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Stochastic demand forecast and inventory management of a seasonal product a supply chain system

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STOCHASTIC DEMAND FORECAST AND INVENTORY MANAGEMENT OF A SEASONAL PRODUCT IN A SUPPLY CHAIN SYSTEM

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ABSTRACT

Estimation of seasonal demand prior to an active demand season is essential in supply chain management. The business cycle of the seasonal demand is divided into two stages: stage-1, the slow-demand period, and stage-2, the peak-demand period. The focus here is to determine an appropriate demand forecast for the peak-demand period. In the first set of forecasting model, a standard gamma and an inverse gamma prior distribution are used to forecast demand. The parameters of the prior model are estimated and updated based on current observation using Bayesian technique. The forecasts are derived for both complete and incomplete datasets. The second set of forecast is derived by ARIMA method using Box-Jenkins approaches. A Bayesian ARIMA is proposed to forecast demand from incomplete dataset. A partial dataset of a seasonal product, collected from the US census bureau, is used in the models.

Missing values in the dataset often arise in various situations. The models are extended to forecast demand from an incomplete dataset by the assumption that the original dataset contains missing values. The forecast by a multiplicative exponential smoothing model is used to compare all the forecast. The performances are tested by several error measures such as relative errors, mean absolute deviation, and tracking signals. A newsvendor inventory model with emergency procurement options and a periodic review model are studied to determine the procurement quantity and inventory costs. The inventory cost of each demand forecast relative to the cost of actual demand is used as the basis to choose an appropriate forecast for the dataset.

This study improves the quality of demand forecasts and determines the best forecast. The result reveals that forecasting models using Bayesian ARIMA model and Bayesian probability models perform better. The flexibility in the Bayesian approaches allows wider variability in the model parameters helps to improve demand forecasts. These models are particularly useful when

past demand information is incomplete or limited to few periods. Furthermore, it was found that improvements in demand forecasting can provide better cost reductions than relying on inventory models.

CHAPTER 1

INTRODUCTION

Demand forecasting includes the prediction, projection or estimation of expected demand of the products over a specified future time period. The demand of seasonal products frequently changes in the marketplace. As soon as the main selling season passes, the excessive inventories of the product are devalued greatly. On the other hand, if the product supplies were relatively short, a direct sale loss occurs. Therefore, demand planning is considered the first step of a supply chain planning process, which provides a continuous link to manage the inventory position and the product demand.

Forecasting is an essential tool for making strategic demand planning. In this study, a number of demand forecasting models are studied to predict demand of a seasonal product for an active sales period. Forecasting accuracy may be measured using several indicators, such as relative error, mean absolute deviation and tracking signals. After forecasts are derived, the inventory quantity for a target business season can be obtained based on these demand forecasts. The total inventory cost of the product can be determined using a dynamic optimization technique. This result can be used as an alternative measure to decide the best forecasting model that provides the minimum inventory cost for the target period.

1.1 Study Context

Market demands of most products remain uncertain until the selling season begins. In most of today’s business environment, seasonality is an important feature. Many products have seasonal effects. The life cycles of these seasonal products are short and the demands are uncertain. It is often found that demand of seasonal products becomes significant only in the specific period in a year. For example, the demand of winter apparel, fashion goods, Christmas gift products are higher during specific seasons and hold seasonality, trends, or cyclic demand pattern. Moreover, future demand may not follow the historical pattern of the past demand, which may imply different predictions at different time period. Therefore, demand planning for seasonal and short life products is considered a vital component for an effective business.

The most known forecasting techniques currently available are based on extrapolation of historical demand data. For accurate forecasting, it is important to estimate the parameters of forecasting models with the most recent demand information and forecast can then be updated as new demand information becomes available. If Bayesian methods are used in forecasting algorithms, the prior knowledge about the future demand and the current sale information can be incorporated to forecast demand. In business, there are always flows of products in the inventories since the products from the stores are demanded constantly. Orders are placed prior to selling season and products are moved to meet demand. A typical supply chain structure is illustrated in Figure 1.1, where the products flow from manufacturers through distributors and retailers to consumers and the demand flows back. The demand forecasting and inventory models can be so constructed that any member of a supply chain may use the models for forecast processing and inventory deployment prior to the selling period.

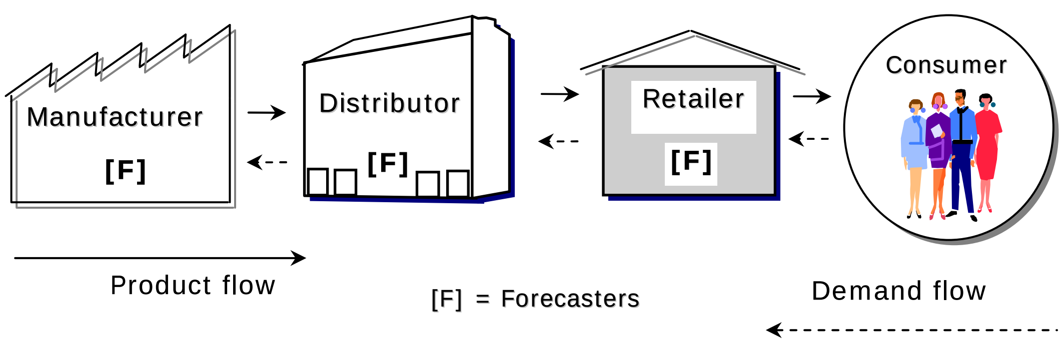


Figure 1.1: Supply-demand flow system in a supply chain

1.2 Problem Statement

In this study, the problem is to find the demand forecast of a seasonal product and to find the best forecast to anticipate the right demand for a target selling season. The demand of the seasonal products increases as the main demand season approaches. Therefore, seasonal demand always occurs in two stages: slow demand period and busy demand period. For the seasonal product considered, the business planning horizon is divided into two stages: s*tage*-1, a prior demand period and *stage*-2, the posterior demand period. The focus is to forecast demand for the stage-2 period. In the forecasting process, the demand data is collected from the past seasons. Current sales of the product are observed at stage-1 of the forecasting year and the forecasts are made prior to the main demand season. The initial sales at a business cycle start at t_1 time. After demand is observed at stage-1, the forecast processing and orders placement are performed prior to t_2 time. The product receiving and peak selling continues throughout stage-2 period. The procurement plans are also anticipated so that demand can be delivered on time during the selling period. The time-related activities at different stages of a business cycle are shown in Figure 1.2.

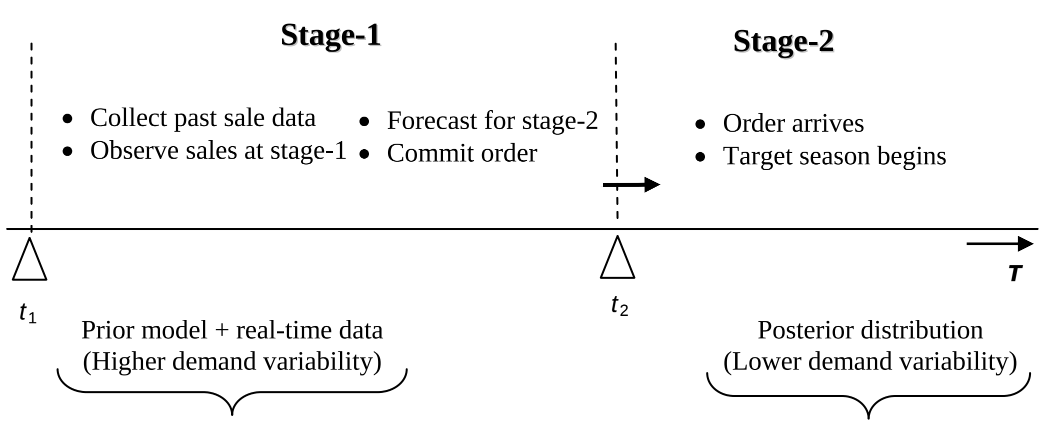


Figure 1.2: A two-stage inventory model

1.3 Research Goal

The goal of this study is to forecast demand of a seasonal product for active demand periods using various forecasting models and to adopt the best forecasting technique resulting in minimum errors and inventory costs. The improvement in demand forecast provides potential cost reductions and assists a decision manager to determine the best demand planning for the active demand period of a seasonal product.

1.4 Research Objectives

This study is to create models to predict future demand of a seasonal product for target sale season using improved forecast information so that demand planning can be performed as precisely as possible with minimum cost. The forecast analysis focuses the following issues:

(a) To forecast seasonal demand of a product using non-negative probability distribution model with Bayesian techniques,

(b) Demand forecast with time-series model using autoregressive integrated moving average (ARIMA), and Bayesian sampling-based ARIMA models. These models are compared with the forecast derived by multiplicative exponential smoothing model, and

(c) To find the best forecast using the inventory models to test the results that provides the minimum inventory cost.

The objectives of the above models are illustrated as following.

- **Model I: Demand Forecasts using Probability Distribution involving Bayesian Techniques**

In this forecasting model, the demand process is described by the probability distribution where distribution parameters are unknown. A prior model is selected to describe the variation of demand over the periods. The objective of this model is two folds: First, to predict the unknown parameters of the demand model using the Bayesian approach and to forecast, using these estimated parameters. Second, as the past data series often contains missing values; the objective here is to extend the model to demonstrate the forecasting approach using data series that contains missing values. The activities of these models are shown in Figure 1.3.

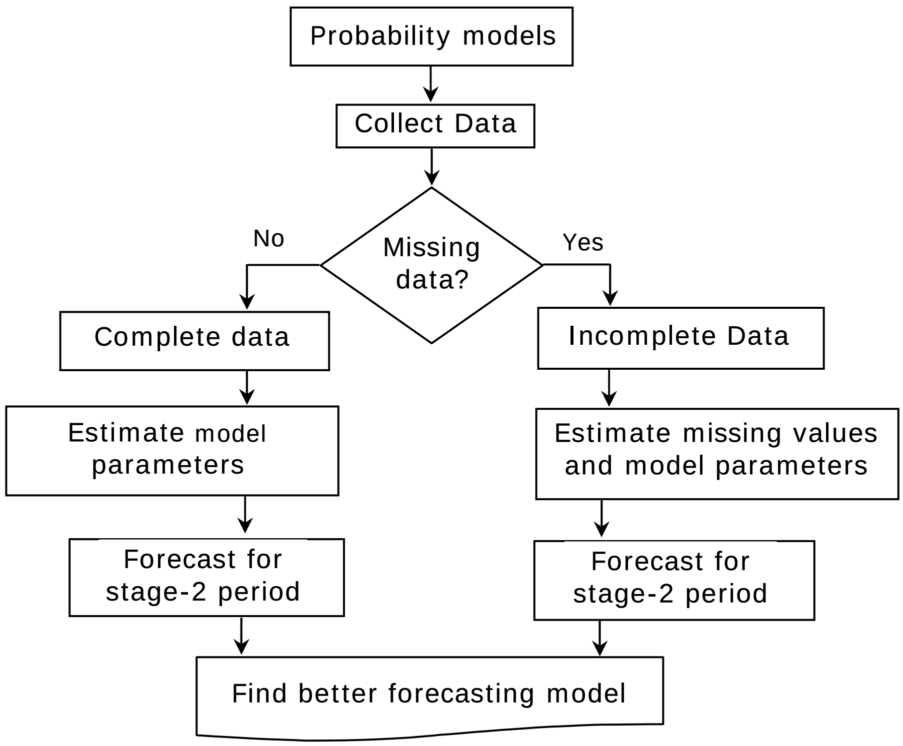


Figure 1.3: Activities of probability distribution models

- **Model II: Demand Forecasts using ARIMA and Bayesian ARIMA Techniques**

In time series forecasting models, past demands can be incorporated as a variable to find the deterministic trend of the seasonal demand. This study is to predict demand of a seasonal product using autoregressive integrated moving average (ARIMA) and Bayesian ARIMA models for the active demand season. The Bayesian ARIMA model is used here to forecast demand from a data series that contains *missing values*. The forecasts computed by these models are then compared with actual demand data. The activities of these models are demonstrated in Figure 1.4. A multiplicative exponential smoothing model is used as the base reference to compare the forecast derived by the probability distribution models and time series (ARIMA and Bayesian ARIMA) models.

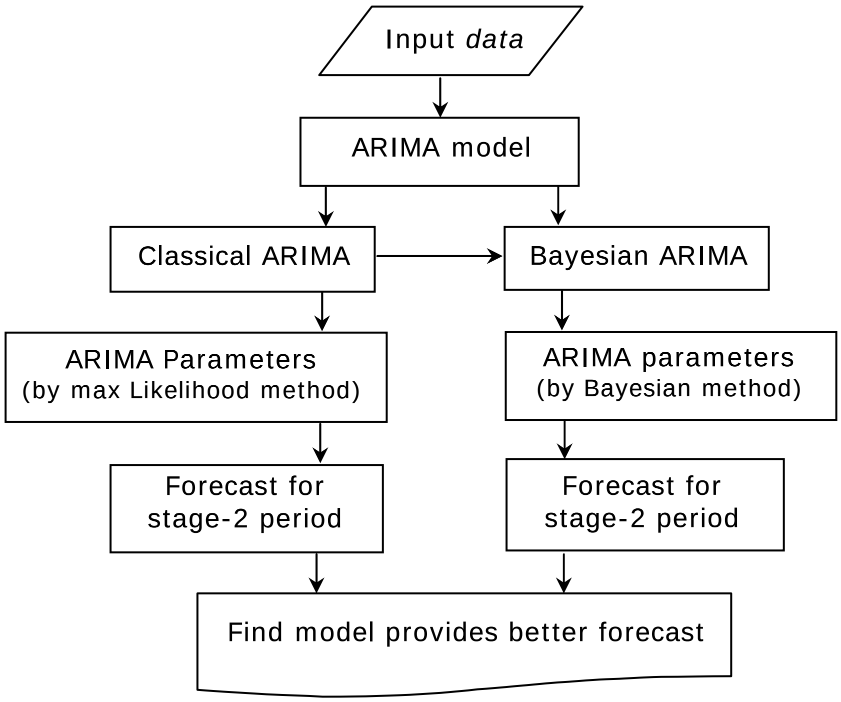


Figure 1.4: Activities of time series forecasting models

- **Model III: Inventory Models Applied to Forecasts**

The objective of this model is to test the best forecast through the application of inventory models by the results that provides minimum inventory cost. Iinventory costs are calculated based on the inventory quantity required for the target business season using each if the forecast derived by the above models. The outcome of this model is also to illustrate potential cost savings utilizing the improved demand forecast.

1.5 Solution Approach

Here, demand forecasts are performed using probability distribution model and ARIMA models. Bayesian statistical techniques are used in the forecasting algorithms, where past information is compiled and knowledge about the future events is gathered into a consistent format to develop the forecasting models. The prior models are used to make inferences about the unknown future demand. The prior models are updated to the posterior models based on the most recent demand observations as they become available. Thus, the updated parameters of the forecasting models improve the precision of the forecast. The forecasts derived by the parameters of these posterior models are then compared with the forecast obtained by the adaptive approaches of exponential smoothing forecasting techniques. The adaptive approach of exponential smoothing techniques is commonly used by the forecasters. A multiplicative exponential smoothing technique is used to serve as the base reference for the forecasting models. Forecasts derived by the above models are verified by comparing actual demand of the product and the forecast accuracy is tested by the results of several forecast measuring indicators such as percentage errors, mean absolute deviations, tracking signals.

Once the demand forecast is achieved, an alternative measure of the forecast is performed through the application of inventory models by determining the total inventory cost of the product for the target business season. A periodic review and an extended newsvendor model with an emergency procurement option are used to find the inventory quantity based on the demand forecasts. The inventory costs are derived by applying the dynamic optimization algorithm, which is then used as a further basis to compare the forecasting techniques. The best forecast is selected as the one that produces minimum error and inventory cost.

1.6 Scope and Opportunities

Accurate measures of demand uncertainty can be important in some applications. The forecasting model studied in this research can be applied to any sale forecasts and inventory management in a supply chain system. The models are especially applicable to forecast sales of seasonal products such as winter jacket, woolen apparel, air conditioner, and Christmas gifts. Products with short-life cycles are widespread in industries. The models can also be applied to forecast demand of products with short-life cycles such as fashion apparel, electronic products, mobile phones; new products (any new model electronic devices such as CD writers or DVD burner), or basic consumable products (gasoline, automobiles, clothing). Forecasting demand and inventory management are common in non-industrial businesses such as art exhibition tickets, or airline tickets prior to any special holydays or sports events. The proposed models can be applied to predict seasonal demand of such non-industrial businesses.

1.7 Actual Time Series Data

The dataset presented in the study was collected from the US corporate business matrices through 'US Census Bureau'. The dataset represents the partial demand of women woolen apparels in the US, supplied a leading apparel manufacturing country (India) over a time period from January 1996 to December 2005. The monthly demand from January 1996 to June 2005 is used to find the parameters of the forecasting models. Using the models, forecasts are made for the period from July to December 2005. For the missing values forecasting models, among the one hundred fourteen observations (January 1996 to June 2005), the demand for the six periods from July to December 2004 were considered unavailable and the forecast are made for the period from July to December 2005. Data series presenting demand from July to December 2005 are used to validate the forecasts obtained from the models.

1.8 Organization of the Dissertation

Apart from the introduction, the study is organized as follows. Chapter 2 provides a description of the relevant literature of demand forecasting and inventory models. In Chapter 3, probability distribution models are studied to forecast the seasonal demand of a product using

Bayesian approaches. Chapter 4 obtains forecast using time series forecasting method. An autoregressive integrated moving average (ARIMA) model and then Bayesian ARIMA models are presented. The performances of ARIMA forecasting models and a multiplicative exponential smoothing model are also presented in this chapter. In Chapter 5, the inventory costs are determined using periodic review and newsvendor inventory policies based on the forecasts attained by all forecasting models. The best forecasting model in terms of minimum inventory costs is established in this chapter. Chapter 6 summarizes the observations and conclusions of this research and possible future research.

CHAPTER 2

LITERATURE REVIEW

There are three topics in the literature that are related to demand forecast and inventory management of seasonal products. First, the literature uses probability distribution and Bayesian approaches to forecast demand - the demand considered here is stochastic, and characterized by the seasonality. The second uses the time series models to forecast demand for the future period. The forecasts are derived by the estimated parameters of the model. In the third, the inventory models are used to determine the order quantity and total inventory cost of the seasonal products prior to an active selling period. Inventory cost may demonstrate potential cost savings due to improved forecast. Following is the literature review of the above directions.

2.1 Demand Forecasts Using Bayesian Procedure

In literature, different aspects of demand forecasting problems with unknown demand distributions and information updates have been studied. For seasonal demand forecasting, starting from the 1990s, a Quick Response (QR) policy was adopted by many researchers. This policy is intended to reduce manufacturers' production time to respond to retailers order in a quicker way so that forecast can be improved by collecting more information about the future demand. Hammond (1990) and Fisher *et. al.* (1994) studied the QR policy with ski apparel (ski suits, ski pants, parkas, etc), and showed that forecast accuracy can be substantially improved by adopting QR policy. Fisher and Raman (1996) developed a forecasting model based on the sale trend using the early stage market sales data to reduce the uncertainty of the future demand under QR ordering system. Iyer and Bergen (1997) studied demand forecast by collecting the demand information of a preseason product to forecast the actual demand of a seasonal product using Bayesian approaches. They proposed that the demand process of the fashion apparel follows normal distribution and presented the improvement of demand forecast due to Bayesian information update in forecasting process. Agrawal and Smith (1996) used negative binomial distribution (NBD) for the demand model and suggested that NBD model provides a better fit than the normal or Poisson distributed data. They developed a parameter estimation method for the demand model in which sales are truncated at a fixed point. Cachon (2000) used the negative binomial distribution model to analyze the demand of the fashion goods where it is assumed that the demand process follows the Poisson distribution and demand rate varies according to a gamma distributed model. Gallego and Ozer (2001) discussed the improvement of demand forecast using early demand data for a regular selling season. Lau and Lau (1997), Gurnani and Tang (1999), Choi *et. al.* (2003) and Choi and Yan (2006) all studied two-stage demand of a fashion product under Bayesian approaches. Gurnani and Tang's (1999) considered a situation in which a retailer can pursue two orders prior to a target selling season. In their model, the forecast was updated by utilizing market information between the first and second orders. Choi *et. al.* (2003) presented a two-stage newsvendor model including Bayesian demand information updating approach. Their work extended Gurnani and Tang's (1999) model by including a cost component during the second ordering option. Choi and Yan (2006) investigated QR policy with two Bayesian models considering that the demand process follows normal distributions. Their first model considered the normal distribution with an unknown mean and a known variance, while, in the second model both an unknown mean and an unknown variance were assumed. The forecasts are then are compared for both models.

In that study, the proposed forecasting model is similar to Iyer and Bergen (1997) and Choi *et. al.* (2006) but different in the following ways: (a) unlike quick response policy, information about prior sales was not collected from the demand of a pre-seasonal product; (b) due to limited production capacity, it may be difficult for manufacturers to apply QR policy to reduce production lead times. A distinct beginning and ending of data collection and demand forecast period (stage-1 and stage-2) are considered; (c) instead of assuming normal demand process and normal prior models, the proposed model uses non-negative probability distributions to model the demand process.

2.2 Time Series Autoregressive Models

Time series forecasting models are increasingly applied to forecast demand and short-life product demand. Under an autoregressive moving average (ARMA) assumption, Kurawarwala and Matsuo (1998) estimated the seasonal variation of PC products demand using demand history of pre-season products and validated the models by checking the forecast performance with respect to actual demand. Miller and Williams (2003) incorporated seasonal factors in their model to improve forecasting accuracy while seasonal factors are estimated from multiplicative model. Hyndman (2004) extended Miller and Williams' (2003) work by applying various relationships between trend and seasonality under seasonal autoregressive integrated moving average (ARIMA) procedure. Forecast from eight different combination of trend and seasonality were compared in the model. The classical approach ARIMA becomes prohibitive, and in many cases it is impossible to determine a model, when seasonal adjustment order is high or seasonal adjustment diagnostics fails to indicate that time series is sufficiently stationary after seasonal adjustment. In such situations, the static parameters of the classical ARIMA model are considered the main restriction to forecasting high variable seasonal demand. Another restriction of the classical ARIMA approach is that it requires a large number of observations to determine the best fit model for a data series.

In the ARIMA model, if the Bayesian approaches are used, the restriction of the static values of the parameters is relieved by imposing the probability distributions to represent the parameters. Although the practices of Bayesian ARIMA models for seasonal forecast are more appropriate, the literature on Bayesian methods applied to ARMA time series is limited. Most of the applications are restricted to simple models such as autoregressive (AR) processes or forecast demand for a single or two future periods. In recent studies, de Alba (1993) derived an autoregressive model under Bayesian approach to forecast the quarterly GNP of Mexico and the quarterly unemployment rate for the United States. Huerta and West (1999) studied autoregressive models where Markov chain Monte Carlo (MCMC) process is used to forecast from AR processes. McCoy and Stephens (2004) extended Huerta and West's work (1999) and proposed ARMA models in which a frequency domain approach is adopted to identify the periodic behavior of time series.

In that study, first a classical ARIMA model is developed for a single dataset, and the Bayesian method is applied to the selected ARIMA model with the purpose of forecasting demand from the dataset that contains missing values. In the proposed model, the Bayesian ARIMA is studied to forecast seasonal demand when there are missing values in the data series. The Monte Carlo integration method based on Gibbs sampling algorithm is used for numerical computation to derive the model parameters. In the proposed model both ARIMA and Bayesian ARIMA models are used to forecast demand for an upcoming season.

2.3 Inventory Models

In several articles, Liau and Lau (1997), Eppen and Iyer (1997), Choi *et al.* (2003, 2006), inventory models were studied to determine the order quantity for a lead time and inventory cost of the seasonal demand. Liao and Shyu (1991) first introduced the concept of crushing cost to variable lead time for a fixed order quantity, where crushing cost is the cost that increases if the procurement lead time is reduced. Ben-Daya and Raouf (1994) extended Liao and Shyu's (1991) work by treating both order quantity and lead time as the decision variables. The inventory problem involving second ordering opportunity was studied by Khouja (1996). In his model, the order quantity is determined for a single period model with an emergency supply option, where he found that the total quantity under emergency supply option is smaller than that of the newsvendor model. Liau and Lau (1997) studied the reordering strategies for a seasonal product under a newsvendor model where a customer receives an order at the beginning of the season and places an additional order at some point during the season. They identified analytical conditions to maximize profits for using the second ordering opportunity. Eppen and Iyer (1997) described an inventory problem of the fashion industry. They determined the initial inventory quantity for a season and adjusted the procurement quantity after information updates using Bayesian techniques. Gurnani and Tang (1999), and Choi *et al.* (2003) investigated the optimal inventory quantity for seasonal products in which a retailer can order twice and the ordering cost at the second time is a variable.

Choi et al. (2004) and Tang *et. al.* (2004) studied multi-stage inventory decisions using the Bayesian process to update demand information in the successive stage. One of the key issues in these investigations is to find the optimal inventory quantity based on a newsvendor model with two supply options. However, the newsvendor model with two supply option may be extended by including two additional cost factors: (i) customer waiting time cost and (ii) expedite shipping cost. In the proposed model, the inventory quantity is determined by using an extended newsvendor model along with a periodic review inventory model based on several forecast datasets. The inventory cost of each forecast under each inventory model is used to demonstrate that improved forecast results in minimum cost. Thus, this study differs from the previous models by incorporating three objectives: (a) providing order quantity with two supply options, (b) deriving optimal inventory cost for each forecast and (c) establishing a basis for comparing demand forecasts.

2.4 Limitations of the Past Research

In most forecasting problems elegant mathematical models such as regression analysis, weighted moving average or exponential smoothing models were developed in which the forecasts are performed either by extrapolation or by averaging demand from the past data. In these historical data-driven forecasting models, forecasts often exhibit the demand trend of the past periods. Besides, the mathematical forecasting models do not permit integrating the subjective information or experts' views about the future demand in the forecasting algorithm. They perform badly if the data series contains mission values. Therefore, forecasts derived by past-data driven models may lead to a wrong conclusion about the future demand.

The demand of seasonal products varies from season to season, from one business cycle to the next. In time series forecasting techniques such as autoregressive models, the parameters of the models are always static. The static coefficient of a time series model cannot capture the uncertainty of the future demand. The imposition of static models implies a fixed relationship between the demand of the past season and the future. This may be considered the inflexibility of the time series forecasting models.

