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Elasto-plastic and damage modeling of reinforced concrete

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ELASTO-PLASTIC AND DAMAGE MODELING OF REINFORCED CONCRETE

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Civil & Environmental Engineering

by

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August 2008
TO THOSE WHO TAUGHT ME EVERYTHING THAT MATTERS

TO MY BELOVED MOTHER & FATHER
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ABSTRACT

Modeling the mechanical behavior of Reinforced Concrete (RC) is still one of the most difficult challenges in the field of structural engineering. The Nonlinear Finite Element Analysis (NFEA) and modeling of the behavior of RC members are the primary goals of this study. The macroscopic components of RC, Concrete material and reinforcing steel, are represented herein by separate material models. These material models are combined together using a model that describes the global effect of interaction between reinforcing steel and concrete in order to simulate the behavior of the composite RC material.

A thermodynamically consistent constitutive model for concrete that incorporates concrete-plasticity and fracture-energy-based continuum damage mechanics is presented. An effective stress space plasticity yield criterion, with multiple hardening functions and a non-associative plasticity flow rule, is used simultaneously with two (tensile and compressive) isotropic damage criteria. The spectral decomposition of the stress tensor into tensile and compressive components is utilized in all criteria in order to simulate different responses of the material under various loading patterns. The damage criteria are based on the hydrostatic-deviatoric sensitive damage energy release rates in tension and compression derived from the Helmholtz free energy function. Three dissipation mechanisms are defined, one for plasticity and two for damage, to control the dissipation process of the material model.

Elastic-plastic models that account for isotropic perfectly-plastic and plastic-strain-hardening (linear, bilinear and nonlinear) of the steel reinforcement are provided as well. The global effect of bond-slip is incorporated into the stress-strain diagram of the reinforcing bars in an attempt to describe this interaction phenomenon in a stress-strain driven environment.

The Numerical implementation and application are important parts of this study. A suitable elastoplasticity-implicit/damage-explicit scheme is adapted here for the integration of the incremental constitutive equations. The elastic-predictor, plastic-corrector and damage-corrector steps are used to facilitate the integration procedure. The constitutive approach is implemented, through numerical algorithms; in the advanced FE software ABAQUS via user defined material subroutine UMAT to analyze and better describe the overall behavior of such a composite material. Concrete and RC beams subjected to static-short-term-monotonic loading are analyzed in an assumed isothermal environment. The simulated results are compared to experimental studies conducted by other researchers.
CHAPTER 1
INTRODUCTION

Reinforced Concrete (RC) is one of the most commonly used building materials nowadays. It is a composite material made of plain concrete, which possesses relatively high compressive strength but low tensile strength, and steel bars embedded in the concrete, which can provide the needed strength in tension. The economy, efficiency, strength and stiffness of RC make it an attractive material for a wide range of structural engineering applications, such as nuclear power-plants, bridges, cooling towers and offshore platforms. For RC to be used as a structural material, it should satisfy special criteria including:

- Strength and Stiffness
- Safety and Appearance
- Economy

By applying the principles of structural analysis, the laws of equilibrium and the consideration of the mechanical properties of the components studied; RC design procedure should yield a sufficient margin of safety against collapse under ultimate loads. Serviceability analysis is conducted to control the deflections under service loads and to limit the crack width to an acceptable level for the structural component to perform and appear safe and inhabitable for the human eye. Economical considerations are satisfied by optimizing the usage of steel/concrete quantities to account for the difference in unit costs of steel and concrete.

The ultimate objective of design is the safety and economy of the RC structural member. The design process is usually based on a linear elastic analysis to calculate the internal forces in the member which are then used to design the reinforcement and the details of the member using some code provision. Codes are usually based on empirical approaches that utilize experimental data and provide design rules to satisfy safety and serviceability requirements. Although the design of RC structures based on linear-elastic stress analysis is adequate and reliable in many cases, the extent and impact of a disaster in terms of human and economical losses in the event of structural failure of large scale modern structures necessitate more careful and detailed structural safety analysis. Thus, Nonlinear Finite Element Analysis (NFEA) is often required to obtain detailed information regarding the ultimate loading capacity and the post-failure behavior of RC structures. The importance and interaction of different nonlinear effects on the response of RC structures can be studied analytically using NFEA.

The complex behavior of concrete, which arises from the composite nature of the material, is characterized by a reduction of the load carrying capacity with increasing deformations after reaching a certain limit load. This global behavior is usually caused by a material behavior which is described as strain softening and occurs in tension and in compression. This necessitates the development of appropriate constitutive models to describe such behavior.
In RC, the response of the structure is even more complicated. In general a number of cracks will develop in the structure due to the bond action between concrete and reinforcement. This results in a redistribution of the tensile loads from concrete to the reinforcement. This phenomenon is called tension-stiffening, because the response is stiffer than the response with a brittle fracture approach.

The behavior of RC is highly nonlinear which is caused by mechanisms such as cracking, crushing, creep and shrinkage of concrete, but also caused by the interaction between reinforcement and concrete, where the load transferring mechanism of the interface between concrete and reinforcement plays an important role. Because all these mechanisms are interacting, it is not realistic to try to formulate a constitutive model which incorporates all these mechanisms, but a model has to be formulated to adequately describe the behavior of a structure within the range of application which has been restricted in advance. Although the constitutive models which are developed within this phenomenological approach are usually simplified representations of the real behavior of the material, it is believed that more insight can be gained by tracing the entire response of a structure in this manner, than modeling a structure with highly sophisticated material models which do not result in a converged solution after failure load and are computationally expensive and complicated.

A large variety of models have been proposed to characterize the stress-strain relation and failure behavior of RC materials. All these models have certain inherent advantages and disadvantages which depend to a large degree on their particular application and complexity. Macroscopic constitutive studies have been conducted with different levels of complexity and applicability in order to address the different aspects of the concrete material behavior. On the other hand, microscopic modeling and multi-scale modeling offer useful ways to model the material behavior, but their applicability to full-scale structural problems is still problematic, due to their requirement for huge amounts of computer resources. Therefore, further development in the macroscopic constitutive modeling of concrete is justified and needed, with the motivation of incorporating contemporary experimentally observed features of the material behavior in the modeling. In this study, constitutive models representing the macro constituents of RC (Concrete and Steel Reinforcement) are developed with emphasis on the rigor and consistency in formulation and implementation into the Finite Element Analysis (FEA) commercial software ABAQUS.

Concrete and reinforcing steel are represented herein by separate material models which are combined together using a model that describes the interaction between reinforcing steel and concrete to simulate the overall behavior of the composite RC material. An elasto-plastic-damage constitutive model is used to describe the behavior of concrete, while steel reinforcement is modeled as an elastoplastic material with strain hardening using the classical von Mises plasticity. Bond considerations are accounted for within the steel reinforcement model. Coupling between damage and plasticity in the constitutive model is employed to capture the observed phenomenological behavior of concrete. In this combined approach, damage theory is used to model the material deterioration, while the permanent deformation and some other behavioral features of
concrete can be captured using plasticity theory. All features of the two theories can be incorporated in this combined approach, making it very promising for use in constitutive modeling or RC structures.

1.1 Scope and Objectives

It is the purpose of this study to introduce a thermodynamically consistent model for the elasto-plastic-damage NFEA of RC beams. Developing a better understanding and representation of the behavior of RC beam structures subjected to short term static loading intensity will be the primary goal. The loading regime will be such that the rotation of the direction of the principal strain vector remains moderate, allowing the use of small strain theory in the Finite Elements (FE) simulations. The mechanical behavior of RC will be studied while isothermal conditions are assumed throughout this work.

The model will be translated into algorithms that will simulate the nonlinear material behavior of concrete, steel reinforcement, and their interaction. They will also facilitate the reproduction of experimentally observed load carrying capacity curves of RC beams. These algorithms will be incorporated into the FEA software ABAQUS via a user-defined material subroutine (UMAT). While ABAQUS performs the standard FE procedure using standard types of finite elements, the UMAT will govern the behavior of these materials during different loading stages, i.e., elastic, inelastic, failure, post-failure loads. The model is intended to be robust, efficient and reliable but by no means vague and complicated.

The complexity of the behavior of RC, and the scatter of the experimental data associated with machine precision, variations in testing techniques, and statistical distributions of material properties from one sample to another is one of the main factors enforcing the notion that the primary goal of any constitutive model should be set in the prediction of essential features of experimentally observed behavior, rather than in exactly replicating the entire history of stress-strain curves. Along this line, it should be emphasized that numerical implementation of a proposed constitutive model into a computer code is almost as important an issue to consider as the model itself. A literature survey can easily reveal models that are mathematically very elegant, but pose overwhelming computational difficulties. It is thus important that a constitutive model, although rigorous in theory, should also be suitable for use in computation and should lend itself well to an efficient implementation in computer codes.

The objective of this study can be cast into the following points:

- Capturing the elasto-plastic-damage behavior of concrete under monotonic loads in tension, compression, tension-tension, compression-compression, and compression-tension stress states. This incorporates defining the continuous damage mechanism in concrete that will represent the strain softening in the post-peak regions and the degradation of elastic stiffness.
- Modeling the behavior of structural steel - embedded in the concrete - as an elastoplastic material with isotropic hardening.
Accounting for the effect of bond between steel and concrete and their interaction on the overall short term material (stress-strain) behavior of RC.

Verifying the applicability of the proposed model by comparing the predicted behaviors with those observed in experimental results obtained by other researchers.

1.2 Outline of the Dissertation

This study starts with a literature review on the physical behaviors of RC and its macroscopic constituents, i.e., steel and concrete, as well as the constitutive models applied to describe such behaviors, all of which are presented in Chapter 2. Emphasis here is placed on the experimental work done by various researchers to study concrete and RC in structural members and on the constitutive models applied to capture the important features of the experimentally observed material behaviors. This emphasis leads to the application of combined approaches employing both damage mechanics and plasticity theory to describe the concrete behavior and classical hardening plasticity theory to describe that of steel reinforcement. The bond effect and material interactions as well as modeling of these phenomena are also addressed.

Chapter 3 addresses the von Mises plasticity-based material constitutive models used to describe the physical behavior of steel reinforcement. This includes perfectly plastic and (linear, bilinear and nonlinear) hardening models. The chapter starts with an introduction discussing the application of the FE method to nonlinear continuum mechanics. The fundamentals of FE analysis procedures for elastic-plastic problems are considered, with algorithms required for convergence at the local (Gauss point) and global (equilibrium) levels. This is followed by a discussion of the incremental constitutive theory of metal plasticity and the numerical techniques employed to integrate the constitutive equations. Examples are provided to verify the implemented algorithms. This chapter is intended to serve as an introduction to the chapters that follow in terms of theoretical formulation and computational implementation methods.

In Chapter 4, the material model for plain concrete is discussed. Rigorous consistent thermodynamic formulation is employed to derive the framework of the elastic-plastic-damage constitutive model. The Helmholtz free energy function is discussed and the dissipation potentials for plastic and damage processes are postulated. To capture the different responses in tension and compression, the approach makes use of the separation of tensile and compressive behaviors, obtained through the decomposition of stress tensor and the introduction of the damage variables, all of which are integrated into the thermodynamic framework. The dissipation process therefore consists of three separate dissipation mechanisms: tensile and compressive damage coupled with plasticity. Detailed description of the theoretical formulation of the constitutive model in terms of the theories of plasticity with multiple hardening rules and continuum damage mechanics is then provided, followed by a step-by-step detailing of the numerical scheme applied to integrate the incremental constitutive equations. Non associative plasticity - with multiple hardening rules - is combined with continuum damage mechanics, where the accumulated damage in the concrete material is represented by two internal damage parameters, one in
tension and one in compression. The constitutive model is formulated as a relation between the undamaged stress and an internal accumulated damage parameter defined using the tensile and compressive damage parameters. The integration scheme of elastic predictor followed by plastic and damage correctors is adopted in order to overcome the difficulties that arise from the combined plastic-damage approach. Several verification examples are provided in order to test the model’s predictions under uniaxial and biaxial stress states, as well as under three point bending test of a single-edge-notched concrete beam. The numerical results are compared to the experimental ones in order to evaluate the performance of the proposed concrete model and to assess the model’s ability to capture the experimentally observed behaviors of concrete materials.

Chapter 5 addresses modeling of RC beams. The bond effect on RC is discussed and a simple experimentally based methodology is applied to account for the effect of steel-concrete interaction on the reinforcement stress level. This discussion is followed by two RC comprehensive examples provided to demonstrate the applicability of the applied constitutive models combined to the NFEA of RC beams. The model is tested using two RC beams with different geometries and reinforcement ratios. The predicted (numerical) results are compared to their experimental counterparts obtained by other researchers. Concrete damage distributions as well as RC stress-strain curves are presented and compared to other works.

Conclusions and further studies are proposed in Chapter 6. A summary of the constitutive model and results is introduced followed by a brief discussion of the merits and weaknesses of the currently proposed model. Results of the current investigation suggest that additional fundamental research is required if computer simulation is to be a viable tool for future research and design of RC structures. A discussion of additional research needs in the area of characterization of material response through experimental investigation, material modeling and non-linear analysis is presented.

Throughout this work, (+) and (−) superscripts are used to indicate tension or compression, respectively. All symbols with an over bar, e.g. \( \bar{\sigma} \), are considered to be in the effective fictitious undamaged configuration. On the other hand, the absence of the over bar indicates the actual damaged configuration.
CHAPTER 2
LITERATURE REVIEW

RC structures are made up of two materials with different characteristics, namely, concrete and steel. Steel can be considered as a homogeneous material with generally well defined material properties. Concrete, on the other hand, is a heterogeneous material made up of cement, mortar and aggregates. Its mechanical properties are widely scattered and cannot be defined easily. For the convenience of analysis and design, however, concrete is often considered a homogeneous material at the macroscopic scale.

The typical stages in the load-deformation behavior of a RC simply supported beam are illustrated in Fig. 2.1. Similar relations are obtained for other types of RC structural elements. This nonlinear response can be roughly divided into three ranges of behavior: the uncracked elastic stage, the crack propagation and the plastic (yielding or crushing) stage (Chen, 1982).

![Figure 2.1 Typical load-displacement curve of RC beams](image)

The nonlinear response is caused by three major effects, namely, cracking of concrete in tension, yielding of the reinforcement or crushing of concrete in compression, and the interaction of the constituents of RC. Interaction includes bond-slip between reinforcing steel and surrounding concrete, aggregate interlock at a crack and dowel action of the reinforcing steel crossing a crack. The time-dependent effects of creep, shrinkage and temperature variation also contribute to the nonlinear behavior. Furthermore, the stress-strain relation of concrete is not only nonlinear, but is different in tension than in compression and the mechanical properties are dependent on concrete age at loading and
on environmental conditions, such as ambient temperature and humidity. The material properties of concrete and steel are also strain-rate dependent to some extent.

The earliest publication on the application of the finite element method to the analysis of RC structures was presented by Ngo and Scordelis (1967). In their study, simple beams were analyzed with a model in which concrete and reinforcing steel were represented by constant strain triangular elements, and a special bond link element was used to connect the steel to the concrete and describe the bond-slip effect. A linear elastic analysis was performed on beams to determine principal stresses in concrete, stresses in steel reinforcement and bond stresses. Ngo and Scordelis (1967) reported that one of the main difficulties in constructing an analytical model for RC member is due to the composite action of steel and concrete. Prefect bonding between steel and concrete can only exist at an early stage under low load intensity. As the load is increased, cracking as well as breaking of bond inevitably occurs, and a certain amount of bond slip will take place in the beam, all of which will in turn affect the stress distributions in concrete and steel.

Since Ngo and Scordelis published their landmark paper in 1967, the analysis of RC structures has enjoyed a growing interest and many publications have appeared. Important progress has also been made in the finite-element-based numerical analysis of plain and RC structures (recent examples: Fantilli et. al., 2002; Sumarac et. al., 2003; Marfia et. al., 2004; Phuvoravan and Sotelino, 2005; Junior and Venturini, 2007; just to mention a few). However, despite this progress, modeling the mechanical behavior of concrete is still one of the most difficult challenges in the field of structural engineering. This is due to the inherent complexity and uncertainty concerning the properties of concrete, which make it excessively difficult to develop accurate constitutive models or algorithms that are sufficiently robust to obtain reliable and converged solutions in numerical analyses.

An adequate numerical analysis of the nonlinear behavior of RC structures is based on the coupled modeling of different inelastic processes in concrete and in reinforcement. Macroscopic representation of crystal dislocation in steel reinforcement within the framework of elastoplasticity yields a reliable prediction of deformation history of reinforcement. The realistic constitutive behavior of concrete is, however, more complex. It has resulted in the appearance of many different concepts to its theoretical description. Concrete failure is usually characterized by many macroscopic cracks. If the discrete cracks are considered, the necessary adaptation of the FE mesh to the trajectory of each crack under new load step makes the FE analysis cumbersome. Even in the two-dimensional case, it limits the applicability of conventional fracture mechanics to simple specimens with one or two cracks. Therefore, the use of models based on continuum damage mechanics and elastoplasticity have found large application in the numerical modeling of concrete fracture.

The development of analytical models for the response of RC structures is complicated due to the following factors (Kwak and Fillipou, 1997):
RC is a composite material made up of concrete and steel, two materials with very different physical and mechanical behavior.

Concrete exhibits nonlinearities even under low level of loading due to nonlinear material behavior, environmental effects, cracking, biaxial stiffening and strain softening.

Reinforcing steel and concrete interact in a complex way through bond-slip and aggregate interlock.

Because of these factors and the differences in short- and long-term behaviors of the constituent materials, it is common practice among researchers to model the short- and long-term response of RC members and structures based on separate material models for reinforcing steel and concrete, which are then combined along with models of interaction between the two constituents to describe the behavior of the composite RC material. This will be the approach adopted in this study of short-term behavior of RC beams. In the following, literature review of the research done to describe the behavior of concrete, reinforcing steel, and their bond-interaction analysis is presented.

2.1 Concrete Materials

Concrete by itself is a composite material. It is made of cement, mortar, and aggregates. The thermo-chemical interaction between these constituents results in a unique building material. One of the most important characteristics of concrete is low tensile strength, which results in tensile cracking at a very low stress compared with compressive stresses. The tensile cracking reduces the stiffness of the concrete component.

Several issues in the current practice of constitutive modeling of concrete material need to be addressed. First, concrete is a non-homogeneous and anisotropic material, the mechanical behavior of which is nonlinear (Kupfer et. al., 1969; Kotsovos and Newman, 1977). Its compressive strength increases as it is loaded in a biaxial compressive state, but decreases as the tensile stress is increased under biaxial compression–tension. Moreover, the ductility of concrete under biaxial stresses is also dependent on the stress state.

Concrete exhibits a large number of micro-cracks, especially at the interface between coarser aggregates and mortar, even before the application of any external loads. The presence of these micro-cracks has a great effect on the mechanical behavior of concrete, since their propagation (concrete damage) during loading contributes to the nonlinear behavior at low stress levels and causes volume expansion near failure. Many of these micro-cracks are initially caused by segregation, shrinkage or thermal expansion of the mortar. Some micro-cracks may develop during loading because of the difference in stiffness between aggregates and mortar. Since the aggregate mortar interface has a significantly lower tensile strength than the mortar; it constitutes the weakest link in the composite system. This is the primary reason for the low tensile strength of concrete.

Furthermore, the relation between the microstructure and mechanical behavior of concrete is quite complex because of the considerable heterogeneity of the distinct phases
Concrete may be treated as a composite material and it may contain porosity in its matrix. The porosity in the matrix is not homogenous and a strong porosity gradient is observed around the inclusions formed by the aggregates (Panoskaltsis and Lubliner, 1994; Ollivier et. al., 1995). This area in the matrix affected by the surface of the aggregate is known as “transition zone”. This transition zone has a damaging effect on the mechanical behavior of concrete. It is clear from the above discussion that the size and texture properties of the aggregates will also have a significant effect on the mechanical behavior of concrete under various types of loading (Chen, 1982).

The nonlinear material behavior of concrete can be attributed to two distinct material mechanical processes; plasticity and damage mechanisms. The cracking process in concrete is distinguished from cracking of other materials in that it is not a sudden onset of new free surfaces but a continuous forming and connecting of micro-cracks. The formation of micro-cracks is presented macroscopically as softening behavior of the material, which causes localization and redistribution of strains in the structure. This phenomenological behavior at the macroscopic level can be modeled by classical plasticity (Pramo and Willam, 1989). On the other hand, the micro-processes such as, micro-cracking, micro-cavities, and their nucleation and coalescence, also cause stiffness degradation, which is difficult to represent with classical plasticity (Lee and Fenves, 1998). This introduces the need for continuum damage mechanics, where the stiffness degradation can be modeled by making use of the effective stress concept (Kachanov, 1958) as will be shown later. Damage mechanics can also be used to represent the post-peak softening behavior of concrete materials; a behavior that cannot be addressed by classical plasticity theory.

The failure behavior of concrete is governed by complex degradation processes starting within the aggregate-matrix interface. These processes are shown in Fig. 2.2. The aggregate-matrix interface contains micro-cracks that exist even before any load is applied to concrete (Fig. 2.2a). The formation of these micro-cracks is primarily due to stress and strain concentrations resulting from the incompatibility of the elastic moduli of the aggregate and cement paste components. Strain concentrations at the aggregate-mortar interface may also occur as a result of volume changes in the concrete due to shrinkage and/or thermal effects resulting from the difference in thermal coefficients of various constituents. Additional micro-cracks can be initiated when concrete is subjected to external loads with magnitudes beyond the micro-crack initiation threshold (Fig. 2.2b). These micro-cracks spread and grow under the effect of continuous loading until they merge into the matrix after a certain threshold is reached (Fig. 2.2c). Cracks in the matrix grow in size and coalesce with each other to form major cracks that eventually lead to failure (Fig. 2.2d).

The nonlinear stress-strain behavior under uniaxial compression of concrete is shown in Fig. 2.3. Investigators (e.g. Kotsovos and Newman, 1977) have shown that concrete compression behavior and fracture characteristics may be explained by the creation and propagation of micro-cracks inside the concrete. It is observed that under different magnitudes of the applied load, the concrete behavior can be summarized in four stages shown in Fig. 2.3. The first stage is observed during 30-60% of the ultimate strength
(shown as 45% in Fig. 2.3). In this initial stage, one can observe the highest tensile strain concentration at particular points where new micro-cracks are initiated as shown in Fig. 2.2b. At this load state, localized cracks are initiated, but they do not propagate (stationary cracks). Hence, the stress-strain behavior is linearly elastic. Therefore, 0.3 \( f'c \) is usually proposed as the limit of elasticity. Beyond this limit, the stress-strain curve begins to deviate from a straight line. Stresses up to 70-90% of the ultimate strength (shown as 85% in Fig. 2.3) characterize the second stage. In this stage, as the applied load is progressively increased, the crack system multiplies and propagates as shown in Fig. 2.2c. The increase of the internal damage in concrete, revealed by deviation from the linear elastic behavior, reduces the material stiffness and causes irreversible deformation in unloading. Although the relief of strain concentration continues during this stage, void formation causes the rate of increase of the tensile strain in the direction normal to that of branching to increase with respect to the rate of increase of the strain in the direction of branching (Kotsovos and Newman, 1977). The start of such deformation behavior is called “onset of stable fracture propagation” (OSFP). In this load stage, the mortar cracks tend to bridge the aggregate-matrix bond cracks.

![Figure 2.2 Aggregate-matrix interface: a) prior to loading, b) 65% of ultimate load, c) 85% of ultimate load, d) failure load, (Kotsovos and Newman, 1977)](image)

A third stage shown in Fig. 2.3 extends up to the ultimate strength. Interface micro-cracks are linked to each other by mortar cracks as shown in Fig. 2.2c, and void formation (dilation) begins to have its effect on deformation at this stage. The start of this stage is called “onset of unstable fracture propagation” (OUFP). This level is easily
defined since it coincides with the level at which the overall volume of the material becomes a minimum. In this stage, the progressive failure of concrete is primarily caused by cracks through the mortar. These cracks merge with bond cracks at the surface of nearby aggregates and form crack zones of internal damage. Following that, a smoothly varying deformation pattern may change and further deformation may be localized.

A fourth stage defines the region beyond the ultimate strength. In this region, the energy released by the propagation of a crack is greater than the energy needed for propagation. Therefore, the cracks become unstable and self-propagating until complete disruption and failure occurs. In this stage, the major cracks form parallel to the direction of the applied load, causing failure of the concrete. The volume of voids increases dramatically causing a rapid dilation of the overall volume of concrete as shown in Fig. 2.2d.

![Uniaxial compression stress-strain relation of concrete](image)

Figure 2.3 Uniaxial compression stress-strain relation of concrete, (Chen, 1982).

All the above mentioned stages are for the uniaxial compression case. Stages I, (II and III), and IV could be categorized into the linear elastic, inelastic, and the localized stages respectively. Understanding these stages is crucial for the development of any concrete model.

Figure 2.4 shows a typical uniaxial tension stress-elongation curve. In general the limit of elasticity is observed to be about 60-80% of the ultimate tensile strength. Above this level, the aggregate-matrix interface micro-cracks start to grow. As the uniaxial tension state of stress tends to arrest the cracks much less frequently than the compressive stage of stress, one can expect the interval of stable crack propagation to be quite short, and the unstable crack propagation to occur much earlier. That is why the deformation behavior of concrete in tension is quite brittle in nature. In addition, the aggregate-matrix
interface has a significantly lower tensile strength than the matrix, which is the primary reason for the low tensile strength of concrete.

Nonlinearities in concrete behavior are well documented and arise from two distinct micro-structural changes that take place in the material: one is the plastic flow; the other is the development of micro-cracks and micro-voids. From a plasticity point of view, the number of bonds between atoms during the plastic flow process is hardly altered; therefore, the elastic compliances remain insensitive to this mode of micro-structural change. On the other hand, micro-cracking destroys the bond between material grains, affects the elastic properties, and may also result in permanent deformations, which can be modeled by damage mechanics.

The idea of combined plasticity and damage mechanics theories through the description of plasticity and damage surfaces has been explored and used in the past (Oritz, 1985; Lubliner et. al. 1989). Many researches attempted to expand the application of plasticity and damage theories to concrete (Chen and Chen, 1975; Lubliner et. al., 1989; Yazdani and Schreyer, 1990; Abu-Lebdeh and Voyiadjis, 1993; Voyiadjis and Abu-Lebdeh, 1994; Luccioni et. al., 1996; Lee and Fenves, 1998, 2001; Faria et. al., 1998; Hansen et. al., 2001; Nechnech et. al., 2002; Gatuingt and Pijaudier-Cabot, 2002; Kratzig and Polling, 2004; Salari et. al., 2004; Shen et. al., 2004; Jankowiak and Lodygowski, 2004; Luccioni and Rougier, 2005; Rabczuk et. al., 2005; Wu et. al., 2006; Grassl and Jirasek, 2006; Nguyen and Korsunsky, 2006; Shao et. al., 2006; Jason et. al., 2006; Contrafatto and Cuomo, 2006; Nguyen and Houlsby, 2007; Mohamad-Hussein and Shao, 2007; Ananiev and Ožbolt, 2007; Voyiadjis et. al., 2008a,b; Yu et. al., 2008; and others).

Plasticity by itself fails to address the softening behavior of concrete under tension and compression caused by damage propagation due to micro-cracking in the strained
material. On the other hand, damage mechanics is only concerned with the description of this progressive weakening of solids due to the development of micro-cracks and micro-voids (Loland, 1980; Ortiz and Popov, 1982; Krajcinovic, 1985; Simo and Ju, 1987a, 1987b; Ju et. al., 1989; Voyiadjis and Kattan, 1989, 1999, 2006). Therefore, a constitutive model should equally address these two distinct physical modes of irreversible changes and should satisfy the basic postulates of mechanics and thermodynamics governing these phenomena. Concrete plasticity and damage will be further discussed in what follows.

2.1.1 Concrete Plasticity

Plasticity theory successfully treated concrete problems in which the material is subjected to primary compressive loads. In situations where tension-compression plays a significant role, plasticity theory is applied to model the compression zones while damage or fracture mechanics is used to model the tensile zones (Lubliner et. al., 1989).

The uniaxial behaviors of plain concrete under tension and compression up to tensile and compressive failure are shown in Fig. 2.5. For tensile failure, the behavior is essentially linear elastic up to the failure load, followed by a strain softening response which was generally neglected or idealized in the past. For compression failure, the material initially exhibits almost linear behavior up to the proportional limit at point A, after which the material is progressively weakened by internal micro-cracking up to the end of the perfectly plastic flow region CD at point D. The nonlinear deformations are basically plastic, since upon unloading only the elastic portion $\varepsilon^e$ can be recovered from the total deformation $\varepsilon$. It is clear that the phenomenon in regions AC and CD corresponds exactly to the behavior of a work-hardening elastoplastic and elastic perfectly plastic solid, respectively. As can be seen from Fig. 2.5, the total strain $\varepsilon$ in a plastic material can be considered as the sum of the reversible elastic strain $\varepsilon^e$ and the permanent plastic strain $\varepsilon^p$. A material is called perfectly plastic or work-hardening according to whether it does or does not admit changes of permanent stain under constant stress (Chen, 1982).

The plastic response of concrete exhibits some characteristics that the classical theory of plasticity cannot describe. It was shown experimentally that there is a lack in simulating the normality rule (Adenaes et. al., 1977). Furthermore, the descending branch of the uniaxial stress-strain diagram of concrete resembles a violation of the Drucker’s stability postulate. On the other hand, and from a macroscopic point of view; classical plasticity can simulate the behavior of concrete particularly in the pre-peak regime such as the nonlinearity of the stress-strain curve and the significant irreversible strain upon loading. Therefore, the plasticity theory can be used in the modeling of strain hardening behavior of concrete. Many works were presented by researchers to modify the classical theory of plasticity in order to make it more suitable for concrete materials (Feenstra and de Borst, 1996; Bicanic and Pearce, 1996; Grassl et. al., 2002; Park and Kim, 2005; and others).
The main characteristics of the plasticity models (Chen and Schnobrich, 1981; Han and Chen, 1985; Ortiz, 1985; Lubliner et al., 1989; Voyiadjis and Abu-Lebdeh, 1994; Lee and Fenves, 1998, 2001; Grassl and Jirásek, 2006; and others) used to describe the behavior of concrete include: pressure and path sensitivity, non-associative flow rules, work or strain hardening, and limited tensile strength. Many of those models have been developed for the finite element analysis of structural elements. Some of these models are associated with high mathematical complexity which renders them undesirable for many engineering applications, especially the analysis and design of simple structural elements, such as beams and columns.

A drawback of the plasticity-based approach is that the stiffness degradation due to progressive damage is not modeled. However, some researcher advocated that experimental evidence (Willam et al., 1986; Hordijk, 1991) shows that the stiffness degradation due to tensile cracking is substantial only when the tensile cracking has developed fully, and the stiffness degradation due to compressive loading is even less pronounced than the stiffness degradation due to tensile loading. Therefore, they argued that under monotonic loading where only local unloading occurs, neglecting the degradation of the elastic stiffness does not seem to entail major errors (Feenstra and de Borst, 1996).

2.1.2 Concrete Damage Mechanics

Concrete contains numerous micro-cracks even before the application of any external loads. These inherent micro-cracks are mainly present at the aggregate-cement interface as a result of shrinkage, and thermal expansion in the cement paste or aggregates. Damage in concrete is primarily caused by the propagation and coalescence of these micro-cracks. The growth of these micro-cracks during loading causes reduction in strength and deterioration in the mechanical properties of the concrete material.
Therefore, modeling of crack initiation and propagation is very important in the failure analysis of concrete structures.

To model damage in concrete, various types of constitutive laws have been presented including different approaches such as the endochronic theory (Bazant, 1978) the plastic fracturing theory (Dragon and Mroz, 1979) the total strain models (Kotsovos 1980), plasticity with decreasing yield limit (Wastiels 1980), microplane models (Bazant and Ozbolt, 1990), and the bounding surface concept (Voyiadjis and Abu-Lebdeh, 1993).

Continuum damage mechanics was first applied to metals and later modified to model different materials. The term damage mechanics has been conventionally used to refer to models that are characterized by a loss of stiffness or a reduction of the secant constitutive modulus. It was first introduced by Kachanov (1958) for creep-related problems and then later applied to the description of progressive failure of metals and composites and to represent the material behavior under fatigue (Kachanov, 1986). The use of continuum damage mechanics in concrete began in 1980’s. Damage models were used to describe the strain-softening behavior of concrete. Since then, researchers developed different damage models to represent concrete (Lemaitre and Mazars, 1982; Krajcinovic, 1986; Mazars and Pijaudier-Cabot, 1989; Voyiadjis and Abu Al-Lebdeh, 1992; and others).

Damage theory in concrete materials can represent the post-peak region (Krajcinovic, 1983b; Mazars and Pijaudier-Cabot, 1989; and others.). The use of fracture mechanics in concrete materials, on the other hand, was debated extensively (Mindess, 1983; Krajcinovic and Fanella, 1986). The debate of the use of fracture mechanics for concrete materials was supported by some experimental evidence (Pak and Trapeznikov, 1981), which showed that energy dissipation and planar crack idealization in concrete make the application of fracture mechanics in concrete materials very arguable. Therefore, continuum damage mechanics is applied for the determination of macroscopic damaging variables and material properties (Krajcinovic, 1979; Dragon and Mroz, 1979; among others).

There are several approaches of how the stiffness degradation is included in a model. Some researchers coupled damage with elastic analysis only (Mazars, 1984; Willam et. al., 2001; Tao and Phillips, 2005; Labadi and Hannachi, 2005; Junior and Venturini, 2007; Khan et. al., 2007), while others coupled damage with plasticity. In the plastic-damage model (Simo and Ju, 1987a,b; Ju, 1989; Lubliner et. al., 1989; Voyiadjis and Kattan, 1989; Luccioni et. al., 1996; Armero and Oller, 2000; Salari et. al., 2004; Voyiadjis et. al., 2008a) stiffness degradation is embedded in a plasticity model. These models introduce the use of elastic and plastic damage variables to represent the stiffness degradation. The damage variables are coupled with the plastic deformation in the constitutive formulations which provide help for calibrating the parameters with the experimental results. Yet, the coupled relations are complex and result in an unstable numerical algorithm. This kind of algorithm may cause unrealistic representation of the plastic behavior of the concrete during numerical implementation and iteration procedures (Lee and Fenves, 1994).
In the effective-plasticity and damage model, plasticity is formulated in the effective stress space to model the plastic irreversible phenomena while continuum damage mechanics is adopted to represent stiffness degradation (Simo and Ju, 1987a, b; Ju, 1989; Hansen and Schreyer, 1992; Yazdani and Schreyer, 1990; Fichant et al., 1999; Voyiadjis and Kattan, 1999; Mazars and Pijaudier-Cabot, 1989; Salari et al., 2004; Lee and Fenves, 1998; Faria et al., 1998; Jefferson, 2003; Voyiadjis and Kattan, 2006; Jason et al., 2006; Voyiadjis et al., 2008b). It has the advantage to decouple the stiffness degradation from the plastic deformation by linearizing the evolution equations (e.g., Jason et al. 2006).

Others considered the interaction of thermal and mechanical damage processes in concrete and concrete-like heterogeneous materials (Nechnech et al., 2002; Willam et al., 2003; and others). They studied the interaction of thermal and mechanical damage at the mesomechanical level of observations, when volumetric and deviatoric degradation take place simultaneously. They also studied the effect of thermal expansion and shrinkage in the two-phase concrete material when thermal softening of the elastic properties leads to massive degradation of the load resistance.

Quasi-brittle material under different kinds of loading undergoes several damage states, such as tensile cracking, compressive failure, and stiffness degradation. Some researchers recently accounted for all of these effects in a concrete model using a single (scalar) damage variable $\varphi$ (Luccioni et al., 1996; Lee and Fenves, 1998, 2001; Willam et al., 2001; Ferrara and di Prisco, 2001; Nechnech et al., 2002; Soh et al., 2003; Shen et al., 2004; Jankowiak and Lodygowski, 2004; Salari et al., 2004; Luccioni and Rougier, 2005; Labadi and Hannachi, 2005; Kukla et al., 2005; Shao et al., 2006; Jason et al., 2006; Grassl and Jirásek, 2006; Nguyen and Korsunsky, 2006; Junior and Venturini, 2007; Sain and Kishen, 2007; Mohamad-Hussein and Shao, 2007; and others). In order to account for different responses of concrete under various loadings, other researchers used multiple hardening scalar damage variables $\varphi^\pm$ (Mazars, 1986; Mazars and Pijaudier-Cabot, 1989; Faria et al., 1998; Comi and Perego, 2001; Gatuingt and Pijaudier-Cabot, 2002; Willam et al., 2003; Jirásek, 2004; Marfia et al., 2004; Tao and Phillips, 2005; Wu et al., 2006; Contrafatto and Cuomo, 2006; Ananiev and Ožbolt, 2007; Nguyen and Houlsby 2008a,b; and others). It was argued that isotropic continuum damage mechanics models with multiple damage variables cannot represent the entire spectrum of damage effects on biaxial tensile and compressive strength of the material because the damage variables eventually contribute to the same isotropic evolution in both strengths. This introduced the need for concrete anisotropic damage models. Yet isotropic damage is very widely used in different types of combinations with plasticity models due to the simplicity of its implementation in a material model for concrete material.

There are models that present thermodynamic theories of anisotropic damage mainly by the use of tensor damage variables (Krajcinovic and Lemaitre, 1986; Krajcinovic and Fonseka, 1981; Krajcinovic, 1983b; Voyiadjis and Kattan, 1989, 1999, 2006). Chow and his co-workers (Chow and Wang, 1988; Chow and Lu, 1989) proposed an energy-based elastic-plastic damage model in order to describe the difference in the observed failure
modes of geological materials under compression and tension, by using a damage tensor identified by the fourth rank order and fourth rank projection tensors. The coupling between elastic-plastic deformation and anisotropic damage using symmetric second order damage tensors is widely presented by several authors (Simo and Ju, 1987a,b; Ju, 1989; Voyiadjis and Kattan, 1989, 1990, 1992a,b; Hansen et. al., 2001; Tikhomirov and Stein, 2001; Voyiadjis et. al., 2008a,b; and others).

In the case of anisotropic damage, different levels of damage are related to different principal directions. This can be done using a symmetric second-order damage tensor, $\varphi_{ij}$ (Murakami and Ohno, 1981; Murakami, 1983; Ortiz, 1985; Murakami, 1988; Voyiadjis and Kattan, 1992a,b, 1999, 2006; Voyiadjis and Abu-Lebdeh, 1993; Voyiadjis and Venson, 1995; Voyiadjis and Park, 1997, 1999; Voyiadjis and Deliktas, 2000, Voyiadjis et. al., 2001, Hansen et. al., 2001; Tikhomirov and Stein, 2001; Voyiadjis et. al., 2008a,b; and others), or fourth order damage tensor $\varphi_{ijkl}$ (Chaboche, 1993, Chaboche et. al., 1995, Taqieddin et. al., 2005, 2006; Voyiadjis et. al., 2007a,b; and others).

The variety in all of these anisotropic models is somewhat puzzling because a) the relation between these models is difficult to establish (except maybe in the case of the isotropic damage) and b) the comparison of damage-induced anisotropy with experimental data is difficult and therefore the characterization of the damage-induced anisotropy of the material requires three-dimensional experimental facilities. Because of this difficulty in the experimental simulation of concrete specimens, together with the inconvenience related to computational aspects, many researchers avoided these complexities by resorting to simplified (single or multiple) isotropic damage models.

