate modules have the following basic properties.

**Lemma 3.2.** Let $V$ be a $G$-module over $\mathbb{C}$. Then

1. $\overline{V} \simeq V$ as $G$-modules.

2. $V$ is an irreducible $G$-module if and only if $\overline{V}$ is an irreducible $G$-module.
3. $V \simeq V^*$ as $G$-modules, where $V^*$ is the dual $G$-module of $V$.

Proof. The first and second statements are obvious. For the third property, we recall that every representation of a finite or compact group over $\mathbb{C}$ is an unitary representation. So there is a $G$-invariant, positive-definite hermitian form $(\ , \ )$ on $V$. Then the map $v \mapsto (\ , v)$ yields a $G$-module isomorphism between $V$ and $V^*$.

Let us write $V_i$ for an irreducible conjugate $G$-module $\overline{V}_i$. Then through the identification $V_i^* \simeq \overline{V}_i$ we also can write $V_i$ for the dual $G$-module $V_i^*$. In the following lemma, we present basic properties of the structure constants $C_{ik}$.  

Lemma 3.3. Let $G$ be a finite (or a compact) group, and $C_{ik}$ be the structure constants defined in (3.8). Then we have the following basic properties:

1. $C_{ik}^l = C_{ki}^l$.

2. $C_{ik}^\tau = C_{ik}^l$.

3. $C_{ik} = C_{il}^\tau$.

4. $\sum_l C_{ik}^l C_{lm}^n = \sum_p C_{km}^p C_{ip}^n$.

Proof. Every property is immediate except for (3). In order to prove the third property, let us first assume that $V_i \otimes V_k = \sum_l C_{ik}^l V_l$. Then we obtain $\chi_{\rho_i \otimes \rho_k}(g) = \sum_l C_{ik}^l \chi_{\rho_l}(g)$ for their characters. Thus we have

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i \otimes \rho_k}(g) \overline{\chi_{\rho_l}}(g) = \langle \chi_{\rho_i \otimes \rho_k}, \chi_{\rho_l} \rangle = C_{ik}^l.$$
which implies that
\[ C_{ik}^j = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \chi_{\rho_k}(g) \chi_{\rho_l}(g). \]
Similarly, we obtain
\[ C_{ik}^F = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \chi_{\rho_j}(g) \chi_{\rho_l}(g). \]
The desire result is now immediate. \qed

### 3.2 Structure Constants of $S_{SU(2)}(\text{End}(V))$

In this section unless otherwise stated $V$ will denote a finite dimensional representation of $SU(2)$ over $\mathbb{C}$.

Recall that in Chapter 1 we parameterized irreducible representations of $SU(2)$ by the half integers $\frac{1}{2} \mathbb{Z}_{\geq 0}$. Thus we can express $V$ as $V = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}_{\geq 0}} r_j V_j$. From this decomposition, we obtain

\[ \text{End}(V) \cong \bigoplus_{j,k} r_j r_k \text{Hom}(V_j, V_k). \quad (3.9) \]

Note that $\text{End}(V)$ has a composition multiplication as a $G$-algebra which can be induced by multiplications over each pair of components in the decomposition (3.9). For this reason, it is enough to consider the composition multiplication

\[ \text{Hom}(V_k, V_l) \otimes \text{Hom}(V_j, V_k) \longrightarrow \text{Hom}(V_j, V_l) \]

for understanding the whole multiplication of $\text{End}(V)$.

Let us now assume that $V_a$ and $V_b$ are subrepresentations of $\text{Hom}(V_j, V_k)$ and $\text{Hom}(V_k, V_l)$ respectively. Then we want to decompose $V_b V_a$ into irreducible representations of $SU(2)$. For this question, we present the following approach which connects the structure constant $S_{SU(2)}(\text{End}(V))$ with the classical $6j$-symbols [S2].

**Theorem 3.4.** Let $V_a$ and $V_b$ be subrepresentations of $\text{Hom}(V_j, V_k)$ and $\text{Hom}(V_k, V_l)$ respectively. Then, $V_c$ is an irreducible components of $V_b V_a$ if and only if the clas-
\[ \{ \begin{array}{ccc} j & k & a \\ b & c & l \end{array} \} \neq 0. \] Moreover we have the decomposition \( V_b V_a = \bigoplus_c V_c \), where the direct sum is taken over \( c \) satisfying \( \{ \begin{array}{ccc} j & k & a \\ b & c & l \end{array} \} \neq 0 \).

**Proof.** Recall that we fixed the standard basis \( \{ e^j_{-j}, \cdots, e^j_j \} \) for each irreducible representation \( V_j \). Let us now consider a \( \mathfrak{sl}(2, \mathbb{C}) \)-module isomorphism \( \varphi_j : V_j \rightarrow V_j^* \) given by \( \varphi_j(e^j_m) = (-1)^m (e^j_{-m})^* \) for convenience of notation.

By assumption, we have an imbedding \( \psi^{jk}_a : V_a \hookrightarrow \text{Hom}(V_j, V_k) \simeq V_j^* \otimes V_k \), \( \psi^{kl}_b : V_b \hookrightarrow V_k \otimes V_l \) and \( \psi^{jl}_c : V_c \hookrightarrow V_j^* \otimes V_l \).

For each imbedding, we have the following corresponding 3\( j \)-symbols.

1. \( \psi^{jk}_a (e^a_m) = \sum_{m_1 + m_2 = m} \begin{pmatrix} j & k & a \\ m_1 & m_2 & m \end{pmatrix} w^j_{m_1} \otimes e^k_{m_2} \).

2. \( \psi^{kl}_b (e^b_m) = \sum_{m_1 + m_2 = m} \begin{pmatrix} k & l & b \\ m_1 & m_2 & m \end{pmatrix} w^k_{m_1} \otimes e^l_{m_2} \).

3. \( \psi^{jl}_c (e^c_m) = \sum_{m_1 + m_2 = m} \begin{pmatrix} j & l & c \\ m_1 & m_2 & m \end{pmatrix} w^j_{m_1} \otimes e^l_{m_2} \).

However, \( V_c \) is also a component of \( V_b V_a \), and the image of the standard basis \( \{ e^c_m \} \) for \( V_c \) into \( V_b V_a \) is given by

\[ \varsigma^c_m = \sum_{p_1, p_2} \begin{pmatrix} b & a & c \\ p_1 & p_2 & m \end{pmatrix} \psi^{kl}_b (e^b_{p_1}) \psi^{jk}_a (e^a_{p_2}). \quad (3.10) \]

Then by Schur’s lemma we have

\[ \varsigma^c_m = R^{jkl}_{abc} \psi^j_c (e^c_m) \quad (3.11) \]
for some scalar multiple $R_{abc}^{jkl}$.

By expanding Equation (3.10) and comparing with the coefficients of Equation (3.11), we have

$$R_{abc}^{jkl} \left( \begin{array}{ccc} j & l & c \\ m_1 & m_2 & m \end{array} \right) = \sum_{p_1,p_2,s} (-1)^s \left( \begin{array}{ccc} b & a & c \\ p_1 & p_2 & m \end{array} \right) \left( \begin{array}{ccc} k & l & b \\ s & m_2 & p_1 \end{array} \right) \left( \begin{array}{ccc} j & k & a \\ m_1 & -s & p_2 \end{array} \right).$$

From Equation (3.180) in [BL], we have

$$\left( \begin{array}{ccc} b & a & c \\ p_1 & p_2 & m \end{array} \right) = (-1)^{c-a-b} \left( \begin{array}{ccc} a & b & c \\ p_2 & p_1 & m \end{array} \right)$$

and

$$\left( \begin{array}{ccc} k & l & b \\ s & m_2 & p_1 \end{array} \right) = (-1)^{2k+b-l-s} \left( \begin{array}{ccc} 2b+1 \\ 2l+1 \end{array} \right)^{1/2} \left( \begin{array}{ccc} k & b & l \\ -s & p_1 & m_2 \end{array} \right).$$

Thus we have

$$(-1)^{a+l-c-2k} \left( \begin{array}{ccc} 2a+1 \\ 2b+1 \end{array} \right)^{1/2} R_{abc}^{jkl} \left( \begin{array}{ccc} j & l & c \\ m_1 & m_2 & m \end{array} \right) = \sum_{p_1,p_2,s} \left( \begin{array}{ccc} a & b & c \\ p_2 & p_1 & m \end{array} \right) \left( \begin{array}{ccc} k & b & l \\ -s & p_1 & m_2 \end{array} \right) \left( \begin{array}{ccc} j & k & a \\ m_1 & -s & p_2 \end{array} \right).$$

However, by Equation (3.267) in [BL] the sum on the right hand side is also equal to

$$[(2a+1)(2l+1)]^{1/2} \left\{ j \atop b \right\} \left\{ k \atop c \right\} \left\{ a \atop l \right\} \left\{ m_1 \atop m_2 \right\}.$$

Hence we have

$$R_{abc}^{jkl} = (-1)^{2k+c-a-l} [(2a+1)(2l+1)]^{1/2} \left\{ j \atop b \right\} \left\{ k \atop c \right\}.$$
Therefore, $V_c$ is a component of $V_bV_a$ precisely when \[
\begin{vmatrix}
j & k & a \\
b & c & l
\end{vmatrix} \neq 0.
\]

\[
\boxed{
\text{Remark 3.5.}
\]

1. Notice that $V_bV_a$ is a quotient of $V_b \otimes V_a$ and hence multiplicity free.

2. According to Definition (1.13), \[
\begin{vmatrix}
j & k & a \\
b & c & l
\end{vmatrix} \neq 0 \text{ if and only if there is a nonzero } G\text{-module map}
\]

\[
V_c \rightarrow V_a \otimes V_b \rightarrow (V_j \otimes V_k) \otimes V_b \simeq V_j \otimes (V_k \otimes V_b) \rightarrow V_j \otimes V_l \rightarrow V_c.
\]
4. Subrepresentation Semirings over Quasi Simply Reducible Groups

4.1 Twisted Dual and Homomorphism Modules

Usually for given $G$-modules $V$ and $W$ of a finite or compact group $G$ we give $G$-module structures to the dual space $V^*$ and the homomorphism space $\text{Hom}(V, W)$ by the rules $(\alpha \cdot f)(v) = f(\alpha^{-1} \cdot v)$ and $(\alpha \cdot g)(v) = \alpha \cdot g(\alpha^{-1} \cdot v)$ respectively, where $f \in V^*$, $g \in \text{Hom}(V, W)$ and $\alpha \in G$. But, in this section we will endow another $G$-module structure with $V^*$ and $\text{Hom}(V, W)$ when $G$ is a group with an involutory anti-automorphism $i$.

**Definition 4.1.** Let $G$ be a finite (or a compact) group with an involutory anti-automorphism $i : G \to G$, and let $V$ be a $G$-module over $\mathbb{C}$. Then a twisted dual $G$-module of $V$ is the dual space $V^*$ equipped with a $G$-module structure given by $\alpha \cdot f(v) = f(i(\alpha) \cdot v)$ for $\alpha \in G$ and $v \in V$. In this case, we denote by $^*V$ the twisted dual $G$-module of $V$.

We present the following theorem which shows a relation between a given $G$-module $V$ and its twisted dual $G$-module $^*V$ when $G$ is a quasi simply reducible group.

**Theorem 4.2.** Let $G$ be a quasi simply reducible group with an involutory anti-automorphism $i$, and let $\rho : G \to \text{GL}(V)$ be a representation of $G$. Then the twisted dual representation $\tilde{\rho} : G \to \text{GL}(^*V)$ satisfies $\tilde{\rho}(g) = {}^t\rho(i(g))$, and $(V, \rho)$ is isomorphic to $(^*V, \tilde{\rho})$.

**Proof.** Let $\{e_1, \cdots, e_n\}$ be a basis of $V$, and $\{e^*_1, \cdots, e^*_n\}$ a corresponding dual basis of $^*V$. Then we have

\[ \tilde{\rho}(g)(e^*_j) = \text{the } j\text{-th column of } \tilde{\rho}(g) = \text{the } j\text{-th row of } \rho(i(g)), \]
which implies $\tilde{\rho}(g) = {}^t\rho(i(g))$.

