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# Dynamic Resonant Scattering of Near-Monochromatic Fields

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# DYNAMIC RESONANT SCATTERING OF NEAR-MONOCHROMATIC FIELDS

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# Abstract

Certain universal features of photonic resonant scattering systems are encapsulated in a simple model which is a resonant modification of the famous Lamb Model for free vibrations of a nucleus in an extended medium. We analyze this “resonant Lamb model” to garner information on dynamic resonant scattering of near-monochromatic fields when an extended system is weakly coupled to a resonator. The transmitted field in a resonant scattering process consists of two distinct pathways: an initial pulse (direct transmission) and a tail of slow decay (resonant transmission). The resonant Lamb model incorporates a two-part scatterer attached to an infinite string with a continuous spectrum. The non-resonant part of the scatterer is associated with direct scattering; and the resonant part is associated with field amplification and delayed transmission. We provide a mathematical characterization of the “direct transmission” and the “resonant transmission” by analyzing the pole structure of the resolvent operator of the system. The coupling constant ($\gamma$), the proximity of resonance to the central frequency of incidence ($\eta$) and the spectral width ($\sigma$) of the incident pulse are three distinguished parameters that are small and affect resonance in the high-Q and near-monochromatic regime. The main objectives of this work are to analyze resonant amplification and transmission anomalies in the simultaneous High-Q and near-monochromatic regime as they depend on the three aforementioned parameters and to quantify the accuracy of coupled mode theory in that same regime.

# Chapter 1
# Introduction

## 1.1 Resonant Scattering

Scattering of electromagnetic plane waves by certain photonic devices give rise to various resonant phenomena under special conditions. Resonant scattering can be described as the phenomenon of resonant amplification of fields in the scatterer while the transmission of energy across the scatterer experiences sharp variations near certain resonant frequencies. Such resonance phenomena and anomalous transmission of energy are observed more generally in problems of scattering of waves by obstacles not only in electromagnetics but also in acoustics, in mechanical waves and matter waves.

In this study we confine our interest to the resonant interaction between electromagnetic waves and photonic scattering devices. These photonic devices include photonic crystals, photonic crystal slabs, photonic crystal fibers and many other devices that have been designed with the ability to control the flow of light [1]. The resonant interaction between electromagnetic waves and a photonic device may materialize in different ways and therein we are concerned only with the resonance of a particular nature.

We consider the situation in which the photonic scatterer supports source-free fields; electromagnetic fields at specific frequencies present inside the photonic device in the absence of any source field originating from the ambient medium outside the scatterer. These fields confined to the scattering device called bound states can be mathematically conceived as an eigen-state whose frequency is embedded in the frequency continuum of the extended states admitted by the scattering system.

These bound states are dynamically decoupled from the energy carrying waves in the ambient medium. If a bound state can be destroyed through radiation loss by coupling to the energy carrying waves in the ambient space under a perturbation of the structure, such a bound state is called a non-robust bound state. When these non-robust bound states are destroyed they become leaky modes radiating energy out of the device. These leaky modes materialize as high amplitude fields in the scatterer that are resonantly excited during the scattering of plane waves.

This type of non-robust bound states in a photonic scattering device referred to as "guided modes" are known to be associated with anomalous scattering behavior in the vicinity of the frequency and wavenumber of the mode. It is well documented that scattering anomalies associated with these non-robust guided modes are useful in the design of photonic devices. Such resonance phenomena are utilized in many photonic devices such as lasers, optical filters and light-emitting diodes and sensors. Guided modes and corresponding resonant phenomena appear in many different photonic structures and numerous studies that has been performed about them are abundant in literature. The existence of such guided modes in certain loss-less two-dimensional electromagnetic structures is discussed in [2] and in loss-less dielectric slabs is discussed in [3]. An analysis on the relation of anomalous transmission of energy across certain photonic slabs to leaky modes are carried out in [4]. Guided modes and resonant scattering under certain types of perturbations in slabs of two-phase dielectric photonic crystal materials have been presented in [5]. Guided modes and corresponding transmission anomalies in scattering of electromagnetic fields by metal films is discussed in [6]. The connection between anomalous transmission and particular types of guided modes on metal films called surface plasmons has been studied in [7, 8].

Resonance in a scattering system can be conceived as the result of the proximity of the system to a certain idealized one. This idealized system admits a discrete set of frequencies corresponding to the bound states of the scatterer. Meanwhile, the system admits a continuum of frequencies corresponding to extended states which are the plane waves in the ambient medium. Under perturbation of certain system parameters, this idealized system is destroyed by the interaction between the bound states inside the scatterer and the propagating plane waves in the ambient space. This interaction causes the bound states to be destroyed and the extended states near the bound state frequency are sharply modified.

When placed in the proper functional analytic setting, a frequency corresponding to a bound state is realized by an eigen-mode, eigenvalues of which are embedded in the continuous spectrum of an operator underlying the system. This continuous spectrum corresponds to the continuum of frequencies of the extended states (plane waves in the ambient medium). The perturbation of the idealized system corresponds to the dissolution of this embedded eigenvalue. The dissolution of the embedded eigenvalue coincides with the frequencies corresponding to the bound states attaining a small imaginary part and moving down to the lower half plane becoming complex resonances.

## 1.2 The Direct and the Resonant Pathways

The transmitted field in resonant scattering by a photonic device consists of two temporal pathways as described by Fan and Joannopoulos in [9]. The time sequence of a transmission process in resonant scattering is observed to consist of two distinct stages: an initial pulse and a tail of long decay. The presence of these two stages demonstrates the existence of two pathways in the transmission process. This phenomenon indicates that the guided modes in a resonant scatterer are resonantly exited only by a portion of the incident energy of the source fields. The initial pathway corresponds to a direct transmission process that is not responsible for the resonant excitation, which instead carries a portion of the incident energy through the scatterer without a time delay. The tail of long decay corresponds to an indirect resonant transmission process in which the remaining portion of the incident energy excites the guided modes, resonantly enhancing the fields in the resonator, and then slowly leaks out. The reflection process in resonant scattering of waves also has two temporal pathways; resonant and direct akin to that of the transmission process.

The Fourier transform of the initial pulse should have a shape which closely resembles that of the incident pulse, whereas the Fourier transform of the decaying tail should be a Lorentzian line shape associated with a resonance. The understanding of this phenomenon in resonant scattering has given new insight into related studies; resonance in scattering of waves is understood to occur from the interference between the direct and the resonance-assisted indirect pathway. Hence, the properties of the resonant transmission process is determined by the interference between the direct and the resonant pathways.

This insight has led to the development of an intuitive theory based on a temporal coupled-mode formalisms to explain complex features of certain resonance phenomena in optical resonant scatterers.

### 1.3 Parameters that Affect Resonance

The resonant features of photonic resonant scattering depend on certain aspects of the scatterer and properties of the incident field originating in the ambient medium. The most prominent resonant features that can be analyzed are field amplification and the sharp dips and peaks in the transmission coefficient characteristic of anomalous transmission of energy across the scatterer. The transmission coefficient is a measure of the portion of the energy transmitted across the scatterer. Analyzing how the transmission coefficient depends on perturbations of system properties will give us detailed insight in to dependence of resonant scattering characteristics of a photonic device on the system parameters.

The tuning of sharp resonant scattering features has been examined in discrete [10, 11] and in continuous [12, 13] models, using a variety of different resonant scatterers. A connection is established between the line shape of resonant transmission across a scatterer in a discrete model and asymmetry of the structure and the field in [14]. The dependance of resonant features on one angle of incidence is analyzed in [15, 16] and on two angles of incidence is presented in [17]. An analysis on the dependence of certain resonant scattering anomalies on non-linearity and the coupling constant is carried out in [18].

Analyzing the delicate behavior of resonance features of a system with a high quality factor and near-monochromatic incidence will be important in providing insight in to resonant scattering characteristics of a photonic device. A high quality factor results in high resonant amplification of the transmitted field and slow energy decay from the resonator within a narrow frequency band while the near-monochromatic regime refers to a resonant system operating under the incidence of a spectrally localized field of finite energy. Under these conditions the resonant features will depend primarily on the constant of coupling between the resonant scatterer and the incident waves, the spectral width of the transmission resonance, the difference between the central frequencies of the resonance and the source field and the angle of incidence. We will focus our study on the three parameters excluding the angle of incidence in the regime when all three parameters are small.

### 1.4 Temporal Coupled Mode Theory for Fano Resonances

The notion of coupling of modes is found extensively in the study of vibrational systems. An electromagnetic mode can generally be viewed as electromagnetic energy that exists independent of other electromagnetic power. Different modes belonging to the same system or to different systems, can exchange energy through a coupling perturbation. Coupled mode theory approximates the solutions to complex problems associated with the interaction of different modes of energy based on known solutions for simpler systems [19].

Temporal coupled mode theory is the application of the coupled mode formalism to coupling of modes in time. Temporal coupled mode theory allows a wide range of devices and systems to be modeled as one or more coupled resonators. The assumptions about the system generally made in applying a temporal coupled mode approach are weak mode coupling, linearity, time-reversal symmetry and energy conservation. Temporal coupled mode formalisms for resonance in optical resonators has been derived in recent years. [20]

We wish to approximately describe the decay rate of resonance in the resonant Lamb model using a temporal coupled mode approach and compare it to the true dynamics of the system.

# Chapter 2
# A Resonant Lamb Model

With the objective of investigating certain sensitive features of dynamic resonance characteristic to a photonic resonant scattering system, we analyze a specific simplified model that encapsulates the essential features of a photonic resonant scattering system. Our model is a resonant modification of the famous Lamb model devised by H. Lamb [21] to study the interaction between an oscillator and a continuum. Our resonant Lamb model consists of a two-part scatterer attached to an infinite string. The non-resonant part of the scatterer is associated with direct scattering while the resonant part is associated with resonant scattering (field amplification and delayed scattering). The infinite string models the ambient medium. Differing versions of Lamb-type models has been used in studies to serve as prototypes for diverse scattering phenomena. Examples include irreversibility [22], state transitions [23], gyroscopic instability [24], and nonlinear bi-stability [18].

## 2.1 Characteristics of a Resonant System

The fundamental mechanism of resonance associated with resonant scattering of waves through a photonic device can be describes by analyzing the interaction between guided electromagnetic modes of the scatterer and plane waves in the ambient medium. Certain photonic devices act both as a guide of electromagnetic waves as well as a scatterer of waves that originate from sources exterior to it. The wave-vector parallel to the waveguide admits a discrete set of frequencies corresponding to guided modes but also admits a continuum of frequencies corresponding to plane waves in the ambient space. An open waveguide is one that is in contact with the ambient space and in this case fields originating in the ambient space interact with the guided modes. The frequency of such a guided mode can be conceived as an eigenvalue that is embedded in the continuous spectrum corresponding to a specific wave-vector.

The interaction between these guided electromagnetic modes in the scatterer and plane waves in the ambient medium results in the dissolution of the eigenvalue and this corresponds to the destruction of the guided mode. In this situation the real frequency of a perfect guided mode attains a small imaginary part and moves to the lower half plane. As a physical manifestation of this phenomenon, the system exhibits delicate resonance phenomena. The most prominent resonant characteristics exhibited are high field amplification and sharp variations of the transmitted energy across the scatterer. The characteristic peak-dip shape of resonant transmission anomalies [By transmission anomalies, we refer to sharp peaks and dips in the graph of the transmission coefficient as a function of frequency] is often called a Fano resonance. The universal features of the type of photonic resonant scattering system we consider can be identified as the following:

- The scatterer admits a set of idealized guided modes
- These guided modes can be conceived as eigen-modes with eigenvalues that are embedded in the continuous spectrum of the extended states corresponding to the plane waves of the ambient medium
- These guided modes are coupled to the plane waves of the ambient medium.
- This coupling results in the destruction of the guided mode and the plane waves with frequencies close to the frequency of the guided mode is resonantly scattered.

In order for us to study characteristics of photonic resonant scattering by analyzing a model we must ensure that the used model encapsulates the essential features of a photonic resonant scattering system described above.

## 2.2 A Resonant Lamb Model

To study certain aspects of a resonant scattering system we will analyze a physical model wherein the modes of the system are not postulated but arise from the model itself. We analyze what we feel is the simplest model that exhibits the essential features of photonic resonance mentioned above. This model is inspired by the famous “Lamb model” proposed by H. Lamb to elucidate the interaction between an extended medium and a nucleus [21]. Lamb’s model consists of an infinite string coupled to a spring-mass system, in which radiation losses are felt by the mass as equivalent to instantaneous linear damping. In our modification, illustrated in Figure 2.1, a point mass is attached to the string, and that point mass is in turn weakly coupled to a separate spring-mass resonator.

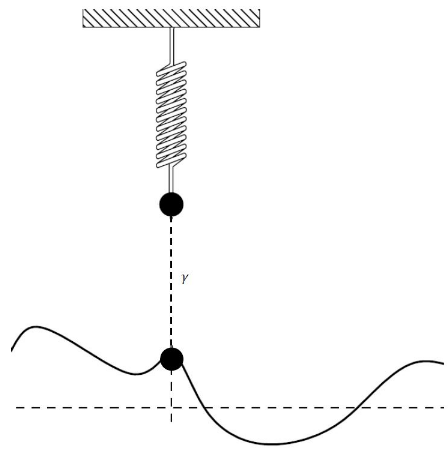


FIGURE 2.1. Resonant Lamb Model

In our model the point-mass along with the spring mass oscillator serves as the scatterer of the system while the string models the ambient medium. Note that even though a photonic scatterer with a complex structure does not have distinguishable resonant and non-resonant parts, our simplified model actually has two such parts with a clear distinction. The point mass on the string serves as the non-resonant part of the scatterer, and the spring-mass serves as the resonant part. The free oscillations of the spring-mass resonator models the guided modes in a photonic resonant scattering system. The natural frequencies $\pm\omega_0$ are eigenvalues for the full system that are embedded in the continuous spectrum of extended states of the string.

The model of resonant scattering that we analyze is illustrated in detail in Figure 2.2. The transmission line (the string) modeling the ambient space, in which disturbances $u(x,\ t)$ travel according to the linear wave equation, has a linear mass density of $\tau$. The two-part scatterer consisting of a non-resonant part and a resonant part is attached to the string at $x = 0$. The displacements of the non-resonant and the resonant parts are denoted consecutively by $y(t)$ and $z(t)$. The spring constant of the resonator is $K_0$. The coupling between the resonant and the non-resonant parts are physically provided by a mass-less spring with spring constant $k_0$.

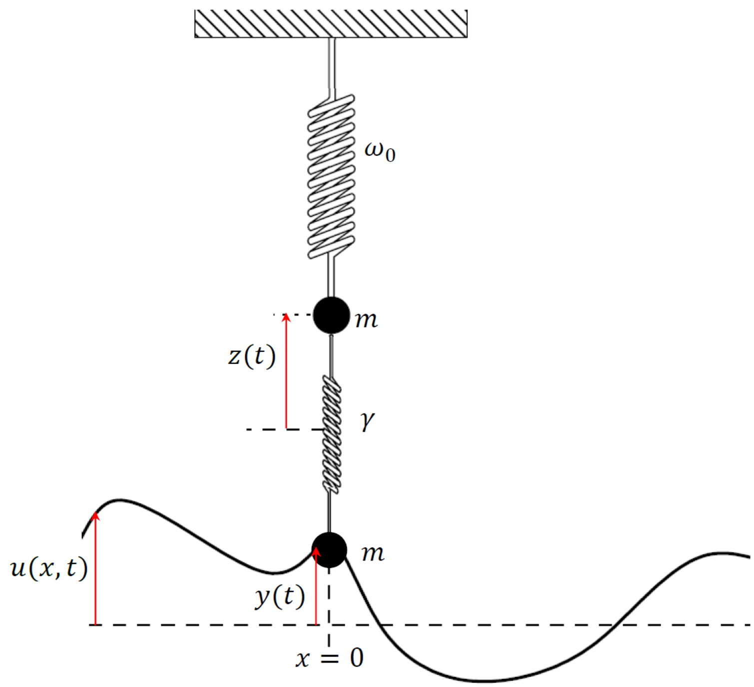


FIGURE 2.2. Detailed Resonant Lamb Model

The equations of motion for the system are the following.

On the string for $x \neq 0$

$$u_{tt} = c^2 u_{xx}$$

For the non-resonant scatterer at $x = 0$ we have,

$$m\ddot{y} = k_0\,(z - y) + \tau[u_x]_0\ , \qquad y\,(t) = u\,(0, t)$$

Letting $\gamma = \frac{k_0}{m}$ and $\beta = \frac{\tau}{m}$ we get

$$\ddot{y} = \gamma \ (z - y) + \beta \ [u_x]_0$$

For the resonant part of the scatterer (the spring-mass oscillator) we have

$$m\ddot{z} = -K_0 z - \gamma_0 (z - y)$$

with $\gamma = \frac{\gamma_0}{m}$ and letting ${\omega_0}^2 = \frac{K_0}{m}$ we get

$$\ddot{z} = -{\omega_0}^2 z + \gamma (y - z)$$

So the equations governing the system becomes;

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad x \neq 0$$

$$\ddot{y}(t) = \gamma \ (z(t) - y(t)) + \beta \ [u_x]_0, \qquad y(t) = u(0,t)$$

$$\ddot{z}(t) = -{\omega_0}^2 z(t) + \gamma (y(t) - z(t))$$

The constant $c$ is the velocity of waves in the string, $\beta$ controls the force exerted by the string on the point-mass, $\gamma$ is the coupling constant between the non-resonant and the resonant parts of the scatterer, and $\omega_0$ is the natural frequency of the resonator.

We have devised this resonant Lamb model so that, when $\gamma = 0$, the string with the point-mass is completely decoupled from the spring-mass. The free oscillation of the spring-mass is then trivially an infinite-lifetime, finite-energy state, and its frequencies $\pm\omega_0$ are eigenvalues for the full system that are embedded in the continuous spectrum of extended states of the string. When the coupling $\gamma$ is turned on, these eigenvalues become resonances in the lower-half complex plane with imaginary part on the order of $\gamma^2$. This system will display sharp resonant behavior for scattering of waves with frequencies in the proximity of $\omega_0$.

It should be noted that the parameter of the system that can be perturbed in order to facilitate resonance in our simple model is the frequency of the incident waves (the central frequency in case of a non-monochromatic incidence). However in a general photonic system, perturbation of any one or more of the parameters such as the structure of the scatterer, frequency of incidence or angle of incidence could result in resonant scattering of waves and anomalous transmission of energy across the scatterer.

## 2.3 The Monochromatic Case

In order to illustrate the resonant behavior of this model we will first demonstrate the scattering of a monochromatic wave across the scatterer. We will consider a monochromatic source field in the string that travel from left to right and are partly reflected by the scatterer and partly transmitted across it as illustrated in figure 2.3.

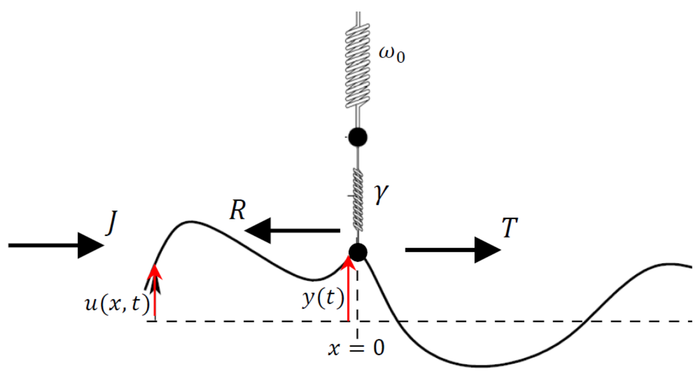


FIGURE 2.3. Monochromatic Incidence

Suppose that the function $J(\xi)$ is a unit monochromatic incident field, and $R(\xi)$ and $T(\xi)$ are consecutively the reflected and transmitted fields. Since the incident field, being monochromatic, is a purely oscillatory wave travelling from left to right we have $J(\xi) = e^{ik(x-ct)}$, $R(\xi) = Re^{ik(x+ct)}$ and $T(\xi) = Te^{ik(x-ct)}$.