Concrete damage can be characterized as a reduction in the material stiffness. As shown in Fig. 2.6 and 2.7, stiffness reduction in tensile loading is less than that under
compressive loading and therefore, more damage occurs in tension than in compression. Thus, the characterization of material damage in tension and in compression should be considered when modeling the response of plain concrete, justifying the use of multiple damage variables.

Development of a damage-based model requires the definition of a damage rule that characterizes the rate at which material damage is accumulated. The identification of this damage rule may also include the definition of a damage surface that defines an initial elastic domain. Various proposed damage models differ in the definition of the damage surface and the corresponding damage rules. Some of the first constitutive relationships for damage characterization were proposed for the isotropic damage case based on the assumption that one defines an effective stress that is larger than the Cauchy stress and accounts for the reduction in the material’s area that results from micro-cracking (Kachanov, 1958).

![Figure 2.7 Stress-deformation history for concrete subjected to cyclic tensile loading (data from Reinhardt, 1984) (Image)](image)

In order to solve the problem of mesh non-objectivity (sensitivity) in the FE analysis, which is often encountered when using models that are based on the strength theory, the fracture energy method is often incorporated in damage mechanics to give an improved mesh-size independent description of the post-peak behavior of concrete (Faria et. al., 1998; Lee and Fenves, 1998, 2001; Salari et. al., 2004; Kratzig and Polling, 2004; Rabczuk et. al., 2005; Luccioni and Rougier, 2005; He et. al., 2006; Contrafatto and Cuomo, 2006; Wu et. al., 2006; Nguyen and Houlsby 2008a,b; and others). The concept of energy dissipation, i.e. fracture energy, has been used extensively in finite element calculations and can be considered as accepted in the engineering community (Feenstra and de Borst, 1996). With the assumption that the fracture energy is uniformly dissipated in a representative area of the structure, the equivalent length, the finite element calculations will lead to results that are insensitive with regard to the global structural response upon mesh refinement, at least below a certain level of refinement.
2.2 Steel Reinforcement

Reinforcement comes in different types and shapes. Those most commonly used are the deformed circular cross-sectional bars. The spiral deformation pattern on the bars strengthens the mechanical bond between the bars and concrete. The properties of reinforcing steel, unlike concrete, are generally not dependent on environmental conditions or time. Thus, the specification of a single stress-strain relation is sufficient to define the material properties needed in the analysis of RC structures.

Typical stress-strain curves for reinforcing steel bars used in concrete construction are obtained from coupon tests of bars loaded monotonically in tension. For all practical purposes steel exhibits the same stress-strain curve in compression as in tension. The steel stress-strain relation exhibits an initial linear elastic portion, a yield plateau, a strain hardening range in which stress again increases with strain and, finally, a range in which the stress drops off until fracture occurs. The extent of the yield plateau is a function of the tensile strength of steel. High-strength, high-carbon steels, generally, have a much shorter yield plateau than relatively low-strength, low-carbon steels (Wang and Salmon, 1998).

Two different idealizations, shown in Fig. 2.8, are commonly used depending on the desired level of accuracy (ASCE 1982). The first idealization neglects the strength increase due to strain hardening and the reinforcing steel is modeled as a linear elastic, perfectly plastic material, as shown in 3.8a (Mazars and Pijaudier-Cabot, 1989; Sumarac et. al., 2003; Kratzig and Polling, 2004; Luccioni and Rougier, 2005; and Wu et. al., 2006). This assumption underlies the design equations of the ACI code. If the strain at the onset of strain hardening is much larger than the yield strain, this approximation yields very satisfactory results. This is the case for low-carbon steels with low yield strength.

![Figure 2.8 Idealizations of the steel stress-strain curves.](image)

If the steel hardens soon after the onset of yielding, this approximation underestimates the steel stress at high strains. In several instances it is necessary to evaluate the steel stress at strains higher than yield to more accurately assess the strength of members at
large deformations. This is, particularly, true in seismic design, where assessing the available ductility of a member requires that the behavior be investigated under strains many times the yield strain. In this case more accurate idealizations which account for the strain hardening effect are required, as shown in Fig. 2.8b for the case of bilinear stress-strain models (Salari and Spacone, 2001; Limkatanyu and Spacone, 2003). The parameters of these models are the stress and strain at the onset of yielding, the strain at the onset of strain hardening and the stress and strain at ultimate (Fig. 2.9). These parameters can be derived from experimentally obtained stress-strain relations.

![Figure 2.9 Linear elastic, linear strain hardening steel stress-strain relation](image)

In this study the reinforcing steel is modeled as a linear elastic, linear strain hardening material with yield stress $\sigma_y$, as shown in Fig. 2.9. The reasons for this approximation are:

1- The computational convenience of the model.
2- The behavior of RC members is greatly affected by the yielding of reinforcing steel when the structure is subjected to monotonic bending moments. Yielding is accompanied by a sudden increase in the deformation of the member. In this case the use of the elastic-perfectly plastic model in Fig. 2.8a leads to numerical convergence problems near the ultimate member strength. It is, therefore, advisable to take advantage of the strain-hardening behavior of steel to improve the numerical stability of the solution. The assumption of a linear strain hardening behavior immediately after yielding of the reinforcement does not adversely affect the accuracy of the results, as long as the slope of the strain hardening branch is determined so that the strain energy of the model is equal to the strain energy of the experimental steel stress-strain relation (Fig. 2.9). Such a model has been successfully used for the analyses of RC structures (Ngo and Scordelis, 1967; Feenstra, 1993; Kwak and Fillippou, 1997; Lowes, 1999; Tikhomirov and Stein,
Different approaches can be used for modeling reinforcing steel in RC structure using the finite element method. Some researchers use two dimensional elements to represent steel reinforcement in their finite element meshes (Ngo and Scordelis, 1967; Mazars and Pijaudier-Cabot, 1989; Pijaudier-Cabot et. al., 1991; Luccioni and Rougier, 2005; and Wu et. al., 2006; and others). On the other hand, one dimensional representation of steel reinforcement is more widely used since it is unnecessary to introduce the complexities of multiaxial constitutive relationships for structural steel (Chen, 1982; Feenstra, 1993; Kwak and Fillippou, 1997; Faria et. al., 1998; Lowes, 1999; Soh et. al., 2003; Sumarac et. al., 2003; Kratzig and Polling, 2004; Rabczuk et. al., 2005; Phuvoravan and Sotelino, 2005; He et. al., 2006; and others). The discrete model, the distributed or smeared model, and the embedded model are examples of one dimensional representation of steel reinforcement in a RC finite element mesh. The most widely used approach is the discrete one dimensional truss element, which is assumed to be pin connected to the concrete elements and posses two degrees of freedom at each node (Feenstra, 1993; Faria et. al., 1998; Kratzig and Polling, 2004; Phuvoravan and Sotelino, 2005; He et. al., 2006; and others). Alternatively, beam elements can also be used where three degrees of freedom are allowed at each end of the bar element (Rabczuk et. al., 2005). In the distributed model, the reinforcing steel is assumed to be distributed over the concrete elements at a certain orientation angle, and in the embedded steel model the reinforcing steel is considered as an axial member built into the isoparametric elements representing the concrete. A significant advantage of the discrete representation, in addition to the simplicity, is that it can include the slip of reinforcing steel with respect to the surrounding concrete in contrast with the other 1D models where perfect bond between concrete and steel is usually assumed (e.g. de Borst et. al., 2004).

2.3 Bond and Interaction at Steel – Concrete Interface

Utilization of RC as a structural material is derived from the combination of concrete and reinforcing steel into structural elements. Concrete is strong and relatively durable in compression while reinforcing steel is strong and ductile in tension and in compression. Maintaining this composite action requires transfer of load between concrete and steel. This load transfer is referred to as bond and is idealized as a continuous stress field that develops in the vicinity of the steel-concrete interface. For RC structures subjected to moderate loading, the bond stress capacity of the system exceeds the demand allowing steel and the surrounding concrete to behave as a unit (full bond). However, further application of external loads results in an increase in the stresses in the interface between concrete and steel. It also results in localized bond demand that exceeds the bond capacity, resulting in the deterioration of that capacity, a localized damage that gradually spreads to the surrounding material, and a significant movement between the reinforcing steel and the surrounding concrete. Therefore, the properties of the interface between concrete and steel is very important in the analysis of RC structures.
The main stress transfer mechanisms between concrete and steel in RC elements are represented by adhesion, mechanical interaction, and friction (Luccioni et. al., 2005). Adhesion is constituted by chemical bonds and stresses developed during the curing process of concrete. This transfer mechanism, schematically represented in Fig. 2.10a, is prevailing in the case of bars of smooth surface and its failure is characterized by the initiation and propagation of cracks in the concrete/steel interface. In the case of corrugated bars, the described mechanism is secondary and the stress transfer is mainly due to the interaction between ribs and the surrounding concrete. Adhesion is relatively soon exhausted in the global response and consequently, the transfer force is transmitted by friction and mechanical interaction between the ribs and the adjacent concrete.

As the force in the reinforcing bar is increased, the transfer forces are dominated by the mechanical interaction concentrating at the faces of the ribs. In this state, the term “adhesion stress” refers to the mean force per unit of surface. At increased loading, the concrete begins to fail near the ribs with two different modes of failure; by failure of the concrete adjacent to the contact area, as illustrated in Fig. 2.10b and by transverse cracking.

Transverse cracking initiates at the ribs presenting a characteristic cone shape, also called secondary cracking or adhesion cracking. The bond zone, constituted by a damaged zone with finite depth surrounding the steel bar, is defined by the extension of the transverse cracking (Fig. 2.11a). With the stable propagation of transverse cracking, the concrete next to the bar seems to form inclined struts that are known as compression cones (Fig. 2.11b). The adhesion stiffness is usually characterized by the stiffness of these struts (Luccioni et. al., 2005).

When the load is further increased, radial splitting forces can be developed. This phenomenon is due to the rotation of the inclined struts (Fig. 2.11b) that produces a considerable radial component of the contact force, the increase of the radial force that is produced by the increase of the effective contact angle between ribs and concrete due to the deposition of the crushed concrete at the faces of the ribs, the wedge action of the ribs, and the radial component of the contact forces. Without an accurate confinement, a splitting failure can occur, spreading the effect of bond outside the bond zone.
The bond zone dilation, due to the longitudinal cracking, takes place when the adhesion stress reaches values next to the limit one. Following this moment, a stress softening is observed in the response. This stress softening, characteristic of the bond–slip behavior, is frequently interpreted as a progressive shear failure of concrete between the ribs. When the confinement stresses are low, the geometric variation in the contact occurring between the ribs and concrete can also contribute to the stress softening. Both the progressive shear failure of concrete and the contact zone geometric variation are stimulated by the reduction of the confinement stresses produced by the propagation of longitudinal cracking. In both cases, the stress softening reveals a strong discontinuity between the reinforcing bar and the surrounding concrete.

Different approaches have been adopted to incorporate the effect of steel–concrete interaction into the analysis of RC structures. One of the earliest attempts to describe the bond in RC beams is that of Ngo and Scordelis, (1967). They used a link element which can be conceptually thought of as consisting of two linear springs parallel to a set of orthogonal axes. These axes can be in sync with the steel bar longitudinal and tangential directions. The link element has no physical dimensions, and only its mechanical properties are of importance, i.e., longitudinal and tangential stiffness. Therefore, link elements can be placed anywhere in the beam without disturbing the beam geometry. This link element represents the bond between concrete and steel and can permit a certain amount of slip to take place during stress transfer through that bond.

Based on the bond zone mentioned above, some researchers combined 2D interface elements with regular 2D elements representing steel and concrete (Rots, 1988; Clement, 1987). The nonlinear response of concrete near the steel bar is lumped into a fictitious interface that has special constitutive equations. In addition some mechanisms such as the friction that depends on the shape of the steel bars (deformed or not) have been included into the behavior of interface elements. The implementation of such relations raises the problem of the identification of additional material properties. Others lumped the damage zone characteristics into an interface of zero thickness. In the work of (Dragosavic and Groneveld, 1984), it is assumed that the thickness of the damaged layer around the reinforcement is equal to the radius of the bar. This technique is justified using homogenization in (Clement et. al., 1985). Thus omitting interface elements and considering that concrete is progressively damaged around the steel bar is strictly
equivalent to using interface elements if the bar presents surface deformations, i.e., if
displacements are continuous at the interface. In finite element applications, omitting the
interface elements should certainly reduce the computational cost. However, it requires
the implementation of accurate constitutive equations for concrete which can be a very
sensitive subject.

Pijaudier-Cabot et. al. (1991) introduced a non-local damage mechanics approach to
analyze the bond between steel and concrete. A scalar damage variable was used to relate
the stresses in the concrete to the strains. Critical state of concrete stress at the interface
will initiate that damage process which will eventually expand to propagate damage to
the surrounding concrete. They compared their results to experiments of pull-out test.

Many researchers recently addressed the problem of steel-concrete interaction (Cox
and Herrmann, 1998, 1999; Ghandehari et. al., 1999; Soh et. al., 1999; Monti and
Spacone, 2000; Ayoub and Filippou, 2000; Salari and Spacone, 2001; Ben Romdhane
and Ulm, 2002; Limkatanyu and Spacone, 2003; Spacone and El-Tawil, 2004; Luccioni
et. al., 2005; Liang et. al., 2005; Rabczuk et. al., 2005; Kwak and Kim, 2006; Ragueneau
et. al., 2006; Wu et. al., 2006; and others). On the other hand, many researchers assumed
full bonding between the reinforcing steel and concrete in order to allow for detailed
modeling of other phenomena observed in RC (Bazant and Pijaudier-Cabot, 1987;
Feenstra, 1993; He et. al., 2006; Junior and Venturini, 2007; and others).

2.4 Methodology

It was mentioned earlier that because of the many factors affecting the nonlinear
behavior of RC and the differences in short- and long-term behavior of the constituent
materials, it is a common practice among researchers to model the short- and long-term
response of RC members and structures based on separate material models for reinforcing
steel and concrete, which are then combined along with models of the interaction
between the two constituents to describe the behavior of the composite RC material. This
will be the adopted approach in this study of short-term behavior of RC beams under
monotonic loads. Time dependent effects such as creep, shrinkage, and temperature
change will not be considered here. Thermodynamically consistent derivation of the
formulations will be shown throughout this work. The RC behavior predicted by the
proposed model will be eventually compared to results obtained through experiments
conducted by other researchers. In the following, the methodologies used to describe the
behavior of concrete, reinforcing steel, and their bond-interaction analysis is presented.

2.4.1 Concrete Plasticity

The plasticity concrete model selected for this study is the model presented originally
by Lubliner et. al., (1989) (also known as the Barcelona model) and later modified by Lee
and Fenves, (1998) and Wu et. al., (2006). The model is based on an internal variable-
formulation of plasticity theory for the nonlinear analysis of concrete. The model makes
use of the fact that concrete eventually exhibits strain-softening in tension and
compression, leading to complete loss of strength. In this regard concrete resembles such
materials as cohesive soils, and may be classified with them as frictional materials with cohesion, where the eventual loss of strength may be thought of as the vanishing of the cohesion (Lubliner et. al., 1989). For such a model to be capable of representing the behavior of concrete materials, the yield criterion must be of the form in which the concept of cohesion is clear, and the hardening rule should be such that it will eventually lead to vanishing of the cohesion (total damage).

Any plasticity model must include three basic components: an initial yield surface that defines the stress level at which plastic deformation begins, a hardening rule that defines the change of loading surface as well as the change of the hardening properties of the material during the course of plastic flow, and a flow rule which gives an incremental plastic stress-strain relation to allow for the plastic strain evolution during the course of loading.

The adopted yield surface accounts for plasticity in tension and compression, and is observed as a successful yield criterion in simulating the behavior of concrete under uniaxial, biaxial, multiaxial, and cyclic loadings (Lee and Fenves, 1998, 2001; We et. al., 2006). Multiple hardening variables are introduced through this yield surface to account for different hardening. The uniaxial strength functions are factored into two parts, corresponding to the effective stress and the degradation of elastic stiffness. The constitutive relations for elastoplastic responses are decoupled from the degradation damage response (discussed in the next section), which provides advantages in the numerical implementation. This criterion is given in the effective (undamaged) configuration as follows:

\[
 f = \sqrt{3 \bar{J}_2} + \alpha \bar{T}_1 + \beta (\kappa^\pm) H(\hat{\sigma}_{\max}) \hat{\sigma}_{\max} - (1 - \alpha) c^- (\kappa^-) = 0
\]  

(2.1)

where \( \bar{J}_2 = \frac{s_y s_y}{2} \) is the second-invariant of the effective deviatoric stress \( s_y = \bar{\sigma}_{ij} - \bar{\sigma}_{kk} \delta_{ij} / 3 \), \( \bar{T}_1 = \bar{\sigma}_{kk} \) is the first-invariant of the effective stress tensor \( \bar{\sigma}_{ij} \), \( \kappa^\pm \) are the equivalent plastic strains, \( H(\hat{\sigma}_{\max}) \) is the Heaviside step function (\( H = 1 \) for \( \hat{\sigma}_{\max} > 0 \) and \( H = 0 \) for \( \hat{\sigma}_{\max} < 0 \)), and \( \hat{\sigma}_{\max} \) is the maximum principal stress. The parameters \( \alpha \) and \( \beta \) were originally defined by Lubliner et. al. (1989) as dimensionless constants. Lee and Fenves, (1998) modified \( \beta \) to be a function of the tensile and compressive cohesions:

\[
 \alpha = \frac{(f_{b0}^- / f_0^-)^{-1} - 1}{2(f_{b0}^- / f_0^-)^{-1} - 1}
\]  

(2.2)

\[
 \beta(\kappa^\pm) = (1 - \alpha) \frac{c^- (\kappa^-)}{c^+ (\kappa^-)} - (1 + \alpha)
\]  

(2.3)

where \( f_{b0}^- \) and \( f_0^- \) are the initial biaxial and uniaxial compressive yield stresses, respectively, \( c^- \) and \( c^+ \) are the tensile and compressive cohesions, respectively.
The tensile and compressive cohesions $c^\pm$ in Eq. (2.3), which are functions of the tensile and compressive equivalent plastic strains $\kappa^\pm$, should be scaled so that their initial
values are $f_0^\pm$, the initial yield strength in uniaxial tension and compression, respectively, and their final values vanishes when the damage parameters reach their maximum. However, unlike the usual plasticity models with isotropic hardening, the cohesions $c^\pm$ themselves are assumed to be internal variables governed by rate equations that simulate their evolution.

It should be noted here that for tensile loading, damage and plasticity are initiated when the equivalent applied stress reaches the uniaxial tensile strength $f_0^+$ as shown in Fig. 2.12a. However, under compressive loading, damage is initiated at a different stage than plasticity. Once the equivalent applied stress reaches $f_0^-$ (i.e. when nonlinear behavior starts) damage is initiated, whereas plasticity occurs once $f_u^-$ is reached (Fig. 2.12b). Therefore, generally $f_0^+ = f_u^+$ for tensile loading, but this is not true for compressive loading (i.e. $f_0^- \neq f_u^-$).

The flow rule gives the relation between the plastic flow direction and the plastic strain rate. In contrast with metals, the nonassociative flow rule is necessary to control the dilatancy in modeling frictional materials (Chen and Han, 1988). Because the yield surface in Eq. (2.1) is a combined geometric shape from two different Drucker-Prager type functions, a Drucker-Prager type function is used as the plastic potential function (Lee and Fenves, 1998, 2001; Wu et al., 2006).

The flow rule also connects the loading surface/function and the stress-strain relation. When the current yield surface $f$ is reached, the material is considered to be in plastic flow state upon increase of the loading. The flow rule is presented as follows:

$$
F^p = \sqrt{3J_2} + \alpha^p T_i
$$

$$
\frac{\partial F^p}{\partial \sigma_y} = \frac{3}{2} \frac{\delta_i}{\sqrt{3J_2}} + \alpha^p \delta_j
$$

$$
\dot{\varepsilon}^p = \lambda \frac{\partial F^p}{\partial \sigma_y}
$$

where $\lambda$ is the plastic loading factor known as the Lagrangian multiplier.

### 2.4.2 Concrete Damage Mechanics

Damage mechanics can be illustrated using the effective stress concept proposed first by Kachanov (1958) as explained below: Consider a uniform bar subjected to a uniaxial uniform tensile stress, $\sigma$, as shown in Fig. 2.13a. The cross-sectional area of the bar in the stressed configuration is $A$ and it is assumed that both voids and cracks appear as damage in the bar. The uniaxial tensile force $T$, acting on the bar is expressed using the relation $T = \sigma A$. In order to use the principles of continuum damage mechanics, one
considers a fictitious undamaged configuration (effective configuration) of the bar as shown in Fig. 2.13b. In this configuration all types of damage, including voids and cracks, are removed from the bar. The effective stressed cross-sectional area of the bar in this configuration is denoted by $\bar{A}$ and the effective uniaxial stress is $\bar{\sigma}_y$. The bars in both the damaged configuration and the effective undamaged configuration are subjected to the same tensile force $T$. Therefore, considering the effective undamaged configuration, one obtains the relation $T = \bar{\sigma}\bar{A}$. Equating the two expressions of $T$ obtained from both configurations, the following expression is derived:

\[ \sigma = \frac{\bar{A}}{A} \bar{\sigma} \]  \hspace{1cm} (2.7)

Moreover, as it is seen from Fig. 2.13, the effective area $\bar{A}$ is obtained from $A$ by removing the surface intersections of the micro-cracks and cavities (Voyiadjis and Kattan, 1999, 2006) and correcting for the micro-stress concentrations in the vicinity of discontinuities and for the interactions between these effects. Therefore, the damage parameter $\varphi$ in case of uniaxial loading can be defined as follows:

\[ \varphi = 1 - \frac{\bar{A}}{A} \]  \hspace{1cm} (2.8)

In the above equation, in the case of no damage (effective state) in the material the damage parameter is equal to zero (i.e. $\varphi = 0$ for $A = \bar{A}$). The critical damage $\varphi = \varphi_{cr}$ corresponds to the rupture of the element. Lemaitre (1984) showed that the damage parameter value ranges between 0.2 and 0.8 ($0.2 \leq \varphi \leq 0.8$) for metals. The theoretical value of damage parameter, $\varphi$, for the general case lies in the range $0 \leq \varphi \leq 1$. 
The effective area $\tilde{A}$ can be obtained through mathematical homogenization techniques (Voyiadjis and Kattan, 1999, 2006). Homogenization techniques can be used when the shape and the size of the defects are known which is somewhat difficult to obtain even with electron microscopes.

Making use of Eqs. (2.7) and (2.8), one obtains the following expression for the effective uniaxial stress $\tilde{\sigma}$ (Kachanov, 1958):

$$\tilde{\sigma} = \frac{\sigma}{1 - \varphi}$$  \hspace{1cm} (2.9)

It should be noted that the undamaged (effective) stress $\tilde{\sigma}$ can be considered as a fictitious stress acting on an undamaged equivalent area $\tilde{A}$.

For a three-dimensional state of stress, Eq. (2.9) can be generalized for isotropic damage as follows:

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - \varphi}$$  \hspace{1cm} (2.10)

where $\sigma_{ij}$ and $\tilde{\sigma}_{ij}$ are the stress tensors in the damaged and effective configurations, respectively.

The transformation equations from the nominal configuration to the effective one can be obtained using either the strain energy equivalence hypothesis or the strain equivalence hypothesis (Voyiadjis and Kattan, 1990, 1999, 2006). The strain equivalence hypothesis states that the strain in the effective configuration is equal to the strain in the nominal configuration such that in the constitutive equations one can simply replace the nominal strain by the corresponding effective strain. On the other hand, the strain energy equivalence hypothesis states that the elastic strain energy density in the damaged configuration is equal to the elastic strain energy density in the effective (undamaged) configuration.

The constitutive equation for the damaged material is written in terms of that of the virgin material; that is, the damaged material is modeled using the constitutive laws of the effective undamaged material in which the Cauchy stress tensor $\sigma_{ij}$ is replaced by the effective stress tensor, $\tilde{\sigma}_{ij}$ (Murakami and Ohno, 1981):

$$\tilde{\sigma}_{ij} = M_{ijkl}\sigma_{kl}$$  \hspace{1cm} (2.11)

where $M_{ijkl}$ is the fourth-order damage effect tensor which has many definitions in literature (isotropic or anisotropic). One of these definitions is given by Voyiadjis and
Venson (1995) and Voyiadjis et. al. (2008a,b) for anisotropic damage as follows (in an effort to symmetrize the stress tensor):

\[
M_{ijkl} = 2\left[ (\delta_{ij} - \varphi_{ij}) \delta_{kl} + \delta_{ij} (\delta_{kl} - \varphi_{kl}) \right]^{-1}
\] (2.12)

In this study, two isotropic damage variables are used to model the tensile and compressive damage phenomena in concrete material. The constitutive equation is written in terms of an accumulated damage variable that is function of the tensile and compressive damage variables and the stress state using the spectral decomposition of the stress tensor. Detailed formulation is presented in Chapter 4.

2.4.3 Steel Reinforcement

In this study the reinforcing steel is modeled as a linear elastic, linear strain hardening material with yield stress \( \sigma_y \), as shown in Fig. 2.9. A \( J_2 \) elasto-plasticity model with linear hardening will be adopted to describe the behavior of steel reinforcement. The von Mises yield criterion, associative flow rule and isotropic hardening are suitable for modeling structural steel. The von Mises yield criterion can be written as:

\[
F(\sigma, R) = \sigma_{eq} - \sigma_y - R(p) \leq 0
\] (2.13)

where \( \sigma_y \) is the yield stress, \( R(p) \) is the isotropic hardening stress (linear function of the accumulated plastic strain \( R = kp \)), and \( \sigma_{eq} \) is the von Mises equivalent stress defined by:

\[
\sigma_{eq} = \sqrt{\frac{3}{2}} S_y S_y
\] (2.14)

where \( S_y \) is the deviatoric part of the Cauchy stress tensor:

\[
S_y = \sigma_{ij} - \frac{1}{3} \sigma_{mm} \delta_{ij}
\] (2.15)

The plastic flow rule that governs the evolution of the plastic strain is given as follows:

\[
\dot{\varepsilon}_p^{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}
\] (2.16)

The scalar \( \dot{\lambda} \) is the plastic multiplier which is equal in this case to the rate of the accumulated plastic strain:
\[
\dot{\lambda} = \dot{\rho} = \sqrt{\frac{2}{3}} \dot{\varepsilon}_p^p \varepsilon_p^p
\]  

(2.17)

where \( \dot{\varepsilon}_p^p \) is the plastic strain rate.

### 2.4.4 Steel-Concrete Bond and Interaction

The importance of bond consideration in analysis of RC structures is shown through the reinforcing bar stress-strain diagram in Fig. 2.14. Significant difference in the initial behavior of the bar can be observed depending on the degree of bond. The initial stiffness in both the normal and weak bond cases is smaller than for the theoretical full bond case (Monti and Spacone, 2000). As for the post-yield phase, the normal bond case is practically identical to the full bond case. It is obvious from the figure that the nature of bond failure drastically changes the stress-strain behavior of the bar.

![Figure 2.14](image)

Figure 2.14 Monotonic stress-strain response of reinforcing bar fiber with different degrees of bond (Monti and Spacone, 2000)

The effect of bond between steel and concrete will be incorporated in this study through the description of the steel constitutive model. The global effect of bond and interaction on the stress-strain diagram will be accounted for by reducing the yield stress and the elastic and hardening moduli of the steel bars according to the parametric studies conducted by Belarbi and Hsu (1994), Kwak and Kim (2006), and the references therein. Further discussion is provided in Chapter 5.

### 2.4.5 Numerical Implementation of the Model

Once the material constitutive models for concrete, steel reinforcement, and bond-interaction are developed, they will be all implemented into the advanced finite element analysis software ABAQUS via a user defined material subroutine (UMAT). While ABAQUS performs the standard finite element procedure using standard types of finite elements, the UMAT will govern the behavior of these materials during different loading
stages, i.e., elastic, inelastic, failure, post-failure loads. Combining a UMAT subroutine with standard finite element procedure will manifest the simplicity and applicability of the proposed model into any engineering problem where the average user will not be exposed to the complexities associated with introducing non standard finite elements.
3.1 Introduction

Steel reinforcement is used in RC structures to provide the tensile strength that concrete lacks. The properties of reinforcing steel, unlike concrete, are generally well known and do not dependent on environmental conditions or time (creep). Reinforcement is usually classified on the basis of geometrical properties, such as size and surface characteristics, and on the mechanical properties, such as characteristic yield stress and ductility.

Typical stress-strain curves for reinforcing steel bars used in RC construction are obtained from coupon tests of bars loaded monotonically in tension. For all practical purposes, reinforcing steel exhibits the same stress-strain curve in compression as in tension. Thus, the specification of a single stress-strain relation is sufficient to define the material properties needed in the analysis of RC structures. The steel stress-strain relation exhibits an initial linear elastic portion, a yield plateau, a strain hardening range in which stress again increases with strain and finally, a range in which the stress drops off (softens) until fracture occurs (Fig. 3.1).

![Figure 3.1 Experimental Stress-Strain curve for reinforcing steel](image)

The extent of the yield plateau is a function of the tensile strength of steel. High-strength, high-carbon steels, generally, have a much shorter yield plateau than relatively low-strength, low-carbon steels (Wang and Salmon, 1998). The experimentally obtained
stress-strain diagram for the reinforcing steel is usually replaced in research by an
idealized characteristic diagram (Fig. 3.2).

Figure 3.2 Idealizations of the steel stress-strain curves.

Two different idealizations, shown in Fig. 3.2, are commonly used in literature
depending on the desired level of accuracy (ASCE 1982). The first idealization neglects
the strength increase due to strain hardening and the reinforcing steel is modeled as a
linear elastic, perfectly plastic material, as shown in Fig. 3.2a. This assumption underlies
the design equations of the ACI code. If the strain at the onset of strain hardening is much
larger than the yield strain, this approximation yields very satisfactory results. This is the
case for low-carbon steels with low yield strength.

If the steel hardens soon after the onset of yielding, this approximation underestimates
the steel stress at high strains. At several instances it is necessary to evaluate the steel
stress at strains much higher than yield to more accurately assess the strength of members
at large deformations. This is, particularly, true in seismic design, where assessing the
available ductility of a member requires that the behavior be investigated under strains
many times the yield strain. In this case, more accurate idealizations which account for
the strain hardening effect are required. The second idealization - the case of the bilinear
stress strain models - would be more appropriate to model the steel behavior. As shown
in Fig. 3.2b, the reinforcing steel is modeled as a linear elastic, linear-plastic-hardening
material. The parameters of these models are the stress and strain at the onset of yielding
and the stress and strain at the ultimate load (Fig. 3.3). These parameters can be derived
from experimentally obtained stress-strain relations.

The first idealization (Fig. 3.2a) is frequently used by researches (Mazars and
Pijaudier-Cabot, 1989; Sumarac et. al., 2003; Kratzig and Polling, 2004; Luccioni and
Rougier, 2005; and Wu et. al., 2006) because of its simplicity, especially in the cases
where most of the interesting activities in the RC member occur before the strain-
hardening of reinforcing steel starts (Vecchio and Collins, 1982). The second idealization
(Fig. 3.2b) is also used in many studies (Ngo and Scordelis, 1967; Feenstra, 1993; Kwak
and Fillippou, 1997; Lowes, 1999; Tikhomirov and Stein, 2001; Rabczuk et. al., 2005;
Phuvoravan and Sotelino, 2005; He et. al., 2006; and Junior and Venturini, 2007). It is claimed by some researchers (e.g. Kwak and Fillippou, 1997 and the references therein) that adopted the second idealization that the use of the elastic-perfectly plastic model shown in Fig. 3.2a leads to numerical convergence problems near the ultimate member strength in situations where yielding is accompanied by a sudden increase in the deformation of the member under monotonic bending moments. In these situations, the RC member is greatly affected by the hardening of reinforcing steel, the fact that encourages the use of bilinear stress-strain models to improve the numerical stability of the solution. They also claimed that the assumption of a linear strain hardening behavior immediately after yielding of the reinforcement does not adversely affect the accuracy of the results, as long as the slope of the strain hardening branch is determined so that the strain energy of the model is equal to the strain energy of the experimental steel stress-strain relation (Fig. 3.3). Despite the discussion, both idealizations have been reported to be successfully implemented into FE codes used to model RC structures.

![Figure 3.3 Linear elastic, linear strain hardening steel stress-strain relation](image)

Some researchers extended the bilinear stress-strain representations to models involving linear elastic behavior followed by nonlinear strain hardening (e.g., Marfia et. al., 2004). A recent and more accurate idealization of the reinforcing steel behavior is the one that accounts for the different phenomena/stages governing the steel behavior (elasticity, perfect-plasticity, and strain-hardening-plasticity) as shown in Fig. 3.4. The material parameters of this idealization are the stress and strain at the onset of yielding, the strain at the onset of strain hardening, and the stress and strain at the ultimate load (Selby and Vecchio, 1997; Fantilli et. al., 2002; and Hoehler and Stanton, 2006). Again, these parameters can be derived from experimentally obtained stress-strain relations. This
idealization requires the development of a new criterion to determine the strains at which hardening starts, a procedure that increases the complexity of this idealization technique.

Different approaches can be used to model reinforcing steel in RC structure using the FE method. Some researchers chose to use two dimensional elements to represent steel reinforcement in their FE meshes (Ngo and Scordelis, 1967; Mazars and Pijaudier-Cabot, 1989; Pijaudier-Cabot et. al., 1991; Luccioni and Rougier, 2005; and Wu et. al., 2006). On the other hand, one dimensional FE representation of steel reinforcement is more widely used to avoid introducing the complexities associated with multiaxial constitutive relationships for structural steel (Chen, 1982; Feenstra, 1993; Kwak and Fillippou, 1997; Faria et. al., 1998; Lowes, 1999; Soh et. al. 2003; Sumarac et. al., 2003; Kratzig and Polling, 2004; Rabczuk et. al., 2005; Phuvoravan and Sotelino, 2005; and He et. al., 2006).

![Figure 3.4 Linear elastic, yield plateau, plastic hardening steel stress-strain relation](image)

The introduction of the steel reinforcing elements into RC FE analysis can be done in many different ways. The discrete element model, the distributed or smeared model, and the embedded model are examples of representing steel reinforcement in an RC FE mesh. The most widely used approach is the discrete element (e.g., Wu et. al., 2006), where steel and concrete are modeled as two distinguished elements with different material properties and behaviors. Whereas, in the distributed model, the reinforcing steel is assumed to be distributed over the concrete elements at a certain orientation angle, and in the embedded steel model the reinforcing steel is considered as an axial member built into the isoparametric elements representing the concrete. A significant advantage of the discrete representation, in addition to its simplicity, is that it eliminates the need to develop more sophisticated types of finite elements needed by the other models (e.g., Soh et. al., 2003), and therefore facilitates the use of classical finite elements available to a wider spectrum of engineers and researchers.

In this study, the steel reinforcing bars are modeled by two-dimensional constitutive relations to comply with the primary objective of this work which is the two-dimensional FE analysis of RC structures. The elastic-plastic with linear strain hardening model (i.e.,
the second idealization, Fig. 3.2b) will be eventually used in the RC beam analysis in a subsequent chapter. Nevertheless, numerical algorithms facilitating the use of different idealizations will be discussed and presented in this chapter.

The following sections describe how metal plasticity may be applied to solve practical boundary-value problems. Emphasis is given to the development of numerical procedures used in solving elastic-plastic boundary-value problems by the nonlinear FE method and the theory of plasticity. The application of the FE method to nonlinear mechanics will be presented first, followed by the discussion of the aspects related to the constitutive equations. Implementation of incremental elastic-plastic constitutive relations in a numerical process where strain and stress increments are no longer infinitesimal but with a finite size, will also be discussed and presented.

3.2 The Application of the FE Method to Nonlinear Continuum Mechanics

The basic formulations and procedures of the FE method for elastic analysis are described briefly as an introduction, followed by a description of the formulations and procedures for elastic-plastic analysis, where elastic-plastic FE analysis procedure results in a set of simultaneous nonlinear equations, the solution of which will also be discussed.

3.2.1 FE Analysis Procedure for Elastic Problems

Recalling the classical statement of a boundary-value problem in quasi-static materials, any arbitrary solid body can be considered. The body before deformation occupies a volume $V$ and is subjected to forces per unit volume $q_i$, traction forces acting per unit surface area $T_i$, and displacement boundary conditions imposed on certain parts of the body $u_i$. The general governing equation of the FE method for a static elastic analysis may be derived from the principle of virtual work (weak form) as follows:

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV = \int_A T_i \delta u_i dA + \int_V q_i \delta u_i dV$$

(3.1)

where $\sigma_{ij}$ and $\epsilon_{ij}$ are the stress and strain tensors, respectively, $\delta u_i$ and $\delta \epsilon_{ij}$ are virtual displacement and virtual strain increments, respectively, where these increments form a compatible set of deformations, and $\sigma_{ij}$ with $T_i$ and $q_i$ form an equilibrium set. In matrix form, which is adopted more frequently for FE formulations than tensorial notations, Eq. (3.1) can be represented as:

$$\int_V \left\{ \delta \epsilon \right\}^T \left\{ \sigma \right\} dV = \int_A \left\{ \delta u \right\}^T \left\{ T \right\} dA + \int_V \left\{ \delta u \right\}^T \left\{ q \right\} dV$$

(3.2)

where the vectors for displacement $\{u\}$, strain $\{\epsilon\}$ (similarly $\{\delta \epsilon\}$ and $\{\delta u\}$) and stress $\{\sigma\}$ are defined in general (3D) terms as follows:
For a small-deformation analysis, the following well-established strain displacement relationship is obtained:

\[
\{\varepsilon\} = [L]\{u\}
\]

\[
\{\delta\varepsilon\} = [L]\{\delta u\}
\]

where \([L]\) is the differential operator matrix defined as:

\[
[L]^T = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 \\
0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
\]

In the displacement-based FE method, the body being analyzed is approximated by an assemblage of discrete finite elements interconnected at nodal points on their boundaries. The nodal points are numbered from 1 to \(N\), and the elements are numbered from 1 to \(M\). The displacement of the body measured in a given coordinate system is approximated as a piece-wise continuous function over the body such that the function is continuous inside each element and the continuity at element boundaries is assumed to be a certain degree. The displacement at the nodal points in an FE system defines the displacement vector \(\{U\}\) as follows:

\[
\{U\}^T = \{u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n\}
\]

Within an element, say element \(m\), the displacement is approximated as:

\[
\{u\}_m = [N]_m \{U\}
\]

where \([N]_m\) is a matrix of 3 by 3N, and is the matrix of displacement interpolation function or the shape function of the element \((m)\). For example, if the element \((m)\) has a
four nodal points, say points i, j, k, l, the shape function matrix will have non-zero sub matrices of 3 by 3 only at columns i, j, k, l, and may be expressed as:

\[
[N]_m = \begin{bmatrix} 0,\ldots,0, N_i I, 0,\ldots,0, N_j I, 0,\ldots,0, N_k I, 0,\ldots,0, N_l I, 0,\ldots,0, 0,\ldots,0 \end{bmatrix}
\] (3.10)

where \([I]\) is a 3 by 3 unit (Identity) matrix, and \([0]\) is a 3 by 3 zero matrix. Substituting Eq. (3.9) into Eqs. (3.6)a,b, one obtains the strain vectors for the element (m) as:

\[
\{\varepsilon\}_m = [B]_m \{U\} \quad (3.11)a,b
\]

\[
\{\delta\varepsilon\}_m = [B]_m \{\delta U\}
\]

where \([B]_m\) is called the strain-displacement matrix, and defined as:

\[
[B]_m = [L][N]_m
\] (3.12)

Substituting Eqs. (3.11)a,b and (3.9) into Eq. (3.2), one obtains the governing equation for small-deformation FE analysis as:

\[
\sum \int [B]_m^T \{\sigma\} dV = \sum \int [N]_m^T \{T\} dA + \sum \int [N]_m^T \{q\} dV
\] (3.13)

where \(m = 1, 2, \ldots, M\), or summation is over every element. This equation may also be written in a simplified form as:

\[
\int [B]_m^T \{\sigma\} dV = \int [N]_m^T \{T\} dA + \int [N]_m^T \{q\} dV
\] (3.14)

where \([B]\) and \([N]\) take the values of \([B]_m\) and \([N]_m\), respectively, when the integrations are performed in the element (m). Next, let:

\[
\{R\} = \int [N]_m^T \{T\} dA + \int [N]_m^T \{q\} dV
\] (3.15)

denote the equivalent external force acting on the nodal points, then accordingly Eq. (3.14) can be reduced to:

\[
\int [B]_m^T \{\sigma\} dV = \{R\}
\] (3.16)

For an elastic analysis, the stress-strain relationship may be generally written as:

\[
\{\sigma\} = [C] \{\varepsilon\}
\] (3.17)
where $[C]$ is the elasticity matrix. The governing Eq. (3.16) may be further written as:

$$
[K] \{U\} = \{R\}
$$

$$
[K] = \int_\Omega [B]^T [C] [B] dV
$$

(3.18)a,b

where $[K]$ is termed the stiffness matrix of the FE system. Equation (3.18)a represents a set of linear simultaneous equations. The displacement vector $\{U\}$ may be determined by solving this set of equations, and the strain and stress of each element can then be determined by Eqs. (3.11)a and (3.17).