In order to show $(V, \rho) \simeq (\ast V, \tilde{\rho})$, let us compute the character of $\tilde{\rho}$. Then $\chi_{\tilde{\rho}}(g) = tr(\tilde{\rho}(g)) = tr({}^t\rho(i(g))) = \chi_{\tilde{\rho}}(i(g)) = \chi_{\rho}(g)$, because $g$ is conjugate to $i(g)$. The theorem now follows.

From the proof of Theorem (4.2), we have the following corollary.

**Corollary 4.3.** Let $G$ be a group with an involutory anti-automorphism $i$, and let $V$ be a $G$-module. Then we have $V \simeq \ast \ast V$ as $G$-modules.

**Proof.** Let $\rho : G \rightarrow GL(V)$ and $\tilde{\rho} \rightarrow GL(\ast V)$ be representations of $V$ and $\ast V$ respectively. Then the representation $\lambda : G \rightarrow GL(\ast \ast V)$ is given by $\lambda(g) = {}^t\tilde{\rho}(i(g))$. Thus

$$\chi_{\lambda}(g) = tr({}^t\tilde{\rho}(i(g))) = tr(\rho(g)) = \chi_{\rho}(g).$$

Now the corollary follows.

**Definition 4.4.** Let $G$ be a finite (or a compact) group with an involutory anti-automorphism $i$, and let $V$ and $W$ be $G$-modules over $\mathbb{C}$. Then a twisted homomorphism $G$-module of $V$ and $W$ is a vector space $\text{Hom}(V, W)$ equipped with a $G$-module structure given by $(\alpha \cdot f)(v) = \alpha \cdot f(i(\alpha) \cdot v)$ for $\alpha \in G$ and $v \in V$. In this case, we denote by $\widetilde{\text{Hom}}(V, W)$ the twisted homomorphism $G$-module of $V$ and $W$.

The following theorem will play an important role in concerning with a construction of Clebsch-Gordan coefficients.

**Theorem 4.5.** Let $G$ be a finite (or a compact) group with an involutory anti-automorphism $i : G \rightarrow G$, and let $V$ and $W$ be $G$-modules over $\mathbb{C}$. Then we have $\ast V \otimes W \simeq \widetilde{\text{Hom}}(V, W)$ as $G$-modules.
Proof. Define \( \phi : V \otimes W \rightarrow \tilde{\text{Hom}}(V, W) \) by \( \phi(f \otimes w)(v) = f(v)w \) for \( f \in *V \), \( w \in W \) and \( v \in V \). Then \( \phi \) gives a \( G \)-module isomorphism. \( \square \)

4.2 Twisted 6j-Symbols

We first want to fix an orthonormal basis for the twisted dual vector space \( *V_r \) which can be obtained canonically from the fixed basis of \( V_r \). The following lemma shows a way how we can do this.

**Lemma 4.6.** Let \( G \) be a finite (or compact) group. Suppose that \( V \) and \( W \) are isomorphic irreducible \( G \)-modules over \( \mathbb{C} \) under the \( G \)-module isomorphism \( \theta : V \rightarrow W \). Let \( (\ , \ )_V \) and \( (\ , \ )_W \) be \( G \)-invariant, positive-definite hermitian inner product on \( V \) and \( W \) respectively, and \( \{v_1, \cdots , v_n\} \) an orthonormal basis of \( V \) with respect to the inner product \( (\ , \ )_V \). Then \( \{\theta(v_1), \cdots , \theta(v_n)\} \) is an orthonormal basis of \( W \) with respect to an inner product \( c(\ , \ )_W \) for some constant \( c \).

**Proof.** Let us define an inner product \( (\ , \ )'_W \) on \( W \) by the formula
\[
(w_1, w_2)'_W := (v_1, v_2)_V \text{ if } w_1 = \theta(v_1) \text{ and } w_2 = \theta(v_2).
\]
Then clearly \( (\ , \ )'_W \) is a \( G \)-invariant, positive-definite hermitian inner product on \( W \). By Lemma (1.2), we have \( (\ , \ )'_W = c(\ , \ )_W \) for some constant \( c \). Thus we have
\[
c(\theta(v_i), \theta(v_j))_W = (\theta(v_i), \theta(v_j))'_W = (v_i, v_j)_V = \delta_{ij}.
\]
\( \square \)

Let \( V_r \) be an irreducible representation of a quasi simply reducible group \( G \) over \( \mathbb{C} \), and let us fix an orthonormal basis \( \{e^r_1, \cdots , e^r_{n_r}\} \) on the duality class of \( V_r \) with respect to the unique (up to a scalar multiplication) \( G \)-invariant, positive-definite hermitian inner product \( (\ , \ )_r \) of \( V_r \). In other words, if \( V_r \simeq \tilde{V}_r \) as \( G \)-modules, then we still choose an orthonormal basis \( \{e^r_1, \cdots , e^r_{n_r}\} \) as a fixed basis on \( \tilde{V}_r \).
which yields an orthonormal dual basis \( \{(e^r_1)^*, \cdots, (e^r_n)^*\} \) on \( V_r^* \) via the \( G \)-module identification \( V_r \simeq V_r^* \). In the case of \( V_r \not\simeq V_r^* \), we choose independently the dual basis \( \{(e^*_1)^*, \cdots, (e^*_n)^*\} \) as a fixed orthonormal basis on \( V_r^* \), and this dual basis yields an orthonormal basis \( \{e^*_1, \cdots, e^*_n\} \) on \( V_r \) through the identification \( V_r \simeq V_r^* \).

The following corollary is immediate from Lemma (4.6).

**Corollary 4.7.** Let \( V_r \) be an irreducible representation of a quasi simply reducible group \( G \) over \( \mathbb{C} \). Suppose that we fix an orthonormal basis \( \{e^r_1, \cdots, e^r_n\} \) for \( V_r \). Let \( \theta_r : V_r \longrightarrow V_r^* \) be a \( G \)-module isomorphism between \( V_r \) and \( V_r^* \). Then the set \( \{\theta_r(e^r_1), \cdots, \theta_r(e^r_n)\} \) is an orthonormal basis of \( V_r^* \). In this case, we denote the orthonormal basis \( \{\theta_r(e^r_1), \cdots, \theta_r(e^r_n)\} \) by \( \{e^*_1, \cdots, e^*_n\} \).

**Remark 4.8.** By Schur’s lemma, we know that \( \dim \mathbb{C} \text{Hom}_G(V_r, V_r) = 1 \). Then this fact implies that we may regard the \( G \)-module isomorphism \( \theta_r : V_r \longrightarrow V_r^* \) as the unique \( G \)-module (up to a constant multiplication) isomorphism between \( V_r \) and \( V_r^* \).

**Definition 4.9.** We call the orthonormal basis \( \{e^*_1, \cdots, e^*_n\} \) defined in Corollary (4.7) a **twisted dual basis** of the twisted dual \( G \)-module \( V_r^* \).

We also fix an orthonormal basis for \( \widetilde{\text{Hom}}(V_j, V_l) \) when \( V_j \) and \( V_l \) are irreducible representations of a quasi simply reducible group \( G \) over \( \mathbb{C} \).

First, note that we can fix an orthonormal basis of \( V_j^* \otimes_v V_l \) which comes from the fixed bases \( \{ e^j_p \} \) of \( V_j \) and \( \{ e^l_q \} \) of \( V_l \). More precisely, if we define an inner product \( (\ , \ )_{jl} \) on \( V_j^* \otimes V_l \) by the rule

\[
(*_{p} \otimes e^j_q, *^j_{p'} \otimes e^l_q)_{jl} := (*_{p} *^j_{p'}, *^j_{p'} \cdot (e^l_q e^l_{q'}))_{l},
\]
then the inner product \((\ , \ )_{jl}\) is a \(G\)-invariant, positive-definite hermitian inner product on \(\ast V_j \otimes V_l\). Clearly the basis \(\{\ast e_p^j \otimes e_q^l\}\) yields an orthonormal basis for \(\ast V_j \otimes V_l\) relative to the inner product \((\ , \ )_{jl}\).

After we fix an orthonormal basis \(\{\ast e_p^j \otimes e_q^l\}\) for \(\ast V_j \otimes V_l\), we fix an orthonormal basis on \(\tilde{\text{Hom}}(V_j, V_l)\) through the \(G\)-module isomorphism given in Theorem (4.5). If no confusion is likely to arise, then we also denote this orthonormal basis on \(\tilde{\text{Hom}}(V_j, V_l)\) by \(\{\ast e_p^j \otimes e_q^l\}\) for convenience.

In the remaining part of this section, a group \(G\) will always mean a quasi simply reducible group with an involutory anti-automorphism \(i\), and every representation of \(G\) is considered over \(\mathbb{C}\).

Let us assume that a representation \(V\) of \(G\) has a decomposition \(V = \oplus_{j \in J} r_j V_j\) into irreducible representations of \(G\). Then we obtain \(\text{End}(V) = \oplus_{j,k} r_j r_k \text{Hom}(V_j, V_k)\). Recall that \(\text{End}(V)\) has a composition multiplication as a \(G\)-algebra which can be induced by multiplications over each pair of components in its decomposition. Thus it is natural to think about the composition multiplication

\[
m : \text{Hom}(V_k, V_l) \times \text{Hom}(V_j, V_k) \longrightarrow \text{Hom}(V_j, V_l).
\]

However, if we give a \(G\)-module structure to \(\text{Hom}(V_j, V_k)\) such as Definition (4.4), then it turns out that we should consider the following \(G\)-module homomorphism for the compatibility of \(G\)-module structures

\[
\tilde{m} : \text{Hom}(V_k, V_l) \otimes \tilde{\text{Hom}}(V_j, V_k) \longrightarrow \tilde{\text{Hom}}(V_j, V_l),
\]

which is induced from the composition multiplication \(m\).

Let \(V_s\) and \(V_t\) be irreducible \(G\)-modules, and let \(V_r\) be an irreducible component of \(\tilde{\text{Hom}}(V_s, V_t)\) as a \(G\)-submodule. Then we will denote by \(\psi_{r}^{st}\) the unique \(G\)-module
imbedding $V_r \hookrightarrow \widetilde{\text{Hom}}(V_s, V_l)$ up to a constant factor. Since we already fixed an orthonormal basis $\{e^r_u\}$ of $V_r$, we obtain

$$\psi^{st}_r(e^r_u) = \sum_{p_1, p_2} C^{str}_{p_1 p_2 u} e^s_{p_1} \otimes e^t_{p_2}$$

for some complex numbers $C^{str}_{p_1 p_2 u}$. (4.15)

**Definition 4.10.** We call the coefficients $C^{str}_{p_1 p_2 u}$ defined in Equation (4.15) Clebsch-Gordan coefficients of $G$.

**Remark 4.11.** We should note that our definition of Clebsch-Gordan coefficients is exactly same as the classical’s because the following diagram commutes

Let us now assume that $V$ is a representation of $G$, and let $V_j$, $V_k$ and $V_l$ be irreducible representations of $G$. We also assume that $V_a$, $V_b$ and $V_c$ are irreducible components of $\widetilde{\text{Hom}}(V_j, V_k)$, $\widetilde{\text{Hom}}(V_k, V_l)$ and $V_b V_a$ respectively, where $V_b V_a$ denotes the multiplication of $V_b$ and $V_a$ in the subrepresentation semiring $S_G(\text{End}(V))$. Next, we consider the following composition

$$\tau = \left((\kappa \otimes \text{id}_{V_l}) \otimes \text{id}_{\widetilde{\text{Hom}}(V_j, V_k)}\right) \circ (\psi^{jl}_{b} \otimes \psi^{jk}_{a}) \circ \text{id}$$

of $G$-module homomorphisms:
where $\kappa : *V_{k} \rightarrow V_{k}^{*}$ is a $G$-module isomorphism given by $*e_{s}^{k} \mapsto (e_{s}^{k})^{*}$.

