Transcendental eigenvalue problems associated with vibration, buckling and control

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TRANSCENDENTAL EIGENVALUE PROBLEMS ASSOCIATED WITH VIBRATION, BUCKLING AND CONTROL

A Thesis

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The Department of Mechanical Engineering

by

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Dedication

I dedicate this work to-

“Maatha, Pitha, Guru, Daivam”

Maatha. My mother, Mrs. Rama Subramanian, my creator, and an embodiment eternal of love encouragement and devotion. I dedicate this thesis as an outcome of the your blessings.

Pitha. I dedicate this work to my father, Mr. K.R. Subramanian, for his unconditional love, his efforts and sacrifices to help me fulfill my academic goals.

Guru. I am forever indebted to Dr. Yitshak Ram whose perseverance and support has made this work possible. Sir, I dedicate this work to you and your good health.

Daivam. God Almighty

I would also like to dedicate this work to my grandparents Mr. and Mrs. Ramachandran and Mr. and Mrs. Muthukrishnan, my family and friends and my motherland India.
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List of Nomenclature

\( \beta \) \hspace{1cm} \text{Damping Co-efficient, Eigenvalue}
\( \epsilon \) \hspace{1cm} \text{Error}
\( \delta \) \hspace{1cm} \text{Convergence tolerance}
\( \lambda, \mu \) \hspace{1cm} \text{Eigenvalue}
\( \rho \) \hspace{1cm} \text{Density}
\( \omega \) \hspace{1cm} \text{Natural Frequency}
\( A \) \hspace{1cm} \text{Area, Transcendental Matrix}
\( A_L \) \hspace{1cm} \text{Transcendental matrix of left boundary condition}
\( A_R \) \hspace{1cm} \text{Transcendental matrix of right boundary condition}
\( A_{Dc} \) \hspace{1cm} \text{Transcendental matrix of displacement continuity condition}
\( A_{Sc} \) \hspace{1cm} \text{Transcendental matrix of left slope continuity condition}
\( A_{Mc} \) \hspace{1cm} \text{Transcendental matrix of left moment continuity condition}
\( A_{Fc} \) \hspace{1cm} \text{Transcendental matrix of left shear force continuity condition}
\( B \) \hspace{1cm} \text{Derivative of Transcendental Matrix wrt. } \omega
\( C \) \hspace{1cm} \text{Damping Co-efficient, constants of integration}
\( D \) \hspace{1cm} \text{Flexural rigidity in plates}
\( E \) \hspace{1cm} \text{Young’s Modulus}
\( I \) \hspace{1cm} \text{Area Moment of Inertia, Modified Bessel function of the first kind}
\( K \) \hspace{1cm} \text{Stiffness matrix}
\( K^{(e)} \) \hspace{1cm} \text{Elemental stiffness matrix}
\( K^{(G)} \) \hspace{1cm} \text{Global stiffness matrix}
\( L \) \hspace{1cm} \text{Length}
\( L(x) \) \hspace{1cm} \text{Interpolation function or Shape Function}
\( M \) \hspace{1cm} \text{Mass Matrix}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^{(e)}$</td>
<td>Elemental mass matrix</td>
</tr>
<tr>
<td>$M^{(G)}$</td>
<td>Global mass matrix</td>
</tr>
<tr>
<td>$N_x$</td>
<td>Compressive force on plate along $x-$ axis</td>
</tr>
<tr>
<td>$b$</td>
<td>Amplification factor</td>
</tr>
<tr>
<td>$c$</td>
<td>Velocity of wave propagation</td>
</tr>
<tr>
<td>$f,g$</td>
<td>force function coefficients</td>
</tr>
<tr>
<td>$h$</td>
<td>Thickness, height, discretization size</td>
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<td>$k$</td>
<td>Spring Stiffness</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass</td>
</tr>
<tr>
<td>$q_i$</td>
<td>Mass per unit length</td>
</tr>
<tr>
<td>$r_i$</td>
<td>Flexural rigidity</td>
</tr>
<tr>
<td>$s$</td>
<td>Poles of the system</td>
</tr>
<tr>
<td>$t$</td>
<td>Time variable</td>
</tr>
<tr>
<td>$u$</td>
<td>Displacement in axial direction</td>
</tr>
<tr>
<td>$v$</td>
<td>Spatial displacement function</td>
</tr>
<tr>
<td>$w$</td>
<td>Displacement in transverse direction</td>
</tr>
<tr>
<td>$x$</td>
<td>Spatial co-ordinate, distance</td>
</tr>
<tr>
<td>$y$</td>
<td>Displacement in $y-$ direction, Spatial co-ordinate</td>
</tr>
<tr>
<td>$z,\hat{z}$</td>
<td>Eigenfunction or eigenvector</td>
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Abstract

The static and dynamic analysis of structures requires us to obtain solutions of their respective governing differential equations subject to appropriate boundary conditions. The dynamic analysis of non-uniform continuous structures is of primary interest, as most traditional methods take the help of discrete models to analyze them. Well established discrete model methods lead to an algebraic eigenvalue problem, the characteristic equation associated with which is a polynomial. The spectral characteristics of a continuous system nevertheless are represented by transcendental functions and cannot be approximated by polynomials efficiently. Hence finite dimensional discrete models are not capable of predicting the response of continuous systems irrespective of the model order used.

In this research, a new low order analytical model is developed which approximates the dynamic behavior of the continuous system accurately. The idea here is to replace a non-uniform continuous system by a set of continuous system with piecewise constant physical properties. Such approximations lead to a transcendental eigenvalue problem, i.e. a problem with transcendental characteristic equation. A numerical method has been developed to solve such eigenvalue problems. The spectrum of non-uniform rods and beams are approximated with fair accuracy by solving the corresponding transcendental eigenvalue problem. This mathematical model is extended to reconstruct non-uniform rods and beams using a linear polynomial approximation of piecewise area. A piecewise tapered approximation of the physical parameters in non-uniform rods and beams leads to better accuracy in the solution. The ability to use higher order area functions as basic building blocks profoundly reduces the model order when using the mathematical model to analyze complex geometries.

To further study the impact of this method in various problems of engineering the buckling of thin rectangular plates with stepped thickness has been analyzed and compared with the finite element solution. The transcendental eigenvalue method leads to the reduction
in matrix sizes when compared with discrete model methods, thus making the solution computationally viable. Finally the transcendental eigenvalue problem associated with the active control of vibration in discrete mass-spring-damper systems has been developed and the proposed mathematical method has been applied.
Chapter 1
Introduction

The design and analysis of many engineering problems involve the use of mathematical models developed from fundamental principles of engineering. Structural design of various systems with mechanical and civil engineering applications is based on analysis of the response of the system to various operating loads. Design of structures subjected to time invariant loads requires the computation of deformation and stresses induced in the system from the principles of static equilibrium. Dynamic analysis of involves the solution of corresponding governing equations of motion to determine the response of the system to transient loads such as its deformation as a function of time and its natural frequencies. In the analysis of response of a system to static and dynamic loads the emphasis is on developing the governing equations from fundamental principles and solving differential equations to obtain solutions in closed form.

Most systems are continuous in nature, are three dimensional in reality and have infinite degrees of freedom. The fundamental development in the field of mechanical engineering is based on the assumption that the behavior of such systems can be closely mapped by modeling them as simpler elements such as rods, beams, plates, etc. The type element chosen to represent a three dimensional structure is based on the applied forces and the aspect ratio of the structure. Over the last few centuries the works of Euler, Newton, Bernoulli and Kirchhoff have established the principles that govern the analysis of such structures. This research focuses on two special areas in mechanical design, vie-à-vie vibration and buckling analysis of structures. Vibration analysis of structures involves the prediction of the response of its natural frequency, its mode shape in order to characterize its motion over a period of time. Buckling analysis involves the determination of the critical load that causes the loss of stability in structures. There is a wealth of literature available that address problems of vibration and buckling of structures, development of corresponding governing
equations and methods used to obtain analytical or approximate solutions. Some of the available literature has been reviewed in the following sections.

1.1 Vibration Analysis

Vibration is defined as a repetitive mechanical displacement or oscillation of a structure about its equilibrium position. The analysis of vibration in a system involves the ability to determine characteristics that define its motion, (a) the frequency, and (b) the amplitude. The harmonic vibration in continuous structures is governed by an eigenvalue problem of the form,

\[(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad (1.1)\]

and,

\[y(0, t) = 0 \]
\[y(L, t) = 0. \quad (1.2)\]

The free vibration of a structure is its displacement characteristics in the absence of an exciting force, can be determined by evaluating its natural frequencies and its mode shapes which are dependent on physical parameters such as elastic modulus, density, cross-sectional area and the moment of inertia. Since continuous structures have infinite degrees of freedom, the spectrum of vibration consists of an infinite number of natural frequencies and modes shapes. Though in reality it is not required to determine the entire spectrum of vibration to design a structure, failure can occur due to excitation of the system at a frequency close any one of the infinite values. Thus the knowledge of the entire spectrum is necessary to have a complete understanding of the system behavior.

Differential equations of the form in Eq.1.1 directly lead to a simple solution of the form \(y = 0\). However this is a trivial solution and is of practical interest. Non-trivial solutions \(y \neq 0\) that satisfy the boundary conditions in Eq.1.2, if they exist, are obtained by solving
a special genre of mathematical problems known as eigenvalue problems. The non trivial solution \( y \) of such a problem is called the eigenvector and \( \lambda \) is known as the eigenvalue. The number of such eigenvalues and eigenvectors that can be computed for a given system is dependent on the number of degrees of freedom of system.

The problem of analyzing the spectrum of non-uniform structures is of special interest as most engineering structures have non-uniform distribution of physical parameters. Non-uniform distribution of physical parameters in structures includes variation in their geometry, rigidity, elastic modulus and density. To analysis such structures the knowledge of their building blocks is of absolute importance. A literature study reveals the existence of analytical solutions to such problems. There are discrete model methods that have been developed, with constant physical properties in each discrete element. Mabie and Rogers (1964) developed a Bessel function solution to the problem of vibration in tapered and double tapered beams. Leissa, A.W. (1987) discussed some of the exact solutions to the vibration of plates and the problems associated with the numerical methods used to obtain their solutions. Numerical methods that are prominent in most available literature have been developed with discrete model methods like the finite element method in mind.

There is evidence however with the development of problems on reconstruction of a discrete model for axially vibrating rods and transversely vibrating beams based on the spectral sets of data investigated by Barcilon (1979, 1982), Gladwell (1986 – b), Gladwell and Williams (1988), Ram and Gladwell (1994 – c) and Ram and Elhay (1998) that these methods cannot accurately describe the spectrum of continuous structures. The discrete model of a rod can be reconstructed from one natural frequency and two mode shapes Ram, (1994 – a). Similarly, the discrete model of a Bernoulli-Euler beam can be reconstructed from two natural frequencies and three mode shapes (Ram, 1994-b). These methods however have been shown to be inaccurate in the identification of physical parameters.

The premise that the mathematical formulation for a continuous system differs from
its discrete approximation, and the evidence provided in the literature demands the development of a low order analytical model to obtain spectral data of non-uniform continuous structures. There is also a need to develop a numerical method that can solve such an analytical model. Singh and Ram, (2002) in their research developed analytical models to solve for the vibration in non-uniform rods and beams. The use of piecewise uniform continuous elements as building blocks and the numerical method developed (Singh, (2002)) leads to better accuracy as compared to the discrete element methods in Cook, R.D. This method also works better in the physical reconstruction of continuous structures.

There is a need however to develop analytical solutions for piecewise continuous models with higher order functions that approximate the physical parameters, i.e. variation in cross-section area, say a quadratic variation, is better approximated with the use of piecewise elements with linearly varying area than with the use of piecewise elements of constant area. There is however, very limited literature available that provide analytical solutions for such higher order approximation. The focus in this research, thus, to develop such analytical solutions for non-uniform rods and beams. The numerical methods that exist to solve such problems also need to be revisited and refined.

1.2 Buckling Analysis

As in the case of vibration of homogeneous and non-homogeneous structures the problems of determining the buckling load, or the load that induces instability in a structure, is a special case of a Strum-Louiville problem or eigenvalue problem. Buckling of non-uniform columns was pioneered by Euler (1759). He considered the buckling of a column with stiffness given as a polynomial \((a + bx/L)\). Extensive study of non-uniform columns was later done by Dinnik (1912, 1929, 1955) and Freudenthal (1966) who provided bessel function solutions to the problem of buckling in columns with uniformly varying cross-section area. Stability analysis of continuous structures can also be carried out using discrete model methods. However the error in predicting the critical load in the case non-uniform
structures again necessitates the need for development of continuous model methods and suitable numerical techniques. A part of this research addresses this problem and attempts to provide a better means of analysis of beam-columns and plates.

1.3 Research Objectives and Scope of the Thesis

This research work in summary primarily focuses on the following issues:

- Develop an analytical model that can approximate continuous vibratory systems.

- Study existing methods such as finite discrete models and the mathematical methods used to solve them. This essentially should provide a good background of the problems associated with such approximations, if any, and provide a basis for the research.

- Develop a numerical method to solve transcendental eigenvalue problems associated with continuous structures, especially structures with non-uniform physical parameters, for example non-uniform rods, beams and plates.

- Investigate the applicability of the developed method in various engineering problems and the need, if any, for refinement of the numerical technique.

Chapter 2 discusses the existing discrete model approximations of continuous systems. The discrepancies associated with such an approximation have been demonstrated. It is shown that discrete models are not a good approximation of continuous systems and the possible reasons for such behavior have been discussed. This chapter concludes that a new mathematical model is required for solving problems associated with vibration in continuous structures.

Chapter 3 introduces the formulation of a mathematical model that analyzes continuous systems using approximate piecewise continuous models that are of very low order. To overcome the difficulty stated in chapter 2, the proposed mathematical model approximates the given continuous system by another continuous system with piecewise constant physical
properties. Such models lead to transcendental eigenvalue problems where the elements of the approximating matrices are transcendental in nature. Further an algorithm based on the Newton’s eigenvalue iteration is developed in this chapter as an effective method of solving transcendental eigenvalue problems.