It should be noted here that in Eqs. (3.9), (3.11), and (3.12), the element shape function matrix $[N]_m$ and the strain-displacement matrix $[B]_m$ are expressed in terms of the global displacement vector $\{U\}$. More compact form of these two matrices may be obtained if they are expressed in terms of the element displacement vector $\{U\}_m$. For a four-node element with nodal points i, j, k, and l, one can write:

$$
\{U\}_m^T = \{u_i^T, u_j^T, u_k^T, u_l^T\} 
$$

(3.19)

where each sub-vector in the equation represents the displacement vector at a nodal point. In terms of $\{U\}_m$, Eqs. (3.9) and (3.10) may be rewritten as:

$$
\{u\}_m = [N]_m \{U\}_m 
$$

(3.20)

$$
[N]_m = \begin{bmatrix} N_i & N_j & N_k & N_l \end{bmatrix}
$$

(3.21)

In an actual FE representation/program, such compact form is usually used. The stiffness matrix and the external equivalent force vector are computed individually for each element, and are then assembled to the global stiffness matrix and the global force vector, respectively.

### 3.2.2 FE Analysis Procedure for Elastic-Plastic Problems

For most elastic-plastic problems, closed-form solutions are not possible and numerical solutions are sought via the FE method. Due to the nonlinear relationship between the stress $\{\sigma\}$ and the strain $\{\varepsilon\}$, the governing equation, Eq. (3.16), is a nonlinear equation of strains, and thus, a nonlinear equation of nodal displacements $\{U\}$. Iterative methods are therefore necessary to solve this equation for a given set of external forces. Moreover, because of the deformation history dependence of an elastic-plastic constitutive relation, an incremental analysis following the actual variation of the external
Forces must be used to trace the history of the displacements, strains and stresses along with the applied external forces.

In an incremental analysis, the external force history may be expressed as a progressive accumulation of external force increments in certain load steps. At the \((n+1)\)th step, the external force may be expressed as:

\[
\{R\}^{n+1} = \{R\}^n + \{\Delta R\}
\]

(3.22)

where the right superscript \((n)\) is used to refer to the \(n\)-th incremental step, and \(\{\Delta R\}\) is the force increment from step \((n)\) to step \((n+1)\). Assuming that the solution at the \(n\)-th step, \(\{U\}^n\), \(\{\sigma\}^n\) and \(\{\varepsilon\}^n\) are known, and at the \((n+1)\)th step, corresponding to the load increment \(\{\Delta R\}\), one obtains:

\[
\begin{align*}
\{U\}^{n+1} &= \{U\}^n + \{\Delta U\} \quad \text{(3.23)a-b} \\
\{\sigma\}^{n+1} &= \{\sigma\}^n + \{\Delta \sigma\} \\
\{F\}^{n+1} &= \{R\}^{n+1}
\end{align*}
\]

(3.24)

Using these equations, Eq. (3.16) may be rewritten as:

\[
\{F\}^{n+1} = \int_B [\mathbf{B}]^T \{\sigma\}^{n+1} dV
\]

(3.25)

where \(\{F\}^{n+1}\) is the stress equivalent force acting on the nodal points. Substituting Eq. (3.23)b into Eq. (3.22), the following can be obtained:

\[
\int_B [\mathbf{B}]^T \{\Delta \sigma\} dV = \{R\}^{n+1} - \int_B [\mathbf{B}]^T \{\sigma\}^n dV
\]

(3.26)

which is another way of noting that the stiffness matrix, at any \((n+1)\) step, is a function of the displacement. Therefore, the structural equation, Eq. (3.18)a, which now can be written as:

\[
[K(U)]\{U\} = \{R\}
\]

(3.27)

cannot be immediately solved for the nodal displacement \(\{U\}\) because information needed to construct the stiffness matrix \([K(U)]\) is not known in advance. An iterative process is required to obtain \(\{U\}\) and its associated \([K(U)]\) such that the product of \([K(U)]\{U\}\) is in equilibrium with \(\{R\}\).
Equation (3.24) in fact represents the equilibrium of the external force \( \{R\}_{n+1} \) with the internal force \( \{F\}_{n+1} \). Equation (3.26) is the governing equation of the incremental FE formulations. Two types of numerical algorithms are involved in solving this equation for the displacement increment \( \{\Delta U\} \) and the stress increment \( \{\Delta \sigma\} \). One is the algorithm used for solving the nonlinear simultaneous equations generated from Eqs. (3.24) or (3.26) for the displacement increment \( \{\Delta U\} \) (global/equilibrium iterations). Another is the algorithm used to determine the stress increment \( \{\Delta \sigma\} \) corresponding to a strain increment \( \{\Delta \varepsilon\} \), which is computed from \( \{\Delta U\} \), at a given stress state and a given deformation history (local iterations). These two algorithms constitute the nonlinear FE procedure for elastic-plastic analysis. Many algorithms exist for solving the nonlinear equations shown above. What follows is a description of two equation-solving techniques applicable to the time-independent nonlinear equations.

3.3 Algorithms for Nonlinear Global/Equilibrium Iterations

The governing equation of an elastic-plastic FE incremental analysis may be generally written in terms of the displacement \( \{U\} \) at the incremental step \((n+1)\) as:

\[
\prod(\{U^{n+1}\}) = \{F(\{U^{n+1}\})\}^{n+1} - \{R\}^{n+1}
\]

This equation represents an equilibrium of the external force, \( \{R\}_{n+1} \), with the internal force, \( \{F\}_{n+1} \). The iterative methods for solving this equation are therefore referred to as (equilibrium iterative methods). They are also known as global iterations because they are defined at the level of the entire body or structure (Doghri, 2000). This concept is very important when dealing with the user defined material subroutine (UMAT) associated with ABAQUS nonlinear FE analysis; this will be discussed later on. If the weak form of equilibrium is satisfied at a time step \( t_{n+1} \), then the solution at \( t_{n+1} \) has been found, and the process can move on to the next time interval \([t_{n+1}, t_{n+2}]\). Otherwise, a new iteration within the same time interval \([t_n, t_{n+1}]\) starts, that is, a new approximation to the nodal displacements at \( t_{n+1} \) is proposed by solving the nonlinear system of equations again.

3.3.1 The Newton-Raphson Method

Assume that the \((i-1)\)th approximation of the displacement in the \((n+1)\)th incremental step has already been obtained, and is denoted by \( \{U\}_{i-1}^{n+1} \). Expanding \( \prod(\{U^{n+1}\}) \) using the Taylor series expansion at \( \{U\}_{i-1}^{n+1} \) and neglecting all higher-order terms than the linear term, one obtains:
\[ \Pi \left( \{U\}^{n+1}_{i-1} \right) + \frac{\partial \Pi}{\partial \{U\}} \left( \{U\}^{n+1} - \{U\}^{n+1}_{i-1} \right) = 0 \]  
(3.29)

where \( \frac{\partial \Pi}{\partial \{U\}} \) is evaluated at \( \{U\}^{n+1}_{i-1} \), which can be written in terms of the external and internal forces as:

\[ \{F\}^{n+1}_{i-1} + \frac{\partial F}{\partial \{U\}} \left( \{\Delta U\}_i - \{R\}^{n+1}_{i-1} \right) = 0 \]  
(3.30)

where \( \frac{\partial F}{\partial \{U\}} \) is evaluated at \( \{U\}^{n+1}_{i-1} \),

\[ \{\Delta U\}_i = \{U\}^{n+1} - \{U\}^{n+1}_{i-1} \]  
(3.31)

and

\[ \{F\}^{n+1}_{i-1} = \left[ F \left( \{U\}^{n+1}_{i-1} \right) \right]^{n+1} = \int_{V} [B]^T \{\sigma\}^{n+1}_{i-1} dV \]  
(3.32)

Recalling that \( \frac{\partial F}{\partial \{U\}} \) evaluated at \( \{U\}^{n+1}_{i-1} \) is equal to:

\[ [K]^{n+1}_{i-1} = \int_{V} [B]^T \left[ C^{ep} \right] [B] dV \]  
(3.33)

where \( [C^{ep}] \) is evaluated at \( \{U\}^{n+1}_{i-1} \) and is the elastic-plastic matrix, and \( [K]^{n+1}_{i-1} \) is the tangential stiffness matrix of the structure. Therefore, the iteration scheme of a Newton-Raphson algorithm is obtained as follows:

\[ [K]^{n+1}_{i-1} \{\Delta U\}_i = \{R\}^{n+1} - \{F\}^{n+1}_{i-1} \]
\[ \{U\}^{n+1}_i = \{U\}^{n+1}_{i-1} + \{\Delta U\}_i \]  
(3.34)a,b

and the iteration scheme starts at: \( \{U\}^{n+1}_0 = \{U\}^n \), \( [K]^{n+1}_0 = [K]^n \), and \( \{F\}^{n+1}_0 = \{F\}^n \). This iteration continues until a proper convergence criterion is satisfied. Convergence criteria will be discussed and demonstrated through the iteration procedure of a one-degree-of-freedom nonlinear system later on in this chapter (section 3.3.3). The Newton-Raphson algorithm has a high quadratic convergence rate. However, it should be noted from Eq. (3.34)a that the tangential stiffness matrix, \( [K]^{n+1}_{i-1} \), is evaluated and factorized in every
iteration step. This can be prohibitively expensive for a large scale system. Moreover for a perfectly-plastic or a strain-softening material, the tangential matrix may become singular or ill-conditioned during the iteration. This may cause difficulty in finding the solution of the nonlinear equation system. Modification of the Newton-Raphson algorithm might therefore become necessary.

3.3.2 The Modified Newton-Raphson Method

To reduce the expensive operations of stiffness matrix evaluation and factorization, one of the modifications of the Newton-Raphson algorithm is to replace the tangential stiffness matrix $[K]_{i-1}^{n+1}$ in Eq. (3.34)a by a matrix $[K]^m$, which is the stiffness matrix evaluated at a load step $m$, where $m < n+1$. If the matrix is evaluated only at the beginning of the first load step, the initial elastic matrix $[K]_0$ is used throughout all load steps. Such a method is referred to as the initial stress method. Most commonly, when the stiffness matrix is evaluated at the beginning of each load step, or for the $(n+1)$th step, the following stiffness matrix is used:

$$[K]^m = [K]_0^{n+1} = [K]^n$$

(3.35)

The iteration scheme for the Modified Newton-Raphson algorithm is expressed as:

$$[K]^n \{ΔU\}_i = \{R\}_i^{n+1} - \{F\}_{i-1}^{n+1}$$

$$\{U\}_{i}^{n+1} = \{U\}_{i-1}^{n+1} + \{ΔU\}_i$$

(3.36)a,b

and the iteration scheme starts with: $\{U\}_0^n = \{U\}_0^n$, and $\{F\}_0^n = \{F\}_0^n$. Similar to the Newton-Raphson scheme, this iteration procedure continues until a proper convergence criterion is satisfied. This modified iteration procedure is illustrated for a one-degree-of-freedom nonlinear system in the example discussed in section (3.3.3).

The Modified Newton-Raphson algorithm involves less stiffness matrix evaluation and factorization operations than the Newton-Raphson algorithm. This leads to a significantly reduced computational effort in one iteration cycle for a large scale system. However, this modified algorithm converges linearly and, in general, more slowly than the original algorithm. Thus, for a specific nonlinear system, more iterations are needed to reach a convergence when the Modified Newton-Raphson algorithm is used. In some situations, such as in the analysis of a stain softening material, it may become prohibitively slow. The convergence rate of this algorithm depends to a large extent on the number of times the stiffness matrix is updated. The more frequently the stiffness matrix is updated, the lesser the number of iterations needed to reach a convergence. Moreover, the problem that the stiffness matrix may become singular or ill-conditioned still exists.
Another problem associated with the Modified Newton-Raphson algorithm is that if a change in the external load causes an unloading in the structure analyzed, this algorithm may not result in a convergent iteration, unless the stiffness matrix is re-evaluated once an unloading is detected. This problem increases the programming complexity of the implementation of the Modified Newton-Raphson.

3.3.3 Examples of Using Nonlinear Equilibrium Algorithms to Solve FE Elastic-Plastic Problems

An FE example is presented here to demonstrate the application of the first algorithm of the two algorithms constituting the nonlinear FE procedure for elastic-plastic analysis, i.e., using global iterations to solve for the displacement increment \( \{ \Delta U \} \). A simple structure consisting of two horizontal bars is shown in Fig. 3.5. Bar (1) and (2) are made of elastic-linear strain hardening materials with stress-strain relationships shown in the figure. The two materials have the same yield stress \( \sigma_y \) and modulus of elasticity \( E \). \( E_t \) is the tangential (elastic-plastic) modulus of the material. \( E_{t1} = E / 4 \) is the tangential modulus of material of bar (1), while \( E_{t2} = E / 2 \) is the tangential modulus of material of bar (2). A horizontal force \( R = 3 A \sigma_y \) is applied at the joint connecting the two bars, where \( A \) is the cross-sectional area of the bars. Two one-dimensional bar elements with linear shape function are used for the analysis to avoid becoming consumed with the FE formulations of more sophisticated elements, and thus concentrate on the concept of using the nonlinear algorithms. The matrix of the shape function of such elements is given as:

\[
[N] = \begin{bmatrix}
\frac{1}{2}(1-r) \\
\frac{1}{2}(1+r)
\end{bmatrix}
\]  

(3.37)

Figure 3.5 Numerical Example 1: Elastic-plastic analysis of a two bar structure

where \( r \) is the natural coordinate with origin at the center of the bar element. The strain-displacement matrix of the bar element is then obtained as:

\[
[B] = \frac{1}{L}[-1, 1]
\]  

(3.38)
The element stiffness matrices of the two-bar elements are:

\[
[K_1] = \frac{E_i(u)A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \quad [K_2] = \frac{E_2(u)A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\] (3.39)a,b

and the global stiffness matrix, using Eq. (3.18)b, is given as:

\[
[K] = \int [B]^T [C][B]A dx = \frac{A}{L} \begin{bmatrix} E_i(u) & -E_i(u) & 0 \\ -E_i(u) & E_i(u) + E_2(u) & -E_2(u) \\ 0 & -E_2(u) & E_2(u) \end{bmatrix}
\] (3.40)

Using the governing equation, Eq. (3.25), and the boundary condition \( u_1 = u_3 = 0 \), the governing equation at \((n+1)\)th step becomes:

\[
f(u) = F(u) - R = 0
\] (3.41)

where \( u = u_2 \) is the displacement of node 2 (see Fig. 3.5), and \( F(u) \) is the stress equivalent force given by:

\[
F(u) = A\left(\sigma_i(u) - \sigma_2(u)\right)
\] (3.42)

Setting up \([K]\{U\} = \{R\}\) at any given step yields the following:

\[
\frac{A}{L} \begin{bmatrix} E_i(u) & -E_i(u) & 0 \\ -E_i(u) & E_i(u) + E_2(u) & -E_2(u) \\ 0 & -E_2(u) & E_2(u) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}
\] (3.43)

After applying the loads and boundary conditions and removing the subscripts of the load and displacement (only one load and one displacement remains), Eq. (3.43) becomes:

\[
\frac{A}{L} (E_i(u) + E_2(u))(u) = R
\] (3.44)

This is a nonlinear equation of the displacement \( u = u_2 \). Once this displacement is determined, the stress and strain of the two bars may be obtained.

3.3.3.1 Solution Using Newton-Raphson Scheme

The tangential stiffness matrix of the two-bar structure may be expresses as:
\[
\frac{\partial F}{\partial u} \text{ (evaluated at } u_i) = K_i = \frac{A}{L} \left( E_i(u_i) + E_2(u_i) \right) \quad (3.45)
\]

where

\[
E_1(u) = \begin{cases} E, & \varepsilon_1 < \varepsilon_y \\ \frac{E}{4}, & \varepsilon_1 > \varepsilon_y \end{cases} \quad \text{and} \quad E_2(u) = \begin{cases} E, & \varepsilon_2 < \varepsilon_y \\ \frac{E}{2}, & \varepsilon_2 > \varepsilon_y \end{cases} \quad (3.46)
\]

The scheme of the iteration is as follows:

\[
K_i \Delta u = R - F(u_i) = \Delta R \quad , \quad u_i = u_{i-1} + \Delta u \quad , \quad u_0 = 0 \quad (3.47)
\]

The iteration converges in three steps \((i = 3)\). These steps are shown in details below:

---

**Step 1**

Start with \( u_0 = 0 \), \( F(u_0) = A \left( \sigma_1(u_0) - \sigma_2(u_0) \right) = 0 \), \( \Delta R = R - F(u_0) = 3A\sigma_y - 0 = 3A\sigma_y \)

when \( u_0 = 0 \), \( \varepsilon_1 \) and \( \varepsilon_2 \) are both equal to zero, i.e., \( \varepsilon < \varepsilon_y \) which means that \( E_1(u_0) = E_2(u_0) = E \), into \( K_0 \Delta u = R - F(u_0) = \Delta R \) gives \( \frac{A}{L} (E + E) \Delta u = \Delta R = 3A\sigma_y \)

which yields \( \Delta u = \frac{3\sigma_y L}{2E} \). The next step starts

---

**Step 2**

\( u_1 = u_0 + \Delta u = \frac{3\sigma_y L}{2E} \), by rearranging \( \varepsilon_1 = -\varepsilon_2 = \frac{u_0}{L} = \frac{3\sigma_y}{2E} = \frac{3}{2} \varepsilon_y \), which means that both materials have yielded, and the tangential matrix must be used for the two bar elements after yielding, \( E_1(u_1) = \frac{E}{4}, E_2(u_1) = \frac{E}{2} \), therefore,

\[
F(u_i) = A \left( \sigma_y + \frac{E}{4} \varepsilon_y \right) - \left( -\frac{\sigma_y}{2} - \frac{E}{2} \varepsilon_y \right) = A \left( \sigma_y + \frac{1}{8} \sigma_y + \sigma_y + \frac{1}{4} \sigma_y \right) = 2.375A\sigma_y
\]

\( K_i \Delta u = R - F(u_i) = \Delta R \) or \( \frac{A}{L} \left( \frac{E}{4} + \frac{E}{2} \right) \Delta u = 3A\sigma_y - 2.375A\sigma_y = 0.625A\sigma_y \) which gives

\( \Delta u = 0.8333 \frac{\sigma_y L}{E} \), the third and final step starts

---

**Step 3**

\( u_2 = u_1 + \Delta u = 2.333 \frac{\sigma_y L}{E} \), by rearranging \( \varepsilon_1 = -\varepsilon_2 = \frac{u_1}{L} = 2.333 \frac{\sigma_y}{E} = 2.333 \varepsilon_y \) which means that both materials have yielded, and the tangential matrix must be used for the
two bar elements after yielding. \( E_{u1}(u_2) = \frac{E}{4} \), \( E_{u2}(u_2) = \frac{E}{2} \) into \( F(u_2) \) gives
\[
F(u_2) = A \left( \sigma_y + \frac{E}{4} \left( \frac{4}{3} \epsilon_y \right) - \left( -\sigma_y - \frac{E}{2} \left( \frac{4}{3} \epsilon_y \right) \right) \right) = A \left( \sigma_y + \frac{1}{3} \sigma_y + \frac{2}{3} \sigma_y \right) = 3A \sigma_y
\]
\[
\Delta R = R - F(u_i) = 3A \sigma_y - 3A \sigma_y = 0 \text{, which means that } K_2 \Delta u = R - F(u_i) = \Delta R = 0
\]
\[
\Delta u = 0 \text{ end of solution procedure.}
\]

The solution procedure using Newton-Raphson scheme was shown here in full details because it consists of only three steps and it forms the basis for all the successive discussions related to solving the nonlinear equilibrium equations. Note that the solution procedure terminates when \( \Delta u = 0 \), which originates from \( \Delta R = 0 \) when the material stiffness in not equal to zero. The term \( \Delta R \) is defined as the out-of-balance, or the difference between the external force \( R \) and the stress equivalent force \( F \). This term, \( \Delta R \), can be used as a convergence criterion to end the iterative procedure.

3.3.3.2 Solution Using the Modified Newton-Raphson Scheme

Using the Modified Newton-Raphson method, the tangential stiffness matrix is evaluated only once in the first iteration step as follows:
\[
K_0 = \frac{\partial F}{\partial u} \text{ (evaluated at } u_0) = \frac{A}{L} \left( E_1(u_0) + E_2(u_0) \right)
\]
(3.48)

The scheme of the iteration then proceeds as follows:
\[
K_0 \Delta u = R - F(u_i) = \Delta R \text{ and } u_i = u_{i-1} + \Delta u \text{ where } u_0 = 0
\]
(3.49)

Only the first three iterations will be shown here to describe the procedure. The results will be shown afterwards:

Step 1
Start with \( u_0 = 0 \), \( F(u_0) = A \left( \sigma_1(u_0) - \sigma_2(u_0) \right) = 0 \), \( \Delta R = R - F(u_0) = 3A \sigma_y - 0 = 3A \sigma_y \)

when \( u_0 = 0 \), \( \epsilon_1 \) and \( \epsilon_2 \) are both equal to zero, i.e., \( \epsilon < \epsilon_y \), which means that:
\[
E_1(u_0) = E_2(u_0) = E \text{, into } K_0 \Delta u = R - F(u_i) = \Delta R \text{ gives } \frac{A}{L} \left( E + E \right) \Delta u = \Delta R = 3A \sigma_y
\]
which yields \( \Delta u = \frac{3 \sigma_y L}{2E} \), where \( K_0 = \frac{2EA}{L} \). The next step starts

Step 2
\[
u_i = u_0 + \Delta u = \frac{3 \sigma_y L}{2E}, \text{ by rearranging } \epsilon_1 = -\epsilon_2 = \frac{u_i}{L} = \frac{3 \sigma_y}{2E} = \frac{3}{2} \epsilon_y \text{, which means that both}
\]
materials have yielded, and the tangential matrix must be used for the two bar elements after yield. \( E_{1u}(u_1) = \frac{E}{4} \), \( E_{2u}(u_1) = \frac{E}{2} \), therefore,

\[
F(u_1) = A\left(\sigma_y + \frac{E}{4}\left( \frac{1}{2}\varepsilon_y \right) - \left(-\sigma_y - \frac{E}{2}\left( \frac{1}{2}\varepsilon_y \right) \right) \right) = A\left(\sigma_y + \frac{1}{8}\sigma_y + \sigma_y + \frac{1}{4}\sigma_y \right) = 2.375A\sigma_y,
\]

the tangent stiffness matrix \( K_0 \) is not updated and remains constant \( K_0 = \frac{2EA}{L} \).

\[
K_0 \Delta u = R - F(u_1) = \Delta R \quad \text{or} \quad \frac{2EA}{L} \Delta u = 3A\sigma_y - 2.375A\sigma_y = 0.625A\sigma_y,
\]

which gives

\[
\Delta u = 0.3125\frac{\sigma_y L}{E},
\]

the third step starts.

\[
\begin{align*}
\text{Step 3} \\
u_2 = u_1 + \Delta u = \frac{3\sigma_y L}{2E} + 0.3125\frac{\sigma_y L}{E} = 1.8125\frac{\sigma_y L}{E}, \quad \text{by rearranging:}
\end{align*}
\]

\[
\varepsilon_1 = -\varepsilon_2 = \frac{u_1}{L} = 1.8125\frac{\sigma_y}{E} = 1.8125\varepsilon_y,
\]

which means that both materials have yielded, and the tangential matrix must be used for the two bar elements after yielding. \( E_{1u}(u_2) = \frac{E}{4} \), \( E_{2u}(u_2) = \frac{E}{2} \) into \( F(u_2) \) gives:

\[
F(u_2) = A\left(\sigma_y + \frac{E}{4}(0.8125\varepsilon_y) - (-\sigma_y - \frac{E}{2}(0.8125\varepsilon_y)) \right) = 2.6094A\sigma_y,
\]

again the tangent stiffness matrix \( K_0 \) is not updated and remains constant \( K_0 = \frac{2EA}{L} \).

\[
\Delta R = R - F(u_1) = 3A\sigma_y - 2.6094A\sigma_y = 0.3906A\sigma_y,
\]

which means that:

\[
K_0 \Delta u = R - F(u_1) = \Delta R \quad \text{or} \quad \frac{2EA}{L} \Delta u = 0.3906A\sigma_y,
\]

which gives \( \Delta u = 0.1953\frac{\sigma_y L}{E} \), the fourth step then starts.

By observing the value of \( \Delta R \) through the first three steps, we realized that as long as the solution is converging, the value for \( \Delta R \) is decreasing. This iterative procedure is repeated until the convergence criterion \( \Delta R \) reaches a practically small value (in this case, when \( \Delta R \leq 0.0001 \)). Twenty one iterations are performed to complete the analysis.

The results of the two iterative procedures used to solve the one-degree-of-freedom nonlinear problem are shown in Fig. 3.6, where the stress equivalent force is normalized in terms of \( A\sigma_y \), i.e., the ordinate is \( F / A\sigma_y \).

The results for the Modified Newton-Raphson method showing the variation of the element normalized stresses in bar (1) and bar (2) along with the iteration steps are demonstrated in Fig. 3.7. In the case of the Newton-Raphson method; only three points would be plotted for each stress.
If the same FE example of the one-degree-of-freedom nonlinear analysis is carried out again but with the material of bar (1) being a strain softening material, $E_{n1} = -E/4$, as shown in Fig. 3.8, while the material of bar (2) remains the same as in the first example.
(strain hardening material with $E_{t_2} = E/2$), the following results are obtained for the Newton-Raphson and the Modified Newton-Raphson methods. Again, only three steps were required for the Newton-Raphson method to converge. On the other hand, the iterative procedure converges after 70 iterations ($\Delta R \leq 0.0001$) for the Modified Newton-Raphson method (see Fig. 3.9).

Figure 3.9 Example 2: Stress equivalent force convergence iterations

Figure 3.10 shows the normalized stresses in bar (1) and bar (2) along with iterations for the Modified Newton-Raphson method. Note that as bar (1) fails to carry anymore load, the entire load will be carried out by bar (2) in compression.

As can be seen from the previous two examples, the numerical algorithms/procedures discussed above are applied to satisfy the weak form of the equilibrium equation. In each step, the stresses are found and used to calculate the residual $\Delta R$ which in turn is used to calculate a new $\Delta u$ until the equilibrium equation, $K \Delta u = 0$, is satisfied. If the equilibrium equation is not satisfied, a new step is applied where a new approximation to the nodal displacement is proposed by solving the linear system. The procedure is repeated until the equilibrium equation is satisfied.

Up to this point, only the global/equilibrium iterations - required to solve the nonlinear governing equation at a specific time step for the entire structure - were discussed. The stresses were calculated using very simple elastic $E$ and elastic-plastic (tangent) $E_t$ moduli. This is straightforward for a one-degree-of-freedom nonlinear problem, but when discussing more complicated problems, the elastic-plastic tangential operator depends on the incremental properties of the material as does the stresses. The displacement increment $\Delta u$ will be used to calculate the strain increment $\Delta \varepsilon$ using an incremental format of Eq. (3.11)a, and the latter, $\Delta \varepsilon$, will then be used to update the stress. This introduces the need for the incremental theory of plasticity and the integration of the constitutive equations to yield the required solution parameters at the local level.
3.4 Algorithms for Nonlinear Local Iterations

What follows is a discussion of the local or Gaussian iterations, i.e., how to integrate the plasticity constitutive model to update the quantities at the point level. The constitutive equations and computational algorithms for rate-independent elastic-plastic analysis of metals are presented. Everything here will be demonstrated for isotropic metal plasticity, which will be used to model the steel reinforcement in a RC beam. Different forms of metal strain-hardening plasticity will be presented.

3.4.1 The Incremental Constitutive Theory of Metal Plasticity

When the stress tensor, $\sigma_{ij}$, and the strain tensor, $\varepsilon_{ij}$, are related by a strain-dependent matrix rather than a matrix of constants, the material studied is said to be nonlinear, i.e., material nonlinearity is observed. Computational challenges arise from the fact that the equilibrium equations (discussed earlier) must be written using material properties that depend on strains, but strains are not known in advance. The loading history and geometry, support conditions, and material properties are assumed to be known (as discussed in the examples of section 3.3.3). The deformations and stresses in the body as a function of the applied load are being sought.

In the theory of plasticity - the mathematical theory of time-independent irreversible deformations - plastic flow is the cause of material nonlinearity, the physical basis of which involves the movement of dislocations without the influence of viscous phenomena or presence of decohesion which damages the material (Lemaitre and Chaboche, 1990). When the material behavior is nonlinear, material properties in a finite element are dictated by material properties at a finite number of sampling points in each element. Typically these points are quadrature stations of a numerical integration rule, i.e., element centroids or Gauss points of isoparametric elements. At each point one must keep a record of the material’s history and update the record in each computational cycle. This is accomplished during an FE analysis using ABAQUS through storing internal variables in the material subroutine to represent the material history parameters. The

Figure 3.10 Example 2: Element stress variation along with iterations
number of sampling points must be small to reduce computational expense. Accordingly, simple elements should be used. A contrary argument is that many sampling points are needed to accurately capture the spread of yielding in individual elements. In practice, the choice should be made between many simple elements and a small number of more sophisticated elements.

An incremental constitutive relation for an elastic-plastic material is presented next. Tensorial notation will be adopted in this section to comply with the tensorial nature of the UMAT files developed in this study. For the case of isotropic elastic-plastic analysis of metals, one should understand the general features of metal inelastic response, and then address the key concepts in modeling this plastic behavior, which are:

- The decomposition of strain into elastic and plastic parts.
- The yield criterion that predicts whether the metal responds elastically or plastically.
- The strain hardening rule that controls the shape of the stress-strain curve in the plastic regime.
- The plastic flow rule that determines the relationship between stress and plastic strain rate.
- The plastic (loading) and elastic (unloading) condition.

The total strain tensor $\epsilon_{ij}$ is well accepted to be composed of two parts: a small recoverable (reversible) elastic strain $\epsilon_{ij}^e$ and a large irreversible plastic strain $\epsilon_{ij}^p$ such that:

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$  \hspace{1cm} (3.50)

This decomposition which is a basic assumption in the classical theory of rate-independent plasticity is justified by the physics of the theory. An elastic deformation corresponds to a variation in the inter-atomic distances without changes of place while plastic deformation implies slip movements with modification of inter-atomic bonds (Lemaitre and Chaboche, 1990). The elastic (reversible) part is related to the stress through the linear elastic constitutive model, where the Cauchy stress tensor $\sigma_{ij}$ is given in terms of the elastic strain tensor $\epsilon_{ij}^e$ by:

$$\sigma_{ij} = C_{ijkl} \epsilon_{ij}^e$$  \hspace{1cm} (3.51)

where $C_{ijkl}$ is the fourth order Hooke’s elastic operator given as:

$$C_{ijkl} = 2G I_{ijkl}^{dev} + K \delta_{ij} \delta_{kl}$$  \hspace{1cm} (3.52)

The constants $G$ and $K$ are the shear and bulk moduli, respectively given by:
\[ G = \frac{2E}{(1 + \nu)} \quad \text{and} \quad K = \frac{E}{3(1-2\nu)} \]  

(3.53)a,b

where \( E \) and \( \nu \) are the Young’s modulus and Poisson’s ratio, respectively, and \( I_{ijkl}^{\text{dev}} \) is the deviatoric part of the fourth-rank identity tensor \( I_{ijkl} \) given as:

\[ I_{ijkl}^{\text{dev}} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) - \frac{1}{3} \delta_{ij} \delta_{kl} \]  

(3.54)

In order to develop a description of the relation between the plastic stresses and strains, one needs to establish a yield criterion that will predict the onset of inelastic deformation. This is done by adopting a yield function \( f(\sigma, R) \) in terms of the stress \( \sigma \) and the hardening equivalent stress \( R \) (linear strain-hardening is assumed).

The yield function defines a yield surface \( f(\sigma, R) = 0 \) (a space of admissible stresses). Satisfying the yield surface equation indicates that the material is yielding. Drifting away from the yield surface either indicates elastic deformation \( f(\sigma, R) < 0 \) or progression of plastic deformation. Progression of plastic deformations alters the strain hardening parameters (hardening state variable \( p \) and its corresponding stress equivalent \( R \)) and therefore modifies (expands) the yield surface enforcing \( f(\sigma, R) = 0 \) at the updated values of \((\sigma, R)\). This is the numerical interpretation of a physical phenomenon. Hardening is due to an increase in the dislocation density. Higher density of dislocations leads to more intermingle and interlock between dislocations, which causes dislocations to block each other. Once these dislocations interlock, and even if the material is unloaded, the updated value of the yield stress \( \sigma_y + R \) is the new yield stress of the material. Thus, the situation where \( f(\sigma, R) > 0 \) is not physically possible, yet, it will be encountered during the numerical iterative procedure used to solve for material nonlinearity. Special techniques are then required to bring the stresses to an updated yield surface, i.e. \( f(\sigma, R) = 0 \).

By studying the general nature of polycrystalline solids, these solids are assumed to be isotropic and their yield criteria can be assumed to be independent of the hydrostatic pressure. Therefore, the yield criterion here will only depend on the deviatoric components of stresses and furthermore, because of isotropy, the yield criterion will only depend on the magnitudes (not the directions) of the deviatoric stresses (\( J_2 \) elastoplasticity or \( J_2 \) flow theory). This leads to the von Mises yield criterion (maximum distortion-energy criterion) where the yield criterion is a function of the invariants of the deviatoric stresses, given here as:

\[ f(\sigma, R) = \sigma_{eq} - \sigma_y - R(p) \leq 0 \]  

(3.55)
where $\sigma_y$ is the yield stress, $R(p)$ is the isotropic hardening stress (linear function of the accumulated plastic strain $p$), and $\sigma_{eq}$ is the von Mises equivalent stress defined by:

$$\sigma_{eq} = \sqrt{\frac{3}{2} S_y S_y} \tag{3.56}$$

where $S_y$ is the deviatoric part of the Cauchy stress tensor, given as:

$$S_y = \sigma_y - \frac{1}{3} \sigma_{mm} \delta_{ij} \tag{3.57}$$

and $\frac{1}{3} \sigma_{mm}$ is the mean stress and $\delta_{ij}$ is the Kronecker delta.

Isotropic strain hardening can be modeled by relating the size of the yield surface to plastic strain in some appropriate way. One of the most commonly used forms of hardening stress $R(p)$ for steel reinforcement is (Ngo and Scordelis, 1967; Feenstra, 1993; Lowes, 1999; Tikhomirov and Stein, 2001; Phuvoravan and Sotelino, 2005; Rabczuk et. al., 2005; He et. al., 2006; and Junior and Venturini, 2007):

$$R(p) = kp \tag{3.58}$$

which is known as linear isotropic hardening law, $k$ is the isotropic hardening constant, and $\sigma_y + R(p)$ represents the radius of the yield surface (a cylinder) in the space of principal stresses. As the accumulated plastic strain $(p)$ increases, the radius of the yield surface increases.

In order to calculate the plastic strains induced by loading beyond yield, we note that the magnitude of the plastic strain can be determined from the hardening behavior of the material. Since the stress must be on the yield surface at all times (the consistency condition), and the radius of the yield surface is related to the magnitude of the accumulated plastic strain, the magnitude of the increment of the plastic strain must be related to the stress increment. This leads to the definition of a plastic flow rule. A plastic flow rule derived from the yield surface (associative plastic flow rule) governs the evolution of the plastic strain, and is given as follows:

$$\dot{\epsilon}_{ij}^p = \dot{\lambda} N_{ij} \tag{3.59}$$

where the scalar $\dot{\lambda}$ is called the plastic multiplier and $N_{ij} = \frac{\partial f}{\partial \sigma_{ij}}$ is the gradient of the yield function with respect to the stress tensor. The geometric interpretation of the flow rule is very simple. The gradient $N_{ij}$ is perpendicular to the yield surface, and therefore,
the flow rule is also referred to as the normality rule. It should be noted here that the plastic flow rule is incremental in nature.

Using the equation for the yield criterion given in Eq. (3.55), the gradient \( N_y \) can be evaluated for a von Mises material as:

\[
N_y = \frac{\partial f}{\partial \sigma_y} = \frac{3}{2} \frac{S_y}{\sigma_{eq}} \tag{3.60}
\]

and the flow rule can now be written as:

\[
\dot{\varepsilon}_y = \frac{3}{2} \lambda \frac{S_y}{\sigma_{eq}} \tag{3.61}
\]

This is known as the Levy-Mises flow rule.

