Magnetic-Field and Screening Effects in Condensed and Ultradense Matter.

Keith Miles Roussel
Louisiana State University and Agricultural & Mechanical College

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in

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by
Keith Miles Roussel
B.S., Louisiana State University, 1969
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**TABLE OF CONTENTS**

<table>
<thead>
<tr>
<th>Acknowledgements</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Tables</td>
<td>v</td>
</tr>
<tr>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td>Abstract</td>
<td>vii</td>
</tr>
<tr>
<td>Chapter I: Introduction</td>
<td>1</td>
</tr>
<tr>
<td>A. Observations of Magnetic Fields in the Universe</td>
<td>1</td>
</tr>
<tr>
<td>B. Production Mechanisms of Magnetic Fields</td>
<td>19</td>
</tr>
<tr>
<td>C. Plasma Screening Effects</td>
<td>29</td>
</tr>
<tr>
<td>Chapter II: Landau Orbital Ferromagnetism</td>
<td>34</td>
</tr>
<tr>
<td>A. Thermodynamics in a Magnetic Field</td>
<td>34</td>
</tr>
<tr>
<td>B. Non-Relativistic Electron Gas</td>
<td>44</td>
</tr>
<tr>
<td>C. Relativistic Electron Gas</td>
<td>47</td>
</tr>
<tr>
<td>D. Non-Relativistic Gas Interacting Through a Coulomb Potential</td>
<td>51</td>
</tr>
<tr>
<td>E. Effective Mass Corrections</td>
<td>57</td>
</tr>
<tr>
<td>F. Non-Zero Temperature and LOFER State Stability</td>
<td>65</td>
</tr>
<tr>
<td>G. General Conclusions About Landau Orbital Ferromagnetism</td>
<td>68</td>
</tr>
<tr>
<td>Chapter III: Continuum Polarization in the Magnetic White Dwarfs</td>
<td>71</td>
</tr>
<tr>
<td>A. Discussion of Polarization</td>
<td>71</td>
</tr>
<tr>
<td>B. Vector Polarization - Stokes Parameters</td>
<td>72</td>
</tr>
</tbody>
</table>
C. Optically Thin Models for Continuum Polarization 86
D. The Optically Thick Model of Continuum Polarization 94

CHAPTER IV: SCREENING EFFECTS IN CONDENSED AND ULTRADENSE MATTER 106

A. Screening Effects in the Absence of External Magnetic Fields 107
B. Screening Effects in the Presence of External Magnetic Fields 145
C. Summary of Screening Effects 153

BIBLIOGRAPHY 155
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Energies of the lowest 45 states for the Debye-Hückel potential</td>
<td>155</td>
</tr>
<tr>
<td>II</td>
<td>Energies of the lowest 45 states for the completely screened Coulomb potential with $A=D/2$</td>
<td>161</td>
</tr>
<tr>
<td>III</td>
<td>Energies of the lowest 45 states for the completely screened Coulomb potential with $A=D$</td>
<td>164</td>
</tr>
<tr>
<td>IV</td>
<td>Transition Probabilities for the Debye-Hückel potential for the transitions $2p\cdot ls, 2p\cdot 2s, 3s\cdot 3p, 3p\cdot ls$</td>
<td>167</td>
</tr>
<tr>
<td>V</td>
<td>Transition Probabilities for the Debye-Hückel potential for the transitions $3p\cdot 2s, 3p\cdot 3s, 3d\cdot 2p, 3d\cdot 3p$</td>
<td>171</td>
</tr>
<tr>
<td>VI</td>
<td>The fitting coefficients for the energy levels of the Debye-Hückel potential</td>
<td>174</td>
</tr>
<tr>
<td>VII</td>
<td>The fitting coefficients for the squares of the radial matrix elements of the Debye-Hückel potential</td>
<td>175</td>
</tr>
<tr>
<td>VIII</td>
<td>The fitting coefficients for the transition probabilities of the Debye-Hückel potential</td>
<td>176</td>
</tr>
<tr>
<td>IX</td>
<td>The critical screening lengths which the nth $s$ state is unbound</td>
<td>177</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
</table>
| 1      | Wavelength dependence of the predicted circular polarization for the various models. Curve 1 corresponds to the optically thin model of Kemp.  
Curve 2 corresponds to the bremsstrahlung model with the parameters 
T=12,000°K, log g = 8, He=0.9, He=0.1, B=1.2x10^7 gauss. Curve 3 corresponds to the bremsstrahlung model with the parameters 
T=14,000°K, log g = 8, He=1.0, B=2x10^7 gauss. The observed circular polarizations of Angel and Landstreet are shown with triangles. | 178  |
| 2      | Wavelength dependence of the predicted linear polarization for the various models. Curve 1 corresponds to the optically thin model of Chanmugam et al.  
Curve 2 corresponds to Model C with the parameters 
T=12,000°K, log g = 8, He=0.9, He=0.1, B=5x10^7 gauss. Curve 3 corresponds to Model C with the parameters 
T=14,000°K, log g = 8, He=1.0, B=5.5x10^7 gauss. |     |
Curve 4 corresponds to Model A with the parameters $T=12,000^\circ K$, $\log g = 8$, $H=0.9$, $He=0.1$, $B=2.1\times10^8$ gauss. Curve 5 corresponds to Model B with the parameters $T=14,000^\circ K$, $\log g = 8$, $H=0.0$, $He=1.0$, $B=2.1\times10^8$ gauss. Curve 6 corresponds to Model B with the parameters $T=12,000^\circ K$, $\log g = 8$, $H=0.9$, $He=0.1$, $B=2.1\times10^8$ gauss.

The observed linear polarizations are indicated as follows: triangles, Angel and Landstreet; open circles, Angel et al.

Wavelength dependence of the predicted circular polarization for the various models. Curve 1 corresponds to the optically thin model of Kemp. Curve 2 corresponds to Model C with the parameters $T=12,000^\circ K$, $\log g = 8$, $H=0.9$, $He=0.1$, $B=1.2\times10^7$ gauss. Curve 3 corresponds to Model C with the parameters $T=14,000^\circ K$, $\log g = 8$, $H=0.0$, $He=1.0$, $B=2\times10^7$ gauss. The observed circular polarizations are indicated by the following: Crosses, Kemp et al.
triangles, Angel and Landstreet; open circles, Angel et al. Where observational points are clustered together, error bars were omitted from several points for clarity. The error bars of these points may be found in their corresponding references.

4 Spontaneous-emission transition probabilities in the dipole approximation for the static screened Coulomb potential is shown as a function of the Debye-Hückel screening length.

5 The critical temperature \( T_c \) at which the last observed Balmer line determined by the Inglis Teller relation is equal to the last observed Balmer line determined by Debye-Hückel screening. The regimes where the two effects dominate are also labelled.

6 Ionization energies (in effective Rydbergs) vs. screening length (in effective Bohr radii) for various values of the magnetic parameter \( \gamma \).
CHAPTER I: INTRODUCTION

The research described here is concerned with three distinct but related topics: (1) the origin of magnetic fields in white dwarf stars and neutron stars, (2) the detection of magnetic fields in white dwarfs, and (3) screening effects, particularly in semiconductors. For this reason, the introduction will be divided into three sections and a separate chapter devoted to each of the topics described above.

A. Observations of Magnetic Fields in the Universe

Magnetism has been both an intriguing and a useful phenomena since its discovery by the Chinese, who noticed that when lodestone was suspended by a fiber its alignment relative to the Earth was always unique. Although they successfully used the fact of this unique alignment in surveying and navigation, it is not known whether the Chinese understood that physical forces were the cause of this unique alignment.\(^1\) The realization that terrestrial magnetism could be explained if one considers the entire Earth as a giant lodestone was first proposed in 1600 by W. Gilbert in his "De Magnete". Gilbert, using a simple model, was able to explain a dip needle's behavior at different latitudes on the Earth.\(^2\) It was, of course, natural for Gilbert to assume that the Earth's interior
tion of a solar magnetic field was confirmed when in 1908 Hale determined that magnetic fields of the order of several thousand Gauss were present in the sunspots. His method was based on the Zeeman splitting of the spectral lines which will be discussed later. Several years later Hale and his co-workers suggested the identification of a general magnetic field of the sun. However this general field, only several Gauss in magnitude, was at the limit of their detectability. The question of a general field thus remained a controversy until H. D. and H. W. Babcock developed their solar magnetograph in 1952 which enabled very accurate measurement of the Zeeman splittings to be made. The Babcocks then used this solar magnetograph to prove the existence of the general magnetic field and show that its magnitude was 1 Gauss within 35° of the poles but highly irregular at the lower latitudes.

Six years previous to the work on the general solar field, H. W. Babcock discovered a magnetic field in a celestial body other than the Earth or the Sun. Using the 200 inch telescope at Mount Palomar, Babcock detected a magnetic field of about 500 Gauss in the star 78 Virginis. This star belongs to a class of main sequence stars generally called the peculiar A stars (Ap stars), which are spectroscopically characterized by their over-abundance with respect to the Sun of certain elements.
either the direction or magnitude of the galactic field. The results obtained with these methods, however, have generally been met with some skepticism due either to the uncertainties in the data or to the validity of the assumptions made in the theory.

Fairly well received as a method of determining the magnitude of the galactic field is one which involves the measurement of both the cosmic ray electrons and the synchrotron emission. Synchrotron emission is caused by relativistic electrons in a magnetic field and is measured as nonthermal radio emission. If the electrons are detected by measuring the cosmic ray electrons near the Earth, the magnetic field strength can then be readily calculated. A value of $2 \times 10^{-5}$ Gauss for the galactic field has been obtained in this manner. Another feature of synchrotron radiation of electrons is that it is polarized. By the measurement of this polarization, maps have been successfully drawn which show the direction of the magnetic field at the source of the radiation. A method which gives simultaneously both the magnitude and the direction of the galactic field is based on the Faraday effect, a rotation of the plane of polarization of the radiation as it passes through a magnetoactive medium. A simple expression for the angle of rotation $\theta$ is given by
in Van Allen belts of the planet. Models developed to explain this nonthermal radiation have been based both on cyclotron radiation, which requires polar fields of as high as $10^4$ G, and synchrotron radiation, which requires only 0.1 to 1 Gauss fields in the emitting region and perhaps fields sixty times greater at the poles. Recent direct measurements taken by the Pioneer 10 space probe have confirmed a magnetic dipole moment of 4.0 Gauss $R_j^3$ (where $R_j$ is the radius of Jupiter) which is inclined 15° to Jupiter's rotation axis, implying a magnetic dipole surface field ranging from 2.3 to 11.7 G.

Magnetic fields have played a dominant role in astrophysical research in the last decade. Probably the most glamorous topic of this era has been the pulsars, discovered in 1968 by Hewish and his co-workers. Pre-dating the actual discovery, Wheeler and Pacini presented some of the basic ideas underlying the present-day pulsar models. Their suggestion was that rotating neutron stars provided the energy source of the supernovae remnants. It was, however, Gold who used the idea of a rotating neutron star possessing a large (~$10^{12}$ Gauss) magnetic field as a model to explain the actual pulsing nature of the pulsars. Gold's original argument has become strongly entrenched not only because there has been no other successful alternative model
proposed, but also his model has been successful in describing a recent observation of a pulsar\textsuperscript{15}; namely, the coincidence between the energy loss by the Crab pulsar, determined from its steady period decrease, and the energy of the radiation emitted by the nebula surrounding the pulsar. Thus there is strong support, although no direct measurement, for the presence of magnetic fields of $10^{12}$ Gauss in compact astronomical bodies.

The magnetic fields postulated for pulsars, although seemingly quite large, were not totally unsuspected. If the magnetic flux of a star is conserved while it contracts, the magnetic field can be increased by a factor of $\sim 10^{10}$ for a neutron star. Thus, to obtain the large magnetic field strength postulated for neutron stars, a star need start contraction with a field of only $10^2$ Gauss, a value we have seen to be far below the maximum field actually observed in main sequence stars. A complete discussion of this "flux freezing", as well as other magnetic field production methods, will be given later. The interesting and important point to be stressed here is that it was probably this concept of "flux freezing" which promoted both observations and research of magnetic fields in white dwarf stars.

Mestel\textsuperscript{16}, in 1965, first suggested that if the central
ratio of the mass of the white dwarf to its radius, thus increasing the calculated radius.

It is somewhat curious that "evidence" for interior magnetic fields in white dwarfs predated the evidence for surface fields since surface fields had been discovered in main sequence stars 19 years previously. However, these surface fields were detected by the measurement of the splitting of the metal lines caused by the magnetic field (linear Zeeman splitting) and as indicated before, these measurements are possible only if the line is not masked by some broadening mechanism. Of the known bright white dwarfs, nearly 75% belong to the class of white dwarfs where only hydrogen lines are found, the DA stars (D indicating a white dwarf and the letter A being one of the temperature correlated letters of the normal stellar classification scheme). Unfortunately, however, nearly all these DA stars have Balmer lines which are extremely Stark broadened (typical half width 20-40 Å) due to the strong static ion electric field present in the dense white dwarfs thus masking the Zeeman splitting. A determination of the strength of the magnetic field by the method previously used for the Ap stars is thus completely impossible. Different methods obviously were needed for the detection of surface fields of white dwarfs. In 1970, a search for magnetic fields in DA white dwarfs using a different technique was made by
Angel and Landstreet. Besides just splitting the normal line, the magnetic field also causes these split lines to be polarized. In the normal Zeeman effect for the magnetic field aligned along the line of sight, two components practically of equal magnitude are equally shifted from the normal line center; one of which is right circularly polarized, the other left circularly polarized. Since, as we have seen, the split components can not be resolved in the DA white dwarfs Angel and Landstreet used the polarized nature of the radiation in an attempt to determine the magnetic field strength in the DA stars. First, they measured the intensity of each component in a line wing and from this a quantity \( q \), called the fractional circular polarization, was then calculated by the equation

\[
q = \frac{I_R - I_L}{I_R + I_L} \quad (1.3)
\]

where \( I_R, I_L \) are the intensities of the right and left circularly polarized radiation respectively. The quantity \( q \) is related to the magnetic field by the relation

\[
q(v_o) = \frac{\Omega L}{\pi} \int f(v_o,v) \frac{\partial G(v)}{\partial v} \, dv \left/ \int f(v_o,v)G(v)\,dv \right. \quad (1.4)
\]
where $\Omega_L$ is the Larmor frequency $= \frac{eB}{2mc}$, $f(v_o, v)$ is the filter transmission curve and $G(v)$ is the stellar energy distribution (the line shape). On the line wings, Eq. (1.4) reduces to

$$q(v_o) = \frac{\Omega_L}{\Gamma}$$

(1.5)

where $\Gamma$ is the line width. Thus, the approximate relation between $q$ and the magnetic field $B$

$$B(\text{Gauss}) \approx 1.3 \times 10^5 q \text{ (per cent)}$$

(1.6)

can be obtained. Of the nine DA stars observed by Angel and Landstreet, the largest polarization measured was the 0.8% for the star L770-3. This value implied an upper limit of about $10^5$ Gauss in the observed stars, somewhat lower than the $10^6$-$10^8$ Gauss anticipated.

A second method used to determine the magnetic fields in white dwarfs was based on the observed shifts of the Balmer series lines, which is caused basically by a combination of four effects:

1. the radial motion of the star
2. the Einstein redshift
3. Stark effect induced redshift
4. the quadratic Zeeman effect.
Once the Balmer line of interest has been corrected for the first three above effects, any residual shift can be interpreted as due to the quadratic Zeeman effect. This procedure was first used by Preston, who analyzed radial velocity data of white dwarfs measured by Greenstein and Trimble. Preston related these quadratic Zeeman shifts to the magnetic field by using an expression obtained by Jenkins and Segré:

$$<\Delta \lambda_{\text{QZ}}> = -7.5 \times 10^{-23} \lambda^2 n^2 B^2$$ (1.7)

and obtained an upper limit of $5 \times 10^5$ Gauss for the DA stars. It was pointed out by Trimble however, that the data used by Preston contained systematic errors; that is, lines which showed discordant shifts were excluded. Reevaluating the original data, she found fields of from 5 to $9 \times 10^5$ Gauss in five of the observed stars. Two further problems exist. First the equation used by Preston and Trimble was obtained by Jenkins and Segré specifically for the Lyman series, not the Balmer series. Using a Balmer series calculation, Lamb and Sutherland showed that the magnetic field strengths obtained by Trimble should be increased by 10 to 15 percent. Secondly, the equation of Jenkins and Segré was calculated by perturbation theory. As pointed out by O'Connell,
the critical magnetic field where perturbation theory breaks down is well within the range of interest \((-10^6 \text{ G})\) for the transitions involving the higher excited states of interest. Lamb and Sutherland's calculation, while using the good quantum numbers of parity and the \(z\) component of the angular momentum \(l_z\), also required that the principal quantum number \(n\) of Bohr theory be a good quantum number which is not the case. A recent variational calculation of the quadratic Zeeman effect by Surmelian and O'Connell,\(^28\) based solely on the good quantum numbers of parity and \(l_z\), has indeed yielded even larger fields than those obtained by Lamb and Sutherland.

