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Numerical simulations of dynamical mass transfer in binaries

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NUMERICAL SIMULATIONS OF DYNAMICAL MASS TRANSFER IN BINARIES

A Dissertation
Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College
in partial fulfillment of the requirements for the degree of Doctor of Philosophy
in
The Department of Physics and Astronomy

by
Patrick M. Motl
B.S., Indiana University, 1993
December, 2001
Dedication

This work is dedicated to my partner Marie for her patience and understanding and for encouraging me to try to reach my dreams.
Acknowledgments

I would like to thank Joel Tohline for a characteristically enthusiastic question that served as the genesis for the work that I am now presenting. To quote from that first conversation, “Wouldn’t it be neat to watch a movie of one star dumping stuff on another star?” I owe a great deal of gratitude to both Juhan Frank and Joel for their support, encouragement, interest, and for providing a wonderful introduction to theoretical astrophysics and computational fluid dynamics.

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Abstract

We present results from investigations of mass transfer instability in close binary star systems. By unstable mass transfer we mean the exchange of material where the response of the binary to the initial Roche lobe overflow causes the donor to loose even more material. Our work is guided by approximate arguments that dictate the stability boundaries for binary star systems. To proceed further one must explicitly treat extended mass and velocity distributions that are both initially, and through their subsequent evolution in time, self-consistent. In this dissertation, we present the first three-dimensional, fully self-consistent treatment of mass transfer in close binary systems. To perform these calculations we have developed and tested a set of tools including a Self-Consistent Field code for generating polytropic binaries executing synchronous rotation upon circular orbits and a parallel, gravitational hydrodynamics code for evolving the binaries in time. We describe, in detail, these tools and their application to the evolution of binary star systems. We present extended simulations of two detached binaries that have been used to examine the accuracy of our computational techniques in addition to the simulations of interacting binaries.
1. Introduction

Binary stars were discovered by the Italian astronomer Giovanni Baptista Riccioli in the year 1650 when he observed Mizar through a telescope and found the single point of light was actually two distinct stars. In 1669 Geminiano Montanari di Bologna noted that Algol (β Persei) varies by about a factor of three in its brightness (Abell et al., 1987). In 1783, the English astronomer John Goodricke carefully measured the period of Algol’s variation and proposed that Algol was in fact a pair of stars orbiting about one another, the variation being due to the darker star occulting the brighter component (Shore et al., 1992). Goodricke applied the same reasoning to two other variable stars he uncovered; β Lyrae (also known as Sheliak) and δ Cephei. William Herschel was the first to conclusively demonstrate that binary stars orbiting one another exist based on his observations of the motion of the components of Castor in 1804 (Abell et al., 1987). In time, Goodricke’s theoretical explanation for both Algol and β Lyrae were verified when spectra were taken of the systems and their motion deduced. In the case of δ Cephei, Goodricke was of course incorrect; the variation in this star and other Cepheid variables being caused by an instability to radial pulsations. It is remarkable that over his brief life (Goodricke died in 1786 at the age of 21) he discovered the important Cepheid variables and also prototypes for what would become two of the primary categories of eclipsing binaries (Algols and β Lyrae systems).

1.1 Importance of Binaries in Astrophysics

Before continuing the discussion of binaries it is appropriate to give an operable definition for binaries. While it may seem obvious that a binary star pair is simply defined as two stars bound together by the force of gravity as a composite system, this definition is slightly misleading. We should remember that interactions between the binary and other stars or gas clouds in a galaxy will impose a limit on the widest binaries that will survive to a given age on average (the age of the Universe for example). Stars that happen to be bound at one point in time but are at a wider separation than the scattering limit for their location in the Galaxy should not really be regarded as binaries and are instead members of the field star population.

An often quoted fact to argue for the importance of binary stars is that about half of the “stars” in the sky are actually multiple star systems and, of these, binaries are the most common systems (Trimble, 1983). In a general population of stars about a fifth will be multiple systems and in the Solar neighborhood, where the sample is more complete, the frequency of binaries is over 50% (Kallrath & Milone, 1999). In truth, the observed binary frequency is likely to be an underestimate as there is a strong observational bias against detecting long period binary systems. Binaries have been discovered with separations as wide as a 0.2 pc ($6 \times 10^{17}$ cm) (Latham, 1984) which imply
orbital periods of millions of years. For comparison, the binary with shortest known period is 4U 1820-30, where it is believed that a neutron star and white dwarf orbit one another once every 685 seconds. This exotic binary, first discovered as a source of X-rays, has an orbital separation of about $1 \times 10^{10}$ cm or about a seventh the radius of the Sun (Rappaport et al., 1987). A candidate vying for the title of shortest period binary is RX J1914+24, another X-ray source, which may have an orbital period of only 569 seconds (Gavin et al., 2000).

In addition to the importance accorded binaries by their majority position amongst stars, binaries provide the only means of directly measuring the mass of stars beyond the Sun. The radii of the components can also be measured in some systems, a feat that is otherwise only possible for a handful of very extended and intrinsically bright stars such as Betelgeuse and Mira. A general theorem, known as the Russell-Vogt theorem, follows from the equations of stellar structure and states that: (Carroll & Ostlie, 1996)

The mass and composition of a star uniquely determine its radius, luminosity, and internal structure, as well as its subsequent evolution.

