Pseudo-Interiors of Hyperspaces.

Nelly Sebilla Kroonenberg

Louisiana State University and Agricultural & Mechanical College

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by

Nelly Sebilla Kroonenberg
B.S., Universiteit van Amsterdam, 1969
M.S., Universiteit van Amsterdam, 1972
May, 1974
The title of this dissertation refers more to the latter two chapters than to the work as a whole. Chapter I gives an exposition of the topology of Z-sets and capsets in $\mathbb{Q}$, such as developed by Anderson in [2], [3] and [4]. However, the proofs and organization are rather different, and various simplifications have been made. Most of what is new in this chapter has been included in the Master's Thesis of the author [15].

An alternative treatment on Z-sets can be found in Chapters I and II of T. A. Chapman's Notes on Hilbert Cube Manifolds (unpublished).

Chapters II and III consist entirely of new material.

My first acquaintance with Infinite-Dimensional Topology was through a course taught in "Texas-style" by Professor R. D. Anderson during his stay in Amsterdam in 1970-1971. I feel very much indebted to him for this most inspiring introduction to his field. Several proofs in Chapter I resulted from work I did for this course.
The material for Chapters II and III was developed during my stay in 1972-1974 at LSU, under partial support of NSF grant GP 34635X. I received much help and encouragement - in the form of discussions, suggestions, comments and readings of various versions of the manuscript - from Professors R. D. Anderson, D. W. Curtis and R. M. Schori.

Finally, I wish to thank the typist, Monica Loftin, for the excellent job she has done.
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ABSTRACT

The Hilbert cube $Q$ is the countable infinite product of intervals $I^\infty$ - where $I = [-1,1]$ , topologized by the product topology and furnished with a suitable metric. Points of $Q$ are denoted by $x = (x_i)_1$, with $x_i \in I$. We consider the following subsets of $Q$:

1. The pseudo-boundary $B(Q) = \{x| \text{for some i, } |x_i| = 1\}$.
2. Its complement $s = (-1,1)^\infty$ which is called the pseudo-interior of $Q$. It is shown by R. D. Anderson that $s$ is homeomorphic to $(\mathbb{N}) \ell_2$.
3. The closed subsets $K$ of $Q$ such that for each $\epsilon > 0$ there exists a map $f:Q \rightarrow Q-K$ with $d(f, \text{id}_Q) < \epsilon$. These are called Z-sets.

In Chapter I, certain well-known theorems about these subsets are proved. We mention especially:

1. The Homeomorphism Extension Theorem: any homeomorphism between two Z-sets in $Q$ can be extended to an auto-homeomorphism of $Q$.
2. The non-empty Z-sets are exactly those closed subsets of
Q, which can be mapped by an autohomeomorphism of Q onto a set which projects onto a point in infinitely many coordinates.

3. A topological characterization of the pseudo-boundary.

Let $2^X$ be the space of all non-empty compact subsets of a metric space X and let $C(X)$ be the space of non-empty compact connected subsets, both with the Hausdorff metric $d_H$ which is defined by $d_H(A,B) = \inf\{\epsilon | A \subset U_\epsilon(B) \text{ and } B \subset U_\epsilon(A)\}$. D. W. Curtis and R. M. Schori showed that $2^X \cong Q$ for X a non-degenerate Peano continuum, and that $C(X) \cong Q$ for X a non-degenerate Peano continuum without free arcs. In particular, it follows that $2^X \cong C(X) \cong Q$ if X $\not\cong Q$ or X is a compact connected Q-manifold. Also, we have $2^I \cong Q$, which was proved earlier by Schori and West.

In Chapter II, it is shown that the collection of non-empty Z-sets in Q (or in a compact connected Q-manifold M) is a topological pseudo-interior for $2^Q$ (or $2^M$). As a corollary one obtains that $2^{l_2} \cong l_2$, and that $2^M \cong l_2$ for M a connected $l_2$-manifold. Corresponding results are obtained for the collection of non-empty Z-sets in $C(Q)$ or $C(M)$, and also it follows that $C(l_2) \cong l_2$ and that $C(M) \cong l_2$ for M a connected $l_2$-manifold.

In Chapter III, it is shown that the collection of
topological Cantor sets in the interval $I$ and the collection of non-empty zero-dimensional subsets of $I$ are topological pseudo-interiors for $2^I$. 
INTRODUCTION

By the **Hilbertcube** $Q$ we mean the countable infinite product of intervals $I^\infty$ or $[-1,1]^\infty$ with the product topology. If $x = (x_1)_{i \geq 1}$ and $y = (y_1)_{i \geq 1}$ are two points of $Q$, then their distance $d(x,y)$ is defined as $\sum_{i \geq 1} 2^{-i} \cdot |x_i - y_i|$. By the **pseudo-interior** of $Q$ we mean the subset $s = (-1,1)^\infty$; its complement $Q-s$ is called the **pseudo-boundary** $B(Q)$ or $RQ$ of $Q$. It is easily seen that $s$ is a dense $G_\delta$ in $Q$. Anderson proved in [1] that $s$ is homeomorphic to the Hilbertspace $l_2$. The complement $B(Q)$ is also dense in $Q$, so the pseudo-boundary of $Q$ is only a restricted infinite-dimensional analogue of the boundary of a finite-dimensional $n$-cell.

We consider two classes of subsets of $Q$, viz. **Z-sets** and **capsets**. Both concepts have played an important role in I-D topology and especially Z-sets are the focus of continued interest. $K$ is a **Z-set** in $Q$ if for every $\epsilon$ there exists a map $f:Q \to Q-K$ such that $d(f,id) < \epsilon$ (where $id$ denotes the identity-mapping; sometimes we shall
write $\text{id}_X$ instead of $\text{id}$ for the identity-mapping on $X$).

The following facts are easy to verify:

1) The property of being a Z-set in $Q$ is topologically invariant, i.e., invariant under autohomeomorphisms of $Q$.

2) A closed subset of a Z-set is a Z-set.

3) A finite or closed countable union of Z-sets is a Z-set.

4) Examples of Z-sets are compact subsets of $\mathbb{R}$ and closed subsets of $Q$ which project onto a point in infinitely many coordinates. For there exist maps $f$ of $Q$ into itself whose image $f(Q)$ is disjoint from such a set, and such that $f$ leaves the lower-numbered coordinates unchanged.

The definition of Z-set can be generalized for a larger class of spaces, in particular for $Q$-manifolds and (manifolds of) infinite-dimensional topological vector spaces. In Geoghegan-Summerhill [11] a version of Z-sets for Euclidean spaces is introduced. See also the remarks after Lemma II.1. Corollary I.9 and Theorem I.10 state the two most important facts about Z-sets: that (1) any Z-set in $Q$ can be mapped by an autohomeomorphism of $Q$ onto a subset which projects onto a point in infinitely many coordinates, and that (2) any homeomorphism between two Z-sets can be extended to an autohomeomorphism of $Q$.

A capset is a subset of $Q$ which is equivalent to
B(Q) under an autohomeomorphism of Q. In Chapter I more practical characterizations will be given. It is a key observation that there exist cap sets which are entirely contained in s (Proposition I.7). Other useful facts are that for any cap set M and any Z-set K, M−K is a cap set (Corollary I.13) and that the union of a cap set with a countable number of Z-sets is again a cap set (Proposition I.15). One use of cap sets is to show that certain spaces are homeomorphic to $l_2$ by exhibiting an embedding into Q with a cap set as remainder. This principle will be applied in Corollaries II.3 and II.5.

In Chapter I the most important facts about Z-sets and cap sets in Q will be proved. Our treatment is rather different from previous ones (most theorems from Chapter I have appeared originally in either of Anderson's papers [1] - [4]). The several stages in the proof of the Homeomorphism Extension Theorem for compact subsets of s are entirely standard. However, the autohomeomorphism of Q which maps B(Q) into s (Proposition I.7) is obtained by a direct geometrical construction, and requires very little preliminary work. This causes changes in the entire organization.

At the end of the chapter several alternative definitions of Z-set, equivalent for Q and s or $l_2$, will be given (Theorem I.18).
For $X$ a metric space, the hyperspace $2^X$ of $X$ is the collection of non-empty compact subsets of $X$ (for non-compact $X$, in other treatments $2^X$ is sometimes understood to be the collection of all non-empty closed subsets), with metric $d_H(A, B) = \inf \{ \varepsilon | A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A) \}$, where $U_\varepsilon(Y)$ denotes the $\varepsilon$-neighborhood of the set $Y$. If $d$ and $d'$ induce the same topology on $X$ then $d_H$ and $d_H'$ induce the same topology on $2^X$. This metric is called the Hausdorff metric. By $C(X)$ we denote the subspace of $2^X$ consisting of all non-empty subcontinua of $X$. We have the following:

**Theorem A** (Curtis-Schori [9]). $2^X$ is homeomorphic to the Hilbert cube iff $X$ is a non-degenerate Peano-continuum.

**Theorem B** (Curtis-Schori [9]). $C(X)$ is homeomorphic to $\mathbb{Q}$ iff $X$ is a non-degenerate Peano-continuum without free arcs (i.e., not having a topological open interval as an open subset).

**Remark 1.** In [16], Schori and West proved Theorem A for the case $X = I$. This result is used in the proof of the general case.

**Remark 2.** It is easily seen that the hyperspace of non-empty subcontinua of an interval, $C(I)$, is homeomorphic to a two-cell.
In Chapter II, we identify pseudo-interiors for $2^X$ and $C(X)$, where $X$ is a Hilbertcube (Theorem II.2) or a compact connected Hilbertcube manifold (Theorem II.4); viz. the collection of non-empty $Z$-sets in $X$ for $2^X$ and the collection of non-empty connected $Z$-sets for $C(X)$. The proofs rest on the aforementioned results of Curtis and Schori, and for the case where $X$ is a manifold also on the Triangulation Theorem for $Q$-manifolds [8]. As a corollary we obtain that, for $X \cong l_2$ or $X$ an $l_2$-manifold, both $2^X$ and $C(X)$ are homeomorphic to $l_2$ (Corollary II.3 and II.5).

In Chapter III, we prove that both the collection of topological Cantor sets in $I$ and the collection of non-empty zero-dimensional closed subsets of $I$ form pseudo-interiors for $2^I$ (Theorem III.4). Here we use Schori-West [16].

Unfortunately, the author has been unable to generalize the above results, even for finite graphs instead of $I$. It might be worth mentioning that the proofs of Chapter II have very little in common with those of Chapter III.
Preliminaries. For each \( n > 0 \), we can write
\[
Q = I^n \times Q_{n+1}, \quad \text{where} \quad Q_{n+1} = \prod_{i>n+1} I_i.
\]
By \( p_n : Q \to I_n \) we mean the projection onto the \( n \)th coordinate; by \( p_n : Q \to I^n \) the projection onto the first \( n \) coordinates. Note the difference between \( I^n \) and \( I_n \). For any non-empty subset \( C \) of \( \mathbb{N} \), we write \( Q_C = \prod_{n \in C} I_n \), and \( s_C = \prod_{n \in C} I_n^\circ \), where \( I_n^\circ \) is the combinatorial interior of \( I_n \), and \( p_C : Q \to Q_C \) the projection onto \( Q_C \). The endfaces \( W^+_n \) and \( W^-_n \) are the sets \( \{x \in Q | x_n = 1\} \) and \( \{x \in Q | x_n = -1\} \) respectively. Trivially \( B(Q) \) is the union of all endfaces of \( Q \). We call a subset \( K \) of \( Q \) deficient in the \( n \)th coordinate if \( p_n(K) \) is a point. We call \( K \) infinitely deficient if \( K \) is deficient in infinitely many coordinates.

Homeomorphisms are always understood to be onto. We write sometimes "\( X \simeq Y \)" instead of "\( X \) is homeomorphic to \( Y \)" , and "\( (X, X') \simeq (Y, Y') \)" instead of "there exists
a homeomorphism $h: X \to Y$ such that $h(X') = Y'$. The distance between two maps or homeomorphisms $f$ and $g: X \to Y$, where $Y$ is compact metric, is defined as $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. If $f$ is a homeomorphism, then obviously $d(f, \text{id}) = d(f^{-1}, \text{id})$ and $d(g, h) = d(gf, hf)$. For $f: X \to X$, we sometimes say that $f$ is small (or $\varepsilon$-small) instead of "$d(f, \text{id}_X)$ is small (or less than $\varepsilon$)". The space of auto-homeomorphisms of a topological space $X$ is denoted by $\text{H}(X)$.

One convenient property of the Hilbertcube is stated in the following

**Proposition 1.1 (Mapping Replacement Theorem).** Let $f: X \to Q$ be a map from a separable metric space into the Hilbertcube. Then for each $\varepsilon > 0$, there exists an embedding $f': X \to Q$ such that $f'(X)$ is an infinitely deficient subset of $s$ and $d(f', f) < \varepsilon$.

**Proof.** It is well-known that every separable metric space can be embedded in $s$. So let $g: X \to s$ be any embedding. Define, for $\delta \in (0, 1)$ and $M$ any integer and $x \in X$, $f_{M, \delta}(x) = (\delta \cdot p_1 f(x), \ldots, \delta \cdot p_M f(x), \ldots, p_M g(x), 0, p_2 g(x), 0, \ldots)$. Then $f_{M, \delta}$ is an embedding because $g$ is, and $f_{M, \delta}$ is $\varepsilon$-close to $f$ if $M$ is sufficiently large and $\delta$ sufficiently close to 1, because in that case $f$ and $f_{M, \delta}$...
"almost" coincide in the most significant coordinates.