Next, by considering a composition of $G$-module homomorphisms

$$V_{c} \leftarrow V_{b} \otimes V_{a} \simeq *V_{b} \otimes V_{a} \simeq \widetilde{Hom}(V_{b}, V_{a})$$

we obtain $G$-module imbedding $V_{c} \leftarrow \widetilde{Hom}(V_{b}, V_{a})$. Moreover, $V_{c}$ is also an irreducible component of $\widetilde{Hom}(V_{j}, V_{l})$ because $V_{b} \leftarrow Hom(V_{k}, V_{l})$, $V_{a} \leftarrow \widetilde{Hom}(V_{j}, V_{k})$ and $V_{c} \leftarrow V_{b}V_{a}$. These two imbeddings of $V_{c}$ imply the following diagram of $G$-module homomorphisms:

$$\begin{array}{ccc}
V_{c} & \xrightarrow{\psi_{c}^{ba}} & \widetilde{Hom}(V_{b}, V_{a}) \simeq V_{b} \otimes V_{a} \\
\eta \downarrow & & \downarrow \tau \\
\widetilde{Hom}(V_{j}, V_{l}) & \xleftarrow{\widetilde{m}} & Hom(V_{k}, V_{l}) \otimes \widetilde{Hom}(V_{j}, V_{k}),
\end{array}$$

where $\eta$ indicates $\widetilde{m} \circ \tau \circ \psi_{c}^{ba}$.

From this diagram, we obtain by Schur’s lemma that

$$\eta = R_{abc}^{jkl} \psi_{c}^{jl} \text{ for some constant } R_{abc}^{jkl} \quad (4.16)$$

**Definition 4.12.** We call the constant $R_{abc}^{jkl}$ which is defined in (4.16) a twisted $6j$-symbol defined over a quasi simply reducible group $G$.

### 4.3 Structure Constants and The Vanishing of Twisted $6j$-Symbols

Let us now return to the problem of the structure constants for the subrepresentation semiring $S_{G}(End(V))$. Since the structure constants $S_{G}(End(V))$ is completely determined by the decomposition of the product of irreducible components, it is enough to consider the case of the product $V_{b}V_{a}$ of two irreducible components $V_{b}$ and $V_{a}$. However, from Equation (4.16) we know that $V_{c}$ is a nonzero irreducible component of $V_{b}V_{a}$ precisely when $R_{abc}^{jkl} \neq 0$.

To summarize what has been done, we have the following theorem.
Theorem 4.13. Let $V_a$, $V_b$ and $V_c$ be nontrivial irreducible components of $\widetilde{\Hom}(V_j, V_k)$, $\Hom(V_k, V_l)$ and $V_b V_a$ respectively. Then the twisted $6j$-symbol $R^{ijkl}_{abc}$ is nonzero. Moreover, we have

$$V_b V_a = \bigoplus_{c | R^{ijkl}_{abc} \neq 0} V_c.$$  

As an immediate application of Theorem (4.13), we have the following corollary concerned with the structure constants of the usual composition multiplication

$$m : \Hom(V_k, V_l) \otimes \Hom(V_j, V_k) \longrightarrow \Hom(V_j, V_l).$$

Corollary 4.14. Let $G$ be a quasi simply reducible group, and let $V_a, V_b$ and $V_c$ be nontrivial irreducible components of $\Hom(V_j, V_k)$, $\Hom(V_k, V_l)$ and $V_b V_a$ respectively. Then the twisted $6j$-symbols $\widetilde{R}^{ijkl}_{abc} \neq 0$, and we have

$$V_b V_a = \bigoplus_{c | \widetilde{R}^{ijkl}_{abc} \neq 0} V_c.$$ 

Now we will give an interesting formula containing Clebsch-Gordan coefficients of $G$ which will be extensively studied in Chapter 6. Before we give our formula, let us first present the following lemma.

Lemma 4.15. Let $V_j, V_k$ and $V_l$ be irreducible representations of $G$. Then a $G$-module homomorphism

$$\widetilde{m} \circ \left( (\kappa \otimes id_{V_l}) \otimes id_{\widetilde{\Hom}(V_j, V_k)} \right) : \widetilde{\Hom}(V_k, V_l) \otimes \widetilde{\Hom}(V_j, V_k) \longrightarrow \widetilde{\Hom}(V_j, V_l)$$

sends $(e^k_{s_1} \otimes e^l_{s_2}) \otimes (e^j_{t_1} \otimes e^k_{t_2})$ to $\delta_{s_1 t_2} e^j_{t_1} \otimes e^l_{s_2}$.

Proof. We first note that $(e^k_{s_1} \otimes e^l_{s_2}) \otimes (e^j_{t_1} \otimes e^k_{t_2})$ corresponds to $(e^k_{s_1})^* \otimes e^l_{s_2}$ \otimes $(e^j_{t_1} \otimes e^k_{t_2})$ under the map $(\kappa \otimes id_{V_l}) \otimes id_{\widetilde{\Hom}(V_j, V_k)}$.

Then, for $x \in V_j$ we have
\[
\tilde{m} \left( \left( (e_{s_1}^k)^* \otimes e_{s_2}^l \right) \otimes (e_{t_1}^j \otimes e_{t_2}^k) \right)(x) = \left( \left( (e_{s_1}^k)^* \otimes e_{s_2}^l \right) \circ (e_{t_1}^j \otimes e_{t_2}^k) \right)(x)
\]
\[
= \left( (e_{s_1}^k)^* \otimes e_{s_2}^l \right) (e_{t_1}^j(x)e_{t_2}^k)
\]
\[
= (e_{s_1}^k)^* (e_{t_1}^j(x)e_{t_2}^k) e_{s_2}^l
\]
\[
= (e_{s_1}^k)^* (e_{t_2}^k) (e_{t_1}^j(x)e_{s_2}^l)
\]
\[
= \delta_{s_1 t_2} (e_{t_1}^j \otimes e_{s_2}^l)(x),
\]
and the lemma follows. \[\square\]

For irreducible components \(V_a, V_b\) and \(V_c\) of \(\widetilde{\text{Hom}}(V_j, V_k), \widetilde{\text{Hom}}(V_k, V_l)\) and \(V_b V_a\) respectively, we obtain by Equation (4.16)

\[
\eta(e_u^c) = \tilde{m} \circ \tau \left( \sum_{p_1, p_2} C_{p_1 p_2 u}^{bac} e_{p_1}^b \otimes e_{p_2}^a \right)
\]
\[
= \left( \tilde{m} \circ \left( (\kappa \otimes \text{id}_{V_i}) \otimes \text{id}_{\widetilde{\text{Hom}}(V_j, V_k)} \right) \right) \left( \sum_{p_1, p_2} C_{p_1 p_2 u}^{bac} \psi_{c}^{bl} (e_{p_1}^b) \otimes \psi_{c}^{ajk} (e_{p_2}^a) \right)
\]
\[
= \tilde{m} \left( \sum_{p_1, p_2, s_1, s_2, t_1, t_2} C_{p_1 p_2 u}^{bac} C_{s_1 s_2 p_1}^{cbl} C_{t_1 t_2 p_2}^{jka} (e_{s_1}^k)^* \otimes e_{s_2}^l \otimes (e_{t_1}^j \otimes e_{t_2}^k) \right)
\]
\[
= \sum_{p_1, p_2, s_1, s_2, t_1, t_2} C_{p_1 p_2 u}^{bac} C_{s_1 s_2 p_1}^{cbl} C_{t_1 t_2 p_2}^{jka} \delta_{s_1 t_2} (e_{t_1}^j \otimes e_{s_2}^l)
\]
\[
= R_{abc}^{jkl} e_{c}^l (e_u^c)
\]
\[
= \sum_{m_1, m_2} R_{abc}^{jkl} C_{m_1 m_2 u}^{jkl} e_{m_1}^j \otimes e_{m_2}^l.
\]

By comparing the coefficients of \(e_{m_1}^j \otimes e_{m_2}^l\), we obtain the following formula

\[
\sum_{p_1, p_2, s_1} C_{p_1 p_2 u}^{bac} C_{s_1 m_2 p_1}^{cbl} C_{m_1 s_1 p_2}^{jka} = R_{abc}^{jkl} C_{m_1 m_2 u}^{jkl}.
\] (4.17)
Remark 4.16. Equation (4.17) is very similar to equation (2.2) of [W2]. This is the reason why we call the coefficients $R_{abc}^{jkl}$ the twisted 6$j$-symbols.
5. Frobenius-Schur Invariants, Even and Odd Representations

5.1 New Frobenius-Schur Invariants

From now on unless otherwise stated $G$ will denote a finite group with an involutory anti-automorphism $i$.

Let us first present the following lemma.

**Lemma 5.1.** Let $V$ be an irreducible $G$-module over $\mathbb{C}$. Then we have

$$\dim_{\mathbb{C}} \text{Hom}_G(^*V, V) = \dim_{\mathbb{C}} \text{Sym}_G(^*V, V) + \dim_{\mathbb{C}} \text{Alt}_G(^*V, V).$$

Here we mean that

$$\text{Sym}(^*V, V) = \{ f \in \text{Hom}(^*V, V) : f^* = f \},$$

$$\text{Alt}(^*V, V) = \{ f \in \text{Hom}(^*V, V) : f^* = -f \}$$

and $f^*$ indicates the dual linear map $f^*: ^*V \rightarrow **V$ of $f \in \text{Hom}(^*V, V)$ given by $f^*(\alpha) = \alpha \circ f$.

**Proof.** For $f \in \text{Hom}_G(^*V, V), \psi \in ^*V, \rho \in ^*V$ and $\alpha \in G$, we first notice that

$$(f^*(\alpha \cdot \psi))(\rho) = \psi(i(\alpha) \cdot f(\rho)) = \psi(f(i(\alpha) \cdot \rho)) = (\alpha \cdot f^*(\psi))(\rho).$$

Thus we have $f^* \in \text{Hom}_G(^*V, V)$ if $f \in \text{Hom}_G(^*V, V)$, and we can express each $f \in \text{Hom}_G(^*V, V)$ as $\frac{f + f^*}{2} + \frac{f - f^*}{2}$.

The result now follows because $\text{Sym}_G(^*V, V) \cap \text{Alt}_G(^*V, V) = \{0\}$. ☐

In general, $\text{Sym}(^*V, V)$ and $\text{Alt}(^*V, V)$ are not $G$-submodules of $\text{Hom}(^*V, V)$, but they are always $G$-submodules of $\text{\widehat{Hom}}(^*V, V)$. We will check that both $\text{Sym}(V, ^*V)$
and $\text{Alt}(V,^*V)$ are $G$-submodules of $\widehat{\text{Hom}}(V,^*V)$. For this, it is enough to show that

$$(\alpha \cdot f)^* = \alpha \cdot f \text{ for } \alpha \in G \text{ and } f \in \text{Sym}(V,^*V).$$

Let $\psi : V \rightarrow ^*V$ be the canonical $G$-module isomorphism given by $\psi(v)(x) = x(v)$ for $v \in V$ and $x \in ^*V$. Then $\psi(v) \in ^*V$ corresponds to $\psi(v) \circ (\alpha \cdot f) \in ^*V$ under the linear map $(\alpha \cdot f)^*$. If we evaluate the value of $(\alpha \cdot f)^*(\psi(v))$ at $w \in V$, then we obtain

$$(\alpha \cdot f)^*(\psi(v))(w) = (\psi(v) \circ (\alpha \cdot f))(w)$$

$$= \psi(v)((\alpha \cdot f)(w))$$

$$= \psi(v)(\alpha \cdot f(i(\alpha) \cdot w))$$

$$= (\alpha \cdot f(i(\alpha) \cdot w))(v)$$

$$= f(i(\alpha) \cdot w)(i(\alpha) \cdot v)$$

$$= f(i(\alpha) \cdot v)(i(\alpha) \cdot w) \quad (\text{since } f \in \text{Sym}(V,^*V))$$

$$= (\alpha \cdot f(i(\alpha) \cdot v))(w)$$

$$= ((\alpha \cdot f)(v))(w),$$

which implies $(\alpha \cdot f)^* = \alpha \cdot f$.

Similarly we can check that $\text{Alt}(V,^*V)$ is a $G$-submodule of $\widehat{\text{Hom}}(V,^*V)$.