The components of binaries thus allow astronomers to test the theory of stellar structure and evolution in a quantitative way.

There are many classification systems in use for binary star systems. The most important of these being the division according to what an observer can detect and deduce. Visual binary stars are systems where both components are resolvable and the orbit can be observed directly. If the distance to a visual binary is known, through a measurement of the parallax for example, then a linear scale can be assigned to the system and the masses and orbital elements are known. Unfortunately, there are relatively few known visual binaries and they are necessarily close to Earth. If orbital motion is detected in a star but no companion can be discerned, perhaps due to a large difference in brightness between the two components, the binary is termed an astrometric binary. Spectroscopic binaries are unresolved systems where one or both components is detected in the spectra. These are termed single-lined or double-lined spectroscopic binaries respectively. From the Doppler shift arising from the orbital motion one can at best measure the mass ratio of the binary and at worst set lower bounds on the sum of the masses and hence the mass of one of the components. To proceed further with spectroscopic binaries one must know the inclination angle of the orbit relative to the plane of the sky. The inclination angle, $i$, is the angle between the normal to the orbital plane and the observer’s line of sight so that at $i = 0^\circ$ the orbit is seen face on and at $i = 90^\circ$ the orbit is seen edge on. Eclipsing binaries are systems that we happen to be viewing sufficiently close to edge on, given the size of the individual components and the orbital separation, for the components to eclipse one another during their orbit. The light curve for eclipsing binaries along with the radial velocity curve, if the spectra can be decomposed successfully, yield the masses of each component and in principle the radii of the two stars as well. Eclipsing binaries are thus highly prized amongst
astronomical discoveries as they provide a relatively large data set to test our understanding of stars.

Binaries also allow theories to be examined beyond the level treated by the Russell-Vogt theorem. A few examples of the detailed information that can be gained from studying binaries are described briefly below.

The degree of central condensation of a star and the quadrupole moment of its density distribution can be estimated by measuring the rate of precession of the line of nodes (or any other reference point in the orbit) using a formula derived by Kopal (1959). For a point mass binary, the orientation of the orbit is fixed in space but as the potential departs from this limit, due to the extended and nonspherical mass distributions, the orbit instead precesses.

A useful class of eclipsing binaries are the \( \zeta \) Aurigae systems where the components are at very different stages in their evolutions. One component is typically a G or K type giant or subgiant with an extended atmosphere and the other component is a B or other early type main sequence star. Since these binaries eclipse one another, astronomers have the chance to shine a known light source (the B type star) through the atmosphere of the giant as it passes into and out of eclipse. Through modeling of the observed spectra as a function of orbital phase one can obtain information about the run of temperature within the giant’s atmosphere and study convective energy transport within the giant (Shore et al., 1992).

RS CVn binaries, named for the prototype RS Canum Venaticorum, are binaries that exhibit unusually high levels of activity in the X-ray and radio and show evidence of starspots. Unlike sunspots, the starspots of RS CVn stars can occupy up to half the surface area presented to the observer at a time. These systems have short periods (between a day and a month) and at least one of the components has a significant region where energy is transported by convection. It is believed that the relatively rapid synchronous rotation combined with convective motion causes enhanced magnetic activity. RS CVn stars serve as testbeds for magnetic dynamo theory (Shore et al., 1992).

Binary systems also serve as probes of general relativistic effects. The binary pulsar PSR 1913+16 (Hulse & Taylor, 1975) exhibits an orbital decay consistent with the loss of orbital angular momentum and energy due to the emission of gravitational radiation in the quadrupole approximation (Weisberg & Taylor, 1984; Schutz, 1990). General relativistic effects can also result in detectable precession rates. The predicted rate of apsidal motion (Newtonian and relativistic contributions combined) has been compared with the observed rate in DI Herculis and has been found to be in conflict with general relativity although the results remain controversial (Guinan & Maloney, 1985).

1.2 The Roche Potential

Another classification of binaries, based on their morphology, was presented by Kopal (1955). In Kopal’s scheme, the binaries are classified by
the distribution of the matter in the stars relative to the potential the stars generate. Strictly speaking, the discussion applies only to those stars that are executing motion along circular orbits and are synchronously rotating so that in a corotating frame of reference the stars are at rest. For close binaries this is a reasonable approximation as tidal forces will tend to circularize the orbits and synchronize the spins over a timescale shorter than the main sequence lifetime for most stars (Counselman, 1973). A further assumption is that the gravitational potential arising from the two components is given by the potential of point masses. The resulting potential given the stated assumptions is familiar from the restricted three body problem and is named in honor of Edouard Roche. In a reference frame rotating with angular frequency, $\Omega$, the effective potential for point masses $M_1$ and $M_2$ located at $r_1$ and $r_2$ is given by

$$
\Phi_{\text{Roche}} (r) = - \frac{GM_1}{|r - r_1|} - \frac{GM_2}{|r - r_2|} - \frac{1}{2} \Omega^2 r^2.
$$

(1.1)