There exists a large amount of literature in both forecasting and inventory models. However, these two streams of research are traditionally separated. The research in forecasting problems usually ignores the inventory plans, while the research in inventory problems generally presumes that forecasts are given. Very little work has been accomplished on demand forecasting and inventory decision together to determine the best forecasting model that provides minimum inventory cost during an active demand season

2.5 Overcoming the Limitations

The forecast of seasonal demand is essential for inventory planning prior to an active selling season. In demand forecasting, a single model may not be adequate to represent a particular demand series for all times. Further, the chosen model may have been restricted to a certain class of time series. Therefore, a number of forecasting models are studied to provide wider choices to find the best demand forecast of a seasonal product.

In the first forecasting model, forecast by extrapolation is avoided by using a non-negative probability distribution to represent the seasonal demand. The Bayesian approach is applied to update the parameters of the forecasting model. Thus, the literature on forecasting models is extended by using probably distribution model involving Bayesian techniques. An ARIMA forecasting model is developed as the second model to forecast the seasonal demand. The parameters of the ARIMA model are static, but the static parameters can be enhanced by using the Bayesian techniques. In this study, the ARIMA model is extended to Bayesian ARIMA to capture the uncertainty of future demand. The use of Bayesian methods in both models provided additional facilities such as the capacity to use pre-designed models, communicating subjective or prior information, forecasting using little data or the data series that contains missing values. In inventory management literature, emergency procurement option is not always included in procurement strategy and inventory cost is not considered as a basis to find the best demand forecast. In the third model, a newsvendor model with emergency procurement option is used to determine the optimal inventory quantity and cost using several demand forecasts where the best one is chosen by the forecast that produces minimum inventory cost.

CHAPTER 3

BAYESIAN FORECASTING MODEL FOR SEASONAL DEMAND

Seasonal demand varies greatly during demand seasons. In this chapter the focus is to predict demand of a seasonal product for an active demand season using the Bayesian procedure. In the forecasting model, the demand process is described by the probability distribution model where the sales records of the past seasons are incorporated in the forecasting algorithm. First, the initial demand for the target selling period is estimated, and the initial demand is then updated using the Bayesian approach. In Bayesian analysis, demand process is viewed in terms of parameters of a probability distribution and forecast are obtained using updated parameters. Actual demand data is used in this forecasting model. The dataset used in the model is collected from US census bureau and is the partial demand of women woolen's apparel supplied by an apparel manufacturer country (India). It is shown in *Appendix A*.1 (Table A.1). A graphical presentation of the demand data is shown in Figure 3.1.

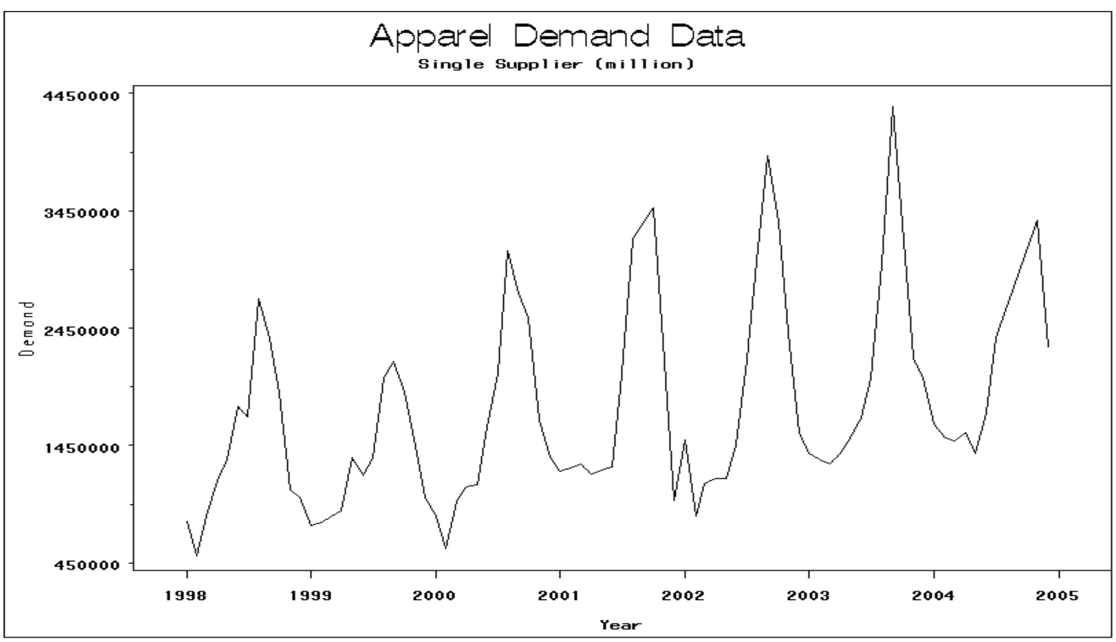


Figure 3.1: Variation of demand data for apparel product (Sources: U.S. Department of Commerce, Office of Textiles and Apparel)

The data series presented in Figure 3.1 includes two sources of demand variations, (a) variation between the periods within a business cycle, (b) variation between the business cycles. Due to the higher variability, the demand from the months of January to June is considered as slow demand period, (stage-1), and demand from the month of July to December, as the busy periods (stage-2). The focus is to forecast demand for the stage-2.

In many forecasting models, the demand process is described by the normal distribution, but the normal distribution may contain negative values. As demand quantity is always a positive number, it is more practical to use a non-negative distribution. In this study, a gamma distribution is chosen to represent the demand process of seasonal product. Comparing the maximum likelihood estimates among a number of non-negative distributions under the same parameterized condition, it is found that gamma distribution is the favored model for the selected data. The maximum likelihood estimate of the probability distributions is presented in *Appendix A*.2. A key feature of the Bayesian analysis is the use of the conjugate prior and posterior distribution for the exponential family parameters. A conjugate prior is mathematically convenient to follow a known posterior distribution as it belongs to same parametric family. An 'inverse gamma' distribution is selected as the conjugate prior for the gamma distribution (Gelman *et. al*., 2004). Following notations are used in this model:

Y_t	Demand at period t, (units/month)
δ	Observed demand rate at stage-1 (January to June at 2005),
μ, σ	Mean, and standard deviation of the demand distribution model
α, β	Shape and scale parameter for the prior inverse gamma distribution model
A, B	Shape and scale parameter for the posterior inverse gamma distribution model

3.1 Demand Model Formulation

Product demand is a continuous process. Y_t is directly dependent on time period t, where $t \geq 0$. The shape parameter of the gamma density is assumed linear in time t as $\alpha(t)$. The gamma density with shape parameter $\alpha(t) > 0$ and scale parameter $\beta > 0$ is given by

$$f_{Y_t}(y) = G(y \mid \alpha\, t, \beta) = \frac{1}{\Gamma(\alpha\, t)\beta}(y/\beta)^{\alpha t - 1}\exp(-y/\beta) \tag{3.1}$$

where $\mathrm{E}(Y_t) = (\alpha\, t)\beta$ and $\mathrm{Var}\,(Y_t) = (\alpha\, t)\beta^2$, (see *Appendix A*.3).

It is assumed that the expected value μ and standard deviation σ of the demand model is linear in time t, (Kallen and van Noortwijk, 2005). Thus, $\mathrm{E}(Y_t) = \mu\, t$ and $\mathrm{Var}\,(Y_t) = \sigma^2 t$. Using the coefficient of variation, $v = \sigma/\mu$, the parameters of the demand model are given by

$$\alpha = \frac{\mu^2}{\sigma^2} = \frac{1}{v^2}\text{, and} \tag{3.2}$$

$$\beta = \frac{\sigma^2}{\mu} = \mu v^2. \tag{3.3}$$

Using Equation (3.2) and (3.3), replacing shape parameter 'α' by $1/v^2$ and scale parameter 'β' by μv^2 in Equation (3.1), the gamma density is given by

$$f(y \mid \mu) = G\left(y \,\middle|\, \frac{t}{v^2}, \mu v^2 \right) = \frac{1}{\Gamma(t/v^2)(\mu v^2)}\left(\frac{y}{\mu v^2}\right)^{(t/v^2 - 1)} \exp\left(\frac{-y}{\mu v^2}\right). \tag{3.4}$$

If the coefficient of variation v is remained fixed, the only unknown variable remaining in Equation (3.5) is the parameter μ. According to Bayesian analysis, a distribution model is assigned for μ to capture the uncertainty of the future demand. An inverse gamma distribution, which is the conjugate family of the gamma distribution, is considered as the prior model. The definition of inverse gamma density (*IG*), $f_0(\mu) = IG(\mu \mid \alpha, \beta)$ is given by

$$f_0(\mu) = IG(\mu \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\left(\frac{1}{\mu}\right)^{\alpha+1} \exp\left(-\frac{\beta}{\mu}\right). \tag{3.5}$$

where μ is a positive random variable. It follows that $1/\mu \sim G(\alpha, \beta)$ with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. The posterior density of parameter μ is described in the next section.

3.2 Bayesian Procedure in Demand Model

The observed demand variable is y, and the prior distribution of the parameter μ is $f_0(\mu)$. According to Bayes' theorem, the posterior density of parameter μ is given by

$$f_1(\mu \mid y) = \frac{f(\mu, y)}{f(y)} \tag{3.6}$$

The joint probability $f(\mu, y)$ can be expressed by conditioning on μ as

$$f(\mu, y) = f(y \mid \mu) f_0(\mu). \tag{3.7a}$$

The marginal density function of y is given by

$$f(y) = \int_0^{\infty} f(y \mid \mu) f_0(\mu) d\mu. \tag{3.7b}$$

Substituting values from Equations (3.6b) and (3.6c) into Equation (3.6a) gives

$$f_1(\mu \mid y) = \frac{f(y \mid \mu) f_0(\mu)}{\int_0^{\infty} f(y \mid \mu) f_0(\mu) d\mu}. \tag{3.7c}$$

The steps to solve Equation (3.7c) are described in Proposition 3.1.

Proposition 3.1: The posterior density $f_1(\mu \mid y)$ may be written as

$$f_1(\mu \mid y) = IG\left(\mu \,\middle|\, \frac{t}{v^2} + \alpha,\ \ \frac{y}{v^2} + \beta\right).$$

Proof: Proposition (3.1) may be proved by following Equations (3.7a to 3.7c) in three steps.

The chronology of events to achieve the posterior density of parameter μ is described as the following:

Step 1: Derivation of joint probability distribution:

The joint probability $f(y \mid \mu).f_0(\mu)$ is expressed conditioning on μ, where $f(y \mid \mu)$ and $f_0(\mu)$ may be found in Equations (3.4) and (3.5), respectively.

$$f(y \mid \mu)\, f_0(\mu) = \frac{1}{\Gamma\left(\frac{t}{v^2}\right)\left(\mu v^2\right)}\left(\frac{y}{\mu v^2}\right)^{\left(t/v^2-1\right)} \exp\left\{\frac{-y}{\mu v^2}\right\} \times \frac{\beta^{\alpha}}{\Gamma(\alpha)}\left(\frac{1}{\mu}\right)^{1+\alpha} \exp\left\{-\frac{\beta}{\mu}\right\}$$

$$= \left(\frac{y}{v^2}\right)^{\left(t/v^2-1\right)} \frac{\beta^{\alpha}}{\Gamma\left(t/v^2\right)\Gamma(\alpha)}\left(\frac{1}{\mu}\right)^{\left(t/v^2+\alpha\right)+1} \exp\left\{-\frac{1}{\mu}\left(\frac{y}{v^2}+\beta\right)\right\} \qquad (3.8a)$$

Equation (3.8a) may be simplified as

$$f(y \mid \mu)\, f_0(\mu) = \frac{C}{\mu^{A+1}} \exp\left\{-\frac{1}{\mu} B\right\}, \qquad (3.8b)$$

where $C = \dfrac{\beta^{\alpha}}{\Gamma(t/v^2)\Gamma(\alpha)}\left(\dfrac{y}{v^2}\right)^{\left(t/v^2-1\right)}$, $A = \left(\dfrac{t}{v^2}+\alpha\right)$, and $B = \left(\dfrac{y}{v^2}+\beta\right)$. (3.9)

Step 2: Derivation of marginal density function:

The marginal density of y is obtained by integrating over μ and using Equation (3.8b):

$$\int_0^{\infty} f(y \mid \mu) f_0(\mu) d\mu = \int_0^{\infty} C\mu^{-A-1} \exp\left(-\frac{B}{\mu}\right) d\mu . \qquad (3.10)$$

Changing $w = B/\mu$ and $\left(-B/w^2\right)dw = d\mu$, then Equation (3.10) transforms to

$$\int_0^{\infty} f(y \mid \mu) f_0(\mu) d\mu \equiv C\int_0^{\infty}\left(\frac{w}{B}\right)^{A+1} \exp(-w)\frac{B}{w^2}\, dw$$

$$= C\frac{1}{B^A}\int_0^{\infty} w^{A-1} \exp(-w) dw \; = C\frac{\Gamma(A)}{B^A} \qquad (3.11)$$

where $\int_0^{\infty} w^{A-1} \exp(-w) dw = \Gamma(A)$ is the gamma function.

Step 3: Derivation of the posterior density using Bayes' theorem:

Substituting the values from Equations (3.8b) and (3.11) into Equation (3.7c), the posterior distribution is given by

$$f_1(\mu \mid y) = \frac{f(y \mid \mu) f_0(\mu)}{\int_0^\infty f(y \mid \mu) f_0(\mu) d\mu} = \left(\frac{B}{\mu}\right)^A \frac{1}{\mu} \frac{1}{\Gamma(A)} \exp\left(-\frac{B}{\mu}\right). \tag{3.12}$$

Equation (3.12) is an inverse gamma function with parameter *A* and *B*,

$$f_1(\mu \mid A, B) = IG(\mu \mid A, B). \tag{3.13}$$

Substituting values of the posterior parameters *A* and *B* from Equation (3.9) into Equation (3.13) yields

$$f_1(\mu \mid y) = IG\left(\mu \,\middle|\, \alpha + \frac{t}{v^2},\ \ \beta + \frac{y}{v^2}\right), \tag{3.14}$$

where α and β are the prior parameters, y is demand for period t, and v is coefficient of variation for the stage-2 period.

3.3 Application of Demand Model

The estimate of future demand for stage-2 period in year 2005 can be determined from the the posterior distribution. The Equation (3.14) is the posterior inverse gamma density with shape parameter $A > 0$ and scale parameter $B > 0$. In Equation (3.14), the component t, y_j, v are known values, which can be obtained from past demand records, but the parameter values of the prior distribution α and β are unknown. The values of α and β may be derived through the application of coefficient of variation (v), and the initial mean demand of each forecast period. The coefficient of variation $v = \sigma/\mu$, where the point estimate for μ is $\bar{y}_n$ and an unbiased estimator of σ is S_n for n data series. For inverse gamma prior distribution, mean is $\beta/(\alpha - 1)$, and variance is $\beta^2/(\alpha-1)^2(\alpha-2)$.

The coefficient of variation (v) is given by,

$$v = \frac{\beta}{(\alpha - 1)\sqrt{(\alpha - 2)}} \Bigg/ \frac{\beta}{(\alpha - 1)} = \frac{1}{\sqrt{(\alpha - 2)}} . \tag{3.15a}$$

After rearrangement, Equation (3.15a) becomes

$$\alpha = \frac{1}{v^2} + 2 . \tag{3.15b}$$

Once parameter α is known, parameter β can be estimated from mean, $\bar{y}_n = \beta/(\alpha - 1)$ as,

$$\beta = \bar{y}_n (\alpha - 1) . \tag{3.15c}$$

From Equation (3.9), the parameters of the posterior distribution are as follows

$$A = \left(\frac{t}{v^2} + \alpha \right), \tag{3.16a}$$

$$B = \left(\frac{y}{v^2} + \beta \right). \tag{3.16b}$$

3.4 Sub-Models

In business, there are many instances where market demand records contain missing values due to natural catastrophe such as hurricane or adverse economical conditions. To demonstrate a forecasting problem with incomplete data, a sub-model is presented with missing value assumption. The original model may be viewed in two sub-models: (a) Bayesian probability model (*B-P Model*), and (b) Bayesian probability model with incomplete data (*BP-I Model*). Both models are used to forecast demand for stage-2 (July to December) in 2005 using the data series from January 1996 to June 2005. The assumption in *BP-I model* is that the demand at stage-2 (July to December) in 2004 is not recorded. Therefore, forecast in BPI model is performed from the data series that contains six missing values. To project the missing values and the initial demand forecast, an approach is described in the following algorithm.

3.4.1 Algorithm 3.1 (Steps to Derive the Prior Values)

(i) Split the business cycle into two stages, (January to June, as stage-1, and July to December, as stage-2).

(ii) Calculate the average demand for stage-1 period, Δ_t and find the demand ratio, r_{tj}, for stage-2 period with respect to average of stage-1, that is, $r_{tj} = y_{tj} / \Delta_t$

(iii) Find the average demand ratios for each period at stage-2, $R_t = \sum_{t=1}^{n} r_{tj} / n$, which may be called demand factor.

(iv) Project missing values using demand factor R_t.

(v) Compute initial estimate of stage-2 demand (before Bayesian update) of the forecast {(n+1)th} year by multiplying Δ_{n+1} with the demand factor, R_{tj} as ($\Delta_{n+1}R_t$) .

The purpose in following steps (*i*) through (*v*) is two fold: (i) project values for missing data, and (ii) project initial demand at stage-2 period of forecast year and this initial estimate is updated in Bayesian techniques. The structure of Algorithm 3.1 is shown in Table 3.1.

Table 3.1: The structure of finding expected prior values

Obs.	Average Stage-1 (Δ_t)	Demand ratio, $r_{tj} = y_{tj} / \Delta_t$ (t = 1,.., n); (j = 7, 8,..., 12)					
		Jul	Aug	-	-	Nov	Dec
1	Δ_1	$r_{7,1}$	$R_{8,1}$	-	-	$r_{11,1}$	$r_{12,1}$
2	Δ_2	$r_{7,2}$	$r_{8,2}$	-	-	$r_{11,2}$	$r_{12,2}$
.	-	-	-	-	-	-	-
.	-	-	-	-	-	-	-
n	Δ_n	$r_{7,n}$	$r_{8,n}$	-	-	$r_{11,n}$	$r_{12,n}$
	$R_{tj} = \sum_{t=1}^{n} r_{tj} / n$	R_7	R_8	-	-	R_{11}	R_{12}
		Expected demand for forecast year 2005					
n+1	$\Delta_{(n+1)}R_t$	$\Delta_{n+1}R_7$	$\Delta_{n+1}R_8$	-	-	$\Delta_{n+1}R_{11}$	$\Delta_{n+1}R_{12}$

The flow diagram for the demand estimation process is shown in Figure 3.2

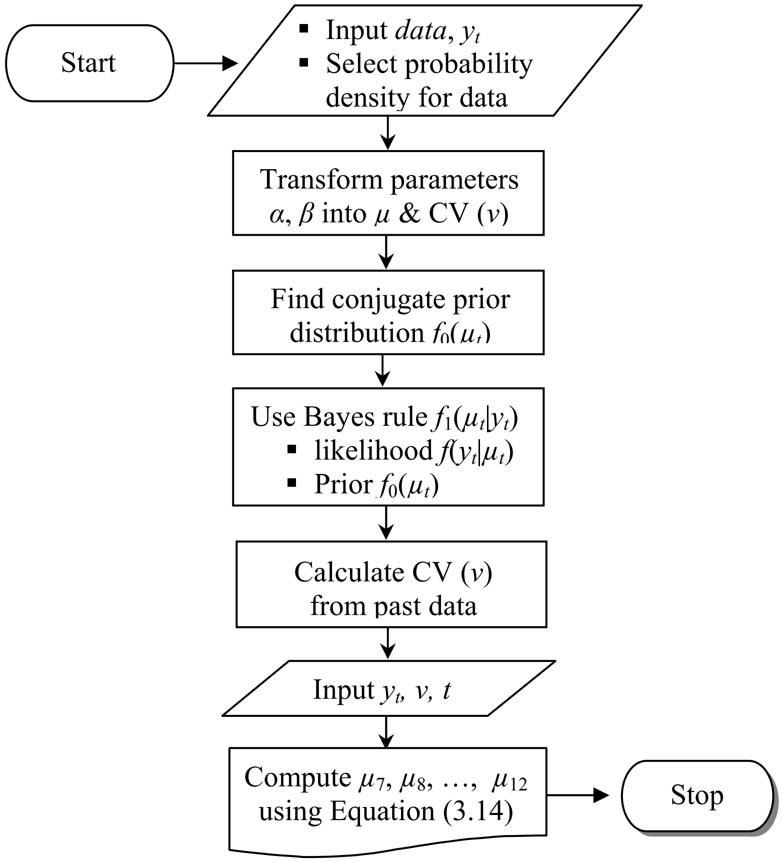


Figure 3.2: Flow diagram for Bayesian computation

3.4.2 Bayesian Probability Model (*B-P* Model)

Following is the structure illustrated in Table 3.1 and Algorithm 3.1, the initial projected demand for the forecast year (before Bayesian update) is shown in Table 3.2.

Table 3.2: Projected demand averaging the sample data (units in million)

Obs.	Year	Average Stage-1 (Δ_t)	Demand ratio at stage-2 periods (t = 1,.., n); (j = 7, 8,..., 12)					
			Jul	Aug	Sep	Oct	Nov	Dec
1	2000	1.03	2.00	3.01	2.65	2.45	1.61	1.31
.	2001	1.25	1.65	2.56	2.68	2.78	1.73	0.78
.	2002	1.21	1.80	2.49	3.24	2.77	1.99	1.29
(n-1)	2003	1.43	1.41	2.09	3.03	3.95	1.54	1.41
n	2004	1.55	1.53	2.00	3.11	3.57	2.17	1.48
	$R_{tj} = \sum_{t=1}^{n} r_{tj} / n$	-	1.68	2.43	2.94	3.11	1.81	1.25
			Expected demand for forecast year by $\Delta_{n+1}R_t$					
(n+1)	$\Delta_{n+1\,(=2005)}$	1.93	3.24	4.69	5.68	5.99	3.49	2.42

The parametric values of the prior and posterior distribution for each period at stage-2, under *B-P* model are shown in Table 3.3.

Table 3.3: Prior and posterior parameters derived by *B-P* model for 2005

Month	Prior parameters (units in million)				Posterior parameters (units in million)			
	Mean $\bar{y}$	CV (v)	α	β	Estimated $\hat{y}$	A	B	*Mean*
Jul	1.96	0.16	41.34	79.19	3.24	80.67	206.51	2.59
Aug	2.87	0.13	59.16	167.34	4.69	116.31	435.28	3.77
Sep	3.38	0.28	14.94	47.24	5.68	27.89	120.74	4.49
Oct	3.47	0.42	7.71	23.35	5.99	13.42	57.56	4.63
Nov	2.04	0.34	10.67	19.77	3.49	19.34	49.98	2.72
Dec	1.46	0.33	10.95	14.53	2.42	19.90	36.17	1.91

3.4.3 Sample Calculation (*B-P* Model)

For the month of July, the mean, $\bar{y}_{July}$ = 1.96, and coefficient of variation, v = 0.16. Using Equation (3.15b), (3.15c), the value of (α, β) is given by

$$\alpha_{Jul} = \frac{1}{v_{Jul}{}^2} + 2 = \frac{1}{0.16^2} + 2 = 41.34 \tag{3.17a}$$

$$\beta_{Jul} = \bar{y}_{Jul}(\alpha - 1) = 1.96(41.34 - 1) = 79.19 \text{ (million)} \tag{3.17b}$$

For the month of July, the initial estimate, $\hat{y}_{July}$ = 3.24. From Equations (3.16a) and (3.16b), the parameters of the posterior distribution are determined as:

$$A_{Jul} = \left(\frac{t}{v_{Jul}{}^2} + \alpha_{Jul} \right) = \frac{1}{0.16^2} + 41.34 = 80.67 \tag{3.18a}$$

$$B_{Jul} = \left(\frac{\hat{y}_{Jul}}{v_{Jul}{}^2} + \beta_{Jul} \right) = \frac{3.24}{0.16^2} + 79.19 = 206.5 \text{ (million).} \tag{3.18b}$$

The demand for the month of July in year 2005 is estimated by the parameters (A and B) of posterior distribution model. The mean of inverse gamma distribution is given by $B/(A-1)$.

By using the values A_{jul} and B_{Jul} from Equations (3.18a) and (3.18b), the mean demand for the month of July is estimated as $206.5/(80.67-1)=2.59$ (million).

3.4.4 Bayesian Probability Model with Incomplete Data (*BP-I* Model)

Missing data often arise in various settings includes market sales, industrial production, shipment arrival, new product trials. The forecast based on missing values can often result in biased and inefficient estimates. In *BP-I* model, the projections of missing values for stage-2 period in year 2004 and the initial demand estimate for stage-2 period in year 2005 are obtained by following Algorithm 3.1. The projected missing values and the initial demand for the forecast year (before Bayesian update) are shown in Table 3.4.

Table 3.4: Projected demand averaging the sample data (units in million)

Obs.	Year	Average Stage-1 (Δ_j)	Stage-2 y_t (t=7, 8, . . . ,12)					
			Jul	Aug	Sep	Oct	Nov	Dec
1	2000	1.03	2.00	3.01	2.65	2.45	1.61	1.31
.	2001	1.25	1.65	2.56	2.68	2.78	1.73	0.78
.	2002	1.21	1.80	2.49	3.24	2.77	1.99	1.29
(n-1)	2003	1.43	1.41	2.09	3.03	3.95	1.54	1.41
$R_t=\sum_{j=1}^{n-1} r_{tj}/(n-1)$		-	1.72	2.54	2.90	2.99	1.72	1.20
		Expected demand for n-th year by $\Delta_n R_t$						
(n) (2004)	Projected Missing values	1.55	2.66	3.94	4.50	4.63	2.66	1.86
	Actual demand	-	2.37	3.09	4.82	5.54	3.36	2.29
	Percentage error	-	-0.12%	-0.27%	0.07%	0.16%	0.21%	0.19%
(n+1) (2005)	Initial estimate	1.93	3.31	4.90	5.60	5.76	3.31	2.31

The mean demand, coefficient of variation, parameters of the prior model and the parameters of the posterior model for each period at stage-2 in 2005 under *BP-I* model are shown in Table 3.5.