For several constructions in this chapter, we obtain a homeomorphism with certain properties as a limit of inductively constructed homeomorphisms. We can ensure convergence to a homeomorphism if at each stage the next homeomorphism can be chosen arbitrarily close to the identity. More formally, let \((f_i)_i\) be a sequence of maps \(f_i:X \to X\) such that the sequence \(f_1 \circ f_1 \circ f_1, f_2 \circ f_1, \ldots\) has a continuous limit; then the limit is denoted \(\varPi f\) and is called the infinite left product of the sequence \((f_i)_i\). We have the following theorem (due to Fort [10] in a slightly different form):

**Theorem 1.2 (The Convergence Criterion).** Let \(X\) be a compact metric space and let \((h_i:X \to X)_i\) be a sequence of autohomeomorphisms. Then \(\varPi h\) is a homeomorphism if for any \(i\):

1. \(d(h_{i+1}, id) < 2^{-i}\) and
2. \(d(h_{i+1}, id) < 3^{-i} \cdot \inf \{|d(h_1 \circ \cdots \circ h_1(x), h_1 \circ \cdots \circ h_1(y))| : |d(x, y) > 1/i\}\).

**Proof.** Convergence to a continuous limit is ensured by the Cauchy condition \(d(h_{i+1}, id) < 2^{-i}\), which is equivalent to \(d(h_{i+1} \circ \cdots \circ h_1, h_1 \circ \cdots \circ h_1) < 2^{-i}\). Because \(X\) is compact, the only other thing we have to show is that \(\varPi h\) is one-to-one. This follows from (2) since points which are
at least 1/\ell_i$ apart are prevented from being mapped onto the same point in the limit by the size-restrictions on $h_{i+1}, h_{i+2}, \ldots$.

The Homeomorphism Extension Theorem for Compact Subsets of $s$. Eventually we want to prove the Homeomorphism Extension Theorem for $Z$-sets (Theorem I.10). For this we will need the concept of basic core set (bcS), to be defined later. It will be seen that any two basic core sets are equivalent under an autohomeomorphism $h \in H(Q)$.

The proof of the general Homeomorphism Extension Theorem is broken up in the following steps:

1) A Homeomorphism Extension Theorem for compact subsets of $s$ (Proposition I.5).

2) It is shown that $(Q, BQ)$ is homeomorphic to $(Q, M)$, where $M$ is a basic core set (Proposition I.7).

3) It is shown that for a bcS $M$ and any Z-set $K$, $(Q, M) \sim (Q, M-K)$, and as a consequence that $(Q, BQ) \sim (Q, BQ-K)$ (Proposition I.8). (In fact, a weaker version of I.8 would suffice, but later on we will need the stronger statement.)

Combining the above results one can easily obtain the general version of the Homeomorphism Extension Theorem.

Lemma I.3 (Anderson [1]). Let $K$ be a compact subset of
s, N a positive integer and ε a positive real number. Then there exists a homeomorphism ϕ: Q → Q such that:

1. for some n > N, p_n(ϕ(K)) is a single point in (-1,1),
2. d(ϕ, id) < ε and
3. for any endface W = \{x|x_1 = 1\} or \{x|x_1 = -1\}, ϕ(W) = W and therefore ϕ(BQ) = BQ.

Proof. The set K is contained in a cube K' = Π_n[a_n, b_n] ⊂ s. Let n > N be so large that for any x, y ∈ Q, if p_{n-1}(x) = p_{n-1}(y), then d(x, y) < ε. First we find a homeomorphism h such that any line in the direction of the n-th coordinate intersects h(K') in at most one point. Let, for any m > n, h_m be a PL autohomeomorphism of the 2-cell I_n x I_m as indicated in Figure I.1 below, which deforms...

\[ \begin{array}{c}
\overset{x_m}{\xrightarrow{h_m}} \\
\overset{x_n}{\xrightarrow{h_n}} \\
\text{Fig. I.1}
\end{array} \]
\([a_n, b_n] \times [a_m, b_m]\) into a slanted figure such that horizontal intersections with it have diameter \(\leq 2^{-m}\), and which leaves the \(x_n\)-coordinate unchanged. Define
\[h(x) = (x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots),\] where \((x_n, y_m) = h_m(x_n, x_m)\). Obviously \(h\) is a homeomorphism, and, for any \(m > n\), by the definition of \(h_m\), the intersection of \(h(K')\) with any interval in the \(x_n\)-direction has diameter at most \(2^{-m}\). Hence the intersection is a point.

For the second and last step we construct a \(g \in H(Q)\) which maps \(h(K')\) into the hyperplane \(p_n^{-1}(0)\). On any interval in the \(x_n\)-direction \(L_x = \{y|m \neq n \Rightarrow y_m = x_m\}\), \(g\) will act as \(f_q: [-1, 1] \to [-1, 1]\), where \(f_q\) is a PL homeomorphism which maps \([-1, q]\) linearly onto \([-1, 0]\) and \([q, 1]\) linearly onto \([0, 1]\), and where \(q \in (-1, 1)\) will be specified later. We write \(Q = I_{n-1} \times Q_{n+1} \times I_n\), and \(x = (\hat{x}, x_n)\), where \(\hat{x} \in I_{n-1} \times Q_{n+1}\). Let \(\hat{K}\) be the projection of \(h(K')\) on \(I_{n-1} \times Q_{n+1}\). We define \(F': \hat{K} \to (-1, 1)\) by \(F'(\hat{x}) = y\), where \(y \in (-1, 1)\) is the unique point such that \((\hat{x}, y) \in h(K')\). By Tietze's lemma, \(F'\) can be extended to \(F: I_{n-1} \times Q_{n+1} \to (-1, 1)\). Define \(g\) by \(g(\hat{x}, x_n) = (\hat{x}, f_{F}(\hat{x})(x_n))\) (see Figure I.2). Then \(g\) is one-to-one onto because \(g\) leaves intervals \(L_x\) invariant and is one-to-one onto on each \(L_x\). Furthermore
$g(h(K'))$ is a subset of the hyperplane $\{x_n = 0\}$ and $g \circ h$ is $\epsilon$-close to $\text{id}_Q$ because it does not alter the first $n-1$ coordinates of any point. Finally, from the construction it follows that $x_i = \pm 1$ iff $p_1 \circ g \circ h(x) = \pm 1$. Therefore $\varphi = g \circ h$ is the desired homeomorphism.

**Corollary 1.4.** Let $K$ be a compact subset of $s$ and $\epsilon$ a positive number. Then there exists a homeomorphism $f: Q \to Q$ such that $f(BQ) = BQ$ and $f(K)$ is infinitely deficient.

**Proof.** We can write $Q = \bigoplus_{i=1}^n Q_{C_i}$ where $C_i \cap C_j = \emptyset$ and $\bigcup_i C_i = N$. We can apply Lemma 1.3 to each of the copies $Q_{C_i}$ of $Q$ and the compact subsets $P_{C_i}(K)$ of $s_{C_i}$ and obtain a homeomorphism $f_1: Q_{C_i} \to Q_{C_i}$ such that $f_1(P_{C_i}(K))$ is deficient in some coordinate $n_i \in C_i$. Then the homeomorphism $f: Q \to Q$ defined by $p_{C_i} \circ f(x) = f_1(x_{C_i})$ maps $K$
onto a set of infinite deficiency. Moreover $f$ is $\epsilon$-close to the identity mapping if the maps $f_{C_1}$ are sufficiently close to the identity.

**Proposition 1.5.** Let $f: K \to f(K)$ be a homeomorphism between two compact subsets of $s$ such that $d(f, \text{id}_K) < \epsilon$. Then $f$ can be extended to an autohomeomorphism $f'$ of $Q$ which maps $B(Q)$ onto $B(Q)$ such that $d(f', \text{id}) < \epsilon$.

**Proof.** The idea of the proof is basically due to Klee [14], and modified by Barit [6] as to satisfy the smallness condition. Let $d(f, \text{id}_K) = \epsilon_1 < \epsilon$ and let $\delta = (\epsilon - \epsilon_1)/5$. Since there exist autohomeomorphisms of $Q$ which map $K \cup f(K)$ onto a subset of $s$ of infinite deficiency and which are arbitrarily close to the identity, we may assume that $K \cup f(K) \subset Q^C \times \{0\}$, where $C$ is such that $d(p_C, \text{id}) < \delta$. For convenience of notation however, we shall write $Q \times Q$ instead of $Q^C \times Q^{1-C}$ but keep in mind that the second copy of $Q$ has a small diameter. We write $p_I(x, y) = x$ and $p_{II}(x, y) = y$. Instead of $p_I(K)$ and $p_I(f(K))$ we write $K$ and $f(K)$. The extension $f$ will be a composition $h_2^{-1}h_3h_1$. (See Figure 1.3.)

For the construction of $h_1$, apply Tietze's theorem to each of the functions $p_i \circ f$ to obtain a function $f^*: Q \to s$ which is an extension of $f: K \to f(K)$. Let, for any point $x \in (-1, 1)$, $\varphi_x: [-1, 1] \to [-1, 1]$ be the PL map
Fig. I.3
which maps \([-1,0] \) linearly onto \([-1,x]\) and \([0,1]\) linearly onto \([x,1]\). Let, for \(x \in \mathbb{R}_{+}\), \(F_x: \mathbb{Q} \to \mathbb{Q}\) be defined by \(p_i F_x(y) = \varphi_{x_i}(y_i)\). Now define \(h_1: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}\) by \(h_1(x,y) = (x,F_{f^*(x)}(y))\). Then on each set \([x] \times \mathbb{Q}\), \(h_1\) is equal to \(F_{f^*(x)}\) and maps \((x,0)\) onto \((x,f^*(x))\). In particular \(K\) is mapped onto the "graph" of \(f\). Furthermore \(d(h_1, id_{\mathbb{Q}\times \mathbb{Q}}) < \delta\). For \(h_2\) we use a similar construction, except that \(f\) is replaced by \(id^*_K\) and \(f^*\) by an extension of \(id^*_K\) to a map \(id^*\) from \(\mathbb{Q}\) into \(\mathbb{R}_{+}\).

We want \(h_3\) to map the graph of \(f\) (not of \(f^*\)) onto the graph of \(id^*_f(K)\). We use a modification of the trick for \(h_1\) and \(h_2\): let, for \(x,y \in (-1,1)\), \(\varphi_{x,y}:[-1,1] \to [-1,1]\) be the PL map which maps \([-1,x]\) linearly onto \([-1,y]\) and \([x,1]\) linearly onto \([y,1]\). Let, for \(x\) and \(y\) in \(\mathbb{R}_{+}\), \(F_{x,y}\) be defined by \(p_i F_{x,y}(z) = \varphi_{x_i,y_i}(z_i)\) for \(z \in \mathbb{Q}\). Let \(id^{**}: \mathbb{Q} \to \mathbb{R}_{+}\) be an extension of \(id^*_f(K)\), and \(f^{**}: \mathbb{Q} \to \mathbb{R}_{+}\) an extension of \(f^{-1}\). We cannot define \(h_3(x,y) = (F_{f^{**}}(y),id^{**}(y)(x,y))\) since we are not sure whether \(f^{**}\) and \(id^{**}\) are sufficiently close, i.e., \(\varepsilon + \delta\)-close together. In other words, if \(d(f^{**},id^{**})\) gets large then so does \(h_3\). But, using a Urysohn function which is 1 on \(f(K)\) and 0 on \([y|d(f^{**}(y),id^{**}(y)) > \varepsilon_1 + \delta]\), we can replace \(f^{**}\) and
id** by f† and id† which coincide with f** and id** respectively on K, and both of which coincide with the average of f** and id** for those x for which they would have been too far apart (i.e., more than ε₁+δ). We define h₃(x,y) = (Ff+(y),id+(y)(x),y). Then h⁻¹h₃h⁻¹:Q x Q → Q x Q is the desired extension of f.

Corollary 1.6. a) In Proposition 1.5, we can furthermore require that f' is the identity outside Uε(K).

b) In addition, if K ∪ f(K) projects onto a point in all but finitely many coordinates, then for some n, f' can be constructed as the product of an automorphism of Iⁿ and idⁿ⁺¹.

Proof. a) Let δ = (ε - ε₁)/8 instead of (ε - ε₁)/5. We can accomplish a) by, in the construction of h₁ and h₂, replacing f* and id* by functions which coincide with f* and id* on K and f(K) respectively, and which are zero outside a δ-neighborhood of K and f(K) respectively, using suitable Urysohn functions. In the construction of h₃, ϕₓ,y has to be replaced by ϕₓ,y' which, if x ≤ y, is the identity outside [x-δ,y+δ], maps [x-δ,x] onto [x-δ,y] and [x,y+δ] onto [y,y+δ]. The case x ≥ y is treated analogously. Furthermore, the
Urysohn function which regulates the "averaging out" of $f**$ and $id**$ has to be zero if either $f**(x)$ or $id**(x)$ is more than $\delta$ away from $f(K)$. b) Left to the reader.