Contours of the Roche potential in the equatorial plane are shown in Fig. 1.1 for a binary in which one component is 3 times more massive than its companion. The Roche potential is plotted along the line of centers (the line connecting the centers of mass of the two components, here taken to
be the $x$ axis) in Fig. 1.2. The center of mass of each component in Fig. 1.1 is marked with an asterisk and the system center of mass is marked with a plus sign labeled COM. There are five stationary points in the Roche potential in the equatorial plane labeled here as $L_1$ through $L_5$. All of these points, the Lagrange points, are local extrema in the potential (points where $\nabla \Phi^{\text{Roche}} = 0$. Close to either star the curves of constant potential are nearly circular and this is also the case far from the binary. In the intermediate region the potential is much more complicated. The point $L_1$ is a saddle point and it lies deepest in the potential at the intersection of a figure eight surface common to both stars. Material at this critical surface, termed the Roche surface, is equally bound to either star and the Roche surface marks the maximal extent for a star that is to remain entirely in hydrostatic equilibrium. If at any time during its evolution a star overflows its Roche lobe, it will lose material to its companion in what is termed Roche lobe overflow (RLOF). The other two Lagrange points lying along the line of centers ($L_2$ and $L_3$) are likewise saddle points and points of unstable equilibrium. Finally, the points $L_4$ and $L_5$ are local maxima, although orbits near these outer Lagrange points may be stabilized by the Coriolis force. For example, the Trojan asteroids lie at the $L_4$ and $L_5$ points for the Sun - Jupiter system.

Figure 1.2: The Roche potential along the line of centers, taken to be the $x$ axis. The three local maxima are labeled as the appropriate Lagrange points.
Figure 1.3: Cartoons of the four types of binary morphologies.

With the Roche picture in mind, binary stars can also be classified in terms of the more physically meaningful relationship between the extent of the components relative to their critical bounding surface. Cartoons of the four possible cases are shown in Fig. 1.3. If both components extend beyond or, as a special case, just up to their Roche lobe, the system is referred to as a contact binary. These systems are also called W UMa binaries after the prototype star W Ursa Majoris. If only one component is in contact with its Roche lobe, the binary is semi-detached and if neither star fills its Roche lobe the system is detached. A semi-detached system, based on the arguments presented above, would be expected to transfer material to its companion. An analysis of the stability of a given binary system to mass transfer will be presented in §2.1.

1.3 Mass Transfer in Binaries

Of the binary star systems about half of them are at small enough separations that at some point in their evolution they will exchange material (Trimble, 1983). Paczyński first studied the possibility of mass transfer in binaries with insight from theoretical calculations of stellar evolution (Kippenhahn & Weigert, 1967). He noted that there are three occasions in the
life of a star that is actively undergoing nuclear fusion for it to swell up to fill
its Roche lobe and begin mass transfer. Paczyński labeled these cases A,
B, and C. For the sake of concreteness the discussion is cast in terms of a
donor star that is 5 times the mass of the sun (hereafter denoted with the
unit $M_\odot$ for solar mass) in a binary with a $2.5M_\odot$ companion. If the binary
period is between 0.65 and 1.5 days the donor will overflow its Roche lobe
while it expands on the main sequence resulting in case A. A star normally
brightens and expands due to its changing composition as hydrogen is fused
into helium in its core. Only about 10% of interacting binaries have periods
short enough to interact in case A. If the period lies between 1.5 days and
about 87 days, the donor will fill its Roche lobe between the end of central
hydrogen burning and helium ignition and be a class B system and transfer
material while it is a red giant star. Case B is the most frequently observed
scenario for interacting binary stars, accounting for about 50% of the donor
stars. This is the case for both Algol and $\beta$ Lyrae. If, after the end of central
Helium burning, the star comes into contact with its Roche lobe the binary
will be in class C. This will be the case for orbital periods from 87 days up
to 4,300 days, although the upper limit is fairly uncertain (Paczyński, 1971;
Trimble, 1983).

The distorted Roche geometry can have effects that are easily detectable
in the light curves of eclipsing binary systems. To lowest order in the tidal
field, the distorted star's surface is described by an ellipsoid. The variation
in the light curve observed outside of eclipse is called ellipsoidal variation
in reference to the non-spherical, and hence time varying, surface area and
nonuniform surface brightness presented to the observer at different orbital
phases. In the limit of W UMa systems the light curve varies continuously
throughout the orbit. For semi-detached systems the distortion is less se-
vere but often still detectable giving rise to the $\beta$ Lyrae class of eclipsing
variables. Finally, for eclipsing binaries where the separation is larger still,
the light curve becomes flat or nearly flat outside of the eclipses. Binaries
exhibiting this character of light curve are Algol binaries. By modeling the
ellipsoidal variation observed in a light curve, one can estimate how close the
components are to their Roche lobe (Kalrath & Milone, 1999).

Binaries and the interaction between the components have become im-
portant theoretical constructs. One of the first applications of mass transfer
in theoretical astrophysics was the resolution of the Algol paradox. Briefly,
in the Algol binary system it is the less massive component that appears to
have evolved most rapidly which contradicts the expectation that the more
massive star will have the higher central temperature and consume its hy-
drogen fuel first. The evolution of a star due to the nuclear reactions that
power the star is characterized by the nuclear timescale. This timescale is
defined as the time it takes for the star to convert a tenth of its hydrogen to
helium and is given approximately by (Hansen & Kawler, 1994),

$$t_{\text{nuclear}} \approx 10^{10} \left( \frac{M}{M_\odot} \right)^{-2.5} \text{ years.}$$

(1.2)
Crawford proposed that Algol, as seen today, is the result of the following evolutionary scenario (Crawford, 1955). The initially more massive star evolved to the point where it underwent Roche lobe overflow and transferred a large fraction of its material to the companion star which then became the more massive, though less evolved star that we see today. A similar scenario is used to explain the blue stragglers, stars that have stayed on the main sequence too long given the evolutionary state of other stars that formed in the cluster at the same time (Hoyle, 1964; McCrea, 1964).