Table 3.5: Prior and posterior parameters derived by *BP-I* model for 2005

	Prior parameters (units in million)		Posterior parameters (units in million)					
	mean	CV(v)	α	β	Initial estimate	A	B	*Mean*
Jul	2.01	0.19	30.26	58.67	3.31	58.52	152.19	2.60
Aug	2.99	0.18	33.69	97.97	4.90	65.38	253.14	3.87
Sep	3.34	0.26	16.58	52.04	5.60	31.15	133.62	4.29
Oct	3.35	0.39	8.65	25.62	5.76	15.29	63.93	4.18
Nov	1.94	0.27	15.63	28.43	3.31	29.26	73.55	2.51
Dec	1.39	0.28	14.34	18.66	2.31	26.69	47.18	1.77

Calculation procedure to obtain the values presented in Table 3.6 is similar to the sample calculation illustrated in sample calculation under Section 3.3.3. The graphical presentation of prior and posterior density for both *B-P* and *BP-I* models are shown in Figures (A.1 and A.2) in *Appendix A*. The results and validation of models are presented in the next section.

3.5 Forecasting Errors and Model Validity

In the forecasting procedure, a portion of the dataset is used to estimate the parameters of the model; the forecasts are then tested on data to validate the model. In the analysis, data points for the 7 years (1998-2004) are used to produce the forecasts for the 8th years (at stage-2 from July to December, 2005). To validate the forecasting models, the forecasts are compared with the original demand for the target forecast periods. The performance of forecasting models can be achieved by a number of error measure indicators such as relative (percentage) errors (PE_t), mean absolute deviation (MAD_t) and tracking signal (TS_t), where index t corresponds to a particular period (month). Tracking signal is the cumulative forecast error (running sum) with respect to *MAD* for a given time period. It measures the limit of *MAD* above or below the actual data. A comparison between *B-P model* and *BP-I model* with respect to the percentage error, mean absolute deviation, and tacking signal are shown in Table 3.6.

Table 3.6: *PE*, *MAD*, *TS* for *B-P* and *BP-I* models (units in million)

Models	Month	Forecast $\hat{y}_j$	Actual y_j	deviation $(y_j - \hat{y}_j)$	PE $\frac{\lvert y_j - \hat{y}_j \rvert}{y_j}$	MAD_t $\frac{\sum \lvert y_j - \hat{y}_j \rvert}{n}$	TS_t $\frac{\sum \lvert y_j - \hat{y}_j \rvert}{MAD_j}$
B-P	Jul	2.59	2.83	0.24	0.09	0.15	1.65
	Aug	3.77	3.33	-0.45	0.13	0.29	-1.50
	Sep	4.49	4.10	-0.39	0.09	0.33	-1.18
	Oct	4.63	5.11	0.48	0.09	0.36	1.31
	Nov	2.73	3.46	0.74	0.21	0.44	1.68
	Dec	1.91	2.28	0.37	0.16	0.43	0.86
			Average error =		0.13		
BP-I	Jul	2.60	2.83	0.23	0.08	0.10	2.35
	Aug	3.87	3.33	-0.54	0.16	0.32	-1.69
	Sep	4.29	4.10	-0.19	0.05	0.28	-0.67
	Oct	4.18	5.11	0.93	0.18	0.44	2.11
	Nov	2.51	3.46	0.95	0.27	0.54	1.75
	Dec	1.77	2.28	0.51	0.22	0.54	0.96
			Average error =		0.16		

The percentage error is computed as $PE_t = \lvert y_t - \hat{y}_t \rvert / \lvert y_t \rvert$, mean absolute deviation as $MAD_t = \sum_{t=1}^{n} \lvert y_t - \hat{y}_t \rvert / n$, and tacking signal as $TS_t = \sum_{t=1}^{n} (y_t - \hat{y}_t) / MAD_t$. The comparison of the forecasts derived from *B-P* and *BP-I* models with respect to the actual demand is shown in Figure 3.3.

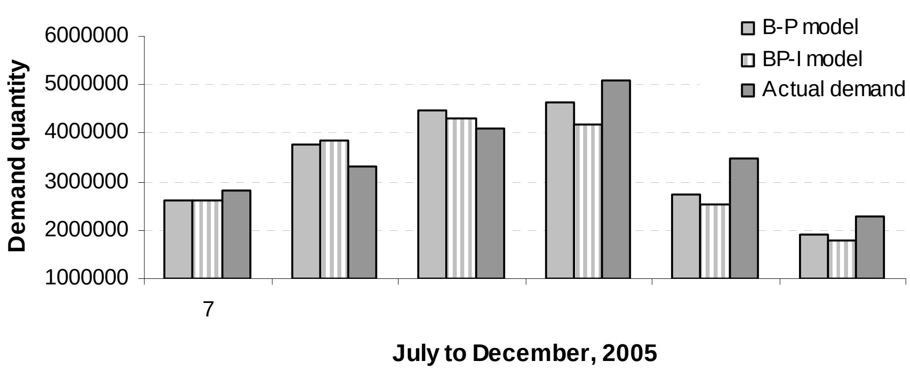


Figure 3.3: Comparison of forecasts with actual demand

The projection of the errors by *B-P* and *BP-I* models for each forecasted period *t*, (with j = 7, 8, …, 12) is shown in Figure 3.4. The 13th point represents the average of errors (*i.e.*, averaged over all 6-periods).

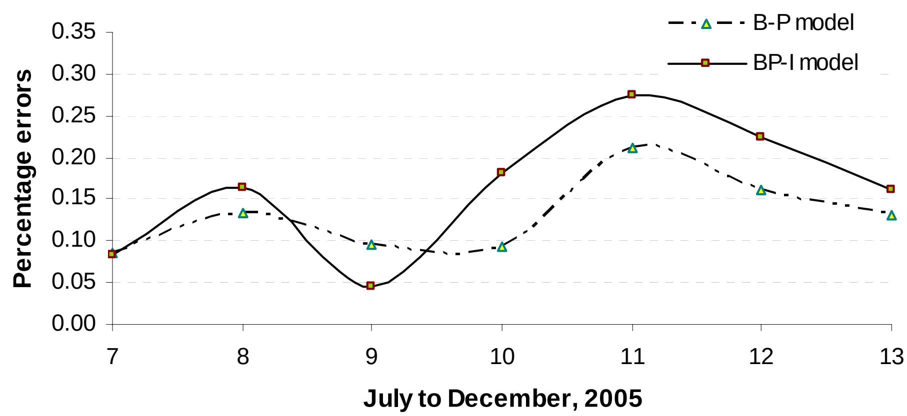


Figure 3.4: Error comparison of all models

The graphical presentation of tracking signals (*TS*) of the forecasting models over the test periods is presented in Figure 3.5.

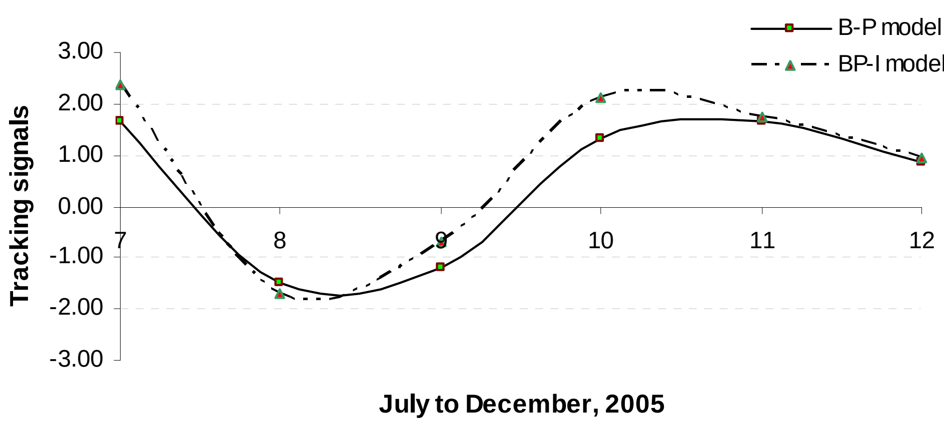


Figure 3.5: Summaries of tracking signals of the models

3.6 Summary

This model is a two-stage demand planning problem, which is particularly useful when the demand of the product is uncorrelated over the period. In this forecasting model, the demand process is described by the probability distribution where the distribution parameters are not known in advance. Based on the maximum likelihood estimate among several heavy tailed probability distributions, an appropriate distribution model is chosen for the data series. A conjugate prior model is selected to describe the variation of the demand rate over the periods. The parameters of the prior distribution are estimated from the data series collected in the past seasons. The model is further applied to forecast demand with the assumption that the data series contains missing values.

The original model was divided into two sub-models: Bayesian probability (*B-P*) model and Bayesian probability for incomplete data (*BP-I*) model. For *BP-I* model, the demand record at stage-2 (July to December) in 2004 was considered unavailable while forecasts are made for stage-2 in 2005. Proposition 3.1 was developed to estimate the missing values and the initial demand forecast. The Bayesian approach was used to update the demand forecasting for the stage-2 period in year 2005. The tracking signals of the forecast indicated that forecasts do not produce trends. Forecast errors for both sub-models were compared with respect to actual data set and it was concluded that *B-P* model provided better forecast.

CHAPTER 4

THE ARIMA APPROACH TO FORECASTING SEASONAL DEMAND

A fundamental element of a supply-chain management is the estimation of future demand. This model is specific to forecast seasonal demand from a time series where the demand of a period is correlated with the demand of the other periods in a business cycle. For example, the first month demand of a seasonal product during an active demand season indicates the progression about the future demand of the other period in that season. To forecast such demand, mathematical models such as autoregressive, moving average, exponential smoothing techniques are used to predict the future values by extrapolating the known data points. The autoregressive integrated moving average (ARIMA) procedure is the most sophisticated forecasting method in time series context (Pankratz, 1983; Vandaele, 1983). The focus of this chapter is to develop an ARIMA model to forecast the demand of a seasonal product.

The ARIMA is a type of time series forecasting technique developed by Box and Jenkins (1970), where the autoregressive (AR) and moving average (MA) terms are used to forecast the demand. To develop an ARIMA model, a sufficiently large dataset is required. The dataset used in this model is an actual time series demand data of a seasonal product collected from US census bureau (Table *A*.1, Appendix *A*). Two sub-models are studied based on ARIMA theory to forecast the demand: (a) Fundamental ARIMA (*F-ARIMA*) and (b) Bayesian sampling-based ARIMA (*BS-ARIMA*). The monthly data from January 1996 through June 2005 is used to construct the model, and forecasts are made for stage-2 period (July to December) in 2005.

A Bayesian approach in the ARIMA model is studied to forecast demand from an incomplete data series. The model is developed with the assumption that among ninety data points, six values at stage-2 period (July to December) in 2004 are unavailable. Forecasts are made for stage-2 period in 2005. The main reason for developing *BS-ARIMA* is to enhance the model capacity to forecast demand from a data series that contains missing values. A number of non-informative prior distributions are used to represent the uncertainty of the parameters of ARIMA model. A Markov chain Monte Carlo sampling algorithm is used to derive the posterior values of model parameters. In the following sections, the fundamental of classic ARIMA model is explained first. After identifying the time series pattern, the parameters of ARIMA model are estimated and applied to forecast demand of the seasonal product.

4.1 Fundamental of the ARIMA Approach (*F-ARIMA*)

A time series is a set of values (observations) represented by a linear combination of independent random variable, y_t (t = 1, 2, ..., n), where index t indicates the intervals of time. For seasonal time series data, the direct scale of time is not always necessary to develop the model. Any mean difference of the series or logarithmic transformation of data can be used to develop the model. The development of the model involves two basic tasks: (a) identifying the nature of the demand represented by the sequence of observations, and (b) predicting future demands of the time series. To achieve the first goal, the following are the steps considered:

- ♦ first, identify the pattern of observed time series data,
- ♦ once the pattern is identified, a model is required to interpret the data, and,
- ♦ then, the parameters of the model are estimated.

The second goal is achieved by extrapolating the model to predict the future demand. In a given time series the following can be recognized as:

- ♦ a long-term component of variability termed as trend represents the pattern of the series,
- ♦ a short-term component, whose shape occurs periodically (at intervals of s lags of the index variable), is known as seasonality,

♦ an autoregressive component of p order, $AR(p)$ relates each value $Z_t = Y_t$ – (trend and seasonality) to the (p) previous Z values, according to following linear relationship

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + \varepsilon_t \tag{4.1}$$

where ϕ_i (i =1, …, p) are parameters to be estimated and ε_t is a residual term; and,

♦ a moving average component of q order, $MA(q)$ relates each Z_t value to the q residuals of the (q) previous Z estimates

$$Z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q} \tag{4.2}$$

where θ_i (i = 1, 2, …, q) are parameters to be estimated.

The theory of time-series analysis has been developed as a set of linear operators. According to Box and Jenkins (1970), a highly useful operator in time-series theory is the lag or backshift linear operator (B) to eliminate the linear or seasonal trend.

4.1.1 Difference Operator to Eliminate Increasing Trend

In time-series analysis, the lag or backshift linear operator (B) is used to eliminate the linear trend. If the operator B makes $BZ_t = Z_{t-1}$, which shifts backward in time by one period, B is called the 1st order delay operator. For example, $BZ_{50} = Z_{49}$. The double application of lag operator is indicated by B^2. Applying the lag operator twice to a series, the result is given by

$$B(BZ_t) = BZ_{t-1} = Z_{t-2}\,.$$

Definition 4.1: The k-th order delay operator is defined as $B^k Z_t = Z_{t-k}$.

Therefore, any integer k is written as $B^k Z_t = Z_{t-k}$. For example, $B^{12}Z_{50} = Z_{50-12} = Z_{38}$.

Using the back operator from Definition 1, the Equation (4.1) can be rewritten as

$$Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \cdots - \phi_p Z_{t-p} = \varepsilon_t = \phi(B) Z_t \tag{4.3}$$

where $\phi(B)$ is the autoregressive operator of p order defined by

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^P$$

Similarly, Equation (4.2) can be written as

$$Z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q} = \theta(B)\varepsilon_t \tag{4.4}$$

where $\theta(B)$ indicates the moving average operator of q order defined by

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q .$$

The autoregressive and moving average components can be combined in an autoregressive moving average (ARMA) (p, q) model as

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

The lag operator used in the above equation is

$$(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_p B^p) Z_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)\varepsilon_t$$

Finally, $\phi(B) Z_t = \theta(B)\varepsilon_t$. (4.5)

Once linear trends of time series are removed, the periodic trends are eliminated as following.

4.1.2 Periodic Difference Operator to Eliminate Periodic Increase

The analysis of a series begins by evaluating the long and short-term periodic components, which are essential to define the regular structure of the series. The trend components are evaluated by fitting a (regular, a polynomial, or a more complicated general) function. According to Box and Jenkins (1970), the seasonal component is estimated by a seasonal decomposition procedure, which calculates a seasonal index based on the ratio of the observed values to the moving average. In the final stage of series modeling, both the trend and the seasonal component are integrated in the ARMA (p, q) process. For the trend, such integration is obtained by using the difference linear operator $(\nabla = 1 - B)$, therefore

$$\nabla Y_t = (1 - B) Y_t .$$

Definition 4.2: A single application of the ∇ operator transforms the data to a linearly increasing trend, and repeated use of the ∇ operator for *d* times (∇^d) transforms the trend to stationary which can be fitted by a *d*-order polynomial.

Stationary series Z_t obtained after the *d*th difference (∇^d) of Y_t, which is given by

$$Z_t = \nabla^d Y_t = (1-B)^d Y_t. \tag{4.6}$$

The combination of ∇ operator in Equation (4.6) and the ARMA (*p*, *q*) process results in an ARIMA (*p*, *d*, *q*) model. Again, ARIMA can be used for the seasonal component of *s* lag period, by using both correlations between Z_t and Z_{t-s} values and those between the corresponding residuals ε_t and ε_{t-s}. A seasonal ARIMA model is an ARIMA (*p*, *d*, *q*) model whose residuals ε_t are further modeled by an ARIMA $(P, D, Q)_s$. The operators of a seasonal ARIMA model is defined as $(p, d, q)\text{x}(P,D,Q)_s$. ARIMA procedure is expressed as follows:

♦ the non-seasonal autoregressive operator of *p* order, *AR*(*p*) is

$$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p,$$

♦ the seasonal autoregressive operator of *P* order, *AR*(*P*) is

$$\phi_P(B^s) = 1 - \phi_{1,s} B^L - \phi_{2,s} B^{2L} - \ldots - \phi_{P,s} B^{PL},$$

♦ the non-seasonal moving average operator of *q* order, *MA*(*q*) is

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q,$$

♦ the seasonal moving average operator of *Q* order, *MA*(*Q*) is

$$\theta_Q(B^s) = 1 - \theta_{1,s} B^s - \theta_{2,s} B^{2s} - \ldots - \theta_{Q,s} B^{Qs},$$

and difference operator of *d* order, $\nabla^d = (1-B)^d$.

The autoregressive term is determined based on autocorrelation and partial autocorrelation statistics. The trend of a time series is converted to stationary by the differencing of the data.

4.2 Application of Box-Jenkins Methodology

The Box-Jenkins methodology defines the strategy for identifying, estimating and forecasting autoregressive integrated moving average models. The methodology consists of a three step iterative cycle, (1) model identification, (2) parameter estimation, and (3) application. The phases of Box-Jenkins methodology to develop *ARIMA* model in three stages. Following Makridakis *et. el.* (1998), the stages of the *F-ARIMA* model is shown in Table 4.1.

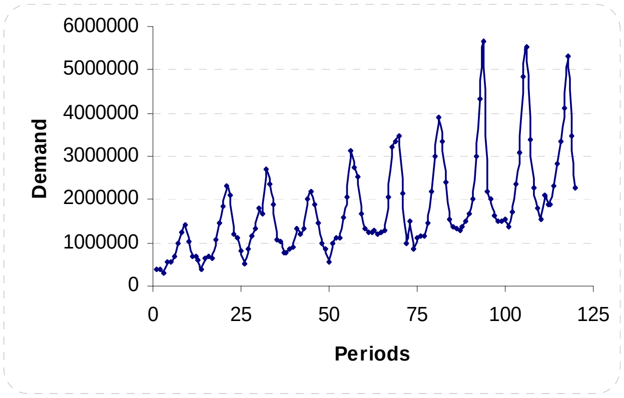
Table 4.1: Steps of *F-ARIMA* methodology for time series modeling

Phase I *Identification*	Data preparation	♦ Transform data to stabilize variance ♦ Difference data to obtain stationary series
	Model Selection	♦ Examine data, ACF and PACF to identify potential models
Phase II *Estimation and testing*	Estimating	♦ Estimate parameters ♦ Select best model if p-value of all model parameters are significant
	Diagnostics	♦ Check AIC/PAIC of residuals ♦ Are residual normally distributed?
Phase III *Application*	Forecasting	♦ Use model to forecast

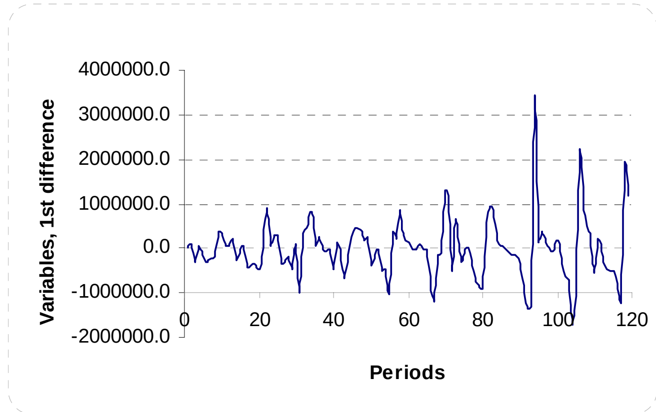
The identification consists of using the data to indicate whether the time series can be described with a moving average model, an autoregressive model, or a mixed autoregressive-moving average model. Estimation consists of using the data to make inferences about the parameters that are needed for the identified model and to estimate parameters of the model. Diagnostic checking involves the examination of residuals from fitted models to indicate the model inadequacy for the data series.

4.2.1 *F-ARIMA* Model Identification

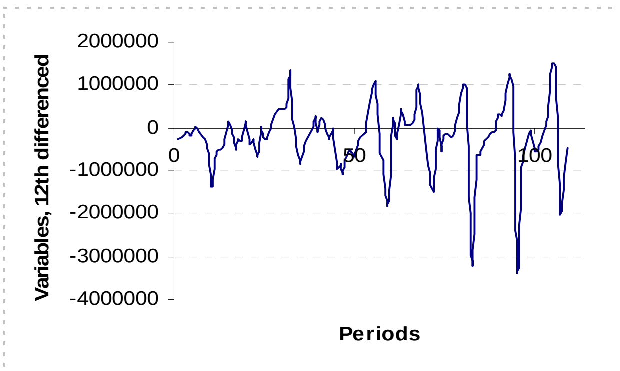
ARIMA model is estimated only after transforming the variable for forecasting into a stationary series. The stationary series is the one whose values vary over time only around a constant mean and constant variance. The stationary of data series after various differencing is shown in Figure 4.1.



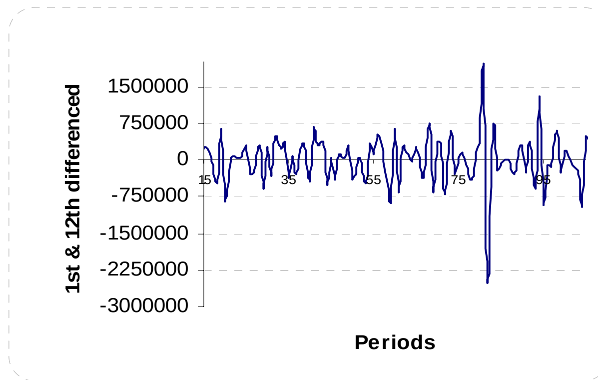
a) Data series with no differencing



b) Stationarity after 1st differencing



c) Stationarity after (12) differencing



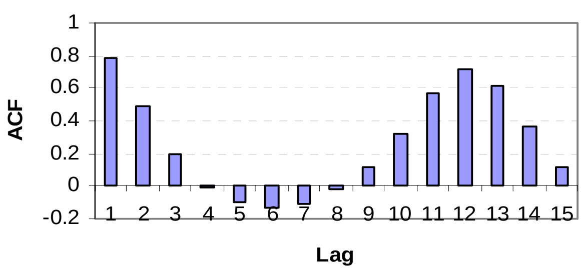
d) Stationarity after (1, 12) differencing

Figure 4.1: Time plot of apparel demand data series after differencing

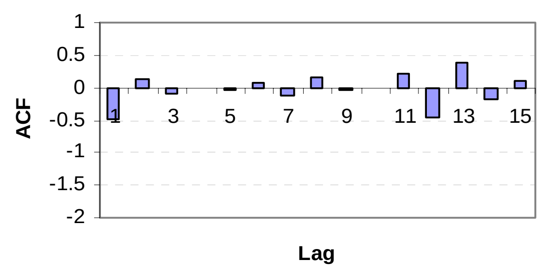
Figure (4.1a) indicates a strong (periodic) seasonal pattern and increasing trend. Figure (4.1b) is the first differenced data series, which appears to be non-stationary. Figure (4.1c) is the seasonal (12th) differenced data series, which does not show enough stationarity but a downward trend. Figure (4.1d) shows the stationarity of data series after first and seasonal (1st and 12th) differencing. Thus, difference of data series ($d_{1,12}$) of order 2 is sufficient to achieve stationarity in mean. After differencing the series, the newly constructed variable is Z_t, which is $Z_t = \nabla^d \nabla^D = (1-B)^d (1-B)^D$. Thus, Z_t is determined after differencing the data, which is

$$Z_t = (y_t - y_{t-1}) - (y_{t-12} - y_{t-13}). \tag{4.7}$$

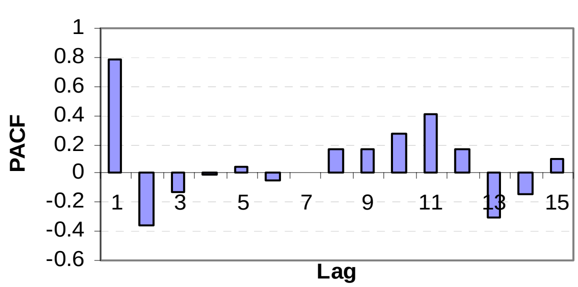
The next part of this step is to identify the values of p and q, which are the AR(p) and MA(q) components for both seasonal and non-seasonal series. In this step, the main analytical tool is the autocorrelation function (ACF) and partial ACF (PACF), which are used to identify the internal structure of the analyzed series. To identify the values of p and q, the autocorrelation and partial autocorrelation coefficients of various orders of Z_t are computed. Both AR(p) and MA(q) are associated with ACF and PACF. The forms of ACF and Partial ACF with differencing of data series and after first and seasonal differencing of the data series are shown in Figure 4.2.



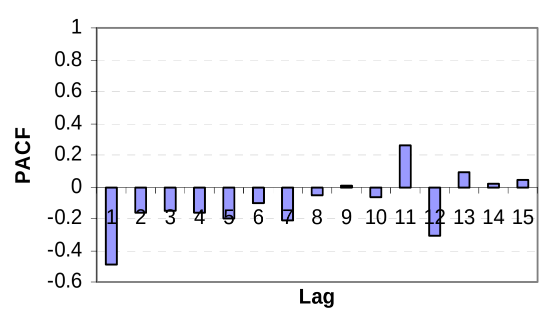
a) ACF with no differencing



b) ACF of differenced data



c) PACF with no differencing



d) PACF of differenced data

Figure 4.2: ACF and Partial ACF of differenced demand data

The patterns of ACF and PACF are usually decreasing exponentially or alternate in sign or decreasing in sinusoidal form. The ACF and Partial ACF show that the order of p and q can at the most be one.

Two goodness-of-fit statistics are most commonly used for model selection, (i) Akaike information criterion (AIC) and (ii) Schwarz Bayesian information criterion (BIC). The AIC and BIC are determined based on a likelihood function. Several (seven) tentative ARIMA models are tested for the data series and the corresponding AIC and BIC values for the models are shown in Table 4.2. The objective here is to select models that provide the minimum AIC and BIC values.