Probably the most remarkable method of measuring magnetic fields in white dwarfs is one recently proposed by Kemp.\(^29\) He suggested that the continuum radiation from any thermal source in a magnetic field is characterized by a circular polarization along the magnetic field. Kemp noted that if the Zeeman-split transitions are broadened to the point of extreme overlapping among all lines in the spectrum, a net polarization should survive since the normal Zeeman splittings \(a_z\) of one sign and there is a subtle difference between oscillator strengths of different Zeeman components. He then formulated a harmonic oscillator model, called the gray-
body magnetoemission theory, which predicted that at a frequency $\omega$ the fractional circular polarization $q$ of the continuum radiation is proportional to $\Omega_L/\omega$ for $\Omega_L \ll \omega$ where $\Omega_L \equiv eB/2mc$ is the electron Larmor frequency. It was also shown that the theory also predicted a fractional linear polarization $q^*$ proportional to $(\Omega_L/\omega)^{2.30}$.

It should be stressed that Kemp's proposal requires polarization measurement of the continuum radiation, not of the line radiation as was the previously mentioned measurements of Angel and Landstreet. Continuum polarization measurements can thus be made on a class of white dwarfs, the DC white dwarfs, which are noted for their complete lack of absorption spectral features, and also on the semi-DC white dwarfs in which those few absorption features which are present are usually not interpreted and are very weak. In fact, after Kemp had proven the existence of the non-zero fractional circular polarization of continuum radiation in the laboratory, the first object which was observed to possess this continuum polarization was the semi-DC white dwarf Grw + 70° 8247, famous for the bands present in its spectrum. These so-called "Minkowski bands" are of unknown origin. These first observations were made with broad pass band filters. The most striking features of the data are the rapid decrease and increase of the
circular polarization in the 3000-4000 Å region, the gradual decrease of the circular polarization with increasing wavelength above 4000 Å and for the linear polarization a decrease above 4000 Å somewhat more pronounced than that of the circular polarization. Kemp's gray-body magnetoemission theory predicted the correct order of magnitude of the circular polarization, however, the theory predicted a spectral dependence of the circular polarization which was directly proportional to the wavelength (as the wavelength increases so should the circular polarization). This is just opposite to the observed decrease in the circular polarization above 4000 Å.

Kemp's theory, however, is an optically thin theory; that is, no effect of radiative transfer on the radiation is considered. To incorporate radiative transfer effects, Shipman considered an optically thick radiating system; that is, he considered the changes in the polarization of the radiation as it traverses an atmosphere. The radiative transfer problem necessitates a knowledge of the opacity of the intervening material. Shipman obtained approximate expressions for the opacity processes in a magnetic field by assuming local thermodynamic equilibrium (LTE), i.e., that all thermodynamic properties of a small volume of material in an atmosphere are obtained by assuming their thermodynamic
Shipman's results for the circular polarization are shown in Figure 1 and agrees well with the observations available at that time and gave strong support for the radiative transfer corrected magnetoemission theory. As mentioned previously, the magnetoemission theory also predicts a linearly polarized component of the radiation, which in the optically thin model is proportional to the square of the wavelength but is observed to show a decrease above 4000 Å even more pronounced than the circular polarization. With the remarkable agreement between the observations and Shipman's radiative transfer corrected magnetoemission theory for the circular polarization, it might be expected that the wavelength dependence of the linear polarization in the radiative transfer corrected model also give good agreement with the observations. Thus, in Chapter III we will consider the wavelength dependence of fractional linear polarization in the radiative transfer corrected model and discuss its agreement with the observed wavelength dependence of linear polarization. We will also reevaluate the circular polarization in the light of new detailed observations and considering the exact results of the optically thin model.
B. Production Mechanisms of Magnetic Fields

The discussion of magnetic field production mechanisms will be confined solely to the density region of ultradense matter. A general review for the condensed matter region has been given by Herring.33

Most of the early (pre 1964) discussions of production mechanisms for the magnetic fields found in the planets, the Sun and the stars center around three mechanisms; the "fossil" field theory, the Battery mechanism, and the Dynamo theory. In 1945 Cowling34 suggested that these observed celestial fields might be "fossil" relics of the field with which the objects were born with since his calculation of the e-folding time for the slowest-decaying dipole component of a solar magnetic field was about $10^{10}$ years, longer than the present age of the universe. However, as Cowling8 himself admits, this suggestion was made somewhat "with my tongue in my cheek" since the origin of the primeval field was left unresolved. The fossil field theory was initially used as the mechanism for the general solar field with the hope that the strong local fields such as in the sunspots could be generated from this residual field by some process such as non-uniform rotation. However, Babcock's discovery of the reversal of the solar magnetic field is considered to generally rule out "fossil" fields as the complete theory of solar magnetism since this type of
variation of the dipole component of the field can not be explained by a quasi-steady theory of magnetic fields, such as the fossil theory.\textsuperscript{35}

Fossil fields did stimulate work in two important areas of stellar structure; namely,

(1) star formation in rotating magnetic clouds, and

(2) the approach to the main sequence of magnetic proto-stars.

The study of star formation by Mestel\textsuperscript{36} can be summarized in two statements. First, if the angular momentum vector $\hat{\mathbf{h}}$ of the rotating cloud is generally parallel to the magnetic field $\hat{\mathbf{B}}$, the limitation to the systematic breakup of the cloud is set by the thermal pressure. The proto-stars formed will have magnetic energies comparable to the gravitational energies; however, they will tend to be less massive than the Sun. Secondly, if the angular momentum is roughly perpendicular to the magnetic field $\hat{\mathbf{B}}$, the magnetic field, not the thermal pressure, fixes the masses of the proto-stars, which are generally strongly magnetic and more massive than those formed with $\hat{\mathbf{h}}$ and $\hat{\mathbf{B}}$ aligned.

If one is to assume that the magnetic fields observed in stars were present at the formation of the stars, it is important to ascertain if these magnetic proto-stars reach the main sequence without the magnetic
field having been destroyed by some process during its
evolution. In particular, it is well known that a non-
magnetic proto-star passes through a completely con-
vective stage, called the Hayashi phase, during its
contraction toward the main sequence. However, a
strong magnetic field will not be greatly affected by
the convection; in fact, the magnetic field modifies the
convection and reduces its efficiency. Thus strongly
magnetic proto-stars pass through this Hayashi phase
with the magnetic field basically unchanged. For weakly
magnetic proto-stars, it is the turbulence which controls
the magnetic field and probably expels the field from
the turbulent zone. Spitzer, however, has argued
that the star's external dipole moment remains unaffected
so that once the Hayashi phase is over, the field could
diffuse back through the star. Thus, weakly magnetic
stars could possibly also reach the main sequence with
magnetic field more or less unchanged.

With the possibility existing that convection during
the Hayashi phase may completely destroy weak magnetic
fields, magnetic field production mechanisms active in
main sequence or post main sequence stars are important.
In 1950 Biermann suggested a mechanism similar to a
simple battery which could generate magnetic flux in
rotating stars. A separation of electrons and ions is
cased by the electron partial pressure gradient $-\n\n$
pression for the energy in terms of the variational parameters $\beta$, $\delta$ and $\theta$. The energy was then minimized numerically. Jaskorzynska recently has improved Wright's calculation by using a linear combination of four of his functions. Fenton and Haering used a similar method, basing their calculation on the variational function $\exp\left[-\left(\frac{x^2+y^2}{a_1^2} + \frac{z^2}{a_n^2}\right)^{1/2}\right]$. In a non-variational calculation, Dyakanov, Mitchell, and Efros used an adiabatic approximation, valid only for very strong magnetic fields, to reduce Schrödinger's three dimensional equation to a one dimensional equation. Approximate solutions of this reduced equation were then found for particular regimes of the screening length.

The solution for a free electron in a magnetic field is well known. Ortenberg and Landwehr used the wavefunction $\psi(r)$ for the free electron in a magnetic field as a basis function for a variational calculation of a particle under the influence of a screened Coulomb potential and in a magnetic field by releasing the constraint on a parameter multiplying the coordinate $\rho$ in the wavefunction and using it as a variational parameter for the minimization of the energy.

A multiparameter variational method for the ground state and excited states, valid for all values of the magnetic field and screening length, will be presented in Chapter IV.
CHAPTER II: LANDAU ORBITAL FERROMAGNETISM

In this chapter we will first derive an expression for the thermodynamic potential per unit volume in a magnetically interacting system; that is, a system where the magnetization $M$ is a function of the magnetic induction $B$, not the magnetic field strength $H$. We then consider the stability of the LOFER states in three electron gas models, (a) a nonrelativistic noninteracting gas, (b) a relativistic noninteracting gas, and (c) a nonrelativistic gas interacting via a Coulomb potential. Finally, we study what effective mass and temperature corrections imply about our previous conclusions.

A. Thermodynamics in a Magnetic Field

Previous to Shoenburg, the magnetization $M$ had been assumed to be a function of the magnetic field $H$, the field produced solely by the source currents. Thus the magnetic induction field, the field produced by both the source currents and the induced currents, was given by

$$\hat{B} = \hat{H} + 4\pi \hat{M}(H).$$  \hspace{1cm} (2.1)

A system in which the magnetic induction $\hat{B}$ is given by Eq. (2.1) we call, following Pippard, a magnetically noninteracting system since the only interaction present
is that between the electrons and the magnetic field \( \mathbf{H} \), which results in the production of the magnetization \( \mathbf{M}(H) \); whereas the interactions between the electrons themselves are neglected. Shoenburg\(^{54}\) proposed that the magnetization is actually a function of the magnetic induction field \( \mathbf{B} \). Thus Eq. (2.1) becomes

\[
\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}(\mathbf{B}) . \tag{2.2}
\]

A system which is described by Eq. (2.2) is called a magnetically interacting system because, besides the interaction between the electrons and the magnetic field \( \mathbf{H} \) resulting in the magnetization \( \mathbf{M}(H) \), the electrons interact magnetically among themselves. The total effect of all the interactions is to produce a magnetization \( \mathbf{M}(\mathbf{B}) \). This magnetically interacting system should not be confused with a directly interacting system, a system in which the electrons would interact directly with each other. A directly interacting system may be considered within the framework of either a magnetically non-interacting or a magnetically interacting system.

Pippard\(^{69}\) credits Holstein with showing that the thermodynamic potential per unit volume in the magnetically interacting system \( \Omega(\mu,T,\mathbf{B}) \), which must be minimal for a stable state in thermal equilibrium, is however, not simply \( \Omega^*(\mu,T,\mathbf{B}) \), that which is obtained
from the noninteracting thermodynamic potential per unit volume \( \Omega^*_0(\mu, T, \hat{H}) \) by replacing \( \hat{H} \) with \( \hat{B} \), but is actually given by

\[
\Omega(\mu, T, \hat{B}) = \Omega^*_0(\mu, T, \hat{B}) + 2\pi M^2(T, \mu, \hat{B}) .
\tag{2.3}
\]

Unfortunately, Holstein's derivation is unpublished and so, because of the importance of this equation to the stability problem, we will reconsider the question of thermodynamics in a magnetic field, explicitly deriving Eq. (2.3) in the process.

The change in the energy per unit volume of the medium when a magnetic field produced by fixed potentials is applied has been shown to be

\[
dW_M = \frac{1}{4\pi} \hat{H} \cdot d\hat{B} .
\tag{2.4}
\]

This must be added to the nonmagnetic portion of the Helmholtz free energy so that the total Helmholtz free energy in differential form becomes

\[
dF(n, T, \hat{B}) = -SdT + \mu dn + \frac{1}{4\pi} \hat{H} \cdot d\hat{B} ,
\tag{2.5}
\]

where \( \mu \) is the chemical potential and \( S \) is the entropy.

The Helmholtz free energy as a function of the magnetic field \( \hat{H} \) rather than the magnetic induction \( \hat{B} \) is:
given by the Legendre transformation:

\[ F^*(T,n,H) = F(T,n,B) - \frac{1}{4\pi} \hat{H} \cdot \hat{B}, \quad (2.6) \]

so that in differential form

\[ dF^*(T,n,H) = -SdT + \mu dn - \frac{1}{4\pi} \hat{B} \cdot d\hat{H}. \quad (2.7) \]

Free energy functions of the parameters \( T, \hat{B} \) or \( \hat{H} \), and \( \mu \) rather than \( T, \hat{B} \) or \( \hat{H} \) and \( n \) are the Gibbs free energy functions defined by the Legendre transformations

\[ G(T,\mu,\hat{B}) = F(T,n,\hat{B}) - \mu n, \quad (2.8) \]

\[ G^*(T,\mu,\hat{H}) = F^*(T,n,\hat{B}) - \mu n, \quad (2.9) \]

so that in differential form

\[ dG(T,\mu,\hat{B}) = -SdT - n d\mu + \frac{1}{4\pi} \hat{H} \cdot d\hat{B}, \quad (2.10) \]

\[ dG^*(T,\mu,\hat{H}) = -SdT - n d\mu - \frac{1}{4\pi} \hat{B} \cdot d\hat{H}. \quad (2.11) \]

From Eqs. (2.6), (2.8), and (2.9) we see \( G(T,\mu,\hat{B}) \) and \( G^*(T,\mu,\hat{H}) \) are related by

\[ G^*(T,\mu,\hat{H}) = G(T,\mu,\hat{B}) - \frac{1}{4} \hat{H} \cdot \hat{B}. \quad (2.12) \]
Microscopic theory provides us with another free energy function of $T$, $\mu$, and $\hat{H}$, called the thermodynamic potential per unit volume, $\Omega^*_0(\mu, T, \hat{H})$, obtained from the expression

$$\Omega^*_0(\mu, T, \hat{H}) = -\frac{kT}{V} \ln\{\text{Tr} \exp[-\frac{1}{kT} (\hat{H} - \mu \hat{N})]\} \quad ,$$

(2.13)

where $\hat{H}$ is the Hamiltonian operator and $\hat{N}$ is the number operator. In differential form the thermodynamic potential per unit volume $\Omega^*_0(T, \mu, \hat{H})$ has the same non-magnetic contribution as does the Gibbs free energy, namely

$$d\Omega^*_0(T, \mu, \hat{H})_{\text{non-mag.}} = -SdT - d\mu \quad .$$

(2.14)

The difference between the thermodynamic potential per unit volume and the Gibbs free energy lies in their magnetic terms. A question that must first be answered is which of the Gibbs free energies, $G^*(\mu, T, \hat{H})$ or $G(\mu, T, \hat{B})$, must we use to compare magnetic contributions to the thermodynamic potential per unit volume. As pointed out by Landau and Lifshitz\textsuperscript{73} if one is considering changes of state in which the potential $\hat{A}$ is kept constant (which is equivalent to keeping $\hat{B}$ constant, since $\hat{B} = \vec{v} \times \vec{A}$) the Gibbs free energy $G(T, \mu, \hat{B})$ must be used
whereas if one is considering changes of state in which
the source currents \( \mathbf{j} \) are kept constant (which is
equivalent to keeping \( \mathbf{H} \) constant, since \( \frac{4\pi}{c} \mathbf{j} = \nabla \times \mathbf{H} \) the
Gibbs free energy \( G^*(T, \mu, \mathbf{H}) \) must be considered. Now the
magnetic contribution to the differential thermodynamic
potential per unit volume must be the work done on the
medium by the magnetic field if the currents producing
the field remain fixed. Thus to compare the thermo-
dynamic potential to the Gibb free energy we must use
the form of the Gibbs free energy in which the source
currents are fixed, namely \( G^*(T, \mu, \mathbf{H}) \). From Eq. (2.11)
we see that the magnetic contribution to the differential
Gibbs free energy, for constant source currents, is

\[
d G^*_{mag} (T, \mu, \mathbf{H}) = - \frac{\mathbf{B}}{4\pi} \cdot d\mathbf{H} . \tag{2.15}
\]

The work done on the medium by the magnetic field if the
currents producing the field remain fixed is the change
in the total energy per unit volume of the medium when a
magnetic field produced by fixed currents is applied
\(- \frac{\mathbf{B}}{4\pi} \cdot d\mathbf{H}\) minus the energy of the magnetic field it-
self, which in differential form is \(- \frac{\mathbf{H}}{4\pi} \cdot d\mathbf{H}\). Thus the
magnetic contribution to the differential thermodynamic
potential per unit volume is given by

\[
d G^*_{0} (T, \mu, \mathbf{H})_{mag} = - \frac{\mathbf{B}}{4\pi} \cdot d\mathbf{H} + \frac{\mathbf{H}}{4\pi} \cdot d\mathbf{H} = -\mathbf{M} \cdot d\mathbf{H} . \tag{2.16}
\]
Combining Eqs. (2.14) and (2.16)

\[ d\Omega^*_0(T,\mu,\hat{H}) = -SdT-n\mu-\hat{M} \cdot d\hat{H} \quad . \tag{2.17} \]

Thus

\[ \left( \frac{\partial \Omega^*_0(T,\mu,\hat{H})}{\partial \hat{H}} \right)_{T,\mu} = -\hat{M}(T,\mu,\hat{H}) \quad , \tag{2.18} \]

and

\[ \left( \frac{\partial \Omega^*_0(T,\mu,\hat{H})}{\partial \mu} \right)_{T,\hat{H}} = -n(T,\mu,\hat{H}) \quad . \tag{2.19} \]

Up to now our discussion was confined to a model of magnetically noninteracting electrons in an external field \( \hat{H} \) and from Eqs. (2.13) and (2.18) we are able to calculate the magnetic moment \( M(T,\mu,\hat{H}) \) of the system.

