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A MONOTONE FOLLOWER CONTROL PROBLEM WITH A NONCONVEX FUNCTIONAL AND SOME RELATED PROBLEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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Contents

Abstract

A generalized one-dimensional monotone follower control problem with a nonconvex functional is considered. The controls are assumed to be nonnegative progressively measurable processes. The verification theorem for this problem is presented. A specific monotone follower control problem with a nonconvex functional is then considered in which the diffusion term is constant. The optimal control for this problem, which is explicitly given, can be viewed as tracking a standard Wiener process by a non anticipating process starting at 0. For some parameters values, the value function for this monotone follower control problem is shown to be C^2 and for other values it is shown not to be C^2. Next, a singular control problem with constant coefficients and bounded controls appearing in both the drift and diffusion terms is shown to be equivalent to an optimal stopping problem. Lastly, other various singular control problems are considered for both smoothness of their value functions and existence of their optimal control processes.

Chapter 1. Introduction

This thesis is concerned with one-dimensional optimal control problems of the monotone follower type with an infinite horizon functional in which the payoff is not convex. Problems of the type considered in this thesis but with convex payoffs have been studied by authors such as Bather and Chernoff [4], Benes, Shepp and Witsenhausen [5], Fleming and Soner [9], and Karatzas and Shreve [11, 12]. For the nonconvex monotone follower problem in this thesis, the existence and nature of both the optimal control and the optimal state process will be discussed. The smoothness of the value function will be investigated. In the case where the value function is C^2, an explicit expression for it is obtained. We also show that for some choices of the parameters of the problem, the value function is not C^2. This is a behavior not encountered in monotone follower problems with convex payoffs (e.g. in [5, 9]) in which the value function is always C^2.

Next, an optimal stopping problem equivalent to a singular control problem with the control appearing in both the drift and diffusion terms and with constant drift and diffusion coefficients is derived. Such equivalence is shown in [12] for the monotone follower problem where the control appears only in the drift term. Also, some singular control problems are considered to examine the existence of optimal controls and smoothness of the value function. Lastly, various singular control problems are considered for both smoothness of their value functions and existence of their optimal control processes. Some of these problems are variations of the monotone follower control problem dealt with in this thesis.

A typical stochastic control problem is now presented. Consider a *stochastic differential equation*

$$dX(t) = f(t, X(t), u(t, X(t)))dt + g(t, X(t), u(t, X(t)))dW(t), \quad X(s) = x, \ t > s \tag{1.1}$$

where $x, X(t), f(t,x,u), g(t,x,u)$ are real valued, and $W(t)$ is a standard 1-dimensional Brownian motion. The *drift* and *diffusion* functions $f(t,x,u), \ g(t,x,u)$ respectively are assumed to be sufficiently smooth. The function $u(t,x)$ is called a *control function* and it belongs to a set $\mathcal{U}$ of admissible control functions. Let $X^{s,x}(t)$ be a solution for the equation above. For a real valued function $G(x)$, write the expectation $EG(X^{s,x}(t))$ as $E^{s,x}G(X(t))$. A functional $V^u(s,x)$, called the *cost functional*, arising from the choice of the control function u up to an instant $\tau \leq \infty$ is defined as

$$V^u(s,x) = E^{s,x}\left(\int_s^{\tau} k(r, X(r), u(r, X(r)))dr + M(\tau, X(\tau))\right). \tag{1.2}$$

The integrand in $V^u(s,x)$ represents the running cost of the process $X(t)$ and the second term represents the cost of stopping the process $X(t)$ at the instant τ. The *stochastic control problem* is to find a control function $u^* \in \mathcal{U}$ so that the solution to the stochastic differential equation gives rise to a minimal value of $V^u(s,x)$ over all possibilities arising from other controls $u \in \mathcal{U}$. Other control problems arising from maximizing $V^u(s,x)$ can also be considered.

It can be shown, as for example in Oksendal [16], that under some assumptions on the functions f, g, k, M and the set $\mathcal{U}$, the minimal cost $V^{u^*}(s,x)$ satisfies a Hamilton-Jacobi equation

$$\min_{u \in U} \left(LV^u(s,x) + k(s,x,u)\right) = 0, \quad s < \tau, \ x \in \mathbb{R}$$

with the end condition $V^u(\tau, x) = M(\tau, x)$ for $\tau < \infty$, where $U \subset \mathbb{R}$ is the range of the control functions in $\mathcal{U}$, and L is the generator of the stochastic process $X(t)$, that is,

$$L = \frac{\partial}{\partial s} + \frac{1}{2}g^2(s,x,u)\frac{\partial^2}{\partial x^2} + f(s,x,u)\frac{\partial}{\partial x}.$$

A control u^* giving rise to the minimal cost $V^{u*}(s,x)$ is called an *optimal* control of the stochastic control problem.

If the *Hamiltonian* $H(s,x,u) = LV^u(s,x) + k(s,x,u)$ in the Hamilton-Jacobi equation is linear in the control variable u, the control problem is said to be a *singular control problem.* When the range U of the controls is unbounded, the optimal controls may take on infinite values or impulses. This causes the underlying stochastic processes $X(t)$ to have instantaneous jumps.

A more special class of control problems we will consider is as follows. Let $W(t)$ be a standard Brownian motion and $\xi(t)$ be an adapted, nondecreasing, and left-continuous process with $\xi(0) = 0$. The singular stochastic control problem of minimizing

$$V^{\xi}(x) = E\left[\int_0^{\tau} h(t, X(t))dt + \int_{[0,\tau)} f(t)d\xi(t) + g(X(\tau))\right]$$

over processes $\xi(t)$ where $h(\cdot,\cdot)$, $f(\cdot)$ and $g(\cdot)$ are given functions, and $X(t)$ satisfies

$$X(t) = x + W(t) - \xi(t), \quad 0 \leq \tau,\ x \in \mathbb{R}$$

is called the *monotone follower control problem* (cf. [12]). The relation between this formulation and the formulation in (1.1)-(1.2) will be explained in Chapter 2. The control is said to be inactive if $d\xi(t) = 0$ and it is active otherwise.

In monotone follower problems, the optimal process can be characterized by its behavior on two regions in the time-space set, an open region of inaction and its complement, a closed region of action. When the state is in the inaction set, no further action by the control is necessary. On the other hand, when the state is in the action set, an impulse control is applied for enough time to take the process to the nearest point of the common boundary. Such problems with $h(t,\cdot)$ and $g(\cdot)$ convex have been the subject of study in Bather and Chernoff [4], Benes, Shepp and Witsenhausen [5], Karatzas and Shreve [11, 12] among others. In this thesis, we consider the above problem with $h(t,x) = e^{-\alpha t}(1 - e^{-x^2})$, a nonconvex function of x, $f(t) = \lambda e^{-\alpha t}$, $g(t) = 0$ for all $t \geq 0$ and constants $\lambda, \alpha > 0$.

Another problem in this thesis is an optimal stopping problem. An optimal stopping problem (cf. [16]) is concerned with finding the optimal (minimal/maximal) value of

$$U^{\tau}(s,x) = E^{s,x}\left(\int_s^{\tau} k(r, X(r))dr + M(\tau, X(\tau))\right),$$

over all stopping times τ with respect to a Brownian motion $W(t)$. The process $X(t)$ satisfies a stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad X(s) = x, \ t > s.$$

In the case of minimization, U^{τ} will be called the stopping risk of the problem. In this thesis, for a generalization of the monotone follower control problem in Karatzas [12], an equivalent optimal stopping problem is found.

Optimal stopping problems arise in many decision problems. The main concern is to stop the process when the reward (or cost) function which depends on the process, becomes maximal (or minimal). At each moment a decision has to be made of either stopping the process or continuing it in the hope of obtaining a bigger reward (or smaller cost, respectively). This is done by choosing an optimal stopping time up to which the reward (or cost) is measured. As in singular control problems, the optimal stopping time partitions the time-space set into an (open) optimal continuation set and its complement, the optimal stopping set.

The equivalence between an optimal control problem and an optimal stopping problem was observed among others by Bather and Chernoff [4, 6], Karatzas [12], and Ocone and Weerasinghe [15] where the control appears either in the drift or the diffusion term (but not in both terms). It was observed that the continuation region in the stopping problem ought to be the region of inaction in the control problem. One of the benefits of this equivalence is that bounds of the continuation region of the stopping problem translate into bounds on the region of inaction of the control problem. In [12], the optimal stopping risk is equal to the gradient of the value function of the optimal control problem. In [16], the optimal stopping value is equal to the second derivative of the optimal value function.

In the singular control problem considered by Ocone and Weerasinghe [15], the diffusion control processes are bounded and the controlled functional is nonconvex. In this paper, they obtained the boundaries of the region of action of the control problem by transforming it to its equivalent stopping problem.

Going back to optimal control problems, Dorroh, Ferreyra and Sundar [8] considered one class of problems with the control appearing both in the drift and diffusion terms as in (1.1). The positive control functions are bounded above and away from 0. They used a parameterization of time technique for the singular control problem in 1 dimension to obtain an extended problem in 2 dimensions in which the controls can be assumed to be bounded.

Benes, Shepp, and Witsenhausen [5] solved several monotone follower type problems, among them a finite-fuel problem of optimally tracking a standard Wiener process by a non-anticipating process with an apriori bounded total variation. The cost functionals considered in [5] are again convex. The optimal process whose control is bounded was characterized using the Laplace transform.

Alvarez [3] considered other singular stochastic control problems where the functional is not necessarily convex, but in which the controlled process was restricted to the positive side on the number line. Among the assumptions made in [3] is one in which the cost functional is absolutely integrable.

In Chapter 2, a general control problem is set up together with its Hamilton-Jacobi-Bellman equation. A verification theorem is stated in which the optimal control and state are characterized. In Chapter 3 some facts which characterize the crossing times of a stochastic state process over a space interval are stated. The expectation of the exit time of the process from the interval of restriction is given. These will be used in the investigation on the smoothness of the value functions of some singular control problems in Chapter 6.

In Chapter 4 a special monotone follower problem with a nonconvex functional is presented. This problem is the main subject of our study. The existence of an optimal control process is verified and the optimal process is determined together with its action-inaction sets. The value function is explicitly computed in the case when it is C^2. Under some assumptions on the problem's parameters, the action region of the optimal control is shown to be a bounded interval. Lastly, some examples of the problem are given for certain choices of the parameters of the problem in which the value function is C^2 and in which it is not C^2.

In Chapter 5 a general control problem with constant coefficients and bounded controls is transformed into an optimal stopping problem. In the control problem considered, the control appears in both the drift and diffusion terms. The two problems are shown to be equivalent in the case the continuation set for the stopping problem (inaction set for the control problem) is a bounded interval. The equivalence is that the optimal stopping risk of the optimal stopping problem is expressed as a second order differential equation of the optimal value function of the optimal control problem. Here, the value function is assumed to be a C^4 function.

Lastly in Chapter 6, the smoothness of the value function for several singular stochastic control problems is investigated. A theorem due to Lions [14] will be used to is this regard. For the nonconvex monotone follower problem, the value function is assumed to be $\mathbb{C}^2(\mathbb{R})$. This turns out to be true under some parameter values and not true under other parameter values. Although the theorem used here does not guarantee smoothness of the value function of such a control problem, it can guarantee smoothness of a similar singular control problem (with the same dynamics and cost functional) but with bounded controls. Lastly, a theorem due to Karatzas and Shreve [12] is used to guarantee the existence of an optimal control process of the monotone follower control problem irrespective of the problem's parameter values but restricting the state process.