Table 4.2: AIC and BIC values for various *F-ARIMA* models

Model	ARIMA (p, d, q)	AIC	BIC
1	$(1,1,1)(0,1,0)_{12}$	2899.48	2907.33
2	$(0,1,1)(1,1,0)_{12}$	2889.60	2897.45
3	$(0,1,1)(0,1,0)_{12}$	2889.61	2897.44
4	$(0,1,1)(1,1,1)_{12}$	2890.63	2901.08
5	$(1,0,1)(1,0,0)_{12}$	2915.60	2926.11
6	$(1,1,1)(1,1,1)_{12}$	2892.20	2905.29
7	$(1,1,0)(1,0,0)_{12}$	3231.58	3242.49

The models that have the lowest AIC and BIC are *F-ARIMA* $(0,1,1)(1,1,0)_{12}$ and $(0,1,1)(0,1,0)_{12}$. Since two models are identified, the most suitable model is selected by checking the residuals of both models and selected the one with the most significant residuals. The AIC, BIC values, residual test, and the estimation of model parameters are performed by the SAS package. The detail of the results of the selected model is shown in Appendix *C*. A chi-square (χ^2) test can be used to evaluate the residual pattern. The results of the residual tests of the selected models are shown in Table 4.3.

Table 4.3: Autocorrelation check of *F-ARIMA* residuals

Lag	Chi-sq	DF	Pr > Chi sq.	Autocorrelation					
ARIMA $(0,1,1)(1,1,0)_{12}$									
6	6.11	4	0.19	-0.066	-0.167	-0.102	-0.125	-0.051	0.003
12	8.53	10	0.58	-0.057	0.089	-0.014	-0.031	0.057	-0.084
18	16.88	16	0.39	0.248	-0.102	-0.01	-0.032	0.003	-0.035
24	29.8	22	0.13	0.026	-0.089	-0.042	-0.199	0.108	0.199
ARIMA $(0,1,1)(0,1,0)_{12}$									
6	8.97	4	0.06	0.126	0.058	-0.133	-0.15	-0.146	-0.057
12	11.81	10	0.30	-0.092	0.046	0.003	-0.014	0.117	0.022
18	26.82	16	0.04	0.237	-0.087	-0.046	-0.213	-0.025	-0.112
24	39.44	22	0.01	-0.066	-0.142	-0.078	-0.175	0.088	0.164

Through examining the autocorrelations and partial autocorrelations and chi-square test of the residuals in Table 4.3, the results indicates that *F-ARIMA* $(0,1,1)(1,1,0)_{12}$ is a significantly better model. The next step is to estimate the parameters of this model.

4.2.2 Parameters Estimation of *F-ARIMA* Model

Once a suitable *F-ARIMA* $(p, d, q)\times(P, D, Q)_{12}$ structure is identified, the second step is the parameter estimation or fitting stage. The parameters are estimated by the maximum likelihood method. The results of parameter estimations are reported in Table 4.4.

Table 4.4: Estimated values of the *F-ARIMA* parameters

Parameter	Estimate	Standard error	t-value	Pr>|t|	lag
MU	1226.80	7889.10	0.16	0.8700	0
MA1,1	0.74	0.07	10.59	0.0001	1
AR1,1	-0.35	0.09	-3.59	0.0003	12

It is also important to check that the parameters contained in the model are significant. This ensures that there is no extra parameters are present in the model and the parameters used in the model have significant contribution, which can provide the best forecast. The estimate of autoregressive and moving average parameters are labeled "MA1,1" and AR1,1, which are 0.735 and -0.35, respectively. Both the moving average and the autoregressive parameters have significant t values. The subsequent step after the parameter estimation is the Diagnostic Checking or model verification. The Box and Jenkins (1970) estimation process for seasonal *F-ARIMA* model is shown in Figure 4.3.

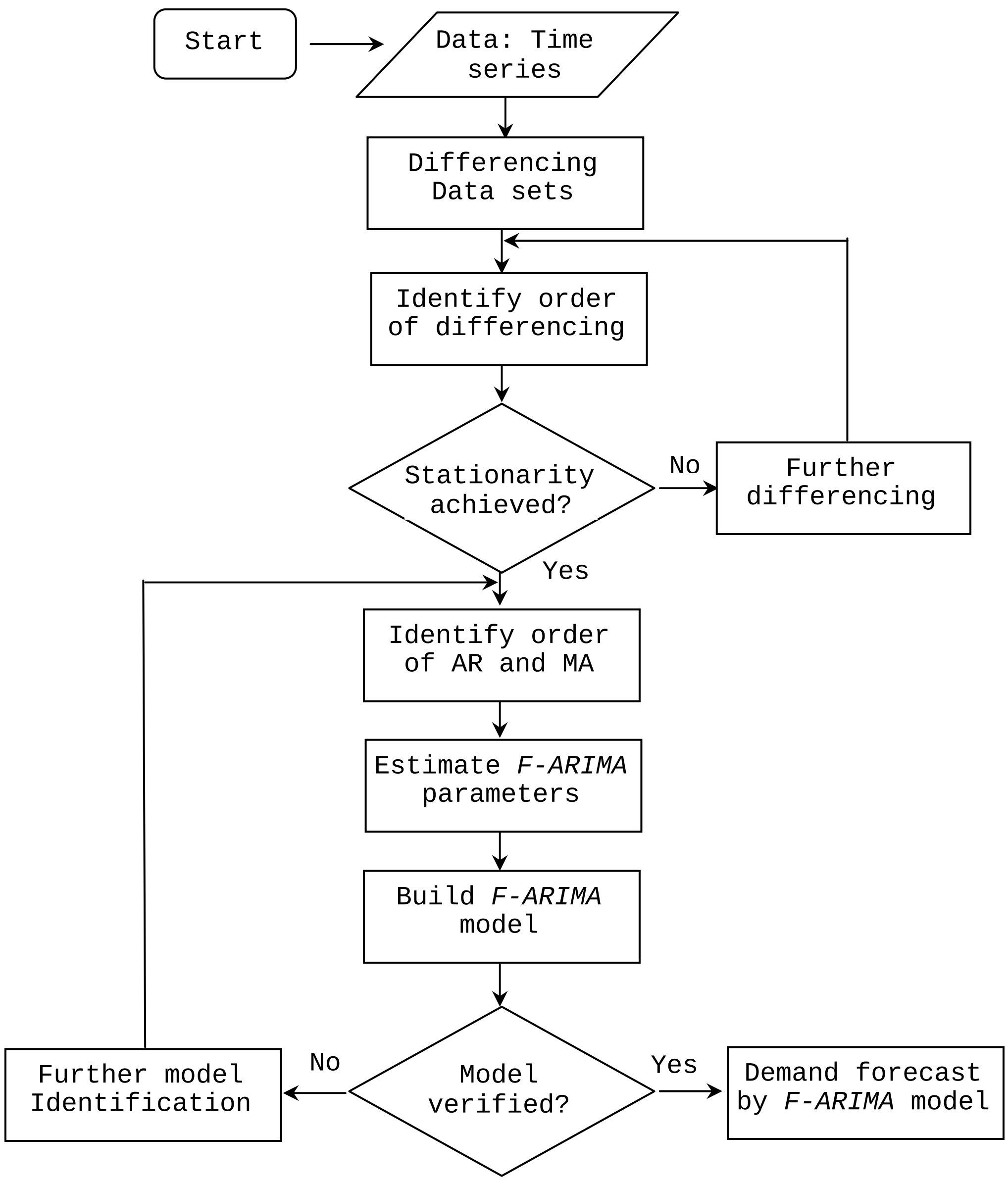


Figure 4.3: Flow chart for *F-ARIMA* estimation process

4.2.3 Diagnostic Checking and Model Validation

The model verification is concerned with checking the residuals of the model to determine if the model contains any systematic pattern which can be removed to improve on the selected ARIMA model. Although the selected model may appear to be the best among a number of models considered, it is also necessary to do diagnostic checking to verify that the model is adequate. Verification of an ARIMA model is tested (i) by verifying the ACF of residuals using the chi squared test, (ii) by verifying the normal probability plot of the residuals.

In Table 4.3, it is revealed that there are no significant autocorrelations or any significant partial autocorrelations of the residuals. The χ^2 tests indicated that the hypothesis cannot be rejected and residuals are uncorrelated. If the residuals are not random, the time series should be further modeled. Since the model diagnostic tests show that all the parameter estimates are significant and the residual series do not follow any pattern, it can be concluded that the ARIMA (0,1,1) $(1,1,0)_{12}$ model is adequate for the demand series. Therefore, ARIMA (0,1,1) $(1,1,0)_{12}$ is used to forecast the demand series.

4.3 Point Forecast with *F-ARIMA* Model

The ARIMA model (0,1,1) $(1,1,0)_{12}$ is selected to forecast the demand variable, where autoregressive term p = 0, P = 1 (seasonal) [that is, $(1-0)(1-\phi_1 B^{12})$]; differencing term d = 1, D = 1 (seasonal difference) [that is, $(1-B)(1-B^{12})$] and moving average term q = 1, Q = 0 (seasonal) [$(1-\theta B)(1-0)$] . For the dataset in Table A.1, the fitted model is given by

$$\underbrace{(1-\phi B^{12})}_{\text{Seasonal AR(1)}}\ \underbrace{(1-B)}_{\text{Non-Seasonal Difference}}\ \underbrace{(1-B^{12})}_{\text{Seasonal Difference}} y_t = C + \underbrace{(1-\theta B)}_{\text{Non-Seasonal MA(1)}} e_t \tag{4.8a}$$

The model has (1, 12) period differencing, the autoregressive factors is $(1-\phi B^{12})$ = (1 + 0.35096 B^{12}), the moving average factors is $(1-\theta B)$ = (1 - 0.73527 B), and the estimated mean, C = 1226.8. Transforming autoregressive terms and coefficient, the form of Equation (4.8a) is given by

$$y_t = \frac{C+(1-\theta B)e_t}{(1-\phi B^{12})\ (1-B)\ (1-B^{12})}. \tag{4.8b}$$

After expanding, Equation (4.8b) becomes

$$y_t = \frac{C+(1-\theta B)e_t}{(1-\phi B^{12})\ (1-B-B^{12}+B^{13})}$$

$$= \frac{C+(1-\theta B)e_t}{(1-B-B^{12}+B^{13}-\phi B^{12}+\phi B^{13}+\phi B^{24}-\phi B^{25})}. \tag{4.8c}$$

Multiplying denominator with left hand side, Equation (4.8d) converts into

$$y_t - y_t(B+B^{12}-B^{13}+\phi B^{12}-\phi B^{13}-\phi B^{24}+\phi B^{25}) = C+(1-\theta B)e_t \tag{4.8d}$$

Changing the sides Equation (4.8d) yields

$$y_t = y_t[B+B^{12}(1+\phi)-B^{13}(1+\phi)-\phi B^{24}+\phi B^{25})]+(1-\theta B)e_t + C \tag{4.8e}$$

Transforming the back operator, the Equation (4.8e) is given by

$$y_t = y_{t-1}+(1+\phi)y_{t-12}-(1+\phi)y_{t-13}-\phi y_{t-24}+\phi y_{t-25}+e_t-\theta e_{t-1}+C. \tag{4.9}$$

4.4 Forecast Results by *F-ARIMA* Model

In order to forecast one period ahead, that is, y_{t+1}, the subscript of the Equation (4.9) is increased by one unit, throughout, as given by

$$y_{t+1} = y_t+(1+\phi)y_{t-11}-(1+\phi)y_{t-12}-\phi y_{t-23}+\phi y_{t-24}+e_{t+1}-\theta e_t+\mu \tag{4.10}$$

The term e_{t+1} is not known because the expected value of future random errors has to be taken as zero. For the forecast of the second period onward, the term e_t is also taken as zero as the actual value is not known so the forecast errors can not be found. There are 114 data points from January 1998 to June 2005 used to build the ARIMA model. Using ϕ = -0.35 and θ = 0.735, the Equation (4.11) is given by

$$y_{t+1} = y_t + 0.65y_{t-11} - 0.65y_{t-12} + 0.35y_{t-23} - 0.35y_{t-24} + e_{t+1} - 0.735e_t + 1226.8 \qquad (4.11)$$

The results *F-ARIMA* forecasting model is shown in Table 4.5 (details in *Appendix B*.1).

Table 4.5: Forecast results by *F-ARIMA* model $(0,1,1)(1,1,0)_{12}$ (units in million)

Month	Demand Forecast	Actual Demand	Std error	95% lower	95% upper
Jul	2.74	2.83	0.38	1.99	3.50
Aug	3.65	3.33	0.40	2.78	4.34
Sep	4.92	4.10	0.41	4.35	5.96
Oct	5.03	5.11	0.42	5.25	6.91
Nov	2.69	3.46	0.44	2.61	4.31
Dec	2.70	2.28	0.45	1.83	3.58

In order to forecast for the period 115 (that is, July 2005), Equation (4.11) is given by

$$y_{115} = y_{114} + 0.65y_{103} - 0.65y_{102} + 0.35y_{91} - 0.35y_{90} + \hat{e}_{115} - 0.735\hat{e}_{114} + 1226.8$$

The value of e_{115} is not known, so $\hat{e}_{115}$ is replaced by zero. The value for $\hat{e}_{114}$ is the difference of actual demand and the forecasted value for the period 114, which is 1.48.

The forecast quantity for period 115 can be calculated as follows:

$\hat{y}_{115}$ = 2.32 + (0.65) 2.37 – (0.65) 1.73 + (0.35) 2.02 - (0.35) 1.68 + ($\hat{e}_{115}$ = 0)

+ (-0.73) 1.48 + 0.001

= 2.74 (million).

Predicting demand for the second period forecast, that is, the period 116, the quantity is

$\hat{y}_{116}$ = 2.83 + (0.65) 3.09 – (0.65) 2.37 + (0.35) 3.0 – (0.35) 2.02+ ($\hat{e}_{116}$ = 0)

+ (-0.73) ($\hat{e}_{115}$ = 0) + 0.001 = 3.65 (million).

4.5 Bayesian Sampling-based ARIMA Model (*BS-ARIMA*)

Bayesian methods have been widely applied in time series context and have played a significant role in prediction processes. In this section, the Bayesian technique is used in ARIMA model to forecast demand of a seasonal product within the framework of sampling theory statistics. The model can be applied in situations where non-standard distributions or nonlinear regression are more realistic for a given data series, Gamerman (1997). The Markov Chain Monte Carlo (MCMC) methods is efficient and flexible algorithms for conducting posterior inference of Bayesian model through simulation. The main reason of using MCMC methods is to make an inference about analytically intractable parameters of the posterior model through generating a Markov chain. Depending of the structure of the time series data, a Markov chain can be constructed in various ways. Gibbs sampler is the most common algorithm used here to derive the posterior parameters of the model. The application of Bayesian ARIMA models in time series with complete values is available in Congdon (2003). The key advantage of developing ARIMA model from Bayesian perspectives is the capacity to forecast future demand from an incomplete data series that contains both observed and unobserved data points.

4.5.1 Bayesian Computation at *BS-ARIMA* Model

A general form of the seasonal ARIMA model is $(p,d,q)(P,D,Q)_{12}$. For the dataset, the pattern of the ARIMA model identified in the previous sections has the form $(0,1,1)(0,1,1)_{12}$. In Equation (4.9), the ARIMA model has expressed with the form as

$$y_t = y_{t-1} + \phi_1 y_{t-12} + \phi_2 y_{t-13} - \phi_3 y_{t-24} + \phi_4 y_{t-25} + e_t - \theta e_{t-1} + C$$

where $\phi_1 = (1+\phi)$, $\phi_2 = -(1+\phi)$, and $\phi_3 = \phi_4 = \phi$.

It is noted that the demand for the stage-2 period from July 2004 to December 2004 are not available. A dummy variable w_t is added to Equation (4.9) to account the missing values of the data series. The form of the *BS-ARIMA* after adding dummy variable is given by

$$y_t = y_{t-1} + \phi_1 y_{t-12} + \phi_2 y_{t-13} - \phi_3 y_{t-24} + \phi_4 y_{t-25} + e_t - \theta e_{t-1} + C + w_t \,. \qquad (4.12)$$

The dummy variable w_t, $0 \le w_t \le 1$ is added to represent the status of past information whether the demand observation is available or not for any period in the past season. A dummy variable is set to 'zero' when demand information of a period is complete. A scaled value of w_t may be set (from 0 to 1) to reflect the partial demand of a period. For example, for a period the value of w_t would be 1.0, if the demand information for the period is unrecorded (missing), while the value 0.50 indicates incomplete demand information which means approximately 50% of the expected demand was observed due to some unnatural event.

For the data series y_t, (t = 1, 2, …, n, n+1, …N), the y_t corresponds to the demand of a period t, where a vector time series from n+1 to N, $\{y_F = (N-n) \ge 1\}$ is the prediction periods. A Bayesian computation is carried out to predict the demand for (N-n) period through the use of sampling-based algorithm. The particular sampling-based approach used in this model is a Markov chain Monte Carlo method based on the Gibbs sampler algorithm.

The likelihood function for n observation y_t, (y_1, y_2, …, y_n) is denoted by $f(y;\psi)$, where $\psi = (\phi_i, \theta, \beta, \tau)$ with $\phi_i = (\phi_1, \ldots, \phi_4)$. The conditional likelihood is then obtained from the factorization theorem (Zellner, 1996) is given by

$$f(y_t | \Psi) = f(y_1 | \Psi) f(y_2 | y_1, \Psi) \ldots f(y_n | y_1, \ldots, y_{n-1}, \Psi).$$

Given the prior distribution for Ψ, $f(\Psi | y_t)$, the posterior density for Ψ is given by

$$f(\Psi | y_t) \propto f(y_t | \Psi). f(\Psi).$$

If $y_F = (y_{n+1}, \ldots, y_N)$, for predicting (*N* - *n*) = *L* period, the predictive density is given by

$$f(y_F \mid y_t) = \int (y_F \mid y_t, \Psi).f(\Psi)d\Psi\,, \tag{4.13}$$

where $\int (y_F \mid y_t, \Psi)$ is the density of the future data y_F.

The *L* steps ahead forecast is then

$$f(y_F \mid y_t, \Psi) = \int (y_{n+1} \mid y_t, \Psi) \int (y_{n+2} \mid y_{n+1}, y_t, \Psi) \ldots \int (y_{n+L} \mid y_{n+1}, y_{n+L-1}, y_t, \Psi) d\Psi$$

To obtain a sample of predictions from the density function in Equation (4.13), for each Ψ_t one needs to draw from $\int (y_F \mid y_t, \Psi)$. The following are the steps to predict the future values of the *BS-ARIMA* model through Win BUGS.

Step 1: Data Definitions

y_t, {for *t* in (1: *n*) }

w_t, {Dummy (*t*), for *t* in 1: *N*}

Step 2: Model Description

$y_t \sim Normal(\mu_t, \tau)$ {for *t* in (2: *n*)}

where

$\mu_t = C + \phi_1 y_{t-12} + \phi_2 y_{t-13} + \phi_3 y_{t-24} + \phi_4 y_{t-25} + e_t + \theta_1 e_t + \beta w_t$

$\tau = 1/\sigma^2$

Step 3: Assigning Priors

$\mu \sim Normal$ (0, 0.001)

ϕ_i ~ Normal(0, 0.001)

θ_i ~ Normal(0, 0.001)

β ~ Normal(0, 0.001)

τ ~ Chi-sq (1)

Step 4: Forecasts Period {*t* = *n* +1 … *N*}

$y_{(new)t} \sim Normal[\mu_{(new)t}, \tau]$

$\mu_{(new)t} = C + \phi_1 y_{t-12} + \phi_2 y_{t-13} + \phi_3 y_{t-24} + \phi_4 y_{t-25} + e_t + \theta_1 e_t + \beta w_t$ {for *t* in (*n* +1: *N*)}

It has been shown in Carlin and Gelfand (1990) that the point estimates arising from $\int(y_F \mid y_t, \Psi)$ perform well and the variance of this estimated predictive distribution is small. To complete the model in '*Step* 3', the following prior distributions are used. The choice of prior distribution is followed by (Gelman *et. al.*, 2004; Congdon, 2003), where the posterior models are derived using MCMC approach through WinBUGS package. For the coefficient ϕ and θ, the non-informative prior distributions *Normal*(0, 0.001) are assumed. Parameter β is expected to follow a relatively informative prior distribution *Normal*(1.0, 0.1). The precision (a reciprocal of variance), τ follows a chi squared distribution with one degree of freedom.

To construct the BS-*ARIMA*, the demand variable y_t is placed with all the values observed from January 1998 to June 2005. An average of stage-1 demand of year 2004 is placed for the missing period from July to December 2004. For dummy variable w_t, the values placed for July to December 2004 are 0.39, 0.59, 0.63, 0.27, 36, 0.11, respectively; and zeros are placed for the remainder of the periods of the series. The values for dummy variable for period (July to December) 2004 are shown in Table 4.6.

Table 4.6: Values of dummy variables for July to December, 2004 (units in million)

Demand	Jul	Aug	Sep	Oct	Nov	Dec
Projected y_p	2.54	3.81	4.14	4.05	2.41	1.75
Stage-1 $y_{st\text{-}1}$	1.55	1.55	1.55	1.55	1.55	1.55
Dummy $w_t = 1 - \left(y_{st-1}/y_p\right)$	0.39	0.59	0.63	0.62	0.36	0.11

4.5.2 Forecast Results of *BS-ARIMA* Model from Incomplete Data

A sampling-based Bayesian approach is adapted to forecast demand from incomplete data series. The WinBUGS code for demand estimate, parameter estimates and forecast results are shown in *Appendix B* (B.2 and B.3). The simulation results of the demand forecast for stage-2 period of 2005 with missing observations are shown in Table 4.7.

Table 4.7: Demand Forecast by *BS-ARIMA* model (units in million)

Month	Actual	Estimate	Std error	2.50%	Median	97.50%
Jul	2.83	2.47	0.71	1.07	2.46	3.88
Aug	3.33	3.62	0.73	2.19	3.61	5.07
Sep	4.10	4.77	0.74	3.32	4.76	6.22
Oct	5.31	5.29	0.73	3.89	5.28	6.76
Nov	3.46	4.28	0.77	2.75	4.28	5.76
Dec	2.28	2.97	0.77	1.53	2.96	4.50

4.6 Forecast Using Adaptive Exponential Smoothing Technique

In this section, an adaptive approaches of Holt-Winters' (H-W) *exponential smoothing* technique is presented to forecast the seasonal demand. For trend and seasonal data, the multiplicative exponential smoothing (*M-ES*) model is commonly used in forecasting. The purpose to study this model is to compare the performances of other forecasting models. Forecasting demand for the next T periods using *M-ES* model, the time series is represented by the model, Askin and Goldberg (2002), Gardner (2006)

$$y_{t+T} = \left(\overline{R}_{t-1} + T\,\overline{G}_{t-1}\right)\overline{S}_{t+T-L},$$

where $\overline{R}_t$ is the estimate of level index, $\overline{G}_t$ is the estimate of the trend, and $\overline{S}_t$ is the estimate of seasonal component (seasonal index). The parameters are estimated as following.

(i) The overall smoothing of level index $\overline{R}_t$ is

$$\overline{R}_t = \alpha\,\frac{y_t}{\overline{S}_{t-L}} + (1-\alpha)\left(\overline{R}_{t-1} + \overline{G}_{t-1}\right),$$

where $0 < \alpha < 1$ is a smoothing constant.

(ii) The smoothing of the trend factor $\overline{G}_t$ is

$$\overline{G}_t = \beta\left(\overline{R}_t - \overline{R}_{t-1}\right) + (1-\beta)\,\overline{G}_{t-1},$$

where $0 < \beta < 1$ is a second smoothing constant.

(iii) The smoothing of the seasonal index $\overline{S}_t$ is

$$\overline{S}_t = \gamma\,\frac{y_t}{\overline{R}_t} + (1-\gamma)\,\overline{S}_{t-L},$$

where $0 < \gamma < 1$ is the third smoothing constant.

The initial values of the parameters α, β, and γ are determined using the data from July 2002 to December 2002. The values are modified in subsequent years. The criterion used for selecting the initial values of the parameters is the value that provides the minimum mean absolute percentage error (MAPE) of the test dataset. The data series from January 2003 to June 2005 are used to adjust the weight of the smoothing parameters and demand forecast is performed for the stage-2 (July to December) in 2005. The demand is estimated using the most recent demand observation y_t, the seasonal factors, and the level index estimates of the time series. The numerical illustration to forecast stage-2 (July to December) in 2005 is shown in *Appendix B*.4. The estimates and actual demand for stage-2 in year 2005 by *M-ES* model are shown in Table 4.8.

Table 4.8: Forecast results by *M-ES* model (units in million)

Month	Seasonal Factor	Level	Trend	Forecast	Actual	Error
	S_t	R_t	G_t	$\hat{y}_t$	y_t	
Jul	0.95	3.02	0.04	2.90	2.83	-0.06
Aug	1.24	2.99	0.03	3.74	3.33	-0.42
Sep	1.75	2.88	0.02	5.07	4.10	-0.97
Oct	2.15	2.82	0.01	6.08	5.31	-0.77
Nov	1.14	2.87	0.01	3.29	3.46	0.17
Dec	0.84	2.85	0.01	2.40	2.28	-0.12

4.7 Forecast Performance and Model Validation

To evaluate the forecasting ability of the models, a number of assessment tools are used to measure the forecast accuracy. Performance measure indicators, such as relative or percentage errors (*PE*), the mean absolute deviation (*MAD*), relative error, tracking signals (*TS*) are used to test the model performances. The equations to calculate relative errors, the mean absolute deviation, and Tracking Signal of the forecasts are shown in Chapter 3 from Equation (3.15) to (3.18). The measures of the forecasting models are shown in Table 4.9.

Table 4.9: Forecast by *F-ARIMA, BS-ARIMA* and *M-ES* models (units in million)

Model	Month	Forecast $\hat{y}_j$	Actual y_j	deviation $(y_j - \hat{y}_j)$	*PE* $\frac{\lvert y_j - \hat{y}_j \rvert}{y_j}$	MAD_t $\frac{\sum \lvert y_j - \hat{y}_j \rvert}{n}$	TS_t $\frac{\sum \lvert y_j - \hat{y}_j \rvert}{MAD_j}$
F-ARIMA	Jul	2.74	2.83	0.01	0.03	0.09	1.00
	Aug	3.65	3.33	-0.32	0.10	0.21	-0.64
	Sep	4.92	4.10	-0.82	0.20	0.34	-0.42
	Oct	5.03	5.11	0.08	0.02	0.11	1.27
	Nov	2.69	3.46	0.77	0.22	0.18	0.23
	Dec	2.70	2.28	-0.42	0.19	0.10	-0.24
				Average =	0.13		
BS-ARIMA	Jul	2.23	2.83	0.60	0.21	0.60	1.00
	Aug	3.27	3.33	0.06	0.02	0.33	0.18
	Sep	3.95	4.10	0.15	0.04	0.27	0.56
	Oct	4.47	5.11	0.64	0.13	0.36	1.77
	Nov	3.47	3.46	-0.004	0.002	0.29	-0.02
	Dec	2.32	2.28	-0.003	0.02	0.25	-0.17
				Average =	0.067		
M-ES	Jul	2.90	2.83	-0.06	0.02	0.06	-1.00
	Aug	3.74	3.33	-0.42	0.12	0.24	-1.73
	Sep	5.07	4.10	-0.97	0.24	0.48	-2.01
	Oct	6.08	5.11	-0.97	0.19	0.61	-1.61
	Nov	3.29	3.46	0.17	0.05	0.52	0.32
	Dec	2.40	2.28	-0.12	0.05	0.45	-0.27
				Average =	0.11		

The graphical presentations of the demand forecast derived by the *F-ARIMA*, *BS-ARIMA* and *M-ES* models with respect to the actual demand are shown in Figure 4.4.