Let us now consider a model of magnetically interacting electrons. According to Shoenburg and Pippard, the magnetic moment \( M(T,\mu,\hat{B}) \) in the interacting system is given by

\[ \left( \frac{\partial \Omega^*_0(T,\mu,\hat{B})}{\partial \hat{B}} \right)_{T,\mu} = -M(T,\mu,\hat{B}) \quad , \tag{2.20} \]

where \( \Omega^*_0(T,\mu,\hat{B}) \) is the thermodynamic potential per unit volume of the noninteracting system with \( \hat{B} \) substituted for \( \hat{H} \). Also the number of particles \( n \) as a function of
\( \mu, T, \) and \( \mathbf{B}, n(T, \mu, \mathbf{B}), \) is given by

\[
n(T, \mu, \mathbf{B}) = - \left( \frac{\partial \Omega^*(T, \mu, \mathbf{B})}{\partial \mu} \right)_{T, \mathbf{B}}.
\]  

(2.21)

It is to be emphasized that \( \Omega^*(T, \mu, \mathbf{B}) \) is not the free energy of the magnetically interacting system. As Pippard\(^{69} \) points out "for a macroscopic body regardless of the mechanism of magnetization" the magnetization of the system is given by the negative of the derivative of the free energy of that system with respect to the magnetic field \( \mathbf{H} \), holding the temperature and the chemical potential constant. Thus

\[
M(T, \mu, \mathbf{B}) = - \left( \frac{\partial \bar{\Omega}}{\partial \mathbf{H}} \right)_{T, \mu},
\]  

(2.22)

where \( \bar{\Omega}(T, \mu, \mathbf{H}) \) is the thermodynamic potential per unit volume in the interacting system and as yet is unknown. Henceforth, when we write \( \mathbf{H} \) we imply that it is given by Eq. (2.2) and not Eq. (2.1). From Eq. (2.22) it follows that

\[
d\bar{\Omega}(T, \mu, \mathbf{H}) = - S dT - nd\mu - \mathbf{H}(\mathbf{B}) \cdot d\mathbf{H}.
\]  

(2.23)

Comparing Eq. (2.23) to (2.11) we see

\[
d\bar{\Omega}(T, \mu, \mathbf{H}) = dG^*(T, \mu, \mathbf{H}) + \frac{\mathbf{H}}{4\pi} \cdot d\mathbf{H}.
\]  

(2.24)
Thus

\[ \bar{\Omega}(T, \mu, \dot{\mathbf{H}}) = G^*(T, \mu, \dot{\mathbf{H}}) + \frac{\mathbf{H}^2}{8\pi}. \]  
(2.25)

From Eqs. (2.17) and (2.20) it is apparent that

\[ d\omega^*_0(T, \mu, \dot{\mathbf{B}}) = -SdT - n d\mu - \dot{M}(B) \cdot d\dot{B}. \]  
(2.26)

Using Eq. (2.10), we obtain

\[ d\Omega^*_0(T, \mu, \dot{\mathbf{B}}) = dG(T, \mu, \dot{\mathbf{B}}) - \frac{\dot{\mathbf{B}}}{4\pi} \cdot d\dot{B}. \]  
(2.27)

Consequently,

\[ \bar{\Omega}_0^*(T, \mu, \dot{\mathbf{B}}) = G(T, \mu, \dot{\mathbf{B}}) - \frac{\dot{\mathbf{B}}^2}{8\pi}. \]  
(2.28)

Equation (2.25) becomes, after using Eq. (2.12),

\[ \bar{\Omega}(T, \mu, \dot{\mathbf{H}}) = G(T, \mu, \dot{\mathbf{B}}) - \frac{\dot{\mathbf{H}} \cdot \dot{\mathbf{B}}}{4\pi} + \frac{\mathbf{H}^2}{8\pi}. \]  
(2.29)

With Eq. (2.28), Eq. (2.29) becomes

\[ \bar{\Omega}(T, \mu, \dot{\mathbf{H}}) = \Omega_0^*(T, \mu, \dot{\mathbf{B}}) + \frac{B^2}{8\pi} - \frac{2\dot{\mathbf{H}} \cdot \dot{\mathbf{B}}}{4\pi} + \frac{H^2}{8\pi}, \]  
(2.30)

or
\( \bar{\Omega}(T, \mu, H) = \Omega^*_0(T, \mu, \hat{B}) + 2\pi M^2(T, \mu, \hat{B}) \)

(2.31)

Therefore we see that the thermodynamic potential per unit volume that must be minimal with respect to \( \hat{B} \) in the interacting system is given by

\[ \Omega(T, \mu, \hat{B}) = \Omega^*_0(T, \mu, \hat{B}) + 2\pi M^2(T, \mu, \hat{B}) \]  \hspace{1cm} (2.32)

Equation (2.32) is the main object of interest in this section. However, we wish to consider the special case of LOFER states, that is: the case where \( H=0 \).

Equation (2.12) simplifies to

\[ G^*(T, \mu, H)|_{H=0} = G(T, \mu, B)|_{H=0} \]  \hspace{1cm} (2.33)

Equation (2.25) simplifies to

\[ \bar{\Omega}(T, \mu, H)|_{H=0} = G^*(T, \mu, H)|_{H=0} = G(T, \mu, B)|_{H=0} \]  \hspace{1cm} (2.34)

that is, the thermodynamic potential is equal to the Gibbs free energy in the magnetically interacting system for the case \( H=0 \).
From Eq. (2.10) it is obvious that

\[
\left( \frac{\partial G(T, \mu, B)}{\partial B} \right)_{T, \mu} = \frac{H}{4\pi} \quad .
\]  

(2.35)

Thus for our special LOFER state case, H=0,

\[
\left( \frac{\partial G(T, \mu, B)}{\partial B} \right)_{T, \mu} = 0 \quad ,
\]  

(2.36)

and it is immediately obvious that the LOFER states correspond to extrema in the Gibbs free energy.

Let us now consider the thermodynamic stability of the LOFER states in three electron gas models.

B. Non-relativistic Electron Gas

The first model we will consider is that of the non-relativistic electron gas. We take the temperature to be zero (complete degeneracy) which corresponds to the most favorable condition for the LOFER state, and use the units \( K = c = m_e \) (mass of the electron) = 1. The non-interacting thermodynamic potential per unit volume \( \mu_0^*(\mu, H) \) has been derived by Wilson. \(^{74}\) Defining the parameter \( b \) by

\[
b = \frac{2\pi \mu}{(B/B_c)} \quad ,
\]  

(2.37)

where \( B_c \) = critical magnetic field = \( \frac{m_e^2 c^3}{e\hbar} \) as a function
of B, \( \Omega_0^* (\mu, B) \) (see Section A) is given by

\[
\Omega_0^* (\mu, B) = - \frac{(2\mu)^{5/2}}{15\pi^2} \left\{ 1 + \frac{3\pi^2}{4} b^{-2} - \frac{15\pi^{1/2}}{4} b^{-5/2} \right\},
\]

(2.38)

where

\[
\gamma_1 = \sum_{r=1}^{\infty} \frac{\cos(b r - \pi/4)}{r^{5/2}}.
\]

(2.39)

We shall consider the case \( b > 1 \) only (for this restriction was imposed \(^7\) in deriving Eq. (2.38) and see Section E of this chapter). The magnetization \( M(\mu, B) \) and number of particles \( n(\mu, B) \) given by Eqs. (2.20) and (2.21) are

\[
M(\mu, B) = - \frac{B^{1/2}}{2\pi} a^{3/4} \mu \gamma_2,
\]

(2.40)

and

\[
n(\mu, B) = \frac{(2\mu)^{3/2}}{3\pi^2} \left\{ 1 + \frac{3\pi^{1/2}}{2} b^{-3/2} \gamma_2 \right\},
\]

(2.41)

where

\[
\gamma_2 = \sum_{r=1}^{\infty} \frac{\sin(br - \pi/4)}{r^{3/2}},
\]

(2.42)
and $\alpha$ is the fine structure constant. Thus, substituting Eq. (2.40) into Eq. (2.2) and taking $H=0$, we find that

$$B + dB^{1/2} = 0 \quad , \quad (2.43)$$

where

$$d = 2\pi^{-2} \alpha^{3/4} \mu \Sigma_2 \quad . \quad (2.44)$$

It follows from Eq. (2.43) that $B=0$ or $B^{1/2}=-d$. The latter implies $B=d^2$ and hence

$$\frac{B}{B_C} = 4\pi^{-4} |\Sigma_2|^2 \alpha^2 \mu^2 \quad . \quad (2.45)$$

We come now to the all-important question of whether or not the $B \neq 0$ solutions are stable, that is, whether they correspond to a minimum value of $\Omega(\mu,B)$, and, in the case where they do correspond to a minimum, whether the corresponding value of $\Omega(\mu,B)$ is greater or less than the value of $\Omega(\mu,B)$ associated with the $B=0$ solution. It will turn out that these questions can actually be answered without explicitly evaluating the value of $\Sigma_2$. Combining Eq. (2.45) with Eqs. (2.3), (2.38), and (2.40) enables us to write, to the order required,
\[ \Omega(\mu, B) = -\frac{(2\mu)^{5/2}}{15\pi^2} \left( 1 - \frac{15\pi^{1/2}}{8} \right) |\Sigma_2| b^{-3/2} \] 

(2.46)

It is clear from this equation that a non-zero value of \( B \) results in a larger value for \( \Omega(\mu, B) \), relative to the \( B=0 \) case. Thus we conclude that the most stable configuration is that for which \( B=0 \).

It should be pointed out, however, that, because of the oscillatory nature of \( \Sigma_2 \), the value of \( \Omega(\mu, B) \) given by Eq. (2.46) goes through successive maxima and minima as the density (and hence \( \mu \) and \( B \)) is slightly changed. This results from the fact that the actual value of \( \Sigma_2 \) is determined not so much by the magnitude of \( b \) (or \( B \)) but by the amount by which \( (b/2\pi) \) differs from an integral number. Although these minima correspond to larger values of \( \Omega(B) \) than that given in the \( B=0 \) case, perhaps there is some way in which the system can get into these metastable states.

C. Relativistic Electron Gas

Lifshitz and Kosevich\(^7\) have derived the magnetic properties of electrons for the case of an arbitrary dispersion law, where the energy levels \( \epsilon \) are given by

\[ \epsilon = \sqrt{P_x^2 + P_y^2 + P_z^2} \] 

(2.47)
For the case of interest here the energy eigenvalues of a relativistic electron in a magnetic field $H$ oriented along the $z$ axis are

$$E = \left\{1 + \frac{p_z^2}{c^2} + 2n(H/H_c)\right\}^{1/2}.$$  \hfill (2.48)

Using the general prescription of Lifshitz and Kosevich, the equation for the non-interacting thermodynamic potential per unit volume, $\Omega_0(\mu, H)$ written as a function of $B$, $\Omega_0(\mu, B)$, for our dispersion law may be written as

$$\Omega_0(\mu, B) = -\frac{1}{24\pi^2} f(x) + \frac{(2\mu)^{5/2}}{4\pi} D(\mu) b^{-5/2} x_1,$$  \hfill (2.49)

where

$$x = \left\{ (\mu + 1)^2 - 1 \right\}^{1/2},$$  \hfill (2.50)

$$f(x) = x(x^2 + 1)^{1/2} (2x^2 - 3) + 3 \sinh^{-1} x,$$  \hfill (2.51)

$$\gamma_1 = \sum_{r=1}^{\alpha} \frac{\cos[A(\mu)br - \pi/4]}{r^{5/2}},$$  \hfill (2.52)

and

$$A(\mu) = 1 + \frac{\mu}{2},$$  \hfill (2.53)
\[
D(\mu) = (1+\mu)^{-1} . \quad (2.54)
\]

From Eqs. (2.21) and (2.20) it follows that

\[
\eta(\mu, B) = \frac{x^3}{3\pi^2} \left\{ 1 + \frac{3\pi^{1/2}}{2} \left( 1 + \frac{\mu}{2} \right)^{-3/2} b^{-3/2} \right\}
\]

and

\[
M(\mu, B) = -\frac{B^{1/2}}{2\pi^{3/4}} \frac{\alpha^{3/4}}{\mu F(\mu)^{1/2}} . \quad (2.55)
\]

where

\[
\gamma_{\mu} \sum_{r=1}^{\infty} \frac{\sin[A(\mu)br^{-\pi/4}]}{r^{3/2}} \quad (2.57)
\]

and

\[
F(\mu) \left[ A(\mu)D(\mu) = \left\{ (1 + \frac{\mu}{2})/(1 + \mu) \right\} . \quad (2.58)
\]

Our expression for \( M \) is the same as we obtained in the non-relativistic case (\( \mu << 1 \)) except for the factor \( F(\mu) \).

Thus, substituting Eq. (2.56) into Eq. (2.2) and taking \( H=0 \), we find that

\[
B + dB^{1/2} = 0 , \quad (2.59)
\]

where
\[- \frac{3}{2\pi^{3/2}} b^{1/2} \Sigma_2 + \frac{9a}{2^{3/2} \pi^{5/2} \mu^{1/2}} b^{1/2} \Sigma_2 + \ldots \] ,

(2.65)

and

\[
n(\mu, B) = \frac{(2\mu)^{3/2}}{3\pi^2} \left[ 1 + \frac{3\alpha}{2^{1/2} \mu^{1/2} \pi^{1/2}} b^{-1/2} \Sigma_3 + \frac{3\pi^{1/2}}{2} b^{-3/2} \Sigma_2 + \ldots \right] ,
\]

(2.66)

where

\[
\Sigma_3 = \sum_{r=1}^{\infty} \frac{\cos(br\pi/4)}{r^{1/2}} .
\]

(2.67)

Since \( b \gg 1 \) we will consider only the dominant term in (2.65) (i.e. the second term) so

\[
M(\mu, B) = -\frac{\alpha^{5/4} 2^{1/2} \mu^{3/2} B^{-1/2}}{\pi^3} \Sigma_3 .
\]

(2.68)

The solutions to Eq. (2.2) with \( H=0 \) are obtained as follows. We note that \( B=0 \) is a self-consistent solution since from Eqs. (2.38) and (2.63) \( \Omega^*_0(\mu, B) \), in the limit \( B=0 \), is independent of \( B \) so that Eq. (2.20) gives...
Thus Eq. (2.2) with $H=0$ is satisfied for the solution $B=0$.

The LOFER state solutions are obtained by substituting Eq. (2.68) into Eq. (2.2) and taking $H=0$. It follows that

$$B + g B^{-1/2} = 0 , \quad (2.70)$$

where

$$g = \frac{2^{5/2}}{\pi} \mu^{3/2} \alpha^{5/4} \Sigma_3 \quad (2.71)$$

Equation (2.70) has the solution

$$B^{1/2} = -g^{1/3} \quad (2.72)$$

which implies

$$B = g^{2/3} \quad (2.73)$$

thus giving

$$\frac{B}{B_C} = \frac{2^{5/3}}{4^{4/3}} \alpha^{4/3} |\Sigma_3|^{2/3} \mu \quad (2.74)$$
any change of state at constant temperature, volume and chemical potential. The change of state of interest here is that due to a change in B. The thermodynamic potential for the LOFER state solution is given by Eq. (2.76). Comparing this to the thermodynamic potential for the non-magnetic state (B=0) given by Eq. (2.75) and again realizing we are considering $b \gg l$, it is apparent that the non-magnetic solution has the lowest thermodynamic potential and thus the non-magnetic solution corresponds to the thermodynamically stable solution for a completely degenerate electron gas interacting through a Coulomb potential.