Chapter 4 addresses the issue of the need for a better analytical model. The models developed in chapter 3 for eigenvalue calculation for non-uniform continuous rods and beams are subject to error due the fact the physical parameters of the non-uniform systems are approximated. This forces an increase in model order for better accuracy. A new analytical model has been developed that assumes a higher order polynomial function to approximate the change in cross-section area. The transcendental eigenvalue method when applied to the new model shows better accuracy with very low model orders.

Chapter 5 presents some numerical examples involving vibration, control and elastic stability in plates to show the effectiveness of the proposed method. The results have been compared with existing work. Chapter 6 summarizes the findings of this research and conclusion that have been drawn from the work presented in this thesis. Recommendations on possible extension of the proposed mathematical model and the developed numerical method have been made.
Chapter 2
Continuous Systems and their Corresponding Discrete Models

2.1 Introduction to Axial Vibration in Rods

Rods or bar elements are structural members that permits displacement in the axial direction alone and offer infinite stiffness along the transverse direction. The axial vibration of such a continuous member that has infinite degrees of freedom often leads to an eigenvalue problem that does not necessarily have a closed form solution. To solve for the problem of vibration of such continuous systems with no closed form solutions; various well established discrete models representations are already in use. Key among such methods are the Finite Difference and Finite Element methods that discretize the system into a finite number of springs and lumped masses to emulate the system as accurately as possible. The spectral analysis of axially vibrating systems, that is the computation of the natural frequency and their corresponding mode shapes, for free vibration are presented in this chapter. For simple systems the analytical solutions are compared with their corresponding discrete models. The chapter also deals with the common issues associated with discretization of continuous structures and provides a piecewise continuous model that provides more accurate solutions when compared to the discrete model counterparts.

2.1.1 Axial Vibration in Uniform Rods: Analytical solution

Figure 2.1 represents a prismatic elastic uniform rod of unit cross-section area $A(x) = 1$ and with unit material properties, Young’s Modulus $E$ and density $\rho$. The rod is of unit length $L = 1$ and is fixed at the end $x = 0$ and is free to displace axially at the other end $x = 1$.

\[
EA \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] = \rho A \left[ \frac{\partial^2 u(x,t)}{\partial t^2} \right], \quad 0 \leq x \leq 1,
\]  
(2.1)
Equation 2.1 represents the equation of motion that governs the displacement of any point on the rod during a given time interval. This can be equivalently written as

\[
\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2}, \quad 0 \leq x \leq 1, \tag{2.2}
\]

where \( c = \sqrt{\frac{E}{\rho}} \) is the velocity of wave propagation through the rod. The system is further subjected to boundary conditions that ensure no displacement in the rod at the fixed end \( x = 0 \) and no axial force at the end \( x = 1 \). This is represented by the following equations.

\[
u(0, t) = 0
\]

\[
EA \frac{\partial u(1, t)}{\partial x} = 0 \tag{2.3}
\]

The analytical solution for displacement of the above system of equations Eqs. 2.2 and 2.3, assuming a harmonic solution in time say,

\[
u(x, t) = v(x) \sin(\omega t), \tag{2.4}
\]

yields a frequency equation of the form (equation 7.122, Meirovitch, 1977),

\[
\cos \left( \frac{\omega}{c} \right) = 0, \tag{2.5}
\]
where the roots of Eq. 2.5 are given by,

$$\omega_i = \left( \frac{(2i - 1) c \pi}{2} \right)$$

Equation 2.6 above gives us $\omega_i$ which is the $i^{th}$ natural frequency of the system and Eq. 2.5 the corresponding mode shape. Since a simple closed form solution exists the need for discretization does not arise. However for the sake of argument and better comprehension the first 15, 30 and 50 natural frequencies of an axially vibrating uniform rod are computed using two common discretization methods, the finite difference and the finite element methods and the results from the corresponding models are compared with the analytical solution from Eq. 2.6.

### 2.1.2 Discrete Element Approximation: Finite Difference Method

One can represent a continuous rod by dividing it into $n$ equal subintervals as a series of springs with masses lumped between consecutive springs. The spring stiffness of such a model represents the axial stiffness of the rod and lumped mass represents the mass density per unit length. For this reason this model is also known as the lumped parameter model. Intuitively with higher number of such discrete springs and masses the one can increase the accuracy of the approximation to converge to the analytical solution. The evident constraint on such a discretization is the computation time and resource available.

![Discrete spring mass approximation of a uniform rod using finite differences](image)

Figure 2.2: Discrete spring mass approximation of a uniform rod using finite differences

Figure 2.2 represents such a discrete model of element size $h$ where,
\[ k_i = \frac{(EA)_i}{h} = \frac{p_i}{h}, \quad (2.7) \]

is the stiffness of the \( i^{th} \) spring and equivalently the axial rigidity of rod and,

\[ m_i = h (\rho A)_i = q_i h, \quad (2.8) \]

represents the \( i^{th} \) mass representing the corresponding mass per unit length of the rod.

The governing equation and boundary conditions of a uniform fixed-free rod have been discussed earlier in Equations. 2.2 and 2.3, which can be re-written, with assumption of harmonic solution in time and using separation of variables, as,

\[
(EAv'(x))' + \lambda \rho Av(x) = 0
\]

\[ v(0) = 0 \]

\[ v'(L) = 0 \quad (2.9) \]

For a given discretization size \( n \) we have displacement of the \( i^{th} \) mass represented by say \( v_i \), then we can compute \( n \) eigenvalues and \( n \) corresponding eigenvectors. As we place the lumped mass at the center of each discrete element the first has half the length and therefore twice the stiffness. Using a central difference scheme(Appendix 1) to represent each derivative in equation 2.9 we have for the \( i^{th} \) element,

\[
\frac{(EA)_i}{h^2} v_{i-1} - \frac{(EA)_i + (EA)_{i+1}}{h^2} v_i + \frac{(EA)_{i+1}}{h^2} v_{i+1} + \lambda (\rho A)_i v_i = 0 \quad (2.10)
\]

Multiplying Eq.2.10 by \(-h\) and using Eqs.2.7 and 2.8 we have an eigenvalue problem of the form

\[
-k_i v_{i-1} + (k_i + k_{i+1}) v_i - k_{i+1} v_{i+1} - \lambda m_i v_i = 0. \quad (2.11)
\]

The boundary conditions in Eq. 2.9 can written as
\( v_0 = 0 \)  
\( v_{n+1} = v_n \)  

Equations 2.11 and 2.12, when written in a matrix form for \( i = 1, 2, 3, \ldots n \) forms a tri-diagonal matrix in \( k \) and a diagonal matrix in \( m \) which form a generalized eigenvalue problem of the form

\[ (K - \lambda M) \mathbf{v} = 0 \]  

(2.13)

where

\[
K = \frac{1}{h} \begin{bmatrix}
p_1 + p_2 & -p_2 \\
-p_2 & p_2 + p_3 & -p_3 \\
& \ddots & \ddots & \ddots \\
& & -p_{n-1} & p_{n-1} + p_n & -p_n \\
& & & -p_n & p_n
\end{bmatrix}
\]  

(2.14)

\[
M = h \begin{bmatrix}
q_1 \\
q_2 \\
& \ddots \\
& & q_{n-1} \\
& & & q_n
\end{bmatrix}
\]  

(2.15)

and

\[ \mathbf{v} = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}^T \]  

(2.16)

Simple mathematical tools can be used to solve for the eigenvalues of Eq.2.13, as all elements of both the \( K \) and \( M \) matrices are algebraic constants.
2.1.3 Discrete Element Approximation: Finite Element Method

In the finite element method the displacement of the system is represented as a finite number of displacements at the nodal points. If a 2-noded element is chosen to represent the rod, the mass density per unit length and the stiffness are averaged over the length of the element and placed on the nodes alone.

Figure 2.3: Discrete model approximation of a uniform rod using finite element method

Figure 2.3 represents such a discrete model of element size \( h \). The displacement function for the \( i^{th} \) element, where \( i = 1, 2, 3, \ldots n \), is given by the interpolation function and the corresponding vector of nodal displacements.

\[
\mathbf{u}(x, t) = L_1(x)u_1(t) + L_2(x)u_2(t) = \mathbf{L}(x)^T\mathbf{u}(t), \tag{2.17}
\]

A simple linear interpolation scheme gives an interpolation function as,

\[
\mathbf{L}(x) = \begin{pmatrix} 1 - \frac{x}{h} \\ \frac{x}{h} \end{pmatrix} \tag{2.18}
\]

Equation 2.18 is also referred to as the shape function of the bar element. Thus the elemental stiffness and mass matrices, \( k_i \) and \( m_i \), for an axially vibrating rod are given by,

\[
K_i^{(e)} = \frac{p_i}{h^2} \int_0^h \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} dx = \frac{p_i}{h^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{2.19}
\]

and,
\[ M_i^{(e)} = q_i \int_0^h \left( 1 - \frac{x}{h} \right) \left( 1 - \frac{x}{h} \frac{x}{h} \right) dx = \frac{q_i h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \] (2.20)

The continuity between discrete elements enables the assembly of the elemental matrices into global \( K^{(G)} \) and \( M^{(G)} \) matrices. On applying the relevant boundary conditions for a uniform fixed-free rod the global stiffness and mass matrices transform to form matrices as shown below.

\[
K^{(G)} = \frac{1}{h} \begin{bmatrix}
p_1 + p_2 & -p_2 & \cdots & \cdots & -p_n \\
-p_2 & p_2 + p_3 & -p_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-p_{n-1} & p_{n-1} + p_n & -p_n & \cdots & \cdots \\
-p_n & p_n & \cdots & \cdots & \cdots
\end{bmatrix}
\] (2.21)

and,

\[
M^{(G)} = \frac{h}{6} \begin{bmatrix}
2 (q_1 + q_2) & q_2 & \cdots & \cdots & q_n \\
q_2 & 2 (q_2 + q_3) & q_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
q_{n-1} & 2 (q_{n-1} + q_n) & q_n & \cdots & \cdots \\
q_n & 2 q_n & \cdots & \cdots & \cdots
\end{bmatrix}
\] (2.22)

The resulting equation of motion leads to the same eigenvalue problem in Eq.2.13

\[(K^{(G)} - \lambda M^{(G)}) \mathbf{v} = 0 \] (2.23)

Each element in the \( K^{(G)} \) and \( M^{(G)} \) matrices is again an algebraic term and hence Eq.2.23 is also an algebraic eigenvalue problem.
2.2 Introduction to Transverse Vibration in Beams

According to Meirovitch, [1977] the governing equations of motion for a transversely vibrating beam can be developed from the deformed shape of general prismatic elastic bar subjected to lateral load \( p(x) \) as shown in Figure.2.4. The governing differential equation was first developed by Euler and is hence known as the Euler-Bernoulli beam theory. From the free-body diagram of an elemental section of the beam shown in Figure.2.5 we have,
\[ M = EI \frac{\partial^2 w}{\partial x^2} \]  

(2.26)

Substituting Eqs. 2.25 and 2.26 in Eq. 2.24 we have the governing equation of motion for a transversely vibrating beam with no external load as,

\[ \frac{\partial^2 EIw''}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \]  

(2.27)

Consider a fixed-free beam shown in Figure 2.6. The governing differential equation is subject to the following boundary conditions.

\[ w(0, t) = 0 \]
\[ \frac{\partial}{\partial x} w(0, t) = 0 \]
\[ \frac{\partial^2}{\partial x^2} w(L, t) = 0 \]  

(2.28)

To derive the last boundary condition we consider the free-body diagram of the end section of the beam at \( x = L \), as shown in Figure 2.7.

\[ EI \frac{\partial^3 w(L, t)}{\partial x^3} - kw(L, t) = 0 \]  

(2.29)
Assuming a harmonic solution in time, say $w(x,t) = w(x)\sin(\omega t)$, Eq.2.27 can be re-written as

$$w''''(x) - \mu^4 w(x) = 0,$$

(2.30)

where, $\mu^4 = \omega^2 \frac{\rho A}{EI}$. The above Eq.2.30 is an eigenvalue problem where $\mu^4$ is the eigenvalue and $w(x)$ is the corresponding mode shape, and $\omega$ is the natural frequency of the system. A generalized solution to the above Eq.2.30 is,

$$w(x) = z_1 \sin(\mu x) + z_2 \cos(\mu x) + z_3 \sinh(\mu x) + z_4 \cosh(\mu x)$$

(2.31)

2.2.1 Discrete Element Approximations: Finite Difference Model

The transversely vibrating Euler-Bernoulli beam can be represented as a series of continuous rods and torsional springs with the mass concentrated at the end of the rods as shown by Gladwell [1986 – a]. Consider the system shown in 2.8. The beam is divided into $n$ elements of length $h_i$. It also consists of $n + 2$ masses lumped at the either end of each rod and $n$ springs $k_i$ connecting the rods.

The discrete systems parameters are determined to be
Figure 2.8: Discrete spring-mass-rod model representing a beam, Gladwell [1986 – a]

\[ m_i = \rho_i A_i h_i \] (2.32)

\[ k_i = \frac{E_i I_i}{h_i} = \frac{\bar{r}_i}{h_i} \]

where \( E_i, I_i, \rho_i \) and \( A_i \) are the modulus of elasticity, moment of inertia, density and cross-sectional area of the beam and \( q_i \) and \( \bar{r}_i \) are the mass per unit length of each element and the flexural rigidity of the beam element. The displacement of the \( i^{th} \) mass from its static equilibrium \( w_i \) and slope \( \theta_i \) of the \( i^{th} \) rod define the motion of the beam structure.