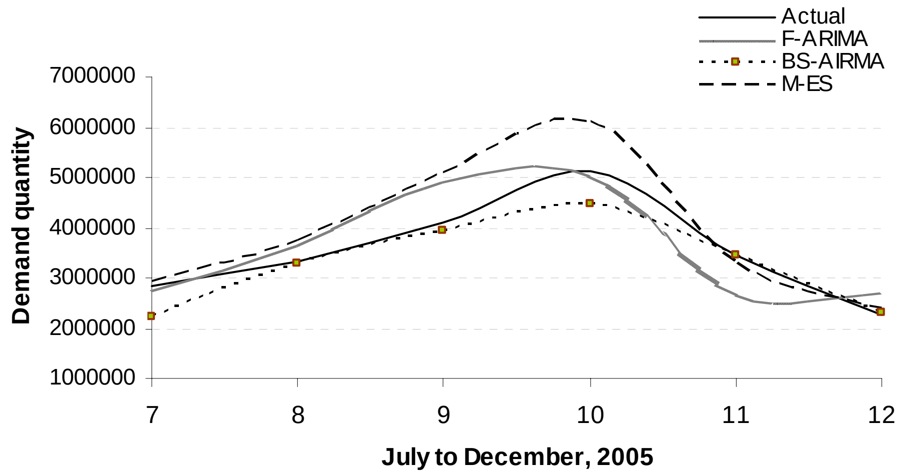


Figure 4.4: Comparison of forecast and actual demand

Tracking Signal (*TS*) is an indicator if the forecast follows any trend that needs to be adjusted. The graphical presentations of the tracking signals fluctuation of the both models are shown in Figure 4.5.

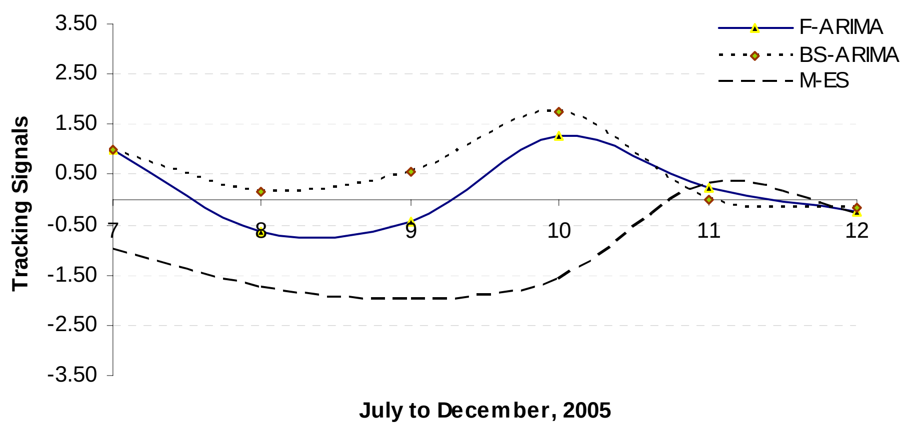


Figure 4.5: Tracking signal of the *F-ARIMA*, *BS-ARIMA* and *M-ES* models

4.8 Summary

In this chapter an ARIMA approach is used to forecast the demand of a seasonal product. It is possible to explore a number of interrelated models where the demand process is correlated across time. Based on the demand pattern, the *F-ARIMA* (0,1,1) $(1,1,0)_{12}$ model was found to be the best fit model for the dataset. For a non stationary stochastic time series such as winter apparel, the forecasting model often becomes complicated. In ARIMA model, forecast errors are incorporated to refine the predicted value, so the model gradually improves toward the end of the time series and provides satisfactory forecasting accuracy.

There are major advantages of using Bayesian methodology to forecast non-stationary demands. As classical ARIMA requires significantly long data series, a Bayesian-sampling based ARIMA model was proposed to model from smaller data or incomplete data with missing values. In this sub-model, it is assumed that data points at stage-2 (July to December) in 2004 were unavailable. A number of non-informative priors were used for the model parameters (α, β, τ). The posterior values of the parameters were computed numerically using the Markov Chain Monte Carlo (MCMC) simulation and BUGS/WinBUGS software.

A multiplicative approach of *exponential smoothing* (*M-ES*) technique is considered as the base reference to forecast seasonal demand and used to measure the forecast performances of the other models developed in the study. Both time series forecasting models and *M-ES* models are used to forecast the demand for stage-2 period in 2005. Test results of the ARIMA (*F-ARIMA* and *BS-ARIMA*) and *M-ES* models showed that both approaches have the advantages of easy modeling and significant accuracy. Errors were less than 13% for all models. Checking the tracking signals of the models, it is found that *M-ES* model has negative trends, which specifies that demand forecast made by *M-ES* model is larger with respect to actual demand. Therefore, the ARIMA models are appropriate for seasonal forecast and the Bayesian ARIMA model is advantageous of the all forecasting models.

CHAPTER 5

INVENTORY COST REDUCTION USING IMPROVED FORECASTING

In a supply chain, the random occurrences of demand are studied to manage the inventory system and the planned customer service at minimum costs. Inventory of a product may stock-out or over-stock if actual demand mismatches the demand forecast. If inventory quantity is determined based on improved forecast, the inventory cost of a product may be reduced. The focus of this chapter is to demonstrate the inventory cost reductions through the application of an appropriate demand forecast of a seasonal product during an active demand season. The best forecast for the product may be selected by comparing the inventory costs derived from several forecasts. The inventory policies are applied to the actual data series and the demand forecasts derived previously.

5.1 Model Description

The inventory policies are used to determine the inventory quantity of a seasonal product for an active demand season. The study has two objectives: (a) to demonstrate the reduction of inventory cost due to improved forecasts, and (b) to determine the procurement quantity of the products to maintain a satisfactory customer service level. A newsvendor inventory model with emergency procurement option is used to determine the inventory quantity. A dynamic programming (DP) algorithm is used to determine the inventory costs of actual demand and the demand forecast. A periodic review policy is used as a model to compare the cost determined by the extended newsvendor model. After applying these inventory policies to the actual demand data, the same inventory policies are applied to each forecast. The percentage above the inventory cost of a forecast, with respect to the inventory cost of actual demand, provides a basis to compare the forecasting models. The demand forecast that provides the minimum percentage is considered as the appropriate forecast for the active demand period. Thus, this chapter presents the cost savings approach in the inventory of a seasonal product by combining the improved forecasting technique and the appropriate inventory policy. The steps to compare the forecasting models are described as follows.

5.2 Procedure to Compute Optimal Inventory Cost

The following are the steps to compute the optimal inventory costs and the relative improvement measures of the forecasting models:

Step 1: Find customer service level by specifying the probability (P_1) of no stock-out from two inventory policies: (a) customer service level using newsvendor model with emergency replenishment policy, and (b) periodic review policy.

Step 2: Select the safety factor z to satisfy $P(Z) = (1- P_1)$.

The value of unit normal variable, $P(Z)$ (with mean 0, standard deviation 1) may be obtained from Z-table or from inverse function of normal distribution.

Step 3: Determine the ordering quantity, Q, from the forecast during lead time L ($\hat{y}_L$) and the safety stock (SS). The safety stock is calculated from standard deviation of the forecast error, SS = $Z\sigma_L$. Therefore, the ordering quantity is

$$Q = \hat{y}_L + Z\sigma_L \, .$$

Ordering quantity may be increased to the next higher integer.

Step 4: Compute the optimal inventory cost of actual demand and demand forecasts using dynamic programming (DP) algorithm. Then, calculate the relative cost of each demand forecast with respect to the cost derived from actual demand.

Step 5: Choose the best forecasting model that gives the minimum costs.

5.3 Cost Components to Determine the Customer Service Level

Businesses often face leadtime delay (shipment time) and cost factors such as transportation cost, set-up costs while procuring products. Inventory is managed to meet demand and maintain relatively low inventory cost during an active selling season. In inventory system, a customer service level is specified as the probability (P_1) of no stock-out during the active demand period. It is assumed that the procurement leadtime is one month. To determine the customer service level, the following is the cost components assumed for the time series. The inventory cost and variable cost per unit per period (holding cost, setup cost, shortage costs etc.) are listed in Table 5.1. The holding cost rate is 30% per year. Therefore, holding cost h_t per month is, h_t = (\$25)(0.30/year)/(12 months/year) = \$0.624/month.

Table 5.1: Unit costs applied to the inventory model

	Parameters	Jul	Aug	Sep	Oct	Nov	Dec
D_t	Actual Demand (in million, \$)	2.87	4.33	5.30	5.46	3.41	2.49
A_t	Fixed cost (in thousand, \$)	15.0	14.0	16.0	16.0	17.0	19.0
c_t	variable cost (\$)	25.0	25.0	24.0	25.0	26.0	30.0
π_t	shortage cost (\$)	5.0	5.0	5.0	5.0	5.0	5.0
h_t	inventory cost (\$)	0.624	0.624	0.624	0.624	0.624	0.624

A newsvendor inventory model with a provision for an urgent shipment is used to demonstrate the customer service level. A general periodic review model is also used as an alternate inventory policy to determine the inventory quantity and cost to compare the extended newsvendor model.

5.4 Newsvendor Procurement Model

In a newsvendor model, if the demand is in excess to the inventory level, then the sales are lost, while, if too much is ordered, inventory remains at the end. To reduce the lost sales, an urgent procurement option is included in the basic newsvendor model. In this model, two types of procurement approaches are considered: (i) standard procurement at regular interval, (ii) urgent (emergency) procurement if shortages occur during the peak selling period.

Products are ordered for a peak season (stage-2) at a variable ordering cost of c per item. For quantity y_t, the total procurement cost C_1 is $y_t c$. Inventory carrying cost occurs when procurement quantities y_t are more than actual demand x_t ($y_t > x_t$). For a unit inventory holding cost of h, the total inventory cost C_2 is

$$C_2 = \int_0^{y_t} h(y_t - x_t) f(x_t) dx_t , \qquad \text{if } y_t > x_t .$$

5.4.1 Expedite Cost Factors

Two additional costs factors are associated with an urgent procurement process: (i) urgent shipping cost, b_1, and (ii) cost due to buyers' waiting time, b_2. The leadtime for urgent shipment is denoted as τ, which may be considered as a decision variable. When y_t is less than the actual demand, a cost factor, ($b_1 \geq 1$) is multiplied with the unit purchasing cost c due to emergency (fast) shipment. The urgent shipment cost, $b_1 c$ may be expressed as function of the urgent leadtime τ as $b_1 c \tau$. For any demand above y_t, the urgent shipment cost, C_3, is given by

$$C_3 = \int_{y_t}^{\infty} b_1 c \tau (x_t - y_t) f(x_t) dx_t , \qquad \text{if } x_t > y_t . \tag{5.1}$$

In the urgent procurement process, another cost factor, (b_2>1) is assigned due to buyers' waiting time cost during the urgent shipment leadtime. The shorter waiting time reduces the buyers' waiting time cost, so the leadtime τ is inversely proportional to the cost, which is expressed as $b_2 c / \tau$. For any demand above y_t, the buyers' waiting time cost at stage-2, C_4 is given by

$$C_4 = \int_{y_t}^{\infty} \frac{b_2 c}{\tau}(x_t - y_t) f(x_t) dx_t\,, \qquad \text{if}\, x_t > y_t\,. \tag{5.2}$$

5.4.2 Urgent Leadtime Cost Function

The urgent transportation cost and the buyers' waiting time costs are a function of urgent procurement lead time τ. By combining the above costs from Equation (5.1) and (5.2), the costs regarding the urgent procurement response, $u(\tau)$, are

$$u(\tau) = c\left(b_1\tau + b_2 / \tau\right). \tag{5.3}$$

A property regarding the urgent shipment function $u(\tau)$ implies that the leadtime τ can be reduced by increasing the shipment cost through expedite shipment process. Thus, the buyers' waiting-time costs can be reduced. The urgent procurement leadtime, $u(\tau)$, is a convex cost function and holds an optimum value for τ >0. Thus, the value for the urgent shipping leadtime τ can be obtained by setting $du(\tau)/d\tau = 0$ as

$$\tau^* = \sqrt{b_2 / b_1}\,. \tag{5.4}$$

In Equation (5.4), the optimal leadtime τ^* decreased if urgent shipment cost factor b_1 is increased, or the buyers' waiting cost factor b_2 decreases. Substituting the value τ^* from Equation (5.4) into Equation (5.3) yields

$$u(\tau^*) = c\left[b_1\tau^* + \frac{b_2}{\tau^*}\right] = c\left[b_1\sqrt{b_2/b_1} + \frac{b_2}{\sqrt{b_2/b_1}}\right] = 2c\sqrt{b_1 b_2}\,. \tag{5.5}$$

Property 5.1: From Equation (5.5), it is implied that if shortages occur during active demand season at stage-2, the unit price of the product with urgent procurement process increases at least twice the original unit cost.

Proof (Property 5.1): If the urgent shipment cost factor b_1 and the buyers' waiting cost factor b_2 are set to 1, *i.e.*, $b_1 = b_2 = 1$, then Equation (5.5) becomes

$$u(\tau) = 2c\sqrt{b_1 b_2} = 2c\sqrt{1.1} = 2c\,,$$

which is equivalent to two times of the product purchasing cost.

5.4.3 Cost Minimization in Newsvendor Model

The total cost function includes regular procurement costs (C_1), inventory costs (C_2), and urgent procurement costs that includes urgent shipment cost (C_3) and leadtime waiting costs (C_4), the total cost is

$$TC(y_t, \tau) = C_1 + C_2 + C_3 + C_4. \text{ or,}$$

$$TC(y_t,\tau) = y_t.c + \int_0^{y_t} h(y_t - x_t) f(x_t) dx_t + \int_{y_t}^{\infty} b_1 c\tau (x_t - y_t) f(x_t) dx_t + \int_{y_t}^{\infty} \frac{b_2 c}{\tau} (x_t - y_t) f(x_t) dx_t \quad (5.6a)$$

$$TC(y_t,\tau) = y_t.c + \int_0^{y_t} h(y_t - x_t) f(x_t) dx_t + \int_{y_t}^{\infty} u(\tau)(x_t - y_t) f(x_t) dx_t\,, \quad (5.6b)$$

where, $u(\tau) = c\left[b_1\tau + \dfrac{b_2}{\tau}\right].$

Proposition 5.1: $TC(y_t)$ is a convex function**.**

Proof (Proposition 5.1): Taking the first order derivatives with respect to y_t, Equation (5.6b) gives

$$\frac{dTC}{dy_t} = c + h\int_0^{y_t} f(x_t) dx_t - u(\tau).\int_{y_t}^{\infty} f(x_t) dx_t\,, \quad (5.7)$$

$$= c + hF(y_i) - u(\tau)[1 - F(y_i)].$$

The second order derivatives of Equation (5.6b), with respect to Q gives

$$\frac{d^2TC}{dy_i^{\,2}} = hf(y_i) + u(\tau)[f(y_i)],$$

$$= f(y_t)[h + u(\tau)]. \quad (5.8)$$

From Equation (5.8), $u(\tau)$ is positive, and inventory holding cost h is a positive quantity. Therefore, ($d^2TC/dy_t^2 > 0$) is positive. Thus, $TC(y_t)$ is a convex function.□

Since $TC(y_t)$ is a convex function, an optimal y_t^* for stage-2 exists.

Now, setting $dTC/dy = 0$ yields

$$c + hF(y_t) - u(\tau^*)[1 - F(y_t)] = 0, \tag{5.9}$$

or,
$$F(y_t) = \frac{u(\tau^*) - c}{u(\tau^*) + h}. \tag{5.10}$$

Substituting $u(\tau^*) = 2c\sqrt{b_1 b_2}$ gives

$$F(y_t) = \left[\frac{2c\sqrt{b_1 b_2} - c}{2c\sqrt{b_1 b_2} + h}\right]. \tag{5.11}$$

5.4.4 Numerical Example (Newsvendor Inventory Policy)

The unit price of a seasonal product (\$$c$) in a distributor's store during stage-2 (July to December) in 2005 is arbitrarily set to \$25.00 and holding cost rate is 15% per year. The possibilities of stock-out during peak sale period may prompt the manager to arrange for an urgent procurement of the product. The estimated cost for urgent procurement is extra \$5 and the waiting cost for end market buyers is \$5 when shortages occur. The holding cost (6-months season) $h = (\$25.0)(1\,\text{month})\dfrac{0.15/\text{year}}{12\,\text{months}/\text{year}} = \$0.313/\text{Season}.$ Costs components using c = \$25.0/per unit, b_1 = \$5.0/per unit, and b_2 = \$5.0/per unit. The customer service level based on demand forecast during peak season may be obtained from Equation (5.11),

$$F(y) = \frac{2c\sqrt{b_1 b_2} - c}{2c\sqrt{b_1 b_2} + h} = \frac{(2)(25)\sqrt{(5)(5)} - 25}{(2)(25)\sqrt{(5)(5)} + 0.313} = 0.90$$

Hence, the customer service level as 90% during the busy demand period.

5.5 Alternate Inventory Policy (Periodic Review)

A periodic review policy is considered as an alternative policy to check the result of newsvendor model. A monthly review plan is considered for periodic inventory replenishment. There are t (t = 1, 2, …, n) forecasting periods at stage-2 and the demand forecast at any period t is y_t, while the actual demand for any period is x_t. Shortages may occur when $x_t > y_t$. The shortage cost is π_t dollars per period. To place an order for procuring y_t items, the fixed ordering cost is A dollars, unit purchasing cost is c dollars and unit holding cost is h dollars. Each unit brings a price of w dollars when it is sold, where $w > c$. Average fixed ordering cost per period is given by A/y_t, while revenue earned per period is $(w-c)y_t$, and average inventory per period is $\frac{h(y_t - x_t)}{2}$. In a periodic (y_t, L) replenishment policy, the expected shortages $S(y_t, L)$ with respect to actual demand x_t is given by

$$S(y_t, L) = \int_{y_t}^{\infty} (x_t - y_t) f(x_t) dx_t$$

$$\frac{dS(y_t, L)}{dy_t} = \overline{S}(y_t, L) = -\int_{y_t}^{\infty} f(x_t) dx_t = -[1 - F(y_t)].$$

The expected shortage cost per replenishment cycle is $\frac{\pi}{L}[1 - F(y_t)]$. Aggregating the cost components, a profit function for a periodic inventory replenishment policy is given by

$$Max\ \ z = (w-c)y_t - \frac{h}{2}(y_t - x_t) - \frac{A}{y_t} - \frac{\pi}{L} S(y_t, L). \tag{5.12}$$

Taking the first derivative of Equation (5.16) and setting to zero, the expected profit function is

$$\frac{dz}{dy_t} = -\frac{h}{2} - \frac{\pi}{L}\frac{d}{dy_t} S(y_t, L) = 0$$

or $$-\frac{h}{2}+\frac{\pi}{L}\left[1-F(y_t)\right]=0 \tag{5.13}$$

After rearranging, Equation (5.13) gives

$$F(y_t)=1-\frac{hL}{2\pi}, \tag{5.14}$$

where $F(y_t)$ is the critical fractile of the demand distribution function. This indicates the probability of no stock-out (customer service level) during the demand period. Following the numerical example in Section 5.4.4 and using L= one-month period, A, c, h, and π from Table 5.1, the Equation (5.14) yields $F(y_t)=0.94$.

The probability of no stock-out (P_1) during busy period is 0.94. Therefore, a 94% customer service level is considered for the periodic review inventory policy.

5.6 Selection of Order Points

In this section, a procedure is discussed to find the order point (inventory replenishment quantity) for each period during an active demand season. The order point is equal to the forecast of the demand for the leadtime period and the safety stock. The safety stock is determined by the standard deviation of the forecast error, σ_L multiplied by a service factor, Z. Therefore, safety stock (SS) is given by SS = $Z\sigma_L$.

5.6.1 Obtaining Order Points by Considering Forecast Errors

The service factor Z is determined from the probability of no stock-out per replenishment cycle. Probability of no stock-out should be no lower than P_i {(i = 1, 2) where P_1 is obtained from newsvendor policy, and P_2 is obtained from periodic review policy}. Conversely, the probability of stock-out should be no more than (1 - P_i). Safety factor is selected from the customer service level, which is given by

$$P(Z) = (1- P_i), \qquad (i = 1, 2) \tag{5.15}$$

where P_i is the probability of no stock-out. The value of Z from $P(Z)$ is obtained from Z-table. For example, if P_1 is 90%, the value for Z is determined by $\Phi^{-1}(0.1)$ from Z-table, which is 1.28. The stock-out probability, P_i and the Z-values corresponding to extended newsvendor and periodic review policies are shown in Table 5.2.

Table 5.2: Probability of stock out and corresponding safety factor

Inventory models	P_i	Z
Extended Newsvendor (P_1)	90%	1.28
Periodic review (P_2)	94%	1.55

Using the forecast $\hat{y}_t$ and the standard deviation of the forecast error, σ_L during the lead time period, the order point (Q) is given by

$$Q = \hat{y}_L + Z\sigma_L . \tag{5.16}$$

Considering the forecast generated from (*B-P*) forecasting model in Chapter 3, the order point Q using extended newsvendor and periodic review policies is shown in Table 5.3.

Table 5.3: Order points derived by (*B-P*) forecasting model

Inventory Policy	Forecast (million) y_t	Std dev (million) σ_L	Newsvendor Policy		Periodic Review	
			Z	Q (million)	Z	Q (million)
Jul	0.69	0.47	1.28	3.29	1.55	3.42
Aug	3.89	0.47	1.28	4.49	1.55	4.62
Sep	4.74	0.47	1.28	5.35	1.55	5.48
Oct	5.01	0.47	1.28	5.62	1.55	5.74
Nov	2.91	0.47	1.28	3.51	1.55	3.64
Dec	2.01	0.47	1.28	2.62	1.55	2.75

5.6.2 Numerical Illustration (Order Points)

For the month of July, the demand forecast by Bayesian probability (*B-P*) model, $\hat{y}$ = 2.69 (million). The standard deviation of the forecast error computed from (*B-P*) model, σ_L = 0.47 (million). In newsvendor policy, the value of Z = 1.28. Safety stock is calculated as

$\text{SS} = Z\sigma_L = (1.28)(0.47) = 0.61$ (million)

The order point Q for the month of July 2005, using Equation (5.16), is given by

$Q_{jul} = \hat{y}_L + Z\sigma_L = 2.69 + 0.61 = 3.29$ (million)

The order quantity for stage-2 demand using the customer service level obtained from newsvendor policy and periodic review inventory policy are shown in Table 5.4 and Table 5.5, respectively. The details calculations are shown in *Appendix C*.1.

Table 5.4: Order quantity for stage-2 by *newsvendor* policy (units in million)

Forecasting Approach	Model	Jul	Aug	Sep	Oct	Nov	Dec
Bayesian Probability Model	*B-P*	2.88	4.34	5.31	5.47	3.42	2.50
	BP-I	2.86	4.57	5.17	5.21	3.37	2.50
ARIMA Model	*FARIMA*	2.93	4.10	5.49	5.49	3.20	3.19
	B-FARIMA	2.88	4.25	5.76	6.45	5.02	3.54
Exponential Smoothing	*M-ES*	3.53	4.80	6.42	7.54	4.46	3.40

Table 5.5: Order quantity for stage-2 by *periodic review* policy (units in million)

Forecasting Approach	Model	Jul	Aug	Sep	Oct	Nov	Dec
Bayesian Probability Model	*B-P*	2.92	4.43	5.44	5.57	3.53	2.60
	BP-I	2.89	4.68	5.29	5.33	3.51	2.63
ARIMA Model	*FARIMA*	2.97	4.20	5.62	5.59	3.31	3.29
	B-FARIMA	2.91	4.33	5.91	6.60	5.19	3.70
Exponential Smoothing	*M-ES*	3.67	5.02	6.70	7.84	4.70	3.61

5.7 Optimal Inventory Cost

In this section, the computations to derive the optimal inventory costs for all demand forecasts by using dynamic programming (DP) algorithm are discussed. It had been shown in the work of Wagner-Whitin (1958) that the DP algorithm guarantees the minimum inventory cost, which comprises the cost for replenishment and carrying inventory. This minimum inventory cost is the optimal cost for inventory. The DP method is based on two main assumptions. The assumptions are as follows:

(i) The required quantity must be available at the beginning of a period.

(ii) No shortages are allowed

5.7.1 Optimal Inventory Cost Using Dynamic Programming

Suppose, F_t is defined as the total cost of the best replenishment strategy that satisfies the demand requirements at stage-2 (periods 1, 2, …, t). To illustrate the procedure for finding F_t and the associated replenishment orders, the forecast obtained by Bayesian probability (*B-P*) model is used. The cost components are shown in Table 5.1. A DP algorithm is applied to determine the inventory quantity and procurement costs, which is shown in Table 5.6.

Table 5.6: Inventory quantity and procurement cost by *B-P* model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	71.91*	183.06	322.55	469.58	563.59	633.82
2		180.37*	316.54	460.16	552.03	620.70
3			307.93*	442.66*	528.98*	593.59*
4				444.73	532.34	597.89
5					531.57	598.06
6						603.92
$j^*(k)$	1	2	3	3	3	4
Min	71.91	180.37	307.93	442.66	528.98	593.59

From Table 5.6, the minimum inventory cost over the six-period planning horizon using demand forecast by the *B-P* model for stage-2 is $593.59. There are four replenishments during stage-2. The inventory quantity and time period for replenishment during stage-2 are shown in Table 5.7.