If the scalar \( \rho \), which is the accumulated plastic strain, is defined as the integration of the rate of the accumulated plastic strain during an iterative procedure, i.e.:

\[
\rho(t) = \int_0^t \dot{\rho} d\tau \tag{3.62}
\]

where \( \tau \) is an integration operator, and the rate itself is defined as:

\[
\dot{\rho} = \sqrt{\frac{2}{3} \dot{\varepsilon}_y^p \dot{\varepsilon}_y^p} \tag{3.63}
\]

where \( \dot{\varepsilon}_y^p \) is the plastic strain rate, the plastic multiplier can then be easily proven to be equal to the rate of the accumulated plastic strain, using Eqs. (3.56), (3.61) and (3.63):

\[
\dot{\lambda} = \dot{\rho} \tag{3.64}
\]

It should be noted here that the plastic multiplier \( \dot{\lambda} \) is a positive scalar \( \dot{\lambda} \geq 0 \) determined through enforcing the consistency condition. Also known as the Kuhn-Tucker complementary or the loading/unloading condition, the consistency condition can be written as:

\[
\dot{\lambda} \geq 0 \ , \ f(\sigma, R) \leq 0 \ , \ \dot{\lambda} f(\sigma, R) = 0 \ , \text{ which gives } \dot{\lambda} \dot{f}(\sigma, R) = 0 \tag{3.65}
\]
leading to the following possible situations in the evaluation of plastic multiplier $\hat{\lambda}$:

\[
\begin{bmatrix}
 f(\sigma, R) < 0 & \Rightarrow \hat{\lambda} = 0 \text{ (elastic)} \\
 f(\sigma, R) = 0 & \Rightarrow \begin{bmatrix}
 \dot{f} < 0 & \Rightarrow \hat{\lambda} = 0 \text{ (elastic unloading)} \\
 \dot{f} = 0 & \Rightarrow \hat{\lambda} = 0 \text{ (neutral loading)} \\
 \dot{f} = 0 & \Rightarrow \hat{\lambda} > 0 \text{ (plastic loading)}
\end{bmatrix}
\end{bmatrix}
\] (3.66)

If the updated yield function $f^{n+1}$ (a scalar) at the end of a step is written in terms of its initial value at the beginning of that step $f^n$ plus the rate of change $df$ (which can be considered equal to $\Delta f$ in a numerical procedure), one can write:

\[ f^{n+1} = f^n + df = 0 \] (3.67)

but $f^n = 0$, which means that $df = 0$ is the only possible outcome. Taking the rate of the yield function, Eq. (3.55), results in the following:

\[ df = \frac{\partial f}{\partial \sigma} \dot{\sigma}_y + \frac{\partial f}{\partial p} \dot{p} = \frac{\partial f}{\partial \sigma} \dot{\sigma}_y + \frac{\partial f}{\partial R} \frac{\partial R}{\partial \sigma} \dot{p} = 0 \] (3.68)

and making use of Eqs. (3.50) and (3.51) along with substitutions as $\frac{\partial f}{\partial R} = -1$ and $\frac{\partial R}{\partial p} = k$, we obtain the following expression for the plastic multiplier ($\dot{\lambda} = \dot{p}$):

\[ \dot{\lambda} = \frac{N_0 C_{ijkl} \dot{\epsilon}_{kl}}{N_0 C_{ijkl} N_{kl} + k} \] (3.69)

Beyond the elastic limit, plastic flow rules are incremental in nature and stresses are related to strains by means of an incremental constitutive relation obtained by considering the rate of Hook’s law (Eq. (3.51)):

\[ \dot{\sigma}_y = C_{ijkl} \dot{\epsilon}_{kl}^e \] (3.70)

as well as the rate of the strain additive decomposition equation, Eq. (3.50):

\[ \dot{\epsilon}_y = \dot{\epsilon}_y^e + \dot{\epsilon}_y^p \] (3.71)

Applying Eq. (3.71) into Eq. (3.70), one obtains:

\[ \dot{\sigma}_y = C_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) \] (3.72)
and substituting for the increment of the plastic strain, $\dot{\varepsilon}_p$, using the expression obtained from the normality rule, Eq. (3.59), along with the expression for the plastic multiplier, Eq. (3.69), the following equation can be obtained:

$$\dot{\sigma}_{ij} = C_{ijkl} \left( \dot{\varepsilon}_{kl} - \frac{N_{mn} C_{pqrs} \dot{\varepsilon}_{pq}}{N_{ab} C_{abcd} N_{cd} + k} N_{kl} \right)$$  \hspace{1cm} (3.73)

In order to find an operator relating $\dot{\sigma}_{ij}$ and $\dot{\varepsilon}_{kl}$, a number of tensorial manipulations and rearrangements (shown below) leads to the following incremental constitutive relation:

$$\dot{\sigma}_{ij} = C_{ijmn} \left( \dot{\varepsilon}_{mn} - \frac{N_{pq} C_{pqrsl} \dot{\varepsilon}_{rs}}{N_{ab} C_{abcd} N_{cd} + k} N_{mn} \right)$$

$$\dot{\sigma}_{ij} = C_{ijmn} \left( \delta_{km} \delta_{ls} \dot{\varepsilon}_{kl} - \frac{N_{pq} C_{pqrsl} \delta_{ls} \delta_{km} \dot{\varepsilon}_{kl}}{N_{ab} C_{abcd} N_{cd} + k} N_{mn} \right)$$

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\varepsilon}_{kl}$$

where

$$C_{ijkl}^{ep} = C_{ijkl} - \frac{C_{ijmn} N_{mn} N_{pq} C_{pqlt}}{N_{ab} C_{abcd} N_{cd} + k}$$  \hspace{1cm} (3.74)

$C_{ijkl}^{ep}$ is the fourth-order tensor known as the elastoplastic tangent operator. It has the same symmetries as $C_{ijkl}$, but unlike $C_{ijkl}$, $C_{ijkl}^{ep}$ is not constant; it depends on the deviatoric stress and the hardening parameter (Doghri, 2000).

If the expression in Eq. (3.75) is simplified to compare with the previously discussed examples of the one-degree-of-freedom FE elastoplastic problem, $C_{ijkl}^{ep}$ simplifies to $E_i$ (section 3.3.3) which is given by:

$$E_i = E \left( 1 - \frac{E}{E + k} \right) = \frac{E k}{E + k}$$  \hspace{1cm} (3.76)

For the first example in section 3.3.3:

$E_{i_1} = E / 4$, but $E_{i_1} = \frac{E k_1}{E + k_1}$ which gives $k_1 = E / 3$ (hardening occurs)

$E_{i_2} = E / 2$ but $E_{i_2} = \frac{E k_2}{E + k_2}$ which gives $k_2 = E$ (hardening occurs)

For the second example in section 3.3.3:
where \( k \) for a given material is a constant property known as the strain-hardening (softening) parameter or the plastic modulus.

The elastic-plastic incremental constitutive equation, Eq. (3.74), relates an infinitesimal stress increment with an infinitesimal strain increment at a given stress state and plastic deformation history. However, the load increment resulting from the FE equilibrium iterative procedure is of a finite magnitude rather than an infinitesimal one. The resulting increments of stress and strain have finite sizes too. Therefore, the incremental constitutive equations need be integrated numerically to compute the stress increment. The numerical algorithm required to integrate the constitutive equations, along with the equilibrium iterative procedure discussed earlier, are the core of the elastic-plastic FE analysis.

It should be noted here that in ABAQUS nonlinear FE analysis associated with the UMAT subroutine, ABAQUS carries out the global/equilibrium iterations while UMAT integrates the material incremental constitutive model (local iterations). Yet, satisfying the equilibrium equations (i.e., convergence) depends on the tangent operator passed back to ABAQUS from the UMAT subroutine. This was illustrated previously in the one-degree-of-freedom examples (3.2.2.2). Recalling that the value of tangential stiffness matrix \( k_i \) was calculated based on the tangential modulus \( E_t \), for both bar elements, obtained in each step. In a multi-degree-of-freedom problem, \( E_t \) will become the elastic-plastic tangent operator \( C^p \) calculated at each integration point and passed back to ABAQUS where it will be used to calculate the stiffness matrix \( K \) shown in Eq. (3.18).

### 3.4.2 Integrating the Incremental Constitutive Equations

The purpose here is to devise accurate and efficient algorithms for the integration of the constitutive relations governing the material behavior. In the context of the FE analysis, the integration of constitutive equations is carried out at the Gauss points for a given deformation increment. The unknowns to be found are the updated stresses and plastic variables.

Efficient algorithms should satisfy three basic requirements: consistency with the constitutive equations to be integrated, numerical stability, and incremental plastic consistency. The soundness and efficiency of the algorithm ensure the effectiveness of the numerical procedure. An improper algorithm may lead not only to an inaccurate stress solution, but may also delay the convergence of the equilibrium iterations or even lead to divergence of the iteration.
In the scope of time-independent plasticity, it is common to use fictitious time increments (pseudo-time) merely as a way of counting successive increments of loads, stresses and strains. Therefore, we denote all quantities at time \( t_n \) with a subscript \( (n) \) while all quantities at time \( t_{n+1} \) are denoted by a subscript \( (n+1) \). Increments within the time step \([t_n, t_{n+1}] (t_{n+1} - t_n = \Delta t)\) are left with no subscripts. Tensorial notation will be dropped whenever an algorithm is introduced to reduce the complication of indices and subscripts. Each term will be written to correspond to its original format shown in the previous sections. The reader can refer to the original format of each equation in the algorithms to figure out what each symbol represents (scalar, vector, or tensor). The procedure starts at time \( t_n \) with knowledge of the stresses \( \sigma_n \) and the total strain increment \( \Delta \varepsilon \) passed to the UMAT file from the previous equilibrium iteration carried out by ABAQUS. The goal now is to find the updated stresses \( \sigma_{n+1} \), plastic strains \( \varepsilon_{p,n+1} \), and the equivalent plastic strain \( p_{n+1} \).

Two schemes for the integration of the constitutive equations will be presented next: The Explicit (Forward-Euler) scheme and a Radial Return form of the Implicit (Backward-Euler) scheme. Other integration methods exist and are numerous with different levels of complexity. But due to the advantage of the presence of high level processors, complicating the integration procedure will not result in any better outputs than those obtained using these two schemes with a reasonable number of time increments.

In both integration schemes, the first step is to use an elastic procedure to update the stresses. If these updated stresses are found to lie within the yield surface, the material at the Gauss point is assumed to have remained elastic. In this case, there is no need to integrate the rate equation; elastic analysis can resume. However, if the elastic stresses are outside the yield surface, an integration scheme is adopted to bring the stresses back to the yield surface.

3.4.2.1 Explicit (Forward-Euler) Integration Algorithm

This integration scheme is famous for its simplicity and being straightforward to implement. By calculating the gradient of the yield function, Eq. (3.60), and thus the plastic multiplier, Eq. (3.69) at the beginning of the increment, the Explicit integration scheme uses the parameters evaluated at the previous time step \( t_n \) to calculate the updated parameters at the end of the current time step \( t_{n+1} \). The following is a block representation of the Explicit (Forward-Euler) integration scheme:

<table>
<thead>
<tr>
<th>Given ( \sigma_n, \varepsilon_{p,n}, p_n ) and ( \Delta \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{If} ( f(\sigma_n, R_n) &lt; 0 ) \textbf{then exit the integration scheme-End If}</td>
</tr>
<tr>
<td>\textbf{If} ( f(\sigma_n, R_n) &gt; 0 ) \textbf{then}</td>
</tr>
</tbody>
</table>

---

60
Calculate the plastic multiplier
\[ \Delta \lambda = \frac{N_n C \Delta \varepsilon}{N_n C N_n + k} \]
Calculate the increment of plastic strain
\[ \Delta \varepsilon^p = \Delta \lambda N_n \]
Calculate the increment of elastic strain using \( \Delta \varepsilon \)
\[ \Delta \varepsilon^e = \Delta \varepsilon - \Delta \varepsilon^p \]
Calculate the increment of stress
\[ \Delta \sigma = C \Delta \varepsilon^e \]
Update all the quantities
Stresses
\[ \sigma_{n+1} = \sigma_n + \Delta \sigma \]
Strains
\[ \varepsilon_{n+1}^p = \varepsilon_n^p + \Delta \varepsilon^p \]
Equivalent plastic strain
\[ p_{n+1} = p_n + \Delta \lambda \]
Calculate the elastic-plastic tangent operator
\[ C_{ep} = C - \frac{CN_n N_n C}{N_n C N_n + k} \]
End If

Associated with its simplicity, however, are three of its disadvantages that need to be discussed: a) Because it is an Explicit procedure, it is conditionally stable. b) The accuracy of the integration depends on the increment size chosen. c) The plastic multiplier was obtained in such a way that the yield condition is satisfied at time \( t_n \). Satisfying the yield condition at time \( t_{n+1} \) was not checked. Therefore, the solution at time \( t_{n+1} \) may drift away from the yield surface over many time steps. This drift can be reduced by decreasing the size of the increment. Furthermore, if the total strain increment \( \Delta \varepsilon_1 \) (see Fig. 3.11) at time \( t_n \) causes the stress to cross the yield surface, then \( f(\sigma_n, R_n) < 0 \), and the increment will be treated elastically, which will over predict the location of the yield stress from \( \sigma_y \) to \( \sigma_{n+1} \). Then the new total strain increment \( \Delta \varepsilon_2 \) and all the consequent increments will result in a shifted stress strain diagram as shown in Fig. 3.11. This leads to accumulation of errors and overestimation of the computed ultimate (collapse) load.

![Figure 3.11 Explicit integration without correction](image-url)
In order to minimize the drift from the yield surface, the total strain increment $\Delta \varepsilon_i$ causing the stress to cross the yield surface for the first time needs to be resolved into two parts (Fig. 3.11). One part of $\Delta \varepsilon_i$ is elastic ($\Delta \varepsilon_i^e$) taking the stress to the first yield $\sigma_y$ while the other part is plastic ($\Delta \varepsilon_i^p$). Only the plastic part needs to be passed into the integration algorithm shown above. The elastic part needs to be considered such that: 

$$\sigma_n + C \Delta \varepsilon_i^e = \sigma_y \quad \text{and} \quad \varepsilon_{n+1}^e = \varepsilon_n^e + \Delta \varepsilon_i^e.$$ 

The remainder plastic part, $\Delta \varepsilon_i^p = \Delta \varepsilon_i - \Delta \varepsilon_i^e$, becomes the new total strain increment passed to the integration algorithm with the known value of the stress being equal to the yield stress, i.e., $\Delta \varepsilon = \Delta \varepsilon_i^p$ and $\sigma_n = \sigma_y$.

Different methods are suggested in literature for finding a scalar scaling factor $\beta$ such that the elastic and plastic part of the strain increment $\Delta \varepsilon_i$ can be found: $\Delta \varepsilon_i^e = \beta \Delta \varepsilon_i$ and $\Delta \varepsilon_i^p = (1-\beta)\Delta \varepsilon_i$. In terms of stresses, the yield function should be satisfied at $\sigma_n + \beta \Delta \sigma = \sigma_n + C \Delta \varepsilon_i^e = \sigma_y$, i.e., $f(\sigma_y) = 0$. Note that $f(\sigma_n) < 0$ as shown in Fig. 3.11. When $\beta$ reaches its maximum value ($\beta = 1$), the elastically predicted stress would be $\sigma_n + \Delta \sigma = \sigma_n + C \Delta \varepsilon_i$, which is equal to $\sigma_{n+1}$. At a given value of $\beta$, say $\beta_i$, the stress would be equal to $\sigma_i = \sigma_n + \beta_i \Delta \sigma$ and the yield function becomes $f(\sigma_i) = f(\sigma_n + \beta_i \Delta \sigma)$. By using a truncated Taylor series, the value of the yield function $f(\sigma_i)$ can be given as: 

$$f(\sigma_i) = f(\sigma_n) + \frac{\partial f}{\partial \sigma_i} \delta \sigma = f(\sigma_n) + \frac{\partial f}{\partial \sigma_i} (\Delta \sigma) \delta \beta = 0.$$ 

Such a scheme might start with an initial estimate $\beta_0 = \frac{-f(\sigma_n)}{f(\sigma_{n+1}) - f(\sigma_n)}$ to calculate $\sigma_o = \sigma_n + \beta_o \Delta \sigma$, evaluate $\frac{\partial f}{\partial \sigma_o}$ and then calculate $\delta \beta_o$ using the truncated Taylor series, 

$$f(\sigma_o) + \frac{\partial f}{\partial \sigma_o} (\Delta \sigma) \delta \beta_o = 0.$$ 

The next iteration then starts with $\beta_1 = \beta_o + \delta \beta_o$ to obtain $\sigma_1 = \sigma_o + \beta_1 \Delta \sigma$. By calculating $\frac{\partial f}{\partial \sigma_1}$, and using the truncated series again, 

$$f(\sigma_1) + \frac{\partial f}{\partial \sigma_1} (\Delta \sigma) \delta \beta_1 = 0,$$ 

the next estimate of $\delta \beta_1$ can be found. The iterations continue until the value of $\delta \beta_1$ practically approaches zero.

Once $\beta$ has been evaluated, the new strain increment $\Delta \varepsilon = \Delta \varepsilon_i^p$ (see Fig. 3.11) will be passed into the Forward-Euler Explicit integration algorithm shown above to complete the analysis of this yield point defining increment. A block representation of the correction to the Explicit (Forward-Euler) integration algorithm is shown below:

Given $\sigma_n$, $\varepsilon_n^p = 0$, $p_n = 0$ and $\Delta \varepsilon$

If $f(\sigma_n) < 0$ then
3.4.2.2 Implicit Integration Algorithm: Radial Return Method

This method is popular because, for the von Mises yield criterion, it takes a particularly simple form. At a certain step in time, \( t_n \), applying the strain increment \( \Delta \varepsilon \) takes the elastically updated stress \( \sigma_n + C \Delta \varepsilon = \sigma_{n+1}^{\text{trial}} \) outside of the yield surface (see Fig. 3.12). This trial stress is known as the elastic predictor. The stress is then updated with a plastic corrector \( C \Delta \varepsilon^p \) to bring it back onto the yield surface at the end of the current time step \( [t_n, t_{n+1}] \) as follows: 

\[
\sigma_{n+1} = \sigma_n + C \Delta \varepsilon = \sigma_n + C \left( \Delta \varepsilon - \Delta \varepsilon^p \right) = \sigma_n + C \Delta \varepsilon - C \Delta \varepsilon^p = \sigma_{n+1}^{\text{trial}} - C \Delta \varepsilon^p .
\]

This plastic correction involves calculating the plastic multiplier \( \Delta \lambda \) using the trial stress \( \sigma_{n+1}^{\text{trial}} \), followed by the calculation of the increment of the plastic strain \( \Delta \varepsilon^p \).

\[
\sigma_{n+1} + C \Delta \varepsilon - C \Delta \varepsilon^p = \sigma_n + C(\Delta \varepsilon - \Delta \varepsilon^p) = \sigma_n + C \Delta \varepsilon^p = \sigma_{n+1} + C \Delta \varepsilon^p = \sigma_{n+1}^{\text{trial}}
\]

\[\]
In contrast to the Explicit scheme, all quantities are written here at the end of the time increment, ensuring that the yield condition is satisfied at the end of the time increment, and therefore, avoiding the drift from the yield surface which was observed in the Explicit scheme. This scheme also leads to a faster (larger time increments) solution.

In the deviatoric stress space, the plane stress von Mises ellipse becomes a circle, and the plastic correction term is always directed towards the center of the yield surface (because of the normality of the flow rule). This fact gives the technique its name, i.e., the radial return method, which can be derived as follows:

Starting with the following expression: (the elastic predictor $\sigma_{\text{trial}}^{n+1}$ will be referred to as $\sigma_{\text{trial}}^n$):

$$\sigma_{ij}^{n+1} = \sigma_{ij}^n - C_{ijkl} \Delta \varepsilon_{kl}^p$$  \hspace{1cm} (3.77)

Expanding the plastic strain increment, using Eq. (3.59), one obtains:

$$\sigma_{ij}^{n+1} = \sigma_{ij}^n - \Delta \lambda C_{ijkl} N_{kl}^{n+1}$$  \hspace{1cm} (3.78)

Only the deviatoric part $S_{ij}$ of the stress tensor $\sigma$ will affect the plastic analysis merely due to the fact that the hydrostatic pressure remains constant. By obtaining the deviatoric parts of each term in Eq. (3.78), the following expression is obtained:

$$S_{ij}^{n+1} = S_{ij}^{\text{trial}} - \Delta \lambda C_{ijkl} N_{kl}^{n+1}$$  \hspace{1cm} (3.79)

However, using Eqs. (3.52) and (3.60), the following can be derived in details for the last term in Eq. (3.79):

$$C_{ijkl} N_{kl}^{n+1} = 2G I_{ijkl}^{\text{dev}} K \delta_{ijkl} = \left(2G I_{ijkl}^{\text{dev}} + K \delta_{ijkl} \right) N_{kl}^{n+1} = \left(2G I_{ijkl}^{\text{dev}} + K \delta_{ijkl} \right) \left(\frac{3 S_{kl}^{n+1}}{2 \sigma_{eq}^{n+1}}\right)$$

$$C_{ijkl} N_{kl}^{n+1} = 2GI_{ijkl}^{\text{dev}} \frac{3 S_{kl}^{n+1}}{2 \sigma_{eq}^{n+1}} + K \delta_{ijkl} \frac{3 S_{kl}^{n+1}}{2 \sigma_{eq}^{n+1}}$$  \hspace{1cm} (3.80)

$$C_{ijkl} N_{kl}^{n+1} = 2G \left(\frac{3 S_{ij}^{n+1}}{2 \sigma_{eq}^{n+1}}\right) = 2GN_{ij}^{n+1}$$

Note that $S_{ij}^{n+1}$ is deviatoric, therefore, $I_{ijkl}^{\text{dev}} S_{kl}^{n+1} = S_{ij}^{n+1}$ and $S_{kk}^{n+1} = 0$ (incompressible plasticity). Substituting Eq. (3.80) back into Eq. (3.79), one obtains the following:

$$S_{ij}^{n+1} = S_{ij}^{\text{trial}} - 2\Delta \lambda GN_{ij}^{n+1} = S_{ij}^{\text{trial}} - 3\Delta \lambda G \frac{S_{ij}^{n+1}}{\sigma_{eq}^{n+1}}$$  \hspace{1cm} (3.81)
Multiplying $S_{ij}^{n+1}$ (Eq. (3.81)) by itself gives:

\[
S_{ij}^{n+1} S_{ij}^{n+1} = \left( S_{ij}^{\text{trial}} - 3\Delta \lambda G \frac{S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} \right) \left( S_{ij}^{\text{trial}} - 3\Delta \lambda G \frac{S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} \right)
\]

\[
S_{ij}^{n+1} S_{ij}^{n+1} = S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} - 6\Delta \lambda G \frac{S_{ij}^{n+1} S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} + 9(\Delta \lambda G)^2 \left( \frac{S_{ij}^{n+1} S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} \right) (3.82)
\]

\[
S_{ij}^{n+1} S_{ij}^{n+1} = S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} - 6\Delta \lambda G \frac{S_{ij}^{n+1} S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} + 6(\Delta \lambda G)^2)
\]

but using Eq. (3.79), one obtains:

\[
S_{ij}^{n+1} S_{ij}^{n+1} = S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} + 3\Delta \lambda G \frac{S_{ij}^{n+1} S_{ij}^{n+1}}{\sigma_{eq}^{n+1}} (3.83)
\]

Substituting Eq. (3.83) back into Eq. (3.82), one obtains:

\[
S_{ij}^{n+1} S_{ij}^{n+1} = S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} - 6\Delta \lambda G \frac{S_{ij}^{n+1} S_{ij}^{n+1} + 2\Delta \lambda G \sigma_{eq}^{n+1}}{\sigma_{eq}^{n+1}} + 6(\Delta \lambda G)^2
\]

\[
S_{ij}^{n+1} S_{ij}^{n+1} = S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} - 4\Delta \lambda G \sigma_{eq}^{n+1} - 12(\Delta \lambda G)^2 + 6(\Delta \lambda G)^2 (3.84)
\]

Rearranging the previous equation gives the following:

\[
S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} = S_{ij}^{n+1} S_{ij}^{n+1} + 4\Delta \lambda G \sigma_{eq}^{n+1} + 6(\Delta \lambda G)^2 (3.85)
\]

Multiplying all the terms in Eq. (3.85) by $\frac{3}{2}$ and taking the square root of both sides gives:

\[
\sqrt{\frac{3}{2}} S_{ij}^{\text{trial}} S_{ij}^{\text{trial}} = \sqrt{\frac{3}{2}} S_{ij}^{n+1} S_{ij}^{n+1} + 6\Delta \lambda G \sigma_{eq}^{n+1} + 9(\Delta \lambda G)^2
\]

\[
\sigma_{eq}^{\text{trial}} = \sqrt{\left( \frac{\sigma_{eq}^{n+1}}{\sigma_{eq}^{n+1}} \right)^2 + 2(3\Delta \lambda G)(\sigma_{eq}^{n+1})^2 + (3\Delta \lambda G)^2} (3.86)
\]

\[
\sigma_{eq}^{\text{trial}} = \sigma_{eq}^{n+1} + 3\Delta \lambda G
\]

and by multiplying all terms of Eq. (3.81) by $\sigma_{eq}^{n+1}$, and rearranging on obtains:
Note that the term between brackets in Eq. (3.87) is equal to $\sigma_{eq}^{trial}$. This is shown in Eq. (3.86). Therefore, Eq. (3.87) can now be written as:

$$\sigma_{eq}^{trial} S_{ij}^{n+1} = \sigma_{eq}^{trial} S_{ij}^{trial}$$

(3.88)

and recalling that $\sigma_{eq}^{trial}$, $\sigma_{eq}^{n+1}$ are scalars, multiplying both sides by $\frac{3}{2}$ and rearranging terms, the above relation reduces to the following:

$$\frac{3}{2} S_{ij}^{n+1} = \frac{3}{2} S_{ij}^{trial} \quad \text{or} \quad N_{ij}^{n+1} = N_{ij}^{trial}$$

(3.89)

which is the proof of the radial return using the von Mises yield criterion.

When the yield criterion, Eq. (3.55), is satisfied, i.e., $\sigma_{eq}^{n+1} = \sigma_y + R(p^{n+1})$, Eq. (3.86) can be rewritten as:

$$\sigma_{eq}^{trial} + 3\Delta\lambda G = \sigma_y + R(p^{n+1}) + 3\Delta\lambda G \quad \sigma_y + R(p^{n+1}) + 3\Delta\lambda G - \sigma_{eq}^{trial} = 0$$

(3.90)

This equation can be used to solve for the plastic multiplier $\Delta\lambda$. For linear hardening, the hardening equivalent stress $R(p^{n+1})$, which is a function of the equivalent plastic strain $p^{n+1}$, can be linearly decomposed into the following:

$$R(p^{n+1}) = kp^{n+1} = k(p^n + \Delta p) = kp^n + k\Delta p$$

(3.91)

which facilitates rewriting Eq. (3.90) in the following format:

$$\sigma_y + kp^n + k\Delta p + 3\Delta\lambda G - \sigma_{eq}^{trial} = 0$$

(3.92)

Rearranging the previous equation, and substituting $\Delta\lambda$ for $\Delta p$ (Eqs. (3.62) to (3.64)), the following expression for the plastic multiplier $\Delta\lambda$ is derived:

$$\Delta\lambda = \frac{\sigma_{eq}^{trial} - \sigma_y + kp^n}{k + 3G} = \frac{f^{trial}}{k + 3G}$$

(3.93)

A block representation of the Radial Return integration scheme is shown below:
Given $\sigma_n$, $\varepsilon_p^n$, $p_n$ and $\Delta \varepsilon$

Calculate a trial elastic stress

$\sigma_{n+1}^{trial} = \sigma_n + C\Delta \varepsilon$

If $f(\sigma_{n+1}^{trial}, R_n) < 0$ then exit integration scheme-End If

If $f(\sigma_{n+1}^{trial}, R_n) > 0$ then

Calculate the plastic multiplier

$\Delta \lambda = \frac{f^{trial}}{3G + k}$

Calculate the increment of plastic strain

$\Delta \varepsilon^p = \Delta \lambda N^{trial}$

Calculate the increment of elastic strain using $\Delta \varepsilon$

$\Delta \varepsilon^e = \Delta \varepsilon - \Delta \varepsilon^p$

Calculate the increment of stress

$\Delta \sigma = C\Delta \varepsilon^e$

Update all the quantities

Stresses

$\sigma_{n+1} = \sigma_n + \Delta \sigma$

Strains

$\varepsilon_{p,n+1} = \varepsilon_p^n + \Delta \varepsilon^p$

Equivalent plastic strain

$p_{n+1} = p_n + \Delta \lambda$

Calculate the elastic-plastic tangent operator

$C^{ep} = C - \frac{CN^{n+1}N^{n+1}\varepsilon}{N^{n+1}CN^{n+1} + k}$

End If

In FE nonlinear material analysis, and when the return mapping algorithm has converged but the weak form of equilibrium, Eq. (3.1), is not satisfied, a new global iteration is needed in order to propose new approximation to the nodal displacements used to calculate the strain field at time $t_{n+1}$ during the same time increment $[t_n, t_{n+1}]$. When the Newton-Raphson method, Eq. (3.29), is used to iterate on the global level, the so called consistent tangent operator $\frac{\partial \sigma}{\partial \varepsilon}$ has been reported to preserve a quadratic rate of convergence, i.e., significantly faster convergence rate than that of the classical tangent operator $C^{ep}$. This results in the substitution of $C^{ep}$ calculated above by a type of material consistent tangent operator $\frac{\partial \sigma}{\partial \varepsilon}$ or one of its approximations, in order to achieve higher convergence rates. Simo and Taylor (1985) demonstrated the importance of the consistent linearization in preserving a quadratic rate of convergence when using the Newton-Raphson’s method.

3.4.2.3 Other Forms of Strain Hardening: Nonlinear Strain Hardening and Linear Strain Hardening Following Perfectly-Plastic Behavior

For nonlinear hardening, the breakdown of $R(p^{n+1})$, Eq. (3.91), is no longer valid, and the above algorithms (Forward-Euler or Radial Return) need to be modified. Equations (3.68) for Explicit and (3.90) for Radial Return, are now nonlinear functions of the plastic multiplier $\Delta \lambda$. These equations are solved using a local Newton-Raphson iteration scheme. The iteration scheme is termed local to distinguish it from the global/equilibrium Newton-Raphson iterations.

67
The algorithms can be easily altered to account for a nonlinear isotropic hardening rule, such as a power hardening law, \( R(p) = kp^n \), where \( n \) is a material constant that can be calibrated to match experimental results. The expressions given by the algorithms for \( \Delta \lambda \), i.e. Eq. (3.69) for Explicit and Eq. (3.93) for Radial Return, are replaced by the following expressions given as:

\[
\Delta \lambda = \frac{2G\nu \Delta \varepsilon_y}{3G + \frac{\partial R}{\partial p}} \quad \text{and} \quad \Delta \lambda = \frac{f^{\text{trial}}}{3G + \frac{\partial R}{\partial p}}
\]

(3.94)a,b

where \( \frac{\partial R}{\partial p} = n k p^{n-1} \) is no longer a constant. This means that the above equations will have to be solved iteratively for \( \Delta \lambda \) using a nonlinear equation solver scheme. The scheme selected here is the Newton-Raphson iterative procedure. Once the plastic multipliers \( \Delta \lambda \) (for both algorithms) are obtained through nonlinear iterations, the rest of the algorithms proceed as shown above for the case of linear hardening except for the elastoplastic tangent operators \( C^{ep} \), where the linear hardening parameter \( k \) in the expressions for \( C^{ep} \) needs to be replaced with \( \frac{\partial R}{\partial p} \).

In order to introduce an algorithm to represent the elastic, perfectly plastic followed by strain hardening behavior of steel, the perfectly plastic behavior is discussed first followed by the introduction of a mechanism to define the strains at the onset of hardening. By considering the simple case of one dimensional analysis for this demonstration, the yield criterion, Eq. (3.55), can be written as \( f = \sigma - \sigma_y \leq 0 \), and the consistency condition (Eq. (3.68) without the hardening term) gives \( \dot{\varepsilon}^p = \dot{\varepsilon} \), which means that once plasticity starts, the strain increment is entirely plastic, which eliminates the need to update the stress \( \dot{\sigma} = E\dot{\varepsilon} \). This explains the meaning of perfect plasticity. In a general 2 or 3 dimensional analysis, however, this argument has to be accounted for numerically.

Beyond the yield point, perfect plasticity dominates the analysis until the onset of strain hardening. At that point and beyond, the strain increment is divided into elastic and plastic parts, where the elastic part is used to update the stress. The issue here is to determine the point at which the strain hardening starts. In a one dimensional analysis, the onset of strain hardening is the experimentally obtained strain value. However, in a two dimensional analysis, a more general term has to be developed to determine the onset of hardening. In this work, the maximum strain criterion is used to define a value for the maximum principal strain in steel, where the latter is used as an indicator of whether the onset of hardening has been reached or not through comparing it to the experimental value of strain at the onset of hardening. This is accomplished through adding IF statements to the UMAT FORTRAN file after the yield condition to determine when the stress is updated. The additional IF statements acquire the value of the maximum
principal strain and allow hardening and stress update to take place if the maximum strain criterion is satisfied.

The following is a block representation of the Explicit (Forward-Euler) integration Algorithm used to model the three phase behavior (elastic, perfectly plastic followed by strain hardening). The same logic can be applied to the Implicit (Radial Return) integration scheme:

```
Given $\sigma_n$, $\varepsilon_n^p$, $p_n$, and $\Delta\varepsilon$
If $f(\sigma_n, R_n) < 0$ then exit the integration scheme-End If
If $f(\sigma_n, R_n) > 0$ then
Update the strain tensor
$\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon$
Calculate the maximum principal strain
$\varepsilon_{\text{max}}$
If $\varepsilon_{\text{max}}^p \geq$ uniaxial strain hardening then allow hardening
Else use perfectly plastic analysis
End If
Calculate the plastic multiplier
$\Delta\lambda = \frac{N_n C \Delta\varepsilon}{N_n C N_n + k}$
Calculate the increment of plastic strain
$\Delta\varepsilon^p = \Delta\lambda N_n$
Calculate the increment of elastic strain using $\Delta\varepsilon$
$\Delta\varepsilon^e = \Delta\varepsilon - \Delta\varepsilon^p$
Calculate the increment of stress
$\Delta\sigma = C \Delta\varepsilon^e$
Update all the quantities
Stresses
$\sigma_{n+1} = \sigma_n + \Delta\sigma$
Strains
$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\varepsilon^p$
Equivalent plastic strain
$p_{n+1} = p_n + \Delta\lambda$
Calculate the elastic-plastic tangent operator
$C^{op} = C - \frac{CN_n N_n C}{N_n C N_n + k}$
End If
```

### 3.5 Implementation and Verification of the Integration Schemes

The two integration schemes discussed above were implemented into the FE commercial code ABAQUS where their applicability, with different post yield behaviors, was verified. The Implicit code (ABAQUS Standard) will be used. In such an Implicit code, provision of both the integration scheme of the plasticity constitutive equations (whether Implicit or Explicit) along with the material tangent operator/matrix is necessary. These will be written using FORTRAN into a material behavior subroutine called UMAT. This UMAT is then linked to ABAQUS during the execution of the command that runs the analysis using the data stored in the input file. While ABAQUS performs the standard FE procedure along with the nonlinear global/equilibrium iterations, the UMAT file will govern the behavior of the materials during different loading stages; elastic and plastic.
A wide range of information and parameters are passed in and out the subroutine UMAT during FE analysis at the beginning and end of each time increment. Data passed into UMAT at the beginning of the time increment would include, but is not limited to, the values of stresses, strains, state variables as well as the strain increments. Data passed back to ABAQUS would include the updated stresses, state variables and material tangent operator (the Jacobian). The Jacobian is required for the global iterative procedure used to minimize the force residual. If convergence occurs after a given number of iterations, the type of the Jacobian does not influence the accuracy of the solution as much as it influences the rate of convergence (Dunne and Petrinic, 2005).

The verification of the UMAT file is carried out under the plane stress condition shown in Fig. 3.13. The plate is subjected to uniaxial displacement control at one of the vertical ends while pinned at the other. Four ABAQUS CPS4R (4 nodded Continuum Plane Stress Reduced Integration) elements are used. The analytical solution to this problem provides guidelines that help in assessing the applicability of the UMAT subroutines. Using the parameters given in Fig. 3.13, the strain at yield will be equal to \( \varepsilon_y = 0.00207 \).

At the beginning of each time increment, ABAQUS passes the stress, strain and strain increment into UMAT. These are all passed in vector form, which can then be transferred into tensors using proper functions/subroutines in UMAT file. The material model is then used to update the stresses and return them back to ABAQUS in vector format. The tangent operator is returned in the form of a matrix.

The results obtained by running the problem shown above with different UMAT files are shown below for the case of linear hardening first, followed by other hardening laws. The number of increments used is indicated on each figure.

Decreasing the size of the total strain increment \( \Delta \varepsilon \) (Fig. 3.11) - causing the stress to cross the yield surface for the first time - will improve the results of the Explicit integration scheme (Fig. 3.14), eliminating the need for a correction procedure. The correction applied to the Explicit integration will have a pronounced effect at a lesser
number of time increments (Fig. 3.15) rather than at high number of increments where the correction would be rendered unnecessary.

![Explicit Integration Scheme](image)

Figure 3.14 Explicit integration scheme using 100 and 1000 increments

![Corrected Explicit .vs. Explicit Integration Schemes](image)

Figure 3.15 Explicit .vs. Corrected Explicit at 100 increments

As was discussed earlier, the Implicit integration scheme requires lesser amount of increments to give good results than the Explicit integration scheme. The differences between the 100 and 1000 increments curves shown below, Fig. 3.16, for the Implicit integration scheme are the smoothness of each line, and the smoothness of the region of transition from elastic to plastic behavior.

![Corrected Explicit Integration Scheme](image)

Fig. 3.17 shows the difference between Explicit and Implicit integrations at different time increments. It shows the difference between the Implicit and Explicit integrations at 100 increments. It also shows that Implicit and Explicit integrations will eventually give the same result at increased numbers of time increments.
If the results of the algorithms of linear strain hardening are compared to a mathematical representation of the experimental results of Grade 60 steel ($\sigma_y = 414$ MPa, $\sigma_u = 720$ MPa), Fig. 3.18, it can be seen that the linear hardening parameter $k$ was chosen such that the ultimate stress of the steel is reached.

If the linear hardening rule is replaced by a nonlinear hardening rule, such as a power hardening law, $R(p) = kp^n$, the following figures (Fig. 3.19 and Fig. 3.20) can be obtained. One should note the difference in the values of the yield stress between the Explicit and Implicit integrations at 100 increments. Reducing the size of these increments results in better agreement between the two integration schemes. The Explicit integration results can be further enhanced by incorporating the correction procedure discussed above.
Figure 3.18 Numerical vs. theoretical stress-strain curve for Reinforcing steel

Figure 3.19 Explicit integration of nonlinear hardening plasticity

It is worthy to note that as the power \( n \) increases from 0.5 to 0.9, the solution approaches that of a linear hardening law. Substituting unity for the value of \( n \) is not admissible. In addition, increasing the number of increments to 1000 will give identical results for both integration schemes, as shown in Fig. 3.21.

A UMAT file was also used to produce an elastic perfectly plastic behavior where the plastic slip in the reinforcing steel equals the applied strain rate which renders the increment of the stress to be equal to zero, resulting with a plateau beyond yielding, see Fig. 3.22. As discussed earlier, this algorithm neglects the strain hardening effect of the steel and therefore, might not be suitable for large deformation analyses such as seismic analysis, etc.
Figure 3.20 Implicit integration of nonlinear hardening plasticity

Figure 3.21 Comparison of integration schemes at 1000 increments

Figure 3.22 Corrected Explicit elastic-perfectly plastic analysis
Another modification was applied to the UMAT file in order to obtain a refined description of the stress-strain behavior of reinforcing steel (Fig. 3.23). Here the analysis beyond the yield point presumes under the perfectly plastic condition, until the point where the strain reaches the strain hardening value of the reinforcing steel, where hardening behavior starts. The analysis changes from a perfectly plastic to a plastic hardening analysis leading to the stress-strain curve shown in Fig. 3.23. It should be mentioned here that this algorithm is sensitive to increment size and sometimes it leads to divergence of the solution. Further work is needed in order to enhance the performance of such an algorithm and to substitute the linear hardening stage with nonlinear hardening to better describe the steel behavior.
CHAPTER 4

CONCRETE MATERIAL MODEL

4.1 Introduction

The analysis and design of a concrete structure requires prior knowledge of its mechanical properties. When continuum mechanics is considered, elastic damage models or elastic plastic constitutive laws are generally the standard approaches to describe the behavior of concrete. In the first case, the mechanical effect of the progressive microcracking and strain softening are represented by a set of internal state variables which act on the elastic behavior (decrease of the stiffness) at the macroscopic level (e.g. Mazars, 1984; Simu and Ju, 1987a,b; Mazars and Piaudier-Cabot, 1989; Labadi and Hannachi, 2005; Tao and Phillips, 2005; Junior and Venturini, 2007; Khan et. al., 2007). In plasticity models, softening is directly included in the expression of a plastic yield surface by means of a hardening–softening function (e.g. Feenstra and de Borst, 1996; Bicanic and Pearce, 1996; Grassl et. al., 2002; Park and Kim, 2005).