An example of a singular stochastic control application is a portfolio selection problem with transaction costs and allowed consumption. In a single agent investment the portfolio consists of two assets, one risk-free low yield asset (bond) with an investment of $x(s)$ dollars and the other a risky high yield asset (stock) with an investment of $y(s)$ dollars. Let $\mu \in (0,1)$ be the cost of transaction from stock to bond and let $\lambda \in (0,1)$ be the cost of transaction from bond to stock. Let $c(s) \geq 0$ be the consumption rate from the bond investment. Let $p_1(s), p_2(s)$ be the prices per share of bonds and stocks respectively. These satisfy $dp_1 = ap_1 ds$ and $dp_2 = p_2(Ads + \sigma dw(s))$ where a, A and σ are positive constants. Then $x(s)$ and $y(s)$ satisfy

$$
\begin{aligned}
dx(s) &= [ax(s) - c(s)]ds + (1-\mu)dM(s) - dN(s), \quad x(0) = x \\
dy(s) &= Ay(s)ds - dM(s) + (1-\lambda)dN(s) + \sigma y(s)dw(s), \quad y(0) = y
\end{aligned}
$$

where $M(s), N(s)$ are the total transactions up to time $s \geq 0$ from stock to bond and bond to stock respectively. For a fixed discount rate $\beta > 0$ and the utility of consumption function $l : [0, \infty) \to [0, \infty)$, the agent's goal is to maximize his discounted total utility of consumption

$$J(x, y; c, M, N) = E_{xy} \int_0^\infty e^{-\beta t} l(c(t)) dt$$

over all progressively measurable $c(\cdot), M(\cdot), N(\cdot)$ such that M, N are left continuous, nondecreasing, $M(0) = N(0) = 0$ and for all $t \geq 0$

$$c(t) \geq 0$$

$$x(t) + (1 - \mu) y(t) \geq 0$$

$$(1 - \lambda) x(t) + y(t) \geq 0.$$

Let $\mathcal{O} = \{(x, y) \in \mathbb{R}^2; x + (1 - \mu) y > 0, (1 - \lambda) x + y > 0\}$. When the utility of consumption function $l(c)$ is of Hara type

$$l(c) = \frac{1}{\gamma} c^\gamma, \; c \geq 0 \text{ and } \gamma \in (0, 1),$$

the problem can be reduced to a 1-dimensional problem. In this case Davis and Norman [7] constructed a $C^2(\mathbb{R})$ value function to the resulting 1-dimensional problem under conditions on α, A, σ, β and γ. Fleming and Soner [9] in Chapter 8 construct an optimal strategy for the problem. The optimal action/inaction (or transaction/no transaction) regions and optimal controls are given below.

Let $r = \frac{x}{x+y}$ be the ratio of the initial investment into bonds. Then the initial investment into stocks is $1 - r$. As part of the optimal strategy there exists a no-transaction region

$$\mathcal{P} = \left\{ (x, y) \in \mathcal{O}; r_1 < \frac{x}{x + y} < r_2 \right\}$$

with $r_1, r_2 \in \left[-\frac{1-\mu}{\mu}, \frac{1}{\lambda}\right]$. Define the sell bond region, SB and sell stock region SSt by

$$SB = \left\{(x,y) \in \mathcal{O}; \frac{x}{x+y} > r_2\right\}$$

$$SSt = \left\{(x,y) \in \mathcal{O}; \frac{x}{x+y} < r_1\right\}.$$

Then when $(x,y) \in SB$ the optimal strategy is to replace bonds with stocks until the processes $(x(t), y(t))$ reaches the no transaction region $\mathcal{P}$ and analogously in SSt, replace stocks with bonds until $(x(t), y(t))$ reaches the no transaction region $\mathcal{P}$. The optimal controls (which carry out this task) $M(\cdot)$ and $N(\cdot)$ turn out to be the local times at $\partial_1\mathcal{P}$ and $\partial_2\mathcal{P}$ respectively where

$$\partial_i\mathcal{P} = \left\{(x,y) \in \mathcal{O}; \frac{x}{x+y} = r_i\right\}$$

for $i = 1, 2$ and the optimal consumption rate is

$$c(x,y) = V_x(x,y)^{\frac{1}{\gamma-1}},$$

where V is the maximized total utility of consumption.

Chapter 2. General Problem

In this chapter, a general singular control problem is stated. In a general singular control problem, the optimal control may take on infinite values. In such a case, the control is replaced by its direction and cumulative magnitude. A Hamilton-Jacobi equation is derived for the value function of this problem. A verification theorem for the Hamilton-Jacobi equation generalized from one in [9] will be stated.

Let $f, g, \sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, with σ positive and bounded away from zero. The control set U is such that either $U = \mathbb{R}$ or $U = [0, \infty)$. Then U satisfies the following forward cone property. If $v \in U, \lambda \geq 0$ then $\lambda v \in U$. We will consider progressively measurable controls in the following sense: Fix an initial time $t \in [0, t_1)$. For $t \leq s \leq t_1$, let $\mathcal{B}_s$ denote the Borel σ-algebra on $[t, s]$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_s\}$ an increasing family of σ-algebras with $\mathcal{F}_s \subset \mathcal{F}$ for all $s \in [t, t_1]$. A process $u : [t, t_1] \times \Omega \to U$ is $\mathcal{F}_s$-progressively measurable if the map $u(r, \omega)$ for $r \in [t, s], \omega \in \Omega$ is $\mathcal{B}_s \times \mathcal{F}_s$-measurable for each $s \in [t, t_1]$.

A reference probability system ν means a 4-tuple

$$\nu = (\Omega, \{\mathcal{F}_s\}, P, W)$$

where $(\Omega, \mathcal{F}_{t_1}, P)$ is a probability space, $\{\mathcal{F}_s\}$ is an increasing family of $\sigma-$algebras and $W(\cdot)$ is a $\mathcal{F}_s$-adapted Brownian motion on $[t, t_1]$. The family $\{\mathcal{F}_s\}$ is called a filtration if $\mathcal{F}_{s_1} \subset \mathcal{F}_{s_2}$ whenever $s_1 < s_2$. Let ν be any reference probability system with a right continuous filtration $\{\mathcal{F}_s\}$. Let $\mathcal{U} = \{u : \mathbb{R}^+ \rightarrow U;\ u$ is progressively measurable and Lebesgue integrable$\}$ be the set of controls.

Let $\alpha, \lambda \in \mathbb{R}$, and $c : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Consider the problem of minimizing

$$F^u(x) = E^x \left(\int_0^\infty e^{-\alpha t}(c(X(t)) + \lambda u(t))dt \right) \tag{2.1}$$

over the set of controls $u \in \mathcal{U}$, where the expectation is taken with respect to the probability measure generated by the one-dimensional process $X(\cdot)$ started at $x \in \mathbb{R}$ given by

$$dX(t) = [f(X(t)) + g(X(t))u(t)]dt + \sqrt{\sigma(X(t))}dW(t), X(0) = x. \tag{2.2}$$

The associated Hamilton-Jacobi equations are linear in the control variable and in general there are no optimal controls and near optimal controls take arbitrarily large values, [8, 9]. For this reason it is convenient to reformulate the above problem by using the integral of $|u(s)|$ as the control. For $u \in \mathcal{U}$, let $\hat{u}$ and ξ be defined by

$$\hat{u}(s) = \begin{cases} |u(s)|^{-1}u(s) & \text{if } u(s) \neq 0 \\ 0 & \text{if } u(s) = 0 \end{cases} \tag{2.3}$$

$$\xi(t) = \int_0^t |u(s)|ds \tag{2.4}$$

($\hat{u}$ registers the direction of u and ξ registers the cumulative magnitude of u.) Then

$$\xi(\cdot) \in \Xi, \tag{2.5}$$

where

$$\Xi = \left\{\xi(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+; \xi(t) \text{ is } \mathcal{F}_t\text{-adapted, non-decreasing,} \right. \\ \left. \text{left-continuous and } \xi(0) = 0\right\}. \tag{2.6}$$

Assume

$$\hat{u}(s) \in U \text{ for } \gamma \text{ a.e. } s \geq 0, \tag{2.7}$$

where γ is the total variation measure of $z(\cdot)$ given by the Stieltjes integral

$$z(t) = \int_{[0,t)} \hat{u}(s) d\xi(s).$$

Further, assume

$$E|z(t)|^m < \infty, \text{ for } m = 1, 2, \dots . \tag{2.8}$$

Let $\mathcal{A}_\nu$ consisting of $(\xi(\cdot), \hat{u}(\cdot))$ satisfying (2.3)-(2.8) be the new control set. The problem is now defined to minimize

$$F(x; \xi, \hat{u}) = E^x \left(\int_0^\infty e^{-\alpha t} (c(X(t))dt + \lambda \hat{u}(t) d\xi(t)) \right), \tag{2.9}$$

with $X(\cdot)$ given by

$$dX(t) = [f(X(t))dt + g(X(t))\hat{u}(t)d\xi(t)] + \sqrt{\sigma(X(t))}dW(t), X(0) = x \tag{2.10}$$

over all $(\xi(\cdot), \hat{u}(\cdot)) \in \mathcal{A}_\nu$ and all reference probability systems ν.

Next, we state the resulting Hamilton-Jacobi equation and the verification theorem for the control problem (2.9)-(2.10). Let

$$\hat{K} = \{u \in U; |u| = 1\}$$

For $x, p \in \mathbb{R}$, define H as

$$H(x,p) = \sup_{v \in \hat{K}} v \left\{ -g(x)p - \lambda \right\} \tag{2.11}$$

and the operator $\mathcal{L}$ as

$$\mathcal{L}V(x) = \alpha V(x) - \frac{1}{2}\sigma(x)V''(x) - f(x)V'(x) - c(x). \tag{2.12}$$

Let $C_p(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ with polynomial growth, $C^1(\mathbb{R})$ the set of continuously differentiable functions on $\mathbb{R}$, $L^\infty(\mathbb{R})$ the set of essentially bounded integrable functions on $\mathbb{R}$ and $W^{k,\infty}_{loc}(\mathbb{R})$ defined as follows:

$$W^{k,\infty}_{loc}(\mathbb{R}) = \left\{ u \in L^\infty(\mathbb{R});\ \text{the weak local derivatives } D^\alpha u \in L^\infty(\mathbb{R}), \alpha \leq k \right\}. \tag{2.13}$$

Definition 2.1. *Let $W \in C_p(\mathbb{R}) \cap C^1(\mathbb{R})$ with $W' \in W^{1,\infty}_{loc}(\mathbb{R})$ be given. Define*

$$\mathcal{P} = \{x \in \mathbb{R}; H(x, W'(x)) < 0\}. \tag{2.14}$$

It is said that W is a classical solution of the Hamilton-Jacobi equation

$$\max\{\mathcal{L}W(x), H(x, W'(x))\} = 0, x \in \mathbb{R} \tag{2.15}$$

if

$$\mathcal{L}W(x) = 0 \text{ on } \mathcal{P},$$

$$H(x, W'(x)) \leq 0 \text{ for } x \in \mathbb{R},$$

$$\mathcal{L}W(x) \leq 0 \text{ for } x \in \mathbb{R}.$$

The verification theorem [9] below, gives conditions for a control process $(\xi(\cdot), \hat{u}(\cdot))$ to be optimal for the problem (2.9)-(2.10). Let $\chi_{a=b}$ be

$$\chi_{a=b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

Theorem 2.2 (Verification Theorem). *Let W be a classical solution of (2.15). Then for every $x \in \mathbb{R}$*

(a) $W(x) \leq F(x; \xi, \hat{u})$ for any $(\xi(\cdot), \hat{u}(\cdot)) \in \mathcal{A}_\nu$ such that

$$\liminf_{t \to \infty} e^{-\alpha t} E^x[W(X(t))\chi_{\tau=\infty}] = 0,$$

where $\tau = \inf\{t; X(t) = \infty\}$.

(b) Assume that $W \geq 0$ and that there exists $(\xi^*(\cdot), \hat{u}^*(\cdot)) \in \mathcal{A}_\nu$ such that w.p. 1

$$X^*(t) \in \mathcal{P}, \text{ Lebesgue a.e. } t$$
$$\int_{[0,t)} [g(X^*(s))W'(X^*(s)) + \lambda]\, \hat{u}^*(s) d\xi^*(s) = 0$$
$$W(X^*(t)) - W(X^*(t^+)) = \lambda \hat{u}^*(t)[\xi^*(t^+) - \xi^*(t)]$$
$$\lim_{t\to\infty} E^x[e^{-\alpha(t \wedge \tau)} W(X^*(t \wedge \tau))\chi_{\tau=\infty}]] = 0$$

with τ as in (a) above and $X^*(\cdot)$ satisfies (2.10). Then $F(x; \xi^*, \hat{u}^*) = W(x)$.

Remark 2.3. *If $U = [0, \infty)$ then $\hat{K} = \{1\}$ and the HJB equation (2.15) becomes*

$$\max\left\{-g(x)V'(x) - \lambda, -\frac{1}{2}\sigma(x)V''(x) - f(x)V'(x) + \alpha V(x) - c(x)\right\} = 0. \quad (2.16)$$

Chapter 3. End Conditions and Crossing Times over an Interval

This chapter contains some theorems from [1, 2, 11] stating conditions for which a stochastic process exits a predefined bounded interval. In particular, conditions for the probability of the process exiting a bounded interval and the expectation of the random exit time is given. These results will be used in Chapter 6 for investigating the smoothness of the value function of a diffusion singular control problem.

Conditions for a process starting in an interval $I = (l, r)$ with $-\infty \leq l < r \leq \infty$ to exit I are given next together with an estimate of the exit time from the interval. These are obtained in Karatzas et al. [11] and Abundo [2, 1].