Table 5.7: Inventory quantity and replenishment time for stage-2 (units in million)

	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Q_t	-	3.29	4.49	5.35	5.62	3.51	2.62
Replenishment	1st	2nd	3rd	-	-	4th	-
Q_{optimal}	3.29	4.49	14.48	-	-	2.62	-

In the first and second replenishment, the inventory quantity required for July and August are 3.29 $\times 10^6$ and 4.49 $\times 10^6$, respectively. The third replenishment quantity is 14.48 $\times 10^6$, which is the total quantity required for September, October and November. This replenishment order is placed at the same time (as $j^*_3 = 3$, $j^*_4 = 3$, $j^*_5 = 3$) at the beginning of August. The fourth replenishment, $j^*_6 = 4$, are placed for the inventory requirement in the month of December. The order is placed at an amount $Q^*_6 = 2.62$ at the beginning of November so that inventory will be at hand prior to December. The numerical illustration of the results presented in Table 5.7 is shown in *Appendix C*.2.

5.7.2 Determining Inventory Costs for All Forecasts

In this section, the average inventory costs per period are determined for each inventory policies using the dynamic programming method. The procedure to calculate inventory cost is described in Section 5.5.2. The cumulative inventory costs based on the quantity required per period for stage-2 period (from July to December) in 2005 implementing newsvendor inventory policy is shown in Table 5.8. In this table, the minimum inventory costs for each period for each of the demand forecasts are shown. The number of replenishment orders for stage-2 period and the corresponding replenishment placement time based on newsvendor inventory policy is shown in Table 5.9. A sample computation of inventory costs and replenishment order placement arrangement is shown in Table 5.6 and 5.7, respectively. The minimum inventory cost for each period, replenishment orders and correspond time period for each demand forecasting method are presented in *Appendix C*.3.

Table 5.8: Inventory cost using *newsvendor policy* for stage-2, 2005

Forecasting Approach	Model	Inventory Cost for stage-2 period, 2005					
		Jul	Aug	Sep	Oct	Nov	Dec
Bayesian Probability Model	*B-P*	71.91	180.37	307.93	442.66	528.98	593.59
	BP-I	86.84	206.46	334.67	467.14	554.66	622.14
ARIMA Models	*F-ARIMA*	86.12	194.96	329.91	470.94	556.56	644.62
	BS-ARIMA	65.66	157.28	261.57	381.29	478.75	549.01
Exponential Smoothing	*M-ES*	87.95	197.03	333.68	498.75	597.51	675.71

Table 5.9: Inventory quantity by *newsvendor policy* at stage-2, 2005 (units in million)

Model		-	Jul	Aug	Sep	Oct	Nov	Dec
B-P	Q		3.29	4.49	5.35	5.62	3.51	2.62
	$j^*(k)$	1st	2nd	3rd	-	4th	-	-
	Q_{rep}	3.29	4.49	14.48	-	2.62	-	-
BP-I	Q		3.47	4.78	5.34	5.38	3.47	2.61
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.47	4.78	16.80	-	-	-	-
FARIMA	Q		3.44	4.35	5.62	5.73	3.39	3.40
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.44	4.35	18.14	-	-	-	-
B-FARIMA	Q		2.63	3.66	4.34	4.86	3.86	2.72
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	2.63	3.66	15.78	-	-	-	-
M-ES	Q		3.52	4.36	5.69	6.70	3.91	3.02
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.52	4.36	19.33	-	-	-	-

The optimal inventory costs using periodic review inventory policy and the replenishment costs per period at stage-2 are shown in Table 5.10. The number of replenishment orders for stage-2 period and the corresponding replenishment order placement using periodic review inventory policy is shown in Table 5.11. Detail computation results, optimal inventory cost, replenishment time period for each demand forecast are presented in *Appendix C*.4.

Table 5.10: Inventory cost based on *periodic review* inventory policy

Forecasting Approach	Model	Inventory Cost for stage-2 period, 2005					
		Jul	Aug	Sep	Oct	Nov	Dec
Bayesian Probability Model	*B-P*	85.59	201.14	332.59	474.04	566.01	637.15
	BP-I	90.72	214.24	346.18	482.49	573.94	645.44
ARIMA Models	*F-ARIMA*	89.81	202.36	340.85	485.52	574.87	666.75
	BS-ARIMA	82.00	192.54	328.45	484.18	606.30	693.60
Exponential Smoothing	*M-ES*	91.21	203.56	343.34	511.62	613.68	695.26

Table 5.11: Inventory quantity by *periodic review* at stage-2, 2005 (units in million)

Model		-	Jul	Aug	Sep	Oct	Nov	Dec
B-P	Q		3.42	4.62	5.48	5.74	3.64	2.75
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.42	4.62	17.61	-	-	-	-
BP-I	Q		3.63	4.94	5.50	5.54	3.62	2.76
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.63	4.94	17.42	-	-	-	-
F-ARIMA	Q		3.59	4.50	5.77	5.88	3.54	3.55
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.59	4.50	18.74	-	-	-	-
B-FARIMA	Q		2.71	3.75	4.43	4.95	3.94	2.80
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	2.71	3.75	16.12	-	-	-	-
M-ES	Q		3.65	4.49	5.82	6.83	4.04	3.15
	$j^*(k)$	1st	2nd	3rd	-	-	-	-
	Q_{rep}	3.65	4.49	19.85	-	-	-	-

5.8 Comparison of Forecasting Methods

In this section, the forecasting models are compared based on total inventory cost (TIC), mean absolute percent error (MAPE) and standard deviation of the forecast error. The average inventory costs are determined for actual demand data and each demand forecast using the extended newsvendor model and a periodic review model for an active demand season. The inventory cost associated with actual demand is the least inventory cost, which is considered the base cost reference to the demand forecasts. The relative percent of inventory cost (RPIC) for each forecast is then determined. These percentages are compared and the minimum percentage value is considered the appropriate forecast for the demand data series. The standard errors of forecast, MAPE, the inventory costs and the percent above the least inventory cost for each demand forecast for the given data set are presented in Table 5.12.

Table 5.12: Inventory cost for each forecasting models and actual demand

Forecast Models	Model	Std. Error (million)	MAPE	Inventory Cost (million)		Relative Percentages	
				Newsvendor	Periodic	Newsvendor	Periodic
Bayesian Probability Model	*B-P*	0.47	11.81%	$617.97	$637.15	17.76%	21.41%
	BP-I	0.58	14.84%	$622.14	$645.44	18.55%	22.99%
ARIMA Model	*F-ARIMA*	0.55	13.18%	$644.62	$666.75	22.84%	27.05%
	BS-ARIMA	0.31	7.43%	$549.01	$561.46	4.62%	6.99%
Exponential Smoothing	*M-ES*	0.48	10.54%	$668.46	$695.26	27.38%	32.49%

5.9 Summary

The inventory costs based on several demand forecasts and the cost savings due to improved forecast with appropriate inventory policy are studied in this chapter. A newsvendor inventory model with emergency procurement option and a periodic review policy as an alternate model are applied to each demand forecast. The dynamic programming algorithm is used to derive the lowest inventory cost for each demand forecast and the actual data. The inventory costs to each inventory policy and the corresponding demand forecast of a seasonal product during an active demand season is shown in Table 5.12. The mean absolute percent error (MAPE) for each forecast and the standard deviation of the forecast errors are also shown. Comparing the cost percentages of each demand forecast above the inventory cost of actual demand data, standard error, and mean absolute percent error, it is observed that the Bayesian sampling-based ARIMA (*BS-ARIMA*) model is well-performed forecasting model for dataset.

CHAPTER 6

CONCLUSION AND FUTURE RESEARCH

In this study the demand forecast of a seasonal product is considered. The demand of a seasonal product such as coat, jacket, and woolen apparel are uncertain so that a manager prefers to procure the product as late as possible so that there is enough time to collect recent information, which helps to improve forecast accuracy. The demand cycle is divided into two stages: *stage*-1 is slow demand period, and *stage*-2 is active demand period. The focus of this study was to forecast the demand for *stage*-2.

Forecasting techniques developed here are: (i) model associated with non-negative probability distribution, and (ii) time series ARIMA model and Bayesian ARIMA model. Bayesian statistical techniques were applied to these models. The advantage of using Bayesian techniques in forecasting models are: (a) flexibility to derive the values of unknown parameters, and (b) ability to forecast from incomplete data series. The forecasting models are extended to forecast demand from an incomplete dataset where the assumption is made that there are missing observations in the actual dataset. A multiplicative exponential smoothing model is used to evaluate the forecast derived by the first two sets of models.

After demand forecasts are made, the model performances are tested by several error measures such as relative errors, mean absolute deviation, and tracking signals. An extended newsvendor model and a periodic review model were studied to determine the inventory costs of the demand forecast and actual demand. The inventory cost helps to determine the improved forecast by comparing the cost of each demand forecast with respect to that of actual demand. General conclusion and future direction of forecasting and inventory research were drawn.

6.1 General Conclusion

In this study, the demand forecasting models and the inventory procurement models were evaluated by using an actual time series demand data of a seasonal product. The data set was collected from the 'US census bureau', http://otexa.ita.doc.gov/scripts/tqmon.exe/catdata, in the Department of Commerce, to demonstrate the models. A data series collected from January 1996 to June 2005 were used in the forecasting models to find the parameters. The models were then applied to forecast demand from July to December, 2005. The missing values in the data series often arise in various situations, including market sales, industrial production, shipment, new product sales. In the data series, six observations from July to December, 2004 were considered unavailable to demonstrate the demand forecast from incomplete data. The forecasts were made for same periods from July to December 2005 based on eighty-four available observations, where demand information is unavailable for six periods. Both the forecasting models were extended to forecast from the incomplete data set. The purpose here is to demonstrate many real-life forecasting problems where the data series contains missing values.

The forecasting methods presented in this study differ in accuracy and complexity for deriving the forecast of the seasonal demand. A forecaster may choose a forecast from a number of models that best fulfils the accuracy requirements at a minimum cost. The first set of forecasting models is based on Bayesian approach associated with non-negative probability distribution models. A gamma distribution demand process and an inverse gamma prior distribution were used to construct the forecasting model. The demand forecasting model was named as Bayesian probability (*B-P*) model. The model was then extended to forecast the demand from the incomplete data series (*BP-I* model). In the *BP-I* model, a data normalization approach was adopted to predict the missing values in the data series, and subsequently, the demand were forecasted by using the Bayesian method. The forecasts derived by both models were competitive, but the results by *B-P* model were superior.

In the second set of forecasting models, the seasonal demand is estimated based on ARIMA model. An ARIMA $(0,1,1)(1,1,0)_{12}$ model was found to be the best time series model for the data series. A sampling-based ARIMA (*BS-ARIMA*) model is used to forecast demand from the incomplete data. The data series used in *BS-ARIMA* model is incomplete since the data points at stage-2 (July to December) in 2004 were assumed unobserved. Finally, a multiplicative exponential smoothing (*M-ES*) model is used to forecast the seasonal demand. The parameters of *M-ES* model are constantly updated using the most recent demand data. This forecast is considered the base reference to compare the demand forecasts made previously. Test results of the ARIMA (*F-ARIMA* and *BS-ARIMA*) and *M-ES* models showed that both approaches are significantly accurate. The errors were less than 13% for all models. Checking the tracking signals of the models, it was found that *M-ES* model has negative trends, which indicates that forecast made by *M-ES* model has larger differences with respect to actual demand. Therefore, ARIMA models are appropriate for seasonal forecast and the Bayesian ARIMA model is advantageous among all forecasting models.

The determination of inventory costs during an active selling season, inventory replenishment rate and the corresponding orders quantities are the valuable control policies in supply chain systems. After the forecasts are obtained, an extended newsvendor inventory policy and a periodic review policy were used to determine the inventory cost of each demand forecast and actual demand. A dynamic programming algorithm was used to derive the inventory costs. Table 5.8 – 5.11 showed the inventory costs, replenishment ordering frequency and the order quantity based on each inventory policy and the demand forecast. The inventory costs associated with each demand forecast with respect to the cost of actual demand are used to compare the forecasting methods. The forecast measuring indicators, namely, standard error for the demand forecast (σ_L), mean absolute percent error (MAPE), total inventory costs (TIC) of based on extended newsvendor and periodic inventory models, and the relative percent of inventory cost (RPIC) above the cost of actual demand for all forecasting models are shown in Table 5.12. From the results, it is noticed that the Bayesian sampling-based ARIMA (*BS-ARIMA*) provided the lowest σ_L, MAPE, TIC, and RPIC. The probability distribution model using Bayesian computation approach (*B-P*) model provided the next lower σ_L, MAPE, TIC, and RPIC. From Table 5.12, it is also seen that the improvements in demand forecasting can provide better cost reductions than relying on inventory models to provide cost reductions. Therefore, the forecasts achieved by Bayesian ARIMA (*BS-ARIMA*) were superior, and the Bayesian probability (*B-P*) model may be considered the next alternative forecasting model for the data series.

The Bayesian approaches are an effective logistic process in the context of seasonal demand forecasting. In this study the analysis has a number of implications for practitioners. It is shown that the seasonal demand variability can be directed to prior distributions and the Bayesian models have the flexibility to combine all the demand information by the means of probability distributions. Since the prior distribution is consistent with the observations, the Bayesian estimates and forecasts are expected to carry smaller errors. The models are also useful in the cases where the past demand information limited or incomplete. The Bayesian approach is also flexible for using new information. For example, the forecast can be modified after observing the demand during the peak-demand season. One disadvantage of the approaches is the computational complexity, but the numerical methods (Markov Chain Monte Carlo) included in the software WinBUGS may be used to overcome the computational difficulties. Forecasts made by the *BS-ARIMA* model followed such numerical computations. It has been illustrated that the Bayesian approaches in time series model is an appropriate forecasting technique when the product has seasonal and trends demand. The models are particularly useful when the past demand information is incomplete. The future direction of this research is described in the next section.

6.2 Future Research Direction

Inventory control problem has long being studied on managing certain specific types and sources of uncertainty in the demand process. There are other important sources of uncertainty which have received relatively little attention. The future research areas of controlling inventory of products with fluctuating demands can be best suggested as follows.

6.2.1 Product Subject to Obsolescence

Many products are subject to obsolescence, that is, the demand of a product is strong at present, but it is quite possible that the current demand will drop sharply in future. The examples of such products in industries are the products with high technical innovation, such as computers and pharmaceuticals. The products in the markets which often change according to consumer tastes including books, CDs, perfumes, and some food items are subjected to this category. Since the timing and rate of obsolescence are uncertain, the standard models with uncertain demands are unable to predict this type of products.

6.2.2 New Products

The problems that inventory managers are facing with new products are the potential arising of new version due to the obsolescence of old version. At the beginning phase, the demand of new version just starts to rise, and if the product is successful, the demand increases rapidly. But the demand volume and time of the increase, however, are not predictable. The difficulties of such demand uncertainty are ignored in standard models.

6.2.3 Products Sensitive to Economic Conditions

The demands of many products extensively fluctuate due to certain basic economic or political situations. The variability of these demands could be the sudden change of economy of a society, such as war, sanctions, or change of GNP, interest rates, production environment. If the economical situations are normal, there are regular demands of the products, but the demands are unpredictable for abnormal circumstances. If the abnormal situations remain, models developed in this study may not be adequate to forecast such fluctuating demand. These examples are widely diverse, and consequently, it is difficult to locate he identifiable factors. The expectation is that here presented models can be further modified to fit many of these extreme demand situations.

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APPENDIX A

BAYESIAN PROBABILITY MODELS

A.1 Selection of Demand Distribution Model

In this study the dataset used in forecasting models is collected from the US census bureau, Department of Commerce, which is the partial demand of women's winter apparel in the US, shown in Table A.1.

Table A.1: Demand record of a seasonal product (woolen apparel, 1996 to 2004)

Month		1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Jan	STAGE 1	400,346	614,704	801,200	771,039	844,005	1,232,589	1,499,544	1,382,481	1,630,893	1,803,997
Feb		387,554	403,570	506,609	790,062	572,093	1,257,118	849,768	1,325,336	1,512,496	1,546,309
Mar		282,266	657,449	850,961	852,246	971,594	1,293,980	1,127,911	1,292,599	1,485,483	2,094,937
Apr		577,188	700,022	1,151,522	888,911	1,100,097	1,205,326	1,168,875	1,380,987	1,559,342	1,901,859
May		536,948	656,937	1,326,139	1,343,017	1,111,883	1,243,011	1,172,453	1,514,573	1,384,160	1,906,783
Jun		678,661	1,081,391	1,784,196	1,193,230	1,603,974	1,270,732	1,449,888	1,682,368	1,725,807	2,315,753
$\sum_{k=1}^{6} y_k / 6$		477,160	685,679	1,070,104	973,084	1,033,941	1,250,459	1,211,406	1,429,724	1,549,697	1,928,273
Jul	STAGE 2	1,005,800	1,466,913	1,692,825	1,349,451	2,070,043	2,059,496	2,185,608	2,018,939	2,366,774	2,834,571
Aug		1,239,992	1,829,723	2,697,061	2,018,771	3,108,957	3,207,157	3,017,227	2,994,918	3,097,687	3,328,028
Sep		1,425,662	2,306,741	2,362,882	2,166,497	2,744,466	3,356,925	3,920,854	4,337,951	4,823,764	4,102,475
Oct		1,030,257	2,091,035	1,894,332	1,906,160	2,535,864	3,475,341	3,359,162	5,651,788	5,536,149	5,110,310
Nov		706,931	1,205,276	1,069,811	1,440,326	1,664,043	2,162,664	2,413,125	2,195,432	3,364,407	3,460,540
Dec		668,230	1,106,473	1,007,821	1,007,062	1,349,740	981,247	1,560,131	2,022,672	2,289,976	2,280,498

Sources: U.S. Department of Commerce, Office of Textiles and Apparel. http://otexa.ita.doc.gov/scripts/tqmon.exe/catdata; [Date: 12/11/2006].

A.2 Selection of Demand Distribution Model

Using the demand dataset illustrated in Table A.1, the maximum likelihood estimate (MLE) of a number of non-negative distribution models is shown in Table A.2. The MLE are estimated using the software package SPLIDA, which is a collection of S-PLUS extensions.

Table A.2: The MLE measures of ten probability distribution models

Probability Distribution	Negative Log Likelihood					
	July	August	September	October	November	December
Exponential	-154.6	-157.9	-159.6	-160	-154.9	-151.7
Lognormal	-146	-149.9	-153.1	-156.6	-150.7	-145.8
Weibull	-145.6	-147.7	-152.9	-156.5	-150.7	-146.1
Gamma	-145.5	-149.3	-152.7	-156.5	-150.6	-145.9

A.3 Moment Generating Function of Gamma Distribution Model

The mean and variance of the gamma density can be derived from the moment generation function. For the gamma distribution, the moment generation function is given by

$$M_{Y_t}(y) = \int_0^\infty \frac{\exp(ty)}{\Gamma(\alpha t)\beta}\left(y/\beta\right)^{\alpha t-1}\exp\left[-y/\beta\right]dy \tag{A.1}$$

Change $\frac{y}{\beta}\left(1-\beta t\right) = u$, then $dy = \frac{\beta}{\left(1-\beta t\right)}du$

$$M_{Y_t}(y) = \frac{1}{\left(1-\beta t\right)^{\alpha t}}\int_0^\alpha \frac{u^{\alpha t-1}\exp(-u)}{\Gamma\left(\alpha t\right)}du$$

where $\Gamma\left(\alpha t\right) = \int_0^a u^{at}\exp\left(-u\right)du$ is a gamma function, therefore $M_{Y_t}(y) = \frac{1}{\left(1-\beta t\right)^{\alpha t}}$, where $t < 1/\beta$. Taking the first and second order derivatives, the log of moment generating function,

$$M = \ln M_{Y_t}(y) = -\alpha t\ln(1-\beta t)$$

$$\frac{dM}{dt} = \frac{\alpha t\beta}{1-\beta t}, \quad \text{and} \quad \frac{d^2M}{dt^2} = \frac{\alpha t\beta^2}{\left(1-\beta t\right)^2}$$

The mean and variance of Y_t are determined as

$$\mathrm{E}\left(Y_t\right) = M'(0) = \alpha(t)\beta \text{, and } \mathrm{Var}\left(Y_t\right) = M''(0) = \alpha(t)\beta^2 .$$

Therefore, $\mathrm{E}(Y_t) = (\alpha\, t)\beta$ and $\mathrm{Var}\,(Y_t) = (\alpha\, t)\beta^2$.

A.4 Prior Demand (Inverted Gamma Model)

The business cycle of the seasonal demand is split into two stages and stage-2 is divided into six periods. The demand ratios of the six periods at stage-2 are taken with respect to the demand at stage-1 collected from the past records from 1996 to 2004. [α, β] = invgamfit(D), where D is the demand rate; The MAT LAB code is shown below:

```
function y = invgamfit(D)

 D= 1928273;          % average demand at quarter II (of 1998-2004).
 n = 100;
U = unifrnd(0,1,n,1);
G = zeros(n,1);
for i =1: n
if U(i)<=0.165
G(i)= 1.67* D;
 elseif U(i)>0.165 & U(i)<=0.33
G(i)= 2.4*D;
 elseif U(i)>0.33 & U(i)<=0.50
G(i)= 2.83*D;
 elseif U(i)>0.50 & U(i)<=0.67
G(i)= 2.84*D;
 elseif U(i)>0.67 & U(i)<=0.84
G(i)= 1.65*D;
 else
G(i)= 1.24*D;         % India_supply at 2005
 end
end
GInv = 1./G;
y = gamfit(GInv);
```

Posterior demand forecast (MATLAB-Code)

Demand forecasting from actual data [forecast for 2005]

```
clear;            % clear the memory
mu_D = 1928273.15;
 cov = 0.16;
% COV for demand rate [COV: 0.21, 0.17, 0.24, 0.27, 0.26, 0.29]
% COV for demand rate [COV: 0.16, 0.13, 0.14, 0.25, 0.19, 0.21,
% Average COV: 0.28]


% Month of July 2005 [Revised woolen apparel]
t = [ 7-6];
D = [(11664953.6-9298179.8)];

% Month of August 2005 [Revised woolen apparel]
%t = [ 8-7 ];
%D = [(14762640.4-11664953.6)];
```

```
% Month of September 2005 [Revised woolen apparel]
%t = [ 9-8 ];
%D = [(19586403.9-14762640.4)];

% Month of October 2005 [Revised woolen apparel]
%t = [10-9];
%D = [(25122553.0-19586403.9)];

% Month of November 2005 [Revised woolen apparel]
%t = [11-10];
%D = [(28486960.3-25122553.0)];

% Month of December 2005 [Revised woolen apparel]
%t = [12-11];
%D=[(30776936.0-28486960.3)];


K = length(t);              % total number of inspections
% Inv_COV = [1.; 1.; 1.];
% COV for each observation

% define the grid for the normal density of the measurement error:
% fit a continuous inverted gamma density prior
n = 10000;
par = invgamfit(mu_D);
a = par(1);                  % Shape parameter
b = 1/par(2);                % Scale parameter

% Define the grid over which the densities are calculated:
GridLenth = mu_D/100;
x = GridLenth:GridLenth:8*mu_D;
N = length(x);

% the inverted gamma distributed prior is given by:
Prior = exp(a*log(b)-gammaln(a)+(-a-1)*log(x)-b./x);

% inverted gamma posterior for 1 perfect inspection:555
A = a + t(K)/cov^2;            % Posterior shape parameter
B = b + D(K)/cov^2;            % Posterior scale parameter
Posterior = exp(A*log(B)-gammaln(A)+(-A-1)*log(x)-B./x);

Figure
plot(x,Prior,'g--',x,Posterior,'r', 'LineWidth',2.5);
%(for Fixed COV, Sub-model B-P)

plot(x,Prior,'r:',x,Posterior,'g', 'LineWidth',2.5);
%(for variable COV, Sub-model BP-I)

 grid
legend('prior density','posterior');
title(['Forecast of woolen apparel, 2005']);
xlabel('Periodic demand [July, 2005]');
ylabel('Density');
```

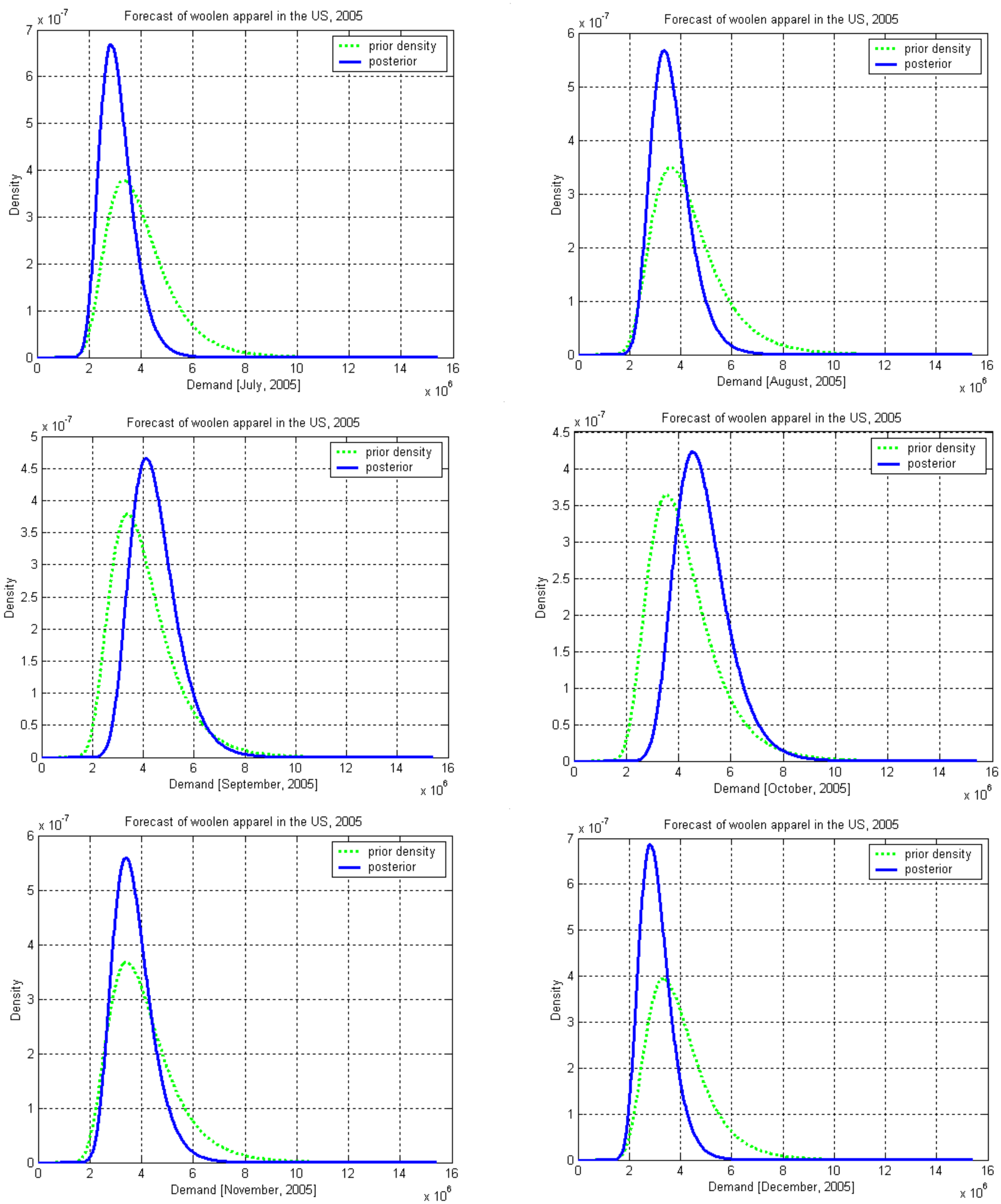

Figure A.1: Graphical presentation of prior and posterior density in (*B*-*P*) model