In addition to rejuvenation, mass transfer is also the death of some stars. Type Ia supernova (categorized by the absence of hydrogen and presence of silicon features in their spectra) are believed to result from the sudden initiation of carbon fusion in the core of a white dwarf that has slowly accreted material from a companion star. As the white dwarf is supported against collapse by electron degeneracy pressure, its equation of state is nearly independent of the temperature and the star does not expand and cool in response to the new energy source in its core. A reaction front of nuclear fusion propagates outward and incinerates the white dwarf (Carroll & Ostlie, 1996).

Compact objects in binaries play a role in many interesting systems beyond supernovae. White dwarfs, due to their relative abundance amongst the stellar remnants, are often part of interacting binary systems. Classical novae, cataclysmic variables and dwarf novae are all the result of mass transfer onto a white dwarf. If the white dwarf is replaced by a neutron star or black hole the energy scale is correspondingly higher due to the deeper potential well of the compact object, giving rise to the X-ray binaries (Kahabka et al., 1999; Iben & Tutukov, 1998). The X-ray binaries are divided into two categories depending on the mass of the donor star. In the High-Mass X-ray binaries the donor typically has a mass between 10 and 40$M_\odot$ and for the Low-Mass X-ray binaries the donor has a mass $\leq 1.2M_\odot$ typically (Shore et al., 1992).

1.4 Fluid Dynamic Simulations of Binaries

The goal of the work presented in this dissertation is the simulation, from first principles, of the self-consistent, dynamical evolution of mass-transferring binaries by applying the techniques of explicit computational fluid dynamics of self-gravitating fluids. No assumptions are made about the symmetry of the binary system and the subsequent flow between components in the calculations are fully three-dimensional. We do assume that the fluid can be treated, from the point of view of thermodynamics, as adiabatic and the initial binary models are restricted to follow circular orbits. Furthermore, we only consider evolutions of binary systems where the components do not differ greatly in size as both components must be resolved adequately to apply the computational techniques described here.

There have been many theoretical investigations of interacting binary systems conducted over the past five decades but few address the problem posed in this dissertation. Most work has been conducted with the tools
appropriate to the theory of stellar evolution and stellar structure. These investigations are, in a sense, one-dimensional in that the donor and receiver (if the receiver’s response is calculated at all) are treated as separate entities which obey certain ordinary differential equations whose form is dictated by the physical effects that one wants to model (c.f., Paczyński (1971); Webbink (1984); King & Kolb (1995); Ritter (1996); McCormick & Frank (1998)). Within this category, the researchers do not solve the full set of partial differential equations that govern the flow of matter and the forces acting on that matter.

Prendergast & Taam (1974) performed pioneering simulations of two-dimensional flow in a system modeled after the U Cephei binary system using a novel particle technique based on the Boltzman equation. Their simulations indicated the possibility of a “hot spot” where the accretion stream self-intersected. More recently, Blondin et al. (1995) performed a two-dimensional simulation of the mass transfer stream in Algol using an Eulerian technique to examine the dependence of the accretion structure on the assumed radiative efficiency of the stream material.

Bisikalo and collaborators have performed a number of three-dimensional, though not self-consistent, simulations of RLOF in systems that model particular binaries of interest including β Lyrae (Bisikalo et al., 2000), the cataclysmic variable Z Cha, and the Low-Mass X-ray binary X1822-371 (Bisikalo et al., 1998). In their work the gravitational potential is given by eq. (1.1) and the Roche lobe of the donor is treated as a boundary condition for the flow where the density is held constant and the velocity field in assumed to be normal to the Roche surface with a magnitude equal to the local speed of sound.

The specific problem of the formation of a common envelope during mass transfer from a donor that is much more massive than its companion has been calculated using Eulerian techniques by Terman et al. (1994, 1995) and Sandquist et al. (1998), and using smoothed particle hydrodynamics by Rasio & Livio (1996) and Taam (1994). These simulations aim to calculate the efficiency of ejection of the common envelope when an unresolved star spirals into a giant. For these common envelope evolution calculations, at most one star is treated as a fluid body, the in-spiraling donor being treated as a rigid sphere.
2. Theoretical Background

2.1 Stability of Mass Transfer

In 1960, Morton made the following observations regarding the collection of eclipsing binaries. For detached binaries composed of main sequence stars, there exist binaries where either the more or less massive star is closest to contact with its Roche lobe. In binaries with stars that have evolved off the main sequence however, no semi-detached systems are known where the more massive star is in contact with its Roche lobe. In all semi-detached systems it is always the less massive star that is transferring mass to its companion (Morton, 1960). This conclusion was based on a set of about 70 semi-detached systems.

The facts discussed above constitute a problem along the lines of the Algol paradox presented in §1.3. In a close, but detached binary system, it must be the case that the more massive component will evolve off the main sequence first and make contact with its Roche lobe first. However, there were no binaries in Morton’s sample where the more massive star was undergoing RLOF. A resolution to this problem is possible if the mass transfer event that reverses the role of more and less massive components in the binary occurs on a short timescale. If this were the case, astronomers would be unlikely to catch any system in this short phase of its evolution. For the stars that Morton considered (those stars more massive than the Sun that have just left the main sequence) he concluded on theoretical grounds that, in fact, the donor will transfer material on a rapid timescale. The mass transfer in this case proceeds on a Kelvin-Helmholtz timescale.