Now since the wave equation governs the motion of disturbances in the infinite string the solution to the scattering problem for our model will be of the form;

$$u(x,t) = e^{ik(x-ct)} + Re^{ik(x+ct)} \ ; \quad x < 0$$

$$u(x,t) = Te^{ik(x-ct)}\;; \quad x>0$$

$$y(t) = u(0,t) = Te^{-ikct}$$

$$z(t) = Ze^{-ikct}$$

Solving for $T,\ R$ and $Z$ we obtain

$$T = \left[\frac{2i\beta k\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)}{\left[\left({\omega_0}^2 - c^2k^2\right)c^2k^2 + \gamma\left(2c^2k^2 - {\omega_0}^2\right)\right]\ +\ i\ \left[2\beta k\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)\right]}\right]$$

$$R = \left[\frac{-\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)\left(c^2k^2 - \gamma\right) - \gamma^2}{\left[\left({\omega_0}^2 - c^2k^2 + \gamma\right)\left(c^2k^2 - \gamma\right) + \gamma^2\right]\ +\ i\ \left[2\beta k\left({\omega_0}^2 - c^2k^2 + \gamma\right)\right]}\right]$$

$$Z = \left[\frac{2i\gamma\beta k}{\left[\left({\omega_0}^2 - c^2k^2\right)c^2k^2 + \gamma\left(2c^2k^2 - {\omega_0}^2\right)\right]\ +\ i\ \left[2\beta k\left({\omega_0}^2 - c^2k^2 + \gamma\right)\right]}\right]$$

To demonstrate the resonant feature of this "resonant Lamb model" we consider the amplitude of the resonator, given by $Z$. When the frequency of the incident harmonic field is away from the natural frequency $\omega_0$ of the resonator the spring-mass is not exited. However, when $\omega$ is in the proximity of $\omega_0$, the resonator is resonantly amplified. This is evident from the plot of $log(z)$ against frequency of the incident field $\omega$ in figure 2.4 (For $\gamma = 0.8, \alpha = 1.5$ and $\omega_0 = 1.2$).

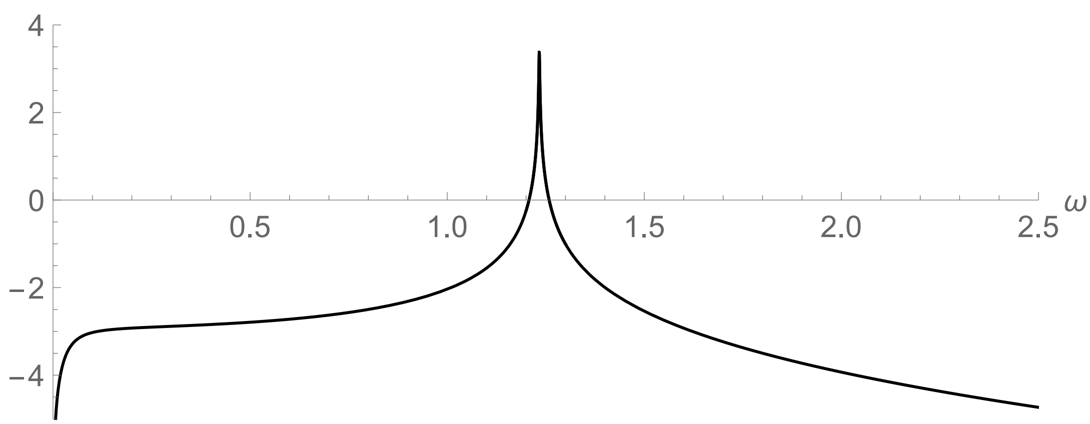


FIGURE 2.4. Resonant amplification of the model

To understand the energy transfer across the scatterer, we consider the monochromatic transmission coefficient defined by $|T|^2$ which is a measure of the portion of energy transmitted across the scatterer. With $k_0 = \frac{\omega_0}{c}$ we get

$$|T(k)|^2 = \left[ \frac{4\tau^2 k^2 \left[c^2\left({k_0}^2 - k^2\right) + \gamma\right]^2}{\left[c^4\left({k_0}^2 - k^2\right)k^2 + \gamma c^2\left(2k^2 - {k_0}^2\right)\right]^2 + 4\tau^2 k^2 \left[c^2\left({k_0}^2 - k^2\right) + \gamma\right]^2} \right]$$

Plotting the transmission coefficient $|T(\omega)|^2$ against frequency $\omega$ we observe sharp resonant behavior near the natural frequency $\omega_0$ of the resonator. Here, $\gamma = 0.8, \alpha = 1.5$ and $\omega_0 = 1.2$. The dashed line represents the Transmission Coefficient of the system with no coupling ($\gamma = 0$) between the resonator and the point-mass scatterer.

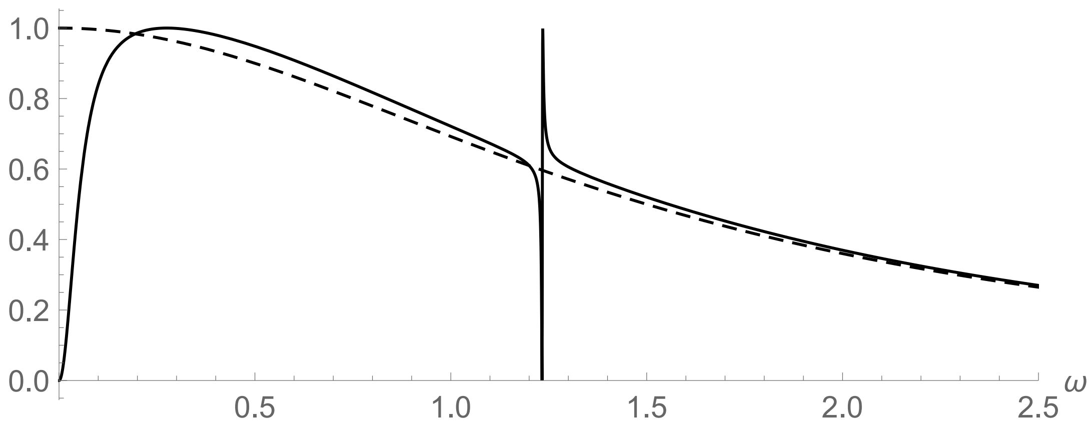


FIGURE 2.5. Transmission Coefficient

The resulting sharp peak-dip anomaly of the transmission coefficient near the resonant frequency is a characteristic of Fano resonance. The term "Fano resonance" which is often used to refer to this type of a double-spiked anomaly observed in many resonant systems. The name "Fano resonance" is attributed to Ugo Fano [25] who derived the following formula for such a line shape:

$$f_q(e) = k\frac{|q+e|}{1+e^2}\,, \qquad e = \frac{E - E_r}{(\Gamma/2)}$$

where $k$ is a constant, $E$ is the incident frequency, $E_r$ is the resonant frequency, $\Gamma$ is the width of the resonance anomaly at half the maximum height and $q$ is a parameter that controls the relative size of the peak and dip as a function of energy.

The formula for the transmission coefficient as a function of the wave number of the incident monochromatic wave in our model is akin to the above formula for the Fano line shape.

$$|T(k)|^2 = \frac{4\tau^2 k^2 \left[c^2\left({k_0}^2 - k^2\right) + \gamma\right]^2}{\left[c^4\left({k_0}^2 - k^2\right)k^2 + \gamma c^2\left(2k^2 - {k_0}^2\right)\right]^2 + 4\tau^2 k^2 \left[c^2\left({k_0}^2 - k^2\right) + \gamma\right]^2}$$

We note that;

$$|T(k)|^2 = \begin{cases} 0;\ \text{when}(c^2\left({k_0}^2 - k^2\right) + \gamma = 0 \\ 1;\ \text{when}(c^4\left({k_0}^2 - k^2\right)k^2 + \gamma c^2\left(2k^2 - {k_0}^2\right) = 0 \end{cases}$$

We explicitly calculate

$$|T(k)|^2 = \begin{cases} 0,\ \text{when} k = \sqrt{{k_0}^2 + \frac{\gamma}{c^2}} \\ 1,\ \text{when} k = \sqrt{\frac{{k_0}^2}{2} + \frac{\gamma}{c^2} + \sqrt{\frac{{k_0}^4}{4} + \frac{\gamma^2}{c^4}}} \end{cases}$$

Since  and $R$ defined above are the transmission and reflection for this system under unit incidence of a monochromatic wave at wave number $k$, $T$ and $R$ are in fact the Fourier coefficients for the transmission and reflection of this system under any incident field. Therefore we denote

$$\hat{t}\,(k) = \left[\frac{2i\beta k\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)}{\left[\left({\omega_0}^2 - c^2k^2\right)c^2k^2 + \gamma\left(2c^2k^2 - {\omega_0}^2\right)\right]\ +\ i\ \left[2\beta k\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)\right]}\right]$$

$$\hat{r}\,(k) = \left[\frac{-\ \left({\omega_0}^2 - c^2k^2 + \gamma\right)\left(c^2k^2 - \gamma\right) - \gamma^2}{\left[\left({\omega_0}^2 - c^2k^2 + \gamma\right)\left(c^2k^2 - \gamma\right) + \gamma^2\right]\ +\ i\ \left[2\beta k\left({\omega_0}^2 - c^2k^2 + \gamma\right)\right]}\right]$$

We can make use of these Fourier coefficients to study the behavior and the resonant features of the resonant lamb model under the incidence of a general non-monochromatic wave pulse in the Fourier domain. This would give us insight in to the anomalous transmission of energy across the scatterer and also enable us to quantify the delicate behavior of resonance features of our model and hence all photonic resonant scattering systems that has the basic resonant features of our model.

In order to better examine and understand the time dynamics of the transmitted pulse of this resonant transmission process (under the incidence of a general non-monochromatic wave pulse), we would have to delve in to the solutions to the scattering problem and analyze it in the spacial domain.

# Chapter 3

# Resonant Scattering of a Non-monochromatic Field

In the previous chapter we analyzed the resonant scattering of a monochromatic wave incident on the scatterer in our Resonant Lamb Model. We observed sharp resonant features when the incident frequency was in close proximity to the natural frequency of the spring-mass oscillator (the resonant part of our model). In reality however, a monochromatic source field is only an idealization which requires the source of this monochromatic wave to have oscillated as a sine wave for infinite past time and to continue to do so for infinite future time. Further in the monochromatic case the total energy of the source field is infinite. In this chapter we will analyze the more realistic situation when a source field with finite energy undergoes resonant scattering. Here we wish to study certain aspects of the resonant scattering of a wave pulse incident upon our Resonant Lamb Model and hence understand certain universal features of resonant scattering of a non-monochromatic field.

Non-monochromatic incidence will cause significant alterations to the crisp frequency domain picture of the resonant features of the scattering process. It also gives rise to further questions in relation to the dynamical nature of the resulting resonance process, such as how the resonant field would build up and decay with time and the corresponding time dynamics of the transmitted field.

One important consequence of non-monochromatic resonant scattering is the partition of energy into direct and resonant scattering channels resulting in two temporal pathways in the transmission and reflection processes.

In this chapter we will first place the scattering problem in a rigorous functional analytic setting. Within this setting we will compute the transmitted field for our Resonant Lamb Model for a general non-monochromatic incidence. We will then provide a rigorous definition of the two distinct temporal pathways in the transmission process of a non-monochromatic field. The transmission coefficient of a scattering system under non-monochromatic incidence will also be discussed.

### 3.1 Functional Analytic Formulation

Here we wish to discuss the mathematical formulation of the solution to the scattering problem for our model.

Recall that the equations for the unforced system is given by

$$u_{tt}\left(x,t\right)=c^{2}u_{xx}\left(x,t\right), \qquad x\neq 0$$

$$\ddot{y}\left(t\right)=\gamma\ \left(z\left(t\right)-y\left(t\right)\right)+\beta\ \left[u_{x}\right]_{0}\ , \qquad y\left(t\right)=u\left(0,t\right)$$

$$\ddot{z}\left(t\right)=-{\omega_{0}}^{2}z\left(t\right)+\gamma\left(y\left(t\right)-z\left(t\right)\right)$$

Applying the Fourier-Laplace transform we obtain

$$-\omega^{2}U\left(x,\omega\right)=c^{2}U_{xx}\left(x,\omega\right), \qquad x\neq 0$$

$$-\omega^{2}Y\left(x,\omega\right)=\gamma\ \left(Z-Y\right)+\beta\ \left[U_{x}\right]_{0}\ , \qquad Y\left(\omega\right)=U\left(0,\omega\right)$$

$$-\omega^{2}Z\left(\omega\right)=-{\omega_{0}}^{2}Z\left(\omega\right)+\gamma\left(Y-Z\right)$$

To place the problem in a proper functional analytic setting we consider the frequency domain problem in the Hilbert space $\mathcal{H}=L^{2}(\mathbb{R})\bigoplus\ \mathbb{C}^{2}$.

We now define the operator $L$ on the domain

$$\mathcal{D}(L)=\left\{\left[U,Y,Z\right]^{T}\in\mathcal{H}\mid U\in H^{2}\left(\mathbb{R}^{*}\right)\ ,\ U\left(0^{-}\right)=U\left(0^{+}\right)=Y\right\}\subseteq\mathcal{H}$$

by

$$L\begin{bmatrix}U\\Y\\Z\end{bmatrix}=\begin{bmatrix}-c^{2}U_{xx}\\-\beta\ \left[U_{x}\right]_{0}-\gamma\ \left(Z-Y\right)\\{\omega_{0}}^{2}Z-\gamma\left(Y-Z\right)\end{bmatrix}$$

Then the unforced system in the frequency domain is given by

$$L\mathcal{U} = \omega^2 \mathcal{U}, \quad \mathcal{U} = \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}$$

**Theorem 3.1.** *The operator $L$ defined on $\mathcal{H} = L^2(\mathbb{R}) \bigoplus \mathbb{C}^2$ by*

$$L \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -c^2 U_{xx} \\ -\beta \left[U_x\right]_0 - \gamma \ (Z - Y) \\ {\omega_0}^2 Z - \gamma \left(Y - Z\right) \end{bmatrix}$$

*is a positive self – adjoint operator in its domain,*

$$\mathcal{D}(L) = \left\{ [U, Y, Z]^T \in \mathcal{H} \mid U \in H^2 \left(\mathbb{R}^*\right) \quad , \ U \left(0^-\right) = U \left(0^+\right) = Y \right\}$$

*with respect to the inner product*

$$\left\langle \begin{bmatrix} U_1 \\ Y_1 \\ Z_1 \end{bmatrix} , \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \beta \int \overline{U_1} U_2 + c^2 \left( \overline{Y_1} Y_2 + \overline{Z_1} Z_2 \right)$$

*Proof.* We first show that $L$ is symmetric.

For $\begin{bmatrix} U \\ Y \\ Z \end{bmatrix} , \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \in \mathcal{D}(L)$

$$\left\langle L \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} , \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} -c^2 U_{xx} \\ -\beta \left[U_x\right]_0 - \gamma \ (Z - Y) \\ {\omega_0}^2 Z - \gamma \left(Y - Z\right) \end{bmatrix} , \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}}$$

$$= \beta \int_{\mathbb{R}^*} -c^2 \overline{U}_{xx} U_2 + c^2 \left[ \left( -\beta \left[ \overline{U}_x \right]_0 - \gamma \left( \overline{Z} - \overline{Y} \right) \right) Y_2 + \left( {\omega_0}^2 \overline{Z} - \gamma \left( \overline{Y} - \overline{Z} \right) \right) Z_2 \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} \overline{U}_x (U_2)_x + c^2 \beta \left[ \overline{U}_x U_2 \right]_0 - c^2 \beta \ \left[ \overline{U}_x \right]_0 Y_2 -$$

$$- c^2 \left[ \left( \gamma \ \left( \overline{Z} - \overline{Y} \right) \right) Y_2 - \left( {\omega_0}^2 \overline{Z} - \gamma \left( \overline{Y} - \overline{Z} \right) \right) Z_2 \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} \overline{U}_x (U_2)_x + c^2 \beta \left( \left[ \overline{U}_x U_2 \right]_0 - \left[ \overline{U}_x \right]_0 Y_2 \right) -$$

$$- c^2 \left[ \ \gamma \overline{Z} Y_2 - \gamma \overline{Y} Y_2 - {\omega_0}^2 \overline{Z} Z_2 + \ \gamma \overline{Y} Z_2 - \gamma \overline{Z} Z_2 \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} \overline{U}_x (U_2)_x - c^2 \left[ \ \gamma \left( \overline{Z} Y_2 + \overline{Y} Z_2 - \overline{Y} Y_2 - \overline{Z} Z_2 \right) - {\omega_0}^2 \overline{Z} Z_2 \right]$$

since $Y_2 = U_2\left(0^-\right) = U_2\left(0^+\right)$

and

$$\left\langle \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, L \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} -c^2 (U_2)_{xx} \\ -\beta \ \left[ (U_2)_x \right]_0 - \gamma \ \left( Z_2 - Y_2 \right) \\ {\omega_0}^2 Z_2 - \gamma \left( Y_2 - Z_2 \right) \end{bmatrix} \right\rangle_{\mathcal{H}}$$

$$= \beta \int_{\mathbb{R}^*} \overline{U} \left( -c^2 (U_2)_{xx} \right) + c^2 \left[ \overline{Y} \left( -\beta \ \left[ (U_2)_x \right]_0 - \gamma \ \left( Z_2 - Y_2 \right) \right) + \right.$$

$$\left. + \overline{Z} \left( {\omega_0}^2 Z_2 - \gamma \left( Y_2 - Z_2 \right) \right) \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} (U_2)_x \overline{U}_x + c^2 \beta \left[ (U_2)_x \overline{U} \right]_0 - c^2 \beta \overline{Y} \left[ (U_2)_x \right]_0 -$$

$$- c^2 \left[ \gamma \ \overline{Y} \left( Z_2 - Y_2 \right) - \overline{Z} \left( {\omega_0}^2 Z_2 - \gamma \left( Y_2 - Z_2 \right) \right) \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} (U_2)_x \overline{U}_x + c^2 \beta \left( \left[ (U_2)_x \overline{U} \right]_0 - \left[ (U_2)_x \right]_0 \overline{Y} \right) -$$

$$- c^2 \left[ \ \gamma Z_2 \overline{Y} - \gamma Y_2 \overline{Y} - {\omega_0}^2 Z_2 \overline{Z} + \ \gamma Y_2 \overline{Z} - \gamma Z_2 \overline{Z} \right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} \overline{U}_x (U_2)_x - c^2 \left[ \ \gamma \left( \overline{Z} Y_2 + \overline{Y} Z_2 - \overline{Y} Y_2 - \overline{Z} Z_2 \right) - {\omega_0}^2 \overline{Z} Z_2 \right]$$

since $Y = U\left(0^-\right) = U\left(0^+\right)$

hence

$$\left\langle L \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, L \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{H}}$$

We will now show that $L$ is a closed operator. Along with the fact that both its deficiency subspaces are trivial this is sufficient for the symmetric operator $L$ to be self-adjoint [26].