The differential equation in matrix form representing such a discrete system is given by

\[ M\ddot{w} + Kw = 0 \] (2.33)

where for \( h_i = h \)

\[ M = h \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \] (2.34)

and

\[ K = EL^{-1}E\Theta E^T L^{-1}E^T - V_R e_n - \tau_R L_n^{-1} E e_n, \] (2.35)

where
\[
\tilde{E} = \begin{bmatrix}
1 & -1 \\
& 1 & -1 \\
& & 1 & -1 \\
& & & \ddots & \ddots & -1 \\
& & & & -1 & 1
\end{bmatrix}, \quad (2.36)
\]

\[
L = \begin{bmatrix}
l_1 \\
l_2 \\
\vdots \\
l_n
\end{bmatrix}, \quad (2.37)
\]

\[
\Theta = \frac{1}{h} \begin{bmatrix}
\bar{r}_1 \\
\bar{r}_2 \\
\vdots \\
\bar{r}_n
\end{bmatrix}, \quad (2.38)
\]

\[
e_n = \begin{pmatrix}
0 & \ldots & 0 & 1
\end{pmatrix} \in (n \times 1). \quad (2.39)
\]

and

\[
u = \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}^T. \quad (2.40)
\]

Assuming harmonic motion, Eq.2.33 we obtain an eigenvalue problem of the form

\[
(K - \lambda M) \mathbf{v} = 0 \quad (2.41)
\]

This mass-spring-rod approximation of beam is frequently used for dynamic analysis. Natural frequencies and mode shapes of the beam can be obtained by solving the eigenvalue
problem Eq.2.41. Mass-spring-rod models can also approximate beams having nonlinear distribution of physical properties and they can be used for approximating their dynamic characteristics.

### 2.2.2 Discrete Element Approximations: Finite Element Model

![Discrete spring-mass-rod model representing a beam, Cook, [2003]](image)

Figure 2.9: Discrete spring-mass-rod model representing a beam, Cook, [2003]

The finite element approximation for transversely vibrating beam as shown in Figure. 2.9 can be obtained, as presented by Cook, [2003] by expressing the bending displacement of any beam element in the following form,

\[
v(x) = L_1(x)w_1 + L_2(x)\theta_1 + L_3(x)w_2 + L_4(x)\theta_2 = \mathbf{L}(x)^T\vec{w}, \quad (2.42)
\]

where \(\mathbf{L}(x)\) is the cubic interpolation function, known as shape function and \(\vec{w}\) is the corresponding vector of nodal displacements. The shape function for a Hermite cubic interpolating function is
and the elemental mass and stiffness matrices for a beam element can be obtained as,

\[ \mathbf{M}^{(e)}_i = q_i h \int_0^h \mathbf{L}(x) \mathbf{L}^T(x) dx, \]  

or

\[ \mathbf{M}^{(e)}_i = \frac{q_i h}{420} \begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \]  

and

\[ \mathbf{K}^{(e)}_i = \frac{4\bar{F}_i}{h^4} \int_0^h \begin{bmatrix} -3 + 6 \frac{x}{h} \\ -2 + 3 \frac{x}{h} \\ 3 - 6 \frac{x}{h} \\ -1 + 3 \frac{x}{h} \end{bmatrix} \begin{bmatrix} -3 + 6 \frac{x}{h} \\ -2 + 3 \frac{x}{h} \\ 3 - 6 \frac{x}{h} \\ -1 + 3 \frac{x}{h} \end{bmatrix}^T dx, \]
or

$$K_{i}^{(e)} = \frac{\vec{r}_{i}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix}$$

(2.47)

After evaluating the elemental matrices the global mass and stiffness matrices $M$ and $K$ can be assembled for fixed-free configuration by using appropriate continuity conditions between the elements and the corresponding eigenvalues can be evaluated by solving, $|K - \lambda M| = 0$.

2.3 Comparison of Continuous Systems with their Discrete Model Representations

The discrete model methods described in the previous section have been used to analyze some simple numerical examples of rods and beams. The solutions obtained have been compared with their respective exact solutions.

2.3.1 Comparison of the Spectrum of Fixed-Free Uniform Rod with their Approximated Discrete Model

In section 2.1.1, an analytical solution for the spectrum of the uniform rod is obtained by Eq. 2.6. The eigenvalues of the uniform rod can also be obtained by finite difference and finite element methods using Eq.2.13 and Eq.2.23 respectively. Assuming, the model order $n = 15$ for finite difference and finite element approximation, the associated mass and stiffness matrices $K$ and $M$ are obtained from Eqs.2.14 - 2.15 and Eqs.2.21 - 2.22 respectively. By solving these eigenvalue problems the lowest 15 eigenvalues of the rod using finite element and finite difference models are compared with the corresponding exact eigenvalues of rod.

The results are expressed in the non-dimensional form $\omega_i^2 = \lambda^2/c^2$ and plotted to-
Figure 2.10: Comparison of the first 15 eigenvalues of axial vibration in a fixed-free uniform rod together in Figure 2.10. It is apparent from Figure 2.10 that with the use of discrete model methods only a few lower eigenvalues are approximated correctly. In order to improve the approximation, model order is increased to $n = 30$ and $n = 50$ and the eigenvalues are compared in Figures 2.11 and 2.12. These results demonstrate that though the accuracy of the first few eigenvalues have improved, only about $n/3$ eigenvalue are approximated accurately. Increasing the model order improves the accuracy of low eigenvalues but at the same time, $\frac{2n}{3}$ inaccurate eigenvalues are also obtained. Thus, the increase in number of elements in the approximated system is necessarily a good method as it does not ensure the spectral consistency between the continuous system and its discrete approximation.

Figure 2.11: Comparison of the first 30 eigenvalues of axial vibration in a fixed-free uniform rod

It is also important to note that the higher eigenvalues obtained from the finite element
Figure 2.12: Comparison of the first 50 eigenvalues of axial vibration in a fixed-free uniform rod

method are higher than the exact solution. This can be attributed to the fact that though the exact solution is obtained by minimization of the Rayleigh quotient, the finite element method uses only a subset of the admissible function. Therefore instead of evaluating the exact minimum the finite element scheme evaluates those eigenvalues that are higher than the true values. In the case of the finite difference method the higher eigenvalues estimated are lower than the exact solution. The reason for such a behavior is the fact that the mass of each discrete element is lumped into the center of the element thus increasing the effective inertia of the system. Further the springs are assumed to be mass-less thus lowering the stiffness of the system leading to the underestimation of the higher eigenvalues.

2.3.2 Comparison of the Spectrum of Fixed-Free Uniform Beam with their Approximated Discrete Model

Consider the discrete models that approximate a fixed-free uniform beam as shown in sections 2.2.1 and 2.2.2 where $k = 0$. The first fifteen eigenvalues are obtained by dividing the beam into 15 equal mass-springs and rods and using the finite element model. The inaccuracy in the predicted eigenvalues as seen in Figure.2.13 show that the discrete model methods are not efficient in analyzing the entire spectrum of transverse vibration in beams either.

It is clear from the examples of uniform rod and beams that the approximation, based upon finite difference and finite element schemes are inaccurate in predicting the eigenval-
Figure 2.13: Comparison of the first 15 eigenvalues of transverse vibration in a fixed-free uniform beam

Figure 2.14: Comparison of the first 50 eigenvalues of transverse vibration in a fixed-free uniform beam

ues. By inspection of Figures 2.12 and 2.14, it can be concluded that the improvements in the results that are achieved by increasing the model order are only in the lower third of the spectrum. The discrepancies that are presented in approximated models of continuous systems are the main motivation of our work presented here. In conclusion, it is observed that the behavior of discrete model of vibrating system leads to an algebraic eigenvalue problem where as the associated continuous model leads to a transcendental eigenvalue problem. Spectral characteristics of the continuous systems are not completely different from their corresponding discrete model but cannot be completely represented by their corresponding discrete models. Hence, this leads to inaccurate approximation of higher eigenvalues. Therefore, a new mathematical model is required that can realistically model the behavior of continuous vibrating systems accurately. The focus in this research is to develop an analytical model which has a closed form solution and is transcendental in
Table 2.1: Comparison of eigenvalues of transverse vibration in an uniform beam.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( n = 15 )</td>
<td>( n = 15 )</td>
</tr>
<tr>
<td>1</td>
<td>1.876</td>
<td>1.7381</td>
<td>1.8751</td>
</tr>
<tr>
<td>2</td>
<td>4.695</td>
<td>4.4178</td>
<td>4.6941</td>
</tr>
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<td>4</td>
<td>10.996</td>
<td>10.2092</td>
<td>10.9966</td>
</tr>
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<td>5</td>
<td>14.138</td>
<td>13.1727</td>
<td>14.1409</td>
</tr>
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<td>6</td>
<td>17.279</td>
<td>15.4531</td>
<td>17.2887</td>
</tr>
<tr>
<td>7</td>
<td>20.421</td>
<td>17.82</td>
<td>20.4429</td>
</tr>
<tr>
<td>8</td>
<td>23.562</td>
<td>20.6821</td>
<td>23.607</td>
</tr>
<tr>
<td>9</td>
<td>26.4633</td>
<td>22.8933</td>
<td>26.7855</td>
</tr>
<tr>
<td>10</td>
<td>29.5318</td>
<td>24.3038</td>
<td>29.9836</td>
</tr>
<tr>
<td>11</td>
<td>32.4417</td>
<td>25.8301</td>
<td>33.2063</td>
</tr>
<tr>
<td>12</td>
<td>35.5506</td>
<td>27.5960</td>
<td>36.4566</td>
</tr>
<tr>
<td>13</td>
<td>38.7687</td>
<td>28.4163</td>
<td>39.7300</td>
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<td>14</td>
<td>41.7768</td>
<td>29.3891</td>
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</tr>
<tr>
<td>15</td>
<td>44.6453</td>
<td>31.4058</td>
<td>45.9735</td>
</tr>
</tbody>
</table>
Chapter 3
Transcendental Eigenvalue Method
Associated with Rods and Beams

3.1 Transcendental Eigenvalue Problem

An eigenvalue problem in general is defined as the problem of determining non-trivial or non-zero solutions to a system of equations that can be represented as

\[ A(\omega)\hat{z} = 0 \]  

(3.1)

where the elements of the \( n \times n \) matrix \( A \) are functions of the eigenvalues \( \omega \) which when determined make the matrix singular and the vector \( \hat{z} \) is a constant non-zero vector called the eigenvector. The following problem of mechanical vibration of a two degrees of freedom system that leads to an eigenvalue problem. This method produces a vector \( \omega \) which is the non-trivial solution to the Eq.3.18, that will make the matrix \( A(\omega) \) singular.

3.1.1 Free Vibration of a Finite vs Infinite-Degree of Freedom System

\[ \begin{align*}
  k_1 & \quad \quad \quad \quad k_2 \\
  m_1 & \quad \quad \quad \quad m_2 \\
  k_2 & \quad \quad \quad \quad k_3
\end{align*} \]

Figure 3.1: Vibration of a two degree of freedom system

The masses \( m_1 \) and \( m_2 \) and spring stiffness \( k_1, k_2 \) and \( k_3 \) can be written in a matrix form as

\[ M = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \]  

(3.2)
The equation of motion that governs the system shown in Figure 3.1 is

\[ M\ddot{x} + Kx = 0 \]  

(3.3)

where \( x \) is the displacement vector. Assuming a general solution to the differential equation as \( x = u\sin(\omega t) \) and separating the variables in Eq. 3.4 we have the eigenvalue problem

\[ (K - \lambda M)u = 0 \]  

(3.4)

where each \( \lambda = \omega^2 \) in the 2 degree of freedom system is a scalar and are the eigenvalues of Eq. 3.4 and \( u \) is the eigenvector defined by

\[ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]  

(3.5)

The eigenvalues \( \lambda \) are obtained by computing the determinant of the singular matrix \( |(K - \lambda M)| \), which is

\[ |(K - \lambda M)| = (k_1 + k_2 - \lambda m_1)(k_2 + k_3 - \lambda m_2) - (k_2)(k_1) = 0 \]  

(3.6)

Notice that the right hand side of Eq. 3.6 is a polynomial equation in \( \lambda \) and thus Eq. 3.4 is called an algebraic eigenvalue problem. On the other hand the problem of axial vibration in a rod as defined in previous section has a governing differential equation of motion given by Eq. 2.2 and is subject to boundary conditions given by Eq. 2.3. The general solution to this problem is

\[ u(x, t) = (b_1 \sin(\mu x) + b_2 \cos(\mu x))\sin(\omega t), \]  

(3.7)

Substituting this solution into the boundary conditions we have,
The eigenvalue problem Eq. 3.8 thus obtained has elements that are transcendental functions of the unknown eigenvalues \( \mu \) and the resulting determinant \( |\cos(\mu)| \) is transcendental in nature as well. Such a problem is called a transcendental eigenvalue problem. In general the transcendental eigenvalue problem is defined as the problem of determining the non-trivial solutions \( \omega \) and \( \hat{z} \neq 0 \) of

\[
A(\omega)\hat{z} = 0
\]  

(3.9)

where the elements of the \( n \times n \) matrix \( A \) are transcendental functions in \( \omega \) and the vector \( \hat{z} \) is a constant non-zero eigenvector. In order to solve the transcendental eigenvalue problem in Eq. 3.9 a rapidly converging algorithm is being used here. This algorithm is based on the Newton eigenvalue iteration method for root finding. Using the Taylor series expansion, Yang (1983) formulated the following method. The matrix \( A(\omega) \) can be expanded in the neighborhood of \( \omega(0) \) using

\[
\omega = \omega^{(0)} + \epsilon
\]

(3.10)

as,

\[
A \left( \omega^{(0)} + \epsilon \right) = A \left( \omega^{(0)} \right) + \epsilon \frac{dA \left( \omega^{(0)} \right)}{d\omega}
\]

(3.11)

In the Taylor series expansion of \( A(\omega) \) the second and higher order terms have been neglected. Now writing,

\[
B(\omega) = -\frac{dA(\omega)}{d\omega}
\]

(3.12)
we have,

\[ A(\omega(0) + \epsilon) = A(\omega(0)) - \epsilon B(\omega(0)) \]  

(3.13)

The determinant of the above Eq. 3.13 gives,

\[ |(A(\omega(0) + \epsilon))| = |(A(\omega(0)) - \epsilon B(\omega(0)))| \]  

(3.14)

The non-trivial solution of Eq. 3.1 can be obtained by solving for \( \omega \) from

\[ |(A(\omega(0)) - \epsilon B(\omega(0)))| \]  

(3.15)

It thus follows from Eq. 3.1 that the above Eq. 3.15 can be written as

\[ (A(\omega(0)) - \epsilon B(\omega(0))) z = 0 \]  

(3.16)

The value of \( \epsilon \) in Eq. 3.16 can be determined by solving an equivalent algebraic eigenvalue problem Eq. 3.16. With an initial guess in the neighborhood of \( \omega(0) \) and the Newton’s eigenvalue iterative method value one can determine the solution to

\[ \omega^{(k)} = \omega^{(k-1)} + \epsilon^{(k)} \]  

(3.17)

where \( \epsilon^{(k)} \) is the eigen-value of the \( k^{th} \) iteration of

\[ (A(\omega) - \lambda B(\omega)) z = 0. \]  

(3.18)

For a given matrix \( A \) of size \( n \times n \) the solution to the above Eq.3.18 gives \( n \) eigenvalues \( \omega_i, i = 1, 2, 3, ..., n \). Since the Taylor series expansion is valid only for small values of \( \epsilon \), the value of \( \epsilon^{(k)} \) is chosen as the smallest eigenvalue, \( \min(\mu_i) \), in the absolute value sense.