The Homeomorphism Extension Theorem for Z-sets. To prove the general Homeomorphism Extension Theorem, we use another copy of $B(Q)$. A basic core set (bcs) structured on the core $\Pi_1[a_1,b_1]$, where $-1 < a_1 < b_1 < 1$, is the set $\{x \in s \mid$ for all but finitely many $i, x_i \in [a_i,b_i]\}$. Notice that one obtains the same bcs from two cores which differ in only finitely many $a_i$ and $b_i$. It is easy to prove, using a coordinatewise defined homeomorphism, that any two basic core sets are homeomorphic under an autohomeomorphism of $Q$. From the definition it follows that any basic core set is invariant under a homeomorphism $h$ with $h(B(Q)) = B(Q)$ such that $h$ changes at most finitely many coordinates of any point. It is easily seen that basic core sets are $\sigma$-compact, e.g., the basic core set structured on the core $[-\frac{1}{2},\frac{1}{2}]^\infty$ can be written as $\bigcup_n \Pi_{i<n}[-1 + \frac{1}{n}, 1 - \frac{1}{n}] \times \Pi_{i>n}[-\frac{1}{2},\frac{1}{2}]$. (The set $\{x \in s \mid$ for all but finitely many $i, x_i = 0\}$, which is the countable union of finite-dimensional compacta, might be considered as a "basic core set structured on a degenerate core"; such sets, f-d capsets, have played an important role in Infinite-Dimensional topology, but we will not concern ourselves with them).

Below we will construct an autohomeomorphism $h = \bigcup \Pi_i h_i$. 
of $Q$ which maps $B(Q)$ onto a basic core set. Convergence to a homeomorphism will be ensured by the convergence criterion. The proof involves two ideas:

1) The terms of the sequence $(h_1^1h_1^{n-1} \cdots h_1^0)$ map any given endface $W_1^+$ into higher and higher indexed endfaces and away from lower indexed endfaces, in such a way that in the limit the endface is mapped disjoint from all of $BQ$. More explicitly, there is an increasing sequence $(n_i)$ such that for any $i$, $U_{j \leq n_i}(W_j^+ \cup W_j^-)$ is mapped successively into $W_{n_i+1}^+, W_{n_i+1}^+, W_{n_i+2}^+, \cdots$ by $h_1^1, h_1^{n+1}h_1^0, h_1^{n+2}h_1^0, \cdots$. We shall write $h(i)$ for $\Pi_{j \geq i} h_j$.

2) By imposing some side-conditions on the $h_i$ (conditions 2) - 4) below), we can accomplish that for any $i$ there exists an infinite product $\prod_{j \leq n_i+1} [a_j, b_j] \times [1] \times \mathbb{Q}_{n_i+2}$ such that $h_i$ maps $W_{n_i+1}^+$ onto $\prod_{j \leq n_i} [a_j, b_j] \times [1] \times \mathbb{Q}_{n_i+2}$, $h_{i+1}^{n+1}h_i^0$ maps $W_{n_i+1}^+$ onto $\Pi_{j \leq n_i+1} [a_j, b_j] \times [1] \times \mathbb{Q}_{n_i+1}+2$, $\cdots$ and therefore that the limit $h(i) = \Pi_{j \geq i} h_j$ maps $W_{n_i+1}^+$ onto $\prod_{j \geq i} [a_j, b_j]$. Up to finitely many $j$, $(a_j)_j$ and $(b_j)_j$ will not depend on the choice of $i$; to be specific, if $i < i'$ then from $j = i'$ on, we will get the same $a_j$ and $b_j$. It will be seen that $U_i h(i)(W_{n_i+1}^+)$ is a basic
core set, where any of the sets $\Pi_j[a_j, b_j]$ referred to
above can be considered as the core. Adding to the
above the observation that $h^{(1)}(\mathbb{W}_{n-1}^{+})$ is con-
tained in $h(h_{1}^{-1} \cdots h_{i-1}^{-1})(BQ) = h(BQ)$, it is not
hard to prove that $h(BQ) = U_i h^{(1)}(\mathbb{W}_{n-1}^{+})$, and there-
fore a basic core set.

Proposition I.7. For every $\varepsilon$ there exists an autohomeomor-
phism $h \in H(Q)$ such that $h(BQ)$ is a basic core set and
$d(h, id_Q) < \varepsilon$.

Proof. Let, for the finite-dimensional cube $I^1$, the
faces $\{x | x_j = \pm 1\}$ be denoted by $F_{1,j}^\pm$. Let, for each
pair $(i,n)$ such that $i \geq n \geq 1$, $h_{i,n}^*$ be a PL-autohomeo-
morphism of $I^{i+1}$ such that the following conditions are
satisfied:

1) $d(h_{i,n}^*, id) < 2^{-i+2}$

2) $h_{i,n}^*$ maps $U_{j<i}(F_{i+1,j}^+ \cup F_{i+1,j}^-)$ into $\{x \in F_{i+1,i+1}^+|$
for all $k \leq i$, $|x_k| \leq 1-2^{-i}\}$. 

3) $h_{i,n}^*(F_{i+1,i+1}^+)$ is a product of intervals (the $i+1$st
interval degenerate) and

4) on $\{x \in F_{i+1,i+1}^+|$ for all $k \neq i, i+1$, $|x_k| \leq 1-2^{-n+1}\}$,
$h_{i,n}^*$ is linear and changes only the $n$th and $i+1$st
coordinate.

Figure I.4 is a diagram for $h_{1,1}^*$ and $h_{2,2}^*$ (the size
Fig. 1.4
of the different coordinates as shown, reflects their relative importance for the metric).

Let \( h_{i,n} = h_i \times 1_{\mathbb{Q}_{i+2}} \). Notice that for each \( i \) and \( n \), \( h_{i,n}(BQ) = BQ \). Select, using the Convergence Criterion, an increasing sequence \( (n_1)_1 \) such that \( n_0+1 = n_1 \) and the infinite left product \( h = \Pi_{i \geq 1} h_{n_i,n_i-1+1} \) is a homeomorphism. Write \( h_i = h_{n_i,n_i-1+1} \). Regarding \( h_0(W^+_{n_0+1}) \), observe that for \( x \in W^+_{n_0+1} \), by 4) \( p_1 \circ h(x) = p_1 \circ h_j \circ \cdots \circ h_1(x) \in (-1,1) \) for the smallest \( j \) such that \( n_j \geq i \). This is because \( x \) is mapped consecutively in \( \{x \in W^+_{n_1+1} \mid \text{for all } k \leq n_1, |x_k| \leq 1-2^{-n_1} \} \), in \( \{x \in W^+_{n_2+1} \mid \text{for all } k \leq n_2, |x_k| \leq 1-2^{-n_2} \} \) etc. Thus \( h(W^+_{n_0+1}) \subset s \).

By 3) and the above (see also the remarks preceding the proposition), \( h(W^+_{n_0+1}) \) is a product of closed subintervals \( [a_{j,1},b_{j,1}] \) of \((-1,1)\). By similar arguments \( h(\Pi_{i \geq 1} W^+_{n_i-1+1}) \) is a product of closed subintervals \( [a_{j,i},b_{j,i}] \) of \((-1,1)\). Moreover for all \( j \geq n_1 \), \( a_{j,i} = a_{j,1} \) and \( b_{j,i} = b_{j,1} \), as can be seen by comparing the sets \( W^+_{n_i-1+1} \) and \( h_{i-1} \circ \cdots \circ h_1(W^+_{n_0+1}) \) and their images under \( h_i, h_{i+1} \circ h_i \) etc.

The set \( \bigcup_i h(\Pi_{i \geq 1} W^+_{n_i-1+1}) \) is easily seen to be a basic core set, structured on the core \( h(W^+_{n_0+1}) = \Pi_{j}[a_{j,1},b_{j,1}] \).
We show that this set equals \( h(BQ) \): write \( h(BQ) = h(U_k W^+_k U W^-_k) \). For any \( k \) there is an \( i \) such that
\[
h^{1-1} \cdots h^{1}(W^+_k U W^-_k) \subseteq W^+_{n-1}.
\]
Therefore
\[
h(BQ) \subseteq \bigcup_i h^{1-1} \cdots h^{1}(W^+_{n-1}+1).
\]
Conversely, \( W^+_{n-1}+1 \subseteq BQ = (h^{1-1} \cdots h^{1}) (BQ) \), and thus, for each \( i \), \( h^{(i)} \subseteq h(BQ) \).

The proof is concluded by the observation that \( h \) can be made arbitrarily small by choosing \( n_1 \) large enough.

**Proposition 1.8.** a) For any basic core set \( M \), any \( Z \)-set \( K \) and any \( \epsilon > 0 \), there exists an \( h \in H(Q) \) such that \( d(h, \text{id}) < \epsilon \) and \( h(M-K) = M \) and \( h \) is the identity outside an \( \epsilon \)-neighborhood of \( K \).

b) For any \( Z \)-set \( K \) and any \( \epsilon > 0 \), there exists an \( h \in H(Q) \) such that \( d(h, \text{id}) < \epsilon \) and \( h(BQ-K) = B(Q) \) and \( h \) is the identity outside an \( \epsilon \)-neighborhood of \( K \).

**Proof.** Obviously b) is a consequence of a) and Proposition I.7. We prove a) in five steps. Let a standard \( n \)-cell in \( Q \) be any set \( \Pi_{1 \leq n}[a_1, b_1] \times \{(0,0,\cdots)\} \), where
\[-1 < a_i < b_i < 1.
\]

**Step 1. Moving \( K \) off a standard \( n \)-cell.** Let
\[
C_n = \Pi_{1 \leq n}[a_1, b_1] \times \{(0,0,\cdots)\}.
\]
For any \( \delta > 0 \), there is
an $f \in H(Q)$ such that (1) $d(f, id) < \delta$, (2) $f(K) \cap C_n = \emptyset$, (3) $f(B(Q)) = B(Q)$, (4) $f$ changes only finitely many coordinates, and therefore $f(M) = M$. For, since $K$ is a $Z$-set, there exists a map $\varphi: Q \to Q-K$ with $d(\varphi, id) < \delta/2$. Let $\eta = \min(\delta/2, d(K, \varphi(C_n)))$. Approximate $\varphi|C_n$ by an embedding $\varphi'$ such that $d(\varphi, \varphi') < \eta$ and $\varphi'(C_n)$ is a compact subset of $s$ which projects onto 0 in all but finitely many coordinates. Then $\varphi'(C_n) \cap K = \emptyset$. By Corollary 1.6, $\varphi': C_n \to \varphi'(C_n)$ can be extended to $f' \in H(Q)$ with $d(f', id) < \delta$ and $f'(B(Q)) = B(Q)$ and which changes only finitely many coordinates, and which therefore maps $M$ onto $M$. Then $f = f'^{-1}$ is the desired homeomorphism.

**Step 2. Moving $K$ off a given infinite-dimensional cube in $s$.** Now let $C = \Pi_{n=1}^\infty [a_n, b_n]$ be any infinite-dimensional subcube of $s$. Then we shall show that for any $\delta$, there exists a homeomorphism $f: Q \to Q$ satisfying (1) - (4) from step 1 with $C$ instead of $C_n$. For let $N$ be so large that $p_N(x) = p_N(y)$ implies $d(x, y) < \delta/2$. Let $C_N = \Pi_{1 \leq i \leq N} [a_i, b_i] \times \{(0,0,\ldots)\}$. Applying step 1, find $g \in H(Q)$ satisfying (1) - (4) from step 1 for $C_N$ and $\delta/2$. Then $g(K)$ is disjoint from an open neighborhood of $C_N$, and in particular disjoint from a set

$\Pi_{1 \leq i \leq N} [a_i, b_i] \times \Pi_{N+1 \leq i \leq M} [a_i, b_i] \times \mathbb{Q}^{M+1}$. Let $h$ be a map, affecting only the $N+1^{th}$ until the $M^{th}$ coordinate, which
maps $\Pi_{i\leq N}[a_i, b_i] \times \Pi_{N<i\leq M}[a'_i, b'_i] \times Q_{M+1}$ onto $\Pi_{i\leq N}[a_i, b_i] \times Q_{M+1}$. Then $hg(K)$ is disjoint from $C$, $hg$ is $\delta$-close to the identity, maps $B(Q)$ onto $B(Q)$ and changes only finitely many coordinates and therefore maps $M$ onto $M$. Therefore $f = hg$ is as desired.

**Step 3.** $f = id$ outside a small neighborhood of $C$.

According to Corollary 1.6 we may suppose that $g$ is the identity outside a small neighborhood of $C_N$. The same can be accomplished for $h$ by making use of Urysohn functions as in the proof of Proposition 1.5: Let $r: I^N \to I$ be 0 outside a small neighborhood of $\Pi_{i\leq N}[a_i, b_i]$ and 1 on $\Pi_{i\leq N}[a_i, b_i]$. Let $\psi: I^M-N \to I^M-N$ be a PL homeomorphism which maps $\Pi_{N<i\leq M}[a'_i, b'_i]$ onto $\Pi_{N<i\leq M}[a'_i, b'_i]$. Write $x = (x_I, x_{II}, x_{III})$ where $x_I = (x_1, \ldots, x_N)$, $x_{II} = (x_{N+1}, \ldots, x_M)$ and $x_{III} = (x_{M+1}, x_{M+2}, \ldots)$. Define $h'(x) = (x_I, r(x_I) \cdot \psi(x_{II}) + (1-r(x_I)) \cdot x_{II}, x_{III})$. If $N$ is chosen sufficiently large, then $f = h' \circ g$ is the identity outside an arbitrarily small neighborhood of $C$.

**Step 4.** $f = id$ outside a small neighborhood of $K$. It is also possible to find an $f \in H(Q)$, satisfying (1) - (4) of step 1 for $C$ and any $\delta > 0$, such that $f$ is the identity outside an arbitrarily small neighborhood of $K \cap C$:

Let $\{C^{(1)}, \ldots, C^{(n)}\}$ be some "canonical" decomposition of $C$ into small closed subcubes. Suppose $C^{(1)}, \ldots, C^{(k)}$ are the subcubes that intersect $K$. Construct $f_1$, such that
$f_1(K) \cap C^{(1)} = \emptyset$, say $d(f_1(K), C^{(1)}) = \delta_1$. Construct $f_2$ such that $d(f_2, \text{id}) < \delta_1/2$ and $f_2f_1(K) \cap C^{(2)} = \emptyset$. Then we still have $f_2f_1(K) \cap C^{(1)} = \emptyset$. Working our way through $C^{(3)} - C^{(k)}$ we get the desired homeomorphism.