Assume that a process $X(\cdot)$ defined on I as follows

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) \tag{3.1}$$

exists in the weak sense (see Karatzas and Shreve [11]). Also assume for each $x \in I$

$$\sigma^2(x) > 0 \tag{3.2}$$

$$\exists \epsilon > 0 \text{ s.t. } \int_{x-\epsilon}^{x+\epsilon} \frac{1 + |b(y)|}{\sigma^2(y)} dy < \infty. \tag{3.3}$$

Define functions $p(x)$ and $v(x)$ by

$$p(x) = \int_c^x \exp\left\{-2\int_c^\xi \frac{b(\eta)}{\sigma^2(\eta)} d\eta\right\} d\xi$$

and

$$v(x) = \int_c^x p'(y) \int_c^{\mathbf{y}} \frac{2dz}{p'(z)\sigma^2(z)} dy.$$

Consider the process $X(t)$ on the interval I. The propositions below give the probabilities of exit of $X(t)$ from I and the expectation of the processes exit time.

Proposition 3.1. *Let $S = \inf\{t \geq 0; X(t) \notin (l,r)\}$. Assume (3.2) and (3.3). $P(S < \infty) = 1$ if and only if one of the following holds:*

(i) $v(r-) < \infty$ and $v(l+) < \infty$.

(ii) $v(r-) < \infty$ and $p(l+) = -\infty$.

(iii) $v(l+) < \infty$ and $p(r-) = \infty$.

In (i) $ES < \infty$.

Proposition 3.2. *Assume (3.2)and (3.3). If a weak solution of the process (3.1) in (l,r) exists. Then*

$$P(S = \infty) = 1 \Leftrightarrow v(l+) = v(r-) = \infty$$

and

$$P(S = \infty) < 1 \Leftrightarrow v(l+) < \infty \text{ or } v(r-) < \infty.$$

Proposition 3.3. *Assume (3.2), (3.3) and that a weak solution of the process (3.1) in (l,r) exists.*

(a) Stay within (l,r): Assume $p(l+) = -\infty$ and $p(r-) = \infty$. Then

$$P(S = \infty) = P(\sup X(t) = r) = P(\inf X(t) = l) = 1.$$

That is the process is recurrent or $P(X(t) = y \text{ for some } t) = 1$ for each $y \in I$.

(b) Exit to the left with probability 1*: Assume* $p(l+) > -\infty$ *and* $p(r-) = \infty$*. Then*

$$P(\lim_{t \uparrow S} X(t) = l) = P(\sup X(t) < r) = 1.$$

(c) Exit to the right with probability 1*: Assume* $p(l+) = -\infty$ *and* $p(r-) < \infty$*. Then*

$$P(\lim_{t \uparrow S} X(t) = r) = P(\inf X(t) > l) = 1.$$

(c) Probability strictly between 0 *and* 1*: Assume* $p(l+) > -\infty$ *and* $p(r-) < \infty$*. Then*

$$P(\lim_{t \uparrow S} X(t) = l) = \frac{p(r-) - p(x)}{p(r-) - p(l+)} = 1 - P(\lim_{t \uparrow S} X(t) = r).$$

Proposition 3.4. *Assume (3.2) and (3.3) and that* $ES < \infty$*. Let* $X(0) = x$ *where* $X(\cdot)$ *satisfies (3.1). Then*

$$ES = v(r-)\frac{p(x)}{p(r-)} - v(x).$$

Chapter 4. A Monotone Follower Problem

In this chapter we obtain new results on a monotone follower control problem. The cost of the monotone follower problem we consider is given by a nonconvex functional. Our results include characterizing the action and the inaction sets and the optimal control process. Also, an explicit form for the value function of the problem is obtained in case it is twice continuously differentiable. The value function is shown to be a classical solution of the Hamilton-Jacobi equation of the control problem. For certain choices of the parameters of this monotone follower control problem, the value function is C^2 and for other choices of the parameters, it is not C^2. This explains why Theorem 6.1 due to Lions [14], stated in Chapter 6 does not guarantee even a C^1 value function for our monotone follower problem.

For any reference probability system $\nu = (\Omega, \{\mathcal{F}_t\}, P, W)$, the set of controls $\mathcal{U}$ is:

$$\mathcal{U} = \{u : [0, \infty) \to [0, \infty);\ u \text{ is integrable and progressively measurable w.r.t. } \nu\}.$$

Assume $c(x) = 1 - e^{-x^2}$. Consider the monotone follower problem of minimizing over the set of controls $u \in \mathcal{U}$ the cost

$$J^u(x) = E^x \left[\int_0^\infty e^{-\alpha t} \left(c(x(t)) + \lambda u(t) \right) dt \right], \tag{4.1}$$

where $\alpha, \lambda > 0$, with the process $x(\cdot)$ satisfying

$$dx(t) = -u(t)dt + dW(t),\ x(0) = x. \tag{4.2}$$

A monotone follower problem of this type but with a convex function $c(x) = x^2$ and with $\lambda = 1$ is studied in Benes et al. [5], Fleming et al. [9], and in Karatzas et al. [12]. In [9], the value function obtained in case the control set is $(-\infty, 0]$ is C^2. Also, the value function obtained in [5] in case the control set is $[0, \infty)$ is C^2. The value function for the problem (4.1)-(4.2) is defined as:

$$\overline{V}(x) = \min_{u \in \mathcal{U}} J^u(x). \tag{4.3}$$

The value function $\overline{V}$ is known [9] to satisfy the Hamilton-Jacobi equation

$$\max\left\{V'(x) - \lambda, \alpha V(x) - \frac{1}{2}V''(x) - c(x)\right\} = 0. \tag{4.4}$$

In this chapter, a solution $V(x)$ of (4.4) is constructed explicitly if $V \in C^2$. Then under some hypothesis on the parameters of the problem, we prove that $\overline{V} = V$ using Theorem 2.2.

4.1 Existence of an Optimal Control Process

Remark 4.1. *The value function $\overline{V}$ in (4.3) is bounded. In particular*

$$0 \leq \overline{V}(x) \leq \frac{1}{\alpha}\|c\|_{L^\infty}.$$

Remark 4.1 follows since

$$\begin{aligned} 0 \le \overline{V}(x) &\le J^0(x) & (4.5) \\ &\le \frac{1}{\alpha}\|c\|_{L^\infty}. & (4.6) \end{aligned}$$

An optimal control for the problem (4.1)-(4.2) does not generally exist. But transforming the control problem (4.1)-(4.2) as in Chapter 2 leads to a problem for which an optimal control exists in most cases. Such existence will be proved with the aid of Theorem 2.2. As in Chapter 2, the problem (4.1)-(4.2) transforms into the problem of minimizing the functional

$$J(x;\xi,\hat{u}) = E^x\left(\int_0^\infty e^{-\alpha t}(c(x(t))dt + \lambda \hat{u}(t)d\xi(t))\right), \tag{4.7}$$

where the state process $x(t)$ satisfies the equation

$$x(t) = x + W(t) - \xi(t), \text{ for } 0 \le t, \tag{4.8}$$

and the control $\xi(t)$ lies in the control set Ξ given by

$$\begin{aligned} \Xi = \big\{\xi(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+; \xi(t) \text{ is } \mathcal{F}_t\text{-adapted, non-decreasing,} \\ \text{left-continuous and } \xi(0) = 0\big\}. \end{aligned} \tag{4.9}$$

To define a candidate optimal control system $(x^*(t), \xi^*_x(t), \hat{u}^*(t))$ for the problem (4.7)-(4.8), its inaction set $\mathcal{P}$ is defined following (2.14). That is

$$\mathcal{P} = \{x \in \mathbb{R}; H(V'(x)) < 0\}, \tag{4.10}$$

where V is a bounded classical solution (in view of Remark 4.1), to this monotone follower problem's Hamilton Jacobi equation (4.4) and

$$H(p) = \sup_{v \in \hat{K}} \{pv - \lambda v\},$$

and $\hat{K} = \{1\}$ is the set of unit controls for those controls in $\mathcal{U}$. Then $H(p) = p - \lambda$ and $\mathcal{P}$ has the form

$$\mathcal{P} = \{x \in \mathbb{R}; V'(x) - \lambda < 0\}.$$

Later it will be shown that there exist numbers a, b such that

$$\mathcal{P} = (-\infty, a) \cup (b, \infty). \tag{4.11}$$

Let ξ_x^* be defined as follows

$$\xi_x^*(t) = \begin{cases} \max\left[0, \max_{0<s\leq t}\{x - a + W(s)\}\right] & \text{for } x \leq b \\ 0 & \text{for } x > b, \end{cases} \tag{4.12}$$

and $\hat{u}^*$ as

$$\hat{u}^*(t) = \begin{cases} 0 & \text{if } \xi_x^*(t) = 0 \\ 1 & \text{if } \xi_x^*(t) > 0. \end{cases} \tag{4.13}$$

where $a < b$ are positive numbers to be defined later. Let the candidate for optimal state process be

$$x^*(t) = x + W(t) - \xi_x^*(t). \tag{4.14}$$

Note that $\xi_x^*(t)$ increases only when $x^*(t) \in [a, b]$.

Next Theorem 2.2 is used to verify that the control system $(x^*(t), \xi_x^*(t), \hat{u}^*(t))$ defined above is optimal for the control problem (4.7)-(4.8). The explosion time τ in Theorem 2.2 is infinite because the Brownian motion $W(t)$ in (4.14) is finite and

$\xi_x^*(t)$ in (4.14) is defined for all times. By Theorem 2.2, the necessary conditions to verify the optimality of the control system $(x^*(t), \xi_x^*(t), \hat{u}^*(t))$ are the following:

(a) $x^*(t) \in \mathcal{P}$ a.e. t.

(b) $\int_{[0,t)}[-V'(x^*(s)) + \lambda]d\xi_x^*(s) = 0$.

(c) $V(x^*(t)) - V(x^*(t+)) = \lambda \hat{u}^*(t)(\xi_x^*(t+) - \xi_x(t))$.

(d) $\lim_{t\to\infty} E^x[e^{-\alpha t}V(x^*(t))] = 0$.

Let us show that the system $(x^*(t), \xi_x^*(t), \hat{u}^*(t))$ satisfies conditions (a)-(d). Notice that $\xi_x^*(\cdot)$ starts at zero. If $x < a$ or $x > b$ then $\xi_x^*(t) = 0$. If $a \le x \le b$ then $\xi_x^*(t) = \max[0, \max_{0<s\le t}\{x - a + W(s)\}]$ which is a left-continuous nondecreasing function. This shows that $\xi_x^* \in \Xi$. An important property of $\xi_x^*(\cdot)$ is that it increases only on a set of Lebesgue measure 0. If $a < x \le b$, then $\xi_x^*(0^+) = x - a$ otherwise $\xi_x^*(t) = 0$. Thus if $a < x \le b$, $x_x^*(0^+) = a$. The process immediately jumps to a. If $x < a$ or $x > b$, then $\xi_x^*(t) = 0$ and $x^*(t) = x + W(t)$ until $x^*(t) = a$ or $x^*(t) = b$. In this case, since $x^*(t)$ is a Brownian motion process starting at x, then $P(x^*(t) = a$ or $x^*(t) = b) = 0$ for all t. This shows that

$$x^*(t) \in \mathcal{P} \text{ a.e. } t$$

and (a) is true. Part (b) is satisfied since $\xi_x^*(t) = 0$ when $x^*(t) \in \mathcal{P}$ and $V'(x^*(t)) - \lambda = 0$ when $x^*(t) \in \mathbb{R} - \mathcal{P}$. To show (c) let $x < a$ or $x > b$. For $t < \tau = \inf\{s; x^*(s) = a, b\}$ then $x^*(t) = x + W(t)$ is a continuous function and $\xi_x^*(t) = 0$. The classical solution V also being continuous means that (c) is trivial. Now let $a < x < b$. The resulting process x^* jumps to a. It is thus enough to prove (c) for $t = 0$. On $a < x < b$, $\hat{u}(0) = 1$ and $V'(x) = \lambda$ so that

$$V(x) = \lambda(x - a) + V(a).$$

Using $\xi_x^*(0^+) = x - a$ and $\xi_x^*(0) = 0$,

$$\begin{aligned} V(x^*(0)) - V(x^*(0^+)) &= V(x) - V(a) && (4.15) \\ &= \lambda(x-a) && (4.16) \\ &= \lambda\hat{u}(0)(x-a) && (4.17) \\ &= \lambda\hat{u}(0)(\xi_x^*(0^+) - \xi_x^*(0)), && (4.18) \end{aligned}$$

proving requirement (c).