For \( n = 1 \) Eq.3.18 reduces to,
\[ \lambda = \frac{A(\omega)}{B(\omega)} = -\frac{f(\omega^{(k-1)})}{f'(\omega^{(k-1)})} \]  

(3.19)

It follows from Eq.3.19 that the Newton’s eigenvalue iteration method is a degenerate case of eigenvalue extraction procedure presented above.

3.1.2 Algorithm for the Solution of the Transcendental Eigenvalue Problem: Newton’s Eigenvalue Iteration Method

A step by step procedure has been outlined here and presented in the form of a flowchart.

**INPUT**

1. The matrix \( A(\omega) \) and compute matrix \( B(\omega) \) as defined in Eq.3.12.

2. The initial estimate of \( \omega^{(0)} \).

3. A tolerance for convergence \( \delta \), a small positive number, is selected.

**ITERATION**

1. Choose the initial guess \( \omega^{(0)} \) and start the iteration.

2. Compute the eigenvalues of system based upon the initial guess \( \omega^{(0)} \).

3. Evaluate the minimum eigenvalue of Eq.3.18 and assign this value to \( \epsilon \).

4. Compute the new estimate \( \omega^{(1)} = \omega^{(0)} + \epsilon \).

5. Compute the matrices \( A(\omega^{(0)}) \) and \( B(\omega^{(0)}) \).

6. Repeat steps (b) – (e) for \( k^{th} \) iteration until the condition \( \epsilon < \delta \) is satisfied.

7. Stop the iteration.

8. Store the value of \( \omega_k \).
9. Repeat steps (a) – (h) to evaluate another \( \omega_k \) for different starting value \( \omega^{(0)} \).

**OUTPUT**

![Flow-chart depicting the Newton’s Eigenvalue Iteration Algorithm.](image)

The algorithm above has been represented in the form of a flow chart as shown in the Figure 3.2. This method produces a vector \( \omega \) which is the non-trivial solution to the Eq.3.18, that will make the matrix \( A(\omega) \) singular. This method can be applied to solve for the free vibration of non-uniform rods and beams. The effectiveness of this method
over the well-established discrete model methods have been demonstrated in the following chapters. Further the application of this algorithm in the case of active control of vibration and buckling of plates have also been presented.

3.2 Axial Vibration in Non-Uniform Rod: Piecewise Continuous Models

3.2.1 Introduction

Consider a non-uniform axially vibrating rod of length $L$, axial rigidity $p(x) = E(x)A(x)$, and mass per unit length $q(x) = \rho(x)A(x)$, which is fixed at $x = 0$ and free to oscillate at $x = L$, as shown in Figure 3.3.

![Image of axially vibrating non-uniform rod](image)

Figure 3.3: Axially vibrating non-uniform rod.

The axial motion of the rod is governed by the following differential equation

$$\frac{\partial}{\partial x} \left( p(x) \frac{\partial u(x,t)}{\partial x} \right) = q(x) \frac{\partial^2 u(x,t)}{\partial t^2}, 0 < x < L, t > 0$$

(3.20)

and boundary conditions

$$u(0,t) = 0$$

$$\frac{\partial}{\partial x} u(L,t) = 0$$

(3.21)

Assuming a harmonic motion of the system during free vibration
\[ u(x, t) = v(x) \sin(\omega t), \]  

separates Eqs.3.20 and 3.21 to the following eigenvalue problem,

\[ (pv')' + \lambda qv = 0, \quad 0 < x < L, \]
\[ v(0) = 0 \quad v'(L) = 0, \]

where \( \lambda = \omega^2 \) and prime denotes the derivatives with respect to \( x \). The analysis of the free vibration of a non-uniform rod has been performed and the need for a piecewise continuous model over discrete representation of continuous systems has been justified.

The non-uniform rod, as shown in Figure.3.3, can also be represented by dividing the rod into \( n \) uniform continuous rods of constant physical parameters \( p_1, p_2, p_3, \ldots, p_n \) and \( q_1, q_2, q_3, \ldots, q_n \) as shown in Figure.3.4. The length of each uniform rod can be represented as, \( L_i = x_i \). Since for each piecewise element the governing differential equation Eq.3.23 holds, we have for an \( n \) order piecewise system, \( n \) equations of motion given by,
\[
\left( p_1 \frac{\partial^2 u_1(x, t)}{\partial x^2} \right) = q_1 \frac{\partial^2 u_1(x, t)}{\partial t^2}, \quad 0 < x < L_1, t > 0
\]
\[
\left( p_2 \frac{\partial^2 u_2(x, t)}{\partial x^2} \right) = q_2 \frac{\partial^2 u_2(x, t)}{\partial t^2}, \quad L_1 < x < L_2, t > 0
\]
\[
\vdots
\]
\[
\left( p_n \frac{\partial^2 u_n(x, t)}{\partial x^2} \right) = q_n \frac{\partial^2 u_n(x, t)}{\partial t^2}, \quad L_{n-1} < x < L_n, t > 0
\]

and the boundary conditions similar to Eq.3.21

\[
u_1(0, t) = 0
\]
\[
\frac{\partial}{\partial x} u_n(L_n, t) = 0
\]

In order for the ensemble to completely represent the non-uniform rod, at \( x = x_1, x_2, ..., x_{n-1} \)

there exist continuity in displacement and axial force. This can be represented as,

\[
u_1(L_1, t) = u_2(L_1, t)
\]
\[
p_1 \frac{\partial u_1(L_1, t)}{\partial x} = p_2 \frac{\partial u_2(L_1, t)}{\partial x}
\]
\[
u_2(L_2, t) = u_3(L_2, t)
\]
\[
p_2 \frac{\partial u_2(L_2, t)}{\partial x} = p_3 \frac{\partial u_3(L_2, t)}{\partial x}
\]
\[
\vdots
\]
\[
u_{n-1}(L_{n-1}, t) = u_n(L_{n-1}, t)
\]
\[
p_{n-1} \frac{\partial u_{n-1}(L_{n-1}, t)}{\partial x} = p_n \frac{\partial u_n(L_{n-1}, t)}{\partial x}
\]

For each piecewise rod in the ensemble, an assumption of harmonic displacement gives,
\[ u_1(x, t) = v_1 \sin(\omega t) \]
\[ u_2(x, t) = v_2 \sin(\omega t) \]
\[ \vdots \]
\[ u_{n-1}(x, t) = v_{n-1} \sin(\omega t) \]

Substituting the above in Eq.3.24, we have the governing differential equations of motion as,
\[ v''_1 + (\omega^2/c_1^2)v_1 = 0, \quad 0 < x < L_1, \]
\[ v''_2 + (\omega^2/c_2^2)v_2 = 0, \quad L_2 < x < L_2, \]
\[ \vdots \]
\[ v''_{n-1} + (\omega^2/c_{n-1}^2)v_n = 0, \quad L_{n-1} < x < L_n, \]

(3.28)

where \( \lambda = \omega^2 \) and \( c_i^2 = p_i/q_i, \ i = 1, 2, ..., n - 1. \) The general solution to the system of equations in 3.28 is

\[ v_1(x) = Q_1 \sin\left(\frac{\omega}{c_1}x\right) + R_1 \cos\left(\frac{\omega}{c_1}x\right), \quad 0 < x < L_1, \]
\[ v_2(x) = Q_2 \sin\left(\frac{\omega}{c_2}x\right) + R_1 \cos\left(\frac{\omega}{c_2}x\right), \quad L_2 < x < L_2, \]
\[ \vdots \]
\[ v_{n-1}(x) = Q_{n-1} \sin\left(\frac{\omega}{c_{n-1}}x\right) + R_{n-1} \cos\left(\frac{\omega}{c_{n-1}}x\right), \quad L_{n-1} < x < L_n, \]

(3.29)

Substituting Eq.3.29 into the boundary conditions in Eq.3.25 and the continuity conditions in Eq.3.26, we get an eigenvalue problem of the form \( \mathbf{A}(\omega)\mathbf{z} = 0. \)
\[ A(\omega) = \begin{bmatrix} \cdots & (A_L)^T & \cdots \\
\cdots & (A_{De})^T & \cdots \\
\cdots & (A_{Fc})^T & \cdots \\
\cdots & (A_R)^T & \cdots \end{bmatrix} \tag{3.30} \]

where \( A(\omega) \) is the transcendental matrix and,

\[ A_L = (0 \ 1 \ 0 \ \cdots \ 0 \ 0)^T \tag{3.31} \]

\[ A_{De} = \begin{bmatrix} \alpha_1 & \beta_1 & -\alpha_1 & -\beta_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \alpha_2 & \beta_2 & -\alpha_2 & -\beta_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \alpha_{n-2} & \beta_{n-2} & -\alpha_{n-2} & -\beta_{n-2} & \cdots & 0 \\
\alpha_{n-1} & \beta_{n-1} & -\alpha_{n-1} & -\beta_{n-1} & & & & & \end{bmatrix} \tag{3.32} \]

\[ A_{Fc} = \begin{bmatrix} \gamma_{11} & \xi_{11} & -\gamma_{12} & -\xi_{12} & 0 & 0 & \cdots & 0 \\
0 & 0 & \gamma_{22} & \xi_{22} & -\gamma_{23} & -\xi_{23} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \gamma_{n-2,n-2} & \xi_{n-2,n-2} & -\gamma_{n-2,n-1} & -\xi_{n-2,n-1} & \cdots & 0 \\
\gamma_{n-1,n-1} & \xi_{n-1,n-1} & -\gamma_{n-1,n} & -\xi_{n-1,n} & -\xi_{n-1,n} & -\xi_{n-1,n} \end{bmatrix} \tag{3.33} \]

and

\[ A_R = \begin{bmatrix} 0 & \cdots & 0 & \frac{\omega}{c_n} \cos \frac{\omega}{c_n} L & -\frac{\omega}{c_n} \sin \frac{\omega}{c_n} L \end{bmatrix}^T \tag{3.34} \]

Further
\[ \alpha_i = \sin \frac{\omega}{c_i} L_i, \quad \beta_i = \cos \frac{\omega}{c_i} L_i, \]

\[ \gamma_{ij} = p_j \frac{\omega}{c_j} \cos \frac{\omega}{c_j} L_i, \quad \xi_{ij} = p_j \frac{\omega}{c_j} \sin \frac{\omega}{c_j} L_i \]

(3.35)

\[ i = 1, 2, \ldots, n - 1, \quad j = 1, 2, \ldots, n. \]

and

\[ z = (Q_1 \quad R_1 \quad Q_2 \quad R_2 \quad \cdots \quad \cdots \quad Q_n \quad R_n^T). \quad (3.36) \]

The Transcendental Eigenvalue Method introduced in section 3.1.2 in conjunction with the piecewise continuous model developed herein can be used to compute the spectral data of a non-uniform vibrating rod. To better understand the implications of the model developed the spectral analysis of axial vibration of rods with stepped cross-sectional area and exponentially varying cross-section area have been discussed and the results are compared with those from available approximate models.

### 3.2.2 Example 1: Axial Vibration in a Stepped Rod

Consider an axially vibrating rod with piecewise uniform cross-sectional area, of unit length as shown in Figure 3.5., fixed at \( x = 0 \) and attached to a spring of stiffness \( k \) at the other end \( x = 1 \). The system is governed by differential equation in Eq.3.20 subject to boundary conditions in Eq.3.21 where \( E \) is the modulus of elasticity, \( \rho \) is the density and \( A(x) \) is the area of cross-section and is defined as

\[ A(x) = \begin{cases} A_1, & 0 \leq x \leq x_1 \\ A_2, & x_1 \leq x \leq 1 \end{cases} \quad (3.37) \]

The axial displacement of the rod is defined as
\( u(x, t) = u_1(x, t), \quad 0 \leq x \leq x_1 \)
\( u_2(x, t), \quad x_1 \leq x \leq 1 \)  \hspace{1cm} (3.38)

Figure 3.5: Axially vibrating rod with piecewise uniform cross-sectional area.