Suppose $\{\mathcal{U}_n\}$, where $\mathcal{U}_n = \begin{bmatrix} U_n \\ Y_n \\ Z_n \end{bmatrix} \in \mathcal{D}(L)$ is a Cauchy sequence in the operator norm of $L$,

$$\|\mathcal{U}\|_L = \left[c^4\|U_{xx}\|^2 + \beta^2 \ |[U_x]_0|^2 + +4\gamma^2|Z-Y|^2 + |Y|^2 + \left(\omega_0{}^4+1\right)|Z|^2 + \|U\|^2\right]^{\frac{1}{2}}$$

Then $\{U_n\}$ is Cauchy in $H^2(\mathbb{R}^*) = H^2(-\infty,0) \bigoplus H^2(0,\infty)$, $\{(U_n)_{xx}\}$ is Cauchy in $L^2(\mathbb{R})$, $\{(U_n)_x\}$ is Cauchy in $L^2(\mathbb{R}^*)$ while $\{Y_n\}$, $\{Z_n\}$ and $\left\{[(U_n)_x]_0\right\}$ are Cauchy in $\mathbb{C}$, with $U_n(0^-) = U_n(0^+) = Y_n$.

Since $L^2(\mathbb{R})$ is complete $(U_n)_{xx}$ converges in $L^2(\mathbb{R})$ while $\{Y_n\}$, $\{Z_n\}$ and $\left\{[(U_n)_x]_0\right\}$ converge in $\mathbb{C}$ since $\mathbb{C}$ is complete.

Consider $U_n{}^+ = U_n|_{(0,\infty)}$ and $U_n{}^- = U_n|_{(-\infty,0)}$

Note that $U_n{}^-(0) = U_n(0^-)$ and $U_n{}^+(0) = U_n(0^+)$

Then $\left\{U_n{}^-\right\}$ is Cauchy in $H^2(\mathbb{R}^-)$ and $\left\{U_n{}^+\right\}$ is Cauchy in $H^2(\mathbb{R}^+)$

Now since $H^2(\mathbb{R}^-)$ is complete, $U_n{}^- \longrightarrow U^-$ in $H^2(\mathbb{R}^-)$

Then by the trace theorem we have the existence of $U^-|_{\partial H^2(\mathbb{R}^-)} = U^-(0)$.

Define $U^-(0) = Y$

Then $Y_n \longrightarrow Y$ in $\mathbb{C}$

Also since $H^2(\mathbb{R}^+)$ complete, $U_n{}^+ \longrightarrow U^+$ in $H^2(\mathbb{R}^+)$.

The trace theorem guarantees the existence of $U^+|_{\partial H^2(\mathbb{R}^-)} = U^+(0)$ and

$$U_n{}^+(0) \longrightarrow U^+(0)\,.$$

Now we have

$$U\left(0^-\right) = U^-(0) = Y = \lim_n Y_n = \lim_n U_n\left(0^+\right) = \lim_n U_n{}^+(0) = U^+(0) = U\left(0^+\right)$$

Define

$$U = \begin{cases} U^- \text{ in } H^2(-\infty, 0) \\ Y \qquad\qquad x = 0 \\ U^+ \quad \text{in } H^2(0, \infty) \end{cases}$$

Then we have $U_n \longrightarrow U$ in $H^2(\mathbb{R}^*)$ with $U(0^-) = Y = U(0^+)$

Hence $\mathcal{U}_n \longrightarrow \mathcal{U}$ in the operator norm and $U \in H^2(\mathbb{R}^*)$.

So we have that $\mathcal{U} = \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} \in \mathcal{D}(L)$

Hence the operator $L$ is closed.

Moreover the solutions to the systems $L\mathcal{U} = i\mathcal{U}$ and $L\mathcal{U} = -i\mathcal{U}$ are trivial. Thus the deficiency subspaces $N_i, N_{-i} = \{0\}$ .

This implies that $L$ has no non-trivial self-adjoint extensions in $\mathcal{H}$.

Hence $L$ is self-adjoint.

Finally we show that $L$ is a positive operator.

For $\begin{bmatrix} U \\ Y \\ Z \end{bmatrix} \in \mathcal{D}(L)$

$$\left\langle L \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} -c^2 U_{xx} \\ -\beta\, [U_x]_0 - \gamma\ (Z - Y) \\ {\omega_0}^2 Z - \gamma (Y - Z) \end{bmatrix}, \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} \right\rangle_{\mathcal{H}}$$

$$= \beta \int_{\mathbb{R}^*} \left(-c^2 \overline{U}_{xx}\right) U + c^2 \left[\left(-\beta\ \left[\overline{U}_x\right]_0 - \gamma\ \left(\overline{Z} - \overline{Y}\right)\right) Y + \left({\omega_0}^2 \overline{Z} - \gamma \left(\overline{Y} - \overline{Z}\right)\right) Z\right]$$

$$= \beta c^2 \int_{\mathbb{R}^*} \left|\overline{U}_x\right|^2 + c^2 \beta \left(\left[\overline{U}_x U\right]_0 - \left[\overline{U}_x\right]_0 Y\right) + c^2 \begin{bmatrix} \overline{Y} & \overline{Z} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \\ -\gamma & {\omega_0}^2 + \gamma \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix}$$

$$= \beta c^2 \int_{\mathbb{R}^*} \left|\overline{U}_x\right|^2 + c^2 \begin{bmatrix} \overline{Y} & \overline{Z} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \\ -\gamma & {\omega_0}^2 + \gamma \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} \geq 0$$

□

Now we wish to present the system as a first order evolution in time. We denote $\mathcal{V} = \frac{\partial \mathcal{U}}{\partial t} = \begin{bmatrix} \dot{U} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}$ Then the system can be expressed as

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -L & 0 \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix}$$

Define $\mathcal{Q}(L)$ by

$$\mathcal{Q}(L) = \left\{(U, Y, Z) \in \mathcal{H} | U \in H^1(\mathbb{R}),\ U(0) = Y\right\},$$

with inner product

$$\left\langle \begin{bmatrix} U_1 \\ Y_1 \\ Z_1 \end{bmatrix}, \begin{bmatrix} U_2 \\ Y_2 \\ Z_2 \end{bmatrix} \right\rangle_{\mathcal{Q}} = \beta c^2 \int \bar{U}_1' U_2' + c^2 \omega_0^2 \bar{Z}_1 Z_2 + \gamma c^2 \left[(\bar{Y}_1 Y_2 + \bar{Z}_1 Z_2) - (\bar{Y}_1 Z_2 + \bar{Z}_1 Y_2)\right]$$

Note that for $\mathcal{U} \in D(L)$

$$\langle \mathcal{U}\ , \mathcal{V} \rangle_{\mathcal{Q}(L)} = \langle L\mathcal{U}\ , \mathcal{V} \rangle_{\mathcal{H}}$$

and for $\mathcal{V} \in D(L)$

$$\langle \mathcal{U}\ , \mathcal{V} \rangle_{\mathcal{Q}(L)} = \langle \mathcal{U}\ , L\mathcal{V} \rangle_{\mathcal{H}}$$

Define the operator $A$ in $\mathcal{Q}(L) \bigoplus \mathcal{H}$ as follows:

On its domain, $\mathcal{D}(A) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L)$ the operator $A$ has the block form

$$A = \begin{bmatrix} 0 & 1 \\ -L & 0 \end{bmatrix}$$

Then our unforced system is equivalent to the two dimensional first order system given by

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix} = A \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix}$$

in which $\mathcal{U} = \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}$, $\mathcal{V} = \frac{\partial \mathcal{U}}{\partial t}$

**Theorem 3.2.** *The operator $A$ defined by*

$$A = \begin{bmatrix} 0 & 1 \\ -L & 0 \end{bmatrix}$$

*is anti-self-adjoint in its domain*

$$\mathcal{D}(A) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L)$$

*Proof.* First we prove that $A$ is anti-symmetric

For

$$\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix}, \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \in \mathcal{D}(A) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L)$$

$$\left\langle A \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix}, \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = \left\langle \begin{bmatrix} \mathcal{V}_1 \\ -L\mathcal{U}_1 \end{bmatrix}, \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

$$= \langle \mathcal{V}_1, \mathcal{U}_2 \rangle_{\mathcal{Q}(L)} + \langle -L\mathcal{U}_1, \mathcal{V}_2 \rangle_{\mathcal{H}}$$

$$= \langle \mathcal{V}_1, L\mathcal{U}_2 \rangle_{\mathcal{H}} - \langle L\mathcal{U}_1, \mathcal{V}_2 \rangle_{\mathcal{H}}$$

Since $\mathcal{U}_2 \in \mathcal{D}(L)$.

$$\left\langle \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix}, A \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = \left\langle \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix}, \begin{bmatrix} \mathcal{V}_2 \\ -L\mathcal{U}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

$$= \langle \mathcal{U}_1 \ , \mathcal{V}_2 \rangle_{\mathcal{Q}(L)} + \langle \mathcal{V}_1 \ , -L\mathcal{U}_2 \rangle_{\mathcal{H}}$$

$$= \langle L\mathcal{U}_1 \ , \mathcal{V}_2 \rangle_{\mathcal{H}} - \langle \mathcal{V}_1 \ , L\mathcal{U}_2 \rangle_{\mathcal{H}}$$

Since $\mathcal{U}_2 \in \mathcal{D}(L)$.

Hence we've proved that

$$\left\langle A \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = -\left\langle \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , A \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

Now we prove that $\mathcal{D}(A^*) = \mathcal{D}(A)$

Let $\begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \in \mathcal{D}(A^*)$

Then the map $l : \mathcal{D}(A) \longrightarrow \mathbb{C}$ given by

$$\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} \longmapsto \left\langle A \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

is bounded.

Then by the Reisz theorem there exists $\begin{bmatrix} \mathcal{U}_* \\ \mathcal{V}_* \end{bmatrix} \in \mathcal{Q}(L) \bigoplus \mathcal{H}$ such that for all $\begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L)$

$$\left\langle A \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = \left\langle \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_* \\ \mathcal{V}_* \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

$$\Rightarrow \left\langle \begin{bmatrix} \mathcal{V}_1 \\ -L\mathcal{U}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = \left\langle \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_* \\ \mathcal{V}_* \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

$$\Rightarrow \langle \mathcal{V}_1 \ ,\mathcal{U}_2 \rangle_{\mathcal{Q}(L)} + \langle -L\mathcal{U}_1 \ , \mathcal{V}_2 \rangle_{\mathcal{H}} = \langle \mathcal{U}_1 \ ,\mathcal{U}_* \rangle_{\mathcal{Q}(L)} + \langle \mathcal{V}_1 \ , \mathcal{V}_* \rangle_{\mathcal{H}}$$

$$\Rightarrow \langle \mathcal{V}_1 \ ,\mathcal{U}_2 \rangle_{\mathcal{Q}(L)} - \langle \mathcal{U}_1 \ , \mathcal{V}_2 \rangle_{\mathcal{Q}(L)} = \langle \mathcal{U}_1 \ ,\mathcal{U}_* \rangle_{\mathcal{Q}(L)} + \langle \mathcal{V}_1 \ , \mathcal{V}_* \rangle_{\mathcal{H}}$$

$$\Rightarrow \langle \mathcal{V}_1 \ ,\mathcal{U}_2 \rangle_{\mathcal{Q}(L)} + \langle \mathcal{U}_1 \ , -\mathcal{V}_2 \rangle_{\mathcal{Q}(L)} = \langle \mathcal{U}_1 \ ,\mathcal{U}_* \rangle_{\mathcal{Q}(L)} + \langle \mathcal{V}_1 \ , \mathcal{V}_* \rangle_{\mathcal{H}}$$

Now we take $\mathcal{V}_1 = 0$ and let $\mathcal{U}_1$ range over all $\mathcal{D}(L)$

Then we have

$$\langle \mathcal{U}_1 \, , -\mathcal{V}_2 \rangle_{\mathcal{Q}(L)} = \langle \mathcal{U}_1 \, , \mathcal{U}_* \rangle_{\mathcal{Q}(L)} \, , \; \forall \, \mathcal{U}_1 \in \mathcal{D}(L)$$

$$\Rightarrow \mathcal{V}_2 = -\mathcal{U}_* \Rightarrow \mathcal{V}_2 \in \mathcal{Q}(L)$$

Now recall that map $l : \mathcal{D}(A) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L) \longrightarrow \mathcal{Q}(L) \bigoplus \mathcal{H}$ given by

$$\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} \longmapsto \left\langle A \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}}$$

is bounded.

Take $\mathcal{U}_1 = 0$ and let $\mathcal{V}_1$ range over all of $\mathcal{Q}(L)$

Then we have $l|_{\{0\} \bigoplus D(L)}$ given by

$$\begin{bmatrix} 0 \\ \mathcal{V}_1 \end{bmatrix} \longmapsto \left\langle A \begin{bmatrix} 0 \\ \mathcal{V}_1 \end{bmatrix} , \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \right\rangle_{\mathcal{Q}(L) \bigoplus \mathcal{H}} = \langle \mathcal{V}_1 \, , \mathcal{U}_2 \rangle_{\mathcal{Q}(L)}$$

is bounded, so that the map $\alpha : D(L) \longrightarrow \mathbb{C}$ given by

$$\mathcal{V}_1 \longmapsto \langle \mathcal{V}_1 \, , \mathcal{U}_2 \rangle_{\mathcal{Q}(L)} = \langle L\mathcal{V}_1 \, , \mathcal{U}_2 \rangle_{\mathcal{Q}(L)}$$

is bounded. But this means that $\mathcal{U}_2 \in D(L^*)$ and since $D(L) = D(L^*)$ we have that

$$\mathcal{U}_2 \in D(L)$$

Since $\mathcal{U}_2 \in D(L)$ and $\mathcal{V}_2 \in \mathcal{Q}(L)$ we conclude $\begin{bmatrix} \mathcal{U}_2 \\ \mathcal{V}_2 \end{bmatrix} \in \mathcal{D}(A) = \mathcal{D}(L) \bigoplus \mathcal{Q}(L)$

Hence $\mathcal{D}(A^*) = \mathcal{D}(A)$

□

Recall that the unforced scattering problem is given by

$$\left(L - \omega^2\right) \mathcal{U} = 0$$

The forced system is given by

$$\left(L-\omega^2\right)\mathcal{U}=\mathcal{F}\ ;\quad \mathcal{U}=\begin{bmatrix}U\\Y\\Z\end{bmatrix},\quad \mathcal{F}=\begin{bmatrix}f\\f_0\\f_*\end{bmatrix}$$

$f(x)$: force at the point $x$ on the string, $x \neq 0$

$f_0$: force on the point-mass scatterer

$f_*$: force on the mass-resonator

When $\gamma = 0$, $\pm\omega_0$ are eigen-values of the operator $L$. Hence the natural frequencies $\pm\omega_0$ of the resonator (the frequencies corresponding to the bound states of the system) are realized as the eigen-value of the operator $L$ underlying the scattering system when the point-mass and the spring-mass resonator are decoupled.

These eigenvalues are embedded in the continuous spectrum of the operator. This continuous spectrum corresponds to the continuum of frequencies of the extended states (waves that travel in the string in our model which models the ambient medium).

When $\gamma > 0$, the point-mass and the mass-spring resonator are coupled and hence bound states of the scatterer are destroyed due to the interaction with the waves travelling on the sting (extended states in the ambient medium). The perturbation of the idealized system via the coupling of the point-mass and the spring-mass resonator is materialized within the operator $L$ underlying the system by the dissolution of these eigenvalues in to the continuous spectrum and the corresponding eigen-values moving down in to the lower half plane.

### 3.2 Solution to The Scattering Problem

Let us now solve the scattering problem for our system under a non-monochromatic incidence. Suppose a wave pulse (non-monochromatic) is incident upon the system from the left. We will consider a wave pulse in the string that travel from left to right and is partly reflected by the scatterer and partly transmitted across it. Suppose that the non-monochromatic incident field is $J(\xi)$ and that $R(\xi)$ and $T(\xi)$ are consecutively the reflected and transmitted fields. Since the infinite string in our Resonant Lamb Model admits the wave equation for its traveling disturbances, the displacement of the string at the point $x$ at time $t$ must be of the form;

$$u(x,t) = \begin{cases} J(x-ct) + R(x+ct), & x \leq 0, \\ T(x-ct), & x \geq 0. \end{cases}$$

If $J(\xi) = 0$ for $\xi > \xi_0$, then for time $t < -\xi_0/c$, the source field $J(x-ct)$ is supported completely to the left of the scatterer and is approaching it. Assuming that there are no disturbances other than $J(\xi)$ affecting the string, the scatterer (point-mass) and the right side of the string are at rest $t < -\xi_0/c$. This means that $R(\xi) = 0$ for $\xi < -\xi_0$ and $T(\xi) = 0$ for $\xi > \xi_0$. By moving $\xi_0$ to $\infty$, one can allow $J(\xi)$ to be an infinitely supported finite energy wave pulse; A Gaussian, for example.

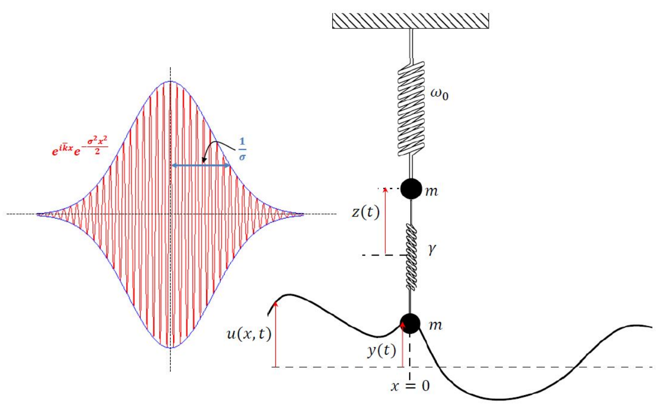


FIGURE 3.1. Non Monochromatic Incidence

To compute the solution for the system under an arbitrary non-monochromatic incidence as described above, we note that the scatterer remains at rest until time $t = -\xi_0/c$ and then it is impulsively forced by the velocity of the incident pulse. The system that we must solve is given by

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) + v(x)\,\delta(t), \quad x \neq 0$$

$$\ddot{y}(t) = \gamma\ (z(t) - y(t)) + \beta\ [u_x]_0 + v_0 \delta(t), \quad y(t) = u(0,t)$$

$$\ddot{z}(t) = -{\omega_0}^2 z(t) + \gamma\,(y(t) - z(t)) + v_* \delta(t)$$

In the Fourier domain we have:

$$-\omega^2 U(x,\omega) = c^2 U_{xx}(x,\omega) + \frac{v(x)}{2\pi}, \quad x \neq 0$$

$$-\omega^2 Y(x,\omega) = \gamma\ (Z - Y) + \beta [U_x]_0 + \frac{v_0}{2\pi}\ , \quad Y(\omega) = U(0,\omega)$$

$$-\omega^2 Z(\omega) = -{\omega_0}^2 Z(\omega) + \gamma\,(Y - Z) + \frac{v_*}{2\pi}$$

where $v(x) = u_t(x,0)$, the velocity of the incident pulse.

Hence we seek solutions of

$$\left(L - \omega^2\right)\mathcal{U} = F\ ; \quad \mathcal{U} = \begin{bmatrix} U \\ Y \\ Z \end{bmatrix}, \quad F = \begin{bmatrix} f \\ f_0 \\ f_* \end{bmatrix}$$

where $f(x) = \frac{v(x)}{2\pi}$ for $x \neq 0$, $f_0 = \frac{v_0}{2\pi}$ and $f_* = \frac{v_*}{2\pi}$ Response of the system to this impulsive forcing will be given by the resolvent of the operator $L$ at $\omega^2$.

$$\mathcal{U} = \left(L - \omega^2 I\right)^{-1} F$$

The response of the system when the forcing is a point source is given by Green's functions. We will use Green's functions to compute the general response of the system.