To show (d), from Remark (4.1)

$$E^x[e^{-\alpha t}V(x^*(t))] \le E^x \frac{1}{\alpha}\|c\|_{L^\infty} e^{-\alpha t}.$$

So that

$$\lim_{t\to\infty} E^x[e^{-\alpha t}V(x^*(t))] = 0.$$

By Theorem 2.2 it follows that $J(x;\xi^*,\hat{u}^*) = V(x)$.

4.2 The Value Function

This section is concerned with finding the value function (4.4) of the problem (4.1)-(4.2) explicitly and to characterize the action set as a single interval. Assume the following:

A1 $\alpha \ge 2/\pi$.

A2 The Hamilton-Jacobi equation (4.4) has a solution $V \in C^2(\mathbb{R})$.

As a consequence of Remark 4.1, the equation

$$V'(x) - \lambda = 0$$

cannot hold on unbounded intervals. Next, we show that there exists an interval (a,b) with $a>0$ such that on $(-\infty,a)$, $V(\cdot)$ satisfies

$$\alpha V(x)-\frac{1}{2}V''(x)-c(x) \;=\; 0, \tag{4.19}$$

$$V'(x)-\lambda\leq 0, \tag{4.20}$$

on (a,b), $V(\cdot)$ satisfies

$$\alpha V(x)-\frac{1}{2}V''(x)-c(x) \;\leq\; 0, \tag{4.21}$$

$$V'(x)-\lambda=0 \tag{4.22}$$

and on (b,∞), $V(\cdot)$ satisfies

$$\alpha V(x)-\frac{1}{2}V''(x)-c(x) \;=\; 0, \tag{4.23}$$

$$V'(x)-\lambda\leq 0. \tag{4.24}$$

That (a,b) is unique will be shown in Lemma 4.4.

Lemma 4.2. *Assume that there exists an interval (a,b) such that*

$$V'(x)-\lambda=0, \tag{4.25}$$

$$\alpha V(x)-c(x)\leq 0, \tag{4.26}$$

with the inequality (4.26) becoming an equality at a. Then a and b are positive.

Proof. From the fact that $c(a)=\alpha V(a)$, $c(x)\geq\alpha V(x)$ on (a,b).

$$\begin{aligned} c'(a) &= \lim_{h\to 0^+}\frac{c(a+h)-c(a)}{h} \\ &\geq \alpha\lim_{h\to 0^+}\frac{V(a+h)-V(a)}{h}=\alpha V'(a)=\alpha\lambda. \end{aligned}$$

So

$$c'(a) \geq \alpha\lambda > 0.$$

But c' is positive only on $(0, \infty)$. Therefore $a > 0$. $\square$

Remark 4.3. *It can be concluded from Lemma 4.2 that for the endpoint a*

$$c'(a) \geq \alpha\lambda. \tag{4.27}$$

This means if $\alpha\lambda > \sqrt{2/e}$*, then there is no solution for the value function* V*.*

Next to obtain an explicit form of the function V solve the equations (4.19), (4.22) and (4.23), so that

$$V(x) = \begin{cases} V_-(x), & x \leq a \\ V_0(x), & a < x < b \\ V_+(x), & b \leq x, \end{cases} \tag{4.28}$$

where

$$\begin{aligned} V_-(x) = c_1^- e^{\sqrt{2\alpha}x} + c_2^- e^{-\sqrt{2\alpha}x} &- \frac{1}{\sqrt{2\alpha}} e^{\sqrt{2\alpha}x} \int_a^x e^{-\sqrt{2\alpha}y} c(y)dy \\ &+ \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}x} \int_a^x e^{\sqrt{2\alpha}y} c(y)dy, \end{aligned} \tag{4.29}$$

$$V_0(x) = V_-(a) + \lambda(x - a), \tag{4.30}$$

$$\begin{aligned} V_+(x) = c_1^+ e^{\sqrt{2\alpha}x} + c_2^+ e^{-\sqrt{2\alpha}x} &- \frac{1}{\sqrt{2\alpha}} e^{\sqrt{2\alpha}x} \int_b^x e^{-\sqrt{2\alpha}y} c(y)dy \\ &+ \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}x} \int_b^x e^{\sqrt{2\alpha}y} c(y)dy. \end{aligned} \tag{4.31}$$

By Assumption **A2**, the inequality (4.21) becomes an equality at both a and b giving

$$\begin{aligned} \alpha V_0(a) &= c(a), && (4.32)\\ \alpha V_0(b) &= c(b). && (4.33) \end{aligned}$$

Using the fact that $V \in C^1(\mathbb{R})$ at a and b yields the four equations

$$\begin{aligned} c(a)/\alpha &= c_1^- e^{\sqrt{2\alpha}a} + c_2^- e^{-\sqrt{2\alpha}a}, && (4.34)\\ \lambda &= c_1^- \sqrt{2\alpha} e^{\sqrt{2\alpha}a} - c_2^- \sqrt{2\alpha} e^{-\sqrt{2\alpha}a}, && (4.35)\\ c(b)/\alpha &= c_1^+ e^{\sqrt{2\alpha}b} + c_2^+ e^{-\sqrt{2\alpha}b}, && (4.36)\\ \lambda &= c_1^+ \sqrt{2\alpha} e^{\sqrt{2\alpha}b} - c_2^+ \sqrt{2\alpha} e^{-\sqrt{2\alpha}b}. && (4.37) \end{aligned}$$

Equations (4.34)-(4.37) give us

$$\begin{aligned} e^{-\sqrt{2\alpha}a}(c(a)/\alpha + \lambda/\sqrt{2\alpha}) &= 2c_1^-, && (4.38)\\ e^{\sqrt{2\alpha}a}(c(a)/\alpha - \lambda/\sqrt{2\alpha}) &= 2c_2^-, && (4.39)\\ e^{-\sqrt{2\alpha}b}(c(b)/\alpha + \lambda/\sqrt{2\alpha}) &= 2c_1^+, && (4.40)\\ e^{\sqrt{2\alpha}b}(c(b)/\alpha - \lambda/\sqrt{2\alpha}) &= 2c_2^+. && (4.41) \end{aligned}$$

Using (4.38)-(4.41) the solution of (4.19) on $(-\infty, a)$ is

$$\begin{aligned} V_-(x) = \frac{c(a)}{\alpha} \cosh \sqrt{2\alpha}(x-a) &+ \frac{\lambda}{\sqrt{2\alpha}} \sinh \sqrt{2\alpha}(x-a) && (4.42)\\ &- \frac{2}{\sqrt{2\alpha}} \int_a^x c(y) \sinh \sqrt{2\alpha}(x-y) dy, \end{aligned}$$

the solution of (4.22) on (a,b) is

$$V_0(x) = \alpha c(a) + \lambda(x-a), \tag{4.43}$$

and the solution of (4.23) on (b,∞) is

$$\begin{aligned} V_+(x) = \alpha c(b)\cosh\sqrt{2\alpha}(x-b) &+ \frac{\lambda}{\sqrt{2\alpha}}\sinh\sqrt{2\alpha}(x-b) \\ &- \frac{2}{\sqrt{2\alpha}}\int_b^x c(y)\sinh\sqrt{2\alpha}(x-y)dy, \end{aligned} \tag{4.44}$$

with a and b to be determined.

4.3 Existence of the Action Set (a,b)

Write V_- on $(-\infty,a)$ in (4.29) as

$$V_-(x) = \sum_{j=1}^{4} I_j(x), \tag{4.45}$$

where the I_j $j=1,2,3,4$ represent the four terms in (4.29). Now

$$\lim_{x\to-\infty} I_1(x) = 0, \tag{4.46}$$

$$\lim_{x\to-\infty} I_3(x) = -\frac{1}{2\alpha} \tag{4.47}$$

and

$$\lim_{x\to-\infty}(I_2(x)+I_4(x)) = \lim_{x\to-\infty} e^{-\sqrt{2\alpha}x}\left(c_2^- + \frac{1}{\sqrt{2\alpha}}\int_a^x e^{\sqrt{2\alpha}y}c(y)dy\right). \tag{4.48}$$

The function V is assumed to be bounded, so that the limit in (4.48) is finite giving us

$$c_2^- = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{a} e^{\sqrt{2\alpha} y} c(y) dy. \tag{4.49}$$

Using the same reasoning with V on (b, ∞) gives us the constant c_1^+ as

$$c_1^+ = \frac{1}{\sqrt{2\alpha}} \int_{b}^{\infty} e^{-\sqrt{2\alpha} y} c(y) dy. \tag{4.50}$$

If no interval (a, b) exists for which (4.21)-(4.22) hold, it will be shown that neither (4.20) nor (4.24) can hold on $(-\infty, \infty)$. First show that (4.20) is not true on $(-\infty, \infty)$. From (4.29)

$$\begin{aligned} V_-'(x) = c_1^- \sqrt{2\alpha} e^{\sqrt{2\alpha} x} - c_2^- \sqrt{2\alpha} e^{-\sqrt{2\alpha} x} &- e^{\sqrt{2\alpha} x} \int_a^x e^{-\sqrt{2\alpha} y} c(y) dy \\ &- e^{-\sqrt{2\alpha} x} \int_a^x e^{\sqrt{2\alpha} y} c(y) dy. \end{aligned} \tag{4.51}$$

Write $V_-'(x) = \sum_{j=1}^4 I_j'(x)$ where I_j' are the terms of equation (4.51). Now

$$\lim_{x \to \infty} I_2'(x) = 0, \tag{4.52}$$

$$\lim_{x \to \infty} I_4'(x) = -\frac{1}{\sqrt{2\alpha}}. \tag{4.53}$$

Write

$$I_1'(x) + I_3'(x) = e^{\sqrt{2\alpha} x} G(x), \tag{4.54}$$

where

$$G(x) = \sqrt{2\alpha} c_1^- - \int_a^x e^{-\sqrt{2\alpha} y} c(y) dy. \tag{4.55}$$

Using (4.38) and the formula for c

$$G(x) = e^{-\sqrt{2\alpha}a}\left(\frac{c(a)}{\sqrt{2\alpha}} + \frac{\lambda}{2}\right) - \int_a^x e^{-\sqrt{2\alpha}y}(1 - e^{-y^2})dy. \tag{4.56}$$

Taking the limit and simplifying the result

$$\lim_{x\to\infty} G(x) = e^{-\sqrt{2\alpha}a}\left(\frac{\lambda}{2} - \frac{e^{-a^2}}{\sqrt{2\alpha}}\right) + e^{\frac{\alpha}{2}}\int_a^\infty e^{-\left(y+\sqrt{\frac{\alpha}{2}}\right)^2}dy. \tag{4.57}$$

As we shall see, Assumption **A1** implies that the limit in (4.57) is positive. Then

$$\lim_{x\to\infty} V_-'(x) = \infty, \tag{4.58}$$

which is contradiction because it violates the inequality (4.20). Showing that the limit in (4.57) is positive is the same as proving that

$$\frac{\lambda}{2} - \frac{e^{-a^2}}{\sqrt{2\alpha}} + e^{\sqrt{2\alpha}a+\frac{\alpha}{2}}\int_a^\infty e^{-\left(y+\sqrt{\frac{\alpha}{2}}\right)^2}dy > 0. \tag{4.59}$$

Completing the square in (4.59) gives the equivalent inequality

$$\frac{\lambda}{2} - e^{-a^2}\left(\frac{1}{\sqrt{2\alpha}} - e^{(a+\sqrt{\frac{\alpha}{2}})^2}\int_a^\infty e^{-\left(y+\sqrt{\frac{\alpha}{2}}\right)^2}dy\right) > 0, \tag{4.60}$$

which after a change of variables is

$$\frac{\lambda}{2} - e^{-a^2}\left(\frac{1}{\sqrt{2\alpha}} - e^{(a+\sqrt{\frac{\alpha}{2}})^2}\int_{a+\sqrt{\frac{\alpha}{2}}}^\infty e^{-y^2}dy\right) > 0. \tag{4.61}$$

Now Assumption **A1** and the fact that $\lambda > 0$ gives

$$\frac{\lambda}{2} > e^{-a^2}\left(\frac{1}{\sqrt{2\alpha}} - \frac{\sqrt{\pi}}{2}\right), \tag{4.62}$$

and for any positive number p

$$e^{p^2}\int_p^\infty e^{-y^2}dy > \frac{\sqrt{\pi}}{2}. \tag{4.63}$$

Combining (4.62)-(4.63) gives

$$\frac{\lambda}{2} > e^{-a^2}\left(\frac{1}{\sqrt{2\alpha}} - e^{p^2}\int_p^\infty e^{-y^2}dy\right), \tag{4.64}$$

which is the condition (4.61) with $p = a + \sqrt{\frac{\alpha}{2}}$. This finishes the proof for the existence of the left end point a of a set, say J, on which (4.21)-(4.22) must hold.