This leads to the following eigenvalue problem

\[
\begin{align*}
  u_1''(x) + \mu^2 u_1(x) &= 0, \quad 0 \leq x \leq x_1 \\
  u_2''(x) + \mu^2 u_2(x) &= 0, \quad x_1 \leq x \leq 1 
\end{align*}
\]  \hspace{1cm} (3.39)

The above Eq. 3.39 is an eigenvalue problem and is subjected to the boundary conditions and continuity conditions

\[
\begin{align*}
  u_1(0) = 0, \quad u_1(x_1) = u_2(x_2), \quad A_1 u_1''(x_1) &= A_2 u_2''(x_1) 
\end{align*}
\]  \hspace{1cm} (3.40)

To derive the last boundary condition we consider the free-body diagram of the end section of the rod at \( x = 1 \), as shown in Figure 3.6

Figure 3.6: Free-body diagram of the rod depicting forces at the end \( x = 1 \)
\[ EA_2 u_2'(1, t) - k u_2(1, t) = 0 \]  

(3.41)

We thus seek the eigenvalues, \( \mu \), of Eq.3.39 that represent the roots of the frequency equation. The generalized solution to this eigenvalue problem that satisfies the boundary and continuity conditions in Eq.3.40 and Eq.3.41 are

\[
\begin{align*}
    u_1(x, t) &= b_1 \sin(\mu x) + b_2 \cos(\mu x), \quad 0 \leq x \leq x_1 \\
    u_2(x, t) &= b_3 \sin(\mu x) + b_4 \cos(\mu x), \quad x_1 \leq x \leq 1
\end{align*}
\]

(3.42)

where \( b_i, \ i = 1, 2, 3, 4 \) are constants of integration. Noting that \( s = 0 \) is not a natural frequency of the rod, but merely a trivial solution the matching and boundary conditions in Eq.3.40 and Eq.3.41 can be written in the matrix form

\[
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    S(\mu x_1) & C(\mu x_1) & -S(\mu x_1) & -C(\mu x_1) \\
    A_1 C(\mu x_1) & -A_1 S(\mu x_1) & -A_2 C(\mu x_1) & A_2 S(\mu x_1) \\
    0 & 0 & EA_2 \mu C(\mu) + k S(\mu) & -EA_2 \mu S(\mu) + k C(\mu)
\end{bmatrix}
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

(3.43)

Let \( k = 300, \ A_1 = 2, \ A_2 = 1 \) and \( x_1 = .5 \). The above Eq.3.43 transcendental eigenvalue problem of the form Eq.3.1. The non-trivial solution of the above equation is obtained using the method discussed in section 3.1.2 is shown in Table.3.2.1 below.

### 3.2.3 Example 2: Axial Vibration in an Exponential Rod

Consider the example of an axially vibrating rod, as shown in the Figure.3.7, with a cross sectional area \( A(x) = e^x \), modulus of elasticity \( E \), density \( \rho \) and length \( L = 1 \), fixed at the end \( x = 0 \) and free to oscillate at \( x = L \). The governing differential equation for the system shown is,
Table 3.1: First five natural frequencies of a stepped rod.

<table>
<thead>
<tr>
<th>i</th>
<th>(\omega_i^{(0)})</th>
<th>Number of iterations</th>
<th>(\omega_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>14</td>
<td>1.8979</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>10</td>
<td>4.34375</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>13</td>
<td>8.13909</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6</td>
<td>10.58616</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>9</td>
<td>14.37982</td>
</tr>
</tbody>
</table>

Figure 3.7: An axially vibrating exponential rod

\[
\frac{\partial}{\partial x} \left( E e^x \frac{\partial u(x,t)}{\partial x} \right) = \rho e^x \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 \leq x \leq 1, \quad t > 0
\]

\[
u(0, t) = 0
\]

\[
u'(1, t) = 0
\]

(3.44)

Assuming a general solution \(u(x,t) = v(x)\sin(\omega t)\) leads to the eigenvalue problem

\[
v''(x) + v'(x) + \lambda v(x) = 0, \quad \lambda = \omega^2 \rho/E
\]

\[
v(0) = 0
\]

\[
v'(1) = 0
\]

(3.45)

The eigenvalues of the system represented by Eq.3.45, are the roots of the frequency
equation given by (Ram, 1994 – a)

\[
\tan \left( \sqrt{\lambda - \frac{1}{4}} \right) = 2 \left( \sqrt{\lambda - \frac{1}{4}} \right) \tag{3.46}
\]

The advantage of applying the transcendental eigenvalue problem is in estimating the spectrum of continuously varying distributed parameter systems. In order to demonstrate the effectiveness of the algorithm, the given continuous system is replaced by an equivalent discrete model of order \( n = 20 \) with uniform elemental length \( h = 1/n \). The eigenvalue problem corresponding to the finite difference and finite element discrete models of the rod are obtained by Eq.2.13 and Eq.2.23 with their corresponding stiffness and mass matrices given by Eq.2.14, Eq.2.15, Eq.2.21 and Eq.2.22 respectively.

![Figure 3.8: Comparison between exact natural frequency of the rod and the approximate solutions obtained with the finite difference and finite element models of order \( n = 20 \).](image)

The eigenvalues from the discrete models of order \( n = 20 \) are shown in Figure 3.8, with apparent deviation of the eigenvalues from the exact solution beyond the first six. The twenty lowest eigenvalues of this problem were estimated using the transcendental eigenvalue approach using a model order of \( n = 3 \), i.e., the rod is divided into three sections of uniform length and cross-sectional area \( A(x) \) is averaged for each section as follows.

41
The matrix \( A(\lambda) \) of Eq.3.1 for this problem takes the form

\[
A(x) = \begin{cases}
(e^0 + e^{1/3})/2 & 0 \leq x \leq 1/3 \\
(e^{1/3} + e^{2/3})/2 & 1/3 \leq x \leq 2/3 \\
(e^{2/3} + e^1)/2 & 2/3 \leq x \leq 1
\end{cases}
\]  

(3.47)

The matrix \( A(\lambda) \) of Eq.3.1 for this problem takes the form

\[
A(\lambda) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sin(\zeta) & \cos(\zeta) & -\sin(\zeta) & -\cos(\zeta) & 0 & 0 & 0 \\
0 & 0 & \sin(\eta) & \cos(\eta) & -\sin(\eta) & -\cos(\eta) & 0 \\
A_1\cos(\zeta) & -A_1\sin(\zeta) & -A_2\cos(\zeta) & -A_2\sin(\zeta) & 0 & 0 & 0 \\
0 & 0 & A_2\cos(\eta) & -A_2\sin(\eta) & -A_3\cos(\eta) & -A_3\sin(\eta) & 0 \\
0 & 0 & 0 & 0 & \cos(\lambda) & -\sin(\lambda) & 0
\end{bmatrix}
\]  

(3.48)

where \( \zeta = \sqrt{\lambda}/3 \) and \( \eta = 2\sqrt{\lambda}/3 \).

Figure 3.9: Comparison between exact natural frequency of the rod and the approximate solution obtained using the transcendental eigenvalue method of order \( n = 3 \). ◦—Exact solution, ×—Transcendental solution

The results are shown in Figure 3.9 together with the exact analytical solution. It has thus been clearly demonstrated that the transcendental eigenvalue method has produced good approximation for entire spectrum. Further, although the model order and hence the size of the matrix (6 × 6) in the case of the transcendental approach is much smaller than
the $(20 \times 20)$ matrices used in the case of finite difference and finite element models, the results obtained are inferior in the case of the latter. With the ability to make an initial guess for $\omega^{(0)}$, the transcendental model, it must be noted can produce unlimited number of eigenvalues, while finite elements and finite difference models of order $n$ can at most produce $n$ eigenvalues, of which only about one-third of the eigenvalues are accurate.

3.3 Transverse Vibration in Non-Uniform Beams

3.3.1 Introduction

A mathematical model representing the transverse vibration in a non-uniform beam can be formulated using the approximation method described in the previous section 3.1.

Consider a transversely vibrating non-uniform Bernoulli-Euler beam, as shown in Figure 3.10. The differential equation of motion for the transversely vibrating non-uniform beam as shown in Figure 3.10 is governed by,

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2}{\partial x^2} w(x, t)\right) + \rho A \frac{\partial^2}{\partial x^2} w(x, t) = 0 \quad (3.49)$$

where $E(x)I(x)$ is the flexural rigidity of the beam and $\rho(x)A(x)$ is the mass per unit length. The non-uniform beam in Figure 3.10 is approximated by another non-uniform beam of order $n$ piecewise constant physical properties as shown in Figure 3.11, i.e. the
Young’s modulus, density, cross-sectional area and moment of inertia for each section of the piecewise beam is constant. The differential equation of motion for such an approximated beam is written as

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} \left( E_1 I_1 \frac{\partial^2}{\partial x^2} w_1(x, t) \right) + \rho_1 A_1 \frac{\partial^2}{\partial x^2} w_1(x, t) &= 0, \quad 0 < x < x_1 \\
\frac{\partial^2}{\partial x^2} \left( E_2 I_2 \frac{\partial^2}{\partial x^2} w_2(x, t) \right) + \rho_2 A_2 \frac{\partial^2}{\partial x^2} w_2(x, t) &= 0, \quad x_1 < x < x_2 \\
& \quad \vdots \\
\frac{\partial^2}{\partial x^2} \left( E_n I_n \frac{\partial^2}{\partial x^2} w_n(x, t) \right) + \rho_n A_n \frac{\partial^2}{\partial x^2} w_n(x, t) &= 0, \quad x_{n-1} < x < x_n
\end{align*}
\] (3.50)

The boundary conditions and matching continuity conditions are given by
\[ w_1(0, t) = 0, \quad \frac{\partial}{\partial x} w_1(0, t) = 0, \]
\[ E_n I_n \frac{\partial^2}{\partial x^2} w_n(1, t) = 0\]
\[ E_n I_n \frac{\partial^3}{\partial x^3} w_n(1, t) = 0\]

and for all \( t > 0 \)

\[ w_1(x) = w_2(x), \quad E_1 I_1 \frac{\partial^2}{\partial x^2} w_1(x) = E_2 I_2 \frac{\partial^2}{\partial x^2} w_2(x), \quad x = x_1 \]
\[ w_2(x) = w_3(x), \quad E_2 I_2 \frac{\partial^2}{\partial x^2} w_2(x) = E_3 I_3 \frac{\partial^2}{\partial x^2} w_3(x), \quad x = x_2 \]
\[ \vdots \]
\[ w_{n-1}(x) = w_n(x), \quad E_{n-1} I_{n-1} \frac{\partial^2}{\partial x^2} w_{n-1}(x) = E_n I_n \frac{\partial^2}{\partial x^2} w_n(x) \quad x = x_{n-1} \]

\[ \frac{\partial}{\partial x} w_1(x) = \frac{\partial}{\partial x} w_2(x), \quad E_1 I_1 \frac{\partial^3}{\partial x^3} w_1(x) = E_2 I_2 \frac{\partial^3}{\partial x^3} w_2(x) \quad x = x_1 \]
\[ \frac{\partial}{\partial x} w_2(x) = \frac{\partial}{\partial x} w_3(x), \quad E_2 I_2 \frac{\partial^3}{\partial x^3} w_2(x) = E_3 I_3 \frac{\partial^3}{\partial x^3} w_3(x) \quad x = x_2 \]
\[ \vdots \]
\[ \frac{\partial}{\partial x} w_{n-1}(x) = \frac{\partial}{\partial x} w_n(x), \quad E_{n-1} I_{n-1} \frac{\partial^3}{\partial x^3} w_{n-1}(x) = E_n I_n \frac{\partial^3}{\partial x^3} w_n(x) \quad x = x_{n-1} \]

Assuming harmonic motion

\[ w_1(x, t) = v_1 \sin(\omega t) \]
\[ w_2(x, t) = v_2 \sin(\omega t) \]
\[ \vdots \]
\[ w_n(x, t) = v_{n-1} \sin(\omega t) \]
and a generalized solution for \( v(x) \) is as shown in Eq. 3.54. Applying the boundary and matching conditions from Eq. 3.51 and matching conditions from Eq. 3.52 in the solution Eq. 3.54 leads to the formulation of transcendental eigenvalue in the form of Eq. 3.1 where,

\[
v_1(x) = P_1 \sin \left( \frac{\beta}{c_1} x \right) + Q_1 \cos \left( \frac{\beta}{c_1} x \right) + R_1 \sinh \left( \frac{\beta}{c_1} x \right) + S_1 \cosh \left( \frac{\beta}{c_1} x \right), \quad 0 < x < L_1,
\]

\[
v_2(x) = P_2 \sin \left( \frac{\beta}{c_2} x \right) + Q_2 \cos \left( \frac{\beta}{c_2} x \right) + R_2 \sinh \left( \frac{\beta}{c_2} x \right) + S_2 \cosh \left( \frac{\beta}{c_2} x \right), \quad L_2 < x < L_2,
\]

\[\vdots\]

\[
v_n(x) = P_n \sin \left( \frac{\beta}{c_n} x \right) + Q_n \cos \left( \frac{\beta}{c_n} x \right) + R_n \sinh \left( \frac{\beta}{c_n} x \right) + S_n \cosh \left( \frac{\beta}{c_n} x \right), \quad L_{n-1} < x < L_n,
\]

(3.54)

\[
A(\omega) = \begin{bmatrix}
- & - & A_{Ld} & - & - \\
- & - & A_{Ls} & - & - \\
- & - & A_{Cd} & - & - \\
- & - & A_{Cs} & - & - \\
- & - & A_{Cm} & - & - \\
- & - & A_{ Cf} & - & - \\
- & - & A_{ Rm} & - & - \\
- & - & A_{ Rf} & - & - 
\end{bmatrix}
\]

(3.55)

and the constant vector
\[ z = \begin{pmatrix} P_1 & Q_1 & R_1 & S_1 & P_1 & Q_2 & \cdots & R_n & S_n \end{pmatrix} \]  

(3.56)

The elements of matrix \( A(\omega) \) are given by the boundary conditions

\[
A_{Ld} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} 
\]  

(3.57)

\[
A_{Ls} = \begin{bmatrix} \frac{1}{c_1} & 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} 
\]  

(3.58)

\[
A_{Rm} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\sin \left( \frac{\beta L c_n}{c_n^2} \right) & -\cos \left( \frac{\beta L c_n}{c_n^2} \right) & -\sinh \left( \frac{\beta L c_n}{c_n^2} \right) & -\cosh \left( \frac{\beta L c_n}{c_n^2} \right) \end{bmatrix} 
\]  