The Kelvin-Helmholtz timescale (hereafter referred to as $t_{KH}$) corresponds to the time it takes for a star to adjust to a change in thermal equilibrium within the star and is defined as the ratio of the energy content of the star to the rate at which it loses energy, that is to say its luminosity. For the Sun, $t_{KH} \approx 2 \times 10^7$ years. Another timescale of interest is the dynamical timescale (hereafter $t_{dyn}$) which is defined as,

$$t_{dyn} = \frac{\sqrt{3 \pi}}{16 G \bar{\rho}},$$

where $\bar{\rho}$ is the mean density of the system. The dynamical time is also roughly the same as the sound crossing time so that a star will readjust to a change in mechanical equilibrium on a timescale of order $t_{dyn}$. For the Sun, $t_{dyn} \approx 42$ minutes.

As first pointed out by Paczyński, mass transfer may occur on a time scale that is even faster than $t_{KH}$ if the donor star has a significant region in its envelope where convection is the dominant heat transport mechanism as is the case for red giants (Paczyński, 1965; Paczyński & Sienkiewicz, 1972). Similar arguments apply to main sequence stars less massive than the Sun.
which have significant convective envelopes or are fully convective and also to compact objects such as white dwarfs.

The condition that a star, which will suggestively be termed the donor, transfers matter is that its radius is equal to or greater than its Roche lobe radius. The stability of the mass transfer event, that is to say whether the rate of mass transfer grows or decays as the result of some small amount of matter being exchanged, is dependent on the change in the orbital separation which dictates the size of the donor’s Roche lobe, the change in the Roche lobe shape due to the changing mass distribution and the response of the donor to a change in its total mass. The condition for stability can be expressed as

$$\dot{R}_d \leq \dot{R}_{d}^{RL}$$  \hspace{1cm} (2.2)

which simply states that the donor’s radius must shrink at least as fast as its Roche lobe or that the donor can expand no faster than its Roche lobe expands.

The first factor we will consider is the change in the orbital separation resulting from a mass transfer event. The orbital angular momentum, $J_{\text{orb}}$ for a binary composed of two point masses is given by,

$$J_{\text{orb}} = \mu a^2 \Omega,$$  \hspace{1cm} (2.3)

where $a$ is the orbital separation, $\Omega$ is the angular frequency of the binary ($\Omega = \frac{2\pi}{P}$ where $P$ is the orbital period) and $\mu$ is the reduced mass given by

$$\mu = \frac{M_r M_d}{M_r + M_d} = \frac{M_r M_d}{M},$$  \hspace{1cm} (2.4)

where the subscripts “r” and “d” label the stars that will play the role of receiver and donor in the mass transfer event. We can use Kepler’s third law,

$$\Omega^2 a^3 = GM,$$  \hspace{1cm} (2.5)

to eliminate the angular frequency from eq. (2.3) to obtain,

$$J_{\text{orb}} = M_r M_d \sqrt{\frac{G a}{M}}.$$  \hspace{1cm} (2.6)

If we logarithmically differentiate eq. (2.6) with respect to time we obtain

$$\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = \frac{\dot{M}_r}{M_r} + \frac{\dot{M}_d}{M_d} + \frac{1}{2} \frac{\dot{a}}{a} - \frac{1}{2} \frac{\dot{M}}{M}.$$  \hspace{1cm} (2.7)

If, for the sake of argument, we assume that the mass transfer event is conservative which means that the orbital angular momentum and total mass are conserved (so that $\dot{M}_d = -\dot{M}_r$ and $J_{\text{orb}} = 0$) eq. (2.7) becomes

$$\frac{\dot{a}}{2a} = -\dot{M}_d \left(\frac{1}{M_d} - \frac{1}{M_r}\right) = \frac{(-\dot{M}_d)}{M_d} (1 - q),$$  \hspace{1cm} (2.8)
where we have introduced the mass ratio, \( q \equiv \frac{M_d}{M_r} \). Eq. (2.8) has been written to suggest that the donor is the star losing mass so that \(-\dot{M}_d\) is positive. We can then see that if the donor is the more massive star \((q > 1)\), the orbital separation will shrink upon mass transfer. Conversely, if \(q < 1\), \(a\) will increase. Physically this follows from the observation that the reduced mass appearing in eq. (2.3) is maximized when the two components are of equal mass. Mass transfer that brings the components closer to equality causes the separation to shrink if the orbital angular momentum is conserved. Mass loss that drives \(\mu\) away from its maximum value likewise causes the separation to increase to conserve angular momentum.