Let $\mathcal{G}(x, x_0; \omega)$ be the influence felt by the system at $x$ due to a unit point source forcing at $x_0$ on the string for $x_0 \neq 0$

$$\mathcal{G}(x, x_0; \omega) = \left(L - \omega^2 I\right)^{-1} F, \quad F = \begin{bmatrix} \delta(x - x_0) \\ 0 \\ 0 \end{bmatrix}$$

Let $\mathcal{G}_0(x;\omega)$ be the influence felt by the system at $x$ due to a unit forcing at the point-mass scatterer

$$\mathcal{G}_0\left(x;\omega\right) = \left(L-\omega^2 I\right)^{-1}F, \quad F = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $\mathcal{G}_*\left(x;\omega\right)$ be the influence felt by the system at $x$ due to a unit forcing at the spring-mass resonator

$$\mathcal{G}_*\left(x;\omega\right) = \left(L-\omega^2 I\right)^{-1}F, \quad F = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We first find $\mathcal{G}\left(x,x_0;\omega\right)$ by solving

$$\left(L-\omega^2\right)\begin{bmatrix} G\left(x,x_0;\omega\right) \\ Y_{x_0}\left(\omega\right) \\ Z_{x_0}(\omega) \end{bmatrix} = \begin{bmatrix} \delta(x-x_0) \\ 0 \\ 0 \end{bmatrix}$$

where

$$\mathcal{G}\left(x,x_0;\omega\right) = \begin{bmatrix} G\left(x,x_0;\omega\right) \\ Y_{x_0}\left(\omega\right) \\ Z_{x_0}(\omega) \end{bmatrix}, \quad Y_{x_0}\left(\omega\right) = G\left(0,x_0;\omega\right)$$

So we have to solve

$$-c^2 G_{xx}\left(x,x_0;\omega\right) - \omega^2 G\left(x,x_0;\omega\right) = \delta(x-x_0), \quad x \neq 0$$

$$-\gamma\ \left(Z_{x_0}(\omega) - Y_{x_0}\left(\omega\right)\right) - \beta\ \left[G_x\left(x,x_0;\omega\right)\right]_0 - \omega^2 Y_{x_0}\left(\omega\right) = 0$$

$${\omega_0}^2 Z_{x_0}\left(\omega\right) - \gamma\left(Y_{x_0}\left(\omega\right) - Z_{x_0}(\omega)\right) - \omega^2 Z_{x_0}(\omega) = 0$$

The solution for $G\left(x,x_0;\omega\right)$ has the form of a field produced by a source concentrated at $x = x_0 \neq 0$ and then modified due to being scattered by the scatterer (point-mass and mass-spring pair) at $x = 0$ given by

$$G\left(x,x_0;\omega\right) = \frac{1}{2i\omega c} e^{\frac{i\omega|x-x_0|}{c}} + g\left(x_0;\omega\right) e^{\frac{i\omega|x|}{c}}$$

Note that

$$[G_x(x,x_0;\omega)]_0 = \left[\frac{\partial}{\partial x}\left(\frac{1}{2i\omega c}e^{\frac{i\omega|x-x_0|}{c}}\right)\right]_0 + \left[\frac{\partial}{\partial x}\left(g(x_0;\omega)\,e^{\frac{i\omega|x|}{c}}\right)\right]_0$$

Since $x_0 \neq 0$, $\lim_{x\longrightarrow 0}(x-x_0) \ \neq 0$ and hence

$$\begin{aligned}
\left[\frac{\partial}{\partial x}\left(\frac{1}{2i\omega c}e^{\frac{i\omega|x-x_0|}{c}}\right)\right]_0 &= \frac{1}{2i\omega c}\left[\lim_{x\longrightarrow 0^+}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x-x_0|}{c}}\right) \ - \lim_{x\longrightarrow 0^-}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x-x_0|}{c}}\right) \ \right] \\
&= \frac{1}{2i\omega c}\frac{i\omega}{c}\left[\lim_{x\longrightarrow 0^+}\left(e^{\frac{i\omega|x-x_0|}{c}}\right) \ - \lim_{x\longrightarrow 0^-}\left(e^{\frac{i\omega|x-x_0|}{c}}\right) \ \right] \\
&= \left(e^{\frac{i\omega|x-x_0|}{c}}\right) - \left(e^{\frac{i\omega|x-x_0|}{c}}\right) = 0
\end{aligned}$$

So we have

$$\begin{aligned}
[G_x(x,x_0;\omega)]_0 &= \left[\frac{\partial}{\partial x}\left(g(x_0;\omega)\,e^{\frac{i\omega|x|}{c}}\right)\right]_0 \\
&= g(x_0;\omega)\left[\lim_{x\longrightarrow 0^+}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right) \ - \lim_{x\longrightarrow 0^-}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right) \ \right] \\
&= g(x_0;\omega)\left[\frac{i\omega}{c} - \left(-\frac{i\omega}{c}\right)\right] = \frac{2i\omega}{c}g(x_0;\omega)
\end{aligned}$$

Since $Y_{x_0}(\omega) = G(0,x_0;\omega)$ we also have

$$Y_{x_0}(\omega) = \frac{1}{2i\omega c}e^{\frac{i\omega|x_0|}{c}} + g(x_0;\omega)$$

Therefore to find $\mathcal{G}(x,x_0;\omega)$ we compute $g(x_0;\omega)$ and $Z_{x_0}(\omega)$ using the equations

$$-\gamma\left(Z_{x_0}(\omega) - g(x_0;\omega)\right) - \frac{2i\beta\omega}{c}g(x_0;\omega) - \omega^2 g(x_0;\omega) = \left(\omega^2 - \gamma \ \right)\frac{1}{2i\omega c}e^{\frac{i\omega|x_0|}{c}}$$

$${\omega_0}^2 Z_{x_0}(\omega) - \gamma\left(g(x_0;\omega) - Z_{x_0}(\omega)\right) - \omega^2 Z_{x_0}(\omega) = \frac{\gamma}{2i\omega c}e^{\frac{i\omega|x_0|}{c}}$$

These equations can be expressed as

$$\begin{bmatrix} \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} & -\gamma \\ -\gamma & {\omega_0}^2 - \omega^2 + \gamma \end{bmatrix}\begin{bmatrix} g(x_0;\omega) \\ Z_{x_0}(\omega) \end{bmatrix} = \frac{1}{2i\omega c}e^{\frac{i\omega|x_0|}{c}}\begin{bmatrix} \omega^2 - \gamma \\ \gamma \end{bmatrix}$$

Denote

$$A = \begin{bmatrix} \gamma - \omega^2 - \frac{2i\beta\omega}{c} & -\gamma \\ -\gamma & {\omega_0}^2 - \omega^2 + \gamma \end{bmatrix}$$

Then $g(x_0;\omega)$ and $Z_{x_0}(\omega)$ is given by

$$\begin{bmatrix} g(x_0;\omega) \\ Z_{x_0}(\omega) \end{bmatrix} = \frac{e^{\frac{i\omega|x_0|}{c}}}{2i\omega c} \frac{1}{\det(A)} \begin{bmatrix} {\omega_0}^2 - \omega^2 + \gamma & \gamma \\ \gamma & \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} \end{bmatrix} \begin{bmatrix} \omega^2 - \gamma \\ \gamma \end{bmatrix}$$

Hence, the solution for $\mathcal{G}(x, x_0;\omega)$ is given by

$$\mathcal{G}(x, x_0;\omega) = \begin{bmatrix} \frac{1}{2i\omega c} e^{\frac{i\omega|x-x_0|}{c}} + g(x_0;\omega)\, e^{\frac{i\omega|x|}{c}} \\ \frac{1}{2i\omega c} e^{\frac{i\omega|x_0|}{c}} + g(x_0;\omega) \\ Z_{x_0}(\omega) \end{bmatrix}$$

with

$$g(x_0;\omega) = \frac{1}{2i\omega c} \frac{\left[(\omega^2 - \gamma)\ \ (\omega^2 - {\omega_0}^2 - \gamma) - \gamma^2\right]}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)({\omega_0}^2 - \omega^2 + \gamma) + \gamma^2\right]} e^{\frac{i\omega|x_0|}{c}}$$

$$Z_{x_0}(\omega) = \frac{\gamma}{c^2} \frac{\beta}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)({\omega_0}^2 - \omega^2 + \gamma) + \gamma^2\right]} e^{\frac{i\omega|x_0|}{c}}$$

We then find $\mathcal{G}_0(x;\omega)$ by solving

$$\left(L - \omega^2\right) \begin{bmatrix} G_0(x;\omega) \\ Y_0(\omega) \\ Z_0(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where

$$\mathcal{G}_0(x;\omega) = \begin{bmatrix} G_0(x;\omega) \\ Y_0(\omega) \\ Z_0(\omega) \end{bmatrix}, \quad Y_0(\omega) = G_0(0;\omega)$$

The solution for $G_0(x;\omega)$ has the form of a field produced by a source concentrated at $x = 0$ given by

$$G_0(x;\omega) = g_0(\omega)\, e^{\frac{i\omega|x|}{c}}$$

Since $Y_0(\omega) = G_0(0;\omega)$ we have

$$Y_0(\omega) = g_0(\omega)$$

and we also compute

$$[G_0(x;\omega)]_0 = g_0(\omega)\left[\lim_{x\longrightarrow 0^+}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right) - \lim_{x\longrightarrow 0^-}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right)\right] = \frac{2i\omega}{c}g_0(\omega)$$

We solve for $Z_0(\omega)$ and $g_0(\omega)$ by using the equations:

$$-\gamma\ \left(Z_0(\omega) - g_0(\omega)\right) - \frac{2i\beta\omega}{c}g_0(\omega) - \omega^2 g_0(\omega) = 1$$

$${\omega_0}^2 Z_0(\omega) - \gamma\left(g_0(\omega) - Z_0(\omega)\right) - \omega^2 Z_0(\omega) = 0$$

These equations can be expressed as

$$\begin{bmatrix} \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} & -\gamma \\ -\gamma & {\omega_0}^2 - \omega^2 + \gamma \end{bmatrix}\begin{bmatrix} g_0(\omega) \\ Z_0(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then $g_0(\omega)$ and $Z_0(\omega)$ are given by

$$\begin{bmatrix} g_0(\omega) \\ Z_0(\omega) \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} {\omega_0}^2 - \omega^2 + \gamma & \gamma \\ \gamma & \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence the solution for $\mathcal{G}_0(x;\omega)$ is given by

$$\mathcal{G}_0(x;\omega) = \begin{bmatrix} g_0(\omega)\, e^{\frac{i\omega|x|}{c}} + g_0(\omega) \\ g_0(\omega) \\ Z_0(\omega) \end{bmatrix}$$

with

$$g_0(\omega) = \frac{\left(\omega^2 - {\omega_0}^2 - \gamma\right)}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2\right]}$$

$$Z_0(\omega) = \frac{-\gamma}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2\right]}$$

We now find $\mathcal{G}_*(x;\omega)$ by solving

$$\left(L - \omega^2\right)\begin{bmatrix} G_*(x;\omega) \\ Y_*(\omega) \\ Z_*(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where

$$\mathcal{G}_*(x;\omega) = \begin{bmatrix} G_*(x;\omega) \\ Y_*(\omega) \\ Z_*(\omega) \end{bmatrix}, \quad Y_*(\omega) = G_*(0;\omega)$$

The solution for $G_*(x;\omega)$ has the form of a field produced by a source concentrated at $x = 0$ given by

$$G_*(x;\omega) = g_*(\omega)\, e^{\frac{i\omega|x|}{c}}$$

Since $Y_*(\omega) = G_*(0;\omega)$ we have

$$Y_*(\omega) = g_*(\omega)$$

and we also compute

$$\left[G_*(x;\omega)\right]_0 = g_*(\omega)\left[\lim_{x\longrightarrow 0^+}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right) - \lim_{x\longrightarrow 0^-}\frac{\partial}{\partial x}\left(e^{\frac{i\omega|x|}{c}}\right)\right] = \frac{2i\omega}{c}g_*(\omega)$$

We solve for $Z_*(\omega)$ and $g_*(\omega)$ by using the equations:

$$-\gamma\ \left(Z_*(\omega) - g_*(\omega)\right) - \frac{2i\beta\ \omega}{c}g_*(\omega) - \omega^2 g_*(\omega) = 0$$

$${\omega_0}^2 Z_*(\omega) - \gamma\left(g_*(\omega) - Z_*(\omega)\right) - \omega^2 Z_*(\omega) = 1$$

These equations can be expressed as

$$\begin{bmatrix} \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} & -\gamma \\ -\gamma & {\omega_0}^2 - \omega^2 + \gamma \end{bmatrix} \begin{bmatrix} g_*(\omega) \\ Z_*(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence $g_*(\omega)$ and $Z_*(\omega)$ are given by

$$\begin{bmatrix} g_*(\omega) \\ Z_*(\omega) \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} {\omega_0}^2 - \omega^2 + \gamma & \gamma \\ \gamma & \gamma - \omega^2 - \frac{2i\beta\ \omega}{c} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore we have

$$\mathcal{G}_*(x;\omega) = \begin{bmatrix} g_*(\omega)\, e^{\frac{i\omega|x|}{c}} \\ g_*(\omega) \\ Z_*(\omega) \end{bmatrix}$$

with

$$g_*(\omega) = \frac{-\gamma}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2\right]}$$

$$Z_*(\omega) = \frac{-\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)}{\left[\left(\frac{2i\beta\omega}{c} + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2\right]}$$

Now we have the solutions for all basis elements that will make up the forcing $F$ and hence we can write down the solution $\mathcal{U}$ in the Fourier domain for the general forced system.

Note that the impulsive forcing $f(x)$ on the string can be expressed as

$$f(x) = \int_{x_0 \neq 0} \delta(x - x_0)\, f(x_0)\, dx_0 \qquad ; x \neq 0$$

The solution for the forced system $(L - \omega^2 I)\mathcal{U} = F$ is given by

$$\mathcal{U} = \int_{x_0 \neq 0} \mathcal{G}(x, x_0; \omega)\, f(x_0)\, dx_0 + \mathcal{G}_0(x;\omega)\, f_0 + \mathcal{G}_*(x;\omega)\, f_*$$

Explicitly for $\mathcal{U} = \begin{bmatrix} U \\ Y \\ Z \end{bmatrix} \in \mathcal{H}$ and $F = \begin{bmatrix} f \\ f_0 \\ f_* \end{bmatrix}$

$$U = \int_{x_0 \neq 0} G(x, x_0; \omega) f(x_0) dx_0 + G_0(x; \omega) f_0 + G_*(x; \omega) f_*$$

$$Y = \int_{x_0 \neq 0} Y_{x_0}(\omega) f(x_0) dx_0 + Y_0(\omega) f_0 + Y_*(\omega) f_*$$

$$Z = \int_{x_0 \neq 0} Z_{x_0}(\omega) f(x_0) dx_0 + Z_0(\omega) f_0 + Z_*(\omega) f_*$$

Now for $f(x) = \frac{v(x)}{2\pi}$, $f_0 = \frac{v_0}{2\pi}$ and $f_* = \frac{v_*}{2\pi}$, the Fourier coefficients $U(x, \omega)$, $Y(x, \omega)$ and $Z(x, \omega)$ are given by,

$$U(x, \omega) = \frac{1}{2\pi} \left[ \int G(x, x_0; \omega) v(x_0) dx_0 + G_0(x; \omega) v_0 + G_*(x; \omega) v_* \right]$$

$$Y(x, \omega) = \frac{1}{2\pi} \left[ \int Y_{x_0}(\omega) v(x_0) dx_0 + Y_0(\omega) v_0 + Y_*(\omega) v_* \right]$$

$$Z(x, \omega) = \frac{1}{2\pi} \left[ \int Z_{x_0}(\omega) v(x_0) dx_0 + Z_0(\omega) v_0 + Z_*(\omega) v_* \right]$$

We can now express the disturbance on the string at a point $x$ at time $t$, $u(x, t)$ as an integral superposition of all harmonic oscillations using the inverse Fourier transform;

$$\begin{aligned} u(x, t) &= \int U(x, \omega) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int \left[ \int G(x, x_0; \omega) v(x_0) dx_0 + G_0(x; \omega) v_0 + G_*(x; \omega) v_* \right] e^{-i\omega t} d\omega \end{aligned}$$

in which (denoting $\alpha = \frac{2\beta}{c}$)

$$G(x, x_0; \omega) = \frac{1}{2i\omega c} \left[ e^{\frac{i\omega|x - x_0|}{c}} + \frac{[(\omega^2 - \gamma)\ (\omega^2 - {\omega_0}^2 - \gamma) - \gamma^2]}{[(i\alpha\omega + \omega^2 - \gamma)({\omega_0}^2 - \omega^2 + \gamma) + \gamma^2]} e^{\frac{i\omega(|x| + |x_0|)}{c}} \right]$$

$$G_0(x; \omega) = \frac{(\omega^2 - {\omega_0}^2 - \gamma)}{[(i\alpha\omega + \omega^2 - \gamma)({\omega_0}^2 - \omega^2 + \gamma) + \gamma^2]} e^{\frac{i\omega|x|}{c}}$$

$$G_*\left(x;\omega\right)=\frac{-\gamma}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}e^{\frac{i\omega|x|}{c}}$$

We now write down the displacement of the point-mass at $x=0$ at time $t$, y($t$);

$$\begin{aligned}y\left(t\right)&=\int Y(x,\omega)e^{-i\omega t}d\omega\\&=\frac{1}{2\pi}\int\left[\int Y_{x_0}\left(\omega\right)v\left(x_0\right)dx_0+Y_0\left(\omega\right)v_0+Y_*\left(\omega\right)v_*\right]e^{-i\omega t}d\omega\end{aligned}$$

where

$$Y_{x_0}\left(\omega\right)=\frac{e^{\frac{i\omega|x_0|}{c}}}{2i\omega c}\left[1+\frac{\left[\left(\omega^2-\gamma\right)\ \left(\omega^2-{\omega_0}^2-\gamma\right)-\gamma^2\right]}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}\right]$$

$$Y_0\left(\omega\right)=\frac{\left(\omega^2-{\omega_0}^2-\gamma\right)}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}$$

$$Y_*\left(\omega\right)=\frac{-\gamma}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}$$

The displacement of the spring-mass at time $t$, y($t$) is given by;

$$\begin{aligned}z\left(t\right)&=\int Z(x,\omega)e^{-i\omega t}d\omega\\&=\frac{1}{2\pi}\int\left[\int Z_{x_0}\left(\omega\right)v\left(x_0\right)dx_0+Z_0\left(\omega\right)v_0+Z_*\left(\omega\right)v_*\right]e^{-i\omega t}d\omega\end{aligned}$$

where

$$Z_{x_0}(\omega)=\frac{\gamma}{c^2}\frac{\beta}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}e^{\frac{i\omega|x_0|}{c}}$$

$$Z_0(\omega)=\frac{-\gamma}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}$$

$$Z_*(\omega)=\frac{-\left(i\alpha\omega+\omega^2-\gamma\right)}{\left[\left(i\alpha\omega+\omega^2-\gamma\right)\left({\omega_0}^2-\omega^2+\gamma\right)+\gamma^2\right]}$$

### 3.3 Poles of the Resolvent Operator of the System

The operator $L$, underlying the scattering system is self-adjoint and positive. Since $L$ is a self-adjoint operator (its spectrum, $\sigma(L)$ is a subset of real numbers [27]), its resolvent, $(L-\omega^2)^{-1}$ of $L$ at $\omega^2$ exists everywhere on the upper-half and the lower-half of the $\omega^2$ plane. This corresponds to quadrant 1 and 2 on the $\omega$ plane, excluding the axes. Further, since $L$ is also a positive operator, the spectrum $\sigma(L) \subseteq [0,\infty)$. Therefore $(L-\omega^2)^{-1}$ exists for all $\omega^2 \in (-\infty, 0)$ or equivalently for all positive imaginary $\omega$ as well. Thus, the resolvent exists everywhere on the upper half $\omega$ plane.