On the set J the equation for the classical solution V is (4.30) which is a linearly increasing function. Since the classical solution V is assumed to be bounded on $\mathbb{R}$, (4.21)-(4.22) cannot hold for an unbounded interval in J. The conclusion is that J is a finite union of bounded intervals. It will be proved in Section 4.6, that J can have at most one interval. Let the right end point for such interval be b. This finishes the existence of the action set (a, b).

4.4 Equations for a and b

The end points of the action set (a, b) are now defined. From (4.39), (4.32) and (4.49), an equation for a follows

$$F_1(a) \triangleq -c(a) + \sqrt{2\alpha}\int_{-\infty}^{a} e^{\sqrt{2\alpha}(y-a)}c(y)dy + \frac{\alpha\lambda}{\sqrt{2\alpha}} = 0, \tag{4.65}$$

and from (4.40), (4.33) and (4.50), an equation for b follows

$$F_2(b) \triangleq c(b) - \sqrt{2\alpha}\int_{b}^{\infty} e^{\sqrt{2\alpha}(b-y)}c(y)dy + \frac{\alpha\lambda}{\sqrt{2\alpha}} = 0. \tag{4.66}$$

Define a and b as the smallest and largest solutions of equations (4.65) and (4.66) respectively.

4.5 The Verification Inequalities

In this section, the verification inequalities (4.20), (4.21) and (4.24) are verified.

To show that (4.20) holds on $(-\infty, a)$ take limits on the I'_j terms in equation (4.51) as $x \to -\infty$

$$\lim_{x \to -\infty} I'_1(x) = 0, \tag{4.67}$$

$$\lim_{x \to -\infty} I'_3(x) = \frac{1}{\sqrt{2\alpha}} c(-\infty), \tag{4.68}$$

$$\lim_{x \to -\infty} (I'_2(x) + I'_4(x)) = -\frac{1}{\sqrt{2\alpha}} c(-\infty), \tag{4.69}$$

where in (4.68) L'Hopital's rule is used and in (4.69) the equation (4.49) and L'Hopital's rule are used. It can be concluded that

$$\lim_{x \to -\infty} (V'_-(x) - \lambda) = -\lambda < 0. \tag{4.70}$$

In the same way

$$\lim_{x \to \infty} (V'_+(x) - \lambda) = -\lambda < 0. \tag{4.71}$$

The solutions a and b in Section 4.4 satisfy

$$V'(a) - \lambda = 0$$

$$V'(b) - \lambda = 0,$$

because the classical solution V satisfies Assumption **A2** and (4.19)-(4.24). Let $c < a$. The equation (4.65) with c replacing a fails to be true only if atleast one of the equations (4.34) or (4.35) from the C^1 property of V or the condition (4.49) from boundedness of V, fails to hold. Since V has to be bounded, (4.49) is always true. The only possibility left is of (4.35) failing because V cannot be differentiable without being continuous at c. Therefore, for $c < a$

$$V'(c) - \lambda < 0.$$

In the same way for $c > b$

$$V'(c) - \lambda < 0.$$

Then for $x < a$ or $x > b$,

$$V'(x) - \lambda < 0$$

which confirms (4.20) and (4.24).

Next, show the inequality (4.21) on (a, b). Recall that

$$V_0(x) = V_-(a) + \lambda(x - a). \tag{4.72}$$

Then

$$V_-(a) + \lambda(b - a) = V_+(b). \tag{4.73}$$

Using equations (4.32)-(4.33) gives a relation between a and b

$$\frac{c(b) - c(a)}{b - a} = \alpha\lambda. \tag{4.74}$$

To show the inequality (4.21) on (a,b) means showing that

$$\alpha V_{-}(a) + \alpha\lambda(x-a) - c(x) \le 0, \ \ x \in (a,b) \tag{4.75}$$

or

$$\frac{c(x)-c(a)}{x-a} \ge \alpha\lambda. \tag{4.76}$$

Using the fact that c'' changes sign once from $+$ to $-$ on $(0,\infty)$, the smallest value of $\frac{c(x)-c(a)}{x-a}$ for x in the interval (a,b) occurs either at a or b. Therefore

$$\frac{c(x)-c(a)}{x-a} \ge \min\left\{c'(a), \frac{c(b)-c(a)}{b-a}\right\}. \tag{4.77}$$

The relation (4.74) and Remark 4.3 imply

$$\begin{aligned}\frac{c(x)-c(a)}{x-a} &\ge \min\{c'(a), \alpha\lambda\} &(4.78)\\ &\ge \min\{\alpha\lambda, \alpha\lambda\} &(4.79)\\ &= \alpha\lambda. &(4.80)\end{aligned}$$

4.6 Uniqueness of (a,b)

Lemma 4.4. *The interval (a,b) where*

$$V'(x) - \lambda = 0 \tag{4.81}$$

and

$$\alpha V(x) - c(x) \le 0, \tag{4.82}$$

with the inequality (4.82) becoming an equality at a and b is unique.

Proof. Existence of such an interval has been shown under Assumptions **A1-A2**. Suppose (4.81) and (4.82) hold on intervals say (a_1, b_1) and (a_2, b_2) with $b_1 < a_2$. Let $I \triangleq (a_1, b_1) \cup (a_2, b_2)$. Since V is linear on I then

$$\frac{V(a_1) - V(b_1)}{a_1 - b_1} = \frac{V(a_2) - V(b_2)}{a_2 - b_2} = \lambda.$$

From the fact that V is continuous and the inequality (4.82) is an equality at a_1, b_1 and a_2, b_2,

$$V(a_i) = \frac{1}{\alpha}c(a_i), \; V(b_i) = \frac{1}{\alpha}c(b_i) \quad \text{for } i = 1, 2, \tag{4.83}$$

meaning that

$$\frac{c(a_1) - c(b_1)}{a_1 - b_1} = \frac{c(a_2) - c(b_2)}{a_2 - b_2} = \alpha\lambda.$$

Using the Mean Value Theorem there exists $k_1 \in (a_1, b_1)$ and $k_2 \in (a_2, b_2)$ such that

$$c'(k_1) = c'(k_2) = \alpha\lambda. \tag{4.84}$$

If $\alpha\lambda \geq \sqrt{2}e^{-\frac{1}{2}}$ then this is a contradiction because $|c'(x)| \leq \sqrt{2}e^{-\frac{1}{2}}$. Lemma 4.2 implies the end points a_1, b_1, a_2, b_2 are positive. If $\alpha\lambda < \sqrt{2}e^{-\frac{1}{2}}$ then the nature of c' implies that

$$c''(k_1) \;>\; 0, \tag{4.85}$$

$$c''(k_2) \;<\; 0. \tag{4.86}$$

Now using the facts that $\alpha V(x) \leq c(x)$ on (a_1, b_1) and the equalities (4.83)

$$\begin{aligned} c'(a_1) &= \lim_{h \to 0^+} \frac{c(a_1 + h) - c(a_1)}{h} \\ &\geq \alpha \lim_{h \to 0^+} \frac{V(a_1 + h) - V(a_1)}{h} = \alpha V'(a_1) = \alpha\lambda, \end{aligned}$$

and

$$\begin{aligned} c'(b_1) &= \lim_{h\to 0^+} \frac{c(b_1) - c(b_1 - h)}{h} \\ &\leq \alpha \lim_{h\to 0^+} \frac{V(b_1) - V(b_1 - h)}{h} = \alpha V'(b_1) = \alpha\lambda. \end{aligned}$$

Using (4.84) there exists $r_1 \in (a_1, k_1)$ and $r_2 \in (k_1, b_1)$ such that

$$c''(r_1) \leq 0, \tag{4.87}$$

$$c''(r_2) \leq 0. \tag{4.88}$$

Since $c'' = 0$ at a single place on $[0, \infty)$, equations (4.85), (4.87)-(4.88) make this a contradiction.

Therefore the pair a, b is unique. $\square$

Remark 4.5. *Lemmas 4.4 and 4.2 are still true for an array of functions say*

$$\begin{aligned} c(x) &= \frac{x^2}{1 + x^2}, \\ c(x) &= arctan(x^2). \end{aligned}$$

4.7 Solutions for a and b.

Here some examples of solutions of equations (4.65)-(4.66) are given. The equations (4.65)-(4.66) for a and b sometimes do not have a solution either when the assumptions are not satisfied or when the bounds in Remark (4.3) are not satisfied. For various values of the parameters of the monotone follower problem, the graphs below represent the functions F_1 and F_2 in the equations (4.65)-(4.66) for a and b respectively.

Example 4.6. *Let $\alpha = 1$ and $\lambda = 0.1$. Assumption **A1** is satisfied. The relevant solutions are $a = 0.58678, b = 1.51763$.*

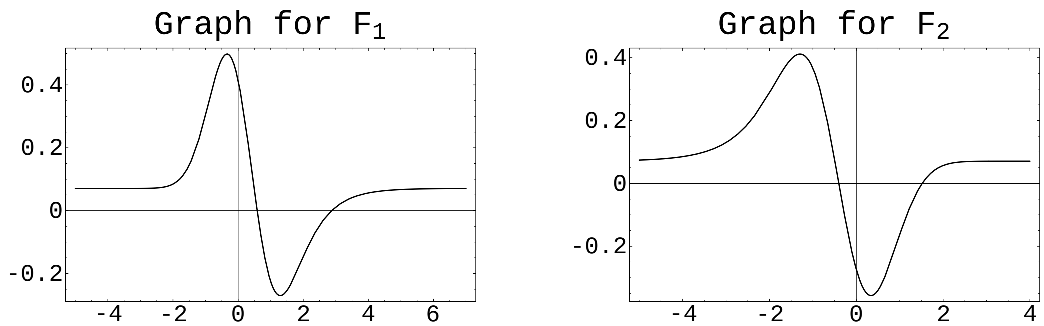

Figure 4.1: Parameter values: $\alpha = 1$, $\lambda = 0.1$.

Example 4.7. *Let $\alpha = 1$ and $\lambda = 1$. Assumption **A1** is still satisfied but there are no solutions for a and b. This implies that Assumption **A2** does not hold, that is, there is no C^2 value function to the monotone follower control problem with these parameters.*

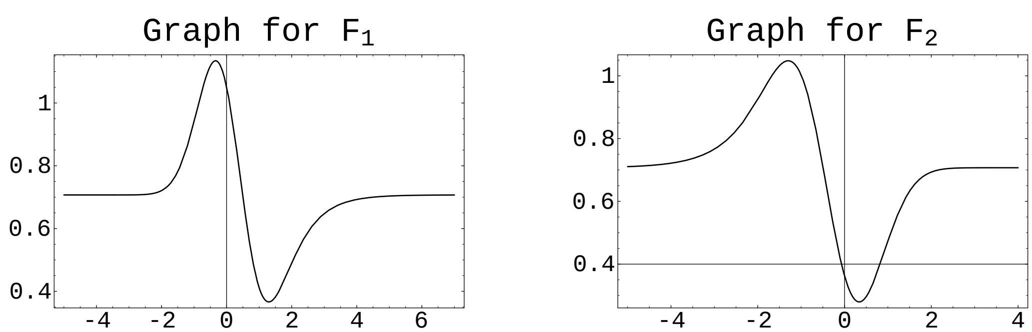

Figure 4.2: Parameter values: $\alpha = 1$, $\lambda = 1$.

Example 4.8. *Let $\alpha = 0.2$ and $\lambda = 4$. Then Assumption **A1** is not satisfied and equations (4.65)-(4.66) have no solutions.*

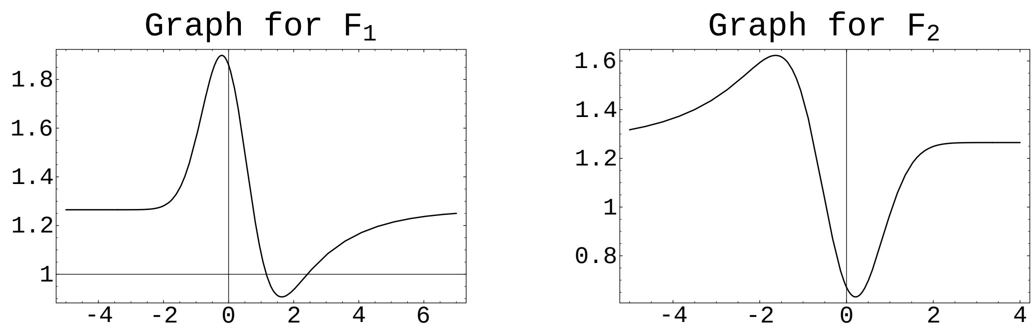

Figure 4.3: Parameter values: $\alpha = 0.2$, $\lambda = 4$.