In concrete material analysis, it is very important to capture the variations (degradation) of the elastic stiffness of the material upon mechanical loading, which cannot be captured by plasticity-based approaches (Feenstra and de Borst, 1996). Continuum damage mechanics is the appropriate theoretical framework for that in order to capture material degradation. However, continuum damage models cannot capture alone the irreversible (plastic) deformations that the material undergoes during loading. Therefore, the combined use of elastic-plastic constitutive equations along with continuum damage mechanics is vital to better describe the mechanical behavior of concrete.

There are several possibilities for coupling plasticity and damage effects in a single constitutive relation. Historically, damage has first been coupled to plasticity (Lemaître and Chaboche, 1984) in the so-called ductile failure approaches for metal alloys. The underlying assumption was that void nucleation is triggered by plastic strains. Applications to concrete were proposed among others, (e.g. Oller et. al., 1990; Voyiadjis and Abu-Lebdeh, 1993, 1994; Abu-Lebdeh and Voyiadjis, 1993; Kratzig and Polling, 2004). In these models, damage growth is a function of the plastic strains. There is a difficulty, however. In uniaxial tension there is little plasticity and quite a lot of damage while in uniaxial compression, the picture is reversed with little damage and important plastic strains. Furthermore, it can be hardly explained how plastic strain may develop in concrete prior to microcracking. A common assumption is that irreversible strains are due to microcrack sliding and internal friction. Such a process requires the prior formation of internal surfaces (microcracks).

The second approach, that is more suited to both tension and compression responses, uses the effective stress. The plastic yield function is written in the effective configuration pertaining to the stresses in the undamaged material. Many authors (Simo
and Ju, 1987a,b; Ju, 1989; Mazars and Pijaudier-Cabot, 1989; Yazdani and Schreyer, 1990; Hansen and Schreyer, 1992; Lee and Fenves, 1998; Faria et. al., 1998; Fichant et. al., 1999; Voyiadjis and Kattan, 1999, 2006; Jefferson, 2003; Salari et. al., 2004; Shen et. al., 2004; Jason et. al., 2006, Voyiadjis et. al., 2008b) applied this approach to isotropic and anisotropic damage coupled to elasto-plasticity. It has been extended to other sources of damage, for instance to thermal damage by Nechnech et. al. (2002) and Willam et. al. (2002).

A last possibility is what could be called the strong coupled approach. As opposed to the above where the plastic yield function is written in terms of the effective stress, the applied stress appears in the plastic process, which becomes coupled to damage. Luccioni et. al. (1996), Armero and Oller (2000) and Voyiadjis et. al. (2008a) provided the thermodynamic consistent backgrounds of such a model.

In this chapter, an elastic plastic damage formulation is proposed to model the nonlinear behavior of concrete materials. The model is intended to circumvent the disadvantages of pure plastic and pure damage approaches applied separately. It is based on an isotropic damage model, with tensile and compressive damage criteria, combined with a plasticity yield criterion with multiple hardening rules. The isotropic damage is responsible for the softening response and the decrease in the elastic stiffness, while hardening plasticity accounts for the development of irreversible strains and volumetric compressive behavior within the effective configuration.

The effective stress approach has been chosen because it provides a simple way to separate the damage and plastic processes. Plastic effects, driven by the effective stresses, can be described independently from damage ones and vice versa. One of the main interests is to ease the numerical implementation which is Implicit/Explicit. The plastic part is Implicit and the damage part is Explicit, same as in classical continuum damage computations. As a consequence, existing robust algorithms for integrating the constitutive relations can be implemented. The calibration of the material parameters is also easier to handle as a consequence of the separation of damage and plasticity processes.

In this contribution, the damage process is (elastic) strain controlled. The isotropic damage model proposed by Tao and Phillips (2005) will be adopted here to describe the damage behavior of concrete. While the Tao and Phillips (2005) model incorporated strain-softening in an elastic-damage framework, it is used in this work simultaneously with the effective stress space plasticity in order to describe the damage and irreversible phenomena in concrete materials under tension and compression. The plastic process shall be described using a yield function inspired from Lubliner et. al. (1989) and later modified by Lee and Fenves (1998) and Wu et. al. (2006). It is an isotropic hardening process in the present model. Softening is controlled by damage, while plasticity controls hardening, in tension and compression. Fracture energy related coefficients have been defined and incorporated in order to achieve a reasonable degree of discretization insensitivity in numerical calculations (Feenstra and de Borst, 1996; Lee and Fenves, 1998; Wu et. al., 2006).
In the following sections, the general framework of the elastic plastic damage model is discussed first, followed by a consistent thermodynamic formulation. The Helmholtz free energy function is then introduced with specific forms to be used in the FE code. The effective stress space plasticity is then introduced followed by the discussion of the damage yield criteria and evolution laws for concrete under tension and compression. Elements of validation and comparisons between the model and existing damage and damage plasticity approaches are also presented. The behavior of the model is then tested using four elementary loading cases: simple tension, simple compression, biaxial tension, and biaxial compression. The model is also used to reproduce the load capacity of a notched beam subjected to three point bending test.

4.2 Framework for the Elastic-Plastic-Damage Model

The model presented in this work is thermodynamically consistent and comes from a generalization of the effective space plasticity theory and isotropic damage theory applied simultaneously under the assumptions of small strains and isothermal conditions.

The transformation from the effective (undamaged) configuration to the damaged one can be done by utilizing the strain equivalence hypothesis, which basically states that the strains in the undamaged (effective) configuration are equal to the strains in the damaged configuration. This hypothesis is commonly applied to the coupling of plasticity and continuum damage mechanics (Lemaitre and Chaboche, 1990; Steinmann et. al., 1994; Lammer and Tsakmakis, 2000; Menzel et. al., 2005; Voyiadjis and Kattan, 2006). Using the additive decomposition of the total strain tensor \( \varepsilon_{ij} \) into elastic (reversible) \( \varepsilon_{ij}^e \) and plastic (irreversible) \( \varepsilon_{ij}^p \) strain tensors, along with the strain equivalence hypothesis, the following arrangements can be assumed:

\[
\begin{align*}
\varepsilon_{ij} &= \varepsilon_{ij}^e + \varepsilon_{ij}^p \\
\overline{\varepsilon}_{ij} &= \overline{\varepsilon}_{ij}^e + \overline{\varepsilon}_{ij}^p
\end{align*}
\]

The equivalence of the elastic strains will be used to obtain an expression for the elasticity tensor \( E_{ijkl}(\Phi) \) in the damaged configuration as well as the damage thermodynamic conjugate forces \( Y^\pm \) used in the damage yield criteria \( g^\pm \). The equivalence of the plastic strains, on the other hand, will be justified through the use of the effective stress space plasticity (Ju, 1989) and the definition of the plastic Helmholtz free energy function. Both equivalences will be discussed later in this chapter.

By taking the time derivative of the arrangements in Eq. (4.1), the following strain rate equations necessary for the plastic-damage incremental procedure are obtained:

\[
\begin{align*}
\dot{\varepsilon}_{ij} &= \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \\
\dot{\overline{\varepsilon}}_{ij} &= \dot{\overline{\varepsilon}}_{ij}^e + \dot{\overline{\varepsilon}}_{ij}^p
\end{align*}
\]
The effective stress tensor (stresses in the undamaged configuration) can now be
written in terms of the strain equivalence hypothesis and using Hook’s law as:

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl} \dot{\varepsilon}_{ijkl} = \bar{E}_{ijkl} (\varepsilon_{ijkl} - \varepsilon_{ijkl}^p)$$  \hspace{1cm} (4.3)$$

where \( \bar{E}_{ijkl} \) is the fourth-order undamaged isotropic elasticity tensor, also known as the
undamaged elastic operator, given as:

$$\bar{E}_{ijkl} = 2\bar{G} I_{ijkl} + \bar{K} \delta_{ijkl}$$  \hspace{1cm} (4.4)$$

where \( I_{ijkl} = I_{ijkl} - \frac{1}{3} \delta_{ij} \delta_{kl} \) is the deviatoric part of the fourth-order identity tensor
\( I_{ijkl} = \frac{1}{2} \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \), and \( \bar{G} \) and \( \bar{K} \) are the effective shear and bulk moduli,
respectively. The tensor \( \delta_{ij} \) is the Kronecker delta, and is equal to one, \( \delta_{ij} = 1 \) when \( i = j \)
or zero, \( \delta_{ij} = 0 \) when \( i \neq j \).

Taking the time derivative of Eq. (4.3), the rate of the constitutive equation in the
effective configuration can be obtained as:

$$\dot{\bar{\sigma}}_{ij} = \bar{E}_{ijkl} \ddot{\varepsilon}_{ijkl} = \bar{E}_{ijkl} (\dot{\varepsilon}_{ijkl} - \dot{\varepsilon}_{ijkl}^p)$$  \hspace{1cm} (4.5)$$

The damage configuration counterpart of Eq. (4.3), i.e. the stress tensor for the
damaged material, can be written as follows:

$$\sigma_{ij} = E_{ijkl} (\Phi) \varepsilon_{ijkl}^p = E_{ijkl} (\Phi)(\varepsilon_{ijkl} - \varepsilon_{ijkl}^p)$$  \hspace{1cm} (4.6)$$

where \( E_{ijkl} \) is the fourth-order elasticity tensor in the damaged configuration.

The stress-strain behavior is affected by the development of micro and macro cracks in
the material body. Concrete contains a large number of micro cracks, especially at
interfaces between coarse aggregates and mortar, even before the application of external
loads. These initial microcracks are caused by segregation, shrinkage, or thermal
expansion in the cement paste. Under applied loading, further micro-cracking may occur
at the aggregate-cement paste interface, which is the weakest link in the composite
system. These microcracks which are initially small (invisible), will eventually lead to
visible cracks that extend as the applied external loads are increased. These cracks
contribute to the generally obtained nonlinear stress-strain behavior. Since a
phenomenological continuum approach is followed in this work, these effects are
smeared out (i.e. averaged) throughout the body where the material is considered as a
mechanical continuum with degraded (damaged) properties.

In this work, the stress-strain relation involves a scalar (isotropic) damage variable \( \Phi \)
which is a weighted average of the tensile and compressive damage scalar variables, \( \varphi^+ \)
and \( \varphi^- \). It was shown in Chapter 2 that the Cauchy stress tensor \( \sigma_{ij} \) is related to the effective stress tensor \( \bar{\sigma}_{ij} \) by:

\[
\sigma_{ij} = (1 - \Phi)\bar{\sigma}_{ij}
\]

(4.7)

where \( \Phi \) is a dimensionless scalar (i.e. isotropic) damage variable interpreted here as averaged the crack density. It is clear from Eq. (4.7) that when the material is in the virgin state (undamaged), \( \Phi = 0 \), the effective stress \( \bar{\sigma}_{ij} \) is equivalent to the Cauchy stress, \( \sigma_{ij} \). In the case of the damaged material, the effective stress is more representative than the Cauchy stress because it acts on the effective area that is resisting the external loads. Furthermore, the scalar damage variable \( \Phi \) is still used in order to represent the macroscopic effect of the material microdamage mechanism discussed above.

By substituting Eqs. (4.3) and (4.6) into Eq. (4.7) one obtains the following relations:

\[
E_{ijkl} = (1 - \Phi)\bar{E}_{ijkl}
\]

(4.8)

\[
\sigma_{ij} = (1 - \Phi)\bar{E}_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p)
\]

(4.9)

The expression for the fourth order elasticity tensor in the damaged configuration \( E_{ijkl} \) in terms of its effective counterpart \( \bar{E}_{ijkl} \) will also be derived from the elastic dissipation potential later on. The damage variable \( \Phi \) has values from zero to one. The value \( \Phi = 0 \) corresponds to the undamaged (effective) material and the value \( \Phi = 1 \) corresponds to the fully damaged material. Damage associated with the failure mechanisms of the concrete (cracking and crushing) results in a reduction in the elastic stiffness (Eq. (4.8)). Within the context of the scalar-damage theory, the stiffness degradation is isotropic (i.e. the same damage evolution is assumed in different directions) and represented by a single degradation value \( \Phi \). The time derivative can now be applied to Eq. (4.9) to obtain the following:

\[
\dot{\sigma}_{ij} = (1 - \Phi)\bar{E}_{ijkl}\dot{\varepsilon}_{kl}^e - \Phi\bar{E}_{ijkl}\varepsilon_{kl}^e
\]

(4.10)

which represents the constitutive relation for the elastic-plastic-damage model used in this work.

4.3 Consistent Thermodynamic Formulation

In this section, a general thermodynamic framework of the elastic-plastic-damage formulation for concrete is developed. Isothermal conditions and rate independence are assumed throughout this work. Irreversible thermodynamic following the internal variable procedure of Coleman and Gurtin (1967) will be applied. The internal variables and potentials used to describe the thermodynamic processes are introduced. The Lagrange minimization approach (calculus of functions of several variables) is used later
on to derive the evolution equations for the proposed model. The constitutive equations are derived from the second law of thermodynamics, the expression of Helmholtz free energy, the additive decomposition of the total strain rate into elastic and plastic components, the Clausius-Duhem inequality, and the maximum dissipation principle.

The Helmholtz free energy can be expressed in terms of a suitable set of internal state variables that characterizes the elastic, plastic, and damage behaviors of concrete. In this work, the following internal state variables are assumed to satisfactorily characterize the behavior of concrete both in tension and compression: the elastic strain tensor $\varepsilon_{ij}$, a set of plastic hardening variables $(\kappa^+, \kappa^-)$ defined as the equivalent plastic strains in tension and compression, respectively, and the scalar damage variables $(\varphi^+, \varphi^-)$ representing the damage densities in the material under tension or compression, respectively, such that:

$$\psi = \psi\left(\varepsilon_{ij}^e, \kappa^+, \kappa^-, \varphi^+, \varphi^-\right)$$  \hspace{1cm} (4.11)

The constitutive model proposed here is based on the hypothesis of uncoupled elasticity (e.g. Lubliner, 1990; Luccioni et al., 1996; Faria et al., 1998; Nechnech et al., 2002; Salari et al., 2004; Kratzig and Polling, 2004; Luccioni and Rougier, 2005; Shao et al., 2006; Wu et al., 2006). According to this hypothesis, the total free energy density per unit volume $\psi$ can be assumed to be formed by two independent parts: an elastic part $\psi^e$ and a plastic part $\psi^p$, corresponding to the elastic and plastic processes respectively (both dissipative). Therefore, the Helmholtz free energy is given as:

$$\psi = \psi^e(\varepsilon_{ij}^e, \varphi^+, \varphi^-) + \psi^p(\kappa^+, \kappa^-)$$  \hspace{1cm} (4.12)

It can be noted from the above decomposition that damage affects only the elastic properties and not the plastic ones. This can be justified by the following: once micro-cracks are initiated during loading of a concrete material, local stresses are redistributed to undamaged material micro-bonds over the effective area ($\tilde{A}$ shown in Fig. 2.13, Chapter 2). Thus, effective stresses of undamaged material points are higher than nominal stresses. Accordingly, it appears reasonable to state that the plastic flow occurs only in the undamaged material micro-bounds by means of effective quantities (Ju, 1989). The plastic response is therefore characterized in the effective stress space and the yield function is no longer written in term of the applied stress, rather, it is a function of the effective stress, i.e., the stress in the undamaged material in between the microcracks. This approach, which is more suited for brittle materials like concrete, has been extensively used by researchers (Simo and Ju, 1987a,b; Ju, 1989; Mazars and Pijaudier-Cabot, 1989; Yazdani and Schreyer, 1990; Hansen and Schreyer, 1992; Lee and Fenves, 1998; Faria et al., 1998; Fichant et al., 1999; Salari et al., 2004; Jefferson, 2003; Jason et al., 2006; and others).

In the following, the thermodynamic conjugate forces associated with the internal state variables in Eq. (4.12) are derived based on the second law of thermodynamics. For
isothermal behavior, the second-law of thermodynamics states that the rate of change in the internal energy is less than or equal to the external expenditure of power such that:

$$\int \rho \dot{\psi} \, dv \leq P_{\text{ext}}$$  \hspace{1cm} (4.13)

where $P_{\text{ext}}$ is the external power which according to the principle of virtual power should be equal to the internal power such that:

$$P_{\text{ext}} = P_{\text{int}} = \int \sigma_{ij} \dot{\varepsilon}_{ij} \, dv$$  \hspace{1cm} (4.14)

Substituting Eq. (4.14) into Eq. (4.13), one obtains the following:

$$\int \rho \dot{\psi} \, dv - \int \sigma_{ij} \dot{\varepsilon}_{ij} \, dv \leq 0 \quad \Leftrightarrow \quad \int (\rho \dot{\psi} - \sigma_{ij} \dot{\varepsilon}_{ij}) \, dv \leq 0$$  \hspace{1cm} (4.15)

In a stepwise sense, the Clausius-Duhem inequality can be inferred from Eq. (4.15) as follows:

$$\sigma_{ij} \dot{\varepsilon}_{ij} - \rho \dot{\psi} \geq 0$$  \hspace{1cm} (4.16)

Taking the time derivative of Eq. (4.12), the following expression can be written:

$$\dot{\psi} = \psi^e + \psi^p = \frac{\partial \psi^e}{\partial E^e_{ij}} \dot{E}^e_{ij} + \left( \sigma_{ij} - \rho \frac{\partial \psi^e}{\partial E^e_{ij}} \right) \dot{\varepsilon}_{ij} - \rho \frac{\partial \psi^e}{\partial \varphi^e} \dot{\varphi}^e - \rho \frac{\partial \psi^e}{\partial \varphi} \dot{\varphi} - \rho \frac{\partial \psi^p}{\partial \kappa^+} \dot{k}^+ - \rho \frac{\partial \psi^p}{\partial \kappa^-} \dot{k}^-$$  \hspace{1cm} (4.17)

By substituting the rate of the Helmholtz free energy density, Eq. (4.17), into the Clausius-Duhem inequality, Eq. (4.16), one can write the following relation:

$$\sigma_{ij} \dot{\varepsilon}_{ij}^p + \left( \sigma_{ij} - \rho \frac{\partial \psi^e}{\partial E^e_{ij}} \right) \dot{E}^e_{ij} - \rho \frac{\partial \psi^e}{\partial \varphi^e} \dot{\varphi}^e - \rho \frac{\partial \psi^e}{\partial \varphi} \dot{\varphi} - \rho \frac{\partial \psi^p}{\partial \kappa^+} \dot{k}^+ - \rho \frac{\partial \psi^p}{\partial \kappa^-} \dot{k}^- \geq 0$$  \hspace{1cm} (4.18)

The above equation is valid for any admissible internal state variable such that the Cauchy stress tensor can be define as:

$$\sigma_{ij} = \rho \frac{\partial \psi^e}{\partial E^e_{ij}}$$  \hspace{1cm} (4.19)

as well as the non-negativeness of intrinsic dissipation:

$$\sigma_{ij} \dot{\varepsilon}_{ij}^p + Y^+ \dot{\varphi}^+ + Y^- \dot{\varphi}^- - c^+ \dot{k}^+ - c^- \dot{k}^- \geq 0$$  \hspace{1cm} (4.20)
where the damage and plasticity conjugate forces that appear in the above expression are defined as follows:

\[
Y^+ = -\rho \frac{\partial \psi^e}{\partial \phi^+}
\]

\[
Y^- = -\rho \frac{\partial \psi^e}{\partial \phi^-}
\]

\[
c^+ = \rho \frac{\partial \psi^p}{\partial \kappa^+}
\]

\[
c^- = \rho \frac{\partial \psi^p}{\partial \kappa^-}
\]

The mechanical dissipation must satisfy the first (Clausius-Duhem) inequality of thermodynamics and can be decomposed in two parts: one part due to the plastic process \(\Pi^p\) and the other due to the damage process \(\Pi^d\). The mechanical dissipation energy function \(\Pi\) can therefore be written as follows:

\[
\Pi = \Pi^p + \Pi^d \geq 0
\]

The plasticity and damage dissipation potentials are given, respectively, as follows:

\[
\Pi^p = \sigma \dot{\epsilon}^p c^+ \kappa^+ - c^- \dot{\kappa}^- \geq 0
\]

\[
\Pi^d = Y^\phi^+ + Y^-\dot{\phi}^- \geq 0
\]

The rate of the internal variables associated with plastic and damage deformations are obtained by utilizing the calculus of functions of several variables with the plasticity and damage Lagrange multipliers \(\dot{\lambda}^p\) and \(\dot{\lambda}^d\), respectively. Thus the following general objective function can be defined:

\[
\Omega = \Pi - \dot{\lambda}^p F - \dot{\lambda}^d g^+ - \dot{\lambda}^- g^- \geq 0
\]

where \(F\) and \(g^\pm\) are the plastic potential function and the tensile and compressive damage potential functions, respectively, to be defined later.

Use is now made of the well known maximum dissipation principle (Simo and Honein, 1990; Simo and Hughes, 1998), which describes the actual state of the thermodynamic forces \((\sigma, Y^\pm, c^\pm)\) as the state that maximizes the dissipation function over all other possible admissible states. Hence, one can maximize the objective function \(\Omega\) by using the following necessary conditions:
Substituting Eq. (4.28) into Eqs. (4.29) along with Eqs. (4.26) and (4.27) yields the following corresponding thermodynamic laws:

\[
\dot{\varepsilon}_{ij}^p = \dot{\lambda}_d^p \frac{\partial F^p}{\partial \sigma_{ij}}
\]

(4.30)

\[
\dot{\phi}^+ = \dot{\lambda}_d^+ \frac{\partial g^+}{\partial Y^+}
\]

(4.31)

\[
\dot{\phi}^- = \dot{\lambda}_d^- \frac{\partial g^-}{\partial Y^-}
\]

(4.32)

\[
\dot{\kappa}^+ = \dot{\lambda}_d^+ \frac{\partial F}{\partial \dot{c}^+}
\]

(4.33)

\[
\dot{\kappa}^- = \dot{\lambda}_d^- \frac{\partial F}{\partial \dot{c}^-}
\]

(4.34)

Note that Eq. (4.30) is defined in terms of a plastic potential \( F^p \) different from \( F \) to indicate the use of a non-associative flow rule. It is also worthy to note that in this work, the damage criteria are characterized with scalar quantities – scalar thermodynamic conjugate forces and scalar damage parameter - therefore, the above general thermodynamic evolution laws for damage, Eqs. (4.31) and (4.32), will be greatly simplified in the implementation procedure.

### 4.4 The Helmholtz Free Energy Function

Based on the additive decomposition of the Helmholtz free energy function into elastic-damage and plastic parts discussed earlier, Eq. (4.12), this section introduces specific forms for the elastic-damage and plastic parts of the Helmholtz free energy function adopted in this work. The elastic-damage part of the Helmholtz free energy function will be defined first, followed by a definition for the plastic part.

#### 4.4.1 The Elastic/Damage Part of the Helmholtz Free Energy Function

In order to define the elastic-damage part of the Helmholtz free energy, the spectral decomposition procedure for obtaining the tensile and compressive parts of the Cauchy stress tensor, as well as the combined scalar damage variable, \( \Phi \), are to be defined first.

To account for the different effects of damage mechanisms on the nonlinear performance of concrete under tension and compression, spectral decomposition of the effective stress tensor \( \sigma_{ij} \) into positive and negative components \( (\sigma_{ij}^+, \sigma_{ij}^-) \) is performed.
(e.g. Ortiz, 1985; Ju, 1989; Lubliner et. al., 1989; Faria et. al., 1998; Lee and Fenves, 1998; Wu et. al., 2006) such that:

$$\bar{\sigma}_{ij} = \sigma_{ij}^+ + \sigma_{ij}^-$$  \hspace{1cm} (4.35)$$

The total effective stress tensor $\sigma_{ij}$ can be written in terms of its principal values $\hat{\sigma}^{(k)}$ and their corresponding principal directions $n_i^{(k)}$ ($k = 1, 2, 3$) as follows:

$$\sigma_{ij} = \sum_{k=1}^{3} \hat{\sigma}^{(k)} n_i^{(k)} n_j^{(k)} = \hat{\sigma}^{(1)} n_i^{(1)} n_j^{(1)} + \hat{\sigma}^{(2)} n_i^{(2)} n_j^{(2)} + \hat{\sigma}^{(3)} n_i^{(3)} n_j^{(3)}$$  \hspace{1cm} (4.36)$$

The positive part $\sigma_{ij}^+$ can be obtained by considering only the tensile principal values as follows:

$$\sigma_{ij}^+ = \sum_{k=1}^{3} H(\hat{\sigma}^{(k)}) \hat{\sigma}^{(k)} n_i^{(k)} n_j^{(k)}$$  \hspace{1cm} (4.37)$$

where $H$ is the Heaviside step function ($H = 1$ for $\max \hat{\sigma} > 0$ and $H = 0$ for $\max \hat{\sigma} < 0$).

The principal stresses $\hat{\sigma}^{(k)}$ in Eqs. (4.36) and (4.37) are defined in the following form:

$$\hat{\sigma}^{(k)} = n_p^{(k)} \bar{\sigma}_{pq} n_q^{(k)}$$  \hspace{1cm} (4.38)$$

By substituting Eq. (4.38) into Eq. (4.37), the tensile stress can be written as:

$$\sigma_{ij}^+ = \sum_{k=1}^{3} H(\hat{\sigma}^{(k)}) n_i^{(k)} \bar{\sigma}_{pq} n_q^{(k)} n_j^{(k)}$$  \hspace{1cm} (4.39)$$

The above equation can be rewritten as follows:

$$\sigma_{ij}^+ = P_{ijpq}^+ \bar{\sigma}_{pq}$$  \hspace{1cm} (4.40)$$

where

$$P_{ijpq}^+ = \sum_{k=1}^{3} H(\hat{\sigma}^{(k)}) n_i^{(k)} n_j^{(k)} n_p^{(k)} n_q^{(k)}$$  \hspace{1cm} (4.41)$$

and substituting Eq. (4.40) into Eq. (4.35), the following expressions are obtained:
\[
\bar{\sigma}_y = P_{ijpq}^+ \bar{\sigma}_{pq}^+ + P_{ijpq}^- \bar{\sigma}_{pq}^-
\]

\[
\bar{\sigma}_y = \bar{\sigma}_{ij} - P_{ijpq}^+ \bar{\sigma}_{pq}^+ = \left[ I_{ijpq} - P_{ijpq}^+ \right] \bar{\sigma}_{pq} = P_{ijpq}^- \bar{\sigma}_{pq}
\]

\[
I_{ijpq} = P_{ijpq}^+ + P_{ijpq}^-
\]

(4.42)

where \(P_{ijpq}^+\) and \(P_{ijpq}^-\) are the tensile and compressive fourth-order projection tensors, respectively. A numerical procedure is provided in Appendix A to illustrate the spectral decomposition concept using the software Maple.

Next, the combined scalar damage variable is defined here similar to that defined by (Tao and Phillips, 2005) as follows:

\[
\Phi = \frac{\left[ \sigma_{ij}^+ \right] \varphi^+ + \left[ \sigma_{ij}^- \right] \varphi^-}{\left[ \sigma_{ij} \right]}
\]

(4.43)

where \(\varphi^+\) and \(\varphi^-\) are the tensile and compressive damage crack densities, respectively, \(\sigma_{ij}^+\) and \(\sigma_{ij}^-\) are the positive and negative spectral decomposition parts of the Cauchy stress tensor, \(\sigma_{ij}\), and \(\left[ X_{ij} \right]\) represents the scalar contraction of the second order tensor, i.e., \(\left[ X_{ij} \right] = X_{ij} X_{ij}\). This definition implies that damage under uniaxial loading is governed by the corresponding damage parameter, while under bi-axial loading two damage parameters, \(\varphi^+\) and \(\varphi^-\), both contribute to the induced damage. The effective contribution is in proportion to the ratio of positive and negative part contractions to the total stress contraction. This definition is slightly different than that given by Tao and Phillips (2005) where they avoided the decomposition of the stress tensor into positive and negative parts by separating the principal values of the stress tensor into positive and negative parts to simplify the implementation in an FE code.

The effective or undamaged elastic free energy \(\rho \psi^e\) of the concrete material is expressed as follows:

\[
\rho \psi^e = \frac{1}{2} \bar{\varepsilon}_{ij} E_{ijkl} \bar{\varepsilon}_{kl} = \frac{1}{2} \bar{\sigma}_{ij} \bar{\varepsilon}_{ij}^e
\]

(4.44)

In order to account for the stiffness degradation induced by the concrete material damage, the elastic free energy in the damaged configuration can be written in terms of the elastic strain equivalence hypothesis as follows:

\[
\rho \psi^e = (1 - \Phi) \rho \psi^e = \frac{1}{2} \varepsilon_{ij} E_{ijkl} (\Phi) \varepsilon_{kl}^e = \frac{1}{2} (1 - \Phi) \bar{\varepsilon}_{ij} \bar{\varepsilon}_{ij}^e = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^e
\]

(4.45)
The previous equation can be substituted into Eq. (4.19) to give the constitutive stress strain relation in the damaged configuration, Eq. (4.9). It also shows that the elasticity tensor in the damaged configuration \( E_{ijkl}(\Phi) \) is given in terms of the elasticity tensor in the effective configuration \( \bar{E}_{ijkl} \) as shown in Eq. (4.8).

Experimental evidence (Resende, 1987) demonstrates that the susceptibility of concrete to damage and failure is different under pure hydrostatic loading than under deviatoric loading. Therefore, in order to distinguish the different contributions of hydrostatic and deviatoric stress/strain components to damage, the above potential is separated into two parts and written as:

\[
\rho \psi^e = \frac{1}{2}(1-\Phi)(\varepsilon_{ij}^e + \frac{1}{3}\varepsilon_{mn}^e \delta_{ij}^e)\bar{E}_{ijkl}(\varepsilon_{kl}^e + \frac{1}{3}\varepsilon_{mn}^e \delta_{kl}^e) + \rho \psi^e
\]

(4.46)

In the above equation, the elastic strain tensor, \( \varepsilon_{uv}^e \), has been additively decomposed into deviatoric, \( \varepsilon_{uv}^e \), and hydrostatic, \( \frac{1}{3}\varepsilon_{pp}^e \), parts such that:

\[
\varepsilon_{uv}^e = \varepsilon_{uv}^e + \frac{1}{3}\varepsilon_{pp}^e \delta_{uv}
\]

(4.47)

Expanding the above equation, one obtains the following:

\[
\rho \psi^e = \frac{1}{2}(1-\Phi)(\varepsilon_{ij}^e \bar{E}_{ijkl} \varepsilon_{kl}^e + \frac{1}{3}\varepsilon_{mn}^e \varepsilon_{ij}^e \bar{E}_{ijkl} \delta_{kl} + \frac{1}{9}(\varepsilon_{mn}^e)^2 \delta_{ij} \bar{E}_{ijkl} \delta_{kl})
\]

(4.48)

The term involving pure hydrostatic strains can be isolated from the rest of the terms as follows:

\[
\rho \psi^e = \frac{1}{2}(1-\Phi)(\varepsilon_{ij}^e \bar{E}_{ijkl} \varepsilon_{kl}^e + \frac{1}{3}\varepsilon_{mn}^e \varepsilon_{ij}^e \bar{E}_{ijkl} \delta_{kl} + \frac{1}{9}(\varepsilon_{mn}^e)^2 \delta_{ij} \bar{E}_{ijkl} \delta_{kl}) + \frac{1}{18}(1-\Phi)(\varepsilon_{mn}^e)^2 \delta_{ij} \bar{E}_{ijkl} \delta_{kl})
\]

(4.49)

To reduce the susceptibility of the hydrostatic part to damage, Tao and Phillips (2005) used a damage multiplier \( \beta \) in the term involving pure hydrostatic strain, \( \frac{1}{18}(1-\Phi)(\varepsilon_{mn}^e)^2 \delta_{ij} \bar{E}_{ijkl} \delta_{kl} \), as follows:
\[
\rho \psi^e = \frac{1}{2} (1 - \Phi) (\epsilon_{ij}^e \epsilon_{kl}^e + \frac{1}{3} \epsilon_{ik}^e \epsilon_{jk}^e \delta_{kl} + \frac{1}{3} \epsilon_{mn}^e \epsilon_{ijkl}^e \delta_{ij}^e + \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e + \frac{1}{2} (1 - \beta \Phi) \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right)
\]

(4.50)

Considering the last term in the previous equation, the following manipulation can be performed in order to reach an objective result:

\[
\frac{1}{2} (1 - \beta \Phi) \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right) = \frac{1}{2} (1 - \Phi + \Phi - \beta \Phi) \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right)
\]

\[
= \left[ \frac{1}{2} (1 - \Phi) + \frac{1}{2} (\Phi - \beta \Phi) \right] \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right)
\]

\[
= \frac{1}{2} (1 - \Phi) \left( \frac{1}{2} (1 - \beta) \Phi \right) \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right)
\]

(4.51)

Substituting the final term in the above equation to Eq. (4.50), one obtains the following relation:

\[
\rho \psi^e = \frac{1}{2} (1 - \Phi) (\epsilon_{ij}^e \epsilon_{kl}^e + \frac{1}{3} \epsilon_{ik}^e \epsilon_{jk}^e \delta_{kl} + \frac{1}{3} \epsilon_{mn}^e \epsilon_{ijkl}^e \delta_{ij}^e + \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e + \frac{1}{2} (1 - \beta \Phi) \left( \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \right)
\]

(4.52)

Comparing the above equation with Eq. (4.48), it can be seen that the first term on the right hand side of Eq. (4.52) is identical to the right hand side of Eq. (4.48), therefore, the elastic free energy function can be now written in terms of the total strain tensor and the hydrostatic strain tensor with the effect reduction factor, \( \beta \), as follows:

\[
\rho \psi^e = \frac{1}{2} (1 - \Phi) \epsilon_{ij}^e \epsilon_{kl}^e \epsilon_{ij} + \frac{1}{2} (1 - \beta \Phi) \frac{1}{9} \epsilon_{mm}^e \epsilon_{ijkl}^e \delta_{ij}^e \]

(4.53)

Based on the failure characteristics of concrete and the experimental fact that the effect of the hydrostatic strain component on damage is less than that of the deviatoric component, Tao and Phillips (2005) designed the damage multiplier (or damage reduction factor) \( \beta \) to provide this effect reduction. Clearly, \( \beta \) is less than or equal to one \( (0 \leq \beta \leq 1) \). For the uniaxial version, they defined the damage multiplier \( \beta \) as the ratio of the average stress \( \sigma_{mm}/3 \) to the maximum principal stress \( \hat{\sigma}_1 \), i.e., \( \left( \frac{\sigma_{mm}}{3} / \hat{\sigma}_1 \right) \).

Whereas, under bi-axial loadings, the response of concrete is dependent on the stress ratio, and since a relationship between damage and stress ratios is not straightforward to establish and due to the fact that whatever the stress ratios are, it is with no doubt that material damage is the consequence of energy dissipation from the damage mechanics point of view, consequently, the bi-axial version of the damage multiplier is proposed to
be a damage energy release rate, \( Y \), dependent parameter (Tao and Phillips, 2005). Different mathematical forms of the damage multiplier can be assumed as long as they can match the corresponding experimental data. Tao and Phillips (2005) adopted the following form of the damage reduction factor \( \beta \):

\[
\beta = 1 - \frac{1}{1 + cY \exp(-dY)} \quad (4.54)
\]

where \( \exp \) is the base of natural logarithms, and \( c \) and \( d \) can be regarded as two material constants to make \( \beta \) dimensionless and be so determined as to match the experimental data. It is worth mentioning here that this form of the damage reduction factor introduces nonlinearity to the definition of the damage energy release rate (damage conjugate force) \( Y \), and requires local iterations when solving for that damage release rate during a given strain increment.

Based on the thermodynamic framework, one can obtain expressions for the damage thermodynamic conjugate forces \( Y^+ \) and \( Y^- \) from Eqs. (4.53), (4.21), and (4.22) in the following form:

\[
Y^+ = -\rho \frac{\partial \psi^e}{\partial \varphi} = \frac{1}{2} \left[ \frac{\sigma_{ij}^e}{\sigma_y^e} \left( \varepsilon_{ijkl}^e \varepsilon_{kl}^e - \frac{1}{9} (1 - \beta) (\varepsilon_{mn}^e)^2 \delta_{ij} E_{ijkl} \delta_{kl} \right) \right] \quad (4.55)
\]

\[
Y^- = -\rho \frac{\partial \psi^e}{\partial \varphi} = \frac{1}{2} \left[ \frac{\sigma_{ij}^e}{\sigma_y^e} \left( \varepsilon_{ijkl}^e \varepsilon_{kl}^e - \frac{1}{9} (1 - \beta) (\varepsilon_{mn}^e)^2 \delta_{ij} E_{ijkl} \delta_{kl} \right) \right] \quad (4.56)
\]

Since the magnitudes of the damage energy release rates \( Y^+ \) and \( Y^- \) are measures of how susceptible the material is to damage, the damage energy release rates \( Y^\pm \) are therefore used to define the damage criteria \( g^\pm \) in tension or compression.

### 4.4.2 The Plastic Part of the Helmholtz Free Energy Function

The plastic part of the Helmholtz free energy is postulated to be a function of the plastic variables \( \kappa^+ \) and \( \kappa^- \) in the following form:

\[
\rho \psi^p = f_0^+ \kappa^+ + \frac{1}{2} h (\kappa^+)^2 + f_0^- \kappa^- + Q \left( \kappa^- + \frac{1}{\omega} \exp(-\omega \kappa^-) \right) \quad (4.57)
\]

where \( f_0^+ \) and \( f_0^- \) are the uniaxial tensile and compressive yield stresses, respectively. The hardening parameters \( \kappa^+ \) and \( \kappa^- \) are introduced as the equivalent plastic strains under tension and compression, respectively, defined as:

\[
\kappa^+ = \int_0^t \kappa^+ dt \quad (4.58)
\]
\[ \kappa^- = \int_0^t \kappa^- \, dt \tag{4.59} \]

where \( \kappa^+ \) and \( \kappa^- \) are the tensile and compressive equivalent plastic strain rates, respectively, which are assumed to be evaluated according to the following expressions (Lee and Fenves, 1998):

\[ \kappa^+ = r(\tilde{\sigma}_i) \hat{\epsilon}^p_{\text{max}} \tag{4.60} \]
\[ \kappa^- = -(1 - r(\tilde{\sigma}_i)) \hat{\epsilon}^p_{\text{min}} \tag{4.61} \]

where \( \hat{\epsilon}^p_{\text{max}} \) and \( \hat{\epsilon}^p_{\text{min}} \) are the maximum and minimum eigenvalues of the plastic strain rate tensor \( \hat{\epsilon}^p_{ij} \) such that \( \hat{\epsilon}^p_1 > \hat{\epsilon}^p_2 > \hat{\epsilon}^p_3 \) and \( \hat{\epsilon}^p_{\text{max}} = \hat{\epsilon}^p_1 \) and \( \hat{\epsilon}^p_{\text{min}} = \hat{\epsilon}^p_3 \). It should be mentioned here that the procedure for obtaining the eigenvalues of a second-order tensor (e.g. stress or strain) is a built-in function in ABAQUS readily available for the UMAT subroutine. Under uniaxial loading, these eigenvalues reduce to \( \hat{\epsilon}^p_{\text{max}} = \hat{\epsilon}^p_{11} \) in tension and \( \hat{\epsilon}^p_{\text{min}} = \hat{\epsilon}^p_{13} \) in compression. The dimensionless parameter \( r(\tilde{\sigma}_i) \) is a weight factor \( 0 \leq r(\tilde{\sigma}_i) \leq 1 \) depending on the effective principal stresses \( \tilde{\sigma}_i \) \((i = 1,2,3)\) and is defined as follows (Lee and Fenves, 1998):

\[ r(\tilde{\sigma}_i) = \frac{\sum_{i=1}^3 \langle \tilde{\sigma}_i \rangle}{\sum_{i=1}^3 |\tilde{\sigma}_i|} \tag{4.62} \]

The symbol \( \langle \ldots \rangle \) is the Macauley bracket, defined as \( \langle x \rangle = \frac{1}{2} (|x| + x) \). Note that \( r(\tilde{\sigma}_i) \) is equal to one if all the eigenstresses \( \tilde{\sigma}_i \) are positive and accordingly equal to zero if they are all negative. The parameters \( Q \) and \( \omega \) are material constants related to the function of isotropic hardening of the material.