An alternate procedure (which is more appealing physically but very cumbersome mathematically) is to hold the number of particles, $n$, fixed rather than the chemical potential during a change of state. In such a case the Helmholtz free energy must be minimized with respect to a change of state, holding the temperature and now the number of particles constant. As we now demonstrate, this will provide us with a verification of our previous results.

The Helmholtz free energy, $F(n,T,B)$, is given by

$$F(n,T,B) = G(\mu,T,B) + \mu n$$  \hspace{1cm} (2.77)
relativistic case is given by

\[ \Omega(\mu,0,B) = \Omega_0^*(\mu,0,B) + 2\pi M^2(\mu,0,B), \tag{2.108} \]

where \( \Omega_0^*(\mu,0,B) \) and \( M(\mu,0,B) \) are given by

\[ \Omega_0^*(\mu,0,B) = -\frac{m^*4}{24\pi^2} f(x) + \left( \frac{1}{2} \chi_{\text{para}} + \frac{1}{2} \chi_{\text{Dia}} \right) B^2 \]
\[ + \frac{(2\mu)^{5/2}}{4\pi^{3/2}} b^{-5/2} D^*(\mu) \Sigma_1^*, \tag{2.109} \]

and

\[ M(\mu,0,B) = -\frac{2^{1/2} m^*3/2}{\pi^3} \mu_0^{3/2} B^{1/2} \mu F^*(\mu) \chi_2^*, \tag{2.110} \]

and where

\[ f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3 \sinh^{-1} x, \tag{2.111} \]
\[ x = \left( \frac{\mu}{m^*} + 1 \right)^2 - 1, \tag{2.112} \]
\[ \chi_{\text{para}} = 2 \mu_0^2 \rho(\mu), \tag{2.113} \]
\[ \chi_{\text{Dia}} = -\frac{2}{3} \mu_0^2 \rho(\mu), \tag{2.114} \]
\[ Q_0(\mu,T,B) = - \frac{(2\mu)^{5/2}}{15\pi^2} \left( 1 + \frac{5}{4} \pi^2 b^{-2} - \frac{15\pi^{1/2}}{4} b^{-5/2} \right) \]  

(2.125)

\[ M(\mu,T,B) = - \frac{(2\mu)^{1/2}}{4\pi^{7/2}} a B^{1/2} \]  

(2.126)

and

\[ n(\mu,T,B) = - \frac{(2\mu)^{3/2}}{3\pi^2} \left[ 1 + \frac{3\pi^{1/2}}{2} b^{-3/2} \right] \]  

(2.127)

where

\[ \Sigma_4 = \sum_{r=1}^{\infty} \frac{\cos(br - \pi/4)}{r^{5/2}} \frac{2\pi^2 kT/(B/B_c)}{\sinh[2\pi^2 kT/(B/B_c)]} \]  

(2.128)

and

\[ \Sigma_5 = \sum_{r=1}^{\infty} \frac{\sin(br - \pi/4)}{r^{3/2}} \frac{2\pi^2 kT/(B/B_c)}{\sinh[2\pi^2 kT/(B/B_c)]} \]  

(2.129)

Thus

\[ 2\pi M^2(\mu,T,B) = - \frac{(2\mu)^{5/2}}{15\pi^2} - \frac{15\alpha}{2^{5/2} \pi^2} \mu^{1/2} b^{-1} |\Sigma_5|^2 \]  

(2.130)
The thermodynamic potential per unit volume from Eq. (2.3) is given by

\[ \Omega(\mu, T, B) = -\frac{(2\mu)^{5/2}}{15\pi^2} \{ 1 - \frac{15\pi^{1/2}}{8} b^{-3/2} |\Sigma| \} + \frac{5}{4} \pi^2 b^{-2} + \ldots \} \quad (2.137) \]

And we see that for a non-zero temperature system it is also obvious that the LOFER state possesses a larger thermodynamic potential than the non-magnetic (B=0) state.

G. General Conclusions

About Landau Orbital Ferromagnetism

We have investigated the LOFER states in the three electron gas models of (a) a non-relativistic non-interacting gas, (b) a relativistic non-interacting gas and (c) a non-relativistic gas interacting via a Coulomb potential and have found that the equations yielding the LOFER solution also admit a non-magnetic (B=0) solution. The free energy of the magnetically interacting system was derived and it was shown that the non-magnetic solution corresponds to the lowest free energy of the system in all three models and is thus the thermodynamically stable state of the system. This leaves only the possibility that LOFER states exist as metastable states of the system which could possibly provide an explanation
of why some but not all white dwarfs possess large magnetic fields. A recent investigation of Landau Orbital Ferromagnetism based on our original publications has shown that Eq. (2.131) sets stringent restrictions on the temperature of the electron gas for which LOFER solutions are possible; namely, the temperature \( T \) of the electron gas must satisfy the inequality

\[
T < T_c
\]

(2.138)

where \( T_c \) is given by

\[
T_c = 1.84 \times 10^2 \frac{(\mu^2-1)^2}{\mu^3} \, ^oK .
\]

(2.139)

For white dwarfs \( \mu \approx 1.5 \) and the limiting temperature \( T_c \) above which the LOFER solutions are not permitted is 85.3\(^o\)K. At this limiting temperature \( T_c \) the maximum LOFER field strength is given by

\[
B = 1.54 \times 10^5 \mu T_c \, Gauss
\]

(2.140)

Therefore, for densities in white dwarfs, magnetic fields of \( 10^7 \) G are not possible above about 100\(^o\)K. Thus there appears to be no correlation between the observed magnetic fields in white dwarfs and the LOFER theory. It
CHAPTER III. CONTINUUM POLARIZATION
IN THE MAGNETIC WHITE DWARFS

A. Discussion of Polarization

Polarization of radiation is usually considered as a "complicating" factor in the study of electromagnetic radiation when in fact the opposite is true; that is, the lack of polarization is not an elementary state of radiation. In general an individual source of radiation, such as an atom or molecule, emits radiation that in a given direction has a well-defined polarization state. It is only the superpositions of many individual contributions that can produce unpolarized radiation. Consider the example of an electron oscillating in the x-y plane. Radiation is emitted perpendicular to the vibration direction. Since the electron does not oscillate forever, the time of oscillation is characterized by a mean lifetime $t$. During this time of vibration, the polarization is a constant since the direction of motion of the electron is constant. At a later time the electron may again be set in motion. However, this motion does not necessarily reproduce the previous motion. Thus the polarization resulting from this second motion bears no relation to the first polarization. After several mean lifetimes, the average excitation energy is the same but the relative phases of the amplitude of the electric
determined from the electric field $\vec{E}(r,t)$ via Maxwell's equations so that it is sufficient to specify only the behavior of the electric field $\vec{E}$. Two orthogonal non-monochromatic plane waves, $E_x(t)$ and $E_y(t)$, may be represented by the equations

$$E_x(t) = E_x^{(0)}(t) \cos(\omega t + \delta_x(t)) ,$$  \hspace{1cm} (3.1)

and

$$E_y(t) = E_y^{(0)}(t) \cos(\omega t + \delta_y(t)) ,$$  \hspace{1cm} (3.2)

where $E_x^{(0)}(t)$ and $E_y^{(0)}(t)$ are the instantaneous amplitudes, $\omega$ is the instantaneous angular frequency, and $\delta_x(t)$ and $\delta_y(t)$ are the instantaneous phase factors of each plane wave. The amplitudes and phase factors are time dependent but we will assume they fluctuate slowly compared to the cosine variation. Rewriting Eqs. (3.1) and (3.2) as

$$\frac{E_x(t)}{E_x^{(0)}(t)} = \cos(\omega t) \cos(\delta_x(t)) - \sin(\omega t) \sin(\delta_x(t)) ,$$  \hspace{1cm} (3.3)

and
\[
\begin{align*}
= E_x^{(0)} E_y^{(0)} \left( \frac{1}{2} \cos \delta_x \cos \delta_y + \frac{1}{2} \sin \delta_y \sin \delta_x \right) \\
\langle E_x(t) E_y(t) \rangle = \frac{1}{2} E_x^{(0)} E_y^{(0)} \cos \delta . \quad (3.25)
\end{align*}
\]

Substituting Eqs. (3.23), (3.24), and (3.25) into (3.22) we have

\[
\begin{align*}
2E_x^{(0)} E_y^{(0)} &+ 2E_x^{(0)} E_y^{(0)} - (2E_x^{(0)} E_y^{(0)} \cos \delta)^2 \\
= (2E_x^{(0)} E_y^{(0)} \sin \delta)^2 . \quad (3.26)
\end{align*}
\]

Now adding and subtracting \(E_x^{(0)} E_y^{(0)}\) to the left hand side of Eq. (3.26) gives

\[
\begin{align*}
(E_x^{(0)} + E_y^{(0)})^2 - (E_x^{(0)} - E_y^{(0)})^2 - (2E_x^{(0)} E_y^{(0)} \cos \delta)^2 \\
= (2E_x^{(0)} E_y^{(0)} \sin \delta)^2 . \quad (3.27)
\end{align*}
\]

Defining the four Stokes parameters \(S_0, S_1, S_2, S_3\) for a monochromatic radiation by

\[
\begin{align*}
S_0 &= E_x^{(0)} + E_y^{(0)} \quad (3.28a) \\
S_1 &= E_x^{(0)} - E_y^{(0)} \quad (3.28b) \\
S_2 &= 2E_x^{(0)} E_y^{(0)} \cos \delta \quad (3.28c)
\end{align*}
\]
we see from Eq. (3.27) that for monochromatic radiation the Stoke's parameters satisfy

\[ S_3 = 2E_x^{(0)}E_y^{(0)} \sin \delta , \quad (3.28d) \]

we see from Eq. (3.27) that for monochromatic radiation the Stoke's parameters satisfy

\[
S_0^2 = S_1^2 + S_2^2 + S_3^2 . \quad (3.29)
\]

We see that \( S_0 \) is the total intensity of the radiation, \( S_1 \) and \( S_2 \) measures the linear polarization along two orthogonal directions, and \( S_3 \) measures the circular polarization. Thus the four quantities which we originally used to describe the radiation may be expressed in terms of the Stokes parameter as:

\[
\text{intensity} = S_0
\]

\[
\text{degree of polarization} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}
\]

\[
\text{angle of polarization} = \frac{1}{2} \tan^{-1} \frac{S_2}{S_1}
\]

\[
\text{eccentricity of polarization} = \frac{2}{\sqrt{\frac{S_1^2 + S_2^2 + S_3^2}{S_1 + S_3}}}
\]
3. Measurement of the Stoke's Parameters

Eq. (3.28) defines the Stoke's parameters for a monochromatic wave. Unfortunately even the sharpest spectral line does not produce strictly monochromatic radiation since it has at least a small variation of frequencies due to the natural broadening associated with the uncertainties in the energy levels. Thus we must first reconsider quasi-monochromatic radiation and generalize the Stokes parameters to incorporate this type of radiation. Since we are interested only in the phase difference $\delta(t)$ here and not specifically either $\delta_x(t)$ or $\delta_y(t)$ let us rewrite Eqs. (3.1) and (3.2) as

$$E_x(t) = E_x^{(0)}(t) \sin \omega t \quad (3.30)$$

$$E_y(t) = E_y^{(0)}(t) \sin(\omega t - \delta(t)) \quad (3.31)$$

Letting the $y$ component be subjected to a constant retardation $\varepsilon$, Eq. (3.31) becomes

$$E_y(t) = E_y^{(0)}(t) \sin(\omega t - \delta(t) - \varepsilon) \quad (3.32)$$

The component of the electric field in a direction which makes an angle $\psi$ with the $x$ axis, is
\[ E(\psi, \varepsilon) = E_x^{(0)}(t) \sin \omega t \cos \psi + E_y^{(0)}(t) \sin(\omega t - \delta(t) - \varepsilon) \sin \psi \]

\[ E(\psi, \varepsilon) = E_x^{(0)}(t) \sin \omega t \cos \psi + E_y^{(0)}(t) \cos(\delta(t) + \varepsilon) \sin \psi \sin \omega t - E_y^{(0)}(t) \sin(\delta(t) + \varepsilon) \sin \psi \cos \omega t \]  

(3.33)

Thus the instantaneous intensity in this direction is

\[ E^2(\psi, \varepsilon) = E_x^{(0)2}(t) \cos^2 \psi + E_y^{(0)2}(t) \sin^2 \psi + 2E_x^{(0)}(t)E_y^{(0)}(t) (\cos \delta \cos \varepsilon - \sin \delta \sin \varepsilon) \sin \psi \cos \psi \]  

(3.34)

The apparent intensity in the direction \( \psi \) is obtained by the time average of Eq. (3.34). Thus

\[ I(\psi, \varepsilon) = \overline{E_x^{(0)2}} \cos^2 \psi + \overline{E_y^{(0)2}} \sin^2 \psi \]
\[ + \sin \psi \cos \psi (2E_x^{(0)} E_y^{(0)}) \cos \delta \cos \epsilon \]
\[ - 2E_x^0 E_y^0 \sin \delta \sin \epsilon \] , \hspace{1cm} (2.35)

where the time average, for example, of \( E_x^{(0)2}(t) \) has been denoted by \( E_x^{(0)2} \). By defining the Stoke's parameters for a quasi-monochromatic wave as simply the time averages of the quantities defined as the Stoke's parameters for monochromatic radiation; namely

\[
S_0 = E_x^{(0)2} + E_y^{(0)2} \hspace{1cm} (3.36a)
\]
\[
S_1 = E_x^{(0)2} - E_y^{(0)2} \hspace{1cm} (3.36b)
\]
\[
S_2 = 2E_x^{(0)} E_y^{(0)} \cos \delta \hspace{1cm} (3.36c)
\]
\[
S_3 = 2E_x^{(0)} E_y^{(0)} \sin \delta \hspace{1cm} (3.36d)
\]

we see that the average intensity in the direction \( \psi \) may be expressed as

\[
I(\psi, \epsilon) = \frac{1}{2} \left[ S_0 + S_1 \cos 2\psi + (S_2 \cos \epsilon - S_3 \sin \epsilon) \right] \sin 2\psi \] . \hspace{1cm} (3.37)
We may now use Eq. (3.37) as a means of measuring the Stoke's parameter since we note that

\[ S_0 = I(0^\circ,0) + I(90^\circ,0) \]  
\[ S_1 = I(0^\circ,0) - I(90^\circ,0) \]  
\[ S_2 = I(45^\circ,0) - I(135^\circ,0) \]  
\[ S_3 = I(45^\circ,\pi/2) - I(135^\circ,\pi/2) \]  

Therefore to measure the Stokes parameters we need only to specify how measurements can be made of the intensity of radiation along a particular axis and how a single component of the wave may be subjected to a constant retardation. These measurements are in fact quite simple. To measure the intensity along a particular axis we need only place a polaroid sheet between the incident radiation and the detector with the easy axis of the polaroid, the only axis along which electronic vibrations are permitted, aligned along the desired axis. To produce the constant retardation of \( \pi/2 \) in one of the electric field components, we simply position a quarter wave plate before the polaroid sheet. The Stoke's parameters can thus be measured with a polaroid sheet and a quarter wave plate.
Besides indicating how the Stoke's parameters may be simply measured, Eq. (3.38) also indicates the physical significance of the Stoke's parameters. We see that $S_0$ is the total intensity of the quasi-monochromatic radiation, $S_1$ is the linearly polarized radiation along the $x$ axis minus the linearly polarized radiation along the $y$ axis. $S_2$ is the linearly polarized radiation along a direction inclined 45° to the $x$ axis minus the linearly polarized radiation along a direction inclined 135° to the $x$ axis, and $S_3$ is the right circularly polarized radiation minus the left circularly polarized radiation.