**Step 5.** Moving $K$ off countably many cubes in $s$. Let $M = \cup_i M_i$ be any basic core set, where $(M_i)_i$ is an increasing sequence of geometrical cubes. Let $h_1 \in H(Q)$ have the properties (1) - (4) of step 1 for $\epsilon/2$ and $M_1$, and such that $h_1$ is the identity outside $U^*_\epsilon(K)$. Let $\delta_2 < \min(\epsilon/4, \frac{1}{2} \cdot d(h_1(K), M_1))$ be small enough with regard to the Convergence Criterion. Let $h_2 \in H(Q)$ satisfy (1) - (4) of step 1 for $\delta_2$ and $M_2$, and be equal to the identity outside $U^*_\epsilon(K) \cap h_1(U^*_\epsilon/2(K))$ (which is a neighborhood of $h_1(K)$). Then $d(h_2h_1(K), M_1) > \frac{1}{2}d(h_1(K), M_1)$. For the inductive step, we let $\delta_n < \min(\epsilon \cdot 2^{-n}, \delta_1 \cdot 2^{-n+1}, \cdots, \delta_{n-1} \cdot 2^{-1})$ and sufficiently small for the convergence criterion, and we let $h_n$ satisfy (1) - (4) from step 1 for $(h_{n-1} \cdots h_1)(K)$ and $M_n$ and $\delta_n$, and we let $h_n$ be the identity outside $U^*_\epsilon(K) \cap h_{n-1}^\circ \cdots h_1^\circ(U^*_\epsilon \cdot 2^{-n+1}(K))$. Then $h = \lim h_n$ has distance less than $\epsilon$ to $\text{id}_Q$ and is the identity outside $U^*_\epsilon(K)$. For any point $x$ not in $K$, $h$ is equal to some finite composition $h_n^\circ \cdots h_1^\circ$ which changes only finitely many coordinates of $x$. Therefore $h$ maps $M-K$ into $M$ and maps $Q-(M \cup K)$ into $Q-M$. Finally, $h(K)$ has positive
distance to every set $M_1$, and therefore $h(K) \cap M = \emptyset$.

But then $h(M-K) = M$, and we have proved a).

**Corollary I.9.** A closed subset $K$ of $Q$ is a $Z$-set in $Q$ iff there is an $h \in H(Q)$ which maps $K$ onto a set of infinite deficiency.

**Proof.** By Proposition I.8 b), $K$ can be mapped into $s$, and by Corollary I.4, the image of $K$ can be made infinitely deficient subsequently.

**Theorem I.10.** Let $f:K \to f(K)$ be a homeomorphism between two $Z$-sets in $Q$. Then there exists an $f'$ in $H(Q)$ which is an extension of $f$. Moreover, if $d(f,\text{id}_K) = \epsilon_1 < \epsilon$ then $f'$ can be chosen in such a way that $d(f',\text{id}_Q) < \epsilon$ and $f'$ is the identity outside an $\epsilon$-neighborhood of $K$.

**Proof.** Let $\delta = (\epsilon-\epsilon_1)/6$. Let $g \in H(Q)$ map $B(Q) \setminus (K \cup f(K))$ onto $B(Q)$ and be $\delta$-close to the identity. Then $d(gfg^{-1},\text{id}_K) < \epsilon_1 + 2\delta$. Let $h$ be an autohomeomorphism of $Q$ which extends $gfg^{-1}:g(K) \to gf(K)$ such that $d(h,\text{id}_Q) < \epsilon_1 + 3\delta$ and $h$ is the identity outside $U_{\epsilon_1+3\delta}(K)$ (which set contains $g(U_{\epsilon_1+3\delta}(K))$). Let $f' = g^{-1}hg$, then $f'$ has the required properties.

**Capsets.** In [3], R. D. Anderson introduced the concept of capset (in Theorem I.12 below, it will be shown that $B(Q)$
and any basic core set are capsets. For the closely related but more general concept of \((G,\mathcal{K})\)-skeletoid, introduced at about the same time, see Bessaga-Pełczyński [6]. A subset \(M\) of \(Q\) is a capset (for \(Q\)) if \(M\) can be written as a countable increasing union \(\bigcup_i M_i\) of \(Z\)-sets such that for each \(\epsilon > 0\), \(n > 0\) and for each \(Z\)-set \(K\) in \(Q\) there exists an \(m \geq n\) and an \(h \in H(Q)\) such that \(h(K) \subset M_m\), \(d(h,\id) < \epsilon\) and \(h|M_n = \id|_{M_n}\). Obviously the concept of capset is topologically invariant. We remark in passing that the property of being a capset, as well as that of being a \(Z\)-set, can be defined for subsets of \(s\) or \(l_2\), and that many of the theorems about \(Z\)-sets and capsets remain valid.

There exists a finite-dimensional analogue, viz. \(f\)-d capsets (see Anderson [3]). These were already briefly touched upon at the discussion of basic core sets.

**Theorem I.11.** Suppose \(M\) and \(N\) are two capsets in \(Q\).

Then for each \(\epsilon > 0\) there exists an \(h \in H(Q)\) such that \(h(M) = N\) and \(d(h,\id_Q) < \epsilon\).

**Proof.** Let the decompositions \(M = \bigcup_i M_i\) and \(N = \bigcup_i N_i\) satisfy the conditions in the definition of capset. We construct \(h\) as a composition \(\cdots g_2^{-1}h_2 g_1^{-1}h_1\). Without further mentioning, it is understood that at each stage the next homeomorphism is constructed in accordance with the con-
vergence criterion.

Applying the definition of capset for $N$, we can find $h_1 \in H(Q)$ such that for some $n_1$, $h_1(M_1) \subset N_{n_1}$. Since $h_1(M)$ is a capset we can find $g_1 \in H(Q)$ such that for some $m_1, g_1(N_{n_1}) \subset h_1(M_{m_1})$ or equivalently $g_1^{-1}h_1(M_{m_1}) \supset N_{n_1}$, and such that moreover $g_1|_{h_1(M_1)} = g_1^{-1}|_{h_1(M_1)} = \text{id}$. Then, since $h_1(M_1) \subset N_{n_1}$, also $g_1^{-1}h_1(M_1) \subset N_{n_1}$. Construct $h_2$ such that for some $n_2, h_2 \circ g_1^{-1}h_1(M_{m_1}) \subset N_{n_2}$ and $h_2|_{N_{n_1}} = \text{id}$. Then again $h_2 \circ g_1^{-1}h_1(M_{m_1}) \supset N_{n_1}$ and $h_2 \circ g_1^{-1}h_1(M_1) \subset N_{n_1}$.

Continuing with the inductive construction of maps $g_2, g_3, \ldots \in H(Q)$ which create and preserve appropriate inclusion-relations, we obtain a sequence of which the infinite left product $h = \Pi_{i=1}^\infty g_i^{-1}h_i$ is a homeomorphism with on the one hand $h|_{M_{m_1}} = h_1 \circ g_1^{-1} \cdots g_i^{-1}h_i|_{M_{m_i}}$, and therefore $h(M_{m_i}) \subset N_{n_{i+1}} \subset N$, and on the other hand $((\Pi_{j>i} g_j^{-1}h_j))|_{N_{n_i}} = \text{id}|_{N_{n_i}}$, or in other words $h^{-1}|_{N_{n_i}} = (g_1^{-1}h_1 \cdots h_i)^{-1}|_{N_{n_i}}$, and therefore $h(M) \supset h(M_{m_1}) \supset N_{n_1}$. Together these show that $h(M) = N$.

**Theorem 1.12.** a) Any basic core set is a capset.

b) The pseudo-boundary is a capset.

**Proof.** By Proposition 1.7, b) follows from a). For the
proof of a) we use the following notion: If $X$ is a countably infinite product $\prod [a_i, b_i]$, where for each $i$, $a_i < b_i$, then we call $\Pi_i(a_i, b_i)$ the pseudo-interior $\text{PsI}(X)$ of $X$. The proof is divided into two sublemmas:

**Sublemma 1.** If $N$ is a countable union of geometric subcubes $N_i$ of $s$ such that for each $i$, $N_i \subset \text{PsI}(N_{i+1})$ and such that $\bigcup_i N_i$ is dense in $Q$, then $N$ is a capset.

**Proof.** Obviously $N$ is a countable union of $Z$-sets. Notice that each $N_i$ can be written as $\prod [a_{i, j}, b_{i, j}]$, where for each $j$, $(a_{i, j})_i$ strictly decreases to $-1$ and $(b_{i, j})_i$ strictly increases to $+1$. Let a $Z$-set $K$, $\epsilon > 0$ and a positive integer $i$ be given. For each $k > i$ there is a coordinatewise defined homeomorphism $f_k : Q \rightarrow N_k$ which leaves $N_i$ pointwise fixed. For sufficiently high $k$, $d(f_k, \text{id}_Q) < \epsilon$. By applying the Homeomorphism Extension Theorem to $f_k | K \cup N_i$ we obtain the autohomeomorphism of $Q$ that proves the capset property for $N$. (This is the only place in the proof of Theorem 1.12 where we need the Homeomorphism Extension Theorem.)

For the proof of the theorem, it clearly suffices to prove the following sublemma.

**Sublemma 2.** For any $bc$ $M$ there exists an $h \in H(Q)$ such
that \( h(M) \) is a union of cubes as described in Sublemma 1.

**Proof.** Since any core can be translated coordinatewise to any other core, it is trivial that for any two basic core sets \( M \) and \( M' \), \( (Q, M) \cong (Q, M') \). Let \( M \) be the particular bcs with core \([-\frac{1}{2}, \frac{1}{2}]^\infty\). Let

\[
M_1 = \prod_{j\leq i}[-1+1/4,1-1/4] \times \prod_{j>1}I_j
\]

then \( M = U_1 M_1 \). We will construct \( h \) as an infinite left product \( \prod_{i \geq 2} h_1 \), which will converge by the Convergence Criterion, and where each \( h_1 \) maps \( M_1 \) into \( P_sI(M_1) \), while satisfying certain side-conditions.

**Step 1.** We construct \( h_2 \) such that \( h_2(M_2) \) is a geometrical cube \( \prod_{j}[a_j, b_j] \) in \( P_sT(M_2) \) and such that for points not in \( M_2 \) only finitely many coordinates are changed. For such an \( h_2 \), \( h_2(M) = M \), as the reader easily checks for himself. Let \( U_1 = [-\frac{1}{2} - 2^{-1}, \frac{1}{2} + 2^{-1}] \times Q_{1+1} \). Then \( (U_1)_1 \) is a neighborhood basis for \( M_2 \). We construct \( h_2 \) as an infinite left product \( \prod_{i \geq 2} f_1 \), where \( f_1 \) is the product of an autohomeomorphism of \( I^1 \) and the identity on \( Q_{1+1} \), and where \( f_1 \) is the identity outside \( f_{i-1} \circ \cdots \circ f_2(U_1) \).

We use the Convergence Criterion to make the left product converge.

Let \( f_2 \) be the product of the identity on \( Q_3 \) and a PL map on \( I^2 \), which maps (see Fig. 1.5) \([-\frac{1}{3}, \frac{1}{3}]^2\).
onto a square \([-\frac{1}{2} + \epsilon_2, \frac{1}{2} - \epsilon_2]\)^2 in its own interior and is the identity outside \([-\frac{3}{4}, \frac{3}{4}\]^2 (the first two coordinates of \(U_2\)). Let \(f_3\) be the product of the identity on \(Q_4\) and a PL map on \(I^3\) which shrinks \([-\frac{1}{2} + \epsilon_2, \frac{1}{2} - \epsilon_2]^2 \times [-\frac{1}{2}, \frac{1}{2}]\) to a set \([-\frac{1}{2} + \epsilon_2, \frac{1}{2} - \epsilon_2]^2 \times [-\frac{1}{2} + \epsilon_3, \frac{1}{2} - \epsilon_3]\), in such a way that \(f_3\) is the identity outside \(f_2(U_3)\), and is small enough for the convergence criterion (see Fig. 1.6). Inductively we construct the remaining \(f_1\) in a similar manner. The left product \(L_1 f_1\) is the desired homeomorphism \(h_2\). Notice that we may assume that for each \(x \in Q\) and each integer \(i\), \(x_i \leq p_i h_2(x) \leq 0\) or \(0 \leq p_i h_2(x) \leq x_i\).

**Second and Inductive Step.** By similar constructions, obtain
Fig. 1.6
a sufficiently small \( h_3 \in H(Q) \) such that
\( h_2(M_2) \subseteq \text{PsI}(h_3(M_3)) \subseteq h_3(M_3) \subseteq \text{PsI}(M_3) \). It is geometrically obvious that we can require that \( h_3|h_2(M_2) = \text{id} \), and that for each \( x \) and \( i \), \( x_i \leq p_i h_3(x) \leq 0 \) or \( 0 \leq p_i h_3(x) \leq x_i \).

Then \( h_3 h_2(M_2) = h_2(M_2) \) is contained in
\( \text{PsI}(h_3 h_2(h_2^{-1}(M_3))) = h_3(M_3) \), and \( h_3(M) = M \).