The above discussion shows that the analytic continuation of the resolvent $(L-\omega^2)^{-1}$ will not have poles in the upper half $\omega$ plane. Poles of the resolvent are obtained when the denominator of the system $D(\omega) = 0$. That is when

$$D(\omega) = \left(i\alpha\omega + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2 = 0$$

When $\gamma = 0$ we have,

$$D(\omega) = \omega\,(i\alpha + \omega)\left({\omega_0}^2 - \omega^2\right)$$

So the roots of $D(\omega)$ when $\gamma = 0$ are $\omega = -\omega_0,\ 0,\ \omega_0,\ -\alpha i$

When $\gamma = 0$, ${\omega_0}^2$ is an eigen-value of $L$ that is embedded in the spectrum of the operator $L$. When $\gamma \neq 0$, the system is perturbed by the coupling between the resonator and the point-mass. This causes the embedded eigen-value to dissolve and move down to the lower-half plane.

When $\gamma \neq 0$ let,

$$D(\omega) = \left(i\alpha\omega + \omega^2 - \gamma\right)\left({\omega_0}^2 - \omega^2 + \gamma\right) + \gamma^2 = (\omega - \omega_*)\,(\omega + \overline{\omega_*})\,(\omega - \omega_1)\,(\omega - \omega_2)$$

Then these poles are given in terms of $\gamma$ by

$$\omega_* = \omega_0 + \frac{1}{2\omega_0}\gamma + \left(\frac{(3{\omega_0}^2 - \alpha^2)}{2{\omega_0}^3\left({\omega_0}^2 + \alpha^2\right)} - \frac{\alpha}{{\omega_0}^2\left({\omega_0}^2 + \alpha^2\right)}i\right)\gamma^2 + O(\gamma^3)$$

$$-\overline{\omega_*} = -\omega_0 - \frac{1}{2\omega_0}\gamma - \left(\frac{(3{\omega_0}^2 - \alpha^2)}{2{\omega_0}^3\left({\omega_0}^2 + \alpha^2\right)} + \frac{\alpha}{{\omega_0}^2\left({\omega_0}^2 + \alpha^2\right)}i\right)\gamma^2 + O(\gamma^3)$$

$$\omega_1 = -\frac{i}{\alpha}\gamma + \frac{2\left({\omega_0}^2 - \alpha^2\right)}{\alpha^3{\omega_0}^2}i\gamma^2 + O(\gamma^3)$$

$$\omega_2 = -i\alpha + \frac{i}{\alpha}\gamma + \frac{2{\omega_0}^2}{\alpha^3\left({\omega_0}^2 + \alpha^2\right)}i\gamma^2 + O(\gamma^3)$$

The corresponding roots of $D(\omega)$, denoted by $\omega_1$, $\omega_2$, $\omega_*$ and $\overline{-\omega_*}$ all moving down to the lower half $\omega$ plane are illustrated in figure 3.2

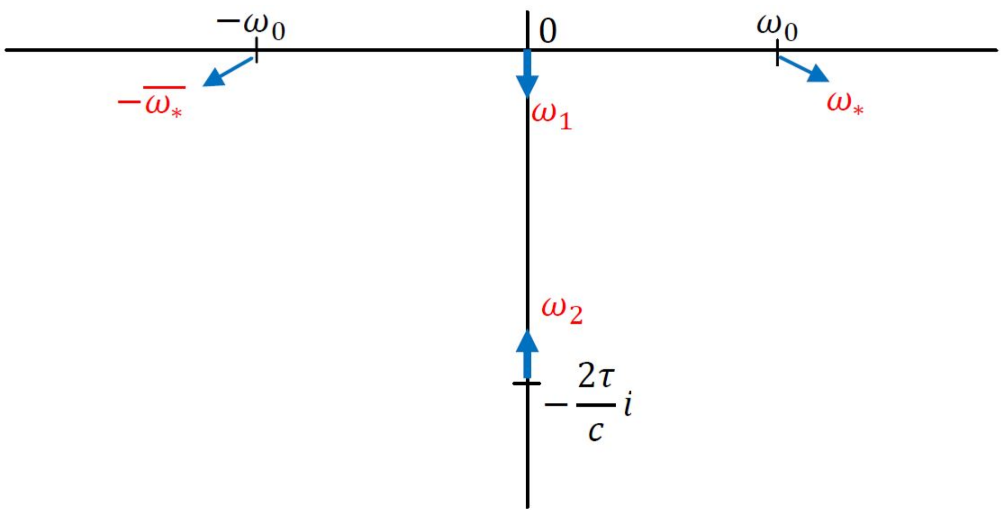


FIGURE 3.2. Poles of the Resolvent

Two of the poles, $\omega_*$ and $-\bar{\omega}_*$, originate from the spring-mass (within the two-part scatterer in the system) and are perturbations of its characteristic frequencies $\omega_0$ and $-\omega_0$ consecutively. These two poles are analytic functions of $\gamma$, with negative imaginary parts on the order of $\gamma^2$. The spring-mass being the resonant part in our two-part scatterer it is natural that these poles become resonances. These two poles contribute to a "resonant pathway" in the transmitted field.

$$\omega_*(\gamma) = \omega_c(\gamma) - i\gamma^2\kappa(\gamma),$$

$$-\bar{\omega}_*(\gamma) = -\omega_c(\gamma) - i\gamma^2\kappa(\gamma),$$

in which $\omega_c$ is the central (real) frequency of the resonance, which is detuned from $\omega_0$ by order $\gamma$,

$$\omega_c(\gamma) = \omega_0 + \frac{1}{2\omega_0}\gamma + O(\gamma^2),$$

and the attenuation term $\kappa(\gamma)$ has an expansion

$$\kappa\left(\gamma\right) = \frac{\alpha}{{\omega_0}^2\left({\omega_0}^2 + \alpha^2\right)} + O\left(\gamma\right).$$

The other two poles originate from the point-mass (within the two-part scatterer in the system) are also analytic functions of $\gamma$, but lies on the negative imaginary axis. Originating from the non-resonant pat in our two-part scatterer, these poles contribute to a "direct pathway" in the transmitted field.

$$\omega_1 = -i\gamma\beta^{(1)}$$

and

$$\omega_2 = i\beta^{(2)}$$

in which

$$\beta^{(1)} = \frac{1}{\alpha} - \frac{2\left({\omega_0}^2 - \alpha^2\right)}{\alpha^3{\omega_0}^2}\gamma + O(\gamma^2)$$

and

$$\beta^{(2)} = \alpha - \frac{1}{\alpha}\gamma - \frac{2{\omega_0}^2}{\alpha^3\left({\omega_0}^2 + \alpha^2\right)}\gamma^2 + O(\gamma^3)$$

### 3.4 The Transmitted Field

The transmitted field of the scattering solution is obtained from $u\left(x,t\right)$ when $x > 0$. We will assume that the support of the impulsive force function $v\left(x\right)$ is to the left of the point mass scatterer at $x = 0$ and that a forcing is applied neither at the point-mass nor at the spring-mass resonator. Therefore in

$$u\left(x,t\right) = \frac{1}{2\pi}\int\left[\int G\left(x,x_0;\omega\right)v\left(x_0\right)dx_0 + G_0\left(x;\omega\right)v_0 + G_*\left(x;\omega\right)v_*\right]e^{-i\omega t}d\omega$$

we have that $supp\ v \subseteq \ (-\infty, 0)$, $v_0 = 0$ and $v_* = 0$. Denoting the transmitted field by $u_T(x,t)$ we have

$$\begin{aligned}
u_T(x,t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} v(x_0)\, G(x,x_0;\omega)\, e^{-i\omega t} dx_0 d\omega \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} v(x_0)\left[\frac{1}{2i\omega c} e^{\frac{i\omega|x-x_0|}{c}} + g(x_0;\omega)\, e^{\frac{i\omega|x|}{c}}\right] e^{-i\omega t} dx_0 d\omega \\
= & \frac{1}{2\pi}\int_{-\infty}^{0}\frac{v(x_0)}{2ic}\int_{-\infty}^{\infty}\left[\frac{e^{i\frac{\omega}{c}(|x-x_0|-ct)}}{\omega} + 2icg(x_0;\omega)\, e^{i\frac{\omega}{c}(|x|+|x_0|-ct)}\right] d\omega dx_0 \\
= & \frac{1}{2\pi}\int_{-\infty}^{0}\frac{v(x_0)}{2ic}\left[\int_{-\infty}^{\infty}\frac{e^{i\frac{\omega}{c}(|x-x_0|-ct)}}{\omega} + 2icg(x_0;\omega)\, e^{i\frac{\omega}{c}(|x|+|x_0|-ct)} d\omega\right] dx_0
\end{aligned}$$

Denote

$$I = \int_{-\infty}^{\infty} A(\omega) d\omega \quad \text{and} \quad J = \int_{-\infty}^{\infty}\frac{e^{i\frac{\omega}{c}(|x-x_0|-ct)}}{\omega} d\omega$$

in which

$$\begin{aligned}
A(\omega) &= 2icg(x_0;\omega)\, e^{i\frac{\omega}{c}(|x|+|x_0|-ct)} \\
&= \frac{\left[(\omega^2-\gamma)\ (\omega^2-{\omega_0}^2-\gamma)-\gamma^2\right]}{\omega\left[(\omega-\omega_*)(\omega+\overline{\omega_*})(\omega-\omega_1)(\omega-\omega_2)\right]} e^{i\frac{\omega}{c}(|x|+|x_0|-ct)}
\end{aligned}$$

Now since $x > 0$ and $x_0 < 0$

$$|x| + |x_0| - ct = |x - x_0| - ct = x - ct - x_0$$

Denote $x - ct = \xi$, then $|x| + |x_0| - ct = \xi - x_0$

Then,

$$J = \int_{-\infty}^{\infty}\frac{e^{i\frac{\omega}{c}(\xi - x_0)}}{\omega} d\omega$$

and

$$I = \int_{-\infty}^{\infty}\frac{\left[(\omega^2-\gamma)\ (\omega^2-{\omega_0}^2-\gamma)-\gamma^2\right]}{\omega\left[(\omega-\omega_*)(\omega+\overline{\omega_*})(\omega-\omega_1)(\omega-\omega_2)\right]} e^{i\frac{\omega}{c}(\xi-x_0)} d\omega$$

Now since $e^{i\frac{\omega}{c}(\xi-x_0)}$ decays exponentially in the upper-half $\omega$ plane for $\xi - x_0 > 0$ and decays exponentially in the lower-half $\omega$ plane for $\xi - x_0 < 0$ residue calculus yields

$$J = \begin{cases} 0, & \xi - x_0 \ > 0 \\ 2\pi i, & \xi - x_0 \ < 0 \end{cases}$$

Also since $A(\omega)$ decays exponentially in the upper-half $\omega$ plane for $\xi - x_0 \ > 0$, we have that

$$I = 0 \text{ when } \xi - x_0 \ > 0$$

and when $\xi - x_0 < 0$, $A(\omega)$ decays exponentially in the lower-half $\omega$ plane and residue calculus gives

$$I = 2\pi i \left[Res\left(A,\ 0\right) + Res\left(A, \omega_1\right) + Res\left(A, \omega_2\right) + Res\left(A, \omega_*\right) + Res\left(A, -\overline{\omega_*}\right)\right]$$

Now we can express the transmitted field, $u_T(x,t)$ in terms of the integrals $I$ and $J$ using the expressions for these residues.

Hence, for $x > 0$, denoting $x - ct = \xi$ and $u_T(x,t) = T(\xi)$

$$\begin{aligned} T(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{v(x_0)\, \chi_{(-\infty,0)}}{2ic} \left[\ J + I\ \right] dx_0 \\ &= \frac{1}{2c} \int_{\xi}^{0} v(x_0) \left[Res\left(A, \omega_1\right) + Res\left(A, \omega_2\right) + Res\left(A, \omega_*\right) + Res\left(A, -\overline{\omega_*}\right)\right] dx_0 \end{aligned}$$

where

$$Res\left(A, \omega_*\right) = \gamma^2 C_{\omega_*} e^{ik_c(\xi - x_0)} . e^{\gamma^2 \kappa^* (\xi - x_0)}$$

$$Res\left(A, -\overline{\omega_*}\right) = \gamma^2 C_{-\overline{\omega_*}} e^{-ik_c(\xi - x_0)} . e^{\gamma^2 \kappa^* (\xi - x_0)}$$

$$Res\left(A, \omega_1\right) = C_{\omega_1} e^{\gamma \kappa^{(1)} (\xi - x_0)}$$

$$Res\left(A, \omega_2\right) = C_{\omega_2} e^{\kappa^{(2)} (\xi - x_0)}$$

with $k_c = \frac{\omega_c}{c}, \kappa^* = \frac{\kappa}{c}$ , $C_{\omega_*},\ C_{-\overline{\omega_*}}, \kappa^{(1)} = \frac{\beta^{(1)}}{c}, \kappa^{(2)} = \frac{\beta^{(2)}}{c},\ C_{\omega_1},\ C_{\omega_2} \ = \mathrm{O}(1)$

We have thus expressed the transmitted field as an integral of the residues of the poles at $\omega_*, -\bar{\omega}_*,\ \ \omega_1,\ \omega_2$ of the analytic continuation of the resolvent $\left(L - \omega^2 I\right)^{-1}$.

### 3.5 Resonant and Modified Direct Pathways

The presence of two temporal pathways in the transmission and reflection processes in resonant scattering by photonic crystal slabs is described in [9]. The time sequence of a transmission process is observed to consist of two distinct stages: an initial pulse and a tail of long decay. The presence of these two stages indicates the existence of two pathways in the transmission process. The initial pulse corresponds to a direct transmission process in which a portion of the incident energy goes straight through the slab. The tail of long decay corresponds to an indirect resonant transmission process: The remaining portion of the incident energy excites the guided resonances, builds up in the resonator and then slowly leaks out.

Intuitively, the Fourier transform of the initial pulse should have a shape which closely resembles that of the incoming pulse and should account for the background in the transmission spectra, while the Fourier transform of the decaying tail should be Lorentzian line shapes associated with resonances.

We can provide a rigorous definition of these two temporal pathways that are intuitively well understood and experimentally observed. The analysis of the pole structure of the resolvent operator underlying the scattering problem for our Resonant Lamb model enables us to mathematically characterize the two transmission processes as "The resonant transmission" and "The direct transmission"

The expression we obtained for the transmitted field using the residues of the poles of the resolvent in the last section enables us to distinguish between the four poles as ones contributing to the direct transmission and the ones contributing to the resonant transmission.

We identify the residues of the poles at $\omega_*$ and $-\bar{\omega}_*$, originating form the resonant part of the two-part scatterer to be contributing to the resonant transmission. The residues of the poles at $\omega_1$ and $\omega_2$ originating form the non-resonant part of the two-part scatterer are identified to be contributing to the direct transmission.

**Definition 3.3.** ***Resonant transmission and Direct transmission***

1. *The part of the transmitted field which is the contribution from the residues at the poles of the resolvent operator originating from the resonant mode of the scatterer*

*is defined as the resonant transmission;*

$$T_R(\xi) = \frac{1}{2c}\int_{\xi}^{0} v(x_0)\left[Res(A,\omega_*) + Res(A,-\bar{\omega_*})\right]dx_0$$

2. *The part of the transmitted field which is the contribution from the residues at the poles of the resolvent operator originating from the non-resonant part of the scatterer is defined as the direct transmission;*

$$T_D(\xi) = \frac{1}{2c}\int_{\xi}^{0} v(x_0)\left[Res(A,\omega_1) + Res(A,\omega_2)\right]dx_0$$

We can express the transmitted field $T(\xi)$as the sum of direst transmission $T_D(\xi)$ and resonant transmission $T_R(\xi)$.

$$T(\xi) = T_D(\xi) + T_R(\xi)$$

Resonance in a scattering process occurs from the interference between the direct pathway and the resonant (resonance-assisted indirect) pathway. Hence, the properties of a resonant transmission process can be determined by analyzing the interference between the direct and the resonant pathways. The same analysis holds for the reflected field of a resonant scattering process as well.

This insight has led to an important development in the theory of Fano resonance for optical resonators: An intuitive theory based on a temporal coupled-mode formalism has been introduced [20] to explain complex features of such resonances. This theory leads to good approximations of resonant behavior of fields.

### 3.6 Examples

In this section we will compute the actual resonant transmission and the direct transmission for our system under the incidence of two types of wave pulses. These computations will better illustrate the resonant feature of the “resonant transmission”.

Before looking at examples under specific forcing fields, let us convert the expressions for the resonant transmission and direct transmission in to a form that can be easily analyzed.