Substituting Eq. (4.57) into Eqs. (4.23) and (4.24) yields the following expressions for the plasticity conjugate forces \( c^+ \) and \( c^- \) (the tensile and compressive hardening functions):

\[ c^+ = \rho \frac{\partial \psi^p}{\partial \kappa^+} = f_0^+ + h \kappa^+ \tag{4.63} \]

and

\[ c^- = \rho \frac{\partial \psi^p}{\partial \kappa^-} = f_0^- + Q \left[ 1 - \exp(-\omega \kappa^-) \right] \tag{4.64} \]
such that by taking the time derivative of the above two expressions, one can easily obtain the following evolution equations of the hardening functions $c^+$ and $c^-$ in terms of the plastic internal state variables $\kappa^+$ and $\kappa^-:$

\[
\dot{c}^+ = h \kappa^+ \quad (4.65)
\]

\[
\dot{c}^- = \omega (Q - c^-_i) \kappa^- \quad (4.66)
\]

Equation (4.66) can be obtained by realizing that the exponential term, $\exp(-\omega \kappa^-)$, in the rate equation, $\dot{c}^- = \omega Q \left( \exp(-\omega \kappa^-) \right) \kappa^-,$ can be replaced by the following term, $\exp(-\omega \kappa^-) = 1 - \frac{c^- - f^-_0}{Q} = 1 - \frac{c^-_i}{Q},$ obtained from rearranging Eq. (4.64).

### 4.5 Plasticity Formulation

In this section, the effective stress space plasticity and its components will be discussed. Owing to the coupling between the damage evolutions and the plastic flow in the elastic plastic damage models, the so-called (effective stress space plasticity) was introduced by Ju (1989). In this approach the effective (undamaged) configuration is resorted in order to establish the evolution laws for the plastic strains governing the plastic irreversible behavior in the material (Wu et al., 2006). To determine the required effective stress tensor $\hat{\sigma}_{ij},$ the evolution law for the irreversible plastic strains tensor $\hat{\varepsilon}^p_{ij}$ has to be established first. The additive decomposition of the effective total strain tensor into elastic and plastic parts is assumed, Eq. (4.1). An effective stress plasticity yield criterion with multiple hardening rules is used along with a non-associative flow rule. Both take into account the dilatation effect of concrete materials. A Kuhn-Tucker consistency condition is applied to obtain the evolution of the magnitude of plastic strains.

#### 4.5.1 Plasticity Yield Surface and Hardening Functions

A crucial component of any material model that involves plasticity theory is the yield surface/criterion. This criterion should address and model the experimentally observed non-symmetrical behavior of concrete under tensile and compressive loadings. Assuming the same yield behavior for both tension and compression in concrete materials leads to over/under estimation of plastic deformations (Lubliner et al., 1989). The yield criterion adopted in this work was first introduced in the Barcelona model by Lubliner et al. (1989), and later modified by Lee and Fenves (1998, 2001) and Wu et al. (2006). These works reported that the yield criterion is successful in simulating the concrete behavior under uniaxial, biaxial, multiaxial, and cyclic loading. This criterion is given in the effective stress space and expressed using the undamaged configuration parameters as follows:

\[
f = \sqrt{3\tilde{J}_2} + \alpha \tilde{T}_1 + \beta (\kappa^z) H(\tilde{\sigma}_{\text{max}})\tilde{\sigma}_{\text{max}} - (1 - \alpha) c^- (\kappa^-) = 0 \quad (4.67)
\]
where $\mathcal{J}_2 = \frac{\overline{\sigma}_y \overline{\sigma}_y}{2}$ is the second-invariant of the effective deviatoric stress $\overline{\sigma}_y = \sigma_{ij} - \sigma_{kk} \delta_{ij}/3$, $\overline{I}_1 = \bar{\sigma}_{kk}$ is the first-invariant of the effective stress $\bar{\sigma}_y$, $\kappa^\pm$ denote a suitable set of plastic variables (Wu et. al, 2006) given as the equivalent plastic strains defined in Eqs. (4.60) and (4.61), $H(\hat{\sigma}_{\text{max}})$ is the Heaviside step function defined in Eq.(4.37), and $\hat{\sigma}_{\text{max}}$ is the maximum principal stress.

The parameter $\alpha$ is a dimensionless constant given by Lubliner et. al. (1989) as follows:

$$\alpha = \frac{(f_{\text{ho}}^- / f_0^-)^{-1} - 1}{2(f_{\text{ho}}^- / f_0^-)^{-1} - 1} \quad (4.68)$$

and the parameter $\beta$, defined as a constant in the Barcelona model, was later modified by Lee and Fenves (1998), and given as a dimensionless function of the tensile and compressive cohesions $c^\pm$ (hardening internal state variables) in the following form:

$$\beta(\kappa^\pm) = (1 - \alpha) \frac{c^-(\kappa^-)}{c^+(\kappa^+)} - (1 + \alpha) \quad (4.69)$$

where $f_{\text{ho}}^-$ and $f_0^-$ are the initial equibiaxial and uniaxial compressive yield stresses, respectively. Experimental values of the ratio $f_{\text{ho}}^- / f_0^-$ lie between 1.10 – 1.20 (Wu et. al., 2006); yielding $\alpha$ to be between 0.08 – 0.14. For further details about the derivation of both parameters, $\alpha$ and $\beta$, the reader is referred to Lubliner et. al. (1989).

The cohesion parameters, $c^+$ and $c^-$, denote evolution stresses (positive quantities) in the effective stress space due to plastic hardening/softening under uniaxial tension and compression, respectively. They are defined as cohesion parameters due to the fact that concrete material behavior resembles that of a frictional material with cohesion (Lubliner et. al., 1989). Since the concrete behavior in compression is more of a ductile behavior, the compressive isotropic hardening function $c^-$ is defined by the following exponential law:

$$c^-(\kappa^-) = f_0^- + Q \left[ 1 - \exp(-\omega \kappa^-) \right] \quad (4.70)$$

where $Q$ and $\omega$ are material constants characterizing the saturated stress and the rate of saturation, respectively. On the other hand, a linear expression is assumed for the tensile hardening function $c^+$ such that:

$$c^+(\kappa^+) = f_0^+ + h \kappa^+ \quad (4.71)$$
where \( h \) is a material constant obtained from the uniaxial tensile stress-strain diagram. The evolution equations of the hardening parameters were shown in Section 4.4.2, Eqs. (4.65) and (4.66).

### 4.5.2 Plasticity Non-associative Flow Rule and Consistency Condition

The flow rule gives the relation between the plastic flow direction and the plastic strain rate. A non-associated flow rule means that the yield function \( f \) and the plastic potential \( F^p \) do not coincide, and therefore, the direction of the plastic flow is not normal to the yield surface. This is important for realistic modeling of the volumetric expansion (dilatancy) under compression for frictional materials such as concrete. Using an associated flow rule for the type of yield surface shown in Eq. (4.67) gives an unrealistically high volumetric expansion in compression, which leads in some cases to an overestimated strength - peak stress (Chen and Han, 1988). Therefore, the shape of the concrete loading surface at any given point in a given loading state should be obtained by using non-associative plasticity. The plastic strain rate can be written in terms of the effective stress \( \bar{\sigma} \) as:

\[
\dot{\epsilon}_p^p = \lambda \frac{\partial F^p}{\partial \bar{\sigma}}
\]

(4.72)

where \( \lambda \) is the plastic flow parameter (consistency factor) known as the Lagrangian multiplier, which can be obtained using the plasticity consistency condition, and the plastic potential function \( F^p \) takes the following Drucker-Prager format as given in Lee and Fenves (1998):

\[
F^p = \sqrt{3J_2} + \alpha^p \bar{I}_1
\]

(4.73)

such that:

\[
\frac{\partial F^p}{\partial \bar{\sigma}} = \frac{3}{2} \frac{\bar{\sigma}}{\sqrt{3J_2}} + \alpha^p \delta
\]

(4.74)

where \( \alpha^p \) is a parameter chosen to provide proper dilatancy with common range between 0.2 and 0.3 for concrete (Lee and Fenves, 1998; Wu et. al., 2006).

The plasticity consistency condition can be obtained by taking the time derivative of the plasticity yield function, \( \dot{f} = 0 \), and satisfying the following Kuhn-Tucker loading/unloading conditions:
If \( f \leq 0 \), \( \dot{\lambda}^p \geq 0 \), \( \dot{\lambda}^p f = 0 \), \( \dot{\lambda}^p \dot{f} = 0 \)

\[
\begin{align*}
\text{If } f &< 0 \quad \text{then} \quad \dot{\lambda}^p = 0; \\
\text{If } f = 0 \text{ and } \dot{f} \leq 0 \quad \text{then} \quad \dot{\lambda}^p = 0; \\
\text{If } f = 0 \text{ and } \dot{f} > 0 \quad \text{then} \quad \dot{\lambda}^p > 0
\end{align*}
\]

This concludes the plastic formulation for the present model. The damage formulation is discussed next where the tensile and compressive damage surfaces are defined.

### 4.6 Damage Formulation

The isotropic damage in this work is responsible for the softening response and the degradation in the elastic stiffness. The tensile and compressive damage surfaces and their hardening functions will be presented first, followed by a brief discussion of the damage consistency conditions.

#### 4.6.1 Tensile and Compressive Damage Surfaces and Hardening Functions

To determine the stress states during a damaging process from the thermodynamic constitutive relations, Eqs. (4.28), (4.31) and (4.32), tensile and compressive damage surfaces and their evolution laws have to be specified. Referring to the definitions of a yield function and the plastic flow rule in plasticity theory, Tao and Phillips (2005) defined for an isothermal process the following two damage surfaces \( g^\pm \) as functions of the damage thermodynamic conjugate forces \( Y^\pm \) and the scalar damage parameters \( \varphi^\pm \), with a similar form to that of La Borderie et. al. (1992):

\[
g^\pm = Y^\pm - Y_0^\pm - Z^\pm \leq 0 \quad \text{(no mixing \( \pm \)) (4.76)}
\]

where \( Y_0^\pm \) are initial damage thresholds (tension and compression) which govern the onset of tensile or compressive damage, respectively. As damage progresses, initial damage surfaces change by means of evolution laws defined by hardening/softening parameters \( Z^\pm \). These parameters \( Z^\pm \) can be expressed mathematically in different forms, such as polynomials, power and exponential functions, etc. Amongst them power and exponential functions have the best match for the shapes of loading curves of concrete (Lubliner et. al., 1989; Lee and Fenves, 1998; Nechnech et. al., 2002; Tao and Phillips, 2005; Wu et. al., 2006 and others). Tao and Phillips (2005) assumed that the softening of damage surfaces follow a power law in the form of:

\[
Z^\pm = \frac{1}{a^\pm} \left( \frac{\varphi^\pm}{1 - \varphi^\pm} \right)^{\frac{1}{b^\pm}} \quad \text{(4.77)}
\]

in which \( a^\pm \) and \( b^\pm \) are four material constants to be calibrated by means of uniaxial tensile and compressive experiments of concrete. Tao and Phillips (2005) studied the
effects of $a^\pm$ and $b^\pm$ on $Z^\pm$ as damage progress. They showed that the shape of damage surface varies with $b^\pm$, whilst $a^\pm$ determine the magnitude of $Z^\pm$. In other words, parameters $a^\pm$ mainly dominate the magnitude of damage surfaces with units of MPa$^{-1}$, whilst $b^\pm$, being dimensionless parameters, influence generally the characteristics of softening/hardening (see Fig. 4.1). The same trend was observed in this work as will be shown in the verification section. Tao and Phillips (2005) claimed that a proper selection of parameters $a^\pm$ and $b^\pm$ tailors $Z^\pm$ to the demands of different types of concrete and their corresponding tensile and compressive strengths. Other researchers that used Eq. (4.76) to model their isotropic damage criteria include Salari et. al. (2004), Shao et. al. (2006), and Grassl and Jirasek (2006). All three used single isotropic damage variable $\varphi$ and thus, a single damage criterion.

![Figure 4.1 Effect of material parameters $a^\pm$ and $b^\pm$ on model behavior](image1)

In this work, and in an effort to reduce the sensitivity of the FE analysis of concrete to the refinement of the FE meshes, the damage magnitude parameters $a^\pm$ are adjusted to include dimensionless embedded coefficients $\gamma^\pm$ that are related to the fracture energies in tension and compression, $G_f^\pm$ (see Fig. 4.2), and to the (geometrical) characteristic length, $\ell$, of the applied FE mesh obtained from ABAQUS. These $\gamma^\pm$ coefficients are given as (Oliver et. al., 1990; Labadi and Hannachi, 2005; Wu. et. al., 2006):

![Figure 4.2 ($\sigma - \varepsilon^p$) relation for uniaxial test, a) tension, b) compression](image2)
\[
\gamma^+ = \left[ \frac{G_1^* E}{f_{\alpha}^* z^2} - \frac{1}{2} \right]^{-1}
\] (4.78)

### 4.6.2 Damage Consistency Conditions

A stress point in principal stress space can be either within or on the current damage surface. When within the damage surface, being tensile or compressive, the stress point may be loading, but it has not violated the current damage criterion yet. Once it is on the damage surface, two damage states are possible. One may be unloading or neutral loading, having \( \phi^+ = 0 \). The other is loading, accompanied by the evolution of damage and defined as \( \phi^+ > 0 \). Mathematically, the above description is expressed as:

\[
\begin{align*}
\text{If } g^+ < 0 & \quad \text{then } \phi^+ = 0; \\
\text{If } g^+ = 0 \text{ and } g^+ \leq 0 & \quad \text{then } \phi^+ = 0; \\
\text{If } g^+ = 0 \text{ and } g^+ > 0 & \quad \text{then } \phi^+ > 0
\end{align*}
\] (4.79)

The above conditions are damage extension of the classical plasticity Kuhn–Tucker conditions (Voyiadjis and Kattan, 1992).

### 4.7 Numerical Integration of the Constitutive Model

The development of a computational algorithm that is consistent with the proposed theoretical formulation is given in details in this section to facilitate the numerical integration of the constitutive equations in the context of the FE method. According to the operator split concept of (Ju, 1989; Simo and Hughes, 1998), Eq. (4.10) can be decomposed into elastic, plastic and damage parts, leading to the corresponding numerical algorithm including elastic-predictor, plastic-corrector and damage-corrector steps (Lee and Fenves, 2001; Wu et. al., 2006, Jason et. al., 2006) as follows:

\[
\begin{align*}
\sigma^e_{ij} &= (1 - \Phi)\tilde{\sigma}_{ij}^e - \Phi \tilde{\sigma}_{ij}^p \\
\Delta \sigma_{ij}^e &= (1 - \Phi)(\tilde{E}^\text{ijkl} \Delta e_{kl}^p - \tilde{E}^\text{ijkl} \Delta e_{kl}^p) - \Delta \Phi \tilde{E}^\text{ijkl} e_{kl}^e \\
\Delta \sigma_{ij}^p &= (1 - \Phi)(\underbrace{\Delta \tilde{\sigma}_{ij}^\text{trial}}_{\text{ELASTIC PREDICTOR}} - \tilde{E}^\text{ijkl} \Delta e_{kl}^p) - \Delta \Phi \tilde{E}^\text{ijkl} e_{kl}^e
\end{align*}
\] (4.80)

In the first step, the elastic-predictor problem is solved with the initial conditions being the converged values of the previous iteration \( (t = t_n) \) along with the new increment while keeping the irreversible variables frozen. This produces a trial stress state, \( \tilde{\sigma}_{ij}^\text{trial} = \tilde{\sigma}_{ij}^e + \Delta \tilde{\sigma}_{ij}^\text{trial} \), which, if outside the plastic surface \( f \) and the damage surfaces \( g^+ \) is taken as the initial conditions for the solution of the plastic-corrector and damage-corrector problems. The scope of the second and third steps is to restore the generalized
plasticity and damage consistency conditions by returning back the trial stress to the plastic surface \( f \) and the damage surfaces \( g \).

During the elastic-predictor \( \Delta \bar{\sigma}_{ij}^{\text{trial}} \) and the plastic-corrector \( -E_{ijkl} \Delta \varepsilon_{kl}^p \) steps the damage variables are fixed, so that the elastic-plastic behavior is decoupled from damage, constituting a standard elastic-plastic problem in the effective stress space. Regarding the adopted plastic yield function, Eq. (4.67), the spectral decomposition form (Lee and Fenves, 2002) of return-mapping algorithm (Simo and Hughes, 1998) is applied to update the effective stress tensor \( \bar{\sigma}_{ij}^{n+1} \). Once the effective stress tensor \( \bar{\sigma}_{ij}^{n+1} \) is updated in the elastic-predictor and plastic-corrector steps, the damage variables \( (\varphi^\pm)^{n+1} \) (and therefore \( \Phi^{n+1} \)) and the Cauchy stress tensor \( \sigma_{ij}^{n+1} \) can then be updated correspondingly in the damage-corrector step (Wu et. al., 2006).

The (degradation) damage-corrector is implemented separately from the plastic-corrector part because \( (\varphi^\pm)^{n+1} \) (and therefore \( \Phi^{n+1} \)) are functions of the updated effective stress tensor \( \bar{\sigma}_{ij}^{n+1} \) and its corresponding elastic strain tensor \( (\varepsilon_{ij}^e)^{n+1} = (\varepsilon_{ij}^e)^n + \Delta \varepsilon_{ij}^e \), which are completely determined during the plastic-corrector step \( \Delta \varepsilon_{kl}^e = \Delta \varepsilon_{kl} - \Delta \varepsilon_{kl}^p \).

A fully Implicit (Backward-Euler) scheme is used for the stress computation problem in the effective space, followed by an explicit integration scheme for the updated damage variables and Cauchy stress tensor. The integration procedure start at the beginning of the \( (n, n+1) \) step where \( \bar{\sigma}_{ij}^n, (\varepsilon_{ij}^p)^n, (\kappa^\pm)^n \) (adopted here for simplicity as \( \kappa^n \)) and \( \Delta \varepsilon_{ij} \) are all known from the previous step \((n-1, n)\).

### 4.7.1 The Effective (Undamaged) Elastic-Plastic Steps

In this section, the elastic-plastic integration procedure is carried out in an undamaged medium. Therefore, the stresses and strains will carry a superimposed dash indicating the effective configuration. The ultimate goal of the elastic-plastic steps is to update the effective stress tensor and the hardening parameters. The updated effective stress is given as:

\[
\bar{\sigma}_{ij}^{n+1} = E_{ijkl} (\bar{\varepsilon}_{kl}^e)^{n+1} = \bar{E}_{ijkl} \left( \bar{\varepsilon}_{kl}^{n+1} - (\bar{\varepsilon}_{kl}^p)^n \right) = \bar{E}_{ijkl} \left( \bar{\varepsilon}_{kl}^n + \Delta \varepsilon_{kl} - (\bar{\varepsilon}_{kl}^p)^n - \Delta \varepsilon_{kl}^p \right) \\
= \bar{E}_{ijkl} \left( \bar{\varepsilon}_{kl}^n - (\bar{\varepsilon}_{kl}^p)^n \right) + \bar{E}_{ijkl} \left( \Delta \varepsilon_{kl} - \Delta \varepsilon_{kl}^p \right) = \bar{E}_{ijkl} (\bar{\varepsilon}_{kl}^e)^n + \bar{E}_{ijkl} \Delta \varepsilon_{kl}^e \\
= \bar{\sigma}_{ij}^n + \Delta \bar{\sigma}_{ij} \\
\tag{4.81}
\]

#### 4.7.1.1 The Elastic - Predictor

The elastic predictor problem defines the stress in the trial state assuming an entirely elastic strain increment \( (\Delta \bar{\varepsilon}_{ij} = \Delta \bar{\varepsilon}_{ij}^e) \) as:
\[
\widetilde{\sigma}_{ij}^{\text{trial}} = \widetilde{E}_{ijkl} \left( \widetilde{e}_{ij}^{n+1} - (\widetilde{e}_{ij}^p)^n \right) = \widetilde{E}_{ijkl} \left( \widetilde{e}_{ij}^n + \Delta \widetilde{e}_{ij} - (\widetilde{e}_{ij}^p)^n \right)
\]
\[
= \widetilde{E}_{ijkl} \left( \widetilde{e}_{kl}^n - (\widetilde{e}_{ij}^p)^n \right) + \widetilde{E}_{ijkl} \Delta \widetilde{e}_{ij} = \widetilde{E}_{ijkl} (\widetilde{e}_{ijkl}^n)^n + \widetilde{E}_{ijkl} \Delta \widetilde{e}_{ij}^n = \sigma_{ij}^n + \Delta \sigma_{ij}^{\text{trial}}
\]

where \( \sigma_{ij}^n \) is calculated in the previous \( (n) \) step.

In order to verify the correctness of this elastic prediction, the trial stress is applied into the yield function \( f(\widetilde{\sigma}_{ij}^{\text{trial}}, \kappa^n) \). If \( f(\widetilde{\sigma}_{ij}^{\text{trial}}, \kappa^n) < 0 \), the process is elastic and the trial state is admissible and accepted as the final state since there is no change in the plastic strain, such that:

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^{\text{trial}}, \quad (\widetilde{e}_{ij}^p)^{n+1} = (\widetilde{e}_{ij}^p)^n, \quad (\kappa^+)^{n+1} = (\kappa^+)^n
\]

On the other hand, if \( f(\widetilde{\sigma}_{ij}^{\text{trial}}, \kappa^n) > 0 \), the Kuhn-Tucker loading/unloading conditions are violated by the trial state which now lies outside the yield surface. Then the consistency is restored by the return-mapping/plastic-corrector step.

**4.7.1.2 The Plastic - Corrector**

If the current step is not an elastic state, \( f(\widetilde{\sigma}_{ij}^{\text{trial}}, \kappa^n) > 0 \), the plastic strain tensor will change. To compute the effective stress along with the plastic strain tensor at the current time increment requires iterations within that time increments for the effective stress in the plastic-corrector step. During these iterations, the discrete version of the plastic consistency condition is imposed as a constraint, \( f(\widetilde{\sigma}_{ij}^{n+1}, \kappa^{n+1}) = 0 \), at the end of each iteration. Therefore, the analysis is transformed into a linear set of equations that depend on the material parameters and on the current coordinates of the integration points within each iteration. The outcomes of the plastic-corrector step are the updated effective stress tensor \( \sigma_{ij}^{n+1} \), plastic strain tensor \( (\widetilde{e}_{ij}^p)^{n+1} \), and plastic variables \( (\kappa^+)^{n+1} \). The plastic-corrector step can be derived as follows:

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^n + \Delta \sigma_{ij} = \sigma_{ij}^n + \widetilde{E}_{ijkl} \Delta \widetilde{e}_{ij}^n = (\sigma_{ij}^n + \widetilde{E}_{ijkl} \Delta \widetilde{e}_{ij}) - \widetilde{E}_{ijkl} \Delta \widetilde{e}_{ij}^p
\]

Within the plastic-corrector step, and in order to compute \( \Delta \widetilde{e}_{ij}^p \) in Eq. (4.84), the radial-return method is used. The flow rule in Eq. (4.72) can be written implicitly for the \( (n+1) \) step as follows:

\[
\Delta \widetilde{e}_{kl}^p = \Delta \lambda^p \frac{\partial F^p}{\partial \sigma_{kl}^{n+1}}
\]
where \( \frac{\partial F^p}{\partial \tilde{\sigma}_{kl}^{n+1}} \) (Eq. (4.74)) is given here as the following:

\[
\frac{\partial F^p}{\partial \tilde{\sigma}_{kl}^{n+1}} = \sqrt{\frac{3}{2} \left[ \frac{S_{kl}^{n+1}}{S_{mn}^{n+1}} \right]} + \alpha_p \delta_{kl} \tag{4.86}
\]

where \( \left\| \tilde{S}_{mn}^{n+1} \right\| = \sqrt{\tilde{S}_{mn}^{n+1} \tilde{S}_{mn}^{n+1}} \) is the norm of the updated effective deviatoric stresses.

By substituting the elasticity tensor in the undamaged configuration, \( \tilde{E}_{ijkl} \) (Eq. (4.4)), into Eq. (4.84), the effective stress can be updated using the return-mapping equation given as follows (derivation shown in Appendix B):

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^{trial} - \left[ 2\tilde{C} \Delta \tilde{\varepsilon}_{ij}^p + \left( \tilde{K} - \frac{2}{3} \tilde{C} \right) \Delta \tilde{\varepsilon}_{kk}^p \delta_{ij} \right] \tag{4.87}
\]

where \( \Delta \tilde{\varepsilon}_{kk}^p \) can now be obtained from Eqs. (4.85) and (4.86) as follows:

\[
\Delta \tilde{\varepsilon}_{kk}^p = 3 \alpha_p \Delta \lambda^p \tag{4.88}
\]

By substituting Eqs. (4.85), (4.86), and (4.88) into Eq. (4.87), one obtains:

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^{trial} - \left[ 2\tilde{G} \Delta \lambda^p \left\{ \sqrt{\frac{3}{2} \left[ \frac{S_{ij}^{n+1}}{S_{mn}^{n+1}} \right]} + \alpha_p \delta_{ij} \right\} + \left( \tilde{K} - \frac{2}{3} \tilde{G} \right)(3 \alpha_p \Delta \lambda^p) \delta_{ij} \right] \tag{4.89}
\]

Expanding and then simplifying the above equation, one can obtain the following form:

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^{trial} - \left[ \sqrt{6\tilde{G}} \Delta \lambda^p \frac{\tilde{S}_{ij}^{n+1}}{\left\| \tilde{S}_{mn}^{n+1} \right\|} + 2\tilde{G} \alpha_p \Delta \lambda^p \delta_{ij} + 3\tilde{K} \alpha_p \Delta \lambda^p \delta_{ij} - 2\tilde{G} \alpha_p \Delta \lambda^p \delta_{ij} \right] \tag{4.90}
\]

\[
\sigma_{ij}^{n+1} = \sigma_{ij}^{trial} - \Delta \lambda^p \left[ \sqrt{6\tilde{G}} \frac{\tilde{S}_{ij}^{n+1}}{\left\| \tilde{S}_{mn}^{n+1} \right\|} + 3\tilde{K} \alpha_p \delta_{ij} \right]
\]

Separating Eq. (4.90) into deviatoric and volumetric parts gives the following:
\[ I_{ijkl}^{\text{dev}} \bar{\sigma}_{kl}^{n+1} = I_{ijkl}^{\text{dev}} \bar{\sigma}_{kl} - \Delta \lambda^p I_{ijkl}^{\text{dev}} \left[ \sqrt{6G} \frac{\bar{S}_{ij}^{n+1}}{\left\| S_{ij}^{n+1} \right\|} + 3K\alpha_p \delta_{kl} \right] \]

\[ \bar{S}_{ij}^{n+1} = S_{ij}^{\text{trial}} - \Delta \lambda^p \left( \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \left[ \sqrt{6G} \frac{S_{ij}^{n+1}}{\left\| S_{ij}^{n+1} \right\|} + 3K\alpha_p \delta_{kl} \right] \quad (4.91) \]

\[ \bar{S}_{ij}^{n+1} = S_{ij}^{\text{trial}} - \sqrt{6\Delta \lambda^p \bar{G}} \frac{S_{ij}^{n+1}}{S_{ij}^{\text{trial}}} \]

and

\[ \bar{\sigma}_{ij}^{n+1} = \bar{\sigma}_{ij}^{\text{trial}} - \Delta \lambda^p \left[ \sqrt{6G} \frac{\bar{S}_{ij}^{n+1}}{\left\| S_{ij}^{n+1} \right\|} + 3K\alpha_p \delta_{ij} \right] \quad (4.92) \]

\[ T_{ij}^{n+1} = T_{ij}^{\text{trial}} - 9\Delta \lambda^p \bar{K}\alpha_p \]

To solve for the plastic multiplier \( \Delta \lambda^p \) from Eq. (4.91), the radial-return method is applied. The proof of the radial-return applied to Eq. (4.91) was provided in Chapter 3, Eq. (3.88) and is stated again here as:

\[ \frac{\bar{S}_{ij}^{n+1}}{\left\| S_{ij}^{n+1} \right\|} = \frac{S_{ij}^{\text{trial}}}{\left\| S_{ij}^{\text{trial}} \right\|} \quad (4.93) \]

Therefore, substituting Eq. (4.93) into Eq. (4.91), the radial-return form of Eq. (4.91) can be provided as follows:

\[ \bar{S}_{ij}^{n+1} = S_{ij}^{\text{trial}} - \sqrt{6\Delta \lambda^p \bar{G}} \frac{S_{ij}^{n+1}}{S_{ij}^{\text{trial}}} \quad (4.94) \]

and rewriting Eq. (3.86) from Chapter 3 using the current notation, one obtains the following:

\[ \left\| S_{ij}^{n+1} \right\| = \left\| S_{ij}^{\text{trial}} \right\| - \sqrt{6\bar{G}\Delta \lambda^p} \quad (4.95) \]

Equation (4.90) can eventually be written as follows:

\[ \bar{\sigma}_{ij}^{n+1} = \bar{\sigma}_{ij}^{\text{trial}} - \Delta \lambda^p \left[ \sqrt{6G} \frac{\bar{S}_{ij}^{n+1}}{\left\| S_{ij}^{n+1} \right\|} + 3K\alpha_p \delta_{ij} \right] \quad (4.96) \]
Next, in order to account for the effective principal stress term in the yield function, Eq. (4.67), the spectral return-mapping algorithm of (Lee and Fenves, 2001) is used. This algorithm has the advantage for yield function which includes principal stress terms in addition to the stress tensor invariants (Wu et. al., 2006). A decoupled version of the return-mapping algorithm is derived (using the spectral decomposition concept) and shown below.

The spectral decomposition of the stress at step \( n+1 \) is given as follows:

\[
\hat{\sigma}_{ij}^{n+1} = l_{ir} \hat{\sigma}_{rs}^{n+1} l_{js}
\] (4.97)

where \( l_{mn} \) is the principal direction tensor corresponding to the effective principle stress tensor. Lee and Fenves, (2001), showed that the plastic potential function \( F^p \) can be written in terms of the effective principal stresses such that the increment of the effective plastic strain \( \Delta \varepsilon^p_{ij} \) can be written as follows:

\[
\Delta \varepsilon^p_{ij} = \Delta \lambda^p l_{ir} \frac{\partial F^p}{\partial \hat{\sigma}_{rs}^{n+1}} l_{js}
\] (4.98)

By substituting Eqs. (4.88), (4.97), and (4.98) into Eq. (4.87), one can write the return-mapping equation in the following form:

\[
l_{ir} \hat{\sigma}_{rs}^{n+1} l_{js} = \hat{\sigma}_{ij}^{\text{trial}} - 2\hat{G} \Delta \lambda^p l_{ir} \frac{\partial F^p}{\partial \hat{\sigma}_{rs}^{n+1}} l_{js} + 3(\overline{K} - \frac{2}{3} \hat{G}) \alpha_p \Delta \lambda^p \delta_{ij}
\] (4.99)

Using the following relation for the Kronecker delta, which holds for any direction tensor \( l_{mn} \):

\[
\delta_{ij} = l_{ir} \delta_{rs} l_{js}
\] (4.100)

one can write Eq. (4.99) in the following form to obtain the effective trial stress directly as:

\[
\hat{\sigma}_{ij}^{\text{trial}} = l_{ir} \hat{\sigma}_{rs}^{n+1} l_{js} + \Delta \lambda^p l_{ir} \left[ 2\hat{G} \frac{\partial F^p}{\partial \hat{\sigma}_{rs}^{n+1}} + 3(\overline{K} - \frac{2}{3} \hat{G}) \alpha_p \right] l_{js}
\] (4.101)

Lee and Fenves (2001) proved that any principal direction tensor for \( \hat{\sigma}_{ij}^{n+1} \) is also a principal direction tensor for \( \hat{\sigma}_{ij}^{\text{trial}} \). This proof is necessary to show that the eigenvector spaces of \( \hat{\sigma}_{ij}^{n+1} \) and \( \hat{\sigma}_{ij}^{\text{trial}} \) are exactly identical despite the fact that symmetric tensors do not have a unique spectral decomposition form if they have repeated eigenvalues. Accordingly, the spectral decomposition of the trial stress tensor is given as:
\[
\hat{\sigma}_{ij}^{trial} = l_p \hat{\sigma}_{rs}^{trial} l_{js}
\]  \quad (4.102)

where \( \hat{\sigma}_{rs}^{trial} \) is the principal trial stress tensor.

From Eq. (4.101) along with Eq. (4.102) one can write the decoupled form of the return-mapping equation as follows:

\[
l_p \hat{\sigma}_{rs}^{trial} l_{js} = l_p \hat{\sigma}_{rs}^{n+1} l_{js} + \Delta \lambda^p l_{ir} \left[ 2 \hat{G} \frac{\partial F^p}{\partial \hat{\sigma}_{rs}^{n+1}} + 3(\hat{K} - \frac{2}{3} \hat{G}) \alpha^p_\sigma \delta_{rs} \right] l_{js}
\]  \quad (4.103)

\[
\hat{\sigma}_{ij}^{n+1} = \hat{\sigma}_{ij}^{trial} - \Delta \lambda^p \left[ 2 \hat{G} \frac{\partial F^p}{\partial \hat{\sigma}_{ij}^{n+1}} + 3(\hat{K} - \frac{2}{3} \hat{G}) \alpha^p_\sigma \delta_{ij} \right]
\]

and from Eq. (4.98), the eigenvalue tensor of the plastic strain increment becomes:

\[
\Delta \hat{\varepsilon}_{ij}^p = \Delta \lambda^p \frac{\partial F}{\partial \hat{\sigma}_{ij}^{n+1}}
\]  \quad (4.104)

where the derivative of the potential function with respect to the principal stress tensor is given as follows:

\[
\frac{\partial F^p}{\partial \hat{\sigma}_{ij}^{n+1}} = \sqrt{2} \frac{3}{2} \hat{\varepsilon}_{mn} + \alpha^p_\sigma \delta_{ij}
\]  \quad (4.105)

However, using Eq. (4.93), one can obtain the following expression (Lee and Fenves, 2001):

\[
\frac{\partial F^p}{\partial \hat{\sigma}_{ij}^{n+1}} = \sqrt{2} \frac{3}{2} \hat{\varepsilon}_{mn} + \alpha^p_\sigma \delta_{ij} = \sqrt{2} \left( \frac{3}{2} \hat{\varepsilon}_{mn} + \alpha^p_\sigma \delta_{ij} \right) \Rightarrow
\]  \quad (4.106)

\[
\Rightarrow = \sqrt{2} \frac{3}{2} \hat{\varepsilon}_{mn} + \left( \alpha^p_\sigma - \sqrt{6} \frac{1}{\hat{\varepsilon}_{mn}} \right) \delta_{ij}
\]

where \( T_{ir}^{trial} = \hat{T}_{ir} \) and \( \| \hat{\varepsilon}_{ij} \| = \| \hat{\varepsilon}_{ij} \| = \sqrt{2(\hat{J}_2)^{trial}} \), \( T_{trial}^{trial} = \hat{\varepsilon}_{mn}^{trial} \).

Now, by substituting the final form of Eq. (4.106) into Eq. (4.103) one can obtain the following form:
\[
\hat{\sigma}_{ij}^{n+1} = \hat{\sigma}_{ij}^{trial} - \Delta \lambda^p \left[ 2G \left( \frac{3}{\sqrt{2}} \frac{\hat{\sigma}_{ij}^{trial}}{S_{mn}} \right) + \left( \alpha_p - \sqrt{\frac{1}{6}} \frac{T_1^{trial}}{S_{mn}} \right) \delta_y \right] + 3\left( K - \frac{2}{3} G \right) \alpha_p \delta_{ij} \\
\hat{\sigma}_{yy}^{n+1} = \hat{\sigma}_{yy}^{trial} - \Delta \lambda^p \left[ \sqrt{6} G \frac{\hat{\sigma}_{yy}^{trial}}{S_{mn}} \right] + 2\hat{\sigma}_{yy}^{trial} \delta_{ij} - \frac{2}{\sqrt{3}} \hat{G} \frac{T_1^{trial}}{S_{mn}} \delta_{ij} + 3\hat{K} \alpha_p \delta_{ij} - 2\hat{G} \alpha_p \delta_{ij} \\
\hat{\sigma}_{yy}^{n+1} = \hat{\sigma}_{yy}^{trial} - \Delta \lambda^p \left[ \sqrt{6} G \frac{\hat{\sigma}_{yy}^{trial}}{S_{mn}} \right] + \left( 3\hat{K} \alpha_p - \frac{2}{\sqrt{3}} \hat{G} \frac{T_1^{trial}}{S_{mn}} \right) \delta_{ij} \right] (4.107)
\]

In order to obtain the expression sought out of the spectral radial-return algorithm for the maximum principal stress \( \hat{\sigma}_{\max}^{n+1} \) used in the yield function, Eq. (4.67), one can contract the above relation such that:

\[
\hat{\sigma}_{\max}^{n+1} = \hat{\sigma}_{\max}^{trial} \left[ 2G \left( \frac{3}{\sqrt{2}} \frac{\hat{\sigma}_{\max}^{trial}}{S_{mn}} \right) + \left( \alpha_p - \sqrt{\frac{1}{6}} \frac{T_1^{trial}}{S_{mn}} \right) \delta_y \right] + 3\left( K - \frac{2}{3} G \right) \alpha_p \delta_{ij} \right] (4.108)
\]

It can be seen from Eqs. (4.103) and their final result, Eq. (4.108), that the eigenvectors are preserved throughout the corrector-steps which basically means that the effective stress eigenvectors are calculated at the predictor step and only the principal stress needs to be determined during the iterations of the plastic-corrector step (Lee and Fenves, 2001). It should also be noted that, if the eigenvalues of the plastic strain increment tensor, \( \Delta \hat{\varepsilon}_{ij}^p \), are obtained by a linear combination of \( \hat{\sigma}_{\max}^{n+1} \) and \( \delta_{ij} \), such as for the Drucker-Prager model, Eq. (4.73), the algebraic order of the effective (undamaged) principal stresses is preserved throughout the corrector-steps. This was shown by Lee and Fenves (2001) through checking Eq. (4.107) and realizing that the updated effective principal stress tensor \( \hat{\sigma}_{ij}^{n+1} \) is obtained only by a scalar multiplication and constant-tensor addition/subtraction on the trial stress. Therefore, the order of the diagonal entries in the trial stress tensor cannot be changed. This argument, however, is not valid for the case where the yield criterion \( f \) given in Eq. (4.67) is used as a plastic potential function. This means that if one takes the derivative with respect to the maximum stresses, \( \hat{\sigma}_{\max}^{n+1} \), the algebraic order in the eigenvalue tensor does not preserve the same order.