Inductively construct sufficiently small \( h_i \) such that
\( h_{i-1}(M_{i-1}) \subseteq \text{PsI}(h_i(M_i)) \subseteq h_i(M_i) \subseteq \text{PsI}(M_i) \) and such that
\( h_i|h_{i-1}(M_{i-1}) = \text{id} \) and for each \( x \) and \( k \),
\( x_k \leq p_k h_i(x) \leq 0 \) or \( 0 \leq p_k h_i(x) \leq x_k \), and \( h_i(M) = M \).

Because of the condition \( |p_k h_i(x)| \leq |x_k| \), we have for each \( i \), \( h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_i) \supseteq M_i \). Let \( h = \text{L} \Pi_i h_i \).

Then \( h(M) = U_i h(M_i) \subseteq U_i h(h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_i)) = U_i h_i(M_i) \) and
\( U_i h_i(M_i) = U_i h(h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_i)) \subseteq h(M) \), and therefore
\( h(M) = U_i h_i(M_i) \). It is easily seen that \( U_i h_i(M_i) \) is a union of geometrical cubes as described in Sublemma 1.

**Corollary I.13.** For each capset \( M \), each \( \epsilon > 0 \) and each \( Z \)-set \( K \) there exists \( h \in H(Q) \) such that
\( d(h, \text{id}_Q) < \epsilon \), \( h(M-K) = M \) and \( h = \text{id} \) outside \( U_\epsilon(K) \).

**Proof.**

By the topological equivalence of all capsets and by Theorem I.12, \( M \) is equivalent to a bcs under some \( h \in H(Q) \); since Proposition I.8 proves the corollary for basic core sets, this completes the proof.
Proposition I.14. Let \( f:K \to f(K) \) be a homeomorphism between two \( Z \)-sets in \( Q \) such that \( f(K) \cap BQ = f(K \cap BQ) \) and \( d(f, id_K) < \epsilon \); then there is an \( f' \in H(Q) \) such that \( f' \) extends \( f \), \( f'(BQ) = BQ \) and \( d(f', id_Q) < \epsilon \).

Proof. Let \( L \subset Q \) be any \( Z \)-set. Using the Homeomorphism Extension Theorem, first find \( g_1 \in H(Q) \) such that \( g_1(L) = s \); next find \( g_2 \in H(Q) \) such that \( g_2g_1(L) \cap g_1(L) = \emptyset \) and \( d(g_1^{-1}g_2g_1, id_Q) < \epsilon/2 \). Then \( g_1^{-1}g_2g_1(L) \cap L = \emptyset \).

Let \( \delta = d(g_1^{-1}g_2g_1(L), L) \). Since both \( BQ \) and \( g_1^{-1}g_2g_1(BQ) \) are cap sets, there exists a \( g_3 \in H(Q) \) such that \( d(g_3, id) < \min(\epsilon/2, \delta) \) and \( g_3g_1^{-1}g_2g_1(BQ) = BQ \). Then \( d(g_3g_1^{-1}g_2g_1(L), L) > 0 \) and \( d(g_3g_1^{-1}g_2g_1, id_Q) < \epsilon \).

Now consider \( K \cup f(K) \) as a \( Z \)-set \( L \) as above. Let \( d(f, id_K) = \epsilon_1 < \epsilon \), and \( \delta = \epsilon_1 - \epsilon \). There exists a \( \delta/4 \)-small \( \varphi \in H(Q) \) such that \( \varphi(K \cup f(K)) \cap (K \cup f(K)) = \emptyset \) and in particular \( K \cap \varphi f(K) = \emptyset \), and such that \( \varphi(BQ) = BQ \).

Define \( f^+: K \cup \varphi f(K) \to K \cup \varphi f(K) \) by \( f^+: K = \varphi f \) and \( f^+: \varphi f(K) = (\varphi f)^{-1} \). Then \( d(f^+, id_K \cup \varphi f(K)) < \epsilon_1 + \delta/4 \).

Let, by Proposition I.8, \( h \) be an autohomeomorphism of \( Q \) such that \( h(BQ - (K \cup f(K))) = BQ \) and \( d(h, id) < \delta/4 \).

Using Proposition I.5, let \( g \in H(Q) \) be such that \( d(g, id_Q) < \delta/4 + \epsilon_1 \), \( g| h(K \cup f^+(K)) = h^{-1}f| h(K \cup f^+(K)) \) and \( g(B(Q)) = B(Q) \). Then \( h^{-1}g \circ h \) is an autohomeomorphism of \( Q \) extending \( f^+ \) such that \( d(h^{-1}g, id) < \epsilon_1 + 3\delta/4 \) and
\[ h^{-1}gh(BQ) = BQ. \] We check the last statement:

\[
\begin{align*}
    h^{-1}gh(BQ) &= h^{-1}gh([BQ - (K \cup f^+(K))] \cup [BQ \cap (K \cup f^+(K))] \\
    &= h^{-1}g(BQ) \cup f^+(BQ \cap (K \cup f^+(K))) \\
    &= h^{-1}(BQ) \cup (BQ \cap (K \cup f^+(K))) \\
    &= BQ.
\end{align*}
\]

Then \( \varphi^{-1} \circ (h^{-1} \circ g \circ h) \) is the desired homeomorphism.

Corollary I.13 states that for any capset \( M \) and any \( \mathbb{Z} \)-set \( K \), \( M - K \) is again a capset. Complementary to this we have the following useful proposition.

**Proposition I.15.** The union of a capset and a \( \mathbb{Z} \)-set (countable union of \( \mathbb{Z} \)-sets) is again a capset.

**Proof.** Let \( M = \bigcup_i M_i \) be a capset, with \( \{M_i\}_i \) having the properties listed in the definition of capset. Let \( K = \bigcup_i K_i \) be a countable increasing union of \( \mathbb{Z} \)-sets.

We show that \( M \cup K \) is a capset by constructing a homeomorphism \( H:Q \to Q \) such that \( H(M \cup K) = M \). This homeomorphism will be an infinite left product \( \prod_i G_i \) such that for some increasing sequence \( \{n_i\}_i \), (1) \( G_i \) is the identity on \( M_{n_i} \), (2) \( G_{i-1} \circ \cdots \circ G_1 \) embeds \( K_{i-1} \) in \( M_{n_i} \) (thus \( G_i \) is also the identity on \( G_{i-1} \circ \cdots \circ G_1(K_{i-1}) \)), and (3) \( G_i(M \cup \bigcup_{j=1}^i K_j) = M \).

Let \( n_1 = 1 \). For the construction of \( G_1 \), choose, for
sufficiently large \( n_2 \), an embedding \( \varphi: K_1 \cup M_{n_1} \to M_{n_2} \) which is the identity on \( M_{n_1} \). Notice that \( \varphi(K_1 \cup M_{n_1}) \) is a Z-set because it is contained in a Z-set. By Corollary I.13, let \( f \in H(Q) \) be such that \( f(M - (K_1 \cup M_{n_1})) = M \) and let \( g \in H(Q) \) be such that \( g(M \cup \varphi(K_1 \cup M_{n_1})) = M \).

Let \( h \in H(Q) \) be such that \( h(M) = M \) and \( hf(K_1 \cup M_{n_1}) = g\varphi(K_1 \cup M_{n_1}) \); then \( G_1 = g^{-1}hf \) maps \( K_1 \) into \( M_{n_2} \) and \( M \cup K_1 \) onto \( M \), leaving \( M_{n_1} \) pointwise fixed. Observe that all the above homeomorphisms can be constructed arbitrarily small.

The inductive step is similar. The reader can verify for himself that, if appropriate size restrictions hold, the infinite left product \( H \) of a sequence of such homeomorphisms \( G_1 \) is an autohomeomorphism of \( Q \) mapping \( M \cup U_1 K_1 \) onto \( M \).

The following two characterizations of capsets will be needed in Chapters II and III. The first characterization (I.17) is known in the folklore, the second (I.18) is especially designed by the author for the proofs in Chapter II.

**Corollary I.16.** Suppose \( M \) is a countable union of compact subsets of \( Q \) such that

1) For every \( \epsilon > 0 \) there exists a map \( h: Q \to Q - M \) such
that \( d(h, id) < \epsilon \)

2) \( M \) contains a set \( U_i M_i \) such that for each \( i \), \( M_i \cong Q \) and \( M_i \) is a Z-set in \( M_{i+1} \)

3) For each \( \epsilon > 0 \), there exists an \( i \) and a map \( h : Q \to M_i \) such that \( d(h, id_Q) < \epsilon \).

Then \( M \) is a capset for \( Q \).

Proof. From 1) it follows that \( M \) is a countable union of Z-sets and that every compact subset of \( M \) is a Z-set. We show that \( U_i M_i \) is a capset. Let \( \epsilon, j \) and a Z-set \( K \) be given. By 3) there exist \( i > j \) and \( h : Q \to M_i \) such that \( d(h, id_Q) < \epsilon/4 \). By the Mapping Replacement Theorem there exists an embedding \( \iota : Q \to M_i \) which maps \( Q \) onto a Z-set in \( M_i \) such that \( d(h, \iota) < \epsilon/4 \). Then \( d(\iota, id_Q) < \epsilon/2 \). By the Homeomorphism Extension Theorem for \( M_i \), there exists a homeomorphism \( f : M_i \to M_i \) which extends \( \iota^{-1} |_{\iota(K \cap M_j)} \) and such that \( d(f, id) < \epsilon/2 \). Then \( f \circ g : K \to M_i \) is an embedding of \( K \) into \( M \) which is the identity on \( K \cap M_j \) and such that \( d(f \circ g, id) < \epsilon \). Extending \( f \circ g \) to an \( \epsilon \)-small \( F \in H(Q) \), we see that \( U_i M_i \) is a capset, and therefore \( M \) as well.

A map \( F = (F_t) : X \times I \to X \) is an isotopy if for each \( t \in I \), \( F_t = F(\cdot, t) : X \to X \) is an embedding. Below \( I \) is replaced by \([1, \infty] \).

Corollary I.17. Suppose \( M \) is a \( c \)-compact subset of \( Q \) such that
1) For every $\varepsilon$ there exists a map $h: Q \to \mathbb{Q} - M$ such that $d(h, \text{id}) < \varepsilon$.

2) There exists an isotopy $F = (F_t)_t: Q \times [1, \infty] \to Q$ such that $F_\infty = \text{id}_Q$ and $F|Q \times [1, \infty]$ is a 1-1 map into $M$. Then $M$ is a capset for $Q$.

Proof. Define $M_1 = F([-1 + 1/i, l - 1/i] \times [1,1])$ and $h_i : Q \to M_1$ by $h_i(x) = F_i((l-1/i)\cdot x)$. Since $\lim_{i \to \infty} d(\text{id}_Q, h_i) = 0$ and $M_1$ is a Z-set in $M_{i+1}$, the conditions of Corollary I.16 are satisfied.

Alternative definitions of Z-sets. Nowadays, the definition of Z-set presented here is the one most commonly used in Infinite-Dimensional Topology. This definition is closely related to (ii) below, which is due to Toruńczyk [11]. (v) below is the original definition of Anderson [3]. Definition (vi) is used in Chapman's "Notes on Hilbert Cube Manifolds."

Theorem I.18. For a closed subset $K$ of $Q$ the following are equivalent:

(i) $K$ is a Z-set.

(ii) For every $n > 0$, the set $\{f \in \mathbb{Q}^n \mid f(I^n) \cap K = \emptyset\}$ is dense in $\mathbb{Q}^n$ (here $X^Y$ denotes the space of all maps from $Y$ to $X$, topologized by the compact-open topology).

(iii) There exists an $h \in \mathcal{H}(Q)$ such that $h(K)$ has
infinite deficiency.

(iv) For every \( \varepsilon > 0 \) there exists an \( h \in H(Q) \) such that \( h(K) \) has infinite deficiency and \( d(h, \text{id}_Q) < \varepsilon \).

(v) For every non-empty homotopically trivial open subset \( O \) of \( Q \), the set \( O-K \) is again non-empty and homotopically trivial.

(vi) For any open subset \( O \) of \( Q \) and any open cover \( \mathcal{O} \) of \( O \), there exists a map \( f:O \to O-K \) such that \( f \) is limited by \( \mathcal{O} \) (i.e., for any \( x \in O \), \( \{x, f(x)\} \) is contained in some element of \( \mathcal{O} \)).

Proof. We show

a) (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (v)

b) (i) \( \Rightarrow \) (iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (i)

c) (i) \( \Rightarrow \) (iv)

a) (i) \( \Rightarrow \) (ii): Suppose \( K \) is a Z-set. Let \( f:T^n \to Q \) and \( \varepsilon > 0 \) be given. Let \( g:Q \to Q-K \) be such that \( d(g, \text{id}_Q) < \varepsilon \), then \( d(f, g \circ f) < \varepsilon \) and \( g \circ f(T^n) \subseteq Q-K \).

(ii) \( \Rightarrow \) (i): Suppose \( K \) satisfies (ii). Let \( n \) be sufficiently large that \( p_n(x) = p_n(y) \) implies \( d(x, y) < \varepsilon/2 \).

Let \( e:T^n \to Q \) be the natural embedding \( e(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots) \). Let \( f:T^n \to Q-K \) such that \( d(f, e) < \varepsilon/2 \).

Then \( d(f \circ p_n, \text{id}_Q) < \varepsilon \) and \( (f \circ p_n)(Q) \subseteq Q-K \).