Denote

$$T_{R^*}(\xi) = \gamma^2 \int_{\xi}^{0} v(x_0)\, Res\,(A, \omega_*) dx_0$$

$$T_{R^{\overline{*}}}(\xi) = \gamma^2 \int_{\xi}^{0} v(x_0)\, Res\,(A, \omega_*) dx_0$$

So that the resonant transmission $T_R(\xi)$ is given by

$$T_R(\xi) = \frac{1}{2c}\left[C_{\omega_*} T_R^{(*)} + C_{-\overline{\omega_*}} T_R^{(\overline{*})}\right]$$

Depending on which frequency out of $k_c = \frac{\omega_c}{c}$ or $-k_c = \frac{-\omega_c}{c}$ lie closer to the forcing frequency $\overline{k} = \frac{\overline{\omega}}{c}$, only one of $T_{R^*}(\xi)$ or $T_{R^{\overline{*}}}(\xi)$ will make a significant contribution to resonant amplification of $T_R(\xi)$. Hence without any loss of generality let us analyze $T_{R^*}(\xi)$ given by

$$T_{R^*}(\xi) = \gamma^2 \int_{\xi}^{0} v(x_0)\, e^{ik_c(\xi - x_0)} . e^{\gamma^2 \kappa^* (\xi - x_0)} dx_0$$

Denoting $h(x) = e^{ik_c x} . e^{\gamma^2 \kappa^* x}$, $T_{R^*}(\xi)$ is given by

$$T_{R^*}(\xi) = \gamma^2 \int_{\xi}^{0} v(x_0)\, h(\xi - x_0) dx_0$$

Let $v_b(x_0) = v(x_0 + b)$ and define

$$T^b{}_{R^*}(\xi) = \gamma^2 \int_{\xi}^{0} v(x_0 + b)\, h(\xi - x_0) dx_0$$

So that

$$T_{R^*}(\xi) = \gamma^2 \int_{\xi}^{b} v(x_0)\, h(\xi - x_0) dx_0$$

Similarly the direct transmission $T_D(\xi)$ is given by

$$T_D(\xi) = \frac{1}{2c} \int_{\xi}^{b} v(x_0) \left[C_{\omega_1} e^{\gamma \kappa^{(1)}(\xi - x_0)} + C_{\omega_2} e^{\kappa^{(2)}(\xi - x_0)}\right] dx_0$$

**Example 1.** As the first example we will consider the situation when the forcing $v(x_0)$ is given by

$$v(x_0) = \chi_{[a,\ b]} cos\left(\overline{k}x_0\right)$$

Considering $v_b(x_0) = v(x_0 + b)$ would give us a forcing that is supported completely to the left of the scatterer and by the preceding discussion for $\xi < b$, $T_{R^*}(\xi)$ is given by

$$T_{R^*}(\xi) = \gamma^2 \int_\xi^b v(x_0)\, h(\xi - x_0) dx_0$$

To analyze $T_{R^*}(\xi)$ when the whole pulse has passed the scatterer, let $\xi < a$. In this case

$$T_{R^*}(\xi) = \gamma^2 \int_a^b v(x_0)\, h(\xi - x_0) dx_0$$

We compute $T_{R^*}(\xi)$, taking $[a,\ b] = [0,\ L]$ (so that $v(x_0) = \chi_{[0,\ L]} cos\left(\overline{k}x_0\right)$)

$$\begin{aligned} T_{R^*}(\xi) &= \mathrm{Re}\left\{ \frac{\gamma^2 C_{\omega_*}}{2c} \int_0^L e^{i\overline{k}x_0} e^{ik_c(\xi - x_0)} . e^{\gamma^2\kappa^*(\xi - x_0)} dx_0 \right\} \\ &= \mathrm{Re}\left\{ \frac{\gamma^2 C_{\omega_*} e^{\gamma^2\kappa^*\xi} e^{ik_c\xi}}{2c} \int_0^L e^{i\left(\overline{k} - k_c\right)x_0 - \gamma^2\kappa^* x_0} dx_0 \right\} \\ &= \mathrm{Re}\left\{ \frac{C_{\omega_*} e^{\gamma^2\kappa^*\xi} e^{ik_c\xi}}{2c} \left( \frac{\gamma^2}{\gamma^2\kappa^* - i\left(\overline{k} - k_c\right)} \right) \left[ 1 - e^{-\gamma^2\kappa^* L} e^{i\left(\overline{k} - k_c\right)L} \right] \right\} \end{aligned}$$

Suppose that the frequency mismatch between the central frequency of the incoming pulse and the resonance frequency, $\overline{k} - k_c$ is of order $\gamma^2$ and the spectral width of the incident pulse, $1/L$ is also of order $\gamma^2$ (so that for a weak coupling we have a spatially broad pulse with spatial width of order $\gamma^{-2}$). Then $T_{R^*}(\xi)$ becomes

$$T_{R^*}(\xi) = \mathrm{Re}\left\{ \frac{C_{\omega_*} e^{\gamma^2\kappa^*\xi} e^{ik_c\xi}}{2c} \left( \frac{1 - e^{(i\Delta - \kappa^*)\tilde{L}}}{\kappa^* - i\Delta} \right) \right\}$$

in which $\gamma^2\Delta = \overline{k} - k_c$ and $\gamma^2 L = \tilde{L}$ with $\Delta,\ \tilde{L}{=}O(1)$

This expression for $T_{R^*}(\xi)$ shows us that when the forcing is concentrated spectrally in the proximity of the resonance frequency $k_c$, the resonant part of the transmitted field $T_R(\xi)$ grows to order 1 despite the factor of $\gamma^2$ in $T_R(\xi)$ coming from the residues of the poles at $\omega_*$ and $-\overline{\omega_*}$. The factor $e^{\gamma^2\kappa^*\xi}$ gives the slow decay of the resonant transmission pathway.

**Example 2.** As the next example we will consider the situation when the forcing $v(x_0)$ is given by

$$v(x_0) = c\left(\sigma^2 x_0 - i\overline{k}\right) e^{i\overline{k}x_0}.e^{-\frac{\sigma^2 x_0{}^2}{2}}$$

so that the incident field is a Gaussian pulse with central frequency $\overline{k}$ and with spectral width $\sigma$. This is not a forcing that satisfies the finite time condition but will serve as an example of an infinitely supported finite energy pulse and we will be able to observe the features of the "resonant" and "direct" transmission pathways by analyzing approximate expressions.

Considering $v_b(x_0) = v(x_0 + b)$ and taking $b \longrightarrow \infty$ gives us a forcing that is supported completely to the left of the scatterer and by the preceding discussion $T_{R^*}(\xi)$ is given by

$$\begin{aligned}
T_{R^*}(\xi) &= \gamma^2 \int_\xi^\infty v(x_0)\, h(\xi - x_0) dx_0 \\
&= \frac{\gamma^2 C_{\omega_*}}{2c} \int_\xi^\infty \left(\sigma^2 x_0 - i\overline{k}\right) e^{i\overline{k}x_0}.e^{-\frac{\sigma^2 x_0{}^2}{2}} e^{ik_c(\xi - x_0)}.e^{\gamma^2\kappa^*(\xi - x_0)} dx_0 \\
&= \frac{\gamma^2 C_{\omega_*}}{2c} e^{ik_c\xi} e^{\gamma^2\kappa^*\xi} \int_\xi^\infty \left(\sigma^2 x_0 - i\overline{k}\right) e^{i\left(\overline{k} - k_c + i\ \gamma^2\kappa^*\right)x_0}.e^{-\frac{\sigma^2 x_0{}^2}{2}} dx_0
\end{aligned}$$

Now define,

$$I_\xi^{(\theta)} = \int_\xi^\infty \left(\sigma^2 x_0 - i\overline{k}\right).e^{i\theta x_0 - \frac{\sigma^2 x_0{}^2}{2}} dx_0$$

$$I_\xi = \sigma^2 \int_\xi^\infty t e^{-\frac{\sigma^2 t^2}{2}} dt$$

$$J_\xi = \int_\xi^\infty e^{-\frac{\sigma^2 t^2}{2}} dt$$

then, using the substitution $t = x_0 - i\frac{\theta}{\sigma^2}$ in $I_\xi^{(\theta)}$ we arrive at

$$I_\xi^{(\theta)} = e^{-\frac{\theta^2}{2\sigma^2}} \left[I_\xi + i\left(\theta - \overline{k}\right) J_\xi\right]$$

Now since $J_\xi$ cannot be analytically evaluated, we express $J_\xi$ in terms of the Gaussian Error function erf$(x)$ given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Let

$$J_\xi{}^0 = \int_\xi^0 e^{-\frac{\sigma^2 {x_0}^2}{2}} dx_0 \quad \text{so that} \quad J_\xi = J_\xi{}^0 + \int_0^\infty e^{-\frac{\sigma^2 {x_0}^2}{2}} dx_0 = J_\xi{}^0 + \sqrt{\frac{\pi}{2}}.\frac{1}{\sigma}$$

The substitution $\sigma x_0 = -\sqrt{2}\, t$ in $J_\xi{}^0$ yields;

$$J_\xi{}^0 = \frac{\sqrt{2}}{\sigma} \int_0^{-\frac{\sigma\xi}{\sqrt{2}}} e^{-t^2} dt = \sqrt{\frac{\pi}{2}}.\frac{1}{\sigma}.erf\left(-\frac{\sigma}{\sqrt{2}}\xi\right)$$

So

$$J_\xi = \sqrt{\frac{\pi}{2}}.\frac{1}{\sigma}\left[\text{erf}\left(-\frac{\sigma}{\sqrt{2}}\xi\right) + 1\right]$$

we also evaluate $I_\xi$ to get, $I_\xi = e^{-\frac{\sigma^2\xi^2}{2}}$

hence

$$I_\xi^{(\theta)} = e^{-\frac{\theta^2}{2\sigma^2}}\left[e^{-\frac{\sigma^2\xi^2}{2}} + i\left(\theta - \overline{k}\right)\sqrt{\frac{\pi}{2}}.\frac{1}{\sigma}.\left[\text{erf}\left(-\frac{\sigma}{\sqrt{2}}\xi\right) + 1\right]\right]$$

We can obtain $T_{R^*}(\xi)$ from $I_\xi^{(\theta)}$ when $\theta = \overline{k} - k_c + i\gamma^2\kappa^*$

$$T_{R^*}(\xi) = \frac{\gamma^2 C_{\omega_*}}{2c} e^{ik_c\xi} e^{\gamma^2\kappa^*\xi}.\ I_\xi^{\left(\overline{k} - k_c + i\gamma^2\kappa^*\right)}$$

$$= \frac{\gamma^2 C_{\omega_*}}{2c} e^{ik_c\xi} e^{\gamma^2\kappa^*\xi} e^{-\frac{\left[\overline{k} - k_c + i\gamma^2\kappa^*\right]^2}{2\sigma^2}} \left[e^{-\frac{\sigma^2\xi^2}{2}} - \frac{\sqrt{\pi}\left(ik_c - \gamma^2\kappa^*\right)}{\sqrt{2}\sigma}\left[\text{erf}\left(-\frac{\sigma}{\sqrt{2}}\xi\right) + 1\right]\right]$$

Let $\eta = \overline{k} - k_c$ be the frequency mismatch between the carrier frequency of the incoming pulse and the resonant frequency. Then

$$T_R^{(*)} = \frac{\gamma^2 C_{\omega_*}}{2c} e^{ik_c\xi} e^{\gamma^2\kappa^*\xi}.e^{-\frac{\left[\eta + i\gamma^2\kappa^*\right]^2}{2\sigma^2}} \left[e^{-\frac{\sigma^2\xi^2}{2}} - \frac{\sqrt{\pi}\left(ik_c - \gamma^2\kappa^*\right)}{\sqrt{2}\sigma}\left[1 - \text{erf}\left(\frac{\sigma}{\sqrt{2}}\xi\right)\right]\right]$$

Now suppose that the frequency mismatch $\eta$ is of order $\gamma^2$ and the spectral width of the incident pulse, $\sigma$ is also of order $\gamma^2$ (so that for a weak coupling we have a spatially broad pulse with spatial width of order $\gamma^{-2}$).

Let $\frac{1}{\sigma} = L$ : Spatial width of the incident Gaussian pulse

Hence $T_{R^*}(\xi)$ becomes,

$$T_R^{(*)} = \frac{\gamma^2 C_{\omega_*}}{2c} e^{ik_c\xi} e^{\gamma^2\kappa^*\xi} e^{-\frac{[\overline{L}(\overline{\eta}+i\kappa^*)]^2}{2}} \left[ e^{-\frac{\sigma^2\xi^2}{2}} - \frac{\sqrt{\pi}\,(ik_c - \gamma^2\kappa^*)}{\sqrt{2}\sigma} \left[ 1 - \operatorname{erf}\left( \frac{\sigma}{\sqrt{2}}\xi \right) \right] \right]$$

in which $\sigma = \gamma^2\,\overline{\sigma}$, $\gamma^2 L = \overline{L}$ and $\overline{k} - k_c = \eta = \gamma^2\overline{\eta}$ ;

$\overline{\sigma}$ : Normalized spectral width

$\overline{\eta}$ : Normalized frequency mismatch

# Chapter 4

# Asymptotic Nature of High-Q and Near-monochromatic Resonance

The resonant features of a resonant scattering process depend on certain aspects of the system, including aspects of the resonant scatterer as well as some aspects of the incident pulse. Quantifying the delicate behavior of resonance features in the simultaneous high-Q (high quality factor as a consequence of a small bandwidth of resonance) and near-monochromatic (spectrally localized incident field) regimes will give us detailed insight in to resonant scattering characteristics of a photonic device. High Q-factor result in high resonant amplification of the transmitted field and slow energy decay from the resonator within a narrow frequency band; and the near-monochromatic regime refers to a resonant system operating under the incidence of a source field with a well-defined central (carrier) frequency that is tapered in time and space (finite energy source).

The scattering characteristics of a photonic device operating at resonance are very sensitive to parameters of the structure and the source field. Specific devices are analyzed with a combination of mathematical and numerical methods, and the analysis often treats the resonant scattering of Gaussian beams and pulses and how it depends on the angle of incidence [28, 29].

The model we devised as our Resonant Lamb Model is not complicated enough to address aspects of angle of incidence. Our aim is to rigorously address universal aspects of resonant amplification with regard to the delicate balance between the spectral widths of the resonance and the source field.

## 4.1 Three Distinguished Parameters that Affect Resonance

We will study the effects of three distinguished parameters on the resonant features of the system. The high-Q near-monochromatic regime involves three simultaneously small physical parameters. The constant of coupling, $\gamma$, between the point-mass and the resonator controls the spectral width of the transmission resonance, which is of order $\gamma^2$, and the Q-factor, which is of order $\gamma^{-2}$. The spectrum of the resonator is centered about a frequency $\omega_c$. The source field will be a wave packet centered about a frequency $\overline{\omega}$ with spectral width $\sigma$. The difference between the central frequencies of the resonance and the source field is denoted by $\eta = \overline{\omega} - \omega_c$. The three parameters

$\gamma^2$ : spectral width of transmission resonance

$\sigma$ : spectral width of source field

$\eta$ : difference between resonant and source frequencies

are considered to be small, and we will analyze the effects of differing their relative sizes.

We will analyze the effects of differing the relative sizes of the three above mentioned parameters has on the resonant features of our resonant Lamb model. However, the discussion will be universal for a resonant system that has the essential features of resonance captured in this model as described in chapter two.

In order to perform the appropriate analysis, we will first reduce the dynamics of the scattering system to the two-part scatterer and analyze the amplitude $y(t)$ of the point-mass (non-resonant part of the scatterer) and the amplitude $z(t)$ of the spring-mass (resonant part of the sactterer). Similar to the notion of resonant and direct pathways of the transmitted field that was defined in chapter 3, we will define resonant and regular parts of the fields $y(t)$ and $z(t)$. The analysis of the effects of the preceding three parameters on the resonant features of our model will then essentially come down to analyzing the effects of differing their relative size has on the resonant part of the field $y(t)$ (amplitude of the point-mass).

### 4.2 Reduction of the system to the scatterer

Recall that the scattering solutions for our resonant Lamb model is of the form

$$u(x,t) = \begin{cases} J(x-ct) + R(x+ct), & x \le 0, \\ T(x-ct), & x \ge 0. \end{cases}$$

where the function $J(\xi)$ is the source field incident upon the scatterer from the left, and $R(\xi)$ and $T(\xi)$ are the reflected and transmitted fields. If $J(\xi) = 0$ for $\xi > \xi_0$, then before time $t = -\xi_0/c$, the excitation $J(x-ct)$ is supported completely to the left of the scatterer. In this case, we assume that the scatterer and the right side of the string are at rest before time $t = -\xi_0/c$ so that $R(\xi) = 0$ for $\xi < -\xi_0$ and $T(\xi) = 0$ for $\xi > \xi_0$.

Letting $\xi = x - ct$ and setting $x = 0$, we observe that the point-mass on the string at $x = 0$ experiences the source field $J(\xi)$ as a time-dependent input $j(t)$ given by

$$j(t) := J(-c\,t)\,.$$

The Fourier-Laplace transform, $\hat{j}(\omega)$ of $j(t)$ is given by

$$\hat{j}(\omega) = \int j(t) e^{i\omega t}\, dt\,.$$

The equations for the dynamics of the string and scatterer;

$$u_{tt}\,(x,t) = c^2 u_{xx}\,(x,t)\,, \qquad x \neq 0$$

$$\ddot{y}\,(t) = \gamma\ \left(z\,(t) - y\,(t)\right) + \beta\ [u_x]_0\,, \qquad y\,(t) = u\,(0,t)$$

$$\ddot{z}\,(t) = -{\omega_0}^2 z\,(t) + \gamma\,(y\,(t) - z\,(t))$$

can be reduced to the two-part scatterer given by the following system of ordinary differential equations for $y$ and $z$,

$$\begin{aligned} \ddot{y}(t) &= -\alpha\,\dot{y}(t) + \gamma\big(z(t) - y(t)\big) + \alpha j'(t) \\ \ddot{z}(t) &= -{\omega_0}^2 z\,(t) + \gamma\big(y(t) - z(t)\big). \end{aligned}$$

in which $\alpha = 2\beta/c$.

Once we have a solution to this system in hand, the transmitted and reflected fields, and therefore the full field $u(x,t)$, can be obtained through

$$\begin{aligned} T(\xi) &= y(-\xi/c) \\ R(\xi) &= y(\xi/c) - J(-\xi). \end{aligned}$$

Under the Fourier-Laplace transform, the above system of equations becomes

$$\begin{bmatrix} \omega^2 + i\alpha\,\omega - \gamma & \gamma \\ \gamma & \omega^2 - \omega_0^2 - \gamma \end{bmatrix} \begin{bmatrix} \hat{y}(\omega) \\ \hat{z}(\omega) \end{bmatrix} = \begin{bmatrix} i\alpha\,\omega\hat{\jmath}(\omega) \\ 0 \end{bmatrix}.$$

The solution is given by

$$\hat{y}(\omega) = i\alpha\,\frac{\omega(\omega^2 - \omega_0^2 - \gamma)}{D(\omega)}\,\hat{\jmath}(\omega) \tag{4.1}$$

$$\hat{z}(\omega) = -i\alpha\gamma\,\frac{\omega}{D(\omega)}\,\hat{\jmath}(\omega), \tag{4.2}$$

in which the determinant of the system is the same denominator $D(\omega)$ we had in chapter three for the resolvent given by

$$\begin{aligned} D(\omega) &= (\omega^2 + i\alpha\omega - \gamma)(\omega^2 - \omega_0^2 - \gamma) - \gamma^2 \\ &= (\omega - \omega_*)\,(\omega + \overline{\omega_*})\,(\omega - \omega_1)\,(\omega - \omega_2) \end{aligned}$$

A partial-fraction decomposition of the rational functions multiplying $\hat{\jmath}(\omega)$ in $\hat{y}(\omega)$ and $\hat{z}(\omega)$ splits the solutions $y(t)$ and $z(t)$ into four parts, so that each root of $D(\omega)$ correspond to one partial fraction of each rational function. Recall that the poles $\omega_*$ and $-\bar{\omega}_*$ correspond to resonant transmission and the poles $\omega_1, \omega_2$ correspond to direct transmission within the transmitted field. This same notion stands for the field $y(t)$ of the scatterer as well as the field $z(t)$ of the resonator. We define the "resonant part" and the "regular part" of these fields;

$$y_{\text{res}}(t) = \frac{i}{2\pi}\,\gamma^2 \int \left( \frac{A(\gamma)}{\omega - \omega_*} - \frac{\bar{A}(\gamma)}{\omega + \bar{\omega}_*} \right) \hat{\jmath}(\omega)\, e^{-i\omega t}\, d\omega\,,$$

$$z_{\text{res}}(t) = \frac{i}{2\pi}\,\gamma \int \left( \frac{B(\gamma)}{\omega - \omega_*} - \frac{\bar{B}(\gamma)}{\omega + \bar{\omega}_*} \right) \hat{\jmath}(\omega)\, e^{-i\omega t}\, d\omega\,.$$

$$y_{\text{reg}}(t) = \frac{i}{2\pi} \int \left( \frac{C_1(\gamma)}{\omega - \omega_1} + \frac{C_2(\gamma)}{\omega - \omega_2} \right) \hat{\jmath}(\omega)\, e^{-i\omega t}\, d\omega\,,$$

$$z_{\text{reg}}(t) = \frac{i}{2\pi}\,\gamma \int \left( \frac{D_1(\gamma)}{\omega - \omega_1} + \frac{D_2(\gamma)}{\omega - \omega_2} \right) \hat{\jmath}(\omega)\, e^{-i\omega t}\, d\omega\,.$$

Powers of $\gamma$ have been extracted from these expressions so that the $\gamma$-dependent residues $A, B, C, D$ are of order 1.