Example 4.9. *Take α close to the minimum allowable value in assumption **A1** and let $\alpha = \frac{2}{\pi} + 0.001$ and $\lambda = 0.1$. Then Assumption **A1** is satisfied and equations (4.65)-(4.66) have solutions. For these parameters, the monotone follower control problem has a C^2 value function. The relevant solutions are $a = 0.63417, b = 1.61325$.*

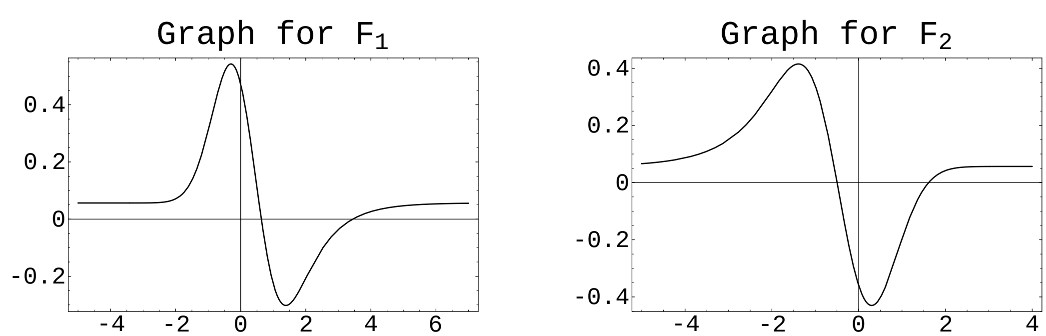

Figure 4.4: Parameter values: $\alpha = \frac{2}{\pi} + 0.001$, $\lambda = 0.1$.

Example 4.10. *Let* $\alpha = \frac{2}{\pi} + 0.001$ *and* $\lambda = 1.5$*. Then Assumption* ***A1*** *is satisfied but equations (4.65)-(4.66) have no solutions. For these parameters, the monotone follower control problem does not have a* C^2 *value function.*

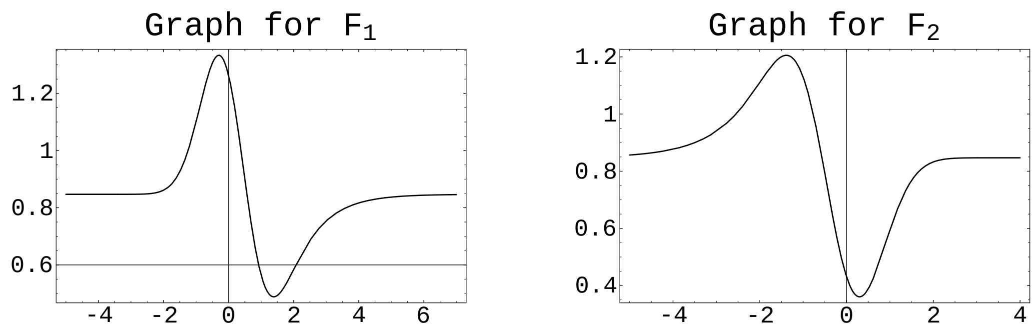

Figure 4.5: Parameter values: $\alpha = \frac{2}{\pi} + 0.001$, $\lambda = 1.5$.

Remark 4.11. *In Examples 4.7 and 4.10, there are no solutions to equations (4.65)-(4.66). The reason is because the product* $\alpha\lambda$ *is not small enough as noted in Remark 4.3.*

Chapter 5. Optimal Stopping Problem: Bounded Case

In this chapter, an equivalent optimal stopping problem to a singular optimal control problem stated in [8] will be presented. Here, in the optimal control problem, the control appears in both the constant drift and constant diffusion terms and the control process has its range as a bounded set. The equivalence will be shown under the assumption that the value function for the control problem is in $C^4(\mathbb{R})$ and in case the continuation set for the stopping problem (inaction set for the control problem) is a bounded interval. The equivalence expresses the optimal stopping value of the optimal stopping problem as a second order differential equation of the optimal value function of the optimal control problem. This control problem is a generalization of the monotone follower control problem in Karatzas [12] where the control appears in the drift term. It is also a generalization of the singular control problem in Ocone [16] where the control appears in the diffusion term.

Next, the control problem is defined from which an optimal stopping problem is derived. For any reference probability system $\nu = (\Omega, \{\mathcal{F}_s\}, P, W)$, let $\mathcal{U} = \{u : \mathbb{R}^+ \to [0, M];\ u$ is progressively measurable w.r.t. ν and $M \geq 0\}$ be the control set. Consider the problem of minimizing over $\mathcal{U}$ the cost functional

$$J(x,u) = E^x \left[\int_0^\infty e^{-\alpha t} \left(c(X(t)) + u(t) \right) dt \right], \tag{5.1}$$

such that for constants $\beta, \gamma, \sigma > 0, \mu > 0$ and $c \in C^2(\mathbb{R})$ the state process $X(t)$

satisfies

$$dX(t) = [\beta + \gamma u(t)]dt + \sqrt{\sigma + \mu u(t)}dW(t). \tag{5.2}$$

The value function $V(x) = \min_{u \in \mathcal{U}} J(x, u)$ for this problem satisfies (c.f. [9]) the Hamilton-Jacobi equation:

$$\inf_{0 \leq u \leq M} \left\{ u \left[\frac{1}{2}\mu V'' + \gamma V' + 1 \right] + \frac{1}{2}\sigma V'' + \beta V' - \alpha V + c \right\} = 0, \tag{5.3}$$

under the assumption that $V \in C^2(\mathbb{R})$.

Assume that $V \in C^4(\mathbb{R})$. To define the equivalent optimal stopping problem to (5.1)-(5.2), first a differential equation whose solution is the stopping problem's minimized value $\overline{V}(x)$ is obtained. Define a function $\overline{V}(x)$ by

$$\overline{V}(x) = \frac{1}{2}\mu V''(x) + \gamma V'(x) + 1,$$

and a set G by

$$\begin{aligned} G &= \left\{ x \in \mathbb{R}; \frac{1}{2}\mu V''(x) + \gamma V'(x) + 1 < 0 \right\} && (5.4) \\ &= \left\{ x \in \mathbb{R}; \overline{V}(x) < 0 \right\}. && (5.5) \end{aligned}$$

It follows from (5.3) that on G

$$M\overline{V} + \frac{1}{2}\sigma V'' + \beta V' - \alpha V + c = 0. \tag{5.6}$$

Differentiating (5.6) with respect to x and multiplying the result by γ gives

$$M\gamma\overline{V}' + \frac{1}{2}\sigma\gamma V''' + \beta\gamma V'' - \alpha\gamma V' + c'\gamma = 0. \tag{5.7}$$

Differentiating (5.6) twice with respect to x and multiplying the result by $\frac{1}{2}\mu$ gives

$$\frac{1}{2}\mu M\overline{V}'' + \frac{1}{4}\mu\sigma V^{iv} + \frac{1}{2}\mu\beta V''' - \frac{1}{2}\mu\alpha V'' + \frac{1}{2}\mu c'' = 0. \tag{5.8}$$

Combining equations (5.7) and (5.8) and using the definition of $\overline{V}$ gives

$$\frac{1}{2}(\sigma + \mu M)\overline{V}'' + (\beta + M\gamma)\overline{V}' - \alpha\overline{V} + \frac{1}{2}\mu c'' + c'\gamma + \alpha = 0. \tag{5.9}$$

Upon inspection of equation (5.9), define an optimal stopping problem:

$$\overline{V}(x) = \inf_{0\leq\tau} E^x\left[\int_0^\tau e^{-\alpha t}\left(\frac{1}{2}\mu c''(Y(t)) + c'(Y(t))\gamma + \alpha\right) dt\right], \tag{5.10}$$

where the process $Y(t)$ satisfies

$$dY(t) = [\beta + M\gamma]\, dt + \sqrt{\sigma + \mu M} dW(t), \quad Y(0) = x \tag{5.11}$$

and τ is a stopping time with respect to the filtration generated by $W(t)$ above. It will be shown in Proposition 5.3 that $\tilde{V}(x)$ satisfies the differential equation (5.9). First a related result is proved.

Proposition 5.1. *Let $Y(t)$ be the process satisfying*

$$dY(t) = [\beta + M\gamma]\, dt + \sqrt{\sigma + \mu M} dW(t), Y(0) = x. \tag{5.12}$$

Let $a > 0$ and define the exit time τ_a^x by $\tau_a^x = \inf\{t \geq 0; |Y(t)| \geq a\}$. Then $\overline{V}_a(x)$ defined as

$$\overline{V}_a(x) = E^x\left[\int_0^{\tau_a^x} e^{-\alpha t}\left(\frac{1}{2}\mu c''(Y(t)) + c'(Y(t))\gamma + \alpha\right) dt\right] \tag{5.13}$$

on $[-a,a]$ is the unique solution to the second order ordinary differential equation

$$\frac{1}{2}(\sigma+\mu M)V''+(\beta+M\gamma)V'-\alpha V+\frac{1}{2}\mu c''+c'\gamma+\alpha=0. \tag{5.14}$$

with $V(-a)=V(a)=0$.

Proof. Let $x\in\mathbb{R}$ and let $\eta(x)$ denote the integral in (5.13) and $\mathcal{F}_t=\sigma\{W(s);s\leq t\}$. Then $\overline{V}_a(x)=E^x\eta(x)$. Using the Markov Property of $Y(t)$:

$$\begin{aligned}\frac{E^x\overline{V}_a(Y(t))-\overline{V}_a(x)}{t} &= \frac{E^xE^{Y(t)}\eta(Y(t))-E^x\eta(x)}{t}\\ &= \frac{E^xE^x[\eta(Y(2t))|\mathcal{F}_t]-E^x\eta(x)}{t}\\ &= \frac{E^x\eta(Y(2t))-E^x\eta(x)}{t}\\ &= \frac{1}{t}E^x\left[\int_{\tau_a^x}^{\tau_a^{Y(2t)}} e^{-\alpha r}\left(\frac{1}{2}\mu c''(Y(r))+c'(Y(r))\gamma+\alpha\right)dr\right].\end{aligned}$$

Taking $t\to 0$

$$\mathcal{A}_{Y(t)}\overline{V}_a(x)=-\left(\frac{1}{2}\mu c''(x)+c'(x)\gamma+\alpha\right), \tag{5.15}$$

which translates to

$$\frac{1}{2}(\sigma+\mu M)\overline{V}_a''+(\beta+M\gamma)\overline{V}_a'-\alpha\overline{V}_a+\frac{1}{2}\mu c''+c'\gamma+\alpha=0. \tag{5.16}$$

The conditions $\overline{V}_a(-a)=\overline{V}_a(a)=0$ are obvious since $\tau_a^a=0$.

Conversely, applying Ito's rule to $e^{-\alpha t}\overline{V}_a(Y(t))$ gives

$$\begin{aligned}d[e^{-\alpha t}\overline{V}_a(Y(t))] = {} & e^{-\alpha t}\left[\tfrac{1}{2}(\sigma+\mu M)\overline{V}_a''(Y(t))+(\beta+M\gamma)\overline{V}_a'(Y(t))-\alpha\overline{V}_a(Y(t))\right]dt\\ & +e^{-\alpha t}\overline{V}_a'(Y(t))\sqrt{\sigma+\mu M}\,dW(t).\end{aligned} \tag{5.17}$$

Let $x \in \mathbb{R}$. Using (5.14)

$$\begin{aligned} e^{-\alpha\tau_a^x}\overline{V}_a(Y(\tau_a^x)) = \overline{V}_a(x) \quad &- \quad \int_0^{\tau_a^x} e^{-\alpha t}\left(\frac{1}{2}\mu c''(Y(t)) + c'(Y(t))\gamma + \alpha\right) dt \\ &+ \quad \int_0^{\tau_a^x} e^{-\alpha t}\sqrt{\sigma + \mu M}\overline{V}_a'(Y(t))dW(t), \end{aligned} \tag{5.18}$$

where τ_a^x is the exit time of the process $Y(t)$ from $(-a, a)$. Using $Y(\tau_a^x) = a$ and $\overline{V}_a(a) = 0$, and taking E^x

$$\overline{V}_a(x) = E^x\left[\int_0^{\tau_a^x} e^{-\alpha t}\left(\frac{1}{2}\mu c''(Y(t)) + c'(Y(t))\gamma + \alpha\right) dt\right]. \tag{5.19}$$

□

Remark 5.2. *In Proposition 5.1, it is possible to replace without loss of generality the even interval $[-a, a]$ with an uneven interval say $[p, q]$ and the exit time τ_a^x from $[-a, a]$ with τ_{pq}^x from $[p, q]$.*

Now, a proposition establishing that the optimal stopping problem (5.10)-(5.11) is equivalent to the optimal control problem (5.1)-(5.2) is stated and proved. First, define a function $f : \mathbb{R} \to \mathbb{R}$, and a differential operator L by

$$\begin{aligned} f(x) &= -\left(\frac{1}{2}\mu c''(x) + \gamma c'(x) + \alpha\right), \\ L &= \frac{1}{2}(\sigma + M\mu)\frac{\partial^2}{\partial x^2} + (\beta + M\gamma)\frac{\partial}{\partial x} - \alpha. \end{aligned}$$

Assume that

$$E^x\left[\int_0^{\infty} e^{-\alpha t}|f(Y(t))|dt\right] < \infty \text{ for all } x \in \mathbb{R} \tag{5.20}$$

Proposition 5.3. *Suppose a function $\overline{V} : \mathbb{R} \to \mathbb{R}$ can be found such that*

(i) $\overline{V} \in C^1(\mathbb{R})$.