### 4.7.1.3 The Effective Configuration Integration Algorithm

The radial-return mapping algorithm is derived next in order to solve for the plastic multiplier \( \Delta \lambda^p \) using the yield function discussed in Eq. (4.67), which can be written here at the end of the \((n+1)\) step as:
\begin{equation}
\sqrt{\frac{3}{2}} \left\| \mathbf{S}_{ij}^{\text{max}} \right\| + \alpha T_{i}^{n+1} + \beta (\kappa^{\pm})^{n+1} H(\hat{\sigma}_{\text{max}}^{n+1}) \hat{\sigma}_{\text{max}}^{n+1} - (1 - \alpha) c^{-}[\kappa^{-}]^{n+1} = 0
\tag{4.109}
\end{equation}

Applying the plasticity consistency condition given in Eq. (4.75) to the previous equation yields the following:

\begin{equation}
f^{n+1} = f^{n} + \frac{\partial f}{\partial \hat{\sigma}_{ij}} \Delta \hat{\sigma}_{ij} + \frac{\partial f}{\partial \hat{\sigma}_{\text{max}}} \Delta \hat{\sigma}_{\text{max}} + \frac{\partial f}{\partial \kappa^{+}} \Delta \kappa^{+} + \frac{\partial f}{\partial \kappa^{-}} \Delta \kappa^{-} = 0
\tag{4.110}
\end{equation}

where \( \Delta \hat{\sigma}_{ij} \) is defined here in the incremental form using Eqs. (4.90) and (4.96) as:

\begin{equation}
\Delta \hat{\sigma}_{ij} = \mathbf{E}_{ijkl} \Delta \varepsilon_{kl} = \mathbf{E}_{ijkl} (\Delta \varepsilon_{kl} - \Delta \varepsilon_{kl}^{p}) = \mathbf{E}_{ijkl} \Delta \varepsilon_{kl} - \mathbf{E}_{ijkl} \Delta \lambda^{p} \frac{\partial F^{p}}{\partial \hat{\sigma}_{kl}}
\end{equation}

\begin{equation}
= \Delta \hat{\sigma}_{ij}^{\text{trial}} - \left[ 2G \Delta \varepsilon_{ij}^{p} + \left( K - \frac{2}{3} G \right) \Delta \varepsilon_{kl}^{p} \delta_{ij} \right] = \Delta \hat{\sigma}_{ij}^{\text{trial}} - \Delta \lambda^{p} \left[ \sqrt{6G} \frac{\hat{\sigma}_{ij}^{\text{trial}}}{\left\| \mathbf{S}_{ij}^{\text{trial}} \right\|} + 3K \alpha_{p} \delta_{ij} \right]
\tag{4.111}
\end{equation}

and the principal increment \( \Delta \hat{\sigma}_{\text{max}} \) can be obtained using Eq. (4.108) as follows:

\begin{equation}
\Delta \hat{\sigma}_{\text{max}} = \Delta \hat{\sigma}_{\text{max}}^{\text{trial}} - \Delta \lambda^{p} \left[ \sqrt{6G} \frac{\hat{\sigma}_{\text{max}}^{\text{trial}}}{\left\| \mathbf{S}_{\text{mn}}^{\text{trial}} \right\|} + \left( 3K \alpha_{p} - \frac{2}{3} G \frac{T_{\text{trial}}^{\text{trial}}}{\left\| \mathbf{S}_{\text{mn}}^{\text{trial}} \right\|} \right) \right]
\tag{4.112}
\end{equation}

The increments of the equivalent plastic strains \( \Delta \kappa^{\pm} \) are expressed using Eqs. (4.60), (4.61) and (4.104) as follows:

\begin{equation}
\Delta \kappa^{+} = r \Delta \lambda^{p} \frac{\partial F^{p}}{\partial \hat{\sigma}_{\text{max}}} \tag{4.113}
\end{equation}

\begin{equation}
\Delta \kappa^{-} = -(1 - r) \Delta \lambda^{p} \frac{\partial F^{p}}{\partial \hat{\sigma}_{\text{min}}} \tag{4.114}
\end{equation}

where \( r \) is defined in Eq. (4.62). Substituting Eqs. (4.111), (4.112), (4.113) and (4.114) into Eq. (4.110), one can obtain the following relation:
\[ f^{n+1} = f^n + \frac{\partial f}{\partial \sigma_{ij}} \left( \Delta \sigma_{ij} - \Delta \lambda^p \left[ \sqrt{6G} \frac{S_{ij}^{\text{trial}}}{\|S_{mn}^{\text{trial}}\|} + \frac{3K}{3} \right] \right) \]

\[ + \frac{\partial f}{\partial \sigma_{\max}} \left( \Delta \sigma_{\max} - \Delta \lambda^p \left[ \sqrt{6G} \frac{\hat{\sigma}_{\max}^{\text{trial}}}{\|S_{mn}^{\text{trial}}\|} + \left( \frac{2}{3} \right) \hat{T}_{\text{trial}} \right] \right) \]

\[ + \frac{\partial f}{\partial \kappa^-} \left( r \Delta \lambda^p \frac{\partial F^p}{\partial \sigma_{\min}} \right) + \frac{\partial f}{\partial \kappa^+} \left( -(1-r) \Delta \lambda^p \frac{\partial F^p}{\partial \sigma_{\min}} \right) = 0 \] (4.115)

The plastic potential function derivatives, \( \frac{\partial F^p}{\partial \sigma_{\max}} \) and \( \frac{\partial F^p}{\partial \sigma_{\min}} \), are given using Eq. (4.105) along with the Eq. (4.93) as functions of the trial state:

\[ \frac{\partial F^p}{\partial \sigma_{\max}} = \sqrt{2} \left( \frac{\hat{\sigma}_{\max}^{\text{trial}} - \frac{1}{3} \hat{T}_{\text{trial}}}{\|S_{mn}^{\text{trial}}\|} + \alpha_p \right) \] (4.116)

\[ \frac{\partial F^p}{\partial \sigma_{\min}} = \sqrt{2} \left( \frac{\hat{\sigma}_{\min}^{\text{trial}} - \frac{1}{3} \hat{T}_{\text{trial}}}{\|S_{mn}^{\text{trial}}\|} + \alpha_p \right) \] (4.117)

and the yield function derivatives, \( \frac{\partial f}{\partial \kappa^-} \) and \( \frac{\partial f}{\partial \kappa^+} \), are obtained using Eqs. (4.67), (4.69), (4.70), and (4.71) as follows:

\[ \frac{\partial f}{\partial \kappa^-} = -\frac{(1-\alpha)c^h}{(c^+)^2} \langle \hat{\sigma}_{\max}^{\tau+1} \rangle \] (4.118)

\[ \frac{\partial f}{\partial \kappa^+} = \frac{(1-\alpha)}{c^+} Q\omega \exp(-\omega \kappa^-) \langle \hat{\sigma}_{\max}^{\tau+1} \rangle - (1-\alpha)Q\omega \exp(-\omega \kappa^-) \] (4.119)

Rearranging Eq. (4.115) to obtain an expression for \( \Delta \lambda^p \) gives the following:

\[ \Delta \lambda^p = \frac{f^n + \frac{\partial f}{\partial \sigma_{ij}} \Delta \sigma_{ij}^{\text{trial}} + \frac{\partial f}{\partial \sigma_{\max}} \Delta \sigma_{\max}^{\text{trial}}}{H} \] (4.120)

where

\[ H = \frac{\partial f}{\partial \sigma_{ij}} \]
\[ H = \partial f / \partial \tilde{\sigma}_y \left[ \sqrt{6} \tilde{G} \frac{\tilde{S}_{ij}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} + 3K \alpha_p \delta_{ij} \right] \]

\[ + \partial f / \partial \tilde{\sigma}_{\text{max}} \left[ \sqrt{6} \tilde{G} \frac{\tilde{\sigma}_{\text{max}}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} + \left( 3K \alpha_p - \frac{2}{3} \tilde{G} \frac{T_{ij}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} \right) \right] \]  \hfill (4.121)

\[-r \partial f / \partial \kappa^+ \partial F^p / \partial \tilde{\sigma}_{\text{max}} + (1 - r) \partial f / \partial \kappa^- \partial F^p / \partial \tilde{\sigma}_{\text{min}} \]

Note that Eq. (4.121) can be further simplified by realizing that:

\[ \frac{\partial f}{\partial \tilde{\sigma}_y} = \sqrt{\frac{3}{2}} \frac{\tilde{S}_{ij}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} + \alpha \delta_{ij} \]  \hfill (4.122)

obtained by using Eq. (4.93) and

\[ \frac{\partial f}{\partial \tilde{\sigma}_{\text{max}}} = \beta (\kappa^\pm)^{n+1} H(\tilde{\sigma}_{\text{max}}^{n+1}) \]  \hfill (4.123)

where \((\kappa^\pm)^{n+1}\) are the updated plastic variables. Applying Eqs. (4.122) and (4.123) into Eq. (4.121) yields the following simplified expression for \(H\):

\[ H = 3\tilde{G} + 9K \alpha_p \alpha + \beta (\kappa^\pm)^{n+1} H(\tilde{\sigma}_{\text{max}}^{n+1}) \left[ \sqrt{6} \tilde{G} \frac{\tilde{\sigma}_{\text{max}}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} + \left( 3K \alpha_p - \frac{2}{3} \tilde{G} \frac{T_{ij}^{\text{trial}}}{\tilde{S}_{mn}^{\text{trial}}} \right) \right] \]

\[-r \partial f / \partial \kappa^+ \partial F^p / \partial \tilde{\sigma}_{\text{max}} + (1 - r) \partial f / \partial \kappa^- \partial F^p / \partial \tilde{\sigma}_{\text{min}} \]  \hfill (4.124)

It should be noted here that Eqs. (4.118), (4.119) and (4.123) are functions of the updated forms of the principal stress in tension \(\langle \tilde{\sigma}_{\text{max}}^{n+1} \rangle\) and the plastic variables \((\kappa^\pm)^{n+1}\). This is the reason why local iterations are required to obtain the plastic multiplier \(\Delta \lambda^p\). In each iteration the new evaluated trial stress will be used to update the plastic variables and the hardening functions in order to obtain an updated plastic multiplier. The process goes on until a convergence tolerance for the yield function is satisfied.

The numerator of Eq. (4.120) can be shown to be equal to \(f_{\text{trial}}\) during the iterative procedure by substituting Eqs. (4.92), (4.95), and (4.108) into Eq. (4.109) and considering only the terms that do not involve the plastic multiplier \(\Delta \lambda^p\) to obtain the following:
allowing Eq. (4.120) to be written during the iterative procedure as follows:

$$\Delta \lambda^n = \frac{f_{\text{trial}}}{H}$$

(4.126)

This concludes the elastic-plastic steps in the effective configuration. Upon convergence, the updated effective stress $\sigma_{ij}^{n+1}$ along with its updated elastic strain tensor $(\varepsilon_{ij})^{n+1}$ are now available to be used in the damage-corrector step in order to update the damage variables $(\varphi)^{n+1}$ (and therefore $\Phi^{n+1}$) and the Cauchy stress tensor $\sigma_{ij}^{n+1}$. The flowchart shown in Fig. 4.3 demonstrates the effective elastic-plastic integration procedure.

### 4.7.2 The Degradation (Damage) Step

The degradation process is termed explicit because the spectral stress ratios used in defining the damage parameter $\Phi$ are assumed to depend on the initial stress states of each degradation increment such that Eq. (4.43) can now be written as:

$$\Phi = \frac{\|\sigma^{+}_{ij}\| \varphi^{+} + \|\sigma^{-}_{ij}\| \varphi^{-}}{\|\sigma_{ij}\|}$$

(4.127)

where $\sigma^{+}_{ij}$ and $\sigma^{-}_{ij}$ are the positive and negative spectral decomposition parts of the effective stress tensor, $\sigma_{ij}$; another modification to the Tao and Phillips (2005) definition for $\Phi$, since their model (only elastic-damage analysis) did not include the plastic effect. Nevertheless, satisfying the damage criteria and updating the damage variables are accomplished by using the projected values (n+1) of the effective elastic strain tensor. Lee and Fenves (2001) showed that the eigenvectors of the current stress $\sigma_{ij}^{n+1}$ are the same as those of the effective stress $\sigma_{ij}^{n+1}$ due to the fact that scalar degradation damage is assumed. Therefore, at the degradation corrector step, the final form of the stress tensor $\sigma_{ij}^{n+1}$ is obtained by computing the degradation damage variable $\Phi$, such that:

$$\sigma_{ij}^{n+1} = (1 - \Phi)\sigma_{ij}^{n+1}$$

(4.128)

### 4.7.2.1 The Damage - Corrector Step

This step starts after the effective elastic-plastic steps pass the updated forms of the effective stress tensor $\sigma_{ij}^{n+1}$ and its corresponding elastic strain tensor $(\varepsilon_{ij})^{n+1}$. The spectral decomposition of the updated effective stress tensor $\sigma_{ij}^{n+1}$ into positive and
negative parts is then performed according to the procedure shown in Section 4.4.1. These spectral components are then used to calculate the damage release rates (damage thermodynamic conjugate forces) $Y^+$ and $Y^-$ by substituting Eq. (4.54) into Eqs. (4.55) and (4.56) to obtain the following two equations:

\[
Y^+ = \frac{1}{2} \left( \frac{\sigma_y}{\varepsilon_{ij}} \right)^{n+1} \left[ (\varepsilon_{ij})^{n+1} \bar{E}_{ijkl} (\varepsilon_{kl})^{n+1} \right] = \frac{1}{9} \left[ 1 + c Y^+ \exp(-dY^+) \right] \left( (\varepsilon_{mm})^{n+1} \right)^{2} \delta_{ij} \bar{E}_{ijkl} \delta_{kl} \tag{4.129}
\]

\[
Y^- = \frac{1}{2} \left( \frac{\sigma_y}{\varepsilon_{ij}} \right)^{n+1} \left[ (\varepsilon_{ij})^{n+1} \bar{E}_{ijkl} (\varepsilon_{kl})^{n+1} \right] = \frac{1}{9} \left[ 1 + c Y^- \exp(-dY^-) \right] \left( (\varepsilon_{mm})^{n+1} \right)^{2} \delta_{ij} \bar{E}_{ijkl} \delta_{kl} \tag{4.130}
\]

Equations (4.129) and (4.130) are nonlinear functions of the conjugate forces $Y^+$ and $Y^-$, respectively. Therefore, Newton-Raphson iterative procedure is used in order to solve each of these two equations independently. The procedure is demonstrated next for the thermodynamic force $Y^i$ for expediency, where $i$ represents (+) or (−) such that:

\[
Y^i - \frac{1}{2} \left( \frac{\sigma_y}{\varepsilon_{ij}} \right)^{n+1} \left[ (\varepsilon_{ij})^{n+1} \bar{E}_{ijkl} (\varepsilon_{kl})^{n+1} \right] = \frac{1}{9} \left[ 1 + c Y^i \exp(-dY^i) \right] \left( (\varepsilon_{mm})^{n+1} \right)^{2} \delta_{ij} \bar{E}_{ijkl} \delta_{kl} = 0 \tag{4.131}
\]

Equation (4.131) can be written as $K(Y^i) = 0$ with function roots $Y^i$ obtained using the Newton-Raphson iterative technique as follows:

\[
Y^i_{m+1} = Y^i_m + \Delta Y^i = Y^i_m - K(Y^i_m) \left[ \frac{\partial K(Y^i)}{\partial Y^i} \right] \tag{4.132}
\]

where the derivative $\frac{\partial K(Y^i)}{\partial Y^i}$ is given as follows (evaluated at $Y^i = Y^i_m$):

\[
\frac{\partial K(Y^i)}{\partial Y^i} = 1 - \frac{1}{2} \left( \frac{\sigma_y}{\varepsilon_{ij}} \right)^{n+1} \left[ \frac{1}{9} \left( \frac{c \exp(-dY^i) - c dY^i \exp(-dY^i)}{1 + c Y^i \exp(-dY^i)} \right) \left( (\varepsilon_{mm})^{n+1} \right)^{2} \delta_{ij} \bar{E}_{ijkl} \delta_{kl} \right] \tag{4.133}
\]

Since the outcome of this iterative process is highly dependent on the “guess-value” initially suggested for $Y^i_m = Y^i_0$, the following form of $Y^i_0$ was observed in this study to be convenient:

\[
Y^i_0 = \frac{1}{2} \left( \frac{\sigma_y}{\varepsilon_{ij}} \right)^{n+1} \left[ (\varepsilon_{ij})^{n+1} \bar{E}_{ijkl} (\varepsilon_{kl})^{n+1} \right] \tag{4.134}
\]
The iterative procedure is terminated upon convergence of a tolerance criterion (e.g. \(\Delta Y' \approx 0\)).

Now that the thermodynamic conjugate forces are determined, \(Y^+\) and \(Y^-\) are substituted into Eqs. (4.76) in order to check the activation of the tensile and compressive damage yield criteria \(g^+\) and \(g^-\), respectively:

\[
g^+ = (Y^+)^{n+} - Y_0^+ - (Z^+)^n \leq 0
\]

(4.135)

If \(g^+ < 0\) and \(g^- < 0\), the damage-corrector step ends and the Cauchy stress tensor takes the form of the updated effective stress tensor. If either of the damage criteria is activated, \(g^+ > 0\), damage evolution takes place and the Cauchy stress tensor is evaluated using Eq. (4.128). An integration algorithm is thus devised as demonstrated here to obtain the updated damage variables. These updated damage variables, along with the thermodynamic conjugate forces \(Y^+\) and \(Y^-\), should satisfy the damage consistency conditions shown in Eq. (4.79), such that:

\[
(Y^+)^{n+} - Y_0^+ - (Z^+)^{n+} = 0
\]

(4.136)

where \((Z^+)^{n+}\) are given using Eq. (4.77) at the \((n+1)\) step as:

\[
(Z^+)^{n+} = \frac{1}{a^+} \left( \phi^+ \right)^{n+} \left( 1 - (\phi^+)^{n+} \right)^{1/b^+}
\]

(4.137)

Substituting Eq. (4.137) into Eq. (4.136), explicit expressions for the damage variables \((\phi^+)^{n+}\) satisfying the consistency conditions can be obtained as follows:

\[
(\phi^+)^{n+} = \frac{a^+ \left[ (Y^+)^{n+} - Y_0^+ \right]^{b^+}}{1 + \left[ a^+ \left[ (Y^+)^{n+} - Y_0^+ \right] \right]^{b^+}} = 1 - \frac{1}{1 + \left[ a^+ \left[ (Y^+)^{n+} - Y_0^+ \right] \right]^{b^+}}
\]

(4.138)

These updated damage variables can now be used to evaluate the combined scalar damage variable \(\Phi\) using Eq. (4.127). The later can then be used to update the Cauchy stress tensor as shown in Eq. (4.128). This concludes the damage-corrector step. Figure 4.3 shows an entire step of the integration scheme demonstrating the effective elastic-predictor plastic-corrector steps followed by the damage-corrector step.

4.8 The Consistent Elastic – Plastic – Damage Tangent Operator

For the global equilibrium, solved by ABAQUS according to a Newton–Raphson algorithm, a consistent tangent operator is computed according to the procedure described in Jason et. al. (2006). It is formulated by applying the derivative of the constitutive
equation, Eq. (4.7), with respect to the strain tensor as follows (all parameters are at the (n+1) state):

\[
\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = (1 - \Phi) \frac{\partial \bar{\sigma}_{ij}}{\partial \varepsilon_{kl}} - \frac{\partial \Phi}{\partial \varepsilon_{kl}} \bar{\sigma}_{ij} \\
(4.139)
\]

where the elasto-plastic tangent operator appears in the first term on the right hand side of this equation. Since damage depends on the elastic strain only, \(\frac{\partial \Phi}{\partial \varepsilon_{kl}}\) can be written as:

\[
\frac{\partial \Phi}{\partial \varepsilon_{kl}} = \frac{\partial \Phi}{\partial \varepsilon_{mn}^e} \frac{\partial \varepsilon_{mn}^e}{\partial \varepsilon_{kl}} \\
(4.140)
\]

and the derivative of the elastic strain tensor with respect to the total strain tensor can be obtained by taking the derivative of the constitutive equation, Eq. (4.3), with respect to the total strain tensor as follows:

\[
\frac{\partial \varepsilon_{mn}^e}{\partial \varepsilon_{kl}} = (E^{-1})_{mnpq} \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}} \\
(4.141)
\]

Substituting Eqs. (4.140) and (4.141) into Eq. (4.139), the following arrangement can be obtained:

\[
\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = (1 - \Phi) \frac{\partial \bar{\sigma}_{ij}}{\partial \varepsilon_{kl}} - \frac{\partial \Phi}{\partial \varepsilon_{mn}^e} (E^{-1})_{mnpq} \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}} - \bar{\sigma}_{ij} \\
(4.142)
\]

After some tensorial manipulations, Eq. (4.142) can be given as:

\[
\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \left[ (1 - \Phi) \delta_{pi} \delta_{qj} - \frac{\partial \Phi}{\partial \varepsilon_{mn}^e} (E^{-1})_{mnpq} \bar{\sigma}_{ij} \right] \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}} \\
(4.143)
\]

Equation (4.143) is equivalent in format to that given by Wu et. al. (2006) as follows:

\[
\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \left[ (1 - \Phi) \delta_{pi} \delta_{qj} - \frac{\partial \Phi}{\partial \varepsilon_{mn}^e} (E^{-1})_{mnpq} \bar{\sigma}_{ij} \right] \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}} \\
(4.144)
\]

In order to obtain \(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\), the derivative of the damage variable with respect to the elastic strain tensor, \(\frac{\partial \Phi}{\partial \varepsilon_{mn}^e}\), needs to be evaluated. This can be accomplished by considering the
Given $\Delta \varepsilon$

\[
\bar{\sigma}_{\text{trial}} = \bar{\sigma} + E \Delta \varepsilon \\
\Delta \lambda_0 = 0
\]

If $f(\bar{\sigma}_{\text{trial}}) < 0$

Update $\Delta \lambda_0 = \Delta \lambda_0 + \frac{f_{\text{trial}}}{H}$

Calculate $\Delta \bar{\varepsilon}^p$, $(\kappa^\pm)$
Update $\bar{\sigma}_{\text{trial}} = \bar{\sigma}_{\text{trial}} - E \Delta \bar{\varepsilon}^p$
Update hardening functions $c^+$, $c^-$

If $f(\bar{\sigma}_{\text{trial}}) = 0$

Update $(\bar{\varepsilon}^p)^{n+1}$, $(\bar{\varepsilon}^s)^{n+1}$
Update $\bar{\sigma}^{n+1} = \bar{\sigma}_{\text{trial}}$

Evaluate $(\bar{\sigma}_\theta)^{n+1}$, $(Y^\pm)^{n+1}$
derivative of Eq. (4.127) as follows:

\[
\frac{\partial \Phi}{\partial \varepsilon_{mn}^e} = \frac{\|\sigma^\|_I}{\|\sigma^\|_I} \frac{\partial \varphi^+}{\partial \varepsilon_{mn}^e} + \frac{\|\sigma^\|_I}{\|\sigma^\|_I} \frac{\partial \varphi^-}{\partial \varepsilon_{mn}^e} \quad (4.145)
\]

Note that the stresses in Eq. (4.127) are used to obtain scalar ratios that are used as weighing factors; therefore, they are not considered as contributing components to the foregoing derivative. Using Eq. (4.138), it can be seen that the damage variables \( \varphi^\pm \) are functions of the thermodynamic conjugate forces \( Y^\pm \), respectively, and \( Y^\pm \) are functions of the elastic strain tensor \( \varepsilon_{mn}^e \), Eqs. (4.129) and (4.130), such that the following expression can be obtained:

\[
\frac{\partial \varphi^\pm}{\partial \varepsilon_{mn}^e} = \frac{\partial \varphi^\pm}{\partial Y^\pm} \frac{\partial Y^\pm}{\partial \varepsilon_{mn}^e} \quad (4.146)
\]

The first term on the right hand side is given as:
\[
\frac{\partial \varphi^\pm}{\partial Y^\pm} = \frac{a^\pm b^\pm \left(a^\pm \left[Y^\pm - Y_0^\pm\right]\right)^{(b^\pm)-1}}{1 + \left(a^\pm \left[Y^\pm - Y_0^\pm\right]\right)^{b^\pm}} \tag{4.147}
\]

and the second term is obtained by applying the linearization technique given by (Simo and Hugues, 1998) using Eqs. (4.129) and (4.130) as (4.152):

\[
\frac{\partial Y^\pm}{\partial e^e_{mn}} = \frac{1 - \frac{1}{2} \frac{\left[\overline{\sigma}^e\right]}{\overline{\sigma}^e} \left[1 + \frac{c \exp(-dY^\pm) - c dY^\pm \exp(-dY^\pm)}{\left(1 + c Y^\pm \exp(-dY^\pm)\right)^2} \right]}{9 \left(e^e_{qq}\right)^2 \delta_j \overline{E}_{ijkl} \delta_{kl}} \tag{4.148}
\]

where the derivatives of \(\beta^\pm\) with respect to the elastic strain tensor were expressed as follows:

\[
\frac{\partial \beta^\pm}{\partial e^e_{mn}} = \frac{\partial \beta^\pm}{\partial Y^\pm} \frac{\partial Y^\pm}{\partial e^e_{mn}} \tag{4.149}
\]

The effective consistent tangent operator \(\frac{\partial \overline{\sigma}}{\partial e_{kl}}\) can be obtained using the linearization technique given in the references mentioned above and is stated here as given by Wu et. al. (2006) as:

\[
\frac{\partial \overline{\sigma}}{\partial e_{kl}} = \left(\overline{E}^{-1}\right)_{ijkl} + \frac{\partial F^p}{\partial \overline{\sigma}_{ij}} \frac{\partial \Delta \lambda^p}{\partial \overline{\sigma}_{kl}} + \Delta \lambda^p \frac{\partial^2 F^p}{\partial \overline{\sigma}_{ij} \partial \overline{\sigma}_{kl}} \tag{4.150}
\]

where \(\frac{\partial \Delta \lambda^p}{\partial \overline{\sigma}_{kl}}\) is given as:

\[
\frac{\partial \Delta \lambda^p}{\partial \overline{\sigma}_{kl}} = - \frac{\partial f}{\partial \Delta \lambda^p} \frac{\partial \Delta \lambda^p}{\partial \overline{\sigma}_{kl}} \tag{4.151}
\]

where \(\overline{\sigma}_i = \overline{\sigma}_{\text{max}}\) when \(\kappa = \kappa^+\) and \(\overline{\sigma}_i = \overline{\sigma}_{\text{min}}\) when \(\kappa = \kappa^-\).
4.9 Implementation and Verification of the Integration Scheme

This section is dedicated to the numerical validation of the concrete model. The numerical algorithm of the proposed model was implemented in the non-linear FE code ABAQUS via the user material subroutine UMAT. Several analytical examples are provided here in order to investigate the capability, applicability, and effectiveness of the proposed elastic-plastic-damage model in capturing material behavior in both tension and compression under uniaxial and bi-axial loadings. The results obtained by the proposed model are compared with corresponding experimental results to evaluate the model’s performance. Tensile and compressive verification tests under unaxial loading are demonstrated first, followed by biaxial tests in tension and compression. Then a three point bending test of a notched beam is investigated.

4.9.1 Identification of the Proposed Model’s Parameters

The proposed model contains 17 parameters: two elastic constants for the undamaged material ($E$ and $\nu$), five parameters for the characterization of plasticity ($\alpha, \alpha^p, h, Q$ and $\omega$), eight parameters for damage characterization ($a^z, b^z, Y_0^z, c$ and $d$), and two parameters for the fracture energy of concrete under tension and compression ($G^T$). All the parameters can be identified from a series of tensile and compressive experimental tests (Tao and Phillips, 2005 and Wu et al., 2006). Since the elastic-plastic-damage model presented in this work is a combination of the effective elastic-plastic constitutive relations presented by (Lee and Fenves, 1998 and Wu et. al., 2006) and the damage model presented by (Tao and Phillips, 2005); the model parameters used in these works are used here with some adjustment to the damage parameters in order to take into consideration the plastic effect introduced in this work.

The initial elastic constants are determined from the linear part of stress–strain curves before the initiation of damage and plastic deformation (Shao et al., 2006). For concrete materials, these averaged elastic parameters are documented in literature. The initial damage thresholds in tension and compression, $Y_0^z$, can be determined by locating the onset points of variation of elastic properties in the unloading paths (Tao and Phillips, 2005). As the damage evolution is coupled with plastic flow, it seems to be reasonable to consider that the damage initiation occurs at the same time as the plastic initiation under tensile loading. They are identified as the end of the linear part of the stress–strain curves in uniaxial tensile tests. Under compression, however damage start at an earlier stage than plasticity depending on the initial damage threshold $Y_0^-$. The plastic hardening parameters ($Q$ and $\omega$) are related to the saturated stress in the plastic regime and the rate of saturation.

Lee and Fenves (1998) and Wu et al. (2006) used the following values for the elastic-plastic material parameters: Poisson’s ratio $\nu = 0.20$; the equibiaxial to uniaxial compressive strength ratio $f_0^e / f_0^u = 1.16$, resulting with $\alpha = 0.12$, and the dilatancy parameter $\alpha^p$ was chosen as 0.20. Tao and Phillips, (2005), used the following values for
the material parameters used to split the strain tensor into hydrostatic and deviatoric components: \( c = 2.0 \text{ MPa}^{-1} \) and \( d = 0.7 \text{ MPa}^{-1} \). These values are used throughout this work.

4.9.2 Monotonic Uniaxial Tensile Test

In the first example of uniaxial tension test, the following material properties are used (Lee and Fenves, 1998; Tao and Phillips, 2005; Wu et al., 2006; Nguyen and Houlsby, 2008a,b) in order to compare the results with the experimental work of Gopalaratnam and Shah (1985): \( E = 3.1 \times 10^4 \text{ MPa}, \ f_0^+ = 3.48 \text{ MPa} \) and \( G_f^+ = 40 \text{ N/m} \). This test is conducted using a single quadrilateral finite element (82.6 mm x 82.6 mm) shown in Fig. 4.5a to comply with the results of the studies mentioned above. The model’s plastic hardening parameter is given here for the tensile case as: \( h = 2.5 \times 10^4 \text{ MPa} \). The model’s damage parameters are provided by Tao and Phillips, (2005) as: \( b^+ = 1.2, Y_0^+ = 1.9 \times 10^{-4} \text{ MPa} \). Only parameter \( a^+ = 14 \times 10^3 \text{ MPa}^{-1} \) was adjusted to account for the plastic effect introduced in this work. Note that the uniaxial tests (in tension and compression) were used to determine the values of the material parameters to be used along with the fracture energy related factors \( \gamma^\pm \) in further examples. The use of the fracture energy factors \( \gamma^\pm \)
Figure 4.6 The behavior of the proposed model under uniaxial tension

Figure 4.7 Effect of material parameters on the model response in tension

in the verification problems is arbitrary and redundant as explained in (Nguyen and Houlsby, 2008a,b).
The stress-strain response is plotted in Fig. 4.6a while damage evolution is plotted in Fig 4.6b. It can be observed from Fig. 4.6a that the predictions obtained from the numerical model agree well with the experimental data (Gopalaratnam and Shah, 1985), especially for the post-peak nonlinear softening branches.

The effect of the model parameters mentioned above on the stress–strain response and damage evolution in tension is shown in Fig. 4.7a-d. Each model parameter in turn is varied, while others are kept fixed, to show the corresponding effect on the stress–strain curve and damage evolution.

Note in Fig. 4.7a, that as the plastic hardening parameter $h$ decreases, the plastic behavior becomes more dominant than the elastic one, resulting in smaller elastic strains, which affect the magnitude of the tensile damage release rate $Y^+$ and therefore negatively affect the damage growth. This shows the coupled effect of damage and plasticity on the predicted behavior.

### 4.9.3 Monotonic Uniaxial Compressive Test

The model’s ability to reproduce the concrete behavior under monotonic uniaxial compression is verified here and compared to the experimental results of Karsan and Jirsa (1969). The material properties used here are (Lee and Fenves, 1998, Tao and Phillips, 2005; Wu et. al., 2006; Nguyen and Houlsby, 2008a,b): $E = 3.1 \times 10^4$ MPa, $f_0^{-} = 10.2$ MPa, $f_c^{-} = 27.6$ MPa and $G_f^{-} = 5690$ N/m. The test is conducted using a single quadrilateral finite element (82.6 mm x 82.6 mm) shown in Fig. 4.5c. The model’s plastic hardening parameters are given here for the compressive case as: $Q = 2.5 \times 10^3$ MPa and $\omega = 200$. The model’s damage parameters are provided as: $a^{-} = 22$ MPa$^{-1}$, $b^{-} = 0.98$, $Y_0^{-} = 3.0 \times 10^{-4}$ MPa$^{-1}$. The model’s parameters $a^{-}$ and $b^{-}$ were adjusted in order to account for the introduction of the plastic effect in this work. The stress-strain response is

![Figure 4.8](image)
Figure 4.9 Effect of material parameters on the model response in compression plotted in Fig. 4.8a while damage evolution is plotted in Fig 4.8b. Whether in the hardening or in the softening regimes, the overall nonlinear numerical performance
predicted by the model and the experimentally obtained stress–strain curve are rather close.

The effect of model parameters on the stress–strain response and damage evolution in compression is shown in the Fig. 4.9. Each model parameter in turn is varied, while others are kept fixed, to show the corresponding effect on the stress–strain curve and damage evolution; Fig. 4.9a-d:

Figures 4.9b shows a trend similar to that observed in Fig. 4.7a. As the magnitudes of the hardening parameters increase, the damage growth and thus the strain softening is more pronounced up to the point where the exponential hardening function becomes saturated. This again shows the coupled effect of damage and plasticity on the response of the proposed model.

4.9.4 Monotonic Biaxial Tests

In this section, the performance of the proposed model subjected to combined loading situations (biaxial tension, biaxial compression, and biaxial tension – compression) is investigated.

In the biaxial tension case, the same material parameters as those for the uniaxial tension test are used to analyze the Quadrilateral FE setup shown in Fig. 4.5b. The numerical results are compared to the experimental ones reported by Kupfer et. al. (1969). In Fig. 4.10a, the ordinate represents the normalized stress $\sigma_{11}$ in terms of the compressive strength $f'_c = 27.6$ MPa for the case ($R = \sigma_{22} / \sigma_{11} = 1$). Note that the results of Kupfer et. al. (1969) cover only the range of ($0 \leq \sigma_{11} / f'_c \leq 0.09$). The results are in good agreement. The full range of the stress ratio ($0 \leq R \leq 1$) is investigated next.

![Figure 4.10 The behavior of the proposed model under biaxial tension ($\sigma_{11} = \sigma_{22}$)](image)
During the biaxial tension test, the total displacements in the horizontal and vertical directions of the setup shown in Fig. 4.5b are specified in the input file. In order to retrieve the full spectrum of stress ratios ($0 \leq R \leq 1$), the displacement in one direction is fixed while the displacement in the other direction is incremented during multiple runs of the input file. The results are shown in Table 4.1.

![Figure 4.11 The tensile quadrant of the biaxial failure surface of concrete](image)

The last two columns of Table 4.1 are plotted against each other to obtain Fig. 4.11 showing the tensile quadrant of the biaxial failure surface of concrete. The experimental values were obtained by Kupfer et. al. (1969) and used by Lee and Fenves (1998) and Wu et. al. (2006).

The tensile biaxial results are compared to those of the uniaxial tension test. Fig 4.12a shows that under the biaxial state ($\sigma_{11} = \sigma_{22}$), the model predicts higher damage growth.
rate than under uniaxial tension. This is also obvious from Fig. 4.12b, where damage starts at an earlier stage and grows at a higher rate.

![Stress - Strain curves](image1)

![Strain vs. Damage Variables](image2)

Figure 4.12 Comparison of the uniaxial and biaxial ($\sigma_{11} = \sigma_{22}$) tension tests

In the case of biaxial compression, the proposed model in its given form is not capable of capturing the trend observed experimentally. A modification that had to be applied to the proposed model in order to enhance its performance under biaxial compression is discussed next. This modification is easily incorporated into the UMAT file using proper IF statements. Experimental results (e.g. Kupfer et. al. (1969)) showed an increase in concrete compressive strength as the biaxial stress ratio ($R = \sigma_{22} / \sigma_{11}$) increases up to the point where the strength in one direction is $1.3 f'_c$, followed by a reduction in strength that reaches $1.16 f'_c$ when $R = 1.0$ (see Fig. 4.13). This is a result of the consolidation of concrete under biaxial compressive loading which leads to reduced damage growth (Wu et. al., 2006). This experimental observation is modeled here through the reduction of the damage encountered by concrete as the stress ratio $R$ is increased. Since the compressive damage parameter $a^{-}$ is responsible for the magnitude of damage endured by concrete as

![The compressive quadrant of the biaxial concrete envelop](image3)

Figure 4.13 The compressive quadrant of the biaxial concrete envelop
was shown in Fig. 4.9c, this parameter is related to the biaxial strain ratio \( \varepsilon_{22}/\varepsilon_{11} \) in an effort to account for damage reduction during biaxial compressive loading in a displacement-controlled environment. The same material parameters as those used for the uniaxial compressive test are used here with \( f^0_c = 15.2 \) MPa (Wu et. al., 2006). The FE setup shown in Fig. 4.5d was used.

Under uniaxial compressive loading, the damage parameter \( a^- = 22 \) MPa\(^{-1} \) was shown to give good experimental fit (Fig. 4.8a, \( f'_c = 27.6 \) MPa). Whereas, under biaxial compression loading with a stress ratio of \( R = 1 \), \( a^- = 11 \) MPa\(^{-1} \) was observed to give acceptable results (\( f'_bc = 32 \) (1.16*27.6) MPa) as shown in Fig. 4.14. The experimental results are those of Kupfer et. al. (1969).

![Figure 4.14: The behavior of the proposed model under biaxial compression (\( \sigma_{11} = \sigma_{22} \))](image1)

![Figure 4.15: Relation between damage hardening (Z\(^-\))/growth (\( \Phi \)) and the damage hardening parameter \( a^- \)](image2)

It is worth mentioning here that the relation between damage growth and the damage parameter \( a^- \) is a proportional relation; the higher the value of \( a^- \), the higher is the magnitude of damage. For example, if the damage magnitude is fixed at \( \Phi = 0.3 \), and
$a^-$ is increased from 11 to 22 MPa$^{-1}$, Fig. 4.15a can be obtained to show the damage hardening function $Z$ (MPa) plotted against $a^-$. Furthermore, if some arbitrary value is assigned to the damage release rate, say $Y^- = 0.01$ MPa, and the value of $a^-$ is increased from 11 to 22 MPa$^{-1}$ again, then damage growth can be plotted against $a^-$ as shown in Fig. 4.15b.