The factor of $\gamma^2$ in $y_{\text{res}}(t)$ is due to the numerator in (4.1) when computing the residues. The factor of $\gamma$ in $z_{\text{res}}(t)$ and $z_{\text{reg}}(t)$ manifests the weak coupling from the string to the resonator, and the $\gamma^2$ in $y_{\text{res}}(t)$ manifests the weak coupling back to the string.

The interesting resonant regime, in which the parameter analysis is applicable, occurs when $\hat{\jmath}(\omega)$ is concentrated near the central frequency $\omega_c = \text{Re}(\omega_*)$ (and/or $-\omega_c$) of the resonance (near-monochromatic). In this case the resonator experiences resonant amplitude enhancement: $z_{\text{res}}$ becomes very large despite the factor of $\gamma$ and then it experiences slow decay. The decay of its energy back into the string through the point-mass scatterer is manifest in $y_{\text{res}}(t)$, which grows to order 1 despite the factor of $\gamma^2$. This phenomenon is referred to as "delayed resonant transmission" since the transmitted field experiences resonant amplification after a certain delay, due to the slow decay of the resonantly amplified resonator field $z(t)$. The analysis of the corresponding resonant enhancements is subtle with regard to the relationships between $\sigma$, $\gamma$, and $\eta = \overline{\omega} - \omega_c$ when all three parameters are small.

### 4.3 Asymptotic Behavior of High-Q and Near-monochromatic Resonance

Let us first clarify the near-monochromatic field that we consider in this analysis. The near-monochromatic field $J(\xi)$ we consider is a pure oscillation at frequency $\overline{\omega}$ modulated by a broad envelope whose width is controlled by $\sigma^{-1}$ with $\sigma$ being a small parameter,

$$J(\xi) = g\big(-\sigma\,\xi/c\big)\exp\left(i\,\frac{\overline{\omega}}{c}\xi\right).$$

By putting $\xi = x - ct$ and setting $x = 0$, we observe that the point-mass on the string experiences this wave temporally as

$$j(t) := J(-c\,t) = g(\sigma t)\exp\left(-i\overline{\omega}t\right).$$

The Fourier-Laplace transform of $j$,

$$\hat{j}(\omega) = \int j(t)e^{i\omega t}\,dt \;=\; \frac{1}{\sigma}\,\hat{g}\left(\frac{\omega-\overline{\omega}}{\sigma}\right).$$

shows us that the spectral width of this source field is $\sigma$.

Suppose that the source field $j(t)$ driving the scatterer at $x = 0$ is "turned on" at $t = 0$ and "turned off" at some later time. We might as well let $g(t)$ be supported in the interval $[0, 1]$, so that $g(t) = 0$ for $t < 0$ and for $t > 1$. According to the definition of $j(t)$, this means that $j(t)$ begins oscillating at time $t = 0$ and stops at time $t = \sigma^{-1}$.

In this case the scatterer is at rest for $t \le 0$, then once it starts being "driven" by the source field, the resonator builds up energy (its amplitude $z(t)$ builds up) during the time interval from $t = 0$ to $t = \sigma^{-1}$, and then once the source field stops "driving" the scatterer at $t = \sigma^{-1}$, the energy stored in the resonator slowly decays back in to the string via the point-mass scatterer ($y(t)$ and the transmitted field $T(\xi)$ experiences delayed resonant enhancement).

Given that the forcing frequency $\overline{\omega}$ is close to the central frequency $\omega_c$ of the resonance (meaning that $\eta$ is small), $\hat{j}(\omega)$ is concentrated near the complex resonance $\omega_*$. Hence only the first of the two terms of $z_{\text{res}}(t)$ contributes significant amplification of the resonator field. We therefore wish to determine the strength of the normal mode corresponding to the resonance at $\omega_*$, as it depends on $\gamma$, $\eta$, and $\sigma$, when all three parameters are considered to be small.

Hence we analyze the quantity

$$z_{\text{res}^*}(t) = \frac{i}{2\pi}\gamma \int \frac{1}{\omega - \omega_*}\hat{\jmath}(\omega)e^{-i\omega t}d\omega$$

to quantify the dependance of resonant enhancement of the spring-mass resonator of our system on the relative sizes of $\gamma$, $\eta$, and $\sigma$.

**Lemma 4.1.** *For* $t > \sigma^{-1}$

$$z_{res^*}(t) \;=\; \frac{\gamma}{\eta + i\kappa\gamma^2}\, h\Big(\frac{\eta + i\kappa\gamma^2}{\sigma}\Big)e^{-i\sigma^{-1}\overline{\omega} - i(t-\sigma^{-1})}e^{-\kappa\gamma^2(t-\sigma^{-1})}$$

*Further when* $\sigma \ll \gamma^2$, $z_{res^*}(t)$ *reduces to*

$$z_{res^*}(t) \;\sim\; \frac{\gamma}{\eta + i\kappa\gamma^2} ig(1)e^{-i\sigma^{-1}}e^{-i(t-\sigma^{-1})}e^{-\kappa\gamma^2(t-\sigma^{-1})}$$

*Proof.* Consider

$$z_{\text{res}^*}(t) = \frac{i}{2\pi}\gamma \int \frac{1}{\omega - \omega_*}\hat{\jmath}(\omega)e^{-i\omega t}d\omega$$

Under the change of variable

$$\omega = \sigma\phi + \overline{\omega},$$

and recalling that $\omega_* = \omega_c - i\kappa\gamma^2$ and $\eta = \overline{\omega} - \omega_c$, the expression for $z_{\text{res}^*}(t)$ becomes

$$z_{\text{res}^*}(t) \;=\; \frac{\gamma}{\eta + i\kappa\gamma^2}\, e^{-i\overline{\omega}t}\frac{i}{2\pi}\int \frac{\eta + i\kappa\gamma^2}{\sigma\phi + \eta + i\kappa\gamma^2}\,\hat{g}(\phi)e^{-i\sigma\phi t}d\phi \tag{4.3}$$

Denote

$$I(t) = \frac{i}{2\pi}\int \frac{\eta + i\kappa\gamma^2}{\sigma\phi + \eta + i\kappa\gamma^2}\,\hat{g}(\phi)e^{-i\sigma\phi t}d\phi \tag{4.4}$$

Since $g(t)$ is bounded and supported in the time interval $[0,1]$, we have

$$|\hat{g}(\phi)| \le C e^{|\mathrm{Im}\,\phi|}/\phi \quad \text{as } \mathrm{Im}\,\phi \to -\infty,$$

$$|\hat{g}(\phi)| \le C \qquad \text{as } \mathrm{Im}\,\phi \to 0, \tag{4.5}$$

$$\hat{g}(\phi) \sim C/\phi \qquad \text{for } \phi \text{ real and } |\phi| \to \infty$$

Thus for $t > \sigma^{-1}$ the integrand of $I(t)$ is exponentially decaying as $\mathrm{Im}\,\phi \to -\infty$ and the integral can be computed by residue calculus. Hence we get

$$I(t) = \frac{\eta + i\kappa\gamma^2}{\sigma}\,\hat{g}\Big(-\frac{\eta + i\kappa\gamma^2}{\sigma}\Big)e^{-\kappa\gamma^2 t}e^{i\eta t}.$$

By defining the function

$$h(\phi) = \phi\, e^{i\phi}\,\hat{g}(-\phi)\,,$$

we obtain the following expression for $z_{\mathrm{res}*}(t)$ for large time, $t > \sigma^{-1}$

$$z_{\mathrm{res}*}(t) \;=\; \frac{\gamma}{\eta + i\kappa\gamma^2}\,h\Big(\frac{\eta + i\kappa\gamma^2}{\sigma}\Big)e^{-i\sigma^{-1}\overline{\omega} - i(t-\sigma^{-1})}e^{-\kappa\gamma^2(t-\sigma^{-1})} \tag{4.6}$$

Because of the bounds (4.5) on $|\hat{g}(\phi)|$, the function $h(\phi)$ is bounded by a constant for $\mathrm{Im}\,\phi < 0$.

Now suppose $\sigma \ll \gamma^2$. In this case, the imaginary part of the argument of $\hat{g}$ becomes unbounded, and we obtain the asymptotic value

$$\hat{g}\Big(-\frac{\eta + i\kappa\gamma^2}{\sigma}\Big) \sim g(1)\frac{i\sigma}{\eta + i\kappa\gamma^2}e^{-i\eta/\sigma}e^{\kappa\gamma^2/\sigma} \quad (\sigma \ll \gamma^2),$$

uniformly in $\eta/\sigma$.

Substituting this expression for $\hat{g}\Big(-\frac{\eta+i\kappa\gamma^2}{\sigma}\Big)$ in $z_{\mathrm{res}*}(t)$ we have,

for $\sigma \ll \gamma^2$ and $t > \sigma^{-1}$,

$$z_{\mathrm{res}*}(t) \;\sim\; \frac{\gamma}{\eta + i\kappa\gamma^2}\;ig(1)e^{-i\sigma^{-1}}\;e^{-i(t-\sigma^{-1})}e^{-\kappa\gamma^2(t-\sigma^{-1})} \tag{4.7}$$

□

Thus the above theorem provides us with an expression for the resonantly enhanced part of the resonator field $z_{\text{res}^*}(t)$ when it starts decaying for $t > \sigma^{-1}$. This expression simplifies further and we get an explicit asymptotic relation for the decaying $z_{\text{res}^*}(t)$ when we consider the case when $\sigma \ll \gamma^2$, which is the regime in which the spectral width of the source field is small compared to the spectral width of the resonance.

We are now in a position to analyze resonant amplification of the spring-mass resonator $z(t)$ in the high-Q ($\gamma$ small) and near-monochromatic ($\sigma$ small) regime. Equations (4.6) and (4.7) for $z_{\text{res}^*}(t)$ give an explicit expression for the coefficient of the slowly decaying normal mode $Ce^{-i(+\kappa\gamma^2)(t-\sigma^{-1})}$ associated to the complex resonance $\omega_*$ for large time $t > \sigma^{-1}$. This is one component of the solution $z(t)$. The other three normal modes in $z(t)$ come from the second term in the integrand of $z_{\text{res}}(t)$ (4.2) and the non-resonant part of $z(t)$ (4.2). All of these are of order $\gamma$.

Any resonant amplification of the spring-mass will come from the pre-factors to the exponential $e^{-i(+\kappa\gamma^2)(t-\sigma^{-1})}$ in (4.6) or (4.7).

The resonant amplification of $z_{\text{res}}(t)$ in different regimes involving the three parameters, $\sigma, \gamma^2$ and $\eta$ is provided by the following theorem.

**Theorem 4.2.** *In the regime $\gamma^2 \ll \sigma$;*

$$|z(t=\sigma^{-1})| \simeq \begin{cases} \gamma/\sigma & (\eta < C\sigma) \\ \gamma/\eta & (\sigma \ll \eta \ll \gamma) \end{cases} \qquad (\gamma^2 \ll \sigma \ll \gamma).$$

*In the regime $\sigma < C\gamma^2$;*

$$|z(t=\sigma^{-1})| \sim C\gamma^{-q}, \qquad q = \min\{p-1,\, 1\}\ .$$

*where*

$$\eta \sim C\gamma^p.$$

*In either regime,*

$$|z(t = \sigma^{-1})| \leq C\gamma^{-1} \quad (\gamma^2 \ll \sigma \ll \gamma).$$

*Proof.* Consider first the regime $\gamma^2 \ll \sigma$.

In this case, the imaginary part in the argument of $h$ in (4.6) tends to zero, and we can use the definition (4.3) of $h$ and the second bound in (4.5) to obtain

$$z_{\mathrm{res}^*}(t) \sim \frac{\gamma}{\sigma}\hat{g}(-\eta/\sigma)e^{-it}e^{-\kappa\gamma^2(t-\sigma^{-1})}$$

for $\gamma^2 \ll \sigma$ and $\eta < C\sigma$

The reason for the constraint $\eta < C\sigma$ is that $\hat{g}(-\eta/\sigma)$ vanishes when the argument becomes unbounded. The condition $\eta < C\sigma$ guarantees that $\hat{g}(-\eta/\sigma)$ is not zero.

In this regime, resonant amplification occurs when the additional relation $\sigma \ll \gamma$ is satisfied.

Now, if $\gamma^2 \ll \sigma \ll \eta$, we get $\hat{g}(-\eta/\sigma) \sim C\sigma/\eta$, so that

$$z_{\mathrm{res}^*}(t) \sim C\frac{\gamma}{\eta}e^{-it}e^{-\kappa\gamma^2(t-\sigma^{-1})}$$

for $\gamma^2 \ll \sigma \ll \eta$

Resonant amplification occurs under the additional condition $\eta \ll \gamma$. Resonance in the regime $\gamma^2 \ll \sigma$ occurs in the following situations:

$$|z(t = \sigma^{-1})| \simeq \begin{cases} \gamma/\sigma & (\eta < C\sigma) \\ \gamma/\eta & (\sigma \ll \eta \ll \gamma) \end{cases} \qquad (\gamma^2 \ll \sigma \ll \gamma).$$

The symbol "$\simeq$" indicates that $|z|$ is bounded from above and below by positive multiples of the right-hand side.

We also note that resonant amplification of $z(t)$ in this regime is always less than $C\gamma^{-1}$,

$$|z(t = \sigma^{-1})| \leq C\gamma^{-1} \quad (\gamma^2 \ll \sigma \ll \gamma).$$

Now we consider the regime $\sigma < C\gamma^2$;

In this situation, the nonzero function $h$ is bounded by a constant, so all the important information is in the amplification factor

$$\mathcal{A} \;:=\; \frac{\gamma}{\eta + i\kappa\gamma^2}.$$

This factor depends only on $\gamma$ and $\eta$, not on $\sigma$.

Given that $|z_{\mathrm{res}^*}(t)| \simeq \mathcal{A}$ for $t = \sigma^{-1}$ and that the other normal modes in $z(t)$ are of order $\gamma$, we have that

$$|z(t=\sigma^{-1})| \simeq \begin{cases} \gamma/\eta & (\gamma^2 \ll \eta < C\gamma) \\ \gamma^{-1} & (\eta < C\gamma^2) \end{cases} \qquad (\sigma < C\gamma^2).$$

To inspect how $\mathcal{A}$ depends on the asymptotic relation between $\eta$ and $\gamma$, we assume a power relation

$$\eta \;\sim\; C\gamma^p.$$

If $p < 2$, then the denominator of $\mathcal{A}$ is dominated by $\eta$, so that $|\mathcal{A}| \sim C\gamma^{1-p}$ (for a different constant $C$). If $p \geq 2$, then the denominator is dominated by $\gamma^2$, and one obtains $|\mathcal{A}| \sim C\gamma^{-1}$. In either case, $|\mathcal{A}| \sim C\gamma^{-q}$ with $q = \min\{p-1,\, 1\}$:

$$|z(t=\sigma^{-1})| \simeq |\mathcal{A}| \sim C\gamma^{-q}, \qquad q = \min\{p-1,\, 1\}$$

□

By the preceding theorem resonance in the regime $\gamma^2 \ll \sigma$ (when the spectral width of the resonance is smaller than the spectral width of the source field) occurs when the additional relations $\sigma \ll \gamma$ and $\eta \ll \gamma$ are satisfied by the parameters.

In the regime $\sigma < C\gamma^2$ (when the spectral width of the source field is at least as small as the spectral width of the resonance) the field in the resonator at time $t = \sigma^{-1}$, given by

$$|z(t=\sigma^{-1})| \simeq |\mathcal{A}| \sim C\gamma^{-q}, \qquad q = \min\{p-1,\, 1\}$$

has experienced resonant amplification over the interval $[0, \sigma^{-1}]$ only if $p > 1$.

If $p < 1$, the forcing frequency is sufficiently far from the resonant frequency so that the spring-mass is practically at rest. However, if $p > 1$, the forcing frequency is close enough to the resonant frequency so that the spring-mass experiences resonant amplification.

The resonantly amplified $z(t = \sigma^{-1})$ does not exceed the order of $\gamma^{-1}$. As time progresses past $t = \sigma^{-1}$, $z(t)$ slowly decays by a factor of $e^{-\kappa\gamma^2(t-\sigma^{-1})}$.

### 4.4 The Transmission Coefficient of a Near-monochromatic Field

The sharp asymmetric anomaly in the monochromatic transmission coefficient depicted in Figure 2.3 is typically known as a Fano resonance, in reference to Ugo Fano's famous paper. It is the result of the resonant interaction between a continuum of extended states and a near-bound state. We will analyze how this anomaly is modified when the source field has a finite lifetime but is nearly monochromatic. It degenerates as the source pulse becomes temporally shorter and spectrally wider.

The transmission number $\mathcal{T}$ for a given incident source field $J(\xi)$ is defined as the ratio of total energy transmitted across the resonator to the energy of $J(\xi)$. Because of the relations $j(t) = J(-ct)$ and $y(t) = T(-ct)$ and unitarity (up to a constant) of the Fourier-Laplace transform, one has

$$\mathcal{T} = \frac{\|T\|^2}{\|J\|^2} = \frac{\|y\|^2}{\|j\|^2} = \frac{\|\hat{y}\|^2}{\|\hat{j}\|^2}, \tag{4.8}$$

in which $\|f\| = (\int |f(s)|^2 ds)^{1/2}$, with integration over the real line, denotes the quadratic norm of any function $f$.

We define a $\sigma$-dependent transmission coefficient $\mathcal{T}_\sigma(\overline{\omega})$ to be the transmission number for a near-monochromatic source field $j(t) = g(\sigma t) \exp(-i\overline{\omega} t)$,

$$\mathcal{T}_\sigma(\overline{\omega}) := \frac{\left\| \mathfrak{t}(\omega)\sigma^{-1}\hat{g}\big(\sigma^{-1}(\omega - \overline{\omega})\big) \right\|^2}{\left\| \sigma^{-1}\hat{g}\big(\sigma^{-1}(\omega - \overline{\omega})\big) \right\|^2} = \frac{\left\| \mathfrak{t}(\omega)\hat{g}\big(\sigma^{-1}(\omega - \overline{\omega})\big) \right\|^2}{\sigma \left\| \hat{g}(\omega) \right\|^2} \tag{4.9}$$

Note that in taking the norms we integrate over $\omega$ with $\overline{\omega}$ fixed.

If the normalization $\|\hat{g}\|^2 = 1$ is taken, then this coefficient simplifies to a convolution of $\mathfrak{t}(\omega)$ with $\sigma^{-1}\left|\hat{g}(\sigma^{-1}\omega)\right|^2$,

$$\mathcal{T}_\sigma(\omega) \;=\; |\mathfrak{t}|^2 * \frac{1}{\sigma}\left|\hat{g}\Big(\frac{\cdot}{\sigma}\Big)\right|^2(\omega), \tag{4.10}$$

in which we have integrated over $\omega$ in (4.9) to compute the norm and then replaced $\overline{\omega}$ with the variable $\omega$. The mollifier $\sigma^{-1}\left|\hat{g}(\sigma^{-1}\omega)\right|^2$ tends to a delta-function as $\sigma$ vanishes, so that

$$\mathcal{T}_\sigma(\omega) \to |\mathfrak{t}(\omega)|^2 \qquad \text{as } \sigma \to 0.$$

The near-monochromatic transmission coefficient $\mathcal{T}_\sigma(\omega)$ is a regularization of the monochromatic one $\mathcal{T}_0(\omega) := |\mathfrak{t}(\omega)|^2$.