(ii) \( \Rightarrow \) (v): Let \( O \subseteq Q \) be open, non-empty and homotopically trivial. Since \( K \) is nowhere dense, \( O-K \) is non-empty. Since \( K \) is an ANR, we only have to show that all
homotopy groups of $0-K$ are trivial. Let $n$ and $f: \partial I^{n-1} \to 0-K$ be given. Extend $f$ to $f': I^n \to 0$, using homotopic triviality of $0$. Because $K$ satisfies (ii), there exists a map $g: I^n \to Q-K$ such that $d(g, f') < \varepsilon$, where $\varepsilon < d(f'(I^n), Q-0)$ and $\varepsilon < d(f(\partial I^{n-1}), K)$. Then $f(I^n) \subset 0-K$, and if a map $F: \partial I^{n-1} \times I \to Q$ is defined by linear interpolation between $f$ and $g|\partial I^{n-1}$ then the image of $F$ lies entirely within $0-K$. Since $I^n$ is homeomorphic to the union of itself and a cylinder $\partial I^n \times I$ attached to its boundary, an extension $f^+: I^n \to 0-K$ of $f$ can be constructed from $F$ and $g$.

(v) $\Rightarrow$ (ii): Let $n$ and $f: I^n \to Q$ be given and let $\varepsilon > 0$. For sufficiently small $\delta$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon \cdot 2^{-n-1} = \varepsilon_1$ for any two points $x$ and $x'$ of $I^n$. Subdivide $I^n$ in equal subcubes $I_1, \ldots, I_k$ of diameter less than $\delta$ and let $T_i$ be the $i$-skeleton of this cellular subdivision. By induction on the skeletons one constructs $f_1: T_1 \to Q-K$, where $d(f|T_0, f_0) < \varepsilon \cdot 2^{-n-1}$ and such that for any $i$-cell $D$ of $C_1$, the diameter of $f_1(D)$ is less than $\varepsilon \cdot 2^{-n+1-1}$. This can be done by applying (v) to an open convex set containing the image of the (combinatorial) boundary of $D$. The details are left to the reader. This completes (a).

(b): (i) $\Rightarrow$ (iv) follows from the Homeomorphism Extension Theorem and the Mapping Replacement Theorem, and
(iv) ⇒ (iii) and (iii) ⇒ (i) are trivial.

(c): (i) ⇒ (vi) can be shown by embedding $K$ in an endface by an autohomeomorphism of $Q$ and applying a simple geometric argument. (vi) ⇒ (i) is trivial.

This proves the theorem.
CHAPTER II

PSEUDO-INTERIORS FOR $2^Q$ AND RELATED RESULTS

First we show (Theorem II.2) that the collection of (connected) $Z$-sets in $Q$ forms a pseudo-interior for $2^Q (C(Q))$ by verifying the conditions of Lemma I.17. Thus we rely heavily on the facts that $2^Q \cong Q$ and $C(Q) \cong Q$ [9]. As a corollary, we show that $2^{l_2} \cong l_2$ (Corollary II.3). Next these results are generalized to the manifold case (Theorem II.4 and Corollary II.5).

Notation. By $X^Y$ we mean the space of all continuous mappings from $Y$ into $X$ endowed with the compact-open topology.

Lemma II.1.  a) The collection of $Z$-sets in $Q$ is a $G_δ$ in $2^Q$.

 b) The collection of connected $Z$-sets in $Q$ is a $G_δ$ in $C(Q)$.

Proof. a) Let $J_1 = \{K \in 2^Q | \exists \epsilon < Q : K(\epsilon) \cap K = \emptyset \}$ and
d(\(g,\text{id}_Q\)) < 1/1). Obviously \(Z_1\) is an open subset of \(2^Q\) and \(Z = \cap Z_1\) is exactly the collection of Z-sets in \(Q\).

b): This is a direct consequence of a).

Remark. Lemma II.1 has a finite-dimensional analogue. In [11], Geoghegan and Summerhill give generalizations to Euclidean n-space \(E^n\) for many infinite-dimensional notions and results. In [11], Section 3, they define what they call \(Z_m\)-sets and strong \(Z_m\)-sets in \(E^n\) for \(0 \leq m \leq n-2\). For \((n,m) \neq (3,0),(4,1)\) or \((4,2)\), the \(Z_m\)-sets and strong \(Z_m\)-sets coincide. A third possible definition is:

"\(K\) is a \(Z^*_m\)-set if for all \(i \leq m+1\), the maps from \(I^i\) into \(E^n\) \(\setminus K\) lie dense in \((E^n)^I^i\)." This definition is easily seen to imply the definition of \(Z^*_m\)-set given in [11] and to be implied by the definition of strong \(Z^*_m\)-set. The collection of \(Z^*_m\)-sets can be written as a countable intersection of open sets: let, for all \(i \leq m+1\), \(\{I^i_k\}_k\) be a countable dense subset of \((E^n)^I^i\). Let

\[Z^*_m = \{K \in 2^{E^n} | \exists g \in (E^n)^I^i : g(I^i) \cap K = \emptyset \text{ and } d(g,i^i_k) < 1/k\}.\]

Then \(\cap_{i=m+1}^{k=1,2,\ldots} Z^*_m\) is exactly the collection of \(Z^*_m\)-sets. Moreover, this set is dense in \(2^{E^n}\) since the collection of finite subsets of \(E^n\) is a subcollection of it. If \(m \leq n-3\), its intersection with \(C(E^n)\) is also dense in \(C(E^n)\) since the collection of
compact connected one-dimensional rectilinear polyhedra is a subcollection and is dense in \( C(\mathbb{E}^n) \).

**Theorem II.2.** a) The collection \( \mathcal{Z} \) of Z-sets in \( Q \) is a pseudo-interior for \( 2^Q \).

b) The collection \( \mathcal{Z}_C \) of connected Z-sets in \( Q \) is a pseudo-interior for \( C(Q) \).

**Proof.** Note that Lemma I.17 is stated in terms of the pseudo-boundary and Theorem II.2 in terms of the pseudo-interior. The maps \( h \) and \( (F_t)_t \), which are asked for in the lemma will map connected sets onto connected sets, so that they prove a) and b) simultaneously.

As remarked in the Introduction, every compact subset of \( s \) is a Z-set in \( Q \). Therefore the map \( h:Q \to s \), defined by \( h(x) = (1-\epsilon) \cdot x = (1-\epsilon) \cdot x_1, (1-\epsilon) \cdot x_2, \ldots \) induces a map \( 2^h:2^Q \to 2 \) as asked for in 1) of Lemma I.17.

We shall construct \( F_t \), so that for \( K \in 2^Q \) and \( t < \infty \) the set \( F_t(K) \) will be the union of two intersecting sets, one of which carries all information about \( K \) and the other of which is not a Z-set. First we consider the case that \( t \) is an integer. We define a sequence of maps \( (f_1)_1:Q \to Q \) by

\[
f_1(x) = (1 - \frac{1}{4}) \cdot (x_1, \ldots, x_{2i}, 0, x_{2i+1}, 0, x_{2i+2}, \ldots).
\]

Obviously \( f_1(Q) \) is contained in \( s \) and projects onto 0.
in all odd coordinates \( \geq 2i+1 \). We define another auxiliary operator \( T_{j,c} : 2^Q \rightarrow 2^Q \), where \( j \geq 1 \) and \( c \in [0,2] \):

\[
T_{j,c}(K) = \{(x_1, \cdots, x_{j-1}, x_j, y, x_{j+1}, \cdots) | |y| \leq c \text{ and } |x_j + y| \leq 1 \text{ and } (x_1)_1 \in K\}.
\]

As \( c \) varies from 0 to 2, \( T_{j,c}(K) \) is transformed continuously from \( K \) into a set which occupies the whole interval in the \( j \)th direction. We have:

\[
T_{j,0}(K) = K \text{ and } T_{j,2}(p_{j-1}^{-1}(p_j(K))) = p_{j-1}^{-1}(p_j^{-1}(K)).
\]

If \( p_j(K) = \{0\} \) then \( c = 2 \) can be replaced by \( c = 1 \) in the above formula. Now we set:

\[
F_{i}(K) = T_{2i+3,\frac{1}{2}} (f_{i}(K)) \cup p_{2i+3}^{-1} (p_{2i+3}(f_{i}(K)))
\]

For every \( K \) this is a non-Z-set since the second term contains a subset of the form \( p_{j-1}^{-1}(x_1, \cdots, x_j) \) with \(-1 < x_i < 1 \) for \( i = 1, \cdots, j \). Furthermore, \( p_{2i+3}^{-1}(\frac{1}{2}) \cap F_i(K) = p_{2i+3}^{-1}(\frac{1}{2}) \cap T_{2i+3,\frac{1}{2}}(f_{i}(K)) \) is a translation of \( f_{i}(K) \) in the direction of the \( 2i+3 \)rd coordinate, and therefore the first term contains all information about \( K \), and \( F_i \) is one-to-one.

Before we describe \( f_t \) for arbitrary \( t \), we restrict ourselves to \( k = i + \frac{n-1}{n} \) where \( i \geq 1 \) and \( n \geq 1 \):


\[ f_1(x) = (1 - \frac{1}{1}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, \ldots) \]

\[ f_{1+\frac{1}{2}}(x) = (1 - \frac{1}{1+\frac{1}{2}}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, \ldots) \]

\[ f_{1+\frac{2}{3}}(x) = (1 - \frac{1}{1+\frac{2}{3}}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, \ldots) \]

\[ f_{1+\frac{3}{4}}(x) = (1 - \frac{1}{1+\frac{3}{4}}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, \ldots) \]

\[ f_{1+\frac{4}{5}}(x) = (1 - \frac{1}{1+\frac{4}{5}}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, \ldots) \]

\[ \ldots \]

\[ f_{1+1}(x) = (1 - \frac{1}{1+\frac{1}{n}}) \cdot (x_1, \ldots, x_{2i+1}, x_{2i+2}, x_{2i+3}, x_{2i+4}, x_{2i+5}, \ldots) \]

For \( t \in (1 + \frac{n-1}{n}, 1 + \frac{n}{n+1}) \), \( f_t \) is defined by linear
interpolation between \( f_{i + \frac{n}{n-1}} \) and \( f_{i + \frac{n}{n+1}} \). This way \( f_t(Q) \) projects onto 0 in all odd coordinates \( \geq 2i + 3 \) if \( t \leq i+1 \).

For \( i \geq 1 \) and \( u \in [0,1] \) we define

\[
F_{i+u}(K) = T_{2i+3}, \frac{1}{2}(1-u)^o T_{2i+5}, \frac{1}{2}u(f_{i+u}(K)) + U T_{2i+4}, 2-2u^o T_{2i+5}, 1-u(p_{2i+5}(p_{2i+5}(f_{i+u}(K)))).
\]

Note that this is consistent with the previous definition of \( F^i(K) \). We check:

1) \( (F_t)_t \) is continuous. For finite \( t \) this follows from the continuity of the operators \( T_{j,c}, 2^t \) and \( p_{j-1}^o p_j^o ; \) for \( t \to \infty \) it is easily seen that \( F_t(K) \to K \).

2) For every \( K, F_t(K) \) is a non-Z-set if \( t \) is finite, for it contains a subset of the form \( p_{j-1}^o (x_1, \ldots, x_j) \) with \(-1 < x_i < 1 \) for \( i=1, \ldots, j \).

3) \( F = (F_t)_t \) is one-to-one on \( 2^Q \times [0, \infty) \) : for the determination of \( t \) from \( F_t(K) \), note that \( t \in (i,i+1] \) iff \( p_j(F_t(K)) = [-1,1] \) for all \( j > 2i+5 \) and for no odd \( j \leq 2i+5 \). Once it is determined that \( t \in (i,i+1] \), then on that interval \( t \) is in one-to-one correspondence with \( p_{2i+5}^o F_t(K) = [-1-(t-1))/2, 1-(t-1))/2] \) (recalling that \( p_{2i+5}^o f_t(x) = 0 \) for \( x \in Q \) and \( t \leq i+1 \)).

Finally, for \( t = i+u \) and \( u \in (0,1] \),
\[ F_t(K) \cap p_{2i+3}^{-1}((1-u)/2) \cap p_{2i+5}^{-1}(u/2) \] is a copy of \( K \) in a canonical way. Note that this set does not intersect the second term

\[ T_{2i+4,2-2u}^{\ast} T_{2i+5,1-u}(p_{2i+5}^{-1}p_{2i+5}(f_{i+5}(K))) \]

since the latter set projects onto \( Q \) in the \( 2i+3 \)rd coordinate, and if \( u = 1 \) also in the \( 2i+5 \)th coordinate.

4) If \( K \) is connected, then \( F_t(K) \) is connected since \( F_t(K) \) is the union of two connected sets which intersect in \( f_t(K) \).

The following corollary answers a question posed by R. M. Schori:

**Corollary II.3.** Both the collection of compact subsets of \( l_2 \) and the collection of compact connected subsets of \( l_2 \) are homeomorphic to \( l_2 \).

**Proof.** According to [1], \( l_2 \) is homeomorphic to \( s = (-1,1)^\infty \). Thus it is sufficient to show that the collection \( \mathcal{L} \) (or \( \mathcal{L}_c \)) of closed (connected) subsets of \( Q \) which are contained in \( s \) forms a pseudo-interior for \( 2^Q \) (or \( C(Q) \)). Since this collection is a subset of \( \mathcal{P} (\mathcal{P}_c) \) we only have to verify condition 1) of Lemma I.17 and to show that \( \mathcal{P} \) and \( \mathcal{P}_c \) are \( G_\delta \)'s. But the map \( 2^h \) from the proof of Theorem II.2 actually maps \( 2^Q \) and
C(Q) in $\mathcal{L}$ and $\mathcal{L}_C$ respectively, showing 1) of Lemma 1.17. Finally, we can write $\mathcal{L}_C$ as a $G_\delta$ by $\cap_1 [K \subset Q | K$ is closed (and connected) and $p_1(K) \subset (-1,1)]$. This completes the proof of the corollary.