Now we note by Schur’s lemma that Lemma (5.1) implies

$$\dim_{\mathbb{C}} \text{Sym}_G(^*V, V) + \dim_{\mathbb{C}} \text{Alt}_G(^*V, V) = 0 \text{ or } 1. \quad (5.18)$$

In the following definition, we define an analogue of the classical Frobenius-Schur invariants for a group with an involutory anti-automorphism.
**Definition 5.2.** Let $G$ be a finite group with an involutory anti-automorphism $i$, and let $V$ be an irreducible $G$-module over $\mathbb{C}$. Then we call the value

$$dim_{\mathbb{C}}\text{Sym}_G(\ast V, V) - dim_{\mathbb{C}}\text{Alt}_G(\ast V, V)$$

a *twisted Frobenius-Schur invariant*, and write $I(\ast V, V)$ for it.

**Remark 5.3.**
1. For a finite group $G$ with an involutory anti-automorphism $i$ and an irreducible $G$-module $V$ over $\mathbb{C}$, the only possible values for $I(\ast V, V)$ are $\{-1, 0, 1\}$. In particular, $I(\ast V, V) = \pm 1$ if $G$ is a quasi simply reducible group.

2. Our twisted Frobenius-Schur invariants coincide with Mackey’s invariants that appeared in [M2] and [SH].

The following definitions are motivated by the classical definitions of *complex*, *real* and *quaternionic* representations.

**Definition 5.4.** Let $G$ be a finite group with an involutory anti-automorphism $i$, and $V$ an irreducible representation of $G$ over $\mathbb{C}$. Then,

1. $V$ is called a *twisted complex representation* of $G$ if $I(\ast V, V) = 0$.

2. $V$ is called a *twisted real representation* of $G$ if $I(\ast V, V) = 1$.

3. $V$ is called a *twisted quaternionic representation* of $G$ if $I(\ast V, V) = -1$.

As the case of the classical Frobenius-Schur invariants, we can express the twisted Frobenius-Schur invariants as characters. We should notice that the following proposition yields the classical Frobenius-Schur invariants if we take the involutory anti-automorphism $i$ to be the multiplication inverse. This is also immediate from Definition (5.2).
Proposition 5.5. Let $G$ be a finite group with an involutory anti-automorphism $i$, and let $V$ be an irreducible representation of $G$ over $\mathbb{C}$ with the character $\chi$. Then

$$I(V, V) = \frac{1}{|G|} \sum_{g \in G} \chi(gi(g)^{-1}).$$

Proof. Let $\rho : G \to GL(V)$ and $\tilde{\rho} : G \to GL(V)$ be the representations corresponding to $V$ and $V$ respectively. Then we have

$$\frac{1}{|G|} \sum_{g \in G} \chi(gi(g)^{-1}) = \frac{1}{|G|} \sum_{g \in G} \text{tr} \left( \rho(g)^t \tilde{\rho}(i(g)) \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{m,m'} \rho(g)_{mm'} \left( \tilde{\rho}(i(g)) \right)_{m' m}. \quad (5.19)$$

If $V \not\simeq *V$, then Equation (5.20) is equal to 0 by the orthogonality formula.

If $V \simeq *V$, then there exists nonzero $\varphi \in \text{Hom}_G(*V, V)$. Thus Equation (5.20) is equal to

$$I(V, V) = \frac{1}{\text{dim} V} \sum_{m,m'} (\varphi^{-1})_{m'_m} \tilde{\varphi}_{m,m'}$$

Now we re-prove the following theorem that was first studied by Wigner through our setting-up of twisted modules.
Theorem 5.6. Let $G$ be a finite quasi simply reducible group. Then,

1. the tensor product of two twisted real representations of $G$ only contains irreducible twisted real representations of $G$ as its irreducible components,

2. the tensor product of two twisted quaternionic representations of $G$ only contains irreducible twisted real representations of $G$ as its irreducible components,

3. the tensor product of a twisted real representation and a twisted quaternionic representation only contains irreducible twisted quaternionic representations of $G$ as its irreducible components.

Proof. Let $V_{j1}$ and $V_{j2}$ be irreducible representations of $G$, and let $V_i$ be an irreducible component of $V_{j1} \otimes V_{j2}$. Then it is enough to show that $I(V_i, V_i) = I(V_{j1}, V_{j1})I(V_{j2}, V_{j2})$.

Suppose that $\varphi_1 : V_{j1} \rightarrow V_{j1}$ and $\varphi_2 : V_{j2} \rightarrow V_{j2}$ are $G$-module isomorphisms. Then we have a $G$-module isomorphism $\varphi_1 \otimes \varphi_2 : (V_{j1} \otimes V_{j2}) \rightarrow V_{j1} \otimes V_{j2}$. Moreover, the restriction $(\varphi_1 \otimes \varphi_2)_V$ of $\varphi_1 \otimes \varphi_2$ to $V_i$ is nonzero because $(\varphi_1 \otimes \varphi_2)_V$ is injective.

Notice by $(\varphi_1 \otimes \varphi_2)^* = \varphi_1^* \otimes \varphi_2^*$ that $(\varphi_1 \otimes \varphi_2) \in Sym((V_{j1} \otimes V_{j2}), V_{j1} \otimes V_{j2})$ if either it is satisfied $\varphi_1 \in Sym(V_{j1}, V_{j1})$ and $\varphi_2 \in Sym(V_{j2}, V_{j2})$, or $\varphi_1 \in Alt(V_{j1}, V_{j1})$ and $\varphi_2 \in Alt(V_{j2}, V_{j2})$. Similarly, $(\varphi_1 \otimes \varphi_2) \in Alt((V_{j1} \otimes V_{j2}), V_{j1} \otimes V_{j2})$ if either it is satisfied $\varphi_1 \in Sym(V_{j1}, V_{j1})$ and $\varphi_2 \in Alt(V_{j2}, V_{j2})$, or $\varphi_1 \in Alt(V_{j1}, V_{j1})$ and $\varphi_2 \in Sym(V_{j2}, V_{j2})$.

If $I(V_{j1}, V_{j1}) = 1$ and $I(V_{j2}, V_{j2}) = 1$, then

$$dim_{C}Sym_{G}(V_{j1}, V_{j1}) = dim_{C}Sym_{G}(V_{j2}, V_{j2}) = 1$$
which implies $\varphi_1 \in Sym_G(V_{j_1}, V_{j_1})$ and $\varphi_2 \in Sym_G(V_{j_2}, V_{j_2})$. Hence $(\varphi_1 \otimes \varphi_2)V_i \in Sym_G(V_i, V_i)$, and we have $I(V_i, V_i) = 1$. Similarly we can obtain $I(V_i, V_i) = I(V_{j_1}, V_{j_1})I(V_{j_2}, V_{j_2})$ for other possible values of $I(V_{j_1}, V_{j_1})$ and $I(V_{j_2}, V_{j_2})$. □

5.2 Even and Odd Representations

The concepts of even and odd representations were first introduced in [W1] and [W2] by Wigner to study his $3j$ and $6j$ symbols over simply reducible groups. In [SH], Sharp defined the notion of twisted even and twisted odd representations ($\ast$-even and $\ast$-odd in his notations) for quasi simply reducible groups, and claimed that the basic properties of even and odd representations also hold for twisted even and odd representations. However, Sharp’s argument has some serious errors. In particular, in the lemma 3, p 192, of [SH] he stated that there are no irreducible representations of a quasi simply reducible group which can be both twisted even and twisted odd. This statement is false and we will give a counterexample for this. For this reason, all of Sharp’s results on $3j$ and $6j$ symbols over a quasi simply reducible group that are based on the lemma 3 of p 192 in [SH] are also false as stated.

Let us now review the definition of even and odd representations given by Wigner in the case of a simply reducible group.

We first recall that there is a $G$-module decomposition

$$W \otimes W = Sym^2(W) \oplus Alt^2(W). \quad (5.21)$$

Here we mean that

$$Sym^2(W) = \{ z \in W \otimes W \mid \theta(z) = z \}$$

and

$$Alt^2(W) = \{ z \in W \otimes W \mid \theta(z) = -z \}$$
for the automorphism \( \theta \) of \( W \otimes W \) given by \( \theta(w_1 \otimes w_2) = w_2 \otimes w_1 \) \((w_1, w_2 \in W)\).

**Definition 5.7.** Let \( G \) be a simply reducible group. Then an irreducible representation \( V \) of \( G \) is called an *even representation* if \( V \) is an irreducible component of \( \text{Sym}^2(W) \) for some real representation (in the classical sense) \( W \) of \( G \), or \( V \) is an irreducible component of \( \text{Alt}^2(W) \) for some quaternionic representation \( W \) of \( G \).

Similarly, we call an irreducible representation \( V \) of \( G \) an *odd representation* if \( V \) is an irreducible component of \( \text{Alt}^2(W) \) for some real representation \( W \) of \( G \), or \( V \) is an irreducible component of \( \text{Sym}^2(W) \) for some quaternionic representation \( W \) of \( G \).

We denote the set of all even representations of \( G \) and the set of all odd representations of \( G \) by \( \mathcal{E} \) and \( \mathcal{O} \) respectively.

**Example 5.8.** If we consider the group \( SU(2) \), then in Chapter 1 we parameterize the irreducible representations of \( SU(2) \) by the non-negative half integers \( \frac{1}{2} \mathbb{Z}_{\geq 0} \). In this case, it is easy to check that the integer representations are real and the half integer representations are quaternionic. Moreover, for a integer (resp. quaternionic) representation \( V_i \), \( \text{Sym}^2(V_i) \) is a direct sum of even (resp. odd) integer representations and \( \text{Alt}^2(V_i) \) is a direct sum of odd (resp. even) integer representations.

**Example 5.9.** In this example, we treat an example of the symmetric group \( S_3 \) over 3 letters which is a finite simply reducible group.

As a basic fact of the representation theory of the group \( S_3 \), the group \( S_3 \) has the three irreducible representations as follows.

- The trivial representation \( V_0 = \mathbb{C} \),

- The signature representation \( V_1 = \mathbb{C} \) (i.e., \( \sigma \cdot x = x \) if \( \sigma \) even and \( \sigma \cdot x = -x \) if \( \sigma \) odd),
• The 2-dimensional irreducible representation

\[ V_2 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\} \]

with the \( S_3 \)-module structure \( \sigma \cdot (x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \) for \( \sigma \in S_3 \).

Let us now consider a decomposition \( V_2 \otimes V_2 \). It is easy to check by Lemma (3.3) that \( V_2 \otimes V_2 = V_0 \oplus V_1 \oplus V_2 \). On the other hand, we also have a decomposition \( V_2 \otimes V_2 = \text{Sym}^2(V_2) \oplus \text{Alt}^2(V_2) \). Recall that

\[ \chi_{\text{Alt}}(s) = \frac{1}{2} \left( \chi_2(s)^2 - \chi_2(s^2) \right) \quad \text{for} \ s \in S_3. \]

Here we write \( \chi_{\text{Alt}} \) and \( \chi_2 \) for the character of \( \text{Alt}^2(V_2) \) and \( V_2 \), respectively. Then we have

\[ \chi_{\text{Alt}}(1) = 1, \ \chi_{\text{Alt}}((12)) = -1 \ \text{and} \ \chi_{\text{Alt}}((123)) = 1. \]

Thus \( \text{Alt}^2(V_2) \simeq V_1 \) and \( \text{Sym}^2(V_2) \simeq V_0 \oplus V_1 \). Moreover,

\[ \dim \text{Sym}^2_G(V_2) = (\chi_{\text{Sym}} | 1) = \frac{1}{6} \sum_{s \in S_3} (\chi_0(s) + \chi_2(s)) = 1, \]

which implies that \( V_2 \) is a real representation. Therefore, \( S_3 \) has one odd representation \( V_1 \) and two even representation \( V_0 \) and \( V_2 \).

Based on the definition of even and odd representations of a simply reducible group, Wigner proved the following properties:

• \( \mathcal{E} \cup \mathcal{O} \subset \mathcal{R} \), where \( \mathcal{R} \) is the set of all real representations.

• \( \mathcal{E} \cap \mathcal{O} = \emptyset \).