We next consider the response of the Roche lobe to mass transfer. Paczyński’s approximation is a convenient expression for the effective radius of a component’s Roche lobe (the radius of a sphere that occupies the same volume as the Roche lobe) (Paczyński, 1971). The Roche lobe radii, \(R_{d}^{RL}\), is given by

\[
\frac{R_{d}^{RL}}{a} = \frac{2}{3^{4/3}} \left( \frac{M_d}{M_d + M_r} \right)^{\frac{1}{3}} = 0.462 \left( \frac{M_d}{M} \right)^{\frac{1}{3}}. \tag{2.9}
\]

This expression is accurate to within two percent over the range \(0 < \frac{M_d}{M_r} < 0.8\). For reference, a more accurate approximation (accurate to within one percent) for the Roche lobe radius is given by

\[
\frac{R_{d}^{RL}}{a} = \frac{0.49 q^{\frac{3}{2}}}{0.6 q^{\frac{7}{2}} + \ln \left( 1 + q^{\frac{1}{3}} \right)}, \tag{2.10}
\]

for \(0 < q < \infty\) (Eggleton, 1983). When we logarithmically differentiate eq. (2.9) with respect to time, we obtain

\[
\frac{\dot{R}_{d}^{RL}}{R_{d}^{RL}} = \frac{\dot{a}}{a} + \frac{1}{3} \frac{\dot{M}_d}{M_d} - \frac{1}{3} \frac{\dot{M}}{M} \tag{2.11}
\]

Again, in the limit of conservative mass transfer, so that \(\dot{M} = 0\), we can see that the donor’s Roche lobe radius will contract upon mass transfer if it is the more massive star and may expand or contract if it is the less massive component depending on the transfer rate.

It is more convenient to eliminate the orbital separation, \(a\), from eq. (2.11) in favor of the orbital angular momentum. By combining eqs. (2.7) and (2.11) we arrive at

\[
\frac{\dot{R}_{d}^{RL}}{R_{d}^{RL}} = 2 \frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} - \frac{5}{3} \frac{\dot{M}_d}{M_d} - 2 \frac{\dot{M}_r}{M_r} + 2 \frac{\dot{M}}{M} \tag{2.12}
\]

Eliminating the accretion rate as \(\dot{M}_r = \dot{M} - \dot{M}_d\), we obtain

\[
\frac{\dot{R}_{d}^{RL}}{R_{d}^{RL}} = 2 \frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} - \frac{5}{3} \frac{\dot{M}_d}{M_d} + 2 \frac{\dot{M}_d}{M} - 2 \frac{\dot{M}}{M} - 2 \frac{\dot{M}}{M_r} \tag{2.13}
\]
which simplifies to
\[
\frac{\dot{R}_{d_{RL}}}{R_{d_{RL}}} = 2 \frac{\dot{J}_{\text{orb}}}{\dot{J}_{\text{orb}}} + \left(2 \frac{M_d}{M_r} - \frac{5}{3}\right) \frac{\dot{M}_d}{M_d} + \left(\frac{2}{3} - 2 \frac{M}{M_r}\right) \frac{\dot{M}}{M} \quad (2.14)
\]

If we assume the mass transfer to be conservative, eq. (2.14) simplifies further to yield the desired expression
\[
\frac{\dot{R}_{d_{RL}}}{R_{d_{RL}}} = \left(2 \frac{M_d}{M_r} - \frac{5}{3}\right) \frac{\dot{M}_d}{M_d}, \quad (2.15)
\]
which we parameterize as
\[
\frac{\dot{R}_{d_{RL}}}{R_{d_{RL}}} = \xi_{R} \frac{\dot{M}_d}{M_d}. \quad (2.16)
\]

In the conservative case eq. (2.15) indicates that if \( M_d > \frac{5}{6} M_r \) the donor’s Roche lobe will contract upon mass loss as \( \dot{M}_d \) is negative.

We next consider the response of the donor’s radius to a change in its mass. We are concerned only with the dynamical response of the star as it tries to regain hydrostatic equilibrium given its new mass and do not treat the possible expansion or contraction of the star on the longer Kelvin-Helmholtz timescale. For stars like the Sun it is well known that the mass and radius are approximately proportional to one another. In the flavor of the previous expressions, the mass radius relation for solar type stars implies
\[
\frac{\dot{R}_{d}}{R_{d}} = \frac{\dot{M}_d}{M_d} \quad (2.17)
\]

If we substitute eq. (2.15) and eq. (2.17) into (2.2) we obtain a limiting stable mass ratio for the binary,
\[
q_{\text{stable}} = \frac{M_d}{M_r} \leq \frac{4}{3}. \quad (2.18)
\]

If the mass ratio \( q \) exceeds this value, the Roche lobe will shrink faster than the star can contract and the mass transfer will proceed on a dynamical timescale until the stability criterion (2.18) is met. If, on the other hand, \( q \leq \frac{4}{3} \) the star will, on a timescale set by the mass transfer rate, detach from its Roche lobe. Mass transfer may continue even for stable mass ratios however. From eq. (2.14) any mechanism that removes orbital angular momentum from the binary will cause the Roche lobe of the donor to contract. Examples of such mechanisms are the emission of gravitational radiation (Schutz, 1990) and magnetic braking (Frank et al., 1992).