We have similar results about hyperspaces of Hilbert cube manifolds. A separable metric space $M$ is a Hilbert cube manifold or Q-manifold if $M$ is locally homeomorphic to $Q$. In [8], Chapman proved that every Q-manifold $M$ is triangulable, i.e., $M \cong |P| \times Q$, where $P$ is a countable locally finite complex. If $M$ is compact, then $P$ can be chosen finite and even such that $|P|$ is a combinatorial manifold with boundary. We denote points of $|P| \times Q$ by $(q,x)$ or $(q, (x_1)_1)$ and define the projection maps $p_1(q,x) = x_1$ and $p_P(q,x) = q$. For a given triangulation $M = |P| \times Q$, a closed subset $K \subset M$ is called i-deficient if $p_1(K)$ is a point, and infinitely deficient if $K$ is i-deficient for infinitely many $i$. A closed subset $K$ of a compact Q-manifold $M$ is a Z-set if for every $\epsilon$ there is a map $f: M \to M-K$ such that $d(f, id_M) < \epsilon$. Only a restricted version of the Homeomorphism Extension Theorem holds, since homotopy conditions have to be met.

**Theorem II.4.** If $M$ is a compact connected Q-manifold, then

a) the collection $2^M$ of Z-sets in $M$ is a pseudo-interior for $2^M$. 
b) the collection $Z_C^M$ of connected $Z$-sets in $M$ is a pseudo-interior for $C(M)$.

Proof. As observed above, by [8], we may write $M = |P| \times Q$, where $|P|$ is a compact finite-dimensional manifold with boundary. Again we apply Lemma I.17, where the $M$ from the lemma is $Z^M_M$ or $C(M)-Z^M_C$ respectively. As before one can prove that $Z^M_C$ and $Z^M_C$ are $G_\delta$-sets in $Z^M_M$ and $C(M)$ respectively. Condition 1) of the lemma is proved by the map $Z^h$, where $h(p,x) = (p,(1-\epsilon)\cdot x)$.

Let $H: |P| \times [1,\infty] \to |P|$ be an isotopy such that $H_\infty = \text{id}$ and $H_t(|P|) \subset |P|-|P|$ for finite $t$ (remember that we assume that $|P|$ is a compact manifold with boundary). Consider the map $F: Z^Q \times [1,\infty] \to Z^Q$ defined in the proof of Theorem II.2. Define, for $q \in |P|$ and $K \subset Q$,

$$G_t(q \times K) = \{H_t(q)\} \times F_t(K).$$

If $L$ is a subset of $|P| \times Q$, then $L$ can be written as a union

$$\bigcup_{q \in P} (q \times L_q).$$

Now define $G_t(L) = \bigcup_{q \in P} G_t(q \times L_q)$ for $q \in P(L)$. Then $G = (G_t)_t$ satisfies 2) of Lemma I.17. We need only show that $G_t(L)$ is a closed set.

From the definition of $F_t(K)$ one readily sees that

$$F_t(K) = \bigcup_{x \in K} F_t(\{x\}).$$

Therefore we can write $G_t(L) = \bigcup_{(q,x) \in L} \{H_t(q)\} \times F_t(\{x\}).$ Let $(r_1, y_1) \in L$ be a sequence in $G_t(L)$ converging to $(r, y)$. We have to show that
$(r,y) \in G_t(L)$. Let $r_1 = H_t(q_1)$ and $y_1 \in F_t(x)$, where $(q_1,x_1) \in L$. There is a subsequence $(q_{i_k},x_{i_k})$ converging to some point $(q,x) \in L$. Then $H_t(q) = \lim_{k} r_{i_k} = r$, and by continuity of $F_t$ we have that $y \in F_t([x])$. Therefore $(r,y) \in G_t(L)$.

**Corollary II.5.** For any connected $\ell_2$-manifold $M$, both the collection $2^M$ of compact subsets of $M$ and the collection $C(M)$ of connected compact subsets of $M$ are homeomorphic to $\ell_2$.

**Proof.** According to [8] we can triangulate $M = |P| \times \ell_2$, where $P$ is a locally finite simplicial complex. Of course, now we cannot assume that $|P|$ is a manifold with boundary.

Let $K$ be a compact (connected) subset of $M$, then $K$ has a closed neighborhood $|P'| \times \ell_2$, where $P'$ is a finite (connected) subcomplex of $P$. The collection $\mathcal{C}^P$ ($\mathcal{C}_C^P$) of compact (connected) subsets of $M$ which are contained in the topological interior of $|P'| \times \ell_2$ is an open neighborhood of $K$. Its closure in $2^M(C(M))$, the set $\{K \subseteq M | K$ is compact (and connected) and $K \subseteq |P'| \times \ell_2\}$, is a pseudo-interior for $2|P'| \times Q$ (for $C(|P'| \times Q)$) if we identify $\ell_2$ with $(-1,1)^\infty \subseteq Q$. This is proved by an argument similar to that in the proof of Corollary II.3.
Therefore $\mathcal{O}_P'$ ($\mathcal{O}_C'$) is an open subset of a copy of $\ell_2$, showing that $2^M(C(M))$ is an $\ell_2$-manifold.

Next we show that $2^M(C(M))$ is homotopically trivial. By [12], this will prove that $2^M(C(M))$ is homeomorphic to $\ell_2$. Let a map $f: \partial I^n \to 2^M$ (or $f: \partial I^n \to C(M)$) be given. Then $Y' = \bigcup_{y \in \partial I^n} f(y)$ is a compact union of compact sets, and therefore a compact subset of $M$. Choose a finite connected subcomplex $P'$ of $P$ and a compact convex subset $D$ of $\ell_2$ such that $Y' \subset |P'| \times D$. Then $f(\partial I^n) \subset 2|P'| \times D$ ($f(\partial I^n) \subset C(|P'| \times D)$). Moreover, $2|P'| \times D$ and $C(|P'| \times D)$ are contractible: define, for $K \in 2|P'| \times D$ ($K \in C(|P'| \times D)$) and for $t \in [0,T]$ where $T$ is sufficiently large, $H(K,t)$ to be the closed $t$-neighborhood of $K$ in some fixed convex metric for $|P'| \times D$. Then $H$ is a contraction of $2|P'| \times D$ (or $C(|P'| \times D)$). Therefore $f$ can be extended to $\bar{f}: I^n \to 2|P'| \times D \subset 2^M$ (to $\bar{f}: I^n \to C(|P'| \times D) \subset C(M)$).
CHAPTER III

PSEUDO-INTERIORS FOR $2^I$

In this chapter we show that both the collection of zero-dimensional subsets of $I$ and the collection $C$ of Cantor sets in $I$ are pseudo-interiors for $2^I$. We use Lemma I.16. It seems reasonable that similar statements are true for the hyperspace of more general spaces, but the author has been unable to prove a comparable statement even for the hyperspace of a finite graph.

In this chapter $I = [0,1]$.

Lemma III.1. a) The collection $\mathcal{C}$ of zero-dimensional closed subsets of a compact metric space $X$ is a $G_6$ in $2^X$.

b) The collection $C$ of Cantor sets in $X$ is a $G_6$ in $2^X$.

Proof. a) The collection $\mathcal{G}_n = \{ A \subset X | A$ is closed and all components of $A$ have diameter less than $1/n \}$ is an open subset of $2^X$. For let $(A_i)_i \to A$, where $A_i \notin \mathcal{G}_n$
for all $i$. We show that $A \notin \mathcal{C}$. For every $i$ there is a component $K_i$ of $A_i$ with diameter at least $1/n$. The sequence $(K_i)_k$ has a subsequence $(K_i)_k$ which converges to a set $K$ which is closed, connected and has diameter at least $1/n$ and is a subset of $A$. Therefore $A \neq \mathcal{C}_n$. b) We write $\mathcal{C}_n = \{A \subset X| A \text{ is closed and for all } x \in A, \text{ there is a } y \neq x \text{ in } A \text{ such that} \quad d(x,y) < 1/n\}$. Since Cantor sets are exactly the compact metric spaces which are zero-dimensional and have no isolated points, it follows that $\mathcal{C} = \mathcal{C}_0 \cap \mathcal{C}_n$. We show that $\mathcal{C}_n$ is an open subset of $2^X$: let $(A_i)_i \rightarrow A$, where $A_i \notin \mathcal{C}_n$ for all $i$. There is a sequence $(q_i)_i$ such that $U_{1/n}(q_i) \cap A_i = \{q_i\}$. This sequence has a limit point $q$ and it is easily seen that $U_{1/n}(q) \cap A = \{q\}$. Therefore $A \notin \mathcal{C}_n$.

**Main Lemma III.2.** There exist arbitrarily small maps $h: 2^I \rightarrow \mathcal{C}$.

**Proof.** The map $h$ will be defined as a composition

$$2^I \xrightarrow{f} \text{FSI} \xrightarrow{\text{FSI}} \text{FSC} \xrightarrow{\text{FSC}} \mathcal{C}$$

where FSI (Finite Sequences of Intervals) is a collection of finite sequences of intervals, to be defined later.
and FSC (Finite Sequences of Cantor sets) is a collection of finite sequences of topological Cantor sets, which will also be defined later on. The map $f$ will be discontinuous, but $g$, $h_N^+$ and $g\circ h_N^+$ are continuous.

In the subsequent discussion we assume a fixed $\varepsilon < \frac{1}{2}$, and $N$ is the largest integer such that $N\cdot\varepsilon \leq 1$. The map $h = g\circ h_N^+$ will have distance less than $3\varepsilon$ to the identity.

**Step 1. The set FSI.** Let $FSI_n$ be the set of all sequences of $n$ terms $<[a_1,b_1],\ldots,[a_n,b_n]>$ such that

1) $0 \leq a_1$ and $b_n \leq 1$
2) $a_{i+1} \geq b_i$, i.e., the intervals do not overlap
3) $b_i - a_i \geq 2n\cdot\varepsilon^2$ if $1 < i < n$
4) $b_i - a_i \geq n\cdot\varepsilon^2$ if $i=1,n$.

The metric on $FSI_n$ is

$$\rho_n([a_1,b_1],\ldots,[a_n,b_n],[a_1',b_1'],\ldots,[a_n',b_n']) = \max_i \max_j (|a_i - a_i'|,|b_i - b_i'|)$$

Define $FSI = \bigcup_{n=1}^{N} FSI_n$, where $N$ is defined as above. Note that for $n > N$, $FSI_n = \emptyset$ since for any element $X$ of $FSI_n$, the sum of the lengths of the intervals of $X$ is at least $(n-1)\cdot 2n\cdot\varepsilon^2 > (n-1)\cdot 2\varepsilon > 2-2\varepsilon > 1$ since $\varepsilon < \frac{1}{2}$, whereas $X$ is a collection of non-overlapping subintervals of $[0,1]$. We choose the following metric on $FSI$:
\[ p(X,Y) = p_n(X,Y) \] if \( (X,Y) \subseteq \text{FS}I \), i.e., if both \( X \) and \( Y \) consist of \( n \) intervals, and \( p(X,Y) = 1 \) if for no \( n \), \( (X,Y) \subseteq \text{FS}I_n \), i.e., if \( X \) and \( Y \) have a different number of terms.

**Step 2.** The function \( f:2^I \rightarrow \text{FS}I \). Let \( A \in 2^I \); then \( U_\varepsilon(A) \), the open \( \varepsilon \)-neighborhood of \( A \), is a finite union of disjoint subintervals of \( I \), open relative to \( I \). Let \( f(A) = <[a_1,b_1],\ldots,[a_n,b_n]> \), where the intervals \([a_1,b_1]\) are the closures of the components of \( U_\varepsilon(A) \), arranged in increasing order; e.g., if \( U_\varepsilon(A) = (a_1,b_1) \cup (b_1,b_2) \) then \( f(A) = <[a_1,b_1],[b_1,b_2]> \), and not \( <[a_1,b_2]> \).

This assignment is not continuous: Let \( A_\delta = \{0,2\varepsilon+\delta\} \). If \( \delta > 0 \), then \( f(A_\delta) = <[0,\varepsilon],[\varepsilon+\delta,3\varepsilon+\delta]> \) but if \( \delta < 0 \) then \( f(A_\delta) = <[0,3\varepsilon+\delta]> \). But apart from this phenomenon \( f \) is continuous in the following sense: Let \( \delta < \varepsilon \) and suppose for some \( A,B \in 2^I \), \( d_H(A,B) < \delta \), where \( d_H \) denotes the Hausdorff distance (see the Introduction). Then each gap of \( U_\varepsilon(A \cup B) \) (including a gap consisting of one point) corresponds to, i.e., is contained in, a gap of \( U_\varepsilon(A) \), since for \( \delta < \varepsilon \) it cannot lie left or right from \( U_\varepsilon(A) \). Conversely, each gap in \( U_\varepsilon(A) \) which has length \( \geq 2\delta \) corresponds to, i.e., contains, a gap of \( U_\varepsilon(A \cup B) \).