In particular, these properties make even and odd representations useful objects in his theory of 3\( j \) and 6\( j \) symbols. However, these nice properties can not be
extended to the case of a quasi simply reducible group under the twisted Frobenius-Schur invariants.

We now give more details about twisted even and odd representations which generalize the concepts of even and odd representations of a simply reducible group.

**Definition 5.10.** Let $G$ be a quasi simply reducible group. An irreducible representation $V$ of $G$ is called a *twisted even representation* if $V$ is an irreducible component of $\text{Sym}^2(W)$ for some twisted real representation $W$, or $V$ is an irreducible component of $\text{Alt}^2(W)$ for some twisted quaternionic representation $W$.

Similarly, we call an irreducible representation $V$ of $G$ an *twisted odd representation* if $V$ is an irreducible component of $\text{Alt}^2(W)$ for some twisted real representation $W$, or $V$ is an irreducible component of $\text{Sym}^2(W)$ for some twisted quaternionic representation $W$.

As a direct application of Theorem (5.6), we have the following proposition.

**Proposition 5.11.** If $G$ is a quasi simply reducible group, then we have

$$\mathcal{E}^* \cup \mathcal{O}^* \subset \mathcal{R}^*,$$

where $\mathcal{E}^*$, $\mathcal{O}^*$ and $\mathcal{R}^*$ denote the sets of twisted even, twisted odd and twisted real representations of $G$ respectively.

**Proof.** If an irreducible representation $V$ of $G$ is even, then $V$ is an irreducible component of $\text{Sym}^2(W)$ (resp. $\text{Alt}^2(W)$) for some twisted real (resp. twisted quaternionic) representation $W$. But by Theorem (5.6) we know that $\text{Sym}^2(W)$ (resp. $\text{Alt}^2(W)$) consists of irreducible components of twisted real representations.

In the case where an irreducible representation $V$ of $G$ is odd, the proof is the same argument as the above. The result now follows. \qed
Even though we consider Proposition (5.11) as a corresponding property of $\mathcal{E} \cup \mathcal{O} \subset \mathcal{R}$, there does not exist a twisted version of the property $\mathcal{E} \cap \mathcal{O} = \emptyset$. Actually, $\mathcal{E}^* \cap \mathcal{O}^* = \emptyset$ is false in the case of a quasi simply reducible group.

Let us now give a counterexample for this.

**Example 5.12.** In this example, we consider the dicyclic group $Q_3$ defined in Example (1.16). The group $Q_3$ has four 1-dimensional irreducible representations $W_0$ (the trivial representation), $W_1$, $W_2$ and $W_3$. It also has two 2-dimensional irreducible representations $V_1$ and $V_2$. The character table of the group $Q_3$ is as follows. The last two columns indicate the information about $Sym$ and $Alt$ in Equation (5.21).

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$R$</th>
<th>$R^2$</th>
<th>$R^3$</th>
<th>$S$</th>
<th>$RS$</th>
<th>Sym</th>
<th>Alt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$W_0$</td>
<td>-</td>
</tr>
<tr>
<td>$W_1$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>$\sqrt{-1}$</td>
<td>$-\sqrt{-1}$</td>
<td>$W_2$</td>
<td>-</td>
</tr>
<tr>
<td>$W_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>$W_0$</td>
<td>-</td>
</tr>
<tr>
<td>$W_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>$-\sqrt{-1}$</td>
<td>$\sqrt{-1}$</td>
<td>$W_2$</td>
<td>-</td>
</tr>
<tr>
<td>$V_1$</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>$W_2 \oplus V_2$</td>
<td>$W_0$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>$W_0 \oplus V_2$</td>
<td>$W_2$</td>
</tr>
</tbody>
</table>

If we apply Proposition (5.5) to each irreducible representation of $Q_3$, then we can easily check that all irreducible representations have the twisted Frobenius-Schur invariant 1. In other words, all irreducible representations of $Q_3$ are twisted real. Thus, we see that $W_0$ and $W_2$ are both twisted even and twisted odd.
6. Some Properties of Twisted 6$\jmath$-Symbols

6.1 Connection with The Classical 3$\jmath$-Symbols

In this section unless otherwise specified $G$ will denote a quasi simply reducible group with an involutory anti-automorphism $i$, and all representations of $G$ will be considered over $\mathbb{C}$.

Let us first present the following basic lemma.

**Lemma 6.1.** Let $G$ be a finite or compact group, and let $V$ be a unitary representation of $G$. Let $W_1$ and $W_2$ be two distinct irreducible subrepresentations of $V$. Then $W_1$ and $W_2$ are orthogonal relative to given any $G$-invariant, positive-definite hermitian inner product on $V$.

**Proof.** Suppose that $V$ is equipped with a $G$-invariant, positive-definite hermitian inner product $(\ ,\ )$. For given two distinct irreducible subrepresentations $W_1$ and $W_2$ of $V$, we consider a $G$-module homomorphism $\phi : W_1 \otimes W_2 \rightarrow \mathbb{C}$ defined by $\phi(x \otimes y) = (x, y)$, where $\mathbb{C}$ is equipped with the trivial $G$-module structure.

If $(w_1, w_2) \neq 0$ for some $w_1 \in W_1$ and $w_2 \in W_2$, then $\phi$ is a nonzero $G$-module homomorphism. Thus we obtain $W_1 \otimes W_2 \simeq \text{Ker} \phi \oplus \mathbb{C}$ as $G$-modules. Hence we have $\text{Hom}_G(W_1, W_2) \simeq (W_1 \otimes W_2)^G \supset \mathbb{C}$ which contradicts to Schur’s lemma. The result now follows. \hfill $\square$

Now we assume that $V_{j_1}$ and $V_{j_2}$ are irreducible representations of $G$, and suppose that $\wtilde{\text{Hom}}(V_{j_1}, V_{j_2})$ has a decomposition $\oplus_i V_i$, where $V_i$ are irreducible representations of $G$.

For a given $G$-module $\wtilde{\text{Hom}}(V_{j_1}, V_{j_2})$ and an irreducible component $V_i$ of $\wtilde{\text{Hom}}(V_{j_1}, V_{j_2})$, we have so far considered an imbedding $V_i \hookrightarrow \wtilde{\text{Hom}}(V_{j_1}, V_{j_2})$. But, we can also consider an imbedding $V_i \hookrightarrow \wtilde{\text{Hom}}(V_{j_1}, V_{j_2})$ as a conjugate imbedding $V_{j_3} \hookrightarrow$
$\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$, where $V_i \simeq V_{j_3}$. Moreover it will be turned out after a while that a conjugate imbedding will play an important role in establishing the link between our twisted $6j$-symbols and Clebsch-Gordan Coefficients. For these reasons, we prefer to treat a conjugate decomposition $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2}) = \oplus V_{j_3}$ and choose a conjugate imbedding $V_{j_3} \hookrightarrow V_{j_1} \otimes V_{j_2}$. Recall that we fixed the orthonormal basis \{e_{j_1}^p \otimes e_{j_2}^q\} on $V_{j_1} \otimes V_{j_2}$, and considered the basis \{e_{j_1}^p \otimes e_{j_2}^q\} as an orthonormal basis of $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$ through the identification $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2}) \simeq V_{j_1} \otimes V_{j_2}$.

On the other hand, for each component $V_{j_3}$ of $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$ we have an imbedding $\psi_{j_1 j_2}^{j_3} : V_{j_3} \rightarrow \widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$. So by Lemma (6.1) we can obtain another orthonormal basis $\bigcup_{j_3} \{\psi_{j_1 j_2}^{j_3} (e_{j_3}^k)\}$ of $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$ relative to a $G$-invariant, positive-definite hermitian inner product $(~,~)_{j_1 j_2}$ if we give proper scaled $G$-invariant, positive-definite hermitian inner products to each component $V_{j_3}$ of $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$. Let us denote these two orthonormal bases of $\widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$ by

$$B = \bigcup_{j_3} \{\psi_{j_1 j_2}^{j_3} (e_{j_3}^k) : 1 \leq k_3 \leq \text{dim} V_{j_3}\}$$

and

$$B' = \{e_{k_1}^{j_1} \otimes e_{k_2}^{j_2} : 1 \leq k_1 \leq \text{dim} V_{j_1} \text{ and } 1 \leq k_2 \leq \text{dim} V_{j_2}\}$$

respectively.

Then for the basis $B'$ we obtain the Clebsch-Gordan coefficients $C_{j_1 j_2 j_3}^{j_1 j_2 j_3}$ through a conjugate imbedding $\psi_{j_1 j_2}^{j_3} : V_{j_3} \rightarrow \widetilde{\text{Hom}}(V_{j_1}, V_{j_2})$ which satisfy an equation

$$\psi_{j_1 j_2}^{j_3} (e_{j_3}^k) = \sum_{k_1, k_2} C_{j_1 j_2 j_3}^{j_1 j_2 j_3} e_{j_1}^{k_1} \otimes e_{j_2}^{k_2}.$$

Next, we assume that $\rho^{j_1} : G \rightarrow GL(V_{j_1})$ and $\rho^{j_2} : G \rightarrow GL(V_{j_2})$ are the representations of $G$ for the given irreducible $G$-modules $V_{j_1}$ and $V_{j_2}$ respectively. By tensoring the representations $\rho^{j_1}$ and $\rho^{j_2}$, we have the tensor product represent-
tation \( \rho^i \otimes \rho^j : G \rightarrow GL(V_j \otimes V_{j_2}) \). Then for each \( g \in G \) we obtain the following base change formula of the linear automorphism \( (\rho^i \otimes \rho^j)(g) \):

\[
[(\rho^i \otimes \rho^j)(g)]^{B'} = \left( C_{i_1i_2i_3}^{j_1j_2j_3} \right) \left[(\rho^i \otimes \rho^j)(g)\right]^B t \left(C_{i_1i_2i_3}^{j_1j_2j_3}\right), \quad (6.22)
\]

where \( [(\rho^i \otimes \rho^j)(g)]^B \) and \( [(\rho^i \otimes \rho^j)(g)]^{B'} \) denote the matrices of the linear map \( (\rho^i \otimes \rho^j)(g) \) relative to the bases \( B \) and \( B' \) respectively. If we describe Equation (6.22) in terms of the matrices’ components, then we obtain the following:

\[
[rho^i_1(g)]_{\kappa_1, \lambda_1}^{B_1} [rho^j_2(g)]_{\kappa_2, \lambda_2}^{B_2} = \sum_{j_3, \kappa_3, \lambda_3} C_{i_1i_2i_3}^{j_1j_2j_3} [rho^j_3(g)]^{B_3'} \left(C_{i_1i_2i_3}^{j_1j_2j_3}\right), \quad (6.23)
\]

where \( B_1 = \{ e_{j_1}^p : 1 \leq p \leq dimV_{j_1}\} \), \( B_2 = \{ e_{j_2}^q : 1 \leq q \leq dimV_{j_2}\} \) and \( B_3' = \{ \psi_{j_3}^{j_1j_2} (e_{j_3}^r) : 1 \leq r \leq dimV_{j_3}\} \).

If we notice that \( V_{j_3} = V_{j_3} \) as a set and \( [rho^j_3(g)]^{B_3'} \), then Equation (6.23) yields

\[
[rho^i_1(g)]_{\kappa_1, \lambda_1}^{B_1} [rho^j_2(g)]_{\kappa_2, \lambda_2}^{B_2} = \sum_{j_3, \kappa_3, \lambda_3} C_{i_1i_2i_3}^{j_1j_2j_3} [rho^j_3(g)]_{\kappa_3, \lambda_3}^{B_3'} \left(C_{i_1i_2i_3}^{j_1j_2j_3}\right), \quad (6.24)
\]

Henceforce, for convenience we will write \( [rho^i(g)] \) for \( [rho^i(g)]^{B} \) when no confusion is likely to arise.