C. Optically Thin Models for Continuum Polarization

In an attempt to explain the observed circular polarization of continuum radiation, Kemp developed an harmonic oscillator model, called the gray body magneto-emission theory, in order to obtain some physical insight into the problem. Kemp's solution for this model was made with the assumption that the cyclotron frequency is much smaller than the frequency of the observed continuum radiation,

$$2\Omega_L \ll \omega \quad (3.39)$$

However, an exact quantum mechanical solution of this harmonic oscillator model has been obtained.
The Hamiltonian $H$ for a non-relativistic three-dimensional particle is

$$H = \frac{\hat{p}^2}{2m} + V(r) \quad (3.40)$$

where $V(r)$ is the potential energy of the particle, $\hat{p}$ is the particle's momentum and $m$ is the particle's mass. In the presence of an external magnetic field $\vec{B}$ specified by the vector potential $\vec{A}$, the Hamiltonian for the particle becomes

$$H = \frac{(\hat{p} - \frac{e}{c} \vec{A})^2}{2m} + V(r), \quad (3.41)$$

or

$$H = \frac{\hat{p}^2}{2m} - \frac{e}{2mc} (\hat{p} \cdot \vec{A} + \vec{A} \cdot \hat{p}) + \frac{e^2}{2mc^2} \vec{A}^2 + V(r). \quad (3.42)$$

This Hamiltonian has eigenenergies

$$E_{n_+ n_0 n_z} = \mathcal{H}(\omega_c + \Omega_L) (n_0 + 1/2) + \mathcal{H}(\omega_c - \Omega_L) (n_0 + 1/2)$$

$$+ \mathcal{H}_\omega (n_z + 1/2) \quad (3.43)$$
where

\[ \Omega_L \equiv \frac{|e| B}{2mc}, \]

\( \omega_o \) is the natural frequency of the oscillator, \( \omega_c \equiv (\omega_o^2 + \Omega_L^2)^{1/2} \) and the \( n_+ \) and \( n_- \) are non-negative integers representing the energy level of the oscillator. Thus, for example, \( E_{000} \) and \( E_{100} \) are given by

\[ E_{000} = \frac{1}{2} \mathcal{M}(\omega_c + \Omega_L) + \frac{1}{2} \mathcal{M}(\omega_c - \Omega_L) + \frac{1}{2} \mathcal{M}\omega_o \quad (3.43) \]

\[ E_{100} = \frac{3}{2} \mathcal{M}(\omega_c + \Omega_L) + \frac{1}{2} \hbar(\omega_c - \Omega_L) + \frac{1}{2} \mathcal{M}\omega_o \quad (3.44) \]

so that the energy differences between the first excited state for the quanta of type + and the ground state is

\[ E_{100} - E_{000} = \mathcal{M}(\omega_c + \Omega_L). \quad (3.45a) \]

Similarly

\[ E_{010} - E_{000} = \mathcal{M}(\omega_c - \Omega_L) \quad (3.45b) \]

Thus far we have neglected the radiation field. To incorporate it in the Hamiltonian the total vector potential \( \vec{A} \) in Eq. (3.41) may be written simply as the sum of the vector potential \( \vec{A}_{ext} \) for the external magnetic field
and the vector potential $\hat{A}_r$ for the radiation field

$$\hat{A} = \hat{A}_{\text{ext}} + \hat{A}_r \quad (3.46)$$

The Hamiltonian becomes, neglecting second order terms in the vector potentials,

$$H = \frac{P^2}{2m} + \Omega_L L_z - \frac{e}{mc} \hat{A}_r (\hat{p} - \frac{e}{c} \hat{A}_{\text{ext}}) + V(r) \quad (3.47)$$

The electric field for a circularly polarized wave may be expressed as

$$\hat{E}(r,t) = E_0 (\hat{x}+i\hat{y}) e^{ik\cdot r - i\omega t} \quad (3.48)$$

where $\pm$ represents right or left circular polarization respectively. The vector potential $\hat{A}_r$, given by

$$\frac{-i}{c} \frac{\partial \hat{E}}{\partial t}, \text{ is thus}$$

$$\hat{A}_r = \frac{-ic}{\omega} E_0 (\hat{x}+i\hat{y}) e^{ik\cdot r - i\omega t} \quad (3.49)$$

Thus the interaction term $H_I$ of the Hamiltonian due to the radiation field is

$$H_I = -\frac{e}{mc} \frac{-ic}{\omega} E_0 (\hat{x}+i\hat{y}) e^{ik\cdot r - i\omega t} (\hat{p} - \frac{e}{c} \hat{A}_{\text{ext}})$$

$$\quad (3.50)$$
Let us assume that the temperature is sufficiently low that only the ground state \( n \) and the lowest excited states need be considered. Denoting the general state vector by \( |n_+ n_- n_z\rangle \), it follows\(^{30}\) that the matrix element \(<100|H_I|000\rangle\) is

\[
<100|H_I|000\rangle = -\frac{ieEe^{-i\omega t}}{2\omega} \left( \frac{\hbar}{\omega_C} \right)^{1/2} \left( \Omega_L + \omega \right)
\]

(3.51)

and \(<010|H_I|000\rangle\) is

\[
<010|H_I|000\rangle = \frac{ieEe^{-i\omega t}}{2\omega} \left( \frac{\hbar}{\omega_C} \right)^{1/2} \left( \Omega_L - \omega \right)
\]

(3.52)

where \( E \) denotes \( E_{\omega}(x+iy)e^{ix \cdot r} \). Using Eqs. (3.45) in (3.35) and (3.36) then

\[
|<100|H_I|000>|^2 \alpha \begin{cases} 
(\omega-\Omega_L)^{-1} & \text{for } \omega > 2\Omega_L \\
0 & \text{for } \omega \leq 2\Omega_L
\end{cases}
\]

(3.53)

and

\[
|<010|H_I|000>|^2 \alpha (\omega+\Omega_L)^{-1} \text{ for all } \omega
\]

(3.54)

Since the emission coefficient is proportional to the square of the matrix elements, the fractional circular
polarization in the harmonic oscillator optically thin model is

\[ q = \frac{j_+ - j_-}{j_+ + j_-} = \frac{\langle 100 | H_I | 000 \rangle \langle 010 | H_I | 000 \rangle - \langle 100 | H_I | 000 \rangle \langle 010 | H_I | 000 \rangle}{\langle 100 | H_I | 000 \rangle \langle 100 | H_I | 000 \rangle + \langle 010 | H_I | 000 \rangle \langle 010 | H_I | 000 \rangle} \]

\[ \alpha \begin{cases} - \frac{\Omega_L}{\omega} & \text{for } \omega > 2\Omega_L \\ +1 & \text{for } \omega < 2\Omega_L \end{cases} \]  

(3.55)

A similar calculation can be carried out for the linear polarization. With the magnetic field aligned along the z axis, we choose the x-axis along the line of sight. The fractional linear polarization of the \( \sigma \) and \( \pi \) components is then defined by

\[ q^* = \frac{1/2(j_{\sigma+} + j_{\sigma-} - j_\pi)}{1/2(j_{\sigma+} + j_{\sigma-} + j_\pi)} \]  

(3.56)

A calculation similar to that of the circular polarization yields

\[ j_{\sigma+} = \langle 100 | H_I | 000 \rangle \langle 010 | H_I | 000 \rangle \alpha (\omega - \Omega_L)^{-1} \]  

(3.57a)

\[ j_{\sigma-} = \langle 010 | H_I | 000 \rangle \langle 100 | H_I | 000 \rangle \alpha (\omega + \Omega_L)^{-1} \]  

(3.57b)

\[ j_\pi = \langle 001 | H_I | 000 \rangle \langle 001 | H_I | 000 \rangle \alpha \omega^{-1} \]  

(3.57c)
where the general state vector is $|n_{\sigma^+}n_{\sigma^-}n_\pi\rangle$. Thus

$$q^* = \begin{cases} \Omega^2_L & \text{for } 2\Omega_L \leq \omega \\ \frac{2\omega^2 - \Omega^2_L}{2\omega - \Omega^2_L} & \text{for } \omega > 2\Omega_L \\ \frac{(\omega + 2\Omega_L)}{(\omega + 3\Omega_L)} & \text{for } \omega < 2\Omega_L \end{cases}$$

(3.58)

Although Eqs. (3.55) and (3.58) were derived with the assumption that the temperature is sufficiently low so that only the first excited states need be considered, these results have been shown to be valid at all temperatures. 

The previous calculations were performed assuming that the net polarization is determined by the strengths of a single-oscillator emissions at $\omega$. However, in a magnetic field the distribution of oscillators is no longer constant and these changes in the densities $\frac{dn_+}{d\omega}$ and $\frac{dn_-}{d\omega}$ must be included. Now

$$\frac{dn_+}{d\omega} = \frac{\omega + \Omega_L}{(\omega^2 + 2\Omega_L \omega)^{1/2}}$$

(3.59)

$$\frac{dn_-}{d\omega} = \frac{\omega - \Omega_L}{(\omega^2 - 2\Omega_L \omega)^{1/2}}$$

(3.60)

The net radiation, $P_+ (\omega)$, of each type of circular polarization is thus
\[ P_\pm(\omega) = j_\pm(\omega) \frac{dn_\pm}{d\omega} \]  

(3.61)

so that

\[ q = \frac{P_+(\omega)-P_-(\omega)}{P_+(\omega)+P_-(\omega)} = \begin{cases} 
(s^2-1)^{1/2} - s & (s>1) \\
1 & (0<s<1) 
\end{cases} 
\]

(3.62)

where \( s = \frac{\omega}{2\Omega_L} \). Similarly for linear polarization

\[ q^* = \begin{cases} 
\frac{[1+(1+t)^{1/2} + (1-t)^{1/2} - 2(1-t^2)^{1/2}]}{[1+(1+t)^{1/2} + (1-t)^{1/2} + 2(1-t^2)^{1/2}]} & \text{for } t = s^{-1} < 1 \\
\frac{s^{1/2} - 2(l+s)^{1/2}}{s^{1/2} + 2(l+s)^{1/2}} & \text{for } l > s > 0 
\end{cases} 
\]

(3.63)

Besides the harmonic oscillator model for continuum polarization, Kemp\textsuperscript{29} also considered the production of circular polarization in a bremsstrahlung (semi-classical electron-atom collision) model. For the circular polarization he found
This result is the same as obtained for the harmonic oscillator model except for the factor of eight. Extending this model for the linear polarization we find

\[ q = - \frac{8\Omega}{\omega} \quad (3.64) \]

\[
q^* = \frac{1/2\left[ (1 - \frac{8\Omega}{\omega} + 24 \frac{\Omega^2}{\omega^2}) + (1 + \frac{8\Omega}{\omega} + 24 \frac{\Omega^2}{\omega^2}) \right] - 1}{1/2\left[ (1 - \frac{8\Omega}{\omega} + 24 \frac{\Omega^2}{\omega^2}) + (1 + \frac{8\Omega}{\omega} + 24 \frac{\Omega^2}{\omega^2}) \right] + 1}
\]

\[ (3.65) \]

or

\[ q^* \approx 12 \frac{\Omega^2}{\omega^2} \quad (3.66) \]

D. The Optically Thick Model of Continuum Polarization

In this section we will briefly discuss Shipman's radiative transfer corrections to Kemp's models for the circular polarization which was a motivating force behind our investigation. We will reconsider the circular polarization in the light of the exact solutions to Kemp's harmonic oscillator model and also new detailed observations now available, but the main emphasis will be to the extent the optically thick model to treat linear polarization. As we shall see, besides serving as a check
on the optically thick model, an investigation of the linear polarization will enable us to distinguish between the harmonic oscillator theory and the bremsstrahlung theory.

1. Shipman's Optically Thick Model for Circular Polarization

The discovery of continuum circular polarization in the peculiar DC white dwarf Grw + 70° 8247 provided the first astrophysical test for Kemp's theory of continuum polarization. Using the magnitude of the observed circular polarization and Kemp's relation between this polarization and the magnetic field strength, a magnetic field strength of about $10^7$ Gauss was implied, consistent with the strength of the fields one expects in magnetic white dwarfs. However, Kemp's theories, both the harmonic oscillator and the bremsstrahlung model, predict a circular polarization linearly proportional to the wavelength, in direct contradiction to the observed decrease in the circular polarization beyond about 4000 Å. Shipman suggested that the wavelength dependence of the circular polarization may be changed if one includes the effects of radiative transfer of the radiation as it traverses through the atmosphere of the white dwarf.
To obtain the results for the circular polarization in this optically thick model, the necessary quantities which one needs are the absorption coefficients for atoms in the presence of the magnetic fields whereas in the optically thin models the important quantities are the emission coefficients\(^{29,30}\) in the presence of the magnetic field. Since the absorption coefficients for atoms in a magnetic field are not available and the calculation of these quantities entails lengthy and laborious calculations, Shipman used an approximation which enabled him to calculate a correction to the zero magnetic field atomic absorption coefficients which used the calculated model magnetic field emission coefficients. The fractional circular polarization has been defined in the optically thin model by

\[
q = \frac{j_+ - j_-}{j_+ + j_-}
\]  

(3.67)

where \(j_+\) are the emission coefficients for + type quanta. Shipman noted that in local thermodynamic equilibrium, the absorption coefficients could be related to the emission coefficients. As we have seen in Eq. (1.8), this enabled Shipman to express the correction in the absorption coefficients \(\Delta k/2\) for one of the circular components in terms of the optically thin fractional circular polarization \(q\) and the absorption coefficient in the absence of a magnetic
field $\kappa_o$ as

$$\frac{\Delta \kappa}{2} = \kappa_o q.$$  \hspace{1cm} (3.68)

Thus the total absorption coefficients for the right and left polarized radiation $\kappa_{R,L}$ is given by

$$\kappa_{R,L} = \kappa_o (1 \pm \frac{\Delta \kappa}{2\kappa_o})$$  \hspace{1cm} (3.69)

Shipman then used the model atmosphere program ATLAS$^85$ to calculate the flux $H_{R,L}$ of each type of circularly polarized radiation using for the absorption coefficients in a magnetic field their approximate values given by Eq. (3.69). The fractional circular polarization in the optically thick model is thus given by

$$q_{\text{thick}} = \frac{H_R - H_L}{H_R + H_L}$$  \hspace{1cm} (3.70)

The results of his calculation along with the observations available at the time of his calculation$^86$ are shown in Fig. (1). The agreement between Shipman's optically thick results and the observations can indeed be seen to be good, particularly in the optical region, although discrepancies occur for the infrared and ultraviolet regions.
To calculate an optically thick model we must first obtain the absorption coefficients in a magnetic field. As did Shipman, we will attempt to use the model emission coefficients in a magnetic field. We note that the magnetic field-corrected continuous absorption coefficients $\kappa_p$, can be related to the zero field coefficients $\kappa_0$ as follows

$$\kappa_p = \kappa_0 \left( \frac{\kappa_p}{\kappa_0} \right)$$

Now using the LTE Kirchoff-Planck relation between the continuous absorption coefficients and the emission coefficients we have then

$$\kappa_p = \kappa_0 \left( \frac{j_p}{j_0} \right)$$

where $j_p$ and $j_0$ are the polarized and zero magnetic field intensities predicted by the various magnetoemission models. The three models to be considered for the fractional linear polarization are:

**Model A.** An exact harmonic oscillator model with a constant density of oscillator states [see Eq. (3.57)].

**Model B.** An exact harmonic oscillator model with the density of states corrected for the magnetic field [see Eq. (3.61) for example];
Model C. A bremsstrahlung model [see Eq. (3.65)] correct to the first non-zero order in the magnetic field $B$.

Thus the continuous absorption coefficients corrected for the magnetic field for the different models of the linear polarization are

Model A:

$$\kappa_{\sigma^+} = \kappa_0 \left(1 + \frac{\Omega L}{\omega} + \frac{\Omega^2 L^2}{\omega^2}\right),$$  \hspace{1cm} (3.74a)

$$\kappa_{\sigma^-} = \kappa_0 \left(1 - \frac{\Omega L}{\omega} + \frac{\Omega^2 L^2}{\omega^2}\right),$$  \hspace{1cm} (3.74b)

$$\kappa_{\pi} = \kappa_0 ;$$  \hspace{1cm} (3.74c)

Model B:

$$\kappa_{\sigma^+} = \kappa_0 \left(1 + \frac{\Omega L}{\omega} + \frac{3}{2} \frac{\Omega^2 L^2}{\omega^2}\right),$$  \hspace{1cm} (3.75a)

$$\kappa_{\sigma^-} = \kappa_0 \left(1 - \frac{\Omega L}{\omega} + \frac{3}{2} \frac{\Omega^2 L^2}{\omega^2}\right),$$  \hspace{1cm} (3.75b)

$$\kappa_{\pi} = \kappa_0 ;$$  \hspace{1cm} (3.75c)
Model C:

\[ \kappa_{\sigma^+} = \kappa_0 \left( 1 + \frac{8\Omega L}{\omega} + 24 \frac{\Omega L^2}{\omega^2} \right) \]  
\[ (3.76a) \]

\[ \kappa_{\sigma^-} = \kappa_0 \left( 1 - \frac{8\Omega L}{\omega} + 24 \frac{\Omega L^2}{\omega^2} \right) \]  
\[ (3.76b) \]

\[ \kappa_\pi = \kappa_0 , \]  
\[ (3.76c) \]

where \( \Omega_L = eB \sin \theta/2mc \).