By checking the biaxial compression stress envelop of concrete, Fig. 4.13, one can easily see that when the stress ratio $R$ is zero, the point on the envelop is (-1,0) and the consolidation effect doesn’t exist. On the other hand, the consolidation effect starts to increase as $R$ moves away from (-1,0) towards higher values of $R$ up to a certain point where the consolidation effect starts to decrease until the stress ratio $R = 1$ is reached. By plotting the values of the strain ratio $\varepsilon_{22}/\varepsilon_{11}$ versus different power evolution equations of the damage parameter $a^-$, it was realized that the equation that best describes the evolution of $a^-$ with respect to $R$ is the one given as follows:

$$a^- = 22 - 11(\frac{\varepsilon_{22}}{\varepsilon_{11}})^n$$

(4.152)

Figure 4.16a shows different curves for the evolution of $a^-$ obtained by plotting Eq. (4.152) over the full range of the strain ratio $\varepsilon_{22}/\varepsilon_{11}$ using different values of the power parameter (n). Figure 4.16b, on the other hand, shows a symmetric half of the biaxial compressive quadrant of the concrete envelop. The results in Fig. 4.16b are obtained by varying the horizontal and vertical displacements in the input file during multiple runs in order to obtain different stress ratios and their corresponding values of the damage variable $a^-$. Table 4.2 shows the numerical values for the series of runs used to obtain the curve representing $(n = 1/6)$, which is the closest fit to experimental results reported by Kupfer et. al., (1969) as shown in Fig. 4.17.
Table 4.2 Biaxial Compression Test (n = 1/6)

<table>
<thead>
<tr>
<th>$\varepsilon_{22}/\varepsilon_{11}$</th>
<th>$\sigma_{11}$</th>
<th>$\sigma_{22}$</th>
<th>$\sigma_{22}/\sigma_{11}$</th>
<th>$\sigma_{11}/f'_c$</th>
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<td>1</td>
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</tr>
</tbody>
</table>

Figure 4.17 The compressive quadrant of the biaxial concrete envelop

Figure 4.18 The mixed (tension/compressive) quadrant of the biaxial concrete envelop

In the case of biaxial tension and compression, the model did not fit the experimental results, even with the modification used for the biaxial compression case mentioned above. The proposed model underestimates the strength of concrete under combined...
tension and compression (Fig. 4.18). Varying the damage parameters $a^+$ and $a^-$ with the strain ratio $\varepsilon_{22}/\varepsilon_{11}$ did not result in any improvements. This region of the concrete strength envelop still requires further research in order for the proposed model to adequately describe the biaxial tensile-compressive behavior of concrete. The results shown in Fig. 4.18 were obtained by varying the magnitude of the damage release rates $Y^\pm$ to account for the different tensile and compressive contributions to damage growth.

Despite the difference between the numerical and experimental results observed here, it is worth mentioning that the combined damage variable $\Phi$ in the mixed quadrants of the concrete envelope is made out of two parts, $\varphi^+$ and $\varphi^-$, unlike the other two quadrants (pure tensile or compressive quadrants). These two damage variables, $\varphi^+$ and
\( \varphi^- \), are combined using Eq. (4.127) to obtain the total damage variable \( \Phi \) (Fig. 4.19), where the latter is used to update the Cauchy stress tensor.

Combining the results from different quadrants discussed above, the biaxial strength envelop of concrete can be plotted as shown in Figure 4.20. The experimental results pertain to Kupfer et. al., (1969).

### 4.9.5 Three Point Bending Test of a Notched Beam

In this section, the monotonic testing of the damaging process for a single-edge-notched plain concrete beam is simulated using experimental data from Malvar and Warren (1988). The square-cross-section beam, with an initial notch depth of 51 mm, is subjected to three-point loading test as illustrated in Fig. 4.21a. The following material properties are used (Lowes, 1999; Lee and Fenves, 2001): \( \bar{E} = 2.17 \times 10^4 \) MPa, \( f'_c = 29.0 \) MPa, \( f'_o = 2.4 \) MPa, \( G'_j = 35 \) N/m. Two-dimensional FE mesh for the symmetric left part of the specimen is shown in Fig. 4.21b. Displacement control is used to apply the loading in the simulation.

![Figure 4.21 Single-edge-notched beam subjected to three point bending test](image)

**Figure 4.21** Single-edge-notched beam subjected to three point bending test

![Figure 4.22 Load versus load-point deflection compared to experimental results](image)

**Figure 4.22** Load versus load-point deflection compared to experimental results
Applying the material parameters given above along with the characteristic length of the FE mesh given by ABAQUS as \( \ell = 12.6e^{-3} \text{m} \) into Eq. (4.78), the mesh sensitivity parameter \( \gamma^+ \) can be calculated and the adjusted value for the damage parameter \( a^+ = 1400 \text{MPa}^{-1} \) can be obtained. Since the elastic modulus, \( E = 2.17 \times 10^4 \text{MPa} \), in this test is lower than that considered in the uniaxial test, the tensile plastic hardening parameter \( h \) is reduced here to \( h = 1.9 \times 10^4 \text{MPa} \) to maintain the same ratio \( (h/E) \). In Fig. 4.22, the load versus load-point deflection curve from the simulation is compared with the experimental result of Malvar and Warren (1988). Good agreement between the numerical and experimental results is observed, which demonstrates the effectiveness of the proposed model.

It should be noted here that the non-smoothness of the curve beyond the peak load is a phenomenon frequently observed in any numerical simulations that do not involve a regularization technique (Feenstra, 1993; Fichant et. al., 1999; Lowes, 1999; Tikhomirov and Stein, 2001; Sumarac et. al., 2003; Rabczuk et. al., 2005; He et. al., 2006; Nguyen and Houlsby, 2008a,b; Yu et. al., 2008; and others). He et. al. (2006) reported that the load-displacement curve beyond the maximum load and during the softening region, may reach a valley which is then followed by a new local peak in the next increment. These local peaks, if not robustly tolerated and accounted for, will cause the algorithm to diverge.

The evolution of damage is demonstrated next using the two dimensional FE mesh shown in Fig. 4.21b. By applying a displacement control of -0.5 mm to the point of load application through 100 time increments, Figs. 4.23, 4.24 and 4.25 show how damage propagates starting at the tip of the notch working its way toward the top of the cross-section.

![Figure 4.23 Evolution of damage (\( \Phi \)) at 20 time increments](image)
Figure 4.23 Evolution of damage (Φ) at 50 time increments

Figure 4.23 Evolution of damage (Φ) at 100 time increments
CHAPTER 5

FINITE ELEMENT ANALYSIS OF REINFORCED CONCRETE BEAMS

5.1 Introduction

Since Ngo and Scordelis published their landmark paper on RC beams in 1967, important progress has been made in the FE-based numerical analysis of RC structures. However, despite this progress, modeling the mechanical behavior of RC is still one of the most difficult challenges in the field of structural engineering. This is due to the inherent complexity and uncertainty concerning the properties of concrete, which make it excessively difficult to develop accurate constitutive models or algorithms that are sufficiently robust to obtain reliable and converged solutions in numerical analyses.

For time-independent problems, the nonlinearity of RC structures is caused by at least the following five factors (He et. al., 2006): the cracking of the concrete, the aggregate interlocking of the cracked concrete, the plasticizing and softening of the compressive concrete, bond slip between the steel and the concrete, and the dowel action of the steel reinforcement. A constitutive model that is able to capture all of these structural nonlinearities and their microscopic as well as macroscopic effects is complex and perhaps computationally inadequate. It was mentioned in the introduction (Chapter 1) that this is not the intended purpose of the proposed model. Because all of the above mentioned mechanisms are interacting, it is not realistic to try to formulate a constitutive model which incorporates all these mechanisms, but a model has to be formulated to adequately describe the behavior of a structure within the range of application which has been restricted in advance. Although the constitutive models which are developed within this phenomenological approach are usually simplified representations of the real behavior of material, it is believed that more insight can be gained by tracing the entire response of a structure in this manner, than modeling a structure with highly sophisticated material models which do not result in a converged solution after failure loads and are computationally expensive and complicated.

In order to ensure that the serviceability requirements are met in any RC structure, it is necessary to predict the cracking and the deflection of the structure under service loads. Therefore, estimation of the ultimate load is indeed essential in assessing the margin of safety of RC structures against failure. Furthermore, it is necessary to predict the load–deformation behavior of the structure for responses ranging from elastic to inelastic as well as under all possible loading conditions. Therefore, tracing the entire response of an RC structure is an essential step in the process of understanding the performance of RC.

In this study, the two dimensional RC beam analysis will consist of three major components: steel reinforcing bars, concrete material, and bond effect. Steel reinforcing bars will be modeled using the discrete representation in the FE mesh. The reinforcing...
steel elastoplastic material model, described in Chapter 3, will be used here with linear strain hardening. The concrete behavior will be described using the elastic-plastic-damage model discussed in Chapter 4. The effect of bond deterioration and the transfer of the stresses from concrete to the reinforcing bars will be accounted for using the work of Belarbi and Hsu (1994) which was modified later on by Kwak and Kim (2006). The bond-slip effect along the reinforcing bars is quantified with the force equilibrium and compatibility condition at the post-cracking stage and its contribution is indirectly implemented into the stress–strain relation of reinforcing steel. The advantage of the analytical procedures proposed by Belarbi and Hsu (1994) and Kwak and Kim (2006) is taking into account the incorporation of the bond slip effect while using the conventional discrete representation of steel, without the need for additional considerations such as using double nodes or interface elements.

5.2 Bond-Slip Effect on the Stress–Strain Relation of the Steel Reinforcement

Reinforcing steel is usually modeled as a linear elastic, linear strain hardening material with a yield stress \( \sigma_y \) (see Chapter 3). However, when reinforcing bars are surrounded by concrete, the average behavior of the stress–strain relation is quite different, as shown in Fig. 5.1 (Belarbi and Hsu, 1994). The most strikingly different feature is the lowering of the yield stress below the value of \( \sigma_y \). Yielding of an RC member occurs when the steel stress at a cracked section reaches the yield strength of the bare bar. However, the average steel stress at a cracked element still maintains an elastic stress that is less than the yield strength, because the concrete matrix located between cracks is still partially capable of resisting tensile forces, owing to the bond between the concrete and the reinforcement.

![Figure 5.1 Stress–Strain Relation for Steel Reinforcement (Belarbi and Hsu, 1994)](image)

Figure 5.1 Stress–Strain Relation for Steel Reinforcement (Belarbi and Hsu, 1994)

Determination of element stiffness on the basis of the yielding of steel at a cracked section where a local stress concentration appears in the steel may result in
overestimation of the structural response at the post-yielding range. Since this phenomenon is accelerated with increased deformation, an analysis of RC members subjected to loading accompanied by relatively large deformations requires the use of average stress–strain relations (Stevens et. al., 1991; Belarbi and Hsu, 1994). Accordingly, the average stress–strain relation of steel needs to be defined so as to trace the cracking behavior of RC beams up to the ultimate limit state. This can be accomplished here by using continuum damage mechanics, in which the local displacement discontinuities at cracks are distributed over the finite element and where the behavior of cracked concrete is represented by the average stress–strain relations (Stevens et. al., 1991). Considering these factors, Belarbi and Hsu (1994) proposed the bilinear average stress–strain relation shown in Fig. 5.1, which was introduced from experimental data.

In Fig. 5.1, $\varepsilon_n$ is the limiting boundary strain defined as follows:

$$\varepsilon_n = \varepsilon_y (0.93 - 2B)$$  \hspace{1cm} (5.1)

where $B$ is defined as $B = (f_y / \rho_y)^{1.3} / \rho_y$, $f_y$ and $\varepsilon_y$ are the yield stress and the corresponding yield strain of the bare bar, $f_y$ is the tensile strength of concrete, and $\rho_y$ is the steel reinforcement ratio.

![Stress – Strain Relation for Steel Reinforcement](Kwak and Kim, 2006)

Kwak and Kim (2006) elaborated on the above mentioned works and emphasized that reinforcing bars transfer tensile stresses to concrete through the bond stresses located along the surface between reinforcements and surrounding concrete. They extracted part of an RC member subjected to uniaxial tension and bounded by two adjacent cracks and used it as a free body diagram to obtain the equilibrium equations for concrete and steel using a linear bond stress-slip relation. A reinforcing bar equivalent stress-strain relation
that incorporates the effect of bond-slip is therefore developed. Their final adopted stress-strain relation for the reinforcing bars is shown in Fig. 5.2. The equivalent elastic modulus $E_{eq}$ is given as:

$$E_{eq} = E_s \frac{\varepsilon_{s1}}{\varepsilon_{eq}}$$

(5.2)

where $\varepsilon_{s1}$ is the strain in the steel bar assuming perfect bond, and $\varepsilon_{eq}^s = \int_0^l \varepsilon_s(x)dx / l_x$.

The parameter $l_x$ represents the transfer length which can be determined following the linear relationship proposed by Somayaji and Shah (1981) on the basis of experimental data obtained from pull-out tests as follows:

$$l_x = K_p \frac{F_c}{\sum_a}$$

(5.3)

where $F_c$ is the transfer load equal to $F_c = A_c E_s \varepsilon_{s1}$, $K_p$ is a constant determined from the pullout tests (Mirza and Houde, 1979), and $\sum_a$ is the perimeter of a reinforcing bar. The equivalent strain $\varepsilon_{eqn}$ (Fig. 5.2) is defined as $\varepsilon_{eqn}^n = \sigma_n / E_{eq}$.

Steel reinforcement can now be included in the FE mesh as individual elements with the equivalent stress-strain relation that accounts for bond effects while assuming perfect bond between steel and concrete elements. Furthermore, the corresponding equivalent modulus of elasticity for steel $E_{eq}$ can be used up to the yielding point in the stress–strain relation of the discrete reinforcing steel elements, as depicted by the solid line in Fig. 5.2. The same ratio of $E_{eq}$ to $E_s$ given by Belarbi and Hsu (1994) was assumed by Kwak and Kim (2006) to be maintained at the post-yielding region.

5.3 Applications of the Constitutive Model to the FE Analysis of RC Beams

The proposed model is applied to study the simply supported RC beam tested experimentally by Burns and Siess (1962) and later modeled by Kwak and Filippou
Figure 5.4 FE discretization of steel and concrete, loading, and support conditions

Figure 5.5 Tensile damage $\phi^+$, coarse mesh 78 elements

Figure 5.6 Tensile damage $\phi^+$, finer mesh 156 elements

Figure 5.7 Tensile damage $\phi^+$, finer mesh 312 elements
Figure 5.8 Idealized tensile damage distribution, coarse mesh 78 elements

Figure 5.9 Idealized tensile damage distribution, finer mesh 156 elements

Figure 5.10 Idealized tensile damage distribution, finer mesh 312 elements

Figure 5.11 Idealized tensile damage distributions
Figure 5.12 Damage distribution, results obtained by Kwak and Filippou (1997)

(1997). The tested specimen consisted of a simply supported beam with a span of 12 ft (3.7 m) which was subjected to a concentrated load at midspan. The geometry and the cross section of the beam are shown in Fig. 5.3 and the material properties are given as follows: The moduli of elasticity for concrete and steel are $E_c = 3800 \text{ Ksi}$ and $E_s = 29500 \text{ Ksi}$, respectively. The tensile and compressive strengths of concrete are $f_{c}^{+} = 0.347 \text{ Ksi}$ and $f_{c}^{-} = 4.82 \text{ Ksi}$, respectively. The yielding stress for the steel bars is $f_{y} = 44.9 \text{ Ksi}$. The reinforcement consists of #6 (0.75 in diameter) rebars with a reinforcement ratio of $\rho = 0.99\%$. The reinforcement is modeled using 2 dimensional elements with a thickness. The constitutive model of these elements accounts for the bond effect as shown in section (5.2). The Poisson’s ratios for concrete and steel are $\nu_c = 0.167$ and $\nu_s = 0.333$, respectively. The fracture energy is given as $G_f = 0.5 \text{ lb/in}$.

Only half of the beam is modeled in this FE simulation by taking advantage of the symmetry in the geometry and loading. The FE discretization, the arrangement of the steel reinforcement, and the loading and support conditions are shown in the Fig. 5.4.

Three analyses were carried out using four-nodded quadrilateral elements and 100 deflection/time increments. The same model parameters used for the concrete beam analysis in chapter 4 are used here. The fracture energy along with the characteristic lengths for different meshes provided by ABAQUS are used to reduce the effect of mesh sensitivity. Figures 5.5, 5.6, and 5.7 show the results of the tensile damage variable $\phi^+$ at the end of the FE incremental procedures involving three different mesh sizes, 78, 156, and 312 elements. Figures 5.8, 5.9, and 5.10 show idealized tensile damage distributions - plotted over undeformed meshes - that are used to compare results to each other (Fig. 5.11) as well as to the damage distribution obtained by Kwak and Filippou (1997) as shown in Fig. 5.12.

The load-deflection curves obtained using the three simulations discussed above are shown in Figs. 5.13 through 5.16. The results are compared to the experimental output of the work of Burns and Siess (1962). It can be seen that the numerical results exhibit fluctuations that are related to convergence issues, especially in the zone where concrete is softening under compression (crushing of concrete). The compressive damage material parameters are the ones that govern the shape of the load-deflection curves beyond the 30
Ksi stress level. The mesh sensitivity is reduced by using the fracture energy related coefficients $\gamma$. 

Figure 5.13 Load-Deflection relations for RC beam (Burns and Siess, 1962)

Figure 5.14 Load-Deflection relations for RC beam (Burns and Siess, 1962)
Figures 5.17, 5.18, and 5.19 show the total damage ($\Phi$) distributions for the three simulations discussed above. The total damage variable $\Phi$ (scalar) is obtained using Eq. 4.129. It should be noted here that the total damage variable is a weighted average function of the stress tensor as well as its spectral decomposition parts, and the tensile $\varphi^+$.
and compressive $\varphi$ scalar damage variables. This averaging technique results in the differences between the tensile damage distributions and the total damage distributions plotted in the previously indicated figures.

The proposed model is also applied to study the simply supported RC beams tested experimentally by Karihaloo (1992) and later modeled by Tikhomirov and Stein (2001). Three point bending of the experimental RC specimens was carried out. Two beams with one and two 12 mm diameter bars are analyzed until failure occurred in a displacement controlled environment. The geometry and the cross sections of the beams are shown in Figs. 5.20 and 5.26. The material properties used are as follows: The moduli of elasticity

Figure 5.17 Total damage $\Phi$, coarse mesh 78 elements

Figure 5.18 Total damage $\Phi$, finer mesh 156 elements

Figure 5.19 Total damage $\Phi$, finer mesh 312 elements
for concrete and steel are $E_c = 30$ GPa and $E_s = 200$ GPa, respectively. The tensile strength of concrete is $f_{c}^{\text{t}} = 2.26$ MPa. The yielding stress for the steel bars is $f_y = 463$ MPa. The reinforcement is again modeled using 2 dimensional elements with a thickness. The constitutive model of these elements did not account for the bond effect, since the results underestimate the RC beam strength, see Figs. 5.25 and 5.27. The Poisson’s ratios for concrete and steel are $\nu_c = 0.2$ and $\nu_s = 0.3$, respectively. Only half of each beam is modeled in these FE simulations by taking advantage of the symmetry in the geometry and loading conditions.

The first analysis was performed using a 360 element mesh with 45 elements representing the steel reinforcement as can be seen in Fig. 5.21. The damage distributions are similar in pattern to those of the previous example. The averaging of the properties used in obtaining the total damage $\Phi$ distribution is again the reason for the differences between the tensile $\varphi^{t}$ and total damage $\Phi$ distributions (see Figs 5.21 and 5.22).

Figure 5.20 Geometry and cross section of RC beam 1 (Karihaloo, 1992)

In Fig. 5.23, damage is plotted over the undeformed mesh in order to compare the proposed model’s damage distribution to that of Tikhomirov and Stein (2001), see Fig. 5.24. The load-deflection curves obtained using the FE simulation discussed above is shown in Fig. 5.25. The results are compared to the experimental output of the work of Karihaloo (1992). The numerical results exhibit fluctuations similar to those in Fig. 5.16,
Figure 5.22 Total Damage $\Phi$ distribution, 360 elements mesh

Figure 5.23 Idealized Tensile damage distribution

Figure 5.24 Damage distribution obtained by Tikhomirov and Stein (2001)

Figure 5.25 Load-central deflection relation for RC beam 1 (Karihaloo, 1992)
and are again believed to be related to convergence issues. The model’s behavior underestimates the strength of the RC beam in the intermediate stage where concrete is deteriorating. This was not observed in the previous example, where all three simulations showed overestimation of the strength in the same region. Nevertheless, the trend of the results is close to the experimental ones demonstrating the ability of the proposed model’s to capture the physical behavior of RC.

Figure 5.26 Geometry and cross section of RC beam 2 (Karihaloo, 1992)

Figure 5.27 Load-central deflection relation for RC beam 2 (Karihaloo, 1992)

Figure 5.27 shows the load-deflection curves obtained using the FE simulation of the second beam with two 12mm reinforcing bars. The same number of elements used in the previous example is adopted here. The simulated results are again compared to the experimental output of the work of Karihaloo (1992). The experimental results show a more brittle failure of the RC beam when compared to the previous one (one 12mm bar)
where the beam fails through yielding of steel. This is a direct result of increasing the amount of reinforcement (2 bars). The numerical behavior, on the other hand, underestimates the strength of the RC beam in the intermediate stage where concrete is deteriorating. The trend of the results is close to the experimental ones, demonstrating again the ability of the proposed model’s to capture the physical behavior of RC.
CHAPTER 6
SUMMARY, CONCLUSIONS AND FUTURE WORK

6.1 Summary and Conclusions

This study introduces a continuum FE approach that is appropriate for predicting the physical behavior of RC members subjected to short term monotonic loading assuming isothermal conditions. The macroscopic components of RC members are modeled using separate constitutive models that are brought together through the consideration of the steel-concrete bond and interaction analysis. Elastoplastic constitutive models with strain hardening are introduced to model the behavior of steel reinforcement, and a continuum approach based on damage mechanics and plasticity theory is adopted to describe the complex behavior of concrete material in structural elements. The concept of energy dissipation (fraction energy) is used in order to reduce the effect of mesh sensitivity on the FE numerical results. Bond and interaction between steel and concrete is accounted for by modifying the global stress-strain curve of steel reinforcement to account for the reduction in the strength (elastic and plastic moduli and yield stress) observed experimentally. All these components are integrated into the proposed approach in an attempt to properly describe the complex physical behavior of RC composite materials.

Several forms of elastic-plastic constitutive relations are implemented in this FE study to describe the nonlinear behavior of steel reinforcement in an RC member, including elastic-perfectly plastic, elastic-plastic strain hardening, and elastic-perfectly plastic-followed by plastic strain hardening. Implicit and explicit integration schemes are provided in order to integrate the constitutive models. These constitutive models are well described in the literature and are presented here to give an idea of the diversity of models used by different researcher to describe the same material. All these models are simplified representations of the actual physical behavior of the steel bars embedded in concrete. Nevertheless, these simplified representations were reported to give acceptable results in the research of RC, a path that is surrounded with uncertainties.

The proposed material model for concrete is derived using rigorous and consistent thermodynamic formulation. The additive decomposition of the Helmholtz free energy concept is used to define the thermodynamic conjugate forces associated with the internal state variables, including the damage thermodynamic conjugate forces (damage energy release rates). The energy dissipation mechanisms are formulated to satisfy the first inequality of thermodynamics, and to postulate the plastic and damage dissipation functions. Three dissipation mechanisms (plasticity, tensile and compressive damage) are present to control the dissipation process of the material model.

The concrete model is a combination of the generalized effective space plasticity theory and isotropic damage theory applied simultaneously under the assumptions of small strains, rate independence, and isothermal conditions. The strain equivalence hypothesis is used for the stress mapping/transformation from the effective (undamaged)
to the damaged configurations. A concrete plasticity yield function with multiple isotropic hardening criteria and a non-associative plasticity flow rule is adopted to represent the irreversible plastic behavior of concrete. The non-associative flow rule includes a hydrostatic term to account for the dilatation effect of concrete materials. On the other hand, two damage growth criteria that are based on the hydrostatic-deviatoric sensitive thermodynamic-conjugate forces are used to model stiffness degradation and material deterioration in concrete. Two damage variables, tensile and compressive, are used to model the different damaging behaviors in concrete. These two damage variables are combined using a relation that incorporates the stress tensor and its spectral decomposition into tensile and compressive components. The combined damage variable is consequently used to map the stress from the effective to the damaged configuration. Fracture energy related coefficients have been defined and incorporated in the damage model to achieve a reasonable degree of discretization insensitivity in numerical calculations.

The computational algorithm for the concrete model is based on the operator split concept, where the incremental constitutive equation is decomposed into elastic, plastic, and damage parts, leading to the corresponding numerical elastic-predictor, plastic-corrector and damage-corrector steps. During the first two steps, the damage variables are fixed, so that the elastic-plastic behavior is decoupled from damage, constituting a standard elastic-plastic problem in the effective stress space. The damage variables and the Cauchy stress tensor can then be updated correspondingly in the damage-corrector step. An elastic-plastic implicit, damage explicit integration scheme is adopted. The consistent elastic-plastic-damage tangent operator is derived and a flow chart of the integration technique is provided.

The proposed concrete model is implemented and tested to verify its capability, applicability, and effectiveness in capturing the material behavior in both tension and compression under uniaxial and bi-axial loadings. The results obtained by the proposed model are compared with corresponding experimental results to evaluate the model’s performance. Tensile and compressive verification tests under unaxial loading are demonstrated first, followed by biaxial tests in tension and compression. Then a three point bending test of a notched beam is investigated.

The numerical results of the verification examples under tensile and compressive unaxial loads demonstrate the effectiveness of the model in capturing the uniaxial behavior of concrete. During all stages of loading, the simulated results match the experimental ones in terms of physical behavior. The experimentally observed softening branch of the stress-strain diagram under uniaxial loading (tension or compression) is reproduced efficiently.

Under biaxial tension, the model easily depicts the experimentally observed phenomenon; concrete strength suffers a reduction as the biaxial stress ratio is increased. Whereas, under biaxial compression, an additional equation governing the damage variables is introduced in order to account for the strengthening of concrete due to
consolidation under biaxial compression. This improvement leads to a more efficient numerical representation of the experimental results under biaxial compression.

The proposed concrete model in its current form is not capable of capturing the experimentally observed behavior under combined tensile and compressive loadings. The reduction in compressive strength was not observed numerically as the tensile stress is increased and vice versa. Further development of the model is therefore required in these regions of loading to enhance the numerically simulated results.

When a three point bending test of a single notched concrete beam is investigated, the computational algorithm demonstrated robustness in simulating the entire stress-strain diagram of concrete and the evolution of damage. Nevertheless, some difficulties are encountered: the non-smoothness in the numerical results beyond the peak point is believed to be related to the global equilibrium iterations. Another possible source is the use of non-associative plastic flow rule, where the direction of the plastic flow is not normal to the yield surface. It is well documented in literature that local damage approaches exhibit mesh sensitivity and non-smooth results in the softening region. Regularization methods should be incorporated in order to overcome such drawbacks.

In order to analyze RC beams, the bond-slip effect along the reinforcing bars is quantified with the force equilibrium and compatibility condition at the post-cracking stage. This effect is indirectly implemented into the stress–strain relation of reinforcing steel. The advantage of such procedure is taking into account the incorporation of the bond slip effect while using the conventional discrete representation of steel, without the need for additional considerations such as using double nodes, interface elements, or modified finite element formulation to produce more complicated element library. This approach is more suited to the material modeling using the UMAT subroutine, where all the numerical procedures are written in terms of stresses and strains.

The implemented algorithms are then used to analyze two simply supported RC beams subjected to concentrated loads applied at mid spans. The proposed model is able to capture the three stages of loading discussed in Chapter 2. An attempt to reduce the variation in the numerical results is carried out by means of adjusting the fracture energy related parameters. As the mesh size is reduced, the results varied yet remained close to one another, a behavior attributed to a non regularized constitutive approach.

In conclusion, the proposed approach of analysis of concrete and RC beams is a meaningful experience. It spots the light on many issues relating to the constitutive modeling of RC; one of the most challenging fields in Civil Engineering. Despite all the research done in the past, a current literature review (Chapter 2) shows that scientists still believe in the potential for further improvement of the way RC materials are studied and designed. This particular study reveals more questions than answers; allowing endless space for future development.
6.2 Future Work

The use of the proposed model to predict the response of plain and RC systems indicates a number of characteristics of the current model that could be enhanced to improve modeling accuracy. It also identifies a number of areas in which additional research focused on the model development and verification, can greatly enhance simulation of RC structural behavior.

The most significant deficiency of the current model is the failure to represent the biaxial behavior under combined tension and compression. Next is the dependence of the numerical results on mesh refinement. Although this subject has been partially accounted for using fracture energy induced parameters, regularization methods should be employed in order to significantly trivialize the effect of mesh dependency. At the global level, results of this study indicate that additional research is required to improve solution algorithms for systems that represent material softening (non-smooth material response).

Although the calibration of the model parameters governing the initial yield and the damage surfaces has been carried out, a proper and unified procedure for the identification and determination of the model parameters and their effect under different loading conditions is an important aspect yet to be addressed. A parametric study may reveal important information regarding the parameters that are most susceptible to mesh sensitivity effect, and therefore, help in calibrating these parameters using the fracture energy concept.

Further enhancement of the proposed model is achieved by incorporating additional data defining the response of concrete and RC under variable reversed cyclic loading. This might include introduction of isotropic damage parameters that couple both tension and compression effects as well as the elastic stiffness recovery during transition from tension to compression loadings and vise versa (crack opening/closing). In addition, it is also expected that damage-induced anisotropy is of particular importance when non-proportional loading is considered. This motivates further development of the constitutive model to include anisotropic damage effects through the introduction of tensorial damage formulation.

Application of the proposed model to represent the response of concrete and RC flexural elements provides significant additional information about the model. From the aspect of constitutive modeling, the model in this study shows its potential features in dealing with mode I cracking problems, such as the direct tensile tests or the standard three-point bending tests. In those circumstances, the constitutive modeling using the fracture energy concept to simulate mode I fracture energy furnishes a good way to describe the post-peak behavior of the material. Since concrete fracture energy is defined by Mode I fracture and shear transfer defines Mode II fracture, the investigation of the proposed model’s capability to simulate the experimental results of combined shear and compression is yet to be explored.
In order to further investigate the effect of bond deterioration on the global behavior of RC composite material, the traction forces at the interfaces between steel and concrete finite elements can be studied and modeled using ABAQUS. This will result in the introduction of the slip concept into the analysis as a result of the decay of the traction forces. The slip was not accounted for in the current study since all the analysis was based on modeling of stresses and strains in the RC composite material using the user defined material subroutine UMAT.
BIBLIOGRAPHY

Part I: By the Author


Part II: By Other Authors


damage constitutive equations for small and finite deformations”. Int. J. Plasticity
16, 495–523.

UCB/SEMM-94/03, Department of Civil Engineering, University of California,
Berkeley, California.


Design, 80(2), 233–245.

Cambridge University Press, Cambridge, UK.

l’endommagement au comportement non lineaire et a la rupture de beton de

Analysis of Steel–Concrete Composite Beams in Combined Bending and Shear”.
J. of Struc. Eng., 131(10), 1593-1600.

non-linear response under cyclic loading of RC frame structures”. Earthquake

89. Loland, K. E., 1980. “Continuous Damage model for load-response estimation of

90. Lowes, L. N., 1999. “Finite Element Modeling of Reinforced Concrete Beam-
Column Bridge Connections”. Dissertation, University of California, Berkeley,
California, USA.


APPENDIX A

NUMERICAL PROCEDURE FOR THE SPECTRAL DECOMPOSITION CONCEPT USING THE MATHEMATICAL SOFTWARE MAPLE (CHAPTER 4)

\[\text{restart;}
\text{with(linalg):}
\]

Warning, the protected names norm and trace have been redefined and unprotected

\[\text{Stress := matrix(3,3,[1,2,3,2,7,5,3,5,9]);}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 7 & 5 \\
3 & 5 & 9
\end{bmatrix}
\]

\[\text{Egvals:=evalf(Eigenvals(Stress,vecs));}
\]

\[
\begin{bmatrix}
-0.02414811352 & 2.940332528 & 14.08381559
\end{bmatrix}
\]

\[\text{print(vecs);}
\]

\[
\begin{bmatrix}
0.9580437551 & 0.110063176 & 0.2646710678 \\
-0.07607796278 & -0.7926997978 & 0.6048540332 \\
-0.2763409264 & 0.5996122645 & 0.7510664580
\end{bmatrix}
\]

\[\text{Egvec1:=vector([vecs[1,1],vecs[2,1],vecs[3,1]]);}
\]

\[
\begin{bmatrix}
0.9580437551 & -0.07607796278 & -0.2763409264
\end{bmatrix}
\]

\[\text{Egvec2:=vector([vecs[1,2],vecs[2,2],vecs[3,2]]);}
\]

\[
\begin{bmatrix}
0.110063176 & -0.7926997978 & 0.5996122645
\end{bmatrix}
\]

\[\text{Egvec3:=vector([vecs[1,3],vecs[2,3],vecs[3,3]]);}
\]

\[
\begin{bmatrix}
0.2646710678 & 0.6048540332 & 0.7510664580
\end{bmatrix}
\]

\[\text{Egmat1:=multiply(Egvec1,transpose(Egvec1));}
\]

\[
\begin{bmatrix}
0.9178478367 & -0.07288601714 & -0.2647466988 \\
-0.07288601714 & 0.005787856421 & 0.02102345471 \\
-0.2647466988 & 0.02102345471 & 0.07636430760
\end{bmatrix}
\]

\[\text{Egmat2:=multiply(Egvec2,transpose(Egvec2));}
\]

\[
\begin{bmatrix}
0.01210138991 & -0.08720134548 & 0.06596113721 \\
-0.08720134548 & 0.6283637424 & -0.4753090311 \\
0.06596113721 & -0.4753090311 & 0.3595348677
\end{bmatrix}
\]

\[\text{Egmat3:=multiply(Egvec3,transpose(Egvec3));}
\]
\[
\begin{bmatrix}
0.07005077413 & 0.1600873628 & 0.1987855614 \\
0.1600873628 & 0.3658484015 & 0.4542855763 \\
0.1987855614 & 0.4542855763 & 0.5641008243
\end{bmatrix}
\]

\[
Egmat3 := \begin{bmatrix}
0.07005077413 & 0.1600873628 & 0.1987855614 \\
0.1600873628 & 0.3658484015 & 0.4542855763 \\
0.1987855614 & 0.4542855763 & 0.5641008243
\end{bmatrix}
\]

\[
Egval1 := \text{multiply} (\text{multiply} (\text{transpose}(Egvec1), \text{Stress}), Egvec1);
\]

\[
Egval1 := -0.02414811403
\]

\[
Egval2 := \text{multiply} (\text{multiply} (\text{transpose}(Egvec2), \text{Stress}), Egvec2);
\]

\[
Egval2 := 2.940332529
\]

\[
Egval3 := \text{multiply} (\text{multiply} (\text{transpose}(Egvec3), \text{Stress}), Egvec3);
\]

\[
Egval3 := 14.08381559
\]

\[
\text{Stress} := \text{matadd} \left( \text{matadd}(Egmat1, Egmat2, Egvals[1], Egvals[2]), Egmat3, 1, Egvals[3]) \right);
\]

\[
\text{Stress} := \begin{bmatrix}
1.000000001 & 2.000000003 & 3.000000000 \\
2.000000003 & 7.000000006 & 5.000000000 \\
3.000000000 & 5.000000000 & 8.999999996
\end{bmatrix}
\]

\[
\text{sigma} := \text{matrix} (3, 3, [0, 0, 0, 0, 0, 0, 0, 0, 0])
\]

\[
\sigma := \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\text{for } k \text{ from 1 to 3 do}
\]
\[
\text{for } i \text{ from 1 to 3 do}
\]
\[
\text{for } j \text{ from 1 to 3 do}
\]
\[
\text{sigma}[i,j] := \text{sigma}[i,j] + \text{Egvals}[k] \cdot \text{vecs}[i,k] \cdot \text{vecs}[j,k]
\]
\[
\text{end do};
\]
\[
\text{end do};
\]
\[
\text{end do};
\]

\[
\text{print} (\text{sigma});
\]

\[
\begin{bmatrix}
1.000000001 & 2.000000003 & 3.000000000 \\
2.000000003 & 7.000000006 & 5.000000000 \\
3.000000000 & 5.000000000 & 8.999999996
\end{bmatrix}
\]
APPENDIX B

UPDATING THE EFFECTIVE STRESS USING THE RETURN MAPPING EQUATION (CHAPTER 4)

\[
\overline{\sigma}_{ij}^{n+1} = \overline{\sigma}_{ij}^{trial} - \bar{E}_{ijkl}\Delta \overline{e}_{kl}^p = \overline{\sigma}_{ij}^{trial} - (2\bar{G}I_{ijkl}^{dev} + \bar{K}\delta_{ij}\delta_{kl})\Delta \overline{e}_{kl}^p
\]

\[
\overline{\sigma}_{ij}^{n+1} = \overline{\sigma}_{ij}^{trial} - \left[2\bar{G}\left(\delta_{j}^{ijkl}\delta_{jl} + \delta_{i}^{ijkl}\delta_{jk}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] + \bar{K}\delta_{ij}\delta_{kl}\Delta \overline{e}_{kl}^p
\]

\[
\overline{\sigma}_{ij}^{n+1} = \overline{\sigma}_{ij}^{trial} - \left[G\delta_{i}^{ijkl}\delta_{jl} + G\delta_{j}^{ijkl}\delta_{jk} - \frac{2}{3}G\delta_{ij}\delta_{kl} + \bar{K}\delta_{ij}\delta_{kl}\right] + \bar{K}\delta_{ij}\delta_{kl}\Delta \overline{e}_{kl}^p
\]

\[
\overline{\sigma}_{ij}^{n+1} = \overline{\sigma}_{ij}^{trial} - 2G\Delta \overline{e}_{ij}^p - \left(K - \frac{2}{3}\bar{G}\right)\Delta \overline{e}_{kl}^p\delta_{ij}
\]

\[
\overline{\sigma}_{ij}^{n+1} = \overline{\sigma}_{ij}^{trial} - \left[2\bar{G}\Delta \overline{e}_{ij}^p + \left(K - \frac{2}{3}\bar{G}\right)\Delta \overline{e}_{kl}^p\delta_{ij}\right]
\]
VITA

Ziad N. Taqieddin was born in Amman, Jordan. Ziad graduated from the Educational College and received a certificate of excellence for his distinguished accumulated average (92.7%) in the Jordanian national high-school examination, 1996. He subsequently joined the Civil and Environmental Engineering Program at the Applied Science University (ASU), Amman, Jordan, to pursue his bachelor degree. Five years later, 2001, he graduated with a bachelor and first degree honors for rating excellent and ranking first among his class. He received the prestigious ASU Golden Watch for his distinguished academic performance. Engineer Ziad was immediately hired by the Civil and Environmental Engineering Department, ASU, as a teaching assistant. He helped in teaching several courses and laboratory practices.

In 2003, Ziad traveled to Baton Rouge, Louisiana, USA, to join the structural/mechanics master’s program in the Civil and Environmental Engineering Department, Louisiana State University (LSU). He was granted a Teaching Assistantship (TA) by the department to help in the courses of statics, dynamics and strength of materials. His academic advisor, Boyd Professor George Z. Voyaidjis, mentored and walked him through his master’s degree. Ziad graduated with a Master of Science degree in Civil Engineering in May 2005 with a 4.0 GPA.

Ziad continued his path of passion for science towards his doctoral degree with his mentor, Boyd Professor George Z. Voyiadjis. Ziad got married to the love of his life Rawan in 2006. Two years later, he completed the research presented in this dissertation and maintained a 4.0 GPA. Ziad graduated with a structures/mechanics Doctor of Philosophy degree in civil engineering is August, 2008.

Throughout his stay at LSU, Ziad was always involved in research in addition to his teaching responsibilities. He indulged himself in the mechanics of solids and structures, mechanics of composites, computational mechanics, plasticity, fracture and damage mechanics. He has several publications in prestigious journals in the field of engineering mechanics. He has also participated in a number of internationally recognized conferences.