Let us now write a symbol \( \begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ j_1 & j_2 & j_3 \end{pmatrix} \) for \( \frac{1}{\sqrt{dimV_{j_3}}} C_{\kappa_1\kappa_2\kappa_3}^{j_1j_2j_3} \). Then we obtain the following equation which is the exactly same equation as equation (2) in [W2]:

\[
[rho^i_1(g)]_{\kappa_1, \lambda_1}^{B_1} [rho^j_2(g)]_{\kappa_2, \lambda_2} = \sum_{j_3, \kappa_3, \lambda_3} dimV_{j_3} \begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ j_1 & j_2 & j_3 \end{pmatrix} \left[C_{\kappa_1\kappa_2\kappa_3}^{j_1j_2j_3}\right] \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ j_1 & j_2 & j_3 \end{pmatrix}, \quad (6.25)
\]
Remark 6.2. In [W2], the symbol \( \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \) was defined in the same way as Equation (6.25) of this thesis, and the symbols are called the 3\( j \)-symbols.

### 6.2 Properties of The Classical 3\( j \)-Symbols

Let us now present a list of properties of 3\( j \)-symbols \( \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \) which we will need in later.

**Lemma 6.3.**

1. \( \sum_{j_3} \dim V_{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \delta_{\kappa_1 \lambda_1} \delta_{\kappa_2 \lambda_2}. \)

2. \( \sum_{\lambda_1, \lambda_2} \begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda' \end{pmatrix} = \frac{1}{\dim V_j} \delta_{\lambda \lambda'}. \)

3. \( \sum_{g \in G} [\rho^{j_1}(g)]_{\kappa_1 \lambda_1} [\rho^{j_2}(g)]_{\kappa_2 \lambda_2} [\rho^{j_3}(g)]_{\kappa_3 \lambda_3} = |G| \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}. \)

**Proof.** Note that the tuple \((\kappa_1, \kappa_2)\) determines the row of the matrix \( (C_{\kappa_1 \kappa_2 \kappa_3}^{j_1 j_2 j_3}) \). Similarly \( j_3 \) and \( \kappa_3 \) determine the column of the matrix \( (C_{\kappa_1 \kappa_2 \kappa_3}^{j_1 j_2 j_3}) \). Thus the first and second properties are immediate from the unitary property of the matrix \( (C_{\kappa_1 \kappa_2 \kappa_3}^{j_1 j_2 j_3}) \).

In order to prove the third property, we will start with Equation (6.25). By multiplying \( [\rho^{j_3'}(g)]_{\kappa_3' \lambda_3'} \) to the both sides of (6.25), we obtain

\[
[\rho^{j_1}(g)]_{\kappa_1 \lambda_1} [\rho^{j_2}(g)]_{\kappa_2 \lambda_2} [\rho^{j_3}(g)]_{\kappa_3' \lambda_3'} = \sum_{j_3, \kappa_3, \lambda_3} \dim V_{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \times [\rho^{j_3}(g)]_{\kappa_3 \lambda_3} [\rho^{j_3'}(g)]_{\kappa_3' \lambda_3'}. \]

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Taking a sum over all \( g \in G \) to both sides, we have

\[
\sum_{g \in G} \left[ \rho^{j_1}(g) \right]_{\kappa_1 \lambda_1} \left[ \rho^{j_2}(g) \right]_{\kappa_2 \lambda_2} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3} = \sum_{j_3, \kappa_3, \lambda_3} \text{dim} V_{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \times \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \sum_{g \in G} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3}.
\]

Note that \( \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3} = \left[ \rho^{j_3}(g^{-1}) \right]_{\lambda_3 \kappa_3} \) because the matrix \( \left[ \rho^{j_3}(g) \right] \) is a unitary matrix. Then Schur's lemma implies that

\[
\sum_{g \in G} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3} \left[ \rho^{j_3}(g) \right]_{\kappa_3' \lambda_3'} = \frac{|G|}{\text{dim} V_{j_3}} \delta_{\kappa_3 \kappa_3'} \delta_{\lambda_3 \lambda_3'} \delta_{j_3 j_3'}.
\]

Then we have

\[
\sum_{g \in G} \left[ \rho^{j_1}(g) \right]_{\kappa_1 \lambda_1} \left[ \rho^{j_2}(g) \right]_{\kappa_2 \lambda_2} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \lambda_3} = \sum_{j_3, \kappa_3, \lambda_3} \text{dim} V_{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \times \frac{|G|}{\text{dim} V_{j_3}} \delta_{\kappa_3 \kappa_3'} \delta_{\lambda_3 \lambda_3'} \delta_{j_3 j_3'}
\]

\[
= \frac{|G|}{\text{dim} V_{j_3}} \begin{pmatrix} j_1 & j_2 & j_3' \\ \kappa_1 & \kappa_2 & \kappa_3' \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ \lambda_1 & \lambda_2 & \lambda_3' \end{pmatrix}.
\]

The desire result is now immediate. \( \square \)

As a special case of (3) of Lemma (6.3), if we choose \( \lambda_1 = \kappa_1, \lambda_2 = \kappa_2 \) and \( \lambda_3 = \kappa_3 \), then we have

\[
\left| \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} \right|^2 = \frac{1}{|G|} \sum_{g \in G} \left[ \rho^{j_1}(g) \right]_{\kappa_1 \kappa_1} \left[ \rho^{j_2}(g) \right]_{\kappa_2 \kappa_2} \left[ \rho^{j_3}(g) \right]_{\kappa_3 \kappa_3},
\]

(6.26)
and this implies the invariance property of the absolute values under the permutation of columns. Explicitly we obtain

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} = c_{j_1j_3j_2} \begin{pmatrix} j_2 & j_3 & j_1 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} = c_{j_2j_1j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_2 & \kappa_1 & \kappa_3 \end{pmatrix} \text{ etc., (6.27)}$$

where $|c_{j_1j_3j_2}| = |c_{j_2j_1j_3}| = 1$.

One important fact concerned with the constants $C_{j_1j_2j_3}$ is that these factors do not depend on $\kappa_1, \kappa_2, \kappa_3$. They only depend on the ordered set $j_1j_2j_3$ (See [DS]).

Now we have the following interesting observation.

**Proposition 6.4.** Let $G$ be a quasi simply reducible group, and let $V_{j_1}, V_{j_2}$ and $V_{j_3}$ be irreducible representations of $G$ over $\mathbb{C}$. Suppose that the imbedding $\psi^{j_1j_2}_j : V_{j_3} \hookrightarrow V_{j_1} \otimes V_{j_2}$ is nontrivial. Then the imbeddings $\psi^{pq}_r : V_r \hookrightarrow V_p \otimes V_q$ are also nontrivial, where $\{p, q, r\} = \{j_1, j_2, j_3\}$.

**Proof.** This proposition is immediate from Equation (6.27).

6.3 1$j$-Symbols

Let us now consider the special case of Equation (6.25) to connect our approach to 3$j$-symbols with what is called 1$j$-symbols.

If we consider $V_{j_2}$ as the trivial representation in Equation (6.25), then the corresponding conjugate imbedding is

$$\overline{V}_{j} \simeq \overline{\text{Hom}}(V_j, \mathbb{C}) \simeq V_j \otimes \mathbb{C}. \quad (6.28)$$

In addition, we also obtain the following corresponding equation:
\[ [\rho^j(g)]_{\kappa_1\lambda_1} \cdot 1 = \sum_{\kappa_3, \lambda_3} C^{j0j}_{\kappa_10\kappa_3} [\bar{\rho}^\ast(g)]_{\kappa_3\lambda_3} C^{j0j}_{\lambda_10\lambda_3} \tag{6.29} \]

\[ \quad = \sum_{\kappa_3, \lambda_3} \dim V_j \left( \begin{array}{ccc} j & 0 & \bar{j} \\ \kappa_1 & 0 & \kappa_3 \end{array} \right) [\bar{\rho}^\ast(g)]_{\kappa_3\lambda_3} \left( \begin{array}{ccc} j & 0 & \bar{j} \\ \lambda_1 & 0 & \lambda_3 \end{array} \right) \tag{6.30} \]

where the index 0 denotes the trivial representation \( \mathbb{C} \) of \( G \).

If we write \( \left( \begin{array}{ccc} j \\ r & s \end{array} \right) \) for \( \sqrt{\dim V_j} \left( \begin{array}{ccc} j & 0 & \bar{j} \\ r & 0 & s \end{array} \right) \), then Equation Equation (6.29) yields

\[ [\rho^j(g)]_{\kappa_1\lambda_1} \cdot 1 = \sum_{\kappa_3, \lambda_3} \left( \begin{array}{ccc} j \\ \kappa_1 & \kappa_3 \end{array} \right) [\bar{\rho}^\ast(g)]_{\kappa_3\lambda_3} \left( \begin{array}{ccc} j \\ \lambda_1 & \lambda_3 \end{array} \right), \tag{6.31} \]

which is the exactly same formula as the definition of 1-j-symbols in [W2].

The following theorem of 1j-symbols originally due to Wigner [W2]. However, in [W2] Wigner only considered 1j-symbols over simply reducible groups. His idea was further generalized in [SH] to quasi simply reducible groups by Sharp.

**Theorem 6.5.**

1. \[ \sum_{m_2} \left( \begin{array}{ccc} j \\ m_1 & m_2 \end{array} \right) \left( \begin{array}{ccc} j \\ m_1 & m_2 \end{array} \right) = \delta_{m_1m_1'}, \]

2. \[ \left( \begin{array}{ccc} j \\ m & m' \end{array} \right) = (-1)^{2V_j} \left( \begin{array}{ccc} j \\ m' & m \end{array} \right), \]

3. the normalized imbeddings \( \alpha_{pqr} \psi^n_{\tau} (\alpha_{pqr} \in \mathbb{C}) \) can be chosen such that it is satisfied

\[ \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) = \sum_{n_1, n_2, n_3} \left( \begin{array}{ccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 \\ n_1 & n_2 & n_3 \end{array} \right) \left( \begin{array}{ccc} \bar{j}_1 \\ m_1 & n_1 \end{array} \right) \left( \begin{array}{ccc} \bar{j}_2 \\ m_2 & n_2 \end{array} \right) \left( \begin{array}{ccc} \bar{j}_3 \\ m_3 & n_3 \end{array} \right). \]

**Proof.** See [SH]. \( \square \)
From now on unless otherwise stated all imbeddings $\psi^{pq}$ are normalized. So $3j$-symbols
$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$ automatically satisfy the third property of Theorem (6.5).

Next, we introduce new symbols which we will call them twisted $3j$-symbols in this thesis.

**Definition 6.6.** For a given $3j$-symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$, we write $\begin{pmatrix} j_1 & j_2 & p_3 \\ p_1 & p_2 & \tilde{j}_3 \end{pmatrix}$ for
$$\sum_{p_3} \begin{pmatrix} \tilde{j}_3 \\ p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix},$$
and we call the symbol $\begin{pmatrix} j_1 & j_2 & p_3 \\ p_1 & p_2 & \tilde{j}_3 \end{pmatrix}$ a twisted $3j$-symbol of $\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$.

Similarly the twisted symbols $\begin{pmatrix} j_1 & p_2 & j_3 \\ p_1 & \tilde{j}_2 & p_3 \end{pmatrix}$ and $\begin{pmatrix} p_1 & j_2 & j_3 \\ \tilde{j}_1 & p_2 & p_3 \end{pmatrix}$ are defined in the same way. We also write
$$\begin{pmatrix} j_1 & p_2 & p_3 \\ p_1 & \tilde{j}_2 & \tilde{j}_3 \end{pmatrix}, \begin{pmatrix} p_1 & p_2 & j_3 \\ \tilde{j}_1 & \tilde{j}_2 & p_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_1 & j_2 & p_3 \\ \tilde{j}_1 & p_2 & \tilde{j}_3 \end{pmatrix}$$
for
$$\sum_{p_2,p_3} \begin{pmatrix} \tilde{j}_2 \\ p_2 \end{pmatrix} \begin{pmatrix} \tilde{j}_3 \\ p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}, \sum_{p_1,p_2} \begin{pmatrix} \tilde{j}_1 \\ p_1 \end{pmatrix} \begin{pmatrix} \tilde{j}_2 \\ p_2 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$
and
$$\sum_{p_1,p_3} \begin{pmatrix} \tilde{j}_1 \\ p_1 \end{pmatrix} \begin{pmatrix} \tilde{j}_3 \\ p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}, \sum_{p_1,p_3} \begin{pmatrix} \tilde{j}_1 \\ p_1 \end{pmatrix} \begin{pmatrix} \tilde{j}_3 \\ p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$
respectively.
Lastly we write
\[
\begin{pmatrix}
p_1 & p_2 & p_3 \\
\tilde{j}_1 & \tilde{j}_2 & \tilde{j}_3
\end{pmatrix}
\]
for
\[
\sum_{p_1',p_2',p_3'} \begin{pmatrix}
\tilde{j}_1' \\
p_1'
\end{pmatrix}
\begin{pmatrix}
\tilde{j}_2' \\
p_2'
\end{pmatrix}
\begin{pmatrix}
\tilde{j}_3' \\
p_3'
\end{pmatrix}
\begin{pmatrix}
j_1 & j_2 & j_3 \\
p_1' & p_2' & p_3'
\end{pmatrix}.
\]

It is easy to check that the symmetric properties of \(3j\)-symbols in (6.27) also hold for the twisted \(3j\)-symbols.