(3.59)

\[
A_{Rf} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\cos \left( \frac{\beta L c_n}{c_n^3} \right) & -\sin \left( \frac{\beta L c_n}{c_n^3} \right) & -\cosh \left( \frac{\beta L c_n}{c_n^3} \right) & -\sinh \left( \frac{\beta L c_n}{c_n^3} \right) \end{bmatrix} 
\]  

(3.60)

and the continuity conditions
\[ A_{Cd} = \begin{bmatrix}
\bar{\alpha}_1 & \bar{\eta}_1 & \bar{\delta}_1 & \bar{\gamma}_1 & -\hat{\alpha}_1 & -\hat{\eta}_1 & -\hat{\delta}_1 & -\hat{\gamma}_1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{\alpha}_2 & \bar{\eta}_2 & \bar{\delta}_2 & \bar{\gamma}_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & -\hat{\alpha}_m & -\hat{\eta}_m & -\hat{\delta}_m & -\hat{\gamma}_m
\end{bmatrix} \] (3.61)

\[ A_{Cs} = \begin{bmatrix}
\bar{\eta}_1 & \bar{\alpha}_1 & \bar{\gamma}_1 & \bar{\eta}_1 & -\hat{r}_1 \bar{\eta}_1 & -\hat{r}_1 \bar{\alpha}_1 & -\hat{r}_1 \bar{\gamma}_1 & -\hat{r}_1 \bar{\delta}_1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{\eta}_2 & -\hat{r}_1 \bar{\eta}_2 & \bar{\gamma}_2 & \bar{\delta}_2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\hat{r}_m \bar{\gamma}_m & -\hat{r}_m \bar{\delta}_m \\
\end{bmatrix} \] (3.62)
\[
\begin{align*}
A_{Cm} &= \begin{bmatrix}
-\tilde{\alpha}_1 & -\tilde{\eta}_1 & \tilde{\delta}_1 & \tilde{\gamma}_1 & r_1 \tilde{\alpha}_1 & r_1 \tilde{\eta}_1 & r_1 \tilde{\delta}_1 & r_1 \tilde{\gamma}_1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\alpha}_2 & \tilde{\eta}_2 & \tilde{\delta}_2 & \tilde{\gamma}_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -r_m \hat{\delta}_m & -r_m \hat{\gamma}_m \\
\end{bmatrix} \\
A_{Cf} &= \begin{bmatrix}
-\tilde{\eta}_1 & \tilde{\alpha}_1 & \tilde{\gamma}_1 & \tilde{\eta}_1 & r_1 \tilde{\alpha}_1 & r_1 \tilde{\gamma}_1 & r_1 \tilde{\delta}_1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\eta}_2 & \tilde{\alpha}_2 & \tilde{\gamma}_2 & \tilde{\delta}_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -r_m \hat{\gamma}_m & -r_m \hat{\delta}_m \\
\end{bmatrix}
\end{align*}
\]

In the above set of equations we have boundary condition vectors \(A_{Ld}, A_{Ls}, A_{Rm}, A_{Rf}\) of size \((1 \times 4n)\), and continuity condition matrices \(A_{Cd}, A_{Cs}, A_{Cm}, A_{Cf}\) of size \((m \times 4n)\). The elements in the matrices are defined as,
\[ r_i = \frac{E_{i+1}I_{i+1}}{E_iI_i}, \]

\[ \bar{\alpha}_i = \sin \left( \frac{\beta x_i}{c_i} \right) \quad \bar{\eta}_i = \cos \left( \frac{\beta x_i}{c_i} \right) \quad \bar{\delta}_i = \sinh \left( \frac{\beta x_i}{c_i} \right) \quad \bar{\gamma}_i = \cosh \left( \frac{\beta x_i}{c_i} \right) \]

\[ \bar{\alpha}_i = \sin \left( \frac{\beta x_i}{c_{i+1}} \right) \quad \bar{\eta}_i = \cos \left( \frac{\beta x_i}{c_{i+1}} \right) \quad \bar{\delta}_i = \sinh \left( \frac{\beta x_i}{c_{i+1}} \right) \quad \bar{\gamma}_i = \cosh \left( \frac{\beta x_i}{c_{i+1}} \right) \]  

(3.65)

Solving the above equation using the Newton’s Iteration procedure described in section 3.1.2 gives us the eigenvalues and the eigenfunction solution to the problem vibration in the non-uniform beam. As in the case of a rod, the examples of a stepped beam and a tapered beam have been solved and compared with well-established results to justify the significance of transcendental eigenvalue method.

### 3.3.2 Example 1: Stepped Beam

Consider the problem of transverse vibration in a stepped beam as shown in Figure.3.12, fixed at the end \( x = 0 \) and attached to a spring of stiffness \( k \) at the end \( x = 1 \), and of a material with Young’s Modulus \( E \), and density \( \rho \). The cross-sectional area \( A(x) \) and the moment of inertia \( I(x) \) of the beam are defined as,
\begin{equation}
A(x) = \begin{cases}
A_1, & 0 \leq x \leq x_1 \\
A_2, & x_1 \leq x \leq 1
\end{cases}
\end{equation}

and

\begin{equation}
I(x) = \begin{cases}
I_1, & 0 \leq x \leq x_1 \\
I_2, & x_1 \leq x \leq 1
\end{cases}
\end{equation}

The governing differential equation of motion of such a system is given by,

\begin{equation}
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 w}{\partial x^2} \right) + \rho A(x) \frac{\partial^2 w}{\partial t^2} = 0
\end{equation}

subject to the boundary conditions

\begin{align*}
w_1(0, t) &= 0, \\
\frac{\partial}{\partial x} w_1(0, t) &= 0, \\
\frac{\partial^2}{\partial x^2} w_2(1, t) &= 0
\end{align*}

\begin{equation}
EI_2 \frac{\partial^3}{\partial x^3} w_2(1, t) - kw_2(1, t) = 0
\end{equation}

The continuity in the beam ensures
\[ w_1(x_1, t) = w_2(x_1, t), \]

\[ \frac{\partial}{\partial x} w_1(x_1, t) = \frac{\partial}{\partial x} w_2(x_1, t), \]

\[ \frac{\partial^2}{\partial x^2} w_1(x_1, t) = \frac{\partial^2}{\partial x^2} w_2(x_1, t) \]

\[ EI_1 \frac{\partial^3}{\partial x^3} w_1(x_1, t) = EI_1 \frac{\partial^3}{\partial x^3} w_2(x_1, t) \]

The general solution to the associated eigenvalue problem is of the form

\[ v_1(x) = z_1 \sin(\mu x) + z_2 \cos(\mu x) + z_3 \sinh(\mu x) + z_4 \cosh(\mu x) \]

\[ v_2(x) = z_5 \sin(\mu x) + z_6 \cos(\mu x) + z_7 \sinh(\mu x) + z_8 \cosh(\mu x) \]

For \( x_1 = 0.5, A_1 = 2, A_2 = 1, k = 10 \) the transcendental matrix in Eq. 3.1 is setup by substituting Eq.3.71 in Eqs. 3.69 and 3.70. The finite difference and finite element models of order \( n = 30 \) and their eigenvalues are compared with those obtained from the transcendental method. The results have been shown in Figure. 3.13
3.3.3 Example 2: Tapered Beam

Figure 3.14: Transverse vibration in a tapered beam

Consider a tapered beam with fixed-free configuration as shown in Figure 3.14. The height of beam varies linearly along the direction of length. Let the height of beam be $h_L$ at the left end and $h_R$ at the right end. Thus, the height distribution of beam along the length can be expressed by the following linear function,

$$h(x) = \left( h_L - \frac{h_L - h_R}{L} x \right)$$

(3.72)
If the thickness of the beam is $b$, the Young’s modulus $E$ and the density is $\rho$, then the mass per unit length and the flexural rigidity of the tapered beam are given by

$$\rho A(x) = \rho b \left( h_L - \frac{h_L - h_R}{L} x \right)$$

(3.73)

$$EI(x) = \frac{Eb}{12} \left( h_L - \frac{h_L - h_R}{L} x \right)^3$$

where $A(x)$ is the cross-sectional area and $I(x)$ is the moment of inertia along the length of the beam. The transcendental eigenvalue problem of model order $n = 5$ and $n = 10$ is as in Eq. 3.1 using the Newton’s eigenvalue iteration. The natural frequency derived from the transcendental eigenvalue approach ($\text{Singh2002} - a$) can be defined as,

$$f_{\text{trans}} = \frac{\beta_i^2}{2\pi L^2} \left( EI_L/\rho A_L \right)^{1/2}$$

(3.74)

where $\lambda_{\text{trans}} = \beta_i$ are the eigenvalues of Eq.3.55.

The exact solution of vibration of a tapered beam ($\text{Blevins,1979}$) has been compared with solution obtained from Eq. 3.74 for various $h_L/h_R$ ratios and is presented in the table 3.3.3 below.
Table 3.2: Eigenvalues of transverse vibration in a fixed free tapered beam.

<table>
<thead>
<tr>
<th>$\frac{h_L}{h_R}$</th>
<th>Exact Sol.</th>
<th>$\lambda_{FEM}$</th>
<th>$\lambda_{FDM}$</th>
<th>$\lambda_{TEP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\lambda_{n=5}$</td>
<td>$\lambda_{n=10}$</td>
<td>$\lambda_{n=5}$</td>
</tr>
<tr>
<td>1</td>
<td>1.87</td>
<td>1.8685</td>
<td>1.869</td>
<td>1.8751</td>
</tr>
<tr>
<td></td>
<td>4.69</td>
<td>4.5927</td>
<td>4.6940</td>
<td>4.6941</td>
</tr>
<tr>
<td></td>
<td>7.85</td>
<td>7.4845</td>
<td>7.8546</td>
<td>7.8548</td>
</tr>
<tr>
<td>2</td>
<td>1.95</td>
<td>1.9500</td>
<td>1.9500</td>
<td>1.8903</td>
</tr>
<tr>
<td></td>
<td>4.27</td>
<td>4.2241</td>
<td>4.2257</td>
<td>4.1245</td>
</tr>
<tr>
<td></td>
<td>6.90</td>
<td>6.6778</td>
<td>6.6821</td>
<td>6.9417</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>2.0007</td>
<td>2.0007</td>
<td>1.9195</td>
</tr>
<tr>
<td></td>
<td>4.10</td>
<td>4.0690</td>
<td>4.0704</td>
<td>3.9366</td>
</tr>
<tr>
<td>4</td>
<td>2.05</td>
<td>1.9419</td>
<td>2.0373</td>
<td>1.9419</td>
</tr>
<tr>
<td></td>
<td>4.00</td>
<td>3.8569</td>
<td>3.9858</td>
<td>3.8569</td>
</tr>
</tbody>
</table>
Chapter 4
Higher Order Area Approximation in Non-Uniform Rods and Beams

The previous chapter discusses how the use of a piecewise continuous model can help in analyzing continuous structures. The transcendental eigenvalue method developed predict results that are accurate and do so for the entire spectrum of vibration unlike the finite dimensional discrete model methods. In the case of uniform rods and beams or in the case of stepped beams the model order does not dictate the level of accuracy that can be obtained. In the case of constantly varying cross-section area this however is not true. The model order required to make accurate predictions in the case of such systems is limited by the method one uses to average the change in area within every piecewise element. Consider the transcendental eigenvalue problem developed in section 3.2.3. Here the exponential rod is divided into piecewise continuous elements of uniform right circular cylinders. The constant cross-sectional area of each cylinder is obtained by averaging the area using Eq. 3.47. The results obtained not only match the exact solution with better accuracy as compared to the discrete model methods but also do so for the entire spectrum. The piecewise continuous model with uniform cylindrical steps also requires a much smaller model order to predict the eigenvalues in the case of an exponential rod. However, intuitively, increasing the number of uniform cylindrical steps would cause the area function to more accurately resemble the exponential rod. Table 3.3.3 compares the first fifteen eigenvalues of axial vibration of an exponential rod obtained using various model orders from $n = 3$ to $n = 10$.

The deviation of the eigenvalues in each case from the exact solution as shown in the Table 3.3.3 demands a higher model order for a more accurate prediction. However, the complexity involved in the setting up and the computation of eigenvalues using the transcendental eigenvalue approach increases as the model order increases, thus rendering
Table 4.1: Eigenvalues of an axial vibration in an exponential rod predicted using transcendental eigenvalue method with various model orders.

<table>
<thead>
<tr>
<th>No.</th>
<th>Exact Sol.</th>
<th>Transcendental Eigenvalue Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3-Steps $\exp\left(\frac{i}{3} + \frac{1}{6}\right)$</td>
</tr>
<tr>
<td>Area $A_i(x) =$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.2693</td>
<td>1.2918</td>
</tr>
<tr>
<td>2</td>
<td>4.6327</td>
<td>4.7129</td>
</tr>
<tr>
<td>3</td>
<td>7.8073</td>
<td>8.1324</td>
</tr>
<tr>
<td>6</td>
<td>17.2529</td>
<td>17.5566</td>
</tr>
<tr>
<td>7</td>
<td>20.4020</td>
<td>20.1403</td>
</tr>
<tr>
<td>8</td>
<td>23.5460</td>
<td>23.5647</td>
</tr>
<tr>
<td>10</td>
<td>29.8326</td>
<td>29.5646</td>
</tr>
<tr>
<td>11</td>
<td>32.9754</td>
<td>32.9906</td>
</tr>
<tr>
<td>12</td>
<td>36.1118</td>
<td>36.4049</td>
</tr>
<tr>
<td>13</td>
<td>39.2604</td>
<td>38.9890</td>
</tr>
<tr>
<td>14</td>
<td>42.4027</td>
<td>42.4164</td>
</tr>
<tr>
<td>15</td>
<td>45.5387</td>
<td>45.8290</td>
</tr>
</tbody>
</table>

the method not as effective as one would expect it to be. To overcome this problem a frustum of a right circular cone has been used as a unit block to approximate the cross-sectional area of the rod. The formulation of the problem and the eigenvalues thus obtained have been discussed in the following sections.
4.1 Tapered Rod Approximation of Axial Vibration in an Exponential Rod.