We have used the program ATLAS\textsuperscript{85} to calculate the model white-dwarf atmospheres. The model atmospheres used were \( T_{\text{eff}} = 12,000^\circ \text{K}, \log g = 8, \text{H} = 0.9, \text{He} = 0.1, \) and \( T_{\text{eff}} = 14,000^\circ \text{K}, \log g = 8, \text{H} = 0.0, \text{He} = 1.0. \) Both models are in LTE and radiative equilibrium, and have solar metal content. The magnetic fields were determined by fitting the theoretical models to the observations at a particular wavelength. For Model C in the linear polarization \( 4300^\circ \text{A} \) was chosen. Since Models A and B could not be fitted to the observations, the magnetic field strength was chosen as that of the best fit possible. This choice produces the correct magnitude of the fractional linear polarization; however, the polarization is \( \pi \)-like rather than \( \sigma \)-like.

In Fig. (2) the wavelength dependence of the linear polarization predicted using the model atmosphere is
agreement of the circular polarization. Some recent observations of the circular polarization made since Shipman's results (see Fig. (3)) suggest the latter alternative since the decrease in the predicted circular polarization in the optically thick model beyond 6000 Å is not large enough when compared to these new observations. This is, in fact, the same problem which occurred for the linear polarization.

To summarize the results for both the linear and the circular polarizations we conclude that for the linear polarization there is a significant discrepancy between all observations and the theoretical predictions of the three models. In addition, the circular polarization predicted by Shipman does not agree as well with the new circular polarization observations as it did with the older observations. As a result, we feel that, although a radiative-transfer correction to Kemp's model does drastically change and greatly improve the correlation between the theoretical wavelength dependence of the circular and linear polarization and their observed wavelength dependence, there still remains a significant discrepancy in the optically thick model as presented here. Probably the most glaring assumption made in the model is the approximation for the continuous absorption coefficients. Thus a calculation of the polarization of
A. Screening Effects in the Absence of External Magnetic Fields

1. Derivation of Screened Potential

Let us consider the screening due to free charged particles of the field of a positive ion with charge $Z e$ at the origin. The scalar potential $\phi(r)$ of the field of the ion and the free particles screening it satisfies the Poisson equation in a steady state

$$\nabla^2 \phi(r) = - \frac{4\pi}{\varepsilon} \rho(r) ,$$  \hspace{1cm} (4.1)

where $\varepsilon$ is the dielectric constant and $\rho$ is the charge density of screening particles and $\phi(r)$ is chosen so that $\phi(r) \to 0$ as $r \to \infty$. The mean density of ions of species $i$, $N_i$, at a point where the potential is $\phi$ is given by the Maxwell Boltzmann formula for non-degenerate particles

$$N_i = N_i^{(0)} \exp\left[-Z_i e \phi(r)/kT\right] ,$$  \hspace{1cm} (4.2)

where $N_i^{(0)}$ is the mean density of ions of species $i$ in all space and $Z_i e$ is the charge on an ion of species $i$. Similarly the electron density is

$$N_e = N_e^{(0)} \exp\left[e \phi(r)/kT\right] ,$$  \hspace{1cm} (4.3)
given by Eq. (4.11). We note that the screening of the potential field of the central ion $Z_e$ is not due solely to the negatively charged electrons but also to the positively charged ions of charge $Z_1^e$ and mean number density $N_1^{(0)}$. For a two component plasma consisting of electrons and singly charged ions of charge $Z_1^e$ and mean density $N_1^{(0)}$, the neutrality condition of the plasma as a whole requires that

$$N_e^{(0)} = Z_1 N_1^{(0)} \quad . \quad (4.26)$$

Using Eq. (4.26) in Eq. (4.11), the screening length $D$ may be written as (with $\varepsilon=1$)

$$D^{-2} = \frac{4\pi e^2}{kT} \left( Z_1 + 1 \right) N_e \quad , \quad (4.27)$$

or

$$D = 6.9 \sqrt{\frac{T}{(Z_1+1)N_e}} \text{ cms} \quad . \quad (4.28)$$

The point which we wish to stress is the effect made by considering the ions as point charges rather than a continuous charge distribution. For a continuous ion charge distribution, 50 Eq. (4.4) becomes
\[ \rho = -e N_e + \rho_{\text{ion}} \quad (4.29) \]

and thus Eq. (4.8) becomes

\[ \rho = -e N_e^{(0)} + \rho_{\text{ion}} - \frac{e^2 N_e^{(0)} \phi(r)}{kT}. \quad (4.30) \]

This implies that for a neutral collection of articles the screening length \( D_c \) with the ions considered as a continuous charge distribution is given by

\[ D_c = 6.9 \sqrt{\frac{T}{N_e}} \quad (4.31) \]

Thus the effect of considering the ions as point charges is to reduce the screening length. For a two component plasma the screening length is reduced by a factor of \( \sqrt{Z_1+1} \). The minimum effect occurs for a singly charged plasma \( (Z_1=1) \) where the screening length is reduced by a factor of \( \sqrt{2} \); that is,

\[ D = 4.9 \sqrt{\frac{T}{N_e}} \text{ cms} \quad (4.32) \]

Also of interest are the two assumptions made during the derivation of the screened potential, namely

(i) that there were enough particles present inside the Debye sphere that statistical averaging is meaningful
(that is, we could use the Maxwell Boltzmann distribution of particles) and

(ii) that the quantity \((e\Phi/kT)\) was small.

The first assumption implies that the mean number \(n\) of charged particles in the Debye sphere is large

\[ n >> 1 \]

(4.33)

In terms of the total particle density \(N\) and the radius of the Debye sphere \(D\), Eq. (4.33) becomes

\[ N\left(\frac{4}{3} \pi D^3\right) >> 1 \]

(4.34)

Thus the Debye radius must satisfy the inequality

\[ D^3 >> \frac{3}{4\pi} \frac{1}{N} \]

(4.35)

or

\[ N >> \left(\frac{4\pi}{3} D^3\right)^{-1} \]

(4.36)

Using Eq. (4.32) we see that for a two-component plasma of electrons and singly ionized ions (so that \(N=2N_e\))

\[ (4.9)^3 \left(\frac{T}{N_e}\right)^{3/2} >> \frac{3}{8\pi} \frac{1}{N_e} \]

(4.37)
where $Y_{\ell m}(\theta, \phi)$ is the usual spherical harmonic.

The radial function $R(r)$ is a solution of the equation

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \left\{ \frac{2m}{\hbar^2} [E-V(r)] - \frac{\ell(\ell+1)}{r^2} \right\} R(r) = 0,
$$

(4.50)

Defining the normal reduced radial function $\chi(r)$ by

$$
\chi(r) = r R(r) ,
$$

(4.51)

Eq. (4.50) may be written as

$$
- \frac{d^2}{dr^2} \chi(r) + \left[ \frac{\ell(\ell+1)}{r^2} + V(r) \right] \chi(r) = E \chi(r) ,
$$

(4.52)

where the units chosen are the Rydberg units so that the unit of length is the Bohr radius and the unit of energy is the ionization energy of hydrogen, 13.6 eV. Our problem is that Eq. (4.52) can be solved analytically only when the potential is Coulomb in nature. Thus approximate methods of solution must be used to obtain the solution of Eq. (4.52) when the potential $V(r)$ is our
Slater-type orbitals and Coulomb radial functions may be written simply as linear combinations of each other.

In principle, completeness demands that the extent of the Hilbert space encompassed by our expansion in Eq. (4.54) should include continuum states. This can be practically achieved by choosing non-integer values for the $\alpha_i^{-1}$. In practice, for the problem under discussion, it was found that, to the accuracy desired, it was permissible to exclude continuum states. The various non-linear parameters $\alpha_i$ are chosen input data whereas the linear parameters $a_{ik}$ are evaluated numerically in the course of diagonalization of the Hamiltonian. With regard to the choice of $\alpha_i$ values, we were guided by the fact that, for the hydrogen Coulomb wave functions, $\alpha = (|E|)^{1/2} = n^{-1}$.

In general, 14 of the Slater-type orbitals were used to obtain at least 7 figure stability, although only two were needed for the ground state to obtain stability to the quoted figures. For the special case of the Debye-Hückel potential, good agreement is obtained between our variational results and the results obtained from the numerical integration techniques of Rogers et al., except for some particular cases noted below. We also find that, as expected, the binding energies of the states decrease with decreasing screening length. Table I gives the binding energies with $Z = 1$ for the values
of the screening length $D$ for which some differences between our method and that of Rogers et al.\textsuperscript{63} have been found. Note that we conform to the eigenfunction and eigenenergy labelling of the Coulomb potential, that is, the lowest $s$ state is denoted by $1s$, the second lowest $s$ state by $2s$ etc., although the numerical value is no longer associated with a "principal quantum number".

The crossing of the energy eigenvalues for different levels in the case of strong screening as described by Rogers et al.\textsuperscript{63} is confirmed by us. Some slight differences are noted in the energies of some of the higher levels but these are usually only discrepancies in the last figure. The only major differences are as follows. Bound states not reported in Rogers et al.\textsuperscript{63} but found in our calculations are: for $D = 100$, the $9i$ and $9k$ states; for $D = 70$, the $8g$ state; and for $D = 15$, the $4s$ and $4p$ states. The apparent discrepancy in the results for the $3s$ state for $D = 7$ which Rogers et al.\textsuperscript{63} quote in their Table I and which we fail to find is obviously a cataloging error made by them as they also quote the critical screening length for the $3s$ state as being 7.171 in their Table III.

In addition, we agree with the conclusion of Rogers et al.\textsuperscript{63} that, contrary to the remark of Rouse,\textsuperscript{62} the number of bound energy states for a particle moving
For example, Eq. (4.58) implies there are no s states possible below $D=1/2$. However, it does not imply that there necessarily is a bound s state above $D=1/2$. As we shall see, the first bound s state appears at $D=.84$.

For the general completely screened Coulomb potential, Eq. (4.25), we have calculated the energy levels for some particular values of the screening length and for the mean minimum ion atmosphere $A$ given by $A=D/2$ and $A=D$. These are given in Tables II and III.

We consider that the variational method described above is more advantageous than the numerical integration method used by Rogers et al.\textsuperscript{63} in the following respects.

(1) Rogers et al. numerically solve the radial differential equation in a transformed space ($\rho$-space) and not in the radial coordinate space ($r$-space). The relation between $\rho$ and $r$ is given by

$$\rho = \left(\frac{2Z}{a_0}\right) \frac{r}{\lambda_{nl}}, \quad (4.59)$$

where $\lambda_{nl}$ is related to the eigenenergies $\epsilon_{nl}$ by

$$\epsilon_{nl} = -\frac{Z^2}{2a_0^2} \frac{a}{\mu} \lambda_{nl}^2 . \quad (4.60)$$

In addition, the screening parameter, $d$, used by Rogers
et al.\(^6\) is related to the Debye-Hückel screening length, \(D\), by

\[
d = \left(\frac{2\pi}{a_o}\right) D/\lambda_{nl}.
\]

The relation between \(D\) and \(d\) is proportional to the unknown eigenvalue \(\epsilon_{nl}\), an initial guess must be made for \(d\) and then using the calculated value of the eigenvalue for that choice of \(d\), the Debye-Hückel screening length \(D\) may then be determined. An iterative procedure then must be followed until the choice of \(d\) with the calculated value of \(\epsilon_{nl}\) yields the desired value of \(D\). This procedure must then be repeated for every eigenenergy.

Since calculations using the above described variational method are done in \(r\) space, the desired Debye-Hückel screening length is immediately used and a single calculation yields the number of eigenenergies determined by the number of Slater orbitals used.

(2) Since our method is based on the variational principle it has the advantage that the exact state energy exists as a lower bound to our eigenenergies. This is not true for numerical integration techniques.

(3) Our calculations were performed on an IBM 360-65 computer at a rate of 150 eigenenergies per minute, whereas Rogers et al. used a CDC 6600 computer.
Due to the difference in machines an exact comparison between the two methods of the computational time involved is not straightforward. However, a crude estimate can be obtained by comparing our calculation rate of 150 energies per minute on the IBM 360-65 with the calculation rate of Rogers et al. which they obtained on a CDC 3600, namely 14 eigenenergies per minute, since these two machines are roughly of comparable speed.

(4) Our method is capable of providing analytic wave functions.

3. Spontaneous-Emission Transition Probabilities

For the Debye-Hückel Potential in the Dipole Approximation

In the electric dipole approximation, the probability that an atom will undergo a transition from a state $\ell$ to a state $\ell'$ and emit a photon of angular frequency $\omega_{\ell,\ell'} = (E_\ell - E_{\ell'})/\hbar$ is given in the dipole-length representation by

$$ A_{\ell',\ell} = \frac{4e^2}{3\hbar c^3} \omega_{\ell,\ell'}^3 |<\ell'|\hat{r}|\ell>|^2. $$

For the screened Coulomb potential, the normal selection rules for spontaneous emission in the electric dipole approximation for a general spherically symmetric potential are valid. To our knowledge, no calculation of transition probabilities has been performed for the
4. Parametric Fitting of the Variational Solutions

Previously we have used the elegant variational method to calculate the energies and transition probabilities of the Debye-Hückel potential for various particular values of the screening length $D$. Using these results for the energy of a specific state for particular values of the screening length we present, in this subsection, a method of obtaining the energy of that state for all values of the screening length without having to perform any further variational calculations. Also a similar procedure allows us to use the transition probabilities already calculated for particular values of the screening length to obtain the transition probabilities for all values of the screening length.

We have seen in Chapter I that to first order the effect of screening on the energy levels is to simply raise them all by the factor $e^2/D$. Thus we are motivated to fit the screening corrections to the energy, which have been obtained by the variational method for particular values of the screening length, to a polynomial in powers of the inverse of the screening length. As an example we have fit the energies for the lowest six states as follows, in units of Rydbergs:

$$E_{n\ell}^{S}(D) = E_{n\ell}^{C} + x_0 + \sum_{k=2}^{5} b_{n\ell}^{k} (x_0)^k, \quad (4.64)$$
where $E_{nl}^S(D)$ is the screened energy for a particular state $n_l$ for the screening length $D$, $E_{nl}^C$ is the Coulomb energy corresponding to that state and $x_0 \equiv 2/D$. We have previously seen (see Eq. (1.21)) that, to first order all the energy levels are raised by an amount $(2/D)$ Ryd. -- this accounts for our choice of all the coefficients $b_{nl}^1$ to be exactly equal to 1.0. The coefficients $b_{nl}^k$ for these various states were obtained using the minimization program MINUIT\(^7\) and are given in Table VI. Energies $E_{nl}^S(D)$ obtained by using Eq. (4.64) are accurate to 1 part in $10^5$.