(ii) $\overline{V} \leq 0$ *on* $\mathbb{R}$. *Define* G *as*

$$G = \left\{x; \overline{V}(x) < 0\right\}.$$

(iii) $E^x[\int_0^\infty \chi_{\partial G}(Y(t))dt] = 0$ *for all* $x \in \mathbb{R}$. *($Y(t)$ spends zero time on ∂G.)*

(iv) $\overline{V} \in C^2(\mathbb{R} \setminus \partial G)$ *and the second order derivative of* $\overline{V}$ *is locally bounded near* ∂G.

(v) $L\overline{V} - f \geq 0$ *on* $\mathbb{R} \setminus \overline{G}$.

(vi) $L\overline{V} - f = 0$ *on* G.

(vii) $\tau_G := \inf\{t > 0; Y(t) \notin G\} < \infty$ *a.s.* x.

(viii) The family $\{\overline{V}(Y(\tau)); \tau \leq \tau_G\}$ *is uniformly integrable w.r.t.* x,

where $Y(t)$ *is defined by*

$$dY(t) = [\beta + M\gamma]\, dt + \sqrt{\sigma + \mu M} dW(t), Y(0) = x. \tag{5.21}$$

Then $\overline{V}$ *is the solution to the optimal stopping problem (5.10)-(5.11). That is*

$$\overline{V}(x) = \inf_{0 \leq \tau} E^x \left[- \int_0^\tau e^{-\alpha t} f(Y(t)) dt \right], \tag{5.22}$$

where τ *is a stopping time with respect to the filtration generated by* $W(t)$, *and the* *optimal stopping time $\hat{\tau}$ is*

$$\hat{\tau}(x) = \begin{cases} 0 & x \notin G \\ \tau_G & x \in G. \end{cases}$$

Conversely, if $\overline{V}$ is defined as in (5.22) and G as in (ii) above is an open interval, then $\overline{V}$ is the unique solution of the equation in (vi) above.

Proof. Write $\overline{V} = -V$, so that showing (5.22) means showing that

$$V(x) = \sup_{0 \leq \tau} E^x \left[\int_0^\tau e^{-\alpha t} f(Y(t))dt \right]. \tag{5.23}$$

Also

$$G = \{x; V(x) > 0\}.$$

It will be shown that if $-V$ satisfies the assumptions of the proposition, then $V = \tilde{V}$ where $\tilde{V}$ is the right hand side of (5.23).

By (i) and (iv) get a sequence of functions $V_j \in C^2(\mathbb{R})$, $j = 1, 2, ...,$ such that

(a) $V_j \to V$ uniformly on compact subsets in $\mathbb{R}$, as $j \to \infty$.

(b) $LV_j \to LV$ uniformly on compact subsets in $\mathbb{R} \setminus \partial G$, as $j \to \infty$.

(c) $\{LV_j\}_{j=1}^\infty$ is locally bounded on $\mathbb{R}$.

For $R > 0$ let

$$T_R = \min(R, \inf\{t > 0; |Y(t)| \geq R\}).$$

Let $\tau > 0$ be a stopping time and $y \in \mathbb{R}$, by Dynkin's formula (see [16])

$$E^x[e^{-\alpha\tau \wedge T_R} V_j(Y(\sigma \wedge T_R))] = V_j(x) + E^x \left[\int_0^{\tau \wedge T_R} e^{-\alpha t} LV_j(Y(t))dt \right].$$

Hence by (a), (b), (c), (iii) and Fatou's Lemma

$$\begin{aligned} V(x) &= \lim_{j\to\infty} E^x\left[-\int_0^{\tau\wedge T_R} e^{-\alpha t}LV_j(Y(t))dt + e^{-\alpha\tau\wedge T_R}V_j(Y(\tau\wedge T_R))\right] \\ &\geq E^x\left[-\int_0^{\tau\wedge T_R} e^{-\alpha t}LV(Y(t))dt + e^{-\alpha\tau\wedge T_R}V(Y(\tau\wedge T_R))\right]. \end{aligned}$$

Therefore by (ii), (iii), (v) and (vi)

$$V(x) \geq E^x\left[\int_0^{\tau\wedge T_R} e^{-\alpha t}f(Y(t))dt\right].$$

Hence by Fatou's Lemma and assumption (5.20)

$$\begin{aligned} V(x) &= \underline{\lim}_{R\to\infty} E^x\left[\int_0^{\tau\wedge T_R} e^{-\alpha t}f(Y(t))dt\right] \\ &\geq E^x\left[\int_0^{\tau} e^{-\alpha t}f(Y(t))dt\right]. \end{aligned}$$

Since τ is arbitrary

$$V(x) \geq \tilde{V}(x) \text{ for all } x. \tag{5.24}$$

If $x \notin G$ then considering $\tau = 0$ gives

$$V(x) = 0 \leq \tilde{V}(x).$$

So $V(x) = \tilde{V}(x)$ and the optimal time $\hat{\tau}(x) = 0$ for $x \notin G$.

If $x \in G$, let $\{G_k\}$ be an increasing sequence of open sets such that $G_k \subset G$, G_k is compact for each $k = 1, 2, \ldots$ and $G = \bigcup_{k=1}^{\infty} G_k$. Let

$$\tau_k = \inf\{t > 0; Y(t) \notin G_k\},\ k = 1, 2, \ldots .$$

By Dynkin's formula (see [16]), for $y \in G_k$ and (vi)

$$\begin{aligned}
V(x) = \lim_{j\to\infty} V_j(x) &= \lim_{j\to\infty} E^x\left[\int_0^{\tau_k\wedge T_R} -e^{-\alpha t}LV_j(Y(t))dt + e^{-\alpha\tau_k\wedge T_R}V_j(Y(\tau_k\wedge T_R))\right] \\
&= E^x\left[\int_0^{\tau_k\wedge T_R} -e^{-\alpha t}LV(Y(t))dt + e^{-\alpha\tau_k\wedge T_R}V(Y(\tau_k\wedge T_R))\right] \\
&= E^x\left[\int_0^{\tau_k\wedge T_R} e^{-\alpha t}f(Y(t))dt + e^{-\alpha\tau_k\wedge T_R}V(Y(\tau_k\wedge T_R))\right].
\end{aligned}$$

By (ii), (vii) and (viii)

$$\begin{aligned}
V(x) &= \lim_{R,k\to\infty} E^x\left[\int_0^{\tau_k\wedge T_R} e^{-\alpha t}f(Y(t))dt + e^{-\alpha\tau_k\wedge T_R}V(Y(\tau_k\wedge T_R))\right] \\
&= E^x\left[\int_0^{\tau_G} e^{-\alpha t}f(Y(t))dt\right] \\
&\le \tilde{V}(x). \qquad (5.25)
\end{aligned}$$

Therefore $V(x) = \tilde{V}(x)$ on G. Moreover the optimal stopping time $\hat{\tau}$ is

$$\hat{\tau}(x) = \begin{cases} 0 & x \notin G \\ \tau_G & x \in G. \end{cases}$$

Since G as in (ii) above is an open interval, the converse is true from Proposition 5.1 and Remark 5.2. $\square$

Chapter 6. Other Singular Control Problems

In this chapter, the smoothness of the value function for several singular stochastic control problems is investigated. Theorem 6.1 due to Lions [14] will be used in this regard. For the monotone follower problem (4.1)-(4.2), the value function was assumed to be $\mathbb{C}^2(\mathbb{R})$. This turned out to be true under some parameter values and not true under other parameter values. Although Theorem 6.1 does not guarantee smoothness of the value function of the monotone follower control problem, it guarantees smoothness of a similar singular control problem (with the same dynamics and cost functional) but with bounded controls.

Lastly in this chapter, the monotone follower control problem (4.1)-(4.2) with the state process restricted to a certain domain is considered. Theorem 6.7 due to Karatzas and Shreve [12] is used to guarantee the existence of its optimal control process irrespective of the problem's parameter values.

6.1 Smoothness of Value Functions

In this section a smoothness theorem for the value function due to Lions [14] is used. An investigation of the smoothness of value functions for various problems is carried out. The theorem due to Lions [14] does not apply to the monotone follower problem (4.1)-(4.2), yet the problem sometimes gives rise to a $C^2(\mathbb{R})$ value function as was seen in Chapter 4.

A general control problem is now stated after which Theorem 6.1 due to Loins [14] and its assumptions are stated. The theorem will then be applied to several singular control problems. The theorem will show smoothness for a control problem having the monotone follower dynamics and cost functional, but with bounded controls. Let $\nu = (\Omega, \{\mathcal{F}_t\}, P, W)$ be any reference probability system. Let U be a closed non-empty, convex set in $\mathbb{R}^m$. Define $\mathcal{U}$ and $W^{k,\infty}(\mathbb{R}^n)$ for $k \geq 1$ as

$$\mathcal{U} = \{u : \mathbb{R}^+ \to U;\ u \text{ is progressively measurable w.r.t. } \nu\}, \tag{6.1}$$

$$W^{k,\infty}(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n);\ \text{the weak derivatives } D^\alpha u \in L^\infty(\mathbb{R}^n), |\alpha| \leq k\}\,. \tag{6.2}$$

Consider the value function

$$V(x) = \inf_{u \in \mathcal{U}} J^u(x),$$

where

$$J^u(x) = E^x\left[\int_0^\infty f(X(t), u(t)) \exp\left(-\int_0^t c(X(s), u(s))ds\right) dt\right]$$

and

$$dX(t) = \sigma(X(t), u(t))dW(t) + b(X(t), u(t))dt, X(0) = x.$$

Assume

(1) $\phi(\cdot, u) \in C_b^2(\mathbb{R}^n)$ and $\|\phi(\cdot, u)\|_{C_b^2(\mathbb{R}^n)} \leq M$ for each $u \in U$ the control set, and $\phi = \sigma_{ij}, b_i, c$, where $\|\phi\|_{C_b^2(\mathbb{R}^n)} = \sup_{|\alpha| \leq 2} \|D^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}$.

(2) $c(x, u) \geq \alpha > 0$ for each $x \in \mathbb{R}^n$, $u \in U$.

(3) There exists $p \in C([0,\infty),[0,\infty))$, with $p(0)=0$ such that

$$|\phi(x,u)-\phi(x,u')| \le p(|u-u'|)$$

for each $x \in \mathbb{R}^n$ and $\phi = \sigma_{ij}, b_i, c$.

(4)

$$|f(x,u)-f(x',u')| \le k|x-x'| + p(|u-u'|)$$

for each $x \in \mathbb{R}, u \in \mathcal{U}$ and there is a constant M such that $|f(x,u)| \le M$ for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, and k a positive constant.

Theorem 6.1. *Under Assumptions (1), (2), (3), (4), if $f(\cdot,u)$ is assumed to remain in a bounded set of $W^{2,\infty}(\mathbb{R}^n)$ and if $\lambda = \inf_{x,v} c(x,v)$ is such that $\lambda > \lambda_0$ (where λ_0 depends explicitly on the $L^\infty(\mathbb{R}^n)$ norms of the first and second derivatives of σ, b, see Krylov [13]), then $V \in W^{1,\infty}(\mathbb{R}^n)$. Further, under the assumptions that there exists $\nu > 0$ such that*

$$\sum_{i,j,k} \frac{1}{2}\theta_k a_{ij}(x,v_k)\xi_j\xi_j \ge \nu|\xi|^2 \tag{6.3}$$

for all $\xi \in \mathbb{R}^n, v_k \in U, \theta_k \in (0,1), \sum \theta_i = 1$ and $a = \sigma\sigma^T$, then $V \in W^{2,\infty}(\mathbb{R}^n)$.