Let \( f_B(A) \) be a function from \( 2^I \) to \( \text{FS}I \) which is obtained from \( f(A) \) by replacing each gap in \( U_\varepsilon(A) \) which
has no counterpart in $U_\varepsilon(A \cup B)$ by a degenerate gap (see Fig. III.1);

$\text{Fig. III.1}$

\[ \text{e.g., if } f(A) = <[a_1, b_1], [a_2, b_2]> \text{ with } a_2 - b_1 < 2\delta \text{ and if } U_\varepsilon(A \cup B) = (a_1', b_2') \text{ with } 0 \leq b_2' - b_2 < \delta \text{ and if } 0 \leq a_1 - a_1' < \delta, \text{ then let } f_B(A) = <[a_1, \frac{b_1 + a_2}{2}], \frac{b_1 + a_2}{2}, b_2]>. \]

Let $f_B(A)$ eliminate the degenerate gaps thus obtained (but not the other degenerate gaps); e.g., in the above example $f_B(A) = <[a_1, b_2]>$. Then for $d_H(A, B) < \delta$ we
have \( d(f^B(A), f^A(B)) < \delta \) and also \( d(f(A), f^B(A)) < \delta \) and \( d(f(B), f^A(B)) < \delta \). These notations will be used in the proof of the continuity of \( g \circ h_N \circ f \).

**Step 3.** The set \( FSC \). Let \( C \) be a topological Cantor set such that \( C \subset I \) and \([0,1] \subset C \) and \( d_H(C, I) < \epsilon \). Let \( C(a,b) \) be the image of \( C \) under the linear map which maps 0 onto \( a \) and 1 onto \( b \). For \([a,b] \subset [0,1]\) we also have \( d_C(C(a,b), [a,b]) < \epsilon \). We define \( FSC_n \) to be the collection of all sequences of \( n \) terms \(< C(a^1_1, b^1_1), \ldots, C(a^n_n, b^n_n) > \) such that

1) \( 0 \leq a_1 \leq \cdots \leq a_n \leq 1 \)
2) \( 0 \leq b_1 \leq \cdots \leq b_n \leq 1 \)
3) \( a_i < b_i \) for \( 1 \leq i \leq n \).

Thus the sets \( C(a_i, b_i) \) may overlap. Define \( FSC = \bigcup_{n=1}^\infty FSC_n \).

The metric of \( FSC \) is somewhat analogous to that on \( FSI \):

If \( X = < C(a^1_1, b^1_1), \ldots, C(a^n_n, b^n_n) > \) and \( Y = < C(a'_1, b'_1), \ldots, C(a'_n, b'_n) > \) then \( \rho(X, Y) = \max_{1 \leq i \leq n} d_H(C(a_i, b_i), C(a'_i, b'_i)) \) and if for no \( n \), \( (X, Y) \subset FSC_n \) then \( \rho(X, Y) = 1 \).

**Step 4.** The map \( g : FSC \rightarrow C \). We simply let \( g(X) \) be the union of the terms of \( X \). Obviously \( g \) is continuous. Notice that by the characterization of Cantor sets given in the proof of Lemma III.1, \( g(X) \) is indeed a Cantor set.
Step 5. Construction of $h_N^+$. From the remark at Step 2 it is easily seen that the function
\[ \varphi: [<a_1, b_1], \ldots, [a_n, b_n>] \rightarrow \langle C(a_1, b_1), \ldots, C(a_n, b_n) \rangle \]
do not yield a continuous composition $g \circ \varphi$. Instead, we construct by induction a map $h_n: FSI_n \rightarrow FSC_n$ and set
\[ h_n^+ = \bigcup_{i=0}^{n} h_i \] (i.e., $h_n^+$ is the function which assigns $h_i(X)$ to $X$ if $X \in FSI_i$ and $i \leq n$). The following induction hypotheses should be satisfied:

1) If $X = [<a_1, b_1], \ldots, [a_n, b_n>]$, then $h_n(X) = <C(a'_1, b'_1), \ldots, C(a'_n, b'_n)>$, where $a'_1 = a_1$ and $b'_n = b_n$.

2) Additivity at "large" gaps. If $X$ can be broken up into $Y$ and $Z$ where $Y = [<a_1, b_1], \ldots, [a_i, b_i>]$ and $Z = [<a_{i+1}, b_{i+1}], \ldots, [a_n, b_n>]$, and $a_{i+1} - b_i \geq 2c^2$ then $h_n(X) = <C(a'_1, b'_1), \ldots, C(a'_n, b'_n)>$, where $h_{n-1}(Y) = <C(a'_1, b'_1), \ldots, C(a'_i, b'_i)>$, and $h_{n-1}(Z) = <C(a'_{i+1}, b'_{i+1}), \ldots, C(a'_n, b'_n)>$. In particular, by 1) $b'_i = b_i$ and $a'_{i+1} = a_{i+1}$.

3) If $X = [<a_1, b_1], \ldots, [a_i, b_i], [b_i, b_{i+1}]], \ldots, [a_n, b_n>]$, that is, if $a_{i+1} = b_i$, and if $Y = [<a_1, b_1], \ldots, [a_i, b_{i+1}], \ldots, [a_n, b_n>]$, and if, moreover, $h_{n-1}(Y) = <C(a'_1, b'_1), \ldots, C(a'_i, b'_{i+1}), \ldots, C(a'_n, b'_n)>$, then $h_n(X) = <C(a'_1, b'_1), \ldots, C(a'_i, b'_{i+1}), C(a'_i, b'_{i+1}), C(a'_{i+2}, b'_{i+2}), \ldots, C(a'_n, b'_n)>$, i.e., $a'_1 = a'_{i+1}$.
and \( b_i^' = b_{i+1} \) and \( g_{h_n}(X) = g_{h_{n-1}}(Y) \).

These induction hypotheses, and especially iii), will be seen to insure continuity of \( g \circ h_n^+ \circ f \). We give now the inductive construction of \( h_n : FS_i_n \rightarrow FSC_n \).

\( n = 1 \): set \( h_1([a_1, b_1]) = <C(a_1, b_1)> \), in accordance with i).

\( n = 2 \): let \( X = <[a_1, b_1], [a_2, b_2]> \) with both \( b_1 - a_1 \geq \epsilon^2 \) and \( b_2 - a_2 \geq \epsilon^2 \) and with \( a_2 - b_1 \geq 0 \). If \( a_2 = b_1 \) then according to iii) we have \( h_2(X) = <C(a_1, b_2), C(a_1, b_2)> \).

If \( a_2 - b_1 \geq 2\epsilon^2 \), then according to ii), we have \( h_2(X) = <C(a_1, b_1), C(a_2, b_2)> \). If \( a_2 - b_1 = t \cdot 2\epsilon^2 \) with \( 0 < t < 1 \), then \( b_1^' \) and \( a_2^' \) are constructed as in Figure III.2 (the pictures show what happens if \( t \) is large (upper pictures), and what happens if \( t \) is small, (lower pictures)).

In formulas: let \( X^* = <[a_1, b_1^*], [a_2^*, b_2]> \) be the result of enlarging the gap \((b_1, a_2)\) symmetrically from its midpoint by a factor \( 1/t \). Thus \( a_2^* - b_1^* = 2\epsilon^2 \). We put \( h_2(X) = <C(a_1, t \cdot b_1^* + (1-t) \cdot b_2), C(t \cdot a_2^* + (1-t) \cdot a_1, b_2)> \).

Note that this is consistent with the case \( a_2 = b_1 \) and \( a_2 - b_1 \geq 2\epsilon^2 \) as treated above.

\( n + 1 \): Suppose \( h_n^+ \) is already defined. Let \( X = <[a_1, b_1], \ldots, [a_{n+1}, b_{n+1}> \in FS_i_{n+1} \). If for all \( i \), \( a_{i+1} - b_i = 0 \), i.e., if all gaps are degenerate, then by repeated application of iii) we find that for all \( i \),
Note that above and below we have different $X$ but the same $X^*$. 

Fig. III.2
\( C(a_i', b_i') = C(a_1, b_{n+1}) \). If \( \max_i (a_{i+1} - b_i) \geq 2\varepsilon^2 \), then \( h_n(X) \) is determined by \( i_i \). If for several \( i \), \( a_{i+1} - b_i \geq 2\varepsilon^2 \) then it is easily seen, using \( i_i \) for \( h_n^+ \), that \( h_n+1(X) \) is independent of the choice of the gap at which \( X \) is broken up into \( Y \) and \( Z \). So let us assume that the length of the largest gap \( \max_i (a_{i+1} - b_i) \geq 2t \cdot \varepsilon^2 \) with \( 0 < t < 1 \). Let \( X^* \) be the result of widening each gap symmetrically from its midpoint by a factor \( 1/t \), so that the largest gap of \( X^* \) has width \( 2\varepsilon^2 \). Now break up \( X^* \) into \( Y \) and \( Z \), where the gap in between \( Y \) and \( Z \) has width \( 2\varepsilon^2 \). The reader may check that \( Y \) and \( Z \) are elements of \( FSI_1 \cup \cdots \cup FSI_n \), in particular that they consist of intervals of sufficient length, noting that since \( Y \) and \( Z \) have less terms than \( X \), they are allowed to consist of smaller intervals. Therefore \( h_n^+(Y) \) and \( h_n^+(Z) \) are defined. Let \( h_n^+(Y) = \langle C(a_1, b_1^*), \cdots, C(a_i, b_i^*) \rangle \) and \( h_n^+(Z) = \langle C(a_i^*, b_i^*) \rangle \). The construction of \( h_n+1(X) \) from \( h_n^+(Y) \) and \( h_n^+(Z) \) is shown in Figure III.3.

In formulas: \( h_n+1(X) = \langle C(a_1, t \cdot b_1^* + (1-t) \cdot b_{n+1}), \\
C(t \cdot a_2^* + (1-t) \cdot a_1, t \cdot b_2^* + (1-t) \cdot b_{n+1}), \cdots, C(t \cdot a_{n+1}^* + (1-t) \cdot a_1, b_{n+1}) \rangle \). Thus each Cantor set is stretched somewhat toward \( C(a_1, b_{n+1}) \): only a little if \( t \) is close to \( 1 \) and almost all the way if \( t \) is close to \( 0 \).

It is an easy exercise to check the induction hypo-
the terms of $h_{n+1}(X)$

Fig. III.3
theses and to prove that $d_H(A, goh_N^+ f(A)) < 3\varepsilon$. To show continuity, we refer to the functions $f_B$ and $f^B$, defined at Step 2. From the remarks there and the continuity of $g$ and $h_N^+$ and the fact that $goh_N^+ f_B(A) = goh_N^+ f^B(A)$ for any two $A, B \in 2^I$, we easily see that $goh_N^+ f$ is continuous.

Let $I^* = \{[t] | t \in I\} \subset 2^I$. Then $I^*$ is a Z-set in $2^I$, since the map $f: 2^I \to 2^I$ defined by $f(K) = Cl(U_e(K))$ is an $\varepsilon$-small map from $2^I$ into $2^I-I^*$. Moreover, $I^* \cap C = \emptyset$. Therefore the inclusion of $I^*$ in Lemma III.3 is harmless according to Corollary I.13.

**Lemma III.3.** The set $(2^I-O) \cup I^*$ contains a family of copies of $Q$ as asked for in Lemma I.16 sub 2).

**Proof.** For $K \subset I$, let $[a_K, b_K]$ be the smallest closed interval containing $K$. Define $M_e \subset 2^I$ by

$$M_e = \{K \subset I | K \text{ is closed and } [a_K + (1-c) \cdot (b_K - a_K), b_K] \subset K\}. $$

Let $K_e$ be the image of $K$ under a linear map which maps $a_K$ onto $a_K$ and $b_K$ onto $a_K + (1-c) \cdot (b_K - a_K)$. In formulas: $K_e = \{a_K + (1-c) \cdot (t-a_K) | t \in K\}$. Let $h_e(K)$

$$= K_e \cup [a_K + (1-c) \cdot (b_K - a_K), b_K]. $$

Then $h_e$ is a homeomorphism of $2^I$ onto $M_e$ with distance $\leq \varepsilon$ to the identity. Since Lemma III.2 and the remark on $I^*$ show that every closed subset of $(2^I-O) \cup I^*$ is a Z-set, it
follows that $M_\varepsilon$ is a Z-set in $2^I$. Because for
$\delta < \varepsilon$, $h_{\delta}^{-1}(M_\varepsilon) = M_{(\varepsilon-\delta)/(1-\delta)}$ is a Z-set in $2^I$ by the
same token, we see that $M_\varepsilon$ is a Z-set in $M_{\delta}$. Therefore the family $\{M_{1/1}\}$ satisfies 2) of Lemma 1.16, both
for $M = (2^I-\mathcal{O}) \cup \mathcal{I}^*$ and for $M = 2^I-\mathcal{O}$.

Combining Lemmas III.2 and III.3, we obtain the main
theorem of this chapter:

Theorem III.4. Both the collection of topological Cantor
sets and the collection of zero-dimensional subsets in $I$
are pseudo-interiors for $2^I$.

Finally, we mention the following conjecture:

Conjecture (R. M. Schori). The collection of finite subsets
of $I$ is an f'd capset for $2^I$. 
BIBLIOGRAPHY


VITA

Nelly Sebilla Kroonenberg was born in 1948 in Leiden, the Netherlands. She received her undergraduate training in Mathematics at the Universiteit van Amsterdam during the years 1966-1969. From 1969 till 1972 she worked at the same institution for her Masters degree, while holding a research assistantship at the Mathematical Centre in Amsterdam. Since the fall of 1972 she has been working toward her doctorate at Louisiana State University in Baton Rouge.
Candidate: Nelly Sebilla Kroonenberg

Major Field: Mathematics

Title of Thesis: Pseudo-Interiors of Hyperspaces

Approved:

R. D. Anderson
Major Professor and Chairman

James F. Irons
Dean of the Graduate School

EXAMINING COMMITTEE:

P.E. Conner

Tyrus Allmann

J. R. Darroh

Richard M. Schoen

Date of Examination:

23 April 1974