If instead of using the empirical mass-radius relation (2.17) we consider a polytrope, we arrive at a different stability criterion. A spherical polytrope of
index $n$ and uniform entropy in mechanical equilibrium obeys a mass-radius relation of the form (Chandrasekhar, 1939)

$$R_d \propto M_d^{\frac{1-n}{3-n}}$$

(2.19)

so that for $1 < n < 3$ the donor will expand upon mass loss. Again, taking the logarithmic time derivative of (2.19) we have

$$\frac{\dot{R}_d}{R_d} = \left(\frac{1-n}{3-n}\right) \frac{\dot{M}_d}{M_d} \equiv \xi \frac{\dot{M}_d}{M_d}$$

(2.20)

so that the stability criterion, eq. (2.2) implies that

$$q_{stable}(n) = \frac{9 - 4n}{3(3 - n)}$$

(2.21)

for a polytrope of index, $n$. For $n = \frac{3}{2}$ the mass transfer will be unstable on a dynamical timescale if $q = \frac{M_d}{M_r} > \frac{2}{3}$. The case of $n = \frac{3}{2}$ is of particular interest as this polytrope serves as a good approximation to low mass, fully convective stars, nonrelativistic white dwarfs and emulates some properties of red giants and similar stars with significant convective envelopes.

It is important to note the approximations that were used to arrive at the results presented in this section. We have assumed that the total mass does not change so that all material that leaves the donor is accreted by the receiver. We have also assumed that the orbital angular momentum is conserved by itself despite the fact that, in reality, the components are extended objects that have their own intrinsic spin angular momentum. We have assumed that the gravitational field is given by the field of point masses so we have at best treated the system as two envelopes of massless fluid that are acted on by the force of point-like cores. We have also neglected the inherent non-spherical geometry of the Roche lobes in constructing the limiting stable mass ratios. While the generality of the relations presented above can be extended somewhat by, for example, introducing a mass transfer efficiency that accounts for mass loss from the system as well as accretion by the receiver the problem of mass transfer between two stars is inherently three-dimensional and should be treated as such. Unfortunately, once we allow the mass and velocity fields to correspond to extended bodies we are well beyond the regime where analytical solutions exist in closed form. It is, however, possible to solve the underlying equations numerically. The proper context for the simple relations presented in this section is to provide insight for examining the three-dimensional hydrodynamical simulations.
2.2 The Nature of \( L_1 \) and the Mass Transfer Rate

When the donor comes into contact with its Roche lobe it will overflow at the inner Lagrange point \( (L_1) \) first as this is the deepest of the stationary points near the donor. If we assume that the flow through the “nozzle” of the bounding Roche lobe at \( L_1 \) can be treated as plane parallel flow (that is we are interested only in an approximate solution in the immediate neighborhood of \( L_1 \)) and if we further concern ourselves only with steady flow, the continuity equation can be written as

\[
\frac{d(\rho v)}{dx} = 0 \tag{2.22}
\]

or

\[
\frac{1}{\rho} \frac{d\rho}{dx} = -\frac{1}{v} \frac{dv}{dx}, \tag{2.23}
\]

where \( \rho \) is the density of the fluid, \( v \) is the velocity and we have chosen a coordinate system where the line of centers coincides with the \( x \)-axis. Euler’s equation takes on the form

\[
\frac{v}{\rho} \frac{dv}{dx} = -\frac{c^2}{\rho} \frac{d\rho}{dx} - \frac{d\Phi_{\text{Roche}}}{dx}, \tag{2.24}
\]

where \( p \) is the pressure of the fluid and \( \Phi_{\text{Roche}} \) is the Roche potential defined in eq. (1.1). We can express the pressure force as

\[
\frac{dp}{dx} = \frac{dp}{d\rho} \frac{d\rho}{dx} = c^2 \frac{d\rho}{dx}, \tag{2.25}
\]

where we have introduced the speed of sound, \( c \). Euler’s equation can then be rewritten as

\[
\frac{v}{\rho} \frac{dv}{dx} - \frac{c^2}{v} \frac{dv}{dx} = -\frac{d\Phi_{\text{R}}}{dx}, \tag{2.26}
\]

or

\[
\left(1 - \frac{c^2}{v^2}\right) \frac{dv}{dx} = -\frac{1}{v} \frac{d\Phi_{\text{R}}}{dx}. \tag{2.27}
\]

If the flow velocity is to be a monotonic function of \( x \), \( L_1 \) must be a sonic point in the flow because upon crossing the stationary point at \( L_1 \), the right-hand side of eq. (2.27) changes sign.

Lubow & Shu (1975) have examined the flow of material from the donor to the receiver in much greater detail than presented above, though still in the limit of an isothermal fluid. In their study, they concluded that the flow of material from donor to receiver spans three distinct regimes. In the first regime, which extends from the neighborhood of \( L_1 \) to the receiver the effects of pressure in the fluid stream are small and the motion of the stream is
given to good approximation by ballistic trajectories. In the surface layer of the donor, material flows to the $L_1$ point. In this regime the fluid is approximately in hydrostatic equilibrium in the vertical direction but in the horizontal direction relatively large Coriolis forces accelerate the fluid. The flow from the poles to the equator was found to be suggestive of the large scale circulation patterns observed in the atmosphere and oceans of the Earth and could be further complicated if the star was convective. Finally, they found that the flow near $L_1$ is transonic, as indicated by the simple argument given above. In the neighborhood of $L_1$ the potential perpendicular to the line of centers is given approximately by $\phi_{\text{Roche}} \approx \frac{1}{2} \Omega^2 y^2$ where we have taken $y$ to be the transverse coordinate. If we equate the thermal speed of the fluid at $L_1$ with the potential energy we see that the width of the stream roughly obeys,

$$\text{width} \propto \frac{c}{\Omega}.$$  \hfill (2.28)