The use of piecewise continuous right circular cylinders to approximate the area of an exponential rod as shown in Figure 3.7 we have seen provides a solution to the problem of axial vibration with better accuracy than the finite difference and finite element methods throughout the spectrum of vibration. To avoid the increase in the model order \( n \) of such piecewise cylinders, consider a tapered rod approximation of model order \( n = 3 \) as shown in Figure 4.1.

![Figure 4.1: Tapered rod approximation of an axially vibrating exponential rod.](image)

The area of cross-section of the rod is approximated by a linear function at any given point \( x \) along the length of the rod and is given by \( A(x) = a + bx = \exp(x) \). Each tapered rod is the frustum of a right circular cone, whose area constants \( 'a' \) and \( 'b' \) are given by,

\[
\begin{align*}
\frac{a_1}{b_1} &= \frac{0.333\exp(0)}{\exp(0.333) - \exp(0)} & 0 \leq x \leq 0.333 \\
\frac{a_2}{b_2} &= \frac{0.333(2\exp(0.333) - \exp(0.666))}{\exp(0.666) - \exp(0.333)} & 0.333 \leq x \leq 0.666 \\
\frac{a_3}{b_3} &= \frac{0.0334(2.994\exp(0.666) - 1.994\exp(1))}{\exp(1) - \exp(0.666)} & 0.666 \leq x \leq 1 \\
\end{align*}
\]

The equations of motion governing the vibration of such a piecewise continuous system,
their continuity and the boundary conditions are given by,

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial x} \left( E(a_i + b_i x) \frac{\partial u(x, t)}{\partial x} \right) = \rho(a_i + b_i x) \frac{\partial^2 u(x, t)}{\partial t^2}, \quad x_{i-1} \leq x \leq x_i, \quad t > 0 \quad i = 1, 2, 3 \\
u(0, t) = 0 \\
u_j(x_j, t) = u_{j+1}(x_j, t) \quad j = 1, 2.
\end{array} \right.
\]

\[A_j u'_j(x_j, t) = A_{j+1} u'_{j+1}(x_j, t) \quad j = 1, 2.\]

\[u'(1, t) = 0\]

(4.2)

The general solution to differential equation in Eq. 4.2, assuming a harmonic solution in time, is of the form,

\[
u_1(x) = C_1 J(0, \lambda(a + x)) + C_2 Y(0, \lambda(a + x)) \quad x_0 \leq x \leq x_1
\]

\[
u_2(x) = C_3 J(0, \lambda(a + x)) + C_4 Y(0, \lambda(a + x)) \quad x_1 \leq x \leq x_2
\]

\[
u_3(x) = C_5 J(0, \lambda(a + x)) + C_6 Y(0, \lambda(a + x)) \quad x_2 \leq x \leq 1
\]

(4.3)

where \(\lambda\) are the eigenvalues and \(J()\) and \(Y()\) are hyper-geometric first order Bessel functions of the first and second kind. Substituting Eq. 4.3 into Eq. 4.2 and setting them up in matrix form as in Eq. 3.30 we have a transcendental eigenvalue problem of the form \(A(\omega)z = 0\). The eigenvalues obtained by solving the system of equations using the Newton’s eigenvalue iteration approach have been tabulated below.

It is evident from Table. 3.3.3 that the accuracy of the eigenvalues predicted using the method stated in the above section is far better than the ones obtained by approximating the area of the rod using piecewise uniform cylinders. With about one third the model order the cumulative root mean square error reduces by an order of magnitude 10. Further, increasing the number of frustum of right circular cones per unit length will help in reducing the error in prediction of the eigenvalues. This method, it is safe to say, in general provides a better means of approximating the spectrum of axial vibration in a non-uniform rod.
Table 4.2: Eigenvalues predicted using transcendental eigenvalue method -Cylinder vs. Frustum of a Cone.

<table>
<thead>
<tr>
<th>No.</th>
<th>Exact Sol.</th>
<th>Transcendental Eigenvalue Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3-Steps</td>
</tr>
<tr>
<td>Area $A_i(x) =$</td>
<td>$\exp\left(\frac{i}{3} + \frac{1}{6}\right)$</td>
<td>$a + bx = \exp(x)$</td>
</tr>
<tr>
<td>1</td>
<td>1.2693</td>
<td>1.2918</td>
</tr>
<tr>
<td>2</td>
<td>4.6327</td>
<td>4.7129</td>
</tr>
<tr>
<td>3</td>
<td>7.8073</td>
<td>8.1324</td>
</tr>
<tr>
<td>4</td>
<td>10.9614</td>
<td>10.7160</td>
</tr>
<tr>
<td>5</td>
<td>14.1106</td>
<td>14.1388</td>
</tr>
<tr>
<td>6</td>
<td>17.2529</td>
<td>17.5566</td>
</tr>
<tr>
<td>7</td>
<td>20.4020</td>
<td>20.1403</td>
</tr>
<tr>
<td>8</td>
<td>23.5460</td>
<td>23.5647</td>
</tr>
<tr>
<td>10</td>
<td>29.8326</td>
<td>29.5646</td>
</tr>
<tr>
<td>11</td>
<td>32.9754</td>
<td>32.9906</td>
</tr>
<tr>
<td>12</td>
<td>36.1118</td>
<td>36.4049</td>
</tr>
<tr>
<td>13</td>
<td>39.2604</td>
<td>38.9890</td>
</tr>
<tr>
<td>14</td>
<td>42.4027</td>
<td>42.4164</td>
</tr>
<tr>
<td>15</td>
<td>45.5387</td>
<td>45.8290</td>
</tr>
</tbody>
</table>

4.2 Exact Solution to the Transverse Vibration of a Tapered beam

Example 2. in section 3.3.3 deals with the transverse vibration in a cantilevered tapered beam of constant width and linearly varying thickness $h$,

$$ h(x) = \left( h_L - \frac{h_L - h_R}{L} x \right) $$

(4.4)
where $h_L$ is the thickness at the cross-section $x = 0$, $h_R$ is the thickness attained at the cross-section $x = L$, where $L$ is the length of the beam as shown in Figure 3.14.

The mathematical model developed in section 3.3.3 uses a piecewise continuous approximation of a series of cuboids to represent the tapered beam. The cross-section area and the moment of inertia of each section of the beam are determined by Eq. 3.73. The use of piecewise cuboids in conjunction with the transcendental eigenvalue method predicts eigenvalues that are more accurate than the finite element model. However, similar to the exponential rod, the reduction of error in predicted eigenvalues increases the model order, thus increasing the matrix size and thus the computation complexity. There is however from Mabie and Rogers 1964 an exact solution available to predict the spectrum of a vibrating tapered beam.

Consider the governing differential equation,

$$
E \frac{d^2}{dx^2} \left[ \frac{b}{12} \left[ h_L - (h_R - h_L) \frac{x}{L} \right]^3 \frac{d^2 W}{dx^2} \right] - \rho b \omega^2 \left[ h_L - (h_R - h_L) \frac{x}{L} \right] W = 0.
$$

(4.5)

Introducing a new variable $X$

$$
X = h_L - (h_R - h_L) \frac{x}{L}
$$

(4.6)

Assuming a harmonic solution in time reduces 4.5 to

$$
\frac{d}{dX} \left( X^2 \frac{d}{dX} \right) \pm \lambda^2 X W = 0,
$$

(4.7)

the general solution of which is

$$
W(x) = \left[ C_1 J \left( 0, 2\lambda \sqrt{X} \right) + C_2 Y \left( 0, 2\lambda \sqrt{X} \right) + C_3 I \left( 0, 2\lambda \sqrt{X} \right) + C_4 K \left( 0, 2\lambda \sqrt{X} \right) \right] \frac{1}{\sqrt{X}}
$$

(4.8)
where

\[ \lambda^4 = \frac{12\rho\omega^2L^4}{E(h_R - h_L)^4} \] (4.9)

In Eq. 4.9 \( J() \) and \( Y() \) are first order Bessel functions of the first and second kind and \( I() \) and \( K() \) are first order modified Bessel functions of the first and second kind. Substituting the above solution into the boundary conditions

\[ W(x) = 0 \quad \frac{d}{dx} W(x) = 0 \quad x = 0 \]

\[ \frac{d^2}{dx^2} W(x) = 0 \quad \frac{d^3}{dx^3} W(x) = 0 \quad x = L \] (4.10)

we have a transcendental eigenvalue problem of the form \( A(\lambda)z = 0 \). Using the numerical method developed we now have the exact solution of a transversely vibrating tapered beam with \( h_L = 2 \) and \( h_R = 1 \). The first 10 eigenvalues obtained using a linearly varying cross-section are shown in the table below.

Table 4.3: Eigenvalues predicted using transcendental eigenvalue method -Step vs. Tapered beam.

<table>
<thead>
<tr>
<th>No.</th>
<th>Transcendental Eigenvalue Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10-Steps</td>
</tr>
<tr>
<td>1</td>
<td>1.9510</td>
</tr>
<tr>
<td>2</td>
<td>4.2690</td>
</tr>
<tr>
<td>3</td>
<td>6.8575</td>
</tr>
<tr>
<td>4</td>
<td>9.5525</td>
</tr>
<tr>
<td>5</td>
<td>12.7660</td>
</tr>
</tbody>
</table>

The ability to use the mathematical model with higher order polynomial approxima-
tions for area enables a possibility of further research in the case of vibration of structures with complex geometry. The results also show that the error is primarily due to the polynomial order used in the approximation of the physical parameters and is not inherent in the mathematical model or the numerical technique developed. Thus the ability to develop closed form solutions to problems involving non-uniform beams and rods makes this method very useful in spectrum analysis of various real-life structures. The following chapter discusses some applications of the transcendental eigenvalue method that has been developed. As an extension of the problems developed in the following chapter, some special problems in engineering that involve non-uniform geometry can be solved. The solution to such problems however need closed form solutions developed in this chapter. The better approximation of area as shown in the proceeding sections when applied to such problems will enable better prediction of eigenvalues and with significantly less computational resources.
Chapter 5
Transcendental Eigenvalue Method in Special Problems of Vibration, Buckling and Active Control

5.1 Application of Transcendental Eigenvalue Method is Active Control

Vibration analysis is an integral part of structural design in applications such as bridge construction and rotating machinery. The design involves determining the natural frequency of the structure so as to eliminate failure to operational loads. It is however necessary to be able to control or suppress any undesirable vibration or noise that might induced to wear and tear of the structure. Vibration can be controlled actively through the use of actuators or passively by means of adding passive elements such as springs and dampers. Frahm dynamic absorber, developed by Den Hartog, (1947) and Inman, (1994) is one such passive control mechanism device to counter steady state response of a harmonically excited system. The absorption of steady state motion of a system is based on the ability to place poles on the stable side of the spectrum as shown by Ram, (1998 – b) and Mottershed, (1998) . To illustrate the application of the transcendental eigenvalue method developed, consider the example of a multi-degree-of-freedom mass-spring damper system as shown in Figure 5.1.

![Figure 5.1: Four-degree-of-freedom Mass-Spring-Damper system](image)
The governing differential equation of motion of such a system is given by

\[ M\ddot{x} + C\dot{x} + Kx = bu(t) \]  \hspace{1cm} (5.1)

where \( M, C, K \) are the mass, damping and stiffness matrices respectively and

\[ u(t) = f^T\dot{x} + g^Tx \]  \hspace{1cm} (5.2)

where \( f \) and \( g \) are vectors that define the applied external force. If we were to induce a small delay \( \Delta \) in the time such that \( t = t - \Delta \) and substitute for \( u(t) \) from 5.2, the governing equation in 5.1 transforms to,

\[ M\ddot{x} + C\dot{x} + Kx = bf^T\dot{x}(t - \Delta) + bg^Tx(t - \Delta) \]  \hspace{1cm} (5.3)

Assuming the solution to displacement of such a system is, \( x(t) = ve^{st} \), we have

\[ (s^2M + sC + K)v = sbf^Tv e^{s(t-\Delta)} + bg^Tv e^{s(t-\Delta)} \]  \hspace{1cm} (5.4)

The above equation 5.4 when rearranged forms an eigenvalue problem that is transcendental in nature.

\[ (s^2M + s(C - f^Tbe^{-s\Delta}) + (K - g^Tbe^{-s\Delta}))v = 0 \]  \hspace{1cm} (5.5)

The four-degree-of-freedom system considered in Figure 5.1 has the following parameters.
\[ M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1.5 & -0.5 & 0 & 0 \\ -0.5 & 2.5 & -2 & 0 \\ 0 & -2 & 2.5 & -0.5 \\ 0 & 0 & -0.5 & 1.5 \end{bmatrix} \]

The amplification factor \( b \) and the force function’s co-efficients \( f \) and \( g \) are assumed to be,

\[ b = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T, \quad f = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad g = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T. \]

Table 3.3.3 shows poles of the system shown in Figure 5.1. The shift in the poles with time delay of \( \Delta = 0.01 \) and \( \Delta = 0.1 \) have also been tabulated.