The width of the Fano resonance in our system is of order $\gamma^2$. In other words, the transmission coefficient deviates significantly from the "background" direct transmission when $\eta < C\gamma^2$. Rigorous analyses are carried out in [15, 17]. Equation (4.10) shows that the transmission anomaly persists for $\mathcal{T}_\sigma(\omega)$ if $\sigma$ is small compared to $\gamma^2$ and becomes smoothed out as $\sigma$ becomes relatively large, so that we have the following;

$$\begin{aligned} &\sigma \ll \gamma^2 && \text{sharp anomaly,}\\ &\sigma \simeq \gamma^2 && \text{weak anomaly,}\\ &\sigma \gg \gamma^2 && \text{no anomaly.} \end{aligned}$$

# Chapter 5

# Comparison of Coupled Mode Formalism to the True Dynamics

## 5.1 Temporal Coupled Mode Theory

The notion of coupling of modes is found extensively in the study of vibrational systems. An electromagnetic mode can generally be viewed as electromagnetic energy that exists independent of other electromagnetic power. Different modes belonging to the same system or to different systems, can exchange energy through a coupling perturbation. Coupled mode theory approximates the solutions to complex problems associated with the interaction of different modes of energy based on known solutions for simpler systems. In this formalism, the complex structure of a system is viewed as a collection of simpler waveguides; the modes corresponding to each simple waveguide are perturbed by the presence of other modes and this in turn causes the modes to be coupled and energy is exchanged among these coupled modes.

Temporal coupled mode theory is the application of the coupled mode formalism to coupling of modes in time. Temporal coupled mode theory allows a wide range of devices and systems to be modeled as one or more coupled resonators (coupling among resonators and with free propagating modes). The following assumptions about the system are generally made in applying a temporal coupled mode approach:

- Linearity
- Time-reversal symmetry
- Time-invariance
- Energy conservation
- Weak mode coupling (small perturbation of modes due to coupling)

This formalism can be applied to diverse cases, including coupling between guided modes and the coupling between free propagating modes and guided modes; thus this theory is applicable to the type of resonant scattering systems for which our resonant Lamb model served as a prototype.

In this chapter we wish to approximately describe the decay rate of resonance in the resonant Lamb model using a temporal coupled mode approach and compare it to the true dynamics of the system.

## 5.2 Coupled Mode Theory for the Resonant Lamb Model

We express the equations for the resonant lamb model using a temporal coupled mode formalism. We consider the situation when a resonator with natural frequency $\omega_o$ is weakly coupled to two propagating modes incident upon the resonator from the left and the right.

Let fields $J_L$ and $J_R$ be incident upon a resonator with natural frequency $\omega_o$ from the left and right respectively. Let the outgoing field to the right be $O_L$ and the outgoing field to the left be $O_R$.

Let $\gamma_{L,R}$ be the coupling constant of the resonator to the incoming fields from the left and right, $\alpha_{L,R}$ the reflection coefficients, $\beta_{L,R}$ the direct transmission coefficients and $\theta_{L,R}$ the coupling constant of the resonator to the outgoing fields. Suppose the amplitude of the resonator is $a$ and that $\tau$ is the lifetime of the resonance ( $\frac{1}{\tau}$ is the resonance width). Then the dynamic equations for the amplitude $a$ of the resonance mode can be written as

$$\frac{d^2a}{dt^2} = -{\omega_o}^2 a - \frac{1}{\tau}\frac{da}{dt} + [\gamma_L \quad \gamma_R] \begin{bmatrix} \dot{J}_L \\ \dot{J}_R \end{bmatrix}$$

$$\begin{bmatrix} \dot{O}_L \\ \dot{O}_R \end{bmatrix} = \begin{bmatrix} \alpha_L & \beta_R \\ \beta_L & \alpha_R \end{bmatrix} \begin{bmatrix} \dot{J}_L \\ \dot{J}_R \end{bmatrix} + \frac{da}{dt} \begin{bmatrix} \theta_L \\ \theta_R \end{bmatrix}$$

Denote

$$\begin{bmatrix} \dot{O}_L \\ \dot{O}_R \end{bmatrix} = S_-, \ [\gamma_L \quad \gamma_R] = \Gamma, \quad \begin{bmatrix} \dot{J}_L \\ \dot{J}_R \end{bmatrix} = S_+, \quad \begin{bmatrix} \theta_L \\ \theta_R \end{bmatrix} = \Theta, \begin{bmatrix} \alpha_L & \beta_R \\ \beta_L & \alpha_R \end{bmatrix} = C$$

Then the temporal coupled mode equations for the system can be written as

$$\frac{d^2a}{dt^2} = -{\omega_o}^2 a - \frac{1}{\tau}\frac{da}{dt} + \Gamma S_+ \tag{5.1}$$

$$S_- = CS_+ + \Theta\frac{da}{dt} \tag{5.2}$$

Temporal coupled mode theory uses assumptions of weak coupling between the incoming modes and the resonator, conservation of energy, reciprocity (time-reversal symmetry) to eliminate all unknowns except for $\omega_o$ and $\tau$ and hence express all coupling coefficients in terms of the natural frequency of the resonator and the lifetime of the resonance.

### 5.3 Comparison of Coupled Mode Approximation to the True Dynamics

We now find the true dynamic equations for our resonant Lamb model, with the objective of comparing them to the temporal coupled mode equations approximating the system.

Consider the system reduced to the two-part scatterer given by

$$\begin{aligned} \ddot{y}(t) &= -\alpha\,\dot{y}(t) + \gamma\big(z(t) - y(t)\big) + \alpha j'(t) \\ \ddot{z}(t) &= -{\omega_0}^2 z\,(t) + \gamma\big(y(t) - z(t)\big). \end{aligned}$$

We can present this as a first order system in time in the following manner:

We denote $\mathcal{Y} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$ and $\mathcal{Z} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$

then

$$\frac{d\mathcal{Y}}{dt} = B\mathcal{Y} + \Gamma\mathcal{Z} + \mathcal{F} \tag{5.3}$$

$$\frac{d\mathcal{Z}}{dt} = C\mathcal{Z} + \Gamma\mathcal{Y} \tag{5.4}$$

where

$$B = \begin{bmatrix} 0 & 1 \\ -\gamma & -\alpha \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -({\omega_0}^2 + \gamma) & 0 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad f\,(t) = -\alpha c\Big({J_1}'\,(-ct) + {J_2}'(ct)\Big)$$

From equation 5.3 $\mathcal{Y}$ is given by

$$\mathcal{Y} = \int_0^t e^{B(t-s)} \left[\gamma \mathcal{Z}\left(s\right) + \mathcal{F}\left(s\right)\right] ds$$

We eliminate $\mathcal{Y}$ from equation 5.4 to get

$$\frac{d\mathcal{Z}}{dt} = C\mathcal{Z} + \gamma^2 \int_0^t e^{B(t-s)} \mathcal{Z}\left(s\right) ds + \gamma \int_0^t e^{B(t-s)} \mathcal{F}\left(s\right) ds$$

So we have

$$\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -({\omega_0}^2 + \gamma) & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \gamma^2 \int_0^t e^{B(t-s)} \begin{bmatrix} z(s) \\ \dot{z}(s) \end{bmatrix} + \gamma \int_0^t e^{B(t-s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix}$$

From equation 5.4 $\mathcal{Z}$ is given by

$$\mathcal{Z} = \int_0^t e^{C(t-s)} \gamma \mathcal{Y}\left(s\right) ds$$

We eliminate $\mathcal{Z}$ from equation 5.3 to get

$$\frac{d\mathcal{Y}}{dt} = B\mathcal{Y} + \gamma^2 \int_0^t e^{C(t-s)} \mathcal{Y}\left(s\right) ds + \mathcal{F}\left(t\right)$$

So we have

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\gamma & -\alpha \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \gamma^2 \int_0^t e^{C(t-s)} \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Hence the true dynamics of the system are given by the following explicit equations for $z\left(t\right)$ and $y(t)$

$$\begin{aligned} \ddot{z}\left(t\right) = -\left({\omega_0}^2 + \gamma\right) z + \gamma^3 \frac{1}{\sqrt{\alpha^2 - 4\gamma}} \int_0^t g\left(t-s\right) z\left(s\right) ds + \\ + \gamma^2 \frac{1}{c\sqrt{\alpha^2 - 4\gamma}} \int_0^t h\left(t-s\right) \dot{z}(s) ds \gamma \frac{\alpha}{\sqrt{\alpha^2 - 4\gamma}} \int_0^t h\left(t-s\right) f\left(s\right) ds \end{aligned} \tag{5.5}$$

in which $g(t)=e^{-\frac{1}{2}\left(\alpha+\sqrt{\alpha^2-4\gamma}\right)t}-e^{\frac{1}{2}\left(\sqrt{\alpha^2-4\gamma}-\alpha\right)t}$ and

$$h(t)=\frac{c}{2}\left(\sqrt{\alpha^2-4\gamma}+\alpha\right)e^{-\frac{1}{2}\left(\alpha+\sqrt{\alpha^2-4\gamma}\right)t}+\frac{c}{2}\left(\sqrt{\alpha^2-4\gamma}-\alpha\right)e^{\frac{1}{2}\left(\sqrt{\alpha^2-4\gamma}-\alpha\right)t}$$

$$\begin{aligned}\ddot{y}(t)=-\gamma y-\gamma^2\sqrt{{\omega_0}^2+\gamma}\int_0^t sin\left(\left(\sqrt{{\omega_0}^2+\gamma}\right)(t-s)\right)y(s)\,ds-\\ -\alpha\dot{y}+\gamma^2\int_0^t cos\left(\left(\sqrt{{\omega_0}^2+\gamma}\right)(t-s)\right)\dot{y}(s)ds+f(t)\end{aligned} \tag{5.6}$$

We note the perturbation of the resonant mode in the order of $\gamma$ and the delayed contribution to the decay rate in equation 5.5 as the most prominent deviations in this true dynamics from the coupled mode approximation for our system.

In order to perform a meaningful comparison between the actual decay rate and the decay rate obtained by the coupled mode formalism for this system, we now express the true dynamics of the two-part scatterer in the Fourier domain. Analyzing the equations in the Fourier domain will be algebraically easier since the convolutions in the time domain will become multiplications while the time derivative will be a multiplication by $(-i\omega)$. This would enable us to analyze the accuracy of the decay rate obtained for the resonant Lamb model by a temporal coupled mode formalism.

Recall that the system reduced to the two-part scatterer is given in the Fourier domain by

$$\left(\omega^2+i\alpha\omega-\gamma\right)\hat{y}+\gamma\hat{z}=i\alpha\omega\hat{j}$$

$$\gamma\hat{y}+\left(\omega^2-{\omega_0}^2-\gamma\right)\hat{z}=0$$

We eliminate $\hat{y}(\omega)$ and obtain an equation in $\hat{z}(\omega)$ .

$$-\omega^2\hat{z}=-\left({\omega_0}^2+\gamma\right)\hat{z}-\left(\frac{\gamma^2}{\omega^2+i\alpha\omega-\gamma}\right)\hat{z}+\gamma\left(\frac{i\alpha\omega}{\omega^2+i\alpha\omega-\gamma}\right)\hat{j}$$

Now since $\frac{1}{\omega^2+i\alpha\omega-\gamma} = \frac{\left(\omega^2-\gamma\right)}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2} - \frac{i\alpha\omega}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2}$ we have,

$$(-i\omega)^2\hat{z} + \gamma^2\left[\frac{\alpha}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2}\right](-i\omega)\,\hat{z} + \left[{\omega_0}^2+\gamma+\gamma^2\left(\frac{\left(\omega^2-\gamma\right)}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2}\right)\right]\hat{z} = \gamma\left(\frac{i\alpha\omega}{\omega^2+i\alpha\omega-\gamma}\right)\hat{j} \tag{5.7}$$

where the terms of $\hat{z}$ multiplied by $(-i\omega)$ and $(-i\omega)^2 = -\omega^2$ have been isolated. Now we consider the homogeneous equation of 5.7:

$$(-i\omega)^2\hat{z} + \left[\frac{\gamma^2\alpha}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2}\right](-i\omega)\,\hat{z} + \left[{\omega_0}^2+\gamma+\left(\frac{\gamma^2\left(\omega^2-\gamma\right)}{\left(\omega^2-\gamma\right)^2+\alpha^2\omega^2}\right)\right]\hat{z} = 0 \tag{5.8}$$

which is the equation for the oscillator with no incident field.

In this situation, the coefficients of equation 5.8 are not constant but are also dependent on frequency, which is the manifestation of the convolutions in the coefficients of equation 5.5 in the Fourier domain.

We define

$$B\left(\omega,\widetilde{\omega}\right) = (-i\omega)^2 + \left(\frac{\gamma^2\alpha}{\left(\widetilde{\omega}^2-\gamma\right)^2+\alpha^2\widetilde{\omega}^2}\right)(-i\omega) + \left({\omega_0}^2+\gamma+\frac{\gamma^2\left(\widetilde{\omega}^2-\gamma\right)}{\left(\widetilde{\omega}^2-\gamma\right)^2+\alpha^2\widetilde{\omega}^2}\right)$$

The real resonant frequency of the oscillator will be obtained when $\omega = \widetilde{\omega} = \omega_*$ in $B\left(\omega,\widetilde{\omega}\right)$ are both variable and we had computed this solution to $B\left(\omega_*,\omega_*\right) = 0$ in section 3.5.

$$\omega_* = \omega_0 + \frac{1}{2\omega_0}\gamma + \left(\frac{\left(3{\omega_0}^2-\alpha^2\right)}{2{\omega_0}^3\left({\omega_0}^2+\alpha^2\right)} - \frac{\alpha}{{\omega_0}^2\left({\omega_0}^2+\alpha^2\right)}i\right)\gamma^2 + O(\gamma^3)$$

From this solution the actual decay rate is given by,

$$\mathrm{Im}\ \omega_* = -\gamma^2\left(\frac{\alpha}{{\omega_0}^2\left({\omega_0}^2+\alpha^2\right)}\right) + O\left(\gamma^3\right) \tag{5.9}$$

Next we proceed to obtain a decay rate that would be approximated using a temporal coupled mode formalism. Basic assumptions made in coupled mode theory [19] are that the perturbation of modes due to weak coupling is negligible and that the coupling coefficients are constants. The latter assumption translates to coefficients in $B(\omega, \widetilde{\omega})$ being constant and since the perturbation of the mode $\omega_0$ is assumed to be negligible, a logical coupled mode approximation for $\omega_*$ can be obtained by the solution to $B({\omega_*}^{cm}, \omega_0) = 0$.

$$
\begin{aligned}
{\omega_*}^{cm} = \frac{i}{2}\Bigg[-&\frac{\gamma^2\alpha}{\left({\omega_0}^2-\gamma\right)^2+\alpha^2{\omega_0}^2} \pm \\
&\pm\sqrt{\left(\frac{\gamma^2\alpha}{\left({\omega_0}^2-\gamma\right)^2+\alpha^2{\omega_0}^2}\right)^2 - 4\left({\omega_0}^2+\gamma+\frac{\gamma^2\left({\omega_0}^2-\gamma\right)}{\left({\omega_0}^2-\gamma\right)^2+\alpha^2{\omega_0}^2}\right)}
\end{aligned}
$$

The decay rate approximated by this coupled mode formalism is then given by

$$
\text{Im } {\omega_*}^{cm} = -\frac{\gamma^2\alpha}{{\omega_0}^2\left({\omega_0}^2+\alpha^2\right)+\gamma\left(-2{\omega_0}^2+\gamma\right)} = -\gamma^2\left(\frac{\alpha}{{\omega_0}^2\left({\omega_0}^2+\alpha^2\right)}\right) + O\left(\gamma^3\right) \quad (5.10)
$$

Comparing the actual decay rate with the rate approximated by the temporal coupled mode formalism we arrive at the following proposition.

**Proposition 5.1.** *The decay rate of the oscillator approximated by the temporal coupled mode formalism* $\text{Im } {\omega_*}^{cm}$ *differs from the actual decay rate* $\text{Im } \omega_*$ *by an order of* $\gamma^3$

# Chapter 6
# Conclusion and Future Work

A resonant modification of the famous Lamb model was used as a prototype of a photonic resonant scattering system to analyse the fine features of Fano resonance under the incidence of near-monochromatic field. Our resonant Lamb model consists of a two-part scatterer attached to an infinite string. The point mass on the string serves as the non-resonant part of the scatterer, and the spring-mass serves as the resonant part. The non-resonant part of the scatterer is associated with direct scattering while the resonant part is associated with resonant scattering (field amplification and delayed scattering). The infinite string models the ambient medium. The free oscillations of the spring-mass resonator models the guided modes in a photonic resonant scattering system. This simplified system models a photonic scattering system in which a resonant mode is weakly coupled to a continuum of radiation states. It allows us to investigate the type of resonant scattering when the interaction between the guided modes inside a scatterer and the propagating plane waves in the ambient space cause the guided modes to be destroyed and the extended states near the bound state frequency are sharply modified.

When placed in the proper functional analytic setting, a frequency corresponding to a guided mode is realized by an eigenvalue embedded in the continuous spectrum of an operator underlying the system. This continuous spectrum corresponds to the continuum of frequencies of the extended states (plane waves in the ambient medium). The operator $L$, underlying the scattering system is self-adjoint and positive and hence its spectrum, $\sigma(L) \subseteq [0, \infty)$. Therefore the resolvent, $(L - \omega^2)^{-1}$ exists everywhere on the upper half $\omega$ plane. The perturbation of the idealized system corresponds to the dissolution of this eigenvalue in to the continuous spectrum. The dissolution of the embedded eigenvalue coincides with the frequencies corresponding to the guided modes attaining a small imaginary part and moving down to the lower half plane becoming resonances.

The time sequence of a resonant transmission process is observed to consist of two distinct stages: an initial pulse and a tail of long decay corresponding to two pathways in the transmission (and reflection) process; "Direct transmission" and "Resonant transmission". We provide a rigorous definition of these two temporal pathways by analyzing the pole structure of the resolvent operator $(L-\omega^2)^{-1}$.

The high-Q near-monochromatic regime involves three simultaneously small physical parameters that affect resonant features: the constant of coupling, $\gamma$, difference between resonant and source frequencies, $\eta$ and the spectral width $\sigma$ of the incident near-monochromatic field. The resonant amplification of the slowly decaying quasi-normal mode associated to the decay of energy out of the resonator depends delicately on the relative sizes of these three parameters and our analysis yields the following about the dependence of the resonant amplitude $z(t)$ of the spring-mass oscillator at time $t=\sigma^{-1}$;

When $\sigma$ is much larger than $\gamma^2$,

$$|z(t)| \simeq \frac{\gamma}{\max\{\sigma,\eta\}} \ll \gamma^{-1} \qquad (\gamma^2 \ll \sigma),$$

and when $\sigma$ is not much larger than $\gamma^2$, one has

$$|z(t)| \simeq \frac{\gamma}{\max\{\eta,\gamma^2\}} \qquad (\sigma < C\gamma^2).$$

After time $t=\sigma^{-1}$, the field decays at a slow exponential rate on the order of $\gamma^2$.

The properties of a resonant scattering system can be determined by analyzing the interference between the direct and the resonant pathways. This insight has led to an important development in the theory of Fano resonance for optical resonators: An intuitive theory based on a temporal coupled-mode formalism has been introduced to explain complex features of these resonances. Such models isolate phenomenological component "modes" of a resonant scattering process and this theory leads to good approximations of resonant behavior of photonic resonators. Evaluation of the accuracy of a coupled-mode theory for a scattering system by comparing it to the true dynamics of the system would be beneficial.

More elaborate versions of the Lamb model can be used to analyze more complex resonance phenomena. The problem of analyzing the dependence of resonant features of a more complex photonic scatterer simultaneously on the constant of coupling, the spectral width of the transmission resonance, the frequency mismatch of the resonance and the source field as well as the angle of incidence would also be beneficial.

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# Vita

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