Let us now consider the spontaneous transition probability from an upper state specified by $n_l$ to a lower state $n'_l$. From Eq. (4.64) we see that the amount $W_{nl}$ that a state $n_l$ is raised due to screening may be written as

$$E_{nl}^S - E_{nl}^C \equiv W_{nl} \equiv C_{nl}(D)x_0,$$  \hspace{1cm} (4.65)

where

$$C_{nl}(D) = 1 + \sum_{k=2}^{5} b_{nl}^k (x_0)^{k-1},$$  \hspace{1cm} (4.66a)

$$\equiv 1 + B_{nl}(D),$$  \hspace{1cm} (4.66b)
From Table I we note that

\[ W_{n'l'} \geq W_{n'l} \quad (4.68) \]

where \( n' < n \). This is what we expect physically since screening effects are greatest for the lowest energy states (i.e. the states with the smallest radii). This can be seen most simply from the fact that

\[ \frac{1}{r} - \frac{\exp(-r/D)}{r} = \frac{1}{D} - \frac{r}{2D^2} + \ldots, \quad (4.69) \]

i.e. the magnitude of the deviation of the screened potential from the Coulomb potential increases with decreasing \( r \). Thus

\[ C_{n'l'}(D) \geq C_{n'l}(D) \quad (4.70) \]

Let \( \omega_s \) be the angular frequency of the transition \( n'l+n'l' \) for the screened potential

\[ \omega_s = E_{n'l}^S - E_{n'l'}^S \quad (4.71) \]

From Eq. (4.64) we see
\[ \omega_s = (E_{n_l}^C + W_{n_l}) - (E_{n'_{l'}}^C + W_{n'_{l'}}), \]  
\text{or} \\
\[ \omega_s = \omega_o - (W_{n'_{l'}} - W_{n_l}), \]

where \( \omega_o = E_{n_l}^C - E_{n'_{l'}}^C \), is the angular frequency of the transition \( n_l \rightarrow n'_{l'} \) for the Coulomb potential. Thus

\[ \frac{\omega_s}{\omega_o} = 1 - \frac{x_o}{\omega_o} \left( C_{n'_{l'}}(D) - C_{n_l}(D) \right), \]

\text{or} \\
\[ \frac{\omega_s}{\omega_o} = 1 - \frac{x_o}{\omega_o} \left( B_{n'_{l'}}(D) - B_{n_l}(D) \right). \]

Using Eq. (4.67) we have

\[ \frac{\omega_s}{\omega_o} = 1 - \frac{x_o}{\omega_o} \left[ (b_{n'_{l'}}^2 - b_{n_l}^2)x_o + (b_{n'_{l'}}^3 - b_{n_l}^3)x_o^2 + \ldots \right], \]

\text{or} \\
\[ \frac{\omega_s}{\omega_o} = 1 - \frac{x_o}{\omega_o} \left[ \sum_{k=2}^S (b_{n'_{l'}}^k - b_{n_l}^k)x_o^{k-1} \right]. \]
We see from Eq. (4.70) and (4.74) that

$$\omega_s < \omega_o.$$  

\[(4.78)\]

From Eq. (4.62) for the spontaneous emission transition probability $A_{n',l'}^n$, we see that, in general,

$$A_{n',l'}^n \propto \omega^3 r_{n',l',n'l}^2$$  

\[(4.79)\]

where $r_{n',l',n'l}$ is the radial matrix element. Thus the ratio of the transition probability for the screened potential $(A_{n',l'}^n,(D))^S$ to the transition probability of the Coulomb potential $(A_{n',l'}^n)^C$ is

$$\frac{(A_{n',l'}^n,(D))^S}{(A_{n',l'}^n)^C} = \left(\frac{\omega_s}{\omega_o}\right)^3 \frac{(r_{n',l',n'l}^2)^S}{(r_{n',l',n'l}^2)^C}. \quad \text{(4.80)}$$

Let us now first assume that the radial matrix element is independent of frequency (as we shall see the matrix element is only weakly dependent on the frequency). Thus Eq. (4.80) may be written

$$\frac{(A_{n',l'}^n,(D))^S}{(A_{n',l'}^n)^C} \approx \left(\frac{\omega_s}{\omega_o}\right)^3. \quad \text{(4.81)}$$

Hence, using Eq. (4.78), it follows immediately that screening reduces the transition probabilities. This is
consistent with our numerical results (see Fig. 4).

Using Eq. (4.75) we see that

\[
\frac{(A^{n \ell}_{n', \ell'}, (D))_{S}}{(A^{n \ell}_{n', \ell'})_{C}} \approx \left[1 - \frac{x_{O}}{\omega_{o}} \left(B_{n' \ell', (D)} - B_{n \ell} (D)\right)\right]^{3},
\]

(4.82)

or from Eq. (4.76)

\[
\frac{(A^{n \ell}_{n', \ell'}, (D))_{S}}{(A^{n \ell}_{n', \ell'})_{C}} \approx \left[1 - \frac{x_{O}}{\omega_{o}} \left((b^{2}_{n' \ell', -b^{2}_{n \ell}}) x_{O} + (b^{3}_{n' \ell', -b^{3}_{n \ell}}) x_{O}^{2} + \ldots\right)\right]^{3}.
\]

(4.83)

Thus we see that the first non-zero correction to the transition probability due to screening is second order in $D^{-1}$. For weak screening (i.e., screening length $>100$) where the first order correction to the energy levels is sufficient, the transition probabilities are unaffected by the screening. For screening lengths $D<100$, Eq. (4.83) gives a good approximation for the transition probability for the screened potential, accurate to better than 3% down to screening length $D=20$. For screening length less than 20 we must consider the frequency dependence of the radial matrix elements to obtain accurate results for the transition probabilities.
To enable us to calculate the transition probabilities accurately for all values of the screening length without further variational calculations we have fit, as an example, the square of the radial matrix elements for transitions between the three lowest levels to a polynomial in powers of $\Delta \omega = \omega - \omega_s$. That is

$$(r^2_{n', \ell', n\ell})^s = (r^2_{n', \ell', n\ell})^c - \frac{3}{\Sigma} \sum_{i=1}^{3} c_{n', \ell', n\ell} (\Delta \omega)^i$$

(4.84)

The coefficients $c_{n', \ell', n\ell}$, obtained in the same manner as the coefficients $b_{n\ell}^k$ for the energy levels, are given in Table VII for the transitions between the three lowest levels. The squares of the radial matrix element obtained by using Eq. (4.84) have an accuracy better than 1%.

Using Eqs. (4.80), (4.76), and (4.84), the transition probability from state $n\ell$ to state $n'\ell'$ for any value of screening length $D$ is given by

$$(A_{n', \ell', (D)}^{n\ell})^s = (A_{n', \ell', (D)}^{n\ell})^c \left[1 - \frac{x_o}{\omega_o} \left\{5 \sum_{k=2}^{5} (b_{n', \ell', n\ell}^k - b_{n\ell}^k)\right\}ight]$$

$$\left\{3 \sum_{i=1}^{3} c_{n', \ell', n\ell} (\Delta \omega)^i\right\}$$

(4.85)
where \((A_{nl}^{n'l',n'l'})^c\) and \((r_{n'l',n'l'})^c\) may be obtained from various authors\(^{96,98}\) or easily calculated, and the coefficients \(b_n^k\) and \(c_{n'l',n'l'}^i\) may be obtained by the procedure outlined above (for transitions between the lowest three levels these are tabulated in Tables VI and VII). As a specific example the 2p-1s transition probability as a function of screening length \(D\) is given by

\[
(A_{10}^{21}(D))^s = 6.27 \times 10^8 \left\{ 1 - \frac{4}{3} \left( \frac{2}{D} \right) \zeta(D) \right\}^3 \nonumber
\]

\[
\{ 1 - \frac{1}{1.665} \left( \frac{2}{D} \right) \zeta(D) \{ 5.1065 + 2.9958 \left( \frac{2}{D} \right) \zeta(D) \nonumber
\]

\[
+ 172.84 \left( \frac{2}{D} \right)^2 (\zeta(D))^2 ] \} \right\} \nonumber
\]

(4.86)

where

\[
\zeta(D) = \sum_{k=2}^{5} (b_{10}^k - b_{21}^k) \left( \frac{2}{D} \right)^{k-1} \nonumber
\]

\[
= (b_{10}^2 - b_{21}^2) \left( \frac{2}{D} \right) + (b_{10}^3 - b_{21}^3) \left( \frac{2}{D} \right)^2 + (b_{10}^4 - b_{21}^4) \left( \frac{2}{D} \right)^3 \nonumber
\]

\[
+ (b_{10}^5 - b_{21}^5) \left( \frac{2}{D} \right)^4 \right\} \nonumber \nonumber
\]

(4.87)

Hence
\[ \zeta(D) = 0.86454 \left( \frac{D}{D} \right) - 0.90613 \left( \frac{D}{D} \right)^2 + 1.08680 \left( \frac{D}{D} \right)^3 - 0.35069 \left( \frac{D}{D} \right)^4 \]  

We have also simply fit the transition probabilities to a polynomial in powers of \( \Delta \omega \). That is

\[ (A_{n'l'}^n) = (A_{n'l'}^n)^C - \sum G_{n'l',nl}^i (\Delta \omega)^i \]  

The coefficients \( G_{n'l',nl}^i \) are given in Table VIII. The transition probabilities obtained by using Eq. (4.89) are accurate to 0.5% of the transition probabilities obtained by using the variational calculation.

5. Applications of the General Results

The wide interest in this problem stems from its application to problems in astrophysics, solid state, and plasma physics.

In dealing with stellar structure, the knowledge of the energy levels of atoms is of importance for several calculations. For example, the occupation number\(^{59}\) of a state is given by the Boltzmann equation which relates the ratio of the number of non-degenerate particles in two different states \( i \) and \( j \) of the state of ionization in statistical equilibrium to the energy levels by the equation
Here $N_i$ and $N_j$ are the number of particles in discrete states $i$ and $j$, the $g$'s are the statistical weights of the two energy levels and the $\chi$'s are the excitation potentials (the energies above the ground state) of the two levels.

The energy levels are also necessary for opacity calculations. As mentioned briefly in Chapter I, the first order effect, called the "lowering of the continuum", for the changes in the energy levels has been used by Cox$^{99}$ in his opacity code for the perturbations of the bound electron by the free electrons. The higher order corrections obtained in this chapter for this perturbation may be easily used for the opacity calculation due to the tractable form of results presented by parametric fitting.

We have also shown the necessity for consideration of the perturbation of the bound electrons by the free electrons and other ions in the calculations of the absorption coefficients, in particular for the bound-bound transitions, for the case of strong screening ($D<100$). Since the screening lengths of less than 300 occur for optical depths less than 10 in white dwarf atmospheres with the parameters, temperature = 12000°K and surface
gravity = \(10^8\) g/cc, calculations of absorption coefficients for the stellar interior model must consider this perturbation.

Let us now turn from astrophysics to consider some applications in solid-state physics. In 1942, Bethe postulated that a donor level or an exciton in a semiconductor might be usefully described by a hydrogenic wave-function, modified by a dielectric constant and by effective mass renormalization. This has proved to be a very fruitful idea in semiconductor research. However, for the impurities in a semiconductor to behave like isolated atoms it is necessary that the spacing between nearest impurity neighbors be greater than about 10 times the orbit radius (whereas for the concept of a dielectric constant to be valid it is necessary that the orbit radius should be large in comparison with the distance between atoms). Let \(\mu_o, \mu\) and \(\varepsilon\) denote the isolated electron mass, effective electron mass, and dielectric constant respectively and define

\[
k = \frac{\mu}{\mu_o} \frac{1}{\varepsilon}
\]  

Then, the natural units of length and energy are \(a_o/k\) (\(a_o\) is the Bohr radius) and \((k/\varepsilon)\) Ry., which we will refer to as the effective Bohr radius and effective Rydberg, respectively.
plasmas for the following reason. The important quantity of interest is the ratio $r/D$. Since the screening length $D$ is only proportional to the square root of the dielectric constant $\varepsilon$ whereas the radius of the electron, essentially the Bohr radius of the system for the ground state, is proportional to $k^{-1}$, we see that the ratio $r/D$ is proportional to $\frac{\mu}{\mu_0} \varepsilon^{-1/2}$. For the example of InSb this ratio is a factor of about 300 larger than for the plasma ($\mu_0=\mu$, $\varepsilon=1$) application.

This perturbation of the bound electron energies by the free charged particles has been considered in plasma physics where of particular interest is the quantum number for the last observed line, $q^*$ say, of the Balmer series. Since the dipole transitions in this series occur from upper energy level with angular momentum values of 0, 1, 2, the last possible dipole transition from an upper level specified by $n$ (recall that $n$ for the screened potential does not correspond to a good quantum number) occurs for the angular momentum value 0, since for each level the state with the smallest angular momentum $l$ lies lowest in energy and is thus the last to be raised into the continuum. To obtain the value of this last quantum number $q^*$ as a function of screening length in general we have calculated the value of the screening length, $D_c$, for which each of the nine lowest $s$ states disappears. These are given in Table IX. We then use
these values to determine a relation between the last quantum number $q^*$ and the screening length. By a simple fitting procedure we found the relation

$$q^* = 1.13 \sqrt{D} = \sqrt{1.28 D} \quad (4.92)$$

to be accurate to about 1%.

The disappearance of the highly excited states of the Balmer series of hydrogen can be caused not only by the previously discussed upward shifting of the energy levels but also by the merging together of the highly excited states due to the broadening of these levels. This merging occurs when the splitting of the highly excited states due, for example, to the linear Stark effect caused by the mean electrostatic field set up by the nearby ions, is equal to about one-half of the normal level separation of hydrogen. This implies that the last discernable level due to the merging effect will have a quantum number $n_s$ determined by the ion density $N$ as follows

$$7.5 \log n_s = \log \left( \frac{2.7 \times 10^{-2}}{a_0^3} \right) - \log N \quad , \quad (4.93)$$

where $a_0$ is the Bohr radius. This is the well-known Inglis-Teller relation.106
Let us now determine analytically the temperature density region where each of the two discussed effects dominates. Note from Eq. (4.11) that for a two component plasma consisting of electrons with number density \( N_e \) and ions of charge \( Z_1 e \) and number density \( N_1 \)

\[
D = 6.9 \sqrt{\frac{T}{N_e(Z_1+1)}} \text{ cms} . \tag{4.94}
\]

Thus we have from Eqs. (4.92) and (4.94)

\[
q^* = 2.97 \left( \frac{T}{(Z_1+1)N_e a_o^2} \right)^{1/4} , \tag{4.95}
\]

and

\[
4 \log q^* = \log \frac{77.8T}{(Z_1+1)a_o^2} - \log N_e . \tag{4.96}
\]

Subtracting Eq. (4.96) from (4.93)

\[
7.5 \log n_s - 4 \log q^* = \log \left( \frac{2.7 \times 10^{-2}}{a_o^3} \right) - \log \frac{77.8T}{a_o^2 (Z_1+1)} , \tag{4.97a}
\]

or

\[
7.5 \log n_s - 4 \log q^* = \log \left( \frac{2.7 \times 10^{-2} (Z_1+1)}{77.8 a_o T} \right) . \tag{4.97b}
\]
Thus

\[ 7.5 \log n_s - 4 \log q^* = \log \left( \frac{(Z_1+1)6.59 \times 10^4}{T} \right) \]  

(4.98)

We note that the critical temperature \( T_o \) for which both effects predict the same maximum value of \( n \) (i.e., \( q^* = n_s \)) occurs for

\[ 3.5 \log n_s = \log \left( \frac{6.59 \times 10^4(Z_1+1)}{T} \right) \]  

(4.99)

Thus

\[ T_o = \frac{6.59 \times 10^4(Z_1+1)}{n_s^{7/2}} \]  

(4.100)

Now from Eq. (4.93) it follows that

\[ n_s^{7/2} = \left(1.8 \times 10^{23}\right)^{7/15} N_e^{-7/15} \]  

(4.101)

Substituting Eq. (4.101) into (4.100), the critical temperature for a specified density at which both effects have the same maximum value of \( n \) is given by

\[ T_o = 9.25 \times 10^{-7} (Z_1+1) N_e^{7/15} \]  

(4.102)
Thus the choice for the variational trial function \( \psi \) should be a function of the three coordinates, (i.e., in spherical coordinate \( r, \theta, \phi \)). Since only inversion and rotation about the \( z \) axis leave the Hamiltonian invariant, the only "good" quantum numbers are the eigenvalues of parity and \( L_z \), which we denote by \( \pm \) and \( m \), respectively. Labelling our trial function with these two quantum numbers, its general form is

\[
\psi_m^\pm (r) = \sum_{\ell} R^\ell_m(r) Y_{\ell m}(\theta, \phi) ,
\]  

(4.107)

where

\[
R^\ell_m(r) = \sum_{k=\ell} a_{i k} r^k \exp(-a_i r) ,
\]

(4.108)

where \( a_i \) are chosen input parameters, and the \( a_{i k} \) are parameters evaluated numerically in the course of diagonalization of the Hamiltonian. States with even parity are obtained by a summation on \( \ell \) over even integers, whereas odd parity states are obtained by a summation on \( \ell \) over odd integers. Figure (6) gives the ionization energy as a function of screening length \( D \) for various values of \( \gamma \). The binding energies of the first five excited states are shown in Fig. (7). This calculation is the first such calculation for these
excited states and its importance will be discussed later.