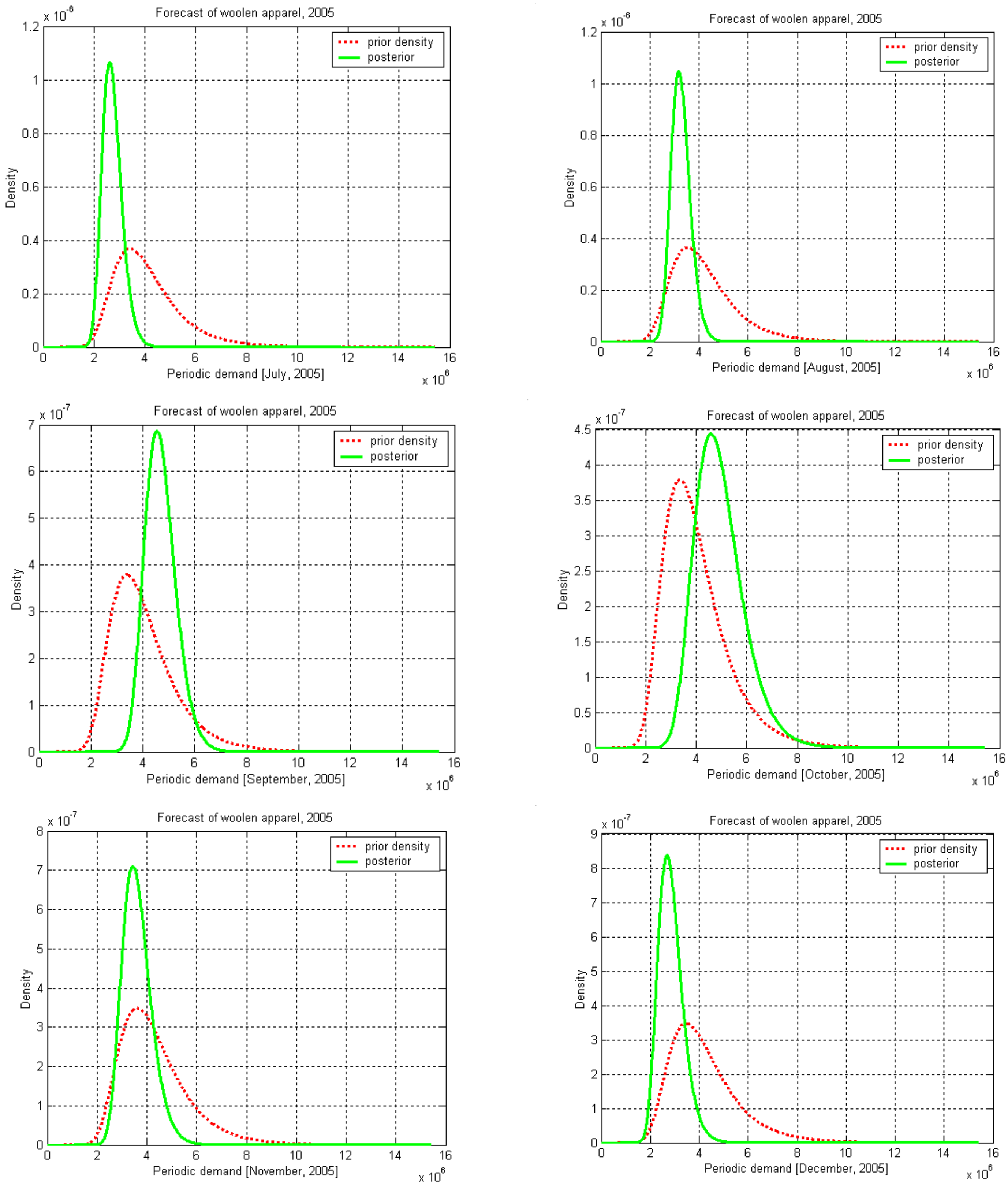

Figure A.2: Graphical presentation of prior and posterior density in (*BP-I*) model

APPENDIX B

ARIMA AND BAYESIAN ARIMA MODELS

B.1 Apparel Group Demand Data (units in millions) for ARIMA (0,1,1) $(1,1,0)_{12}$

Period(s) of Differencing	1,12
Mean of Working Series	3,718.7
Standard Deviation	496,691
Number of Observations	101
Observation(s) eliminated by differencing	13

```
                                Autocorrelations

Lag     Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1      Std Error

  0         1.000     |                    |********************|              0
  1        -0.496     |          **********|   .                |          0.099
  2         0.121     |              .     |**  .               |          0.121
  3        -0.099     |              .   **|    .               |          0.122
  4        -0.015     |              .     |    .               |          0.123
  5        -0.050     |              .    *|    .               |          0.123
  6         0.063     |              .     |*   .               |          0.123
  7        -0.111     |              .   **|    .               |          0.124
  8         0.153     |              .     |*** .               |          0.125
  9        -0.033     |              .    *|    .               |          0.126
 10        -0.023     |              .     |    .               |          0.126
 11         0.204     |              .     |****.               |          0.126
 12        -0.447     |           *********|    .               |          0.130
 13         0.382     |             .      |********            |          0.144
 14        -0.188     |             . ****|     .               |          0.154
 15         0.097     |             .      |**    .             |          0.156
```

```
                          Inverse Autocorrelations

              Lag     Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

                1         0.646     |                .   |*************        |
                2         0.515     |                .   |**********           |
                3         0.436     |                .   |*********            |
                4         0.354     |                .   |*******              |
                5         0.295     |                .   |******               |
                6         0.171     |                .   |***.                 |
                7         0.117     |                .   |** .                 |
                8        -0.001     |                .   |   .                 |
                9        -0.019     |                .   |   .                 |
               10        -0.043     |                .  *|   .                 |
               11        -0.021     |                .   |   .                 |
               12         0.059     |                .   |*  .                 |
               13        -0.076     |                . **|   .                 |
               14        -0.035     |                .  *|   .                 |
               15        -0.021     |                .   |   .                 |
```

Partial Autocorrelations

```
Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

  1       -0.496    |          **********|   .               |
  2       -0.165    |                .***|   .               |
  3       -0.151    |                .***|   .               |
  4       -0.165    |                .***|   .               |
  5       -0.201    |                ****|   .               |
  6       -0.105    |                . **|   .               |
  7       -0.217    |                ****|   .               |
  8       -0.058    |                .  *|   .               |
  9        0.004    |                .   |   .               |
 10       -0.062    |                .  *|   .               |
 11        0.261    |                .   |*****              |
 12       -0.311    |              ******|   .               |
 13        0.085    |                .   |** .               |
 14        0.022    |                .   |   .               |
 15        0.044    |                .   |*  .               |
```

Autocorrelation Check for White Noise

To Lag	Chi-Square	DF	Pr > ChiSq	--------------------Autocorrelations--------------------					
6	28.86	6	<.0001	-0.496	0.121	-0.099	-0.015	-0.05	0.06
12	61.21	12	<.0001	-0.111	0.153	-0.033	-0.023	0.20	-0.45

Maximum Likelihood Estimation

Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|	Lag
MU	1226.80	7889.10	0.16	0.87	0
MA1,1	0.735	0.069	10.59	<.0001	1
AR1,1	-0.35	0.098	-3.59	0.0003	12

Constant Estimate	1657.36
Std Error Estimate	384482.40
AIC	2889.60
SBC	2897.49
Number of Residuals	101.00

Correlations of Parameter Estimates

Parameter	MU	MA1,1	AR1,1
MU	1.000	-0.030	0.013
MA1,1	-0.030	1.000	0.086
AR1,1	0.013	0.086	1.000

Autocorrelation Check of Residuals

To Lag	Chi-Square	DF	Pr > ChiSq	--------------------Autocorrelations--------------------					
6	6.11	4	0.191	-0.066	-0.167	-0.102	-0.125	-0.051	0.003
12	8.53	10	0.577	-0.057	0.089	-0.014	-0.031	0.057	-0.084
18	16.88	16	0.393	0.248	-0.102	-0.010	-0.032	0.003	-0.035
24	29.80	22	0.123	0.026	-0.089	-0.042	-0.199	0.108	0.199

Model for variable D

Estimated Mean 1226.80
Period(s) of Differencing 1,12

Autoregressive Factors

Factor 1: 1 + 0.35 B**(12)

Moving Average Factors
Factor 1: 1 - 0.735 B**(1)

The ARIMA Procedure
(Women Apparel Demand Forecast)

OBS	FORECAST	STD	L95	U95
115	2,743,345	384,482	1,989,132	3,496,275
116	3,652,484	397,727	2,781,752	4,340,812
117	4,921,266	410,544	4,349,932	5,959,235
118	5,265,951	422,973	5,250,699	6,908,722
119	2,690,180	435,047	2,606,103	4,311,455
120	2,702,920	446,795	1,826,754	3,578,157

Detail ARIMA Results
(Women Apparel Demand Forecast)

OBS	DEMAND	FORECAST	STD	L95	U95	RESIDUAL
1	400,346	.	.	.	.	.
2	387,554	.	.	.	.	.
3	282,266	.	.	.	.	.
4	577,188	.	.	.	.	.
5	536,948	.	.	.	.	.
6	678,661	.	.	.	.	.
7	1,005,800	.	.	.	.	.
8	1,239,992	.	.	.	.	.
9	1,425,662	.	.	.	.	.
10	1,030,257	.	.	.	.	.
11	706,931	.	.	.	.	.
12	668,230	.	.	.	.	.
13	614,704	.	.	.	.	.
14	403,570	603,139	509,645	395,748	1,602,025	199,569
15	657,449	394,754	447,858	483,032	1,272,540	262,695
16	700,022	791,246	427,933	47,488	1,629,980	91,224
17	656,937	722,760	419,317	99,086	1,544,606	65,823
18	1,081,391	846,283	415,142	32,620	1,659,946	235,108
19	1,466,913	1,240,650	413,008	431,168	2,050,131	226,263
20	1,829,723	1,537,901	411,889	730,614	2,345,188	291,822
21	2,306,741	1,803,391	411,293	997,271	2,609,511	503,350
22	2,091,035	1,543,709	410,974	738,215	2,349,203	547,326
23	1,205,276	1,367,233	410,802	562,076	2,172,390	161,957
24	1,106,473	1,286,767	410,709	481,792	2,091,743	180,294
25	801,200	1,200,814	398,470	419,827	1,981,801	399,614
26	506,609	934,892	391,585	167,399	1,702,385	428,283
27	850,961	939,677	388,201	178,817	1,700,536	88,716
28	1,151,522	1,047,742	386,459	290,297	1,805,188	103,780
29	1,326,139	1,035,565	385,541	279,918	1,791,212	290,574
30	1,784,196	1,440,540	385,052	685,852	2,195,229	343,656
31	1,692,825	1,898,952	384,790	1,144,778	2,653,125	206,127
32	2,697,061	2,163,470	384,648	1,409,573	2,917,367	533,591
33	2,362,882	2,681,488	384,572	1,927,741	3,435,236	318,606
34	1,894,332	2,319,920	384,531	1,566,253	3,073,586	425,588
35	1,069,811	1,520,466	384,509	766,843	2,274,089	450,655
36	1,007,821	1,325,068	384,497	571,468	2,078,667	317,247
37	771,039	1,025,804	384,490	272,217	1,779,391	254,765
38	790,062	694,710	384,487	58,870	1,448,290	95,352
39	852,246	1,034,211	384,485	280,635	1,787,787	181,965
40	888,911	1,197,713	384,482	444,141	1,951,285	308,802
41	1,343,017	1,215,835	384,482	462,263	1,969,406	127,182
42	1,193,230	1,697,424	384,482	943,853	2,450,996	504,194

0BS	DEMAND	FORECAST	STD	L95	U95	RESIDUAL
43	1,349,451	1,641,607	384,482	888,035	2,395,179	292,156
44	2,018,771	2,345,044	384,482	1,591,472	3,098,616	326,273
45	2,166,497	2,210,846	384,482	1,457,275	2,964,418	44,349
46	1,906,160	1,820,951	384,482	1,067,379	2,574,523	85,209
47	1,440,326	999,153	384,482	245,581	1,752,724	441,173
48	1,007,062	1,042,691	384,482	289,119	1,796,262	35,629
49	844,005	774,097	384,482	20,525	1,527,668	69,908
50	572,093	703,218	384,482	50,354	1,456,790	131,125
51	971,594	831,376	384,482	77,805	1,584,948	140,218
52	1,100,097	999,435	384,482	245,863	1,753,007	100,662
53	1,111,883	1,383,757	384,482	630,185	2,137,329	271,874
54	1,603,974	1,376,983	384,482	623,412	2,130,555	226,991
55	2,070,043	1,508,058	384,482	754,486	2,261,629	561,985
56	3,108,957	2,445,350	384,482	1,691,778	3,198,921	663,607
57	2,744,466	2,601,279	384,482	1,847,707	3,354,851	143,187
58	2,535,864	2,307,430	384,482	1,553,859	3,061,002	228,434
59	1,664,043	1,777,842	384,482	1,024,270	2,531,414	113,799
60	1,349,740	1,446,412	384,482	692,840	2,199,983	96,672
61	1,232,589	1,233,546	384,482	479,974	1,987,118	957
62	1,257,118	1,065,144	384,482	311,573	1,818,716	191,974
63	1,293,980	1,398,739	384,482	645,167	2,152,310	104,759
64	1,205,326	1,468,935	384,482	715,364	2,222,507	263,609
65	1,243,011	1,567,830	384,482	814,259	2,321,402	324,819
66	1,270,732	1,750,317	384,482	996,745	2,503,889	479,585
67	2,059,496	1,982,340	384,482	1,228,769	2,735,912	77,156
68	3,207,157	2,913,624	384,482	2,160,053	3,667,196	293,533
69	3,356,925	2,808,264	384,482	2,054,692	3,561,836	548,661
70	3,475,341	2,728,408	384,482	1,974,836	3,481,980	746,933
71	2,162,664	2,198,463	384,482	1,444,891	2,952,034	35,799
72	981,247	1,834,590	384,482	1,081,018	2,588,161	853,343
73	1,499,544	1,477,082	384,482	723,510	2,230,654	22,462
74	849,768	1,405,176	384,482	651,604	2,158,748	555,408
75	1,127,911	1,423,935	384,482	670,364	2,177,507	296,024
76	1,168,875	1,334,786	384,482	581,214	2,088,358	165,911
77	1,172,453	1,321,118	384,482	567,546	2,074,690	148,665
78	1,449,888	1,474,116	384,482	720,544	2,227,687	24,228
79	2,185,608	2,144,870	384,482	1,391,299	2,898,442	40,738
80	3,017,227	3,266,807	384,482	2,513,236	4,020,379	249,580
81	3,920,854	3,171,678	384,482	2,418,106	3,925,250	749,176
82	3,359,162	3,375,309	384,482	2,621,737	4,128,880	16,147
83	2,413,125	2,214,737	384,482	1,461,165	2,968,309	198,388
84	1,560,131	1,391,818	384,482	638,246	2,145,389	168,313
85	1,382,481	1,733,313	384,482	979,741	2,486,885	350,832
86	1,325,336	1,228,973	384,482	475,401	1,982,545	96,363

OBS	DEMAND	FORECAST	STD	L95	U95	RESIDUAL
87	1,292,599	1,449,604	384,482	696,032	2,203,175	157,005
88	1,380,987	1,405,171	384,482	651,599	2,158,743	24,184
89	1,514,573	1,415,974	384,482	662,403	2,169,546	98,599
90	1,682,368	1,633,529	384,482	879,957	2,387,101	48,839
91	2,018,939	2,402,452	384,482	1,648,880	3,156,023	383,513
92	2,994,918	3,245,120	384,482	2,491,548	3,998,691	250,202
93	4,337,951	3,819,595	384,482	3,066,023	4,573,167	518,356
94	5,651,788	3,635,473	384,482	2,881,902	4,389,045	2,016,315
95	2,195,432	3,096,191	384,482	2,342,620	3,849,763	900,759
96	2,022,672	1,891,136	384,482	1,137,564	2,644,708	131,536
97	1,630,893	1,994,214	384,482	1,240,642	2,747,785	363,321
98	1,512,496	1,634,556	384,482	880,984	2,388,128	122,060
99	1,485,483	1,680,270	384,482	926,698	2,433,842	194,787
100	1,559,342	1,702,106	384,482	948,534	2,455,678	142,764
101	1,384,160	1,753,928	384,482	1,000,357	2,507,500	369,768
102	1,725,807	1,863,972	384,482	1,110,400	2,617,544	138,165
103	2,366,774	2,305,709	384,482	1,552,138	3,059,281	61,065
104	3,097,687	3,248,847	384,482	2,495,275	4,002,419	151,160
105	4,823,764	4,399,307	384,482	3,645,736	5,152,879	424,457
106	5,536,149	5,168,933	384,482	4,415,361	5,922,504	367,216
107	3,364,407	2,692,466	384,482	1,938,894	3,446,038	671,941
108	2,289,976	2,460,510	384,482	1,706,938	3,214,082	170,534
109	1,803,997	2,100,394	384,482	1,346,822	2,853,966	296,397
110	1,546,309	1,926,687	384,482	1,173,115	2,680,259	380,378
111	2,094,937	1,798,626	384,482	1,045,054	2,552,198	296,311
112	1,901,859	1,957,683	384,482	1,204,111	2,711,255	55,824
113	1,906,783	1,877,745	384,482	1,124,173	2,631,317	29,038
114	2,315,753	2,167,722	384,482	1,414,150	2,921,293	148,031
115	.	2,743,345	384,482	1,989,132	3,496,275	.
116	.	3,652,484	397,727	2,781,752	4,340,812	.
117	.	4,921,266	410,544	4,349,932	5,959,235	.
118	.	5,265,951	422,973	5,250,699	6,908,722	.
119	.	2,690,180	435,047	2,606,103	4,311,455	.
120	.	2,702,920	446,795	1,826,754	3,578,157	.

B.2 WinBUGS Code: Bayesian-sampling ARIMA (*BS-ARIMA*)

```
model {
c ~ dnorm(0,0.001)
alpha1 ~ dnorm (0,0.001)
alpha2 ~ dnorm (0,0.001)
alpha3 ~ dnorm (0,0.001)
alpha4 ~ dnorm (0,0.001)
beta ~ dnorm (1,0.1)
tau ~ dchisqr(1)
# priors for innovation and measurement error variances
tau2 ~ dgamma(1,0.001)
```

```
 # Data definitions

for (t in 1:n) { y[t] <- (D[t]) }
for (t in 1:N) { z[t] <- Dummy[t] }
for (t in 1:N) {u[t] ~ dnorm(0,tau2)}
for (j in 1:2) {RM[j] ~ dbeta(1,1)
              rm[j] <- 2*RM[j]-1}
              theta1 <- rm[1]-rm[2]*rm[1]; theta2 <- rm[2]

# Model

for (t in 26:n) { mu[t] <- c + alpha1*y[t-12] + alpha2*y[t-13] + alpha3*y[t-24] +
alpha4*y[t-25] + u[t]-theta1*u[t-1] -theta2*u[t-2] + beta * z[t]
                        y[t] ~ dnorm(mu[t], tau)
                        y.new[t] ~ dnorm(mu[t], tau)
                        sqresid[t] <- pow(mu[t] - y[t],2) }

#ss <- sum(sqresid[2:n])

# Forecasts for t = n+1 ... N

for (t in n+1:N) { mu.new[t] <- c + alpha1*y.new[t-12] + alpha2*y.new[t-13] +
alpha3*y.new[t- 24] + alpha4*y.new [t-25] + u[t]-theta1*u[t-1] -theta2*u[t-2] +
beta*z[t]

   y.new[t] ~ dnorm (mu.new[t], tau)}

}

# Data

list(D=c(400346, 387554, 282266, 577188, 536948, 678661, 1005800, 1239992, 1425662,
1030257, 706931, 668230, 614704, 403570, 657449, 700022, 656937, 1081391, 1466913,
1829723, 2306741, 2091035, 1205276, 1106473, 801200, 506609, 850961, 1151522, 1326139,
1784196, 1692825, 2697061, 2362882, 1894332, 1069811, 1007821, 771039, 790062, 852246,
888911, 1343017, 1193230, 1349451, 2018771, 2166497, 1906160, 1440326, 1007062, 844005,
572093, 971594, 1100097, 1111883, 1603974, 2070043, 3108957, 2744466, 2535864, 1664043,
1349740, 1232589,1257118, 1293980, 1205326, 1243011, 1270732, 2059496, 3207157, 3356925,
3475341, 2162664, 981247, 1499544, 849768, 1127911, 1168875, 1172453, 1449888, 2185608,
3017227, 3920854, 3359162, 2413125, 1560131, 1382481, 1325336, 1292599, 1380987, 1514573,
1682368, 2018939, 2994918, 4337951, 5651788, 2195432, 2022672, 1630893, 1512496, 1485483,
1559342, 1384160, 1725807, 1549696, 1549696, 1549696, 1549696, 1549696, 1549696,
1803997, 1546309, 2094937, 1901859, 1906783, 2315753),


# Dummy Data

Dummy=c(0,0,0,0,0,0,0,0,0,0,0,0,0, 0,0,0,0,0,0,0,0,0,0,0,0,0,
           0,0,0,0,0,0,0,0,0,0,0,0,0,0,   0,0,0,0,0,0,0,0,0,0,0,0,0,0,
           0,0,0,0,0,0,0,0,0,0,0,0,0,0,   0,0,0,0,0,0,0,0,0,0,0,0,0,0,
           0,0,0,0,0,0,0,0,0,0,0,0,0,0,   0,0,0,0,0,0,0,0,0,0,0,0,0,0,
           0,0,0,0,0,0, 0.39, 0.59, 0.63, 0.27, 36, 0.11,
           0,0,0,0,0,0,0,0,0,0,0,0,0), n= 114, N=120)
```

B.3 Parameter Estimates and Demand Forecast of the *BS-ARIMA* Model

Table B.1: Parameter estimate of Bayesian ARIMA model (units in million)

node	mean	St.dev	2.50%	median	97.50%
alpha1	0.15	0.16	-0.17	0.15	0.48
alpha2	0.23	0.15	-0.073	0.23	0.53
alpha3	0.68	0.19	0.29	0.68	1.05
alpha4	0.03	0.22	-0.39	0.03	0.44
theta1	-0.01	0.67	-1.45	-0.008	1.38
beta	1.00	3.22	-5.49	1.05	7.30

Table B.2: Mean, standard errors, and 95% CI using *BS-ARIMA* model

Node	mean	sd	2.50%	median	97.50%
26	481,800	588,700	678,600	482,600	1,618,000
27	400,200	585,400	744,900	398,800	1,544,000
28	654,900	588,500	493,900	659,000	1,817,000
29	660,600	587,700	495,600	665,900	1,821,000
30	793,900	586,300	353,800	792,500	1,943,000
31	1,182,000	591,500	14,010	1,190,000	2,341,000
32	1,496,000	597,800	345,700	1,489,000	2,670,000
33	1,800,000	593,800	636,700	1,800,000	2,988,000
34	1,605,000	608,000	396,700	1,606,000	2,779,000
35	1,181,000	606,200	3,500	1,179,000	2,367,000
36	931,800	594,000	224,100	930,800	2,103,000
37	828,500	591,100	349,800	829,100	1,980,000
38	554,500	583,500	577,800	557,600	1,698,000
39	717,300	589,800	450,300	717,600	1,879,000
40	872,800	591,600	289,600	871,700	2,038,000
41	937,700	594,900	241,300	942,100	2,104,000
42	1,346,000	602,200	156,100	1,349,000	2,524,000
43	1,709,000	595,300	543,600	1,710,000	2,873,000
44	2,105,000	605,500	902,200	2,107,000	3,284,000
45	2,609,000	608,700	1,398,000	2,612,000	3,786,000
46	2,337,000	596,200	1,163,000	2,336,000	3,520,000
47	1,507,000	605,600	303,100	1,513,000	2,709,000
48	1,202,000	587,700	35,370	1,207,000	2,375,000
49	933,300	590,800	232,400	934,500	2,082,000
50	682,700	584,000	456,100	683,300	1,846,000
51	912,100	592,800	236,200	912,000	2,082,000
52	1,140,000	589,400	18,430	1,140,000	2,291,000
53	1,352,000	594,700	185,800	1,350,000	2,515,000
54	1,748,000	599,900	563,900	1,757,000	2,926,000
55	1,703,000	604,700	509,200	1,700,000	2,916,000
56	2,510,000	621,500	1,290,000	2,514,000	3,737,000
57	2,510,000	610,000	1,324,000	2,512,000	3,713,000