Table 5.1: Variation of poles of a spring-mass-damper system with time delay

<table>
<thead>
<tr>
<th>No.</th>
<th>Poles</th>
<th>( \Delta = 0 )</th>
<th>( \Delta = 0.01 )</th>
<th>( \Delta = 0.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( -0.7859 + 1.2767i )</td>
<td>( -0.7859 - 0.245i )</td>
<td>( -1.8600 - 2.4341i )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( -0.8542 + 1.1410i )</td>
<td>( -0.8542 + 0.9787i )</td>
<td>( 1.8542 - 0.5647i )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( -0.0891 + 0.4632i )</td>
<td>( -0.0891 - 1.6466i )</td>
<td>( -2.0019 - 3.6712i )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( -1.0208 + 0.4342i )</td>
<td>( -1.0208 - 1.3321i )</td>
<td>( -0.1181 - 2.9912i )</td>
<td></td>
</tr>
</tbody>
</table>

### 5.2 Buckling of Rectangular Plate with Stepped Thickness

Consider a thin stepped plate of length \( a \) and breadth \( b \) as shown in Figure 5.2. The plate has stepped thickness along its breadth; one half of the breadth is of thickness \( h_1 \), while the other half is of thickness \( h_2 \). Sides 1 and 3 of the plate are simply supported while
sides 2 and 4 are unconstrained. Let $N_x$ be the constant force per unit length applied on the sides 1 and 3 as shown in the figure. The non-dimensional partial differential equations of the buckled plates are

$$
\begin{align*}
\frac{\partial^4 w_1}{\partial \bar{x}^4} + \frac{2a^2}{b^2} \frac{\partial^4 w_1}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{a^4}{b^4} \frac{\partial^4 w_1}{\partial \bar{y}^4} + \frac{N_x}{a \eta_1 \bar{D}} \frac{\partial^2 w_1}{\partial \bar{x}^2} &= 0, \quad 0 < \bar{y} < 1/2; \\
\frac{\partial^4 w_2}{\partial \bar{x}^4} + \frac{2a^2}{b^2} \frac{\partial^2 w_2}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{a^4}{b^4} \frac{\partial^4 w_2}{\partial \bar{y}^4} + \frac{N_x}{a \eta_2 \bar{D}} \frac{\partial^2 w_2}{\partial \bar{x}^2} &= 0, \quad 1/2 < \bar{y} < 1;
\end{align*}
$$

where,

$$\bar{x} = \frac{x}{a}, \quad \bar{y} = \frac{y}{a}, \quad \bar{h} = \frac{h}{a}, \quad \eta_1 = 1, \quad \eta_2 = \frac{h_2}{h_1} \text{ and } \bar{D} = \frac{E \bar{h}}{12 (1 - \nu^2)}$$

Assuming the plate buckles in $m$ sinusoidal waves in the $x$-direction, the solution to Eq.5.8 is of the form

$$w_i = \sin (m \pi \bar{x}) \tilde{v}_i, \quad i = 1, 2.$$
Substituting the solution (5.10) in (5.8) and simplify we obtain the differential equation

\[
\frac{d^4v_i}{d\bar{y}^4} - \frac{2m^2\pi^2b^2}{a^2} \frac{d^2v_i}{d\bar{y}^2} + \left( \frac{m^4\pi^4b^4}{a^4} - \frac{\lambda^2}{\eta_i^2} \right) v_i = 0, \quad i = 1, 2. \tag{5.11}
\]

where

\[
\lambda^2 = \frac{N_x m^2\pi^2b^4}{a^5}. \tag{5.12}
\]

The general solution of 5.12 is of the form

\[
v_i = A_i e^{-\alpha_i y} + B_i e^{\alpha_i y} + C_i e^{-\beta_i y} + D_i e^{\beta_i y}, \quad i = 1, 2. \tag{5.13}
\]

where

\[
\alpha_i = \sqrt{\frac{m^2\pi^2b^2}{a^2} - \frac{\lambda}{\eta_i^{3/2}}}, \quad \beta_i = \sqrt{\frac{m^2\pi^2b^2}{a^2} + \frac{\lambda}{\eta_i^{3/2}}}, \quad i = 1, 2 \tag{5.14}
\]

The boundary conditions of the plate at \(\bar{y} = 0\) are

\[
\left( \frac{1}{b^2} \frac{\partial^2 w_1}{\partial \bar{y}^2} \right)_{\bar{y}=0} + \left( \frac{v}{a^2} \frac{\partial^2 w_1}{\partial \bar{x}^2} \right)_{\bar{y}=0} = 0
\]

\[
\left( \frac{1}{b^3} \frac{\partial^3 w_1}{\partial \bar{y}^3} \right)_{\bar{y}=0} + \left( \frac{2 - v}{a^2b} \frac{\partial^3 w_1}{\partial \bar{x}^2 \partial \bar{y}} \right)_{\bar{y}=0} = 0
\]

and at \(\bar{y} = 1\) are

\[
\left( \frac{1}{b^2} \frac{\partial^2 w_2}{\partial \bar{y}^2} \right)_{\bar{y}=1} + \left( \frac{v}{a^2} \frac{\partial^2 w_2}{\partial \bar{x}^2} \right)_{\bar{y}=1} = 0
\]

\[
\left( \frac{1}{b^3} \frac{\partial^3 w_2}{\partial \bar{y}^3} \right)_{\bar{y}=1} + \left( \frac{2 - v}{a^2b} \frac{\partial^3 w_2}{\partial \bar{x}^2 \partial \bar{y}} \right)_{\bar{y}=1} = 0
\]

The continuity conditions at \(y = 1/2\) are
\( w_1 = w_2 \)

\[ \frac{\partial w_1}{\partial \bar{y}} = \frac{\partial w_2}{\partial \bar{y}} \]

\[ \frac{1}{b^2} \frac{\partial^2 w_1}{\partial \bar{y}^2} + \frac{v}{a^2} \frac{\partial^2 w_1}{\partial \bar{x}^2} = \eta_2 \left( \frac{1}{b^2} \frac{\partial^2 w_2}{\partial \bar{y}^2} + \frac{v}{a^2} \frac{\partial^2 w_2}{\partial \bar{x}^2} \right) \]

(5.17)

\[ \frac{1}{b^3} \frac{\partial^3 w_1}{\partial \bar{y}^3} + \frac{2 - v}{a^2 b} \frac{\partial^3 w_1}{\partial \bar{x}^2 \partial \bar{y}} = \eta_2 \left( \frac{1}{b^3} \frac{\partial^3 w_2}{\partial \bar{y}^3} + \frac{2 - v}{a^2 b} \frac{\partial^3 w_2}{\partial \bar{x}^2 \partial \bar{y}} \right) \]

Substituting Eqs. 5.10 and 5.13 in the boundary conditions and continuity conditions in 5.15, 5.16 and 5.17 and when written in the matrix form yields a transcendental eigenvalue problem of the form \( \mathbf{A}(\lambda) \phi = 0 \) where,

\[
\mathbf{A} = \begin{bmatrix}
\cdots & A_{\text{Top}} & \cdots \\
\cdots & A_{\text{Left}} & \cdots \\
\cdots & A_{\text{Disp}} & \cdots \\
\cdots & A_{\text{Slope}} & \cdots \\
\cdots & A_{\text{Moment}} & \cdots \\
\cdots & A_{\text{Force}} & \cdots \\
\cdots & A_{\text{Right}} & \cdots \\
\cdots & A_{\text{Bottom}} & \cdots 
\end{bmatrix}
\]

(5.18)

and

\[
\phi = \begin{bmatrix}
A_1 & B_1 & C_1 & D_1 & A_2 & B_2 & C_2 & D_2
\end{bmatrix}^T
\]

(5.19)

The variation of first critical load given by the first non-dimensional \( \lambda \) as a function of \( h_2/h_1 \) has been plotted in Figure 5.3 for \( a = b \) and a Poisson’s ratio \( \nu = 0.3 \). The
Figure 5.3: Variation of the first non-dimensional $\lambda$ as a function of $h_2/h_1$ for the case when $a = b$ and $\nu = 0.3$.

The solution obtained using the procedure developed in section 3.1.2 has been compared with that obtained using ANSYS with a $20 \times 20$ mesh size and '8 node Shell-93' elements. The solution obtained is void of discretization error as the transcendental eigenvalue problem developed represents a continuous model. Further to obtain a good level of accuracy using the discrete model method requires matrices of much larger size as compared to matrices of size $8 \times 8$ for above approach.
Chapter 6
Concluding Remarks and Future Work

The study of the dynamic behavior of systems is necessary in the prediction and control of the response of structures. There have been various mathematical models that have been developed to analyze these characteristics analytically. Being continuous systems in real life, structures with non-homogeneous distribution of physical parameters such as geometry, rigidity and density may not lend themselves to solutions that are in closed form. Well established mathematical models that have been developed in the literature researched through the course of the study attempt to approximate the behavior of continuous structures based on finite dimensional discretization. Mathematical models that are based upon the finite element and the finite difference techniques when used to approximate continuous systems require large computational capability and yet do not accurately estimate their entire spectrum. It is therefore necessary to develop mathematical models that can approximate the static and dynamic behavior of non-uniform continuous systems accurately and in totality.

The literature reviewed present governing differential equations of motion of various structures and their behavior at their boundaries. Careful consideration of problems in vibration, the active control of vibration and the buckling of these structures show that the governing equations of motion lead to a special class of problems known as eigenvalue problems. In the case of discrete structures these eigenvalue problems are algebraic in nature, i.e., the characteristic equation is polynomial in nature. There exist fairly simple techniques to determine the roots of such polynomial equations that can be used to determine the spectral behavior in such problems. However, in the case of continuous structures the characteristic equation is transcendental in nature.
Chapter 2. discusses, in detail, spectral analysis using classical finite dimensional mathematical models such as the finite difference and the finite element models for some continuous systems such as axially vibrating rods and beams. Finite dimensional mass and stiffness matrices representing continuous systems are developed by lumping their density and rigidity parameters at regular intervals in order to evaluate eigenvalues and eigenvectors of uniform and distributed parameter systems. It is shown with examples that such approximation techniques cannot capture the true spectral characteristic of the continuous system in consideration. Since these methods approximate the continuous structure into a finite number of discrete elements, using these matrix approximation techniques leads to an algebraic eigenvalue problem. The approximate the solution of continuous system obtained by using trial functions in the form of polynomials does not show sufficient accuracy. This is because no polynomial function can approximate the strong variation of transcendental equations accurately. It has been shown that though models of order \( n \) used in finite dimensional approximation provide about \( n/3 \) eigenvalues that are sufficiently accurate, they at the same time give inaccurate eigenvalues for almost two-third of the spectrum.

Difficulties associated in obtaining the solution for non-uniform rods and beams are demonstrated. Reasons for such inaccurate estimation were identified and studied. It is concluded that an accurate mathematical model is required to solve the associated direct and inverse eigenvalue problems. Considering the difficulties associated with existing research associated with continuous systems, a new low order analytical model is developed in Chapter 3. In this approximation technique, a non-uniform continuous system is approximated by a set of continuous systems with piecewise constant physical parameter distribution. Due to the availability of the closed-form solution for an individual element, approximated matrix eigenvalue problem can be developed, by applying the appropriate continuity conditions between the elements. Such an eigenvalue problem is termed as transcendental eigenvalue problem where the elements of the matrices involved are tran-
scendental functions. For non-uniform axially vibrating rods and transversely vibrating beams, transcendental eigenvalue problems were developed for arbitrary model order \( n \). Obtaining the solution for such eigenvalue problems generally involves the evaluation of determinants of the matrices. Evaluations of such determinant should be circumvented, as they require symbolic computations. In order to solve transcendental eigenvalue problems, a Newtons eigenvalue iteration method is developed. New mathematical models along with the algorithm were successfully used for evaluating the spectrum of non-uniform rod and beam structures. It has been shown that compared to finite element and finite difference methods, these mathematical models contain information of all the eigenvalues of the system. Moreover, all the eigenvalues obtained are of uniform accuracy.

The piecewise approximation of non-uniform rods and beams in Chapter 3. considers each piecewise element to be of constant cross-section area. Though the entire spectrum of eigenvalues predicted using this approximation are fairly accurate as compared to the finite dimensional discrete models, the determination of the \( A \) and its derivative becomes relatively complex as the model order increases. The error that exists in the solution obtained the transcendental eigenvalue approach is due to the error in approximation of the total area. To overcome this problem the physical parameter is refined by the use of tapered elements in Chapter 4. Piecewise elements with linearly varying cross-section area are used as building blocks in the mathematical model and the solution obtained in the case of vibration analysis of an exponential rod shows much better accuracy for a very low model order. This concept has been extended to the analysis of a tapered beam and the results show good accuracy as well. The ability to use the piecewise continuous approach to analyze non-uniform structures shows that the transcendental eigenvalue method is a truly effective method and has practical application in analyzing real-life structures with complex geometries.

The effectiveness of this mathematical model is evident with the various problems that
have been analyzed in Chapter 5. The applicability of this method in the active control of steady state vibration in multi-degree of freedom mass-spring damper systems, provides a scope for further research in the case of vibration control of continuous structures. Finally the mathematical model was used to solve buckling problems associated with stepped plates. The results show that the use of the transcendental eigenvalue approach reduces the model order required to solve such a problem and thus increase computational efficiency.

The research work can be summarized as follows:

1. A problems associated with finite discrete model representation of continuous systems e.g. rods and beams and plates have been researched.

2. An effective algorithm based on Newtons eigenvalue iteration method is developed. It uses an algebraic eigenvalue solver to find the eigenvalues of the transcendental eigenvalue problem iteratively.

3. Newtons eigenvalue iteration method along with transcendental mathematical model is solved to evaluate spectra of the non-uniform rods, which may not be accurately evaluated by other numerical methods.

4. This mathematical model is extended towards the solution of the vibration in non-uniform beams.

5. As a building block to approximate the physical parameters a piecewise continuous element with linearly varying cross-section area has been used in the case of non-uniform rods and beams.

6. The mathematical model developed using the tapered element shows better approximation of the eigenvalues with a drastic reduction in model order.

7. Special problems in vibration, buckling and active control of vibration have been solved using the developed mathematical model.
8. Based upon the observation of the results, a conjecture has been established by which a unique solution is identified.

9. Possible future contribution of the developed mathematical model and algorithms has been demonstrated by solving problems for vibration control and plate buckling analysis. The method of using higher order functions to represent the physical parameters can be extended to solve an array of problems involving non-uniform continuous structures. The analysis of vibration in plates and shells can also be performed using the transcendental approach.
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