Remark 6.2. *If $\sigma(x,u)$ and $b(x,u)$ do not depend on the state x (see Remark at the end of Section 2 in [13]) λ_0 can be chosen arbitrarily as long as $\lambda_0 > 0$.*

Remark 6.3. *Observe $V \in W^{1,\infty}(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ implies $V \in C_b^1(\mathbb{R}^n)$. The first space means V' is Lipschitz continuous, hence it is absolutely continuous and differentiable a.e. with V'' being essentially bounded. The second one shows V' is essentially bounded and Lipschitz continuous and hence differentiable a.e. This shows that $V \in C_b^1(\mathbb{R}^n)$.*

Next, the smoothness of the value function is investigated for some singular control problems.

Example 6.4. *Consider the dynamics of the monotone follower problem but with a bounded control set. Let* $V(x) = \min_{u \in \mathcal{U}} J^u(x)$ *where*

$$J^u(x) = E^x \left[\int_0^\infty e^{-\alpha t} \left(c(\xi(t)) + \lambda u(t) \right) dt \right] \tag{6.4}$$

over $\mathcal{U}$ *in (6.1) with* $m = 1$, $U = [0, M]$, $c(x) = 1 - e^{-x^2}$ *with the state equation*

$$d\xi(t) = -u(t)dt + dW(t), \xi(0) = x. \tag{6.5}$$

Using Theorem 6.1 (with $\sigma(x, u) = 1, b(x, u) = -u, c = \alpha, f(x, u) = c(x) + \lambda u, p(x) = 1$*), the value function* V *is in* $C^1(\mathbb{R})$*.*

On the other hand with $U = [0, \infty)$*, (giving the monotone follower control problem (4.1)-(4.2)),* f *is not necessarily bounded above with respect to the* u *variable (violating Assumption 4 in Theorem 6.1). Then it cannot be concluded that the value function is* $C^1(\mathbb{R})$*. Under some values of* α *and* λ*, a* $C^2(\mathbb{R})$ *value function was obtained in Chapter 4 and under some other parameter values, the value function is not* C^2*.*

Example 6.5. *Consider the minimization of*

$$J^u(x) = E^x \left[\int_0^\infty e^{-t} u(t) f(X(t)) dt \right] \tag{6.6}$$

over $\mathcal{U}$ *in (6.1) with* $U = [0, 1]$ *or* $U = [0, \infty)$*,* $X(t) = x$ *for all* $t > 0$*,* $x \in \mathbb{R}$*,* $f \in D(\mathbb{R})$ *and* $f^+ \notin C^1(\mathbb{R})$*. Here, Assumption (6.3) in Theorem 6.1 is violated. Let* $V(x) = \min_{u \in \mathcal{U}} J^u(x)$ *be the value function. The Hamilton-Jacobi equation of this* *singular control problem is*

$$\sup_{u \in \mathcal{U}} \{uV - uf(x)\} = 0 \text{ a.e. } x$$

gives us $V(x) = f^+(x)$ *and thus* $V \notin C^1(\mathbb{R})$.

Example 6.6. *Another singular control example (an exercise in [16]) in which the value function is not in* $C^1(\mathbb{R})$*, is as follows. Let*

$$f(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1 \\ \sqrt{x} & \text{for } x > 1. \end{cases}$$

The problem is to minimize

$$J^u(s,x) = E^{s,x}\left[\int_s^T e^{-\rho t} f(X^u(t))dt\right]$$

where the process $X^u(t)$ *satisfies*

$$dX^u(t) = u(t)dW(t), \ t \geq s, X^u(s) = x, \ \ x \geq 0 \text{ and } W(t) \in \mathbb{R}, \tag{6.7}$$

the control $u(\cdot) \in \mathcal{U}$ *with* $U = \mathbb{R}$ *and*

$$T = \inf\{t \geq s; X^u(t) \leq 0\}.$$

Let the value function be

$$\Phi(s,x) = \sup_{u \in \mathcal{U}} J^u(s,x).$$

Then $\Phi \notin C^1$.

Proof. First, the Hamilton-Jacobi equation for the problem is stated together with a candidate for the value function. Then, a sequence of controls will be defined and the corresponding value functions will be shown to converge to the candidate value functional. Lastly, the candidate value functional will be shown to satisfy the Hamilton Jacobi equation.

The Hamilton-Jacobi equation (see [9]) is

$$\max\left\{e^{-\rho s}f(x) + V_s(s,x), \frac{1}{2}V_{xx}(s,x)\right\} = 0. \tag{6.8}$$

Define a function $\phi(s,x)$ for $x \geq 0, s \in \mathbb{R}$ as

$$\phi(s,x) = \frac{1}{\rho}e^{-\rho s}\hat{f}(x),$$

where

$$\hat{f}(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ \sqrt{x} & \text{for } x > 1. \end{cases}$$

Define a sequence $\{u_k(x)\}$ of bounded controls as follows

$$\hat{u}_k(x) = \begin{cases} k & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases}$$

and let X_k be the process (6.7) with $u = u_k$. Define

$$\begin{aligned} T_k &= \inf\{t > s; X_k(t) \leq 0\} \\ \tau_k &= \inf\{t > s; X_k(t) \notin (0,1)\}. \end{aligned}$$

If $0 \leq x \leq 1$ then from Proposition 3.3(c) and Proposition 3.4,

$$\begin{aligned} P(\lim_{t\uparrow\tau_k} X_k(t) = 0) &= 1 - x, \\ P(\lim_{t\uparrow\tau_k} X_k(t) = 1) &= x, \end{aligned}$$

and

$$E^x \tau_k = \frac{1}{k} x(1-x).$$

As k increases, $X_k(t)$ exits the interval $[0,1]$ faster. In fact both $T_k, \tau_k \downarrow s$ where s is the initial time of the process. The reward to u_k is

$$\begin{aligned}
J^{u_k}(s,x) &= E^x\left[\int_s^{T_k} e^{-\rho t} f(X_k(t))dt\right] \\
&= E^x\left[\int_s^{\infty} e^{-\rho t} f(X_k(t)) 1_{\{X_k(\tau_k)=1\}} dt\right] \\
&\quad + E^x\left[\int_s^{T_k} e^{-\rho t} f(X_k(t)) 1_{\{X_k(\tau_k)=0\}} dt\right],
\end{aligned} \tag{6.9}$$

so that

$$\begin{aligned}
\lim_{k\to\infty} J^{u_k}(s,x) &= \int_s^{\infty} e^{-\rho t} f(1) \lim_{k\to\infty} P(X^k(\tau_k)=1)dt + 0 \\
&= \int_s^{\infty} e^{-\rho t} x dt \\
&= \phi(s,x).
\end{aligned}$$

If $x > 1$ observe that $u_k(s,x) = 0, X_k(t) = x, T_k = \infty$ so that

$$\begin{aligned}
J^{u_k}(s,x) &= E^x\left[\int_s^{\infty} e^{-\rho t} f(x)dt\right] \\
&= \frac{1}{\rho} e^{-\rho s} f(x) \\
&= \frac{1}{\rho} e^{-\rho s} \sqrt{x} \\
&= \phi(s,x).
\end{aligned}$$

Now showing that $\phi(s,x)$ satisfies the Hamilton-Jacobi equation (6.8) will imply that $\Phi = \phi$. We will show that

$$\max\left\{e^{-\rho s} f(x) + \phi_s(s,x), \frac{1}{2}\phi_{xx}(s,x)\right\} = 0. \tag{6.10}$$

On $0 \leq x \leq 1$ equation (6.10) is true since

$$\max\left\{x(x-1)e^{-\rho s}, \frac{1}{2\rho}e^{-\rho s} \cdot 0\right\} = 0,$$

and on $x > 1$, equation (6.10) is true since

$$\max\left\{e^{-\rho s}\sqrt{x} - e^{-\rho s}\sqrt{x}, -\frac{1}{2\rho\sqrt{x}}e^{-\rho s}\right\} = 0.$$

It can be concluded that ϕ is the optimal control and $\Phi = \phi \notin C^2$. In fact $\phi \notin C^1$. $\square$

6.2 Existence of Optimal Control Processes

In this section, the existence of an optimal control for the monotone follower control problem (4.1)-(4.2) with no restriction on the parameters, but with a restricted process is shown. Theorem (6.7) due to Karatzas and Shreve [12] will be the used here.

A generalized monotone follower control problem stated by Karatzas and Shreve [12] is first presented after which Theorem 6.7 is presented. A domain of restriction for the state process of the monotone follower control problem (4.1)-(4.2) is required to use Theorem 6.7. Let $\nu = (\Omega, \{\mathcal{F}_s\}, P, W)$ be any reference probability system. Consider the monotone follower control problem

$$v(x) = \inf_{\xi \in \Xi} E\left[\int_0^{\tau_x} h(t, x(t))dt + \int_{[0,\tau_x)} f(t)d\xi(t) + g(x(\tau_x))\right], \tag{6.11}$$

with

$$\Xi = \{\xi(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ ; \xi(t) \text{ is } \mathcal{F}_t\text{-adapted, non-decreasing,} \\ \text{left-continuous and } \xi(0) = 0\} . \tag{6.12}$$

$$x(t) = x + W(t) - \xi(t), \quad x \in \mathbb{R} \text{ and } 0 \leq t \leq \tau_x, \tag{6.13}$$

where τ_x is a fixed stopping time with respect to $W(t)$ for each x in (6.13). The following assumptions are necessary in stating Theorem (6.7) due to Karatzas and Shreve [12].

Assumptions

(a) The running cost of effort per unit time $f(t)$ is real valued on $[0, \tau_x]$.

(b) The terminal cost $g(x)$ is continuously differentiable, with $g'(x)$ non-decreasing.

(c) The running cost per unit time $h(t, x)$ and $h_x(t, x)$ are continuous in t, x with $h_x(t, x)$ non-decreasing in x.

(d) $0 < c \leq f(t) \leq C$.

(e) $\sup_x g'(x) \leq f(\tau_x)$.

(f) $h(t, x) \geq 0$, $g(x) \geq 0$.

Theorem 6.7. *Under Assumptions (a)-(f) on the cost functions, there exists an optimal process for the monotone follower control problem.*

Corollary 6.8. *Consider the monotone follower problem:*

$$V(x) = \min_{u \in \mathcal{U}} J^u(x) = \min_{u \in \mathcal{U}} E^x \left[\int_0^{\tau_x} e^{-\alpha t} \left(c(x(t)) + \lambda u(t) \right) dt \right], \tag{6.14}$$

where

$$\mathcal{U} = \{u : \mathbb{R}^+ \to [0,\infty);\ u \text{ is progressively measurable w.r.t. } \nu\}, \tag{6.15}$$

the cost function $c(x) = 1 - e^{-x^2}$,

$$dx(t) = -u(t)dt + dW(t), x(0) = x. \tag{6.16}$$

and $\tau_x = \inf\{t \geq 0; c''(x(t)) \leq 0\}$. *Then there exists an optimal process for this monotone follower control problem.*

Proof. When a transformation of the controls as in Chapter 2 is carried out, the control problem transforms into

$$v(x) = \inf_{\xi \in \Xi} E\left[\int_0^{\tau_x} e^{-\alpha t} c(x(t))dt + \int_{[0,\tau_x)} \lambda e^{-\alpha t} d\xi(t)\right] \tag{6.17}$$

with

$$x(t) = x + W(t) - \xi(t),\ \text{and } 0 \leq t \leq \tau_x. \tag{6.18}$$

To apply Theorem 6.7, consider $h(t,x) = e^{-\alpha t}c(x), f(t) = \lambda e^{-\alpha t}$ and $g(x) = 0$. If $x \in (-1/\sqrt{2}, 1/\sqrt{2})$, then $\tau_x > 0$. Otherwise $\tau_x = 0$ so that $h_x(t,.)$ is non decreasing up to the exit time τ_x. Theorem 6.7 guarantees the existence of an optimal process to the problem (6.17)-(6.18).

□

Chapter 7. Conclusions

A monotone follower control problem with a nonconvex cost function is considered for which an optimal control is obtained. On this problem, some assumptions on the parameters were made to ensure the existence of a single region action optimal control. It remains to be seen what happens if some (or all) of the assumptions do not hold. The optimal control had a single region of action because the even functional changes convexity once on positive numbers. It is worthwhile investigating how many regions of action can one have if the cost function changes convexity more than once. An equivalent optimal stopping problem corresponding to a singular control problem has been stated. This problem needs further pursuing in the case that both the drift and diffusion terms are not constants.

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