2. Applications

As in the previous case of no external field, screening effects are also most important for the solid state and for the same reason as before; namely, the ratio of the screening length to the average radius of the electron is much smaller in the solid state than in plasmas. One might wonder why simple perturbation theory cannot successfully treat the magnetic field effects since the magnitude of laboratory fields ($10^4$-$10^5$ G) is much smaller than the fields one usually associates with the breakdown of perturbation theory ($10^6$-$10^9$ G). The actual requirement for the breakdown of perturbation theory is that the magnetic force must become comparable to the force derived from the potential. We see from Eq. (4.105) that this occurs as $\gamma$ approaches unity. For plasmas the critical field $B_c$ where $\gamma=1$ is

$$B_c = B_o = 2.35 \times 10^9 \text{ Gauss} \quad (4.109)$$

since $k=1$ for plasmas. However, recall for our example of InSb, $k^2 = 6.6 \times 10^{-7}$ thus

$$B_c = 1.55 \times 10^3 \text{ Gauss} \quad (4.110)$$
\[
\frac{(N_A + n)n}{(N_D - N_A - n)} = N_B \exp(-\frac{\varepsilon_I}{kT}), \quad (4.111)
\]

where \( N_A \) and \( N_D \) denote the concentration of acceptors and donors, respectively and

\[ N_B = (2\pi\mu kT)^{1/2} \left( \frac{eB}{h^2c} \right) \quad (4.112) \]

is the effective density of states in the Landau level, and \( \mu \) is the chemical potential. In the derivation of this formula it was assumed (a) that \( \hbar\omega_c > kT \) (where \( \omega_c \) is the cyclotron frequency) so that only the lowest Landau level need be considered in the evaluation of the density of states in the conduction band and (b) that the free electron concentration is sufficiently small that classical statistics may be used. For example, these conditions readily hold for n-type InSb with a total impurity concentration \( \sim 10^{14} \text{ cm}^{-3} \), at a temperature \( \sim 4^\circ \text{K} \), for \( B \sim 3,000 \text{ G} \).

Hall effect measurements were used at 77\(^\circ\)K to determine a conduction band electron concentration \( n \) of \( 4.9 \times 10^{13} \text{ cm}^{-3} \). This is in the "exhaustion range" where \( n \) has approached to a limiting maximum value of \( N_D - N_A \). To determine the ionization energy \( \varepsilon_I \) and the number of acceptors \( N_A \), it was then assumed that these quantities were independent of the conduction band electron con-
centration \( n \) and Eq. (4.111) could then be fitted to the data between 2° and 4°K to obtain \( \varepsilon_I \) and \( N_A \). For example,\textsuperscript{108} for a magnetic field of 8230 G, it was found that \( N_D = 4.3 \times 10^{14} \text{ cm}^{-3} \), \( N_A = 3.81 \times 10^{14} \text{ cm}^{-3} \) and \( \varepsilon_I = 7.5 \times 10^{-4} \text{ eV} \). However, in the above analysis, screening effects were not considered since such effects imply that the ionization energy will depend both on the conduction band electron concentration \( n \) and on the temperature. For the neglect of screening effects to be a valid assumption, the screening length must be large compared to the Bohr radius of the donor,

\[
D \gg 1 \quad (4.113)
\]

We have used the data determined above for a magnetic field of 8230 G to determine \( n \) for temperatures in the range 2° - 8°K. These values of \( n \) at the particular temperature implied screening lengths which varied from 1.5 to 0.6 in the temperature range 2-8°K. These small values of the screening length imply that the neglect of screening effects in the calculation of the ionization energies of the donors from the Hall effect measurements is indeed not justified. Thus a self-consistent method for including screening effects should be developed.

Although we have thus far confined our discussion to the application of the calculation for the ground state
of the donors, our calculation of the excited states is also of importance since it has been recently suggested that the broadening of these excited states may cause them to merge together to form impurity bands\textsuperscript{110} and also possibly form part of, and hence lower, the conduction band\textsuperscript{111}.

The solution of the screened exciton problem in a magnetic field is identical with the solution for the screened donor levels in a magnetic field in the case where the mass of the hole is infinite. In general, if $\mu_e$ denotes the effective mass of the electron, as before, and if $\mu_h$ denotes the effective mass of the hole, then the generalization of Eq. (4.72) to the case of the exciton is,\textsuperscript{112}

$$H = p^2 - \frac{2}{r} e^{-r/D} + \frac{(\frac{\mu_h - \mu_e}{\mu_h})}{\gamma} L_z + \frac{(\frac{\mu_h + \mu_e}{\mu_h})}{4} \frac{1}{2} r^2 \sin^2 \theta.$$  

(4.114)

We see that when $\mu_h$ is infinite, Eq. (4.114) is the same as Eq. (4.105) and the solution to the exciton problem is identical to that of the impurity level problem. In the case where $\mu_e = \mu_h$, the solution to the exciton is identical to that of the impurity problem in the case where $m=0$ and $B^+(B/\sqrt{2})$. In the more general case, the calcula-
tion is similar to that outlined previously.

In this section we have considered the effect of screening on the energy levels of a particle in a magnetic field. The potential chosen to study the screening effects was the screened potential derived in the absence of a magnetic field, Debye-Hückel potential. In general one should consider the effects of the magnetic field on the screening which could possibly modify the screening potential. However, Horing\textsuperscript{113} has shown that for the case of non-degenerate free electrons, the anisotropic effects are not important when $\mathcal{W}(eB/\mu c) \ll kT$, which is easily fulfilled for the case discussed above.

C. Summary of Screening Effects

In this chapter we have investigated the effects of the screening of the two particle electrostatic interaction by free charged particles. We have presented a multiparameter variational solution of Schrödinger's equation for a general screened potential, the completely screened Coulomb potential. The variational solution yielded not only accurate energy levels but also tractable analytic forms of the wave functions. These analytic forms of the wave function were used to obtain the transition probabilities for a special case of the completely screened Coulomb potential; namely, the Debye-Hückel
potential. The variational method was also used to calculate the energy levels for a particle which is under the influence of the Debye-Hückel potential and in the presence of an external magnetic field. We then discussed the implications of our results for various applications.
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Binding Energies of States 1s-9\ell for Static Screened Potential
with Z=1

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Binding Energies of States 1s-9l for Static Screened Potential with Z=1

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Binding Energies of States 1s-9i for Static Screened Potential
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Binding Energies of States 1s-9l for Static Screened Potential with Z=1

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TABLE III

Binding Energies of States 1s-9f for the Completely Screened Coulomb Potential

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Binding Energies of States 1s–9i for the Completely Screened Coulomb Potential

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Binding Energies of States 1s-9l for the Completely Screened Coulomb Potential

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TABLE IV (continued)
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<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wavelength</td>
<td>1263.8Å</td>
<td>13.425μ</td>
<td>1.0516</td>
<td>1123.7Å</td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>5.06(0)</td>
<td>9.30(-5)</td>
<td>1.84(-2)</td>
<td>5.26(-1)</td>
</tr>
<tr>
<td>9</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>Wavelength</td>
<td>1274.9Å</td>
<td>11.384μ</td>
<td>1.1817</td>
<td>1146.5Å</td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>4.81(0)</td>
<td>1.60(-4)</td>
<td>1.24(-2)</td>
<td>2.70(-1)</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wavelength</td>
<td>1290.3Å</td>
<td>9.5472μ</td>
<td>1.4074</td>
<td></td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>4.47(0)</td>
<td>2.91(-4)</td>
<td>5.74(-3)</td>
<td></td>
</tr>
<tr>
<td>7</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Wavelength</td>
<td>1313.1Å</td>
<td>7.9314μ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>4.00(0)</td>
<td>5.62(-4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wavelength</td>
<td>1348.9Å</td>
<td>6.5883μ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>3.30(0)</td>
<td>1.17(-3)</td>
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</tr>
</tbody>
</table>
### TABLE IV (continued)

Transition Probabilities $A_{\ell\ell'}$ for the Debye-Hückel Potential

<table>
<thead>
<tr>
<th>Transitions</th>
<th>$2p+1s$</th>
<th>$2p+2s$</th>
<th>$3p+2p$</th>
<th>$3p+1s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 Wavelength</td>
<td>1411.9 Å</td>
<td>5.6910 μ</td>
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<td></td>
</tr>
<tr>
<td>$A_{\ell\ell'}$</td>
<td>2.16(0)</td>
<td>2.61(-3)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* $1.0(-1) \equiv 1.0 \times 10^{-1}$.
<table>
<thead>
<tr>
<th>Transitions</th>
<th>D</th>
<th>3p+2s</th>
<th>3p+3s</th>
<th>3d+2p</th>
<th>3d+3p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength</td>
<td>1000</td>
<td>6561.4Å</td>
<td>6561.4Å</td>
<td>6.48(-1)</td>
<td>6.48(-1)</td>
</tr>
<tr>
<td></td>
<td>700</td>
<td>6561.9Å</td>
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<td>6.48(-1)</td>
<td>6.48(-1)</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>6562.3Å</td>
<td>6562.2Å</td>
<td>6.48(-1)</td>
<td>6.48(-1)</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>6564.5Å</td>
<td>6564.0Å</td>
<td>6.47(-1)</td>
<td>6.47(-1)</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>6568.6Å</td>
<td>6567.4Å</td>
<td>6.46(-1)</td>
<td>6.46(-1)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>6590.0Å</td>
<td>995.86μ</td>
<td>6585.8Å</td>
<td>497.14μ</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.24(-1)</td>
<td>1.12(-9)</td>
<td>6.39(-1)</td>
<td>6.77(-9)</td>
</tr>
</tbody>
</table>
### TABLE V (continued)

Transition Probabilities $A_{ll'}$, for the Debye-Hückel Potential

<table>
<thead>
<tr>
<th>D</th>
<th>Wavelength</th>
<th>$3p\leftrightarrow 2s$</th>
<th>$3p\leftrightarrow 3s$</th>
<th>$3d\leftrightarrow 2p$</th>
<th>$3d\leftrightarrow 3p$</th>
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</thead>
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<tr>
<td>80</td>
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<tr>
<td></td>
<td>Wavelength</td>
<td>6605.7Å</td>
<td>651.50μ</td>
<td>6599.4Å</td>
<td>324.96</td>
</tr>
<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>2.19(-1)</td>
<td>4.02(-9)</td>
<td>6.35(-1)</td>
<td>2.45(-8)</td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wavelength</td>
<td>6638.8Å</td>
<td>380.12μ</td>
<td>6628.1Å</td>
<td>189.26</td>
</tr>
<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>2.15(-1)</td>
<td>2.05(-8)</td>
<td>6.26(-1)</td>
<td>1.26(-7)</td>
</tr>
<tr>
<td>40</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wavelength</td>
<td>6730.5Å</td>
<td>181.82μ</td>
<td>6708.5Å</td>
<td>90.084</td>
</tr>
<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>2.04(-1)</td>
<td>1.94(-7)</td>
<td>6.01(-1)</td>
<td>1.21(-6)</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wavelength</td>
<td>6855.4Å</td>
<td>110.21μ</td>
<td>6819.3Å</td>
<td>54.125</td>
</tr>
<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>1.91(-1)</td>
<td>9.05(-7)</td>
<td>5.67(-1)</td>
<td>5.85(-6)</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wavelength</td>
<td>7207.9Å</td>
<td>57.327μ</td>
<td>7138.8Å</td>
<td>27.746μ</td>
</tr>
<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>1.57(-1)</td>
<td>7.21(-6)</td>
<td>4.82(-1)</td>
<td>5.11(-5)</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wavelength</td>
<td>7716.0Å</td>
<td>38.464μ</td>
<td>7615.8Å</td>
<td>18.250μ</td>
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<tr>
<td></td>
<td>$A_{ll'}$</td>
<td>1.19(-1)</td>
<td>2.82(-5)</td>
<td>3.73(-1)</td>
<td>2.27(-4)</td>
</tr>
</tbody>
</table>
TABLE V (continued)

Transition Probabilities $A_{\ell\ell'}$ for the Debye-Hückel Potential

<table>
<thead>
<tr>
<th>Transitions</th>
<th>$D$</th>
<th>$3p+2s$</th>
<th>$3p+3s$</th>
<th>$3d+2p$</th>
<th>$3d+3p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Wavelength</td>
<td>8425.4Å</td>
<td>30.424μ</td>
<td>8321.9Å</td>
<td>14.279μ</td>
</tr>
<tr>
<td></td>
<td>$A_{\ell\ell'}$</td>
<td>7.97(-2)</td>
<td>7.22(-4)</td>
<td>2.41(-1)</td>
<td>6.96(-4)</td>
</tr>
<tr>
<td>10</td>
<td>Wavelength</td>
<td>9425.7Å</td>
<td>28.142μ</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{\ell\ell'}$</td>
<td>4.29(-2)</td>
<td>1.32(-4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Wavelength</td>
<td>1.0347μ</td>
<td>30.942μ</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{\ell\ell'}$</td>
<td>1.92(-2)</td>
<td>1.47(-4)</td>
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</tr>
</tbody>
</table>

* $1.0(-1) \equiv 1.0 \times 10^{-1}$.
<table>
<thead>
<tr>
<th>$k_{nl}$</th>
<th>$b_{10}$</th>
<th>$b_{20}$</th>
<th>$b_{21}$</th>
<th>$b_{30}$</th>
<th>$b_{31}$</th>
<th>$b_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>-0.37312</td>
<td>-1.48618</td>
<td>-1.23766</td>
<td>-3.33778</td>
<td>-3.06741</td>
<td>-2.64033</td>
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<tr>
<td>3</td>
<td>0.10958</td>
<td>1.47339</td>
<td>1.01572</td>
<td>7.13993</td>
<td>5.53222</td>
<td>5.64375</td>
</tr>
<tr>
<td>4</td>
<td>-0.04011</td>
<td>-1.55863</td>
<td>-1.12691</td>
<td>-15.4488</td>
<td>-9.11709</td>
<td>-25.50050</td>
</tr>
<tr>
<td>5</td>
<td>0.00741</td>
<td>0.89553</td>
<td>0.35810</td>
<td>18.15150</td>
<td>3.03280</td>
<td>57.87756</td>
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</tbody>
</table>
TABLE VII

The Fitting Coefficients $C_{n'l',nl}^i$ for the Square of the Radial Matrix Element

<table>
<thead>
<tr>
<th>$i$</th>
<th>$C_{10,21}$</th>
<th>$C_{10,31}$</th>
<th>$C_{20,31}$</th>
<th>$C_{21,32}$</th>
<th>$C_{21,30}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.1065</td>
<td>1.8116</td>
<td>61.227</td>
<td>63.634</td>
<td>2.4730</td>
</tr>
<tr>
<td>2</td>
<td>2.9958</td>
<td>-4.1201</td>
<td>-645.20</td>
<td>694.29</td>
<td>5.151</td>
</tr>
<tr>
<td>3</td>
<td>172.84</td>
<td>87.155</td>
<td>37745.0</td>
<td>106012.0</td>
<td>6.7722</td>
</tr>
</tbody>
</table>
### TABLE VIII

The Fitting Coefficients $G_{n',i',n'}^i$ for the Transition Probabilities

<table>
<thead>
<tr>
<th>i</th>
<th>$G_{10,21}$</th>
<th>$G_{10,31}$</th>
<th>$G_{21,30}$</th>
<th>$G_{20,31}$</th>
<th>$G_{21,32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.625</td>
<td>17.339</td>
<td>1.0840</td>
<td>6.0278</td>
<td>15.886</td>
</tr>
<tr>
<td>2</td>
<td>-71.058</td>
<td>-67.968</td>
<td>-4.7317</td>
<td>-47.966</td>
<td>-109.71</td>
</tr>
<tr>
<td>3</td>
<td>283.45</td>
<td>457.60</td>
<td>8.0278</td>
<td>177.57</td>
<td>1380.3</td>
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</tbody>
</table>
TABLE IX

The Critical Screening Length $D_c$ (in units of the Bohr radius) Below Which the State $n$ is No Longer Bound

<table>
<thead>
<tr>
<th>$n$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.84</td>
</tr>
<tr>
<td>2</td>
<td>3.23</td>
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<tr>
<td>3</td>
<td>7.17</td>
</tr>
<tr>
<td>4</td>
<td>12.69</td>
</tr>
<tr>
<td>5</td>
<td>19.77</td>
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<tr>
<td>6</td>
<td>28.42</td>
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<tr>
<td>7</td>
<td>38.65</td>
</tr>
<tr>
<td>8</td>
<td>50.45</td>
</tr>
<tr>
<td>9</td>
<td>63.82</td>
</tr>
</tbody>
</table>
Figure 4
Figure 5

INGLIS-TELLER REGIME

DEBYE-HÜCKEL REGIME
Figure 6
Figure 7
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EXAMINATION AND THESIS REPORT

Candidate: ROUSSEL, Keith Miles

Major Field: Physics

Title of Thesis: Magnetic Field and Screening Effects in Condensed and Ultradense Matter

Approved:

Robert F. O'Connell
Major Professor and Chairman

James H. Trahant
Dean of the Graduate School

EXAMINING COMMITTEE:

Ronald J. Henry
William J. Berry
Paul Allen
Bruce M. Barker

Date of Examination:

April 19, 1974