### 6.4 Properties of The Twisted \(6j\)-Symbols

Now we are ready to present the following main results of Chapter 6.

**Theorem 6.7.** Let \(G\) be a quasi simply reducible group. For given irreducible representations \(V_j, V_k\) and \(V_l\) of \(G\), let \(V_a, V_b\) and \(V_c\) be irreducible components of \(\widetilde{\text{Hom}}(V_j, V_k), \text{Hom}(V_k, V_l)\) and \(V_b V_a\) respectively. Then we have

1. \[
\sum_{p_1,p_2,s_1,m_1,m_2} C_{p_1 p_2 u}^{\text{bac}} C_{s_1 m_2 p_1}^{\text{klb}} C_{m_1 s_1 p_2}^{\text{jka}} C_{m_1 m_2 u}^{\text{jlc}} = R_{abc}^{jkl} C_{m_1 m_2 u}^{\text{jlc}}.
\]
2. \[
\frac{1}{\dim V_a \dim V_b \dim V_j} \sum_c R_{abc}^{jkl} R_{abc}^{jkl} = 1.
\]
3. \[
| R_{abc}^{jkl} | = | R_{bac}^{jkl} |.
\]

**Proof.** In order to prove the first formula, we start with Equation (4.17)
\[
\sum_{p_1,p_2,s_1} C_{p_1 p_2 u}^{\text{bac}} C_{s_1 m_2 p_1}^{\text{klb}} C_{m_1 s_1 p_2}^{\text{jka}} C_{m_1 m_2 u}^{\text{jlc}} = R_{abc}^{jkl} C_{m_1 m_2 u}^{\text{jlc}}.
\]

By multiplying \(C_{m_1 m_2 u}^{\text{jlc}}\) and taking a sum over \(m_1\) and \(m_2\) to the both sides of Equation (4.17), we obtain
\[
\sum_{p_1,p_2,s_1,m_1,m_2} C_{p_1 p_2 u}^{\text{bac}} C_{s_1 m_2 p_1}^{\text{klb}} C_{m_1 s_1 p_2}^{\text{jka}} C_{m_1 m_2 u}^{\text{jlc}} = \sum_{m_1,m_2} R_{abc}^{jkl} C_{m_1 m_2 u}^{\text{jlc}} C_{m_1 m_2 u}^{\text{jlc}} = R_{abc}^{jkl}.
\]
For the second formula, we use the first formula.

Then we have

\[
\sum_{c} R_{abc}^{jkl} R_{abc}^{-jkl} = \sum_{c_{1},c_{2},s_{1},m_{1},m_{2}} C_{p_{1}p_{2}u}^{bac} C_{s_{1}m_{1}p_{2}}^{klb} C_{m_{1}s_{1}p_{1}}^{jka} C_{m_{2}s_{2}p_{2}}^{jlc} \times C_{p_{1}p_{2}u}^{bac} C_{s_{1}m_{1}p_{2}}^{klb} C_{m_{1}s_{1}p_{1}}^{jka} C_{m_{2}s_{2}p_{2}}^{jlc} = \sum_{p_{1},p_{2},s_{1},m_{1},m_{2}} C_{m_{1}s_{1}p_{1}}^{jka} C_{s_{1}m_{1}p_{2}}^{klb} C_{s_{1}m_{1}p_{1}}^{jka} C_{s_{1}m_{1}p_{1}}^{klb} = \sum_{p_{1},p_{2},s_{1},m_{1},m_{2}} C_{m_{1}s_{1}p_{1}}^{jka} C_{s_{1}m_{1}p_{2}}^{klb} C_{s_{1}m_{1}p_{1}}^{jka} C_{s_{1}m_{1}p_{1}}^{klb} = \sum_{p_{1},p_{2},s_{1},m_{1},m_{2}} \dim V_{a} \dim V_{j} \dim V_{b}.
\]

In order to show the last formula, we take the complex conjugate to the both sides of the first formula.

Then, we have

\[
\sum_{p_{1},p_{2},s_{1},m_{1},m_{2}} C_{p_{1}p_{2}u}^{bac} C_{s_{1}m_{1}p_{1}}^{klb} C_{m_{1}s_{1}p_{2}}^{jka} C_{m_{2}s_{2}p_{2}}^{jlc} = R_{abc}^{jkl}. (6.32)
\]

If we write \( \left\{ \begin{array}{c} j \\ k \\ a \end{array} \right\} = \{ \begin{array}{c} b \\ c \\ l \end{array} \} \) for \( R_{abc}^{jkl} \), then Equation (6.32) yields

\[
\left\{ \begin{array}{c} j \\ k \\ a \end{array} \right\} = \sum_{p_{1},p_{2},s_{1},m_{1},m_{2},u} \sqrt{\dim V_{a} \dim V_{j} \dim V_{b}} \left\{ \begin{array}{c} b \\ a \\ \bar{c} \end{array} \right\} \left\{ \begin{array}{c} p_{1} \\ p_{2} \\ u \end{array} \right\} \times \left( \begin{array}{c} k \\ l \\ b \\ s_{1} \\ m_{2} \\ p_{1} \end{array} \right) \left( \begin{array}{c} j \\ k \\ \bar{a} \\ m_{1} \\ s_{1} \\ p_{2} \end{array} \right) \left( \begin{array}{c} j \\ l \\ \bar{c} \\ m_{1} \\ m_{2} \\ u \end{array} \right).
\]
Let us now write $Z$ for $\sqrt{\text{dim} V_a \text{dim} V_b \text{dim} V_c}$. Then by Theorem (6.5) and Definition (6.6), we have

\[
\left\{ \begin{array}{c}
  j & k & a \\
  b & c & l
\end{array} \right\} = \sum_{p_1,p_2,s_1, p_1, p_2, u, \lambda_1, \lambda_2, \lambda_3} Z \left( \begin{array}{ccc}
  b & a & c \\
  p_1 & p_2 & u
\end{array} \right) \left( \begin{array}{ccc}
  \bar{k} & \bar{l} & \bar{b} \\
  s_1 & m_2 & p_1
\end{array} \right) \\
	imes \left( \begin{array}{ccc}
  j & k & \bar{a} \\
  m_1 & s_1 & p_2
\end{array} \right) \left( \begin{array}{ccc}
  \bar{j} & \bar{l} & c \\
  \lambda_1 & \lambda_2 & \lambda_3
\end{array} \right) \left( \begin{array}{ccc}
  \bar{j} \\
  m_1 & \lambda_1
\end{array} \right) \left( \begin{array}{ccc}
  \bar{l} & c \\
  m_2 & \lambda_2 & u
\end{array} \right) \left( \begin{array}{ccc}
  \bar{l} & c \\
  \lambda_3
\end{array} \right)
\]

\[
= \sum_{p_1,p_2,s_1, \lambda_1, \lambda_2, \lambda_3} Z \left( \begin{array}{ccc}
  b & a & \lambda_3 \\
  p_1 & p_2 & c
\end{array} \right) \left( \begin{array}{ccc}
  \bar{k} & \lambda_2 & \bar{b} \\
  s_1 & \bar{l} & p_1
\end{array} \right) \\
	imes \left( \begin{array}{ccc}
  \lambda_1 & k & \bar{a} \\
  \bar{j} & s_1 & p_2
\end{array} \right) \left( \begin{array}{ccc}
  \bar{j} & \bar{l} & c \\
  \lambda_1 & \lambda_2 & \lambda_3
\end{array} \right)
\]

\[
= C_{abc} C_{k\bar{j}n} C_{l\bar{k}b} C_{\bar{j}l c} \sum_{p_1,p_2,s_1, \lambda_1, \lambda_2, \lambda_3} Z \left( \begin{array}{ccc}
  a & b & \lambda_3 \\
  p_2 & p_1 & c
\end{array} \right) \left( \begin{array}{ccc}
  k & \lambda_1 & \bar{a} \\
  s_1 & \bar{j} & p_2
\end{array} \right) \\
	imes \left( \begin{array}{ccc}
  \lambda_2 & \bar{k} & \bar{b} \\
  \bar{l} & s_1 & p_1
\end{array} \right) \left( \begin{array}{ccc}
  \bar{l} & \bar{j} & c \\
  \lambda_2 & \lambda_1 & \lambda_3
\end{array} \right)
\]

\[
= C_{abc} C_{k\bar{j}n} C_{l\bar{k}b} C_{\bar{j}l c} \left\{ \begin{array}{c}
  l & k & b \\
  a & c & j
\end{array} \right\}.
\]

The theorem now follows.

\[\square\]

**Remark 6.8.** It is possible to take a similar approach to this thesis for studying the structure constants of $S_G(\text{End}(V))$ over a simply reducible group. In this case, the situation becomes better than the case of a quasi simply reducible group because every irreducible representation is isomorphic to its dual representation. In this direction, Sage showed that the structure constants of $S_G(\text{End}(V))$ over a simply
reducible group are related to the vanishing of Wigner’s $6j$-symbols defined over a simply reducible group.
7. A Computational Example

In this example, we treat an example of the symmetric group $S_3$ and follow the notations that were used in Example (5.9).

First, we will endow an inner product with each irreducible representation of $S_3$. For the trivial representation $V_0$ and the signature representation $V_1$, we endow the usual complex inner product $(\cdot, \cdot)$ with these spaces as the $G$-invariant, positive-definite hermitian inner product. For the irreducible representation $V_2$, we give the $G$-invariant, positive-definite hermitian inner product $(x, y)_2 := \frac{1}{6} \sum_{\sigma \in S_3} (\sigma \cdot x, \sigma \cdot y)$ to $V_2$, where $(\cdot, \cdot)$ is the standard inner product on $\mathbb{C}^3$. Then it is easy to check that the inner product $(\cdot, \cdot)_2$ is the same inner product as the standard inner product $(\cdot, \cdot)$ on $\mathbb{C}^3$.

Next, we fix orthonormal bases

\[
\{e_0^0 = 1\}, \{e_1^1 = 1\} \quad \text{and} \quad \left\{ e_1^2 = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right), e_2^2 = \left( -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \right\} \tag{7.33}
\]

for $V_0, V_1$ and $V_2$ respectively, and we select the following $S_3$-module isomorphism $\theta_i$ between $V_i$ and $V_i^*$:

- $\theta_0 : V_0 \longrightarrow V_0^*$ defined by $\theta_0(e_1^0) = e_1^{0*}$, where $e_1^{0*}(e_1^0) = 1$,

- $\theta_1 : V_1 \longrightarrow V_1^*$ defined by $\theta_1(e_1^1) = e_1^{1*}$, where $e_1^{1*}(e_1^1) = 1$,

- $\theta_2 : V_2 \longrightarrow V_2^*$ defined by $\theta_2(e_1^2) = e_1^{2*}$ and $\theta_2(e_2^2) = e_2^{2*}$, where $e_1^{2*}(e_1^2) = \delta_{ij}$.

With the orthonormal basis of $V_2$ described in (7.33), we can explicitly describe $S_3$-module structure on $V_2$ as follows:
We remark that for an irreducible representation \( V \) of \( S_3 \) two \( S_3 \)-modules \( V_i \) and \( V_i^* \) have the same \( S_3 \)-module structure induced by the \( S_3 \)-module isomorphisms \( \theta_0, \theta_1 \) and \( \theta_2 \).