Node	mean	sd	2.50%	median	97.50%
58	2,158,000	598,400	978,900	2,162,000	3,330,000
59	1,467,000	596,000	292,300	1,462,000	2,620,000
60	1,216,000	590,500	67,050	1,209,000	2,382,000
61	914,400	587,500	251,400	913,200	2,059,000
62	841,000	594,300	333,300	829,200	2,027,000
63	881,100	585,700	270,800	872,300	2,014,000
64	1,031,000	590,200	137,100	1,031,000	2,199,000
65	1,364,000	601,200	185,700	1,373,000	2,535,000
66	1,369,000	597,000	181,900	1,375,000	2,550,000
67	1,642,000	589,500	504,700	1,640,000	2,799,000
68	2,379,000	619,800	1,153,000	2,375,000	3,583,000
69	2,678,000	609,400	1,499,000	2,667,000	3,902,000
70	2,393,000	584,600	1,258,000	2,391,000	3,551,000
71	1,892,000	592,300	727,200	1,890,000	3,048,000
72	1,335,000	591,900	168,700	1,338,000	2,486,000
73	1,109,000	582,200	28,720	1,117,000	2,261,000
74	905,600	595,600	262,800	905,400	2,061,000
75	1,179,000	594,300	27,290	1,172,000	2,353,000
76	1,267,000	589,300	114,100	1,265,000	2,412,000
77	1,267,000	588,500	99,170	1,261,000	2,437,000
78	1,613,000	596,600	422,700	1,612,000	2,786,000
79	2,075,000	601,000	874,400	2,081,000	3,274,000
80	3,145,000	603,300	1,952,000	3,138,000	4,334,000
81	3,240,000	605,700	2,054,000	3,238,000	4,428,000
82	3,139,000	600,500	1,968,000	3,141,000	4,332,000
83	2,358,000	604,800	1,172,000	2,362,000	3,532,000
84	1,632,000	605,800	437,600	1,626,000	2,826,000
85	1,342,000	593,400	188,200	1,344,000	2,487,000
86	1,379,000	595,500	213,500	1,378,000	2,559,000
87	1,289,000	588,300	147,600	1,287,000	2,441,000
88	1,304,000	590,100	136,400	1,302,000	2,473,000
89	1,347,000	590,800	176,900	1,346,000	2,485,000
90	1,402,000	595,100	231,100	1,398,000	2,559,000
91	2,124,000	598,600	957,900	2,125,000	3,303,000
92	3,218,000	619,600	1,995,000	3,218,000	4,424,000
93	3,694,000	600,900	2,514,000	3,694,000	4,898,000
94	3,901,000	613,200	2,706,000	3,904,000	5,129,000
95	2,738,000	628,300	1,512,000	2,740,000	3,974,000
96	1,548,000	613,500	335,000	1,544,000	2,756,000
97	1,625,000	596,500	445,800	1,631,000	2,774,000
98	1,157,000	590,400	13,950	1,154,000	2,309,000
99	1,312,000	592,200	178,200	1,313,000	2,464,000
100	1,355,000	590,700	210,600	1,355,000	2,532,000
101	1,390,000	591,000	232,900	1,391,000	2,541,000
102	1,624,000	588,700	473,400	1,626,000	2,773,000
103	2,226,000	591,900	1,075,000	2,220,000	3,388,000
104	3,067,000	615,600	1,870,000	3,071,000	4,266,000
105	4,121,000	615,300	2,903,000	4,120,000	5,302,000
106	4,313,000	670,800	2,986,000	4,317,000	5,624,000

Node	mean	sd	2.50%	median	97.50%
107	3,399,000	704,600	2,054,000	3,395,000	4,798,000
108	1,968,000	607,100	781,900	1,967,000	3,152,000
109	1,720,000	587,800	553,600	1,725,000	2,877,000
110	1,568,000	589,300	409,900	1,568,000	2,733,000
111	1,507,000	590,400	345,600	1,501,000	2,674,000
112	1,564,000	579,800	416,300	1,565,000	2,705,000
113	1,653,000	594,300	484,600	1,654,000	2,834,000
114	1,775,000	591,300	629,000	1,774,000	2,941,000
115	2,232,000	798,000	712,400	2,214,000	3,822,000
116	3,270,000	848,500	1,631,000	3,255,000	4,975,000
117	3,951,000	839,000	2,276,000	3,950,000	5,612,000
118	4,467,000	820,700	2,885,000	4,464,000	6,109,000
119	3,466,000	875,100	1,700,000	3,489,000	5,152,000
120	2,322,000	846,900	643,800	2,328,000	3,985,000

B.4 Initial Parametric Values of Multiplicative Exponential Smoothing (*M-ES*) Model

As a rule of thumb, a minimum of two full seasons (or 2*L* periods) of historical data is needed to initialize a set of seasonal factors. Suppose data from m seasons are available and let $\bar{x}_j$, $j = 1, 2, \cdots, mL$ denote the average of the observations during the jth season.

(i) *Estimation of trend component*: The initial estimate of the trend component $\bar{G}_0$ is given by

$$\bar{G}_0 = \frac{\bar{y}_m - \bar{y}_1}{(m-1)L} \tag{B.1}$$

(ii) *Estimation of Index level*: The initial estimate the index level $\bar{R}_0$ (which represents the average level of time series at time 0) is given by

$$\bar{R}_0 = \bar{y}_1 - (L/2)\,\bar{G}_0 \tag{B.2}$$

(iii) *Estimation of seasonal components*: Seasonal factors are computed for each time period *t* = 1, 2, · · ·, *mL* as the ratio of actual observation to the average seasonally adjusted value for that season, further adjusted by the trend, that is,

$$\bar{S}_t = \frac{\bar{y}_t}{\bar{y}_i - [(L+1)/2 - j]\,\bar{G}_0} \tag{B.3}$$

where $\bar{x}_i$ is the average for the season corresponding to the t index, and *j* is the position of the period t within the season. The above equation will produce *m* estimates of the seasonal factor for each period.

$$\bar{S}_t = \frac{1}{m}\sum_{k=0}^{m-1}\bar{S}_{t+kL} \qquad t = 1, 2, \ldots, L \tag{B.4}$$

$$\bar{S}_t(0) = \bar{S}_t \frac{L}{\sum_{t=1}^{L}\bar{S}_t} \qquad t = 1, 2, \ldots, L \tag{B.5}$$

B.5 Numerical Illustration (*M-ES*) Model

In Equation (B.1), the initial estimate of the trend, $\bar{G}_0 = (\bar{y}_m - \bar{y}_1)/[(m-1)L]$, where $\bar{y}_m$ is the average of the observations in year *m*, measures the average level of the time series. Similarly, $\bar{y}_1$ measures the average level of the time series for year 1. Thus ($\bar{y}_m - \bar{y}_1$) measures the difference in these average levels. The total number of seasons between year 1 and the year *m* is (*m*-1)*L*. The initial estimate of $\bar{G}_0$ is the change in average level per season from year 1 to the year *m*. In this model, the proposed apparel data series (in Table A.1, Appendix A) consists of three year of monthly data (from 2002 to 2004) to forecast demand in year 2005, then *m* = 3 and *L* = 12. The average sales for year 1 is $\bar{y}_1$ = 2,316,670, while the average sales for year 3 is $\bar{y}_3$ = 2,740,505. Thus the initial estimate of the trend component is given by

$$\bar{G}_0 = (2{,}740{,}505 - 2{,}316{,}670) / [(3-1)12] = 17{,}660$$

The number of seasons that have elapsed from the start of year 1 to the middle of year is *L*/2. Following Equation (B.2), the initial estimate of index level $\bar{R}_0$ is therefore,

$$\bar{R}_0 = \bar{y}_1 - (L/2)\,\bar{G}_0 = 2{,}316{,}670 - (12/2)\,(17{,}660) = 2{,}210{,}711.$$

The initial estimate of seasonal factors is obtained following Equation (B.3). The calculation of S_t value for the first June (6th month) for three-year sale history is illustrated as follows. The seasonal factor for June (6th month) for the year 1 (*i.e.*, 2003) is given by

$$\overline{S}_6 = \frac{y_6}{\overline{y}_1 - [(L+1)/2 - j]\,\overline{G}_0}$$

$$= \frac{1{,}682{,}368}{2{,}316{,}670 - [(12+1)/2 - 6](17{,}660)} = 0.73$$

where demand at June 2003, y_5 is 1,682,368, which the average demand of year 2003, $\overline{y}_1$ is 2,316,670; L = 12, and initial estimate of trend, $\overline{G}_0$ is 17,660. Similarly, the seasonal factor of the month of June for the year 2 (*i.e.*, 2004) is given by

$$\overline{S}_{18} = \frac{y_{18}}{\overline{y}_2 - [(L+1)/2 - 6]\,\overline{G}_0}$$

$$= \frac{1{,}725{,}807}{2{,}564{,}745 - [(12+1)/2 - 18](17{,}660)} = 0.68\,.$$

The seasonal factor of June for the year 3 (*i.e.*, 2005) is given by

$$\overline{S}_{30} = \frac{y_{30}}{\overline{y}_3 - [(L+1)/2 - 6]\,\overline{G}_0}$$

$$= \frac{2{,}315{,}753}{2{,}740{,}505 - [(12+1)/2 - 18](17{,}660)} = 0.85\,.$$

The average seasonal index for each different season, $\overline{S}_t(0)$ is given by Equation (B.5). After obtaining the initial seasonal factor, the seasonal indices for the next years are adapted by

$$\overline{S}_t = \gamma \frac{y_t}{\overline{R}_t} + (1-\gamma)\,\overline{S}_{t-L} \qquad \text{(B.6)}$$

where $0 < \gamma < 1$ is the third smoothing constant. Using Equation (B.6), the initial average seasonal index for different seasons is illustrated in Table B.3.

Table B.3: Initial average seasonal index for different season

Month	S_1	S_2	S_3	$\bar{S}_t$	$L/\sum_{t=1}^{L}\bar{S}_t$	$\bar{S}_t(0)$
1	0.62	0.66	0.68	0.66	1.007	0.660
2	0.59	0.61	0.58	0.59	1.007	0.598
3	0.57	0.59	0.78	0.65	1.007	0.654
4	0.61	0.62	0.71	0.64	1.007	0.649
5	0.66	0.55	0.70	0.64	1.007	0.641
6	0.73	0.68	0.85	0.75	1.007	0.756
7	0.87	0.92	1.03	0.94	1.007	0.946
8	1.28	1.20	1.20	1.23	1.007	1.234
9	1.84	1.85	1.47	1.72	1.007	1.732
10	2.38	2.11	1.89	2.13	1.007	2.142
11	0.92	1.27	1.23	1.14	1.007	1.147
12	0.84	0.86	0.80	0.83	1.007	0.840
			Total =	11.91		

Since $\bar{G}_0$ = 17,660, $\bar{R}_0$ = 2,210,711, $S_1(0)$ = 0.66, then $\hat{y}_1(0) = [\bar{R}_0 + \bar{G}_0] S_1(0)$ = [2,210,711 + 17,660] [0.11] = 1,471,073. Updating equations for level, trend, and seasonal factors are

$$\bar{R}_1 = \alpha\,[(y_1 / \bar{S}_1(0)] + (1-\alpha)\left(\bar{R}_0 + \bar{G}_0\right)$$

$$= 0.2[1{,}382{,}481 / 0.66] + 0.8\,[17{,}660 + 2{,}210{,}711] = 2{,}201{,}532$$

$$\bar{G}_1 = \beta\left(\bar{R}_1 - \bar{R}_0\right) + (1-\beta)\,\bar{G}_0$$

$$= 0.1[2{,}059{,}193 - 2{,}210{,}711] + 0.9\,[17{,}660] = 742$$

$$\bar{S}_1 = \gamma\,(y_1 / \bar{R}_1) + (1-\gamma)\,\bar{S}_0$$

$$= 0.05[1{,}382{,}481 / 2{,}059{,}193] + 0.95\,[0.66] = 0.66$$

The estimate for period 2 (February, 2002) is

$$\hat{y}_2(1) = \left[\bar{R}_1 + \bar{G}_1\right]\bar{S}_2(0) = [2{,}059{,}193 + 741.97]\ 0.59 = 1{,}326{,}191.$$

Therefore, y_2 = 1,325,336, and the forecast error of [$y_2 - \hat{y}_2(0)$] = 1,325,336 – 1,326,191 = - 855. The updated demand estimates and actual demand for stage-2 at year 2005 obtained by using 30 observations are shown in Table B.4.

Table B.4: Forecast results by *M-ES* model

Year	Month	Seasonal Factor S_t	Level R_t	Trend G_t	Forecast $\hat{y}_t$	Actual y_t	Error $E = \hat{y}_t - y_t$	RSE $(E/y_t)100$
			2,210,711	17,660				
2003	Jan	0.660	2,201,532	14,976	1,471,073	1,382,481	88,592	-6.41
	Feb	0.598	2,216,222	14,947	1,326,191	1,325,336	855	-0.06
	Mar	0.654	2,180,041	9,834	1,459,866	1,292,599	167,267	-12.94
	Apr	0.649	2,177,770	8,624	1,420,241	1,380,987	39,254	-2.84
	May	0.641	2,221,673	12,152	1,401,502	1,514,573	113,071	7.47
	Jun	0.756	2,232,086	11,978	1,688,943	1,682,368	6,575	-0.39
	Jul	0.946	2,221,906	9,762	2,123,789	2,018,939	104,850	-5.19
	Aug	1.234	2,270,610	13,656	2,754,587	2,994,918	240,331	8.02
	Sep	1.732	2,328,234	18,053	3,957,118	4,337,951	380,833	8.78
	Oct	2.142	2,404,815	23,906	5,025,037	5,651,788	626,751	11.09
	Nov	1.147	2,325,849	13,619	2,785,308	2,195,432	589,876	-26.87
	Dec	0.840	2,353,162	14,988	1,965,159	2,022,672	57,513	2.84
2004	Jan	0.659	2,389,821	17,155	1,585,105	1,630,893	45,788	2.81
	Feb	0.598	2,431,171	19,575	1,466,303	1,512,496	46,192	3.05
	Mar	0.651	2,416,800	16,180	1,584,445	1,485,483	98,962	-6.66
	Apr	0.648	2,427,791	15,661	1,582,936	1,559,342	23,594	-1.51
	May	0.643	2,385,262	9,842	1,540,163	1,384,160	156,004	-11.27
	Jun	0.756	2,372,671	7,599	1,799,386	1,725,807	73,579	-4.26
	Jul	0.945	2,405,377	10,110	2,281,466	2,366,774	85,308	3.60
	Aug	1.239	2,432,601	11,821	3,027,542	3,097,687	70,145	2.26
	Sep	1.739	2,510,351	18,414	4,397,216	4,823,764	426,547	8.84
	Oct	2.152	2,537,494	19,287	5,502,506	5,536,149	33,643	0.61
	Nov	1.137	2,637,398	27,349	3,028,953	3,364,407	335,454	9.97
	Dec	0.841	2,676,394	28,513	2,274,773	2,289,976	15,203	0.66
Stage-1 2005	Jan	0.660	2,710,806	29,103	1,807,630	1,803,997	3,633	-0.20
	Feb	0.599	2,707,794	25,892	1,638,844	1,546,309	92,535	-5.98
	Mar	0.649	2,832,133	35,736	1,862,414	2,094,937	232,523	11.10
	Apr	0.648	2,881,697	37,119	1,890,081	1,901,859	11,778	0.62
	May	0.640	2,931,007	38,338	1,900,110	1,906,783	6,673	0.35
	Jun	0.755	2,989,304	40,334	2,285,950	2,315,753	29,803	1.29
Stage-2 2005	Jul	0.95	3,022,677	39,638	2,898,443	2,834,571	63,871	-2.25
	Aug	1.24	2,986,503	32,057	3,743,902	3,328,028	415,873	-12.50
	Sep	1.75	2,884,235	18,624	5,074,236	4,102,475	971,761	-23.69
	Oct	2.15	2,815,444	9,883	6,084,632	5,310,310	774,322	-14.58
	Nov	1.14	2,865,449	13,895	3,292,890	3,460,540	167,650	4.84
	Dec	0.84	2,845,346	10,495	2,403,797	2,280,498	123,299	-5.41

APPENDIX C

INVENTORY COST REDUCTION MODELS

C.1: Order Quantities for Each Forecasting Model

Table C.1: Order points using extended newsvendor and periodic review policies

Models	Newsvendor				Periodic review		
	Q	σ_L	K	OP	σ_L	K	OP
Bayesian probability model (*B-P*)	2,687,466	474,452	1.28	3,294,765	474,452	1.55	3,422,867
	3,886,368	474,452	1.28	4,493,666	474,452	1.55	4,621,768
	4,740,960	474,452	1.28	5,348,258	474,452	1.55	5,476,360
	5,008,807	474,452	1.28	5,616,106	474,452	1.55	5,744,208
	2,907,290	474,452	1.28	3,514,589	474,452	1.55	3,642,691
	2,014,313	474,452	1.28	2,621,612	474,452	1.55	2,749,714
	Q	σ_L	K	OP	σ_L	K	OP
Bayesian probability, incomplete data model (*BP-I*)	2,735,165	576,280	1.28	3,472,804	576,280	1.55	3,628,400
	4,046,625	576,280	1.28	4,784,263	576,280	1.55	4,939,859
	4,603,972	576,280	1.28	5,341,611	576,280	1.55	5,497,206
	4,642,154	576,280	1.28	5,379,792	576,280	1.55	5,535,388
	2,728,824	576,280	1.28	3,466,463	576,280	1.55	3,622,058
	1,870,595	576,280	1.28	2,608,233	576,280	1.55	2,763,829
	Q	σ_L	K	OP	σ_L	K	OP
Fundamental ARIMA model (*F-ARIMA*)	2,743,345	547,498	1.28	3,444,144	547,498	1.55	3,591,968
	3,652,484	547,498	1.28	4,353,282	547,498	1.55	4,501,107
	4,921,267	547,498	1.28	5,622,065	547,498	1.55	5,769,890
	5,026,596	547,498	1.28	5,727,394	547,498	1.55	5,875,218
	2,690,180	547,498	1.28	3,390,978	547,498	1.55	3,538,803
	2,702,920	547,498	1.28	3,403,718	547,498	1.55	3,551,543
	Q	σ_L	K	OP	σ_L	K	OP
Bayesian-sampling ARIMA model (*BS-ARIMA*)	2,232,000	307,759	1.28	2,625,932	307,759	1.55	2709,027
	3,270,000	307,759	1.28	3,663,932	307,759	1.55	3747,027
	3,951,000	307,759	1.28	4,344,932	307,759	1.55	4428,027
	4,468,000	307,759	1.28	4,861,932	307,759	1.55	4945,027
	3,466,000	307,759	1.28	3,859,932	307,759	1.55	3943,027
	2,322,000	307,759	1.28	2,715,932	307,759	1.55	2799,027
	Q	σ_L	K	OP	σ_L	K	OP
Multiplicative Exponential Smoothing (*M-ES*)	2,898,443	483,472	1.28	3,517,286	483,472	1.55	3,647,824
	3,743,902	483,472	1.28	4,362,746	483,472	1.55	4,493,283
	5,074,236	483,472	1.28	5,693,080	483,472	1.55	5,823,618
	6,084,632	483,472	1.28	6,703,476	483,472	1.55	6,834,013
	3,292,890	483,472	1.28	3,911,734	483,472	1.55	4,042,271
	2,403,797	483,472	1.28	3,022,640	483,472	1.55	3,153,178

C.2 Numerical Illustration of Inventory Cost (units in million)

Step 1: One choice, just satisfy order point, Q_1.

$$F_1^* = A_1 + C_1 Q_1 = 0.015 + 25(2.87) = 71.91$$

$$j_2^* = 1$$

Step 2: Two choices, either j_1* = 1 or j_2* = 2.

$$F_2^* = \min \begin{cases} A_1 + C_1 Q_1 + h_1 Q_2, & \text{procure by 1} \\ F_1^* + A_2 + C_2 Q_2, & \text{procure by 2} \end{cases}$$

$$= \min \begin{cases} 0.015 + 25(2.88 + 4.34) + .624(4.34) = 183.06 \\ 71.91 + 25(4.34) + 0.014 \qquad\qquad = 180.37 \end{cases}$$

$$= 180.37$$

$$j_2^* = 2$$

Step3: Two choices, j_3* = 2, 3.

$$F_3^* = \min \begin{cases} A_1 + C_1(Q_1 + Q_2 + Q_3) + h_1(Q_2 + Q_3) + h_2)Q_3, & \text{procure by 1} \\ F_1^* + A_2 + C_2(Q_2 + Q_3) + h_2 Q_3, & \text{procure by 2} \\ F_2^* + A_3 + C_3(Q_3), & \text{procure by 3} \end{cases}$$

$$= \min \begin{cases} 0.015 + 313.20 + 6.02 + 3.32 = 322.55 \\ 719.09 + 0.014 + 241.31 + 3.32 = 316.54 \\ 180.37 + 0.016 + 127.54 \qquad = 307.93 \end{cases}$$

$$= 307.93$$

$$j_3^* = 3$$

Step 4: Two choices, j_4* = 3, 4.

$$F_4^* = \min \begin{cases} F_2^* + A_3 + C_3(Q_3 + Q_4) + h_3 Q_4, & \text{procure by 3} \\ F_3^* + A_4 + \mathrm{C}_4 \mathrm{Q}_4, & \text{procure by 4} \end{cases}$$

$$= \min \begin{cases} 180.37 + 0.016 + 258.86 + 3.41 = 442.66 \\ 307.93 + 0.016 + 136.78 \qquad = 444.73 \end{cases}$$

$$= 442.66$$

$$j_4^* = 3$$

Step 5: Three choices, j_5* = 3, 4, 5.

$$F_5^* = \min \begin{cases} F_2^* + A_3 + C_3(Q_3 + Q_4 + Q_5) + h_3(Q_4 + Q_5) + h_4 Q_5, & \text{procure by 3} \\ F_3^* + A_4 + C_4(Q_4 + Q_5) + h_4 Q_5, & \text{procure by 4} \\ F_4^* + A_5 + \mathrm{C}_5 \mathrm{Q}_5, & \end{cases}$$

$$= \min \begin{cases} 180.37 + 0.016 + 340.91 + 55.46 + 2.13 = 528.98 \\ 307.93 + 0.016 + 222.26 + 2.13 = 532.34 \\ 442.65 + 0.017 + 88.89 = 581.56 \end{cases}$$

$$= 528.98$$

$$j_5^* = 3$$

Step 6: Four choices, j6* = 3, 4, 5, 6.

- And so on.

C.3 Optimal Inventory Cost Using Newsvendor Policy (units in million)

Table C.2: Inventory cost based on (*B-P*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	71.91*	183.06	322.55	469.58	563.59	633.82
2		180.37*	316.54	460.16	552.03	620.70
3			307.93*	442.66*	528.98*	593.59*
4				444.73	532.34	597.89
5					531.57	598.06
6						603.92
Min	71.91	180.37	307.93	442.66	528.98	593.59
$j^*(k)$	1	2	3	3	3	4

Table C.3: Inventory cost based on forecasts for (*BP-I*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	86.84*	209.43	349.63	494.20	589.51	662.86
2		206.46*	343.33	484.54	577.69	649.40
3			334.67*	467.14*	554.66*	622.14*
4				469.18	558.01	626.47
5					557.29	626.73
6						632.93
Min	86.84	206.46	334.67	467.14	554.66	622.14
$j^*(k)$	1	2	3	3	3	3

Table C.4: Inventory cost based on (*F-ARIMA*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	86.12*	197.67	345.24	499.14	592.38	688.09
2		194.96*	339.02	489.36	580.48	674.07
3			329.91*	470.94*	556.56*	644.62*
4				473.11	560.00	649.34
5					559.12	649.74
6						658.69
Min	86.12	194.96	329.91	470.94	556.56	644.62
$j^*(k)$	1	2	3	3	3	3

Table C.5: Inventory cost based on (*BS-ARIMA*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	65.66*	159.55	273.59	404.24	510.38	586.75
2		157.28*	268.61	396.23	499.95	574.63
3			261.57*	381.29*	478.75*	549.01*
4				383.13	482.04	553.33
5					481.67	553.97
6						560.24
Min	65.66	157.28	261.57	381.29	478.75	549.01
$j^*(k)$	1	2	3	3	3	3

Table C.6: Inventory cost based on (*M-ES*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	87.95*	199.74	349.17	529.31	636.86	721.86
2		197.03*	342.91	518.86	623.98	707.09
3			333.68*	498.75*	597.51*	675.71*
4				501.28	601.52	680.86
5					600.47	680.94
6						688.21
Min	87.95	197.03	333.68	498.75	597.51	675.71
$j^*(k)$	1	2	3	3	3	3

C.4: Optimal Inventory Cost Using Periodic Review Policy (units in million)

Table C.7: Inventory cost based on (*B-P*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	85.59*	204.01	347.76	502.12	602.28	679.60
2		201.14*	341.47	492.25	590.13	665.74
3			332.59*	474.04*	566.01*	637.15*
4				476.21	569.56	641.73
5					568.77	641.97
6						648.52
Min	85.59	201.14	332.59	474.04	566.01	637.15
$j^*(k)$	1	2	3	3	3	3

Table C.8: Inventory cost based on (*BP-I*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	90.72*	217.30	361.59	510.34	609.93	687.65
2		214.24*	355.10	500.39	597.72	673.71
3			346.18*	482.49*	573.94*	645.44*
4				484.59	577.40	649.94
5					576.68	650.26
6						656.87
Min	90.72	214.24	346.18	482.49	573.94	645.44
$j^*(k)$	1	2	3	3	3	3

Table C.9: Inventory cost based on (*F-ARIMA*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	89.81*	205.15	356.60	514.48	611.78	711.65
2		202.36*	350.20	504.42	599.51	697.16
3			340.85*	485.52*	574.87*	666.75*
4				487.75	578.42	671.64
5					577.55	672.10
6						681.43
Min	89.81	202.36	340.85	485.52	574.87	666.75
$j^*(k)$	1	2	3	3	3	3

Table C.10: Inventory cost based on (*BS-ARIMA*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	82.00*	195.28	343.91	513.85	646.85	741.74
2		192.54*	337.63	503.62	633.61	726.39
3			328.45*	484.18*	606.30*	693.60*
4				486.57	610.52	699.08
5					609.96	699.79
6						707.55
Min	82.00	192.54	328.45	484.18	606.30	693.60
$j^*(k)$	1	2	3	3	3	3

Table C.11: Inventory cost based on (*M-ES*) Model (units in million)

F_j	Cost of ordering at stage-2, 2005					
	Jul	Aug	Sep	Oct	Nov	Dec
1	91.21*	206.35	359.20	542.85	653.99	742.66
2		203.56*	352.78	532.16	640.78	727.48
3			343.34*	511.62*	613.68*	695.26*
4				514.21	617.78	700.55
5					616.74	700.69
6						708.29
Min	91.21	203.56	343.34	511.62	613.68	695.26
$j^*(k)$	1	2	3	3	3	3

VITA

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