Let us now present the decompositions of \( \text{Hom}(V_i, V_j) \) into their components.

By virtue of Lemma (3.3), we can easily obtain the following decompositions of \( \text{Hom}(V_i, V_j) \):

\[
\begin{align*}
V_0^* \otimes V_0 & \simeq V_0, \quad V_0^* \otimes V_1 \simeq V_1, \quad V_0^* \otimes V_2 \simeq V_2, \\
V_1^* \otimes V_0 & \simeq V_1, \quad V_1^* \otimes V_1 \simeq V_0, \quad V_1^* \otimes V_2 \simeq V_2, \\
V_2^* \otimes V_0 & \simeq V_2, \quad V_2^* \otimes V_1 \simeq V_2, \quad V_2^* \otimes V_2 \simeq V_0 \oplus V_1 \oplus V_2.
\end{align*}
\]

(7.34)

For each decomposition in (7.34), we choose the following \( S_3 \)-module imbeddings:

- \( \psi_0^{00} : V_0 \hookrightarrow V_0^* \otimes V_0 = \mathbb{C} \langle e_1^{0*} \otimes e_1^0 \rangle \) defined by \( \psi_0^{00}(e_1) = e_1^{0*} \otimes e_1^0 \),
- \( \psi_1^{01} : V_1 \hookrightarrow V_0^* \otimes V_1 = \mathbb{C} \langle e_1^{0*} \otimes e_1^1 \rangle \) defined by \( \psi_1^{01}(e_1) = e_1^{0*} \otimes e_1^1 \),
- \( \psi_2^{02} : V_2 \hookrightarrow V_0^* \otimes V_2 = \mathbb{C} \langle e_1^{0*} \otimes e_1^2 \rangle + \mathbb{C} \langle e_1^{0*} \otimes e_2^2 \rangle \) defined by \( \psi_2^{02}(e_1) = e_1^{0*} \otimes e_1^2 \) and \( \psi_2^{02}(e_2) = e_1^{0*} \otimes e_2^2 \),
- \( \psi_1^{10} : V_1 \hookrightarrow V_1^* \otimes V_0 = \mathbb{C} \langle e_1^{1*} \otimes e_1^0 \rangle \) defined by \( \psi_1^{10}(e_1) = e_1^{1*} \otimes e_1^0 \),
- \( \psi_0^{11} : V_0 \hookrightarrow V_1^* \otimes V_1 = \mathbb{C} \langle e_1^{1*} \otimes e_1^1 \rangle \) defined by \( \psi_0^{11}(e_1) = e_1^{1*} \otimes e_1^1 \),
- \( \psi_2^{12} : V_2 \hookrightarrow V_1^* \otimes V_2 = \mathbb{C} \langle e_1^{1*} \otimes e_1^2 \rangle + \mathbb{C} \langle e_1^{1*} \otimes e_2^2 \rangle \) defined by \( \psi_2^{12}(e_1) = e_1^{1*} \otimes e_2^2 \) and \( \psi_2^{12}(e_2) = -e_1^{1*} \otimes e_1^2 \).
• \( \psi_{20} : V_2 \hookrightarrow V_2^* \otimes V_0 = \mathbb{C} (e_1^{2*} \otimes e_1^0) + \mathbb{C} (e_2^{2*} \otimes e_1^0) \) defined by \( \psi_{20}^0(e_1^0) = e_1^{2*} \otimes e_1^0 \) and \( \psi_{20}^0(e_2^0) = e_2^{2*} \otimes e_1^0. \)

• \( \psi_{21} : V_2 \hookrightarrow V_2^* \otimes V_1 = \mathbb{C} (e_1^{2*} \otimes e_1^1) + \mathbb{C} (e_2^{2*} \otimes e_1^1) \) defined by \( \psi_{21}^1(e_1^1) = e_2^{2*} \otimes e_1^1 \) and \( \psi_{21}^1(e_2^1) = -e_1^{2*} \otimes e_1^1. \)

• \( \psi_{22} : V_0 \hookrightarrow V_2^* \otimes V_2 = \mathbb{C} (e_1^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^2) + \mathbb{C} (e_2^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^1) \) defined by \( \psi_{22}^2(e_1^2) = e_1^{2*} \otimes e_2^2 + e_2^{2*} \otimes e_2^2, \)

• \( \psi_{22} : V_1 \hookrightarrow V_2^* \otimes V_2 = \mathbb{C} (e_1^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^2) + \mathbb{C} (e_2^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^1) \) defined by \( \psi_{22}^2(e_1^1) = e_1^{2*} \otimes e_2^2 - e_2^{2*} \otimes e_2^1, \)

• \( \psi_{22} : V_2 \hookrightarrow V_2^* \otimes V_2 = \mathbb{C} (e_1^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^2) + \mathbb{C} (e_2^{2*} \otimes e_1^2) + \mathbb{C} (e_2^{2*} \otimes e_2^1) \) defined by

\[
\psi_{22}^2(e_1^2) = 2\sqrt{-1} (e_1^{2*} \otimes e_2^2) + 2\sqrt{-1} (e_2^{2*} \otimes e_1^2)
\]

and

\[
\psi_{22}^2(e_2^2) = 2\sqrt{-1} (e_1^{2*} \otimes e_1^2) - 2\sqrt{-1} (e_2^{2*} \otimes e_2^2).
\]
Then our chosen imbeddings \( \psi_{ij}^k \) give the following Clebsch-Gordan coefficients of the symmetric group \( S_3 \):

\[
\begin{align*}
C_{111}^{000} &= 1, & C_{111}^{001} &= 0, & C_{112}^{002} &= 0, & C_{112}^{010} &= 0, \\
C_{111}^{011} &= 1, & C_{112}^{012} &= 0, & C_{111}^{020} &= 0, & C_{121}^{020} &= 0, \\
C_{111}^{021} &= 0, & C_{121}^{022} &= 0, & C_{111}^{022} &= 1, & C_{121}^{022} &= 0, \\
C_{112}^{022} &= 1, & C_{111}^{100} &= 0, & C_{111}^{101} &= 0, & C_{112}^{102} &= 0, & C_{112}^{112} &= 0, \\
C_{111}^{110} &= 1, & C_{111}^{111} &= 0, & C_{112}^{112} &= 0, & C_{112}^{120} &= 0, \\
C_{121}^{120} &= 0, & C_{121}^{121} &= 0, & C_{111}^{122} &= 0, & C_{121}^{122} &= 1, \\
C_{112}^{122} &= -1, & C_{122}^{122} &= 0, & C_{211}^{200} &= 0, & C_{211}^{200} &= 0, & C_{211}^{201} &= 0, \\
C_{211}^{201} &= 0, & C_{211}^{202} &= 1, & C_{211}^{202} &= 0, & C_{211}^{202} &= 0, \\
C_{211}^{210} &= 0, & C_{211}^{210} &= 0, & C_{211}^{211} &= 0, & C_{211}^{212} &= 0, \\
C_{211}^{221} &= 0, & C_{222}^{212} &= 0, & C_{222}^{212} &= 0, & C_{222}^{212} &= 0, \\
C_{222}^{221} &= 1, & C_{212}^{222} &= -1, & C_{212}^{222} &= 0, & C_{221}^{222} &= 0. \\
\end{align*}
\]

With these Clebsch-Gordan coefficients of the group \( S_3 \), let us calculate the twisted 6j-symbols \( R_{abc}^{jkl} \) of \( S_3 \). We will not present the whole list of \( R_{abc}^{jkl} \). Instead, we will examine two important examples.

Let us first consider the case of \( Hom(V_1, V_2) \simeq V_1^* \otimes V_2 \).

We already checked that there is only one imbedding \( \psi_2^{12} : V_2 \hookrightarrow V_1^* \otimes V_2 \) given by \( \psi_2^{12}(e_2^2) = e_1^1 \otimes e_2^2 \) and \( \psi_2^{12}(e_1^2) = -e_1^1 \otimes e_1^2 \) up to a scalar multiplication.

Thus from the following diagram

\[
\begin{array}{ccc}
V_c & \xrightarrow{\psi_2^{12}} & V_1 \otimes V_2 \\
\downarrow & & \downarrow \\
Hom(V_2, V_1) & \xrightarrow{\psi_1^{01} \otimes \psi_2^{20}} & Hom(V_0, V_1) \otimes Hom(V_2, V_0),
\end{array}
\]
we immediately obtain that $R_{210}^{201} = R_{211}^{201} = 0$ because $\psi_{0}^{12} = \psi_{1}^{12} = 0$. Actually the values of $R_{210}^{jkl}$ and $R_{211}^{jkl}$ are equal to 0 for all $j, k$ and $l$.

If we let $V_c = V_2$, then we have
\[
(\tilde{m} \circ (\psi_{1}^{01} \otimes \psi_{2}^{20}) \circ \psi_{12}^{12})(e_1^2) = \mu_{11}^{0} e_2^{2*} \otimes e_1^1 = e_2^{2*} \otimes e_1^1,
\]
and
\[
(\tilde{m} \circ (\psi_{1}^{01} \otimes \psi_{2}^{20}) \circ \psi_{12}^{12})(e_2^2) = -\mu_{11}^{0} e_1^{2*} \otimes e_1^1 = -e_1^{2*} \otimes e_1^1.
\]
Hence from Equation (4.16) we have $R_{212}^{201} = 1$.

Therefore we can see that
\[
V_1 V_2 = V_2 = \bigoplus_{\{c \in R_{210}^{201} \neq 0\}} V_c
\]
as stated in Theorem (4.13).

Another important example is the case where $Hom(V_2, V_2) \simeq V_2^* \otimes V_2$. In this case, all possible imbeddings are

1. $\psi_{0}^{22} : V_0 \hookrightarrow V_2^* \otimes V_2$ given by $\psi_{0}^{22}(e_1^0) = e_1^{2*} \otimes e_1^2 + e_2^{2*} \otimes e_2^2$,
2. $\psi_{1}^{22} : V_1 \hookrightarrow V_2^* \otimes V_2$ given by $\psi_{1}^{22}(e_1^1) = e_1^{2*} \otimes e_2^2 - e_2^{2*} \otimes e_1^2$,
3. $\psi_{2}^{22} : V_2 \hookrightarrow V_2^* \otimes V_2$ given by $\psi_{2}^{22}(e_1^2) = 2\sqrt{-1}(e_1^{2*} \otimes e_1^2) - 2\sqrt{-1}(e_2^{2*} \otimes e_1^2)$ and $\psi_{2}^{22}(e_2^2) = 2\sqrt{-1}(e_1^{2*} \otimes e_2^2) - 2\sqrt{-1}(e_2^{2*} \otimes e_2^2)$.

Similarly, by considering the following diagram
\[
\begin{array}{ccc}
V_2 & \xrightarrow{\psi_{2}^{22}} & V_2 \otimes V_2 \\
\downarrow & & \downarrow \\
Hom(V_2, V_1) & \leftarrow \tilde{m} & Hom(V_2, V_1) \otimes Hom(V_2, V_2),
\end{array}
\]
we have
\[
(\tilde{m} \circ (\psi_{2}^{21} \otimes \psi_{2}^{22}) \circ \psi_{2}^{22})(e_1^2) = 8^* e_2^2 \otimes e_1^1,
\]
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which implies $R^{221}_{222} = 8$.

On the other hand, if we consider the following diagram

\[
\begin{array}{ccc}
V_2 & \xrightarrow{\psi_2^{22}} & V_2 \otimes V_2 \\
\downarrow & & \downarrow \\
\Hom(V_1, V_2) & \xleftarrow{\tilde{m}} & \Hom(V_2, V_2) \otimes \tilde{\Hom}(V_1, V_2),
\end{array}
\]

we obtain

\[
(\tilde{m} \circ (\psi_2^{22} \otimes \psi_2^{12}) \circ \psi_2^{22}) (e_1^2) = 8^* e_1^1 \otimes e_2^2,
\]

which implies $R^{122}_{222} = 8$. 

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References


Vita

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