Given the results presented in this section thus far, the steady-state mass transfer rate for a polytropic donor star can be estimated from the product of the volume swept out by the flow in unit time and the density of the fluid near $L_1$. The volume swept out by the flow is given by the cross section of the flow, which scales as $c^2 P^2$, times the flow velocity which is approximately given by the sound speed. The volume then scales as the cube of the sound speed. The sound speed for a polytrope obeys

$$c \propto \sqrt{\frac{p}{\rho}} \propto \rho^\frac{1}{2n},$$  \hfill (2.29)

as

$$p = \kappa \rho^{1+\frac{1}{n}},$$  \hfill (2.30)

for a polytrope of index $n$ and polytropic constant $\kappa$. The density near $L_1$ can be estimated from the equation of hydrostatic equilibrium near the edge of a spherical polytrope

$$\frac{1}{\rho} \frac{d\rho}{dr} \approx -\frac{GM_s}{R^2_*},$$  \hfill (2.31)

where $M_*$ and $R_*$ are the mass and radius of the polytrope. From the polytropic equation of state, eq. (2.30), where we use the polytropic exponent $\gamma = 1 + \frac{1}{n}$, we can eliminate the pressure to obtain

$$\frac{\kappa \gamma}{\rho} \rho^n \frac{d\rho}{dr} = n \gamma \kappa \frac{d\rho}{dr} = -\frac{GM_s}{R^2_*};$$  \hfill (2.32)

which can be integrated to yield

$$\rho(r) = \left[ \frac{GM_s}{\kappa (n + 1)} \frac{(R_* - r)}{r^2_*} \right]^n.$$  \hfill (2.33)
If we change variables from \( r \) to \( \Delta r \), the width of a thin shell near \( R^* \), we see that
\[
\rho (\Delta r) \propto (\Delta r)^n.
\] (2.34)

If we instead interpret \( \Delta r \) to be the amount that the donor star exceeds its Roche lobe radius (termed the degree of over-contact), \( \Delta r = R^* - R_{dRL}^* \), the mass transfer rate will behave as
\[
\dot{M} \propto \rho^{1+\frac{3}{2n}} \propto (\Delta r)^{n+\frac{3}{2}},
\] (2.35)
or
\[
\dot{M} \propto M^* \frac{\rho}{P} \left( \frac{\Delta r}{R^*} \right)^{n+\frac{3}{2}}.
\] (2.36)

For a polytrope of index, \( n = \frac{3}{2} \), the mass transfer rate should scale as the cube of the degree of overcontact.

### 2.3 A Self-Consistent Roche Potential

One important step towards a consistent evolution of a binary star system is the use of a gravitational field that is derived from the density field itself. In Fig. 2.1 we show a surface and contour plot of the self-consistent Roche potential in the equatorial plane for a binary system where the less massive component is 0.84 times the mass of its companion. The construction of such binary systems is the topic of the next chapter. For now suffice it to say that the gravitational potential is found by explicitly solving Poisson’s equation for a self-consistent matter distribution. The angular frequency of the binary is calculated to be consistent with both the density and potential fields and corresponds to motion of the binary components along circular orbits. To be clear about terminology, we will hereafter refer to the self-consistent Roche potential as simply the Roche potential. When needed in future discussions, the potential given by eq. (1.1) will be referred to as the point mass Roche potential.

For comparison, in Fig. 2.2 the Roche potential and the point mass Roche potential are plotted along the line of centers for the same model. The point mass Roche potential is constructed with the same masses, placed at the same separation as the self-consistent model and uses the Keplerian angular velocity. The bottom plot shows a magnified version of the potential curves in the neighborhood of \( L_1 \).

The potentials obviously disagree towards the center of either star where the point mass potential diverges. The location and value of the \( L_1 \) point are slightly different for the two potentials and the point mass approximation is noticeably worse for the less massive star (the right minimum in the potential). Though not apparent from a plot of the potential alone, it is the less massive star that is closest to its Roche lobe and hence has more mass nearer to its Roche surface. This results in a more significant deviation from the point mass approximation. In general, the disagreement between the self-consistent Roche potential and point mass Roche potential will be greater for systems with stiffer equations of state (closer to being uniform density) and for systems that are closer to contact.
Figure 2.1: A surface plot and contours for a self-consistent Roche potential in the equatorial plane of the binary. The system has a mass ratio of 0.84 and is presented as Model 6 in Chapter 3.
Figure 2.2: The self-consistent Roche potential (solid curve) and point mass Roche potential (dashed curve) arising from point mass components on a Keplerian orbit for the same binary system are shown in the top plot. The plot is made along the line of centers joining the two components. The bottom figure shows a magnified version of the potentials in the immediate neighborhood of the inner Lagrange point.
3. Self-Consistent Field Technique

In this chapter we consider the equilibrium density distributions in binary star systems. The discussion begins with a review of past treatments of the problem. Roche was able to obtain an analytical solution for the density field of an incompressible fluid that is distorted by the tidal field of a rigid spherical companion. The equilibrium figure is an ellipsoid. To obtain the solution, Roche expanded the tidal field in terms of the ratio of the distance from the center of mass of the fluid star to the orbital separation and had to truncate the tidal potential at the quadratic term in this expansion. Darwin later extended Roche’s work to treat two incompressible fluid bodies but consistent solutions are possible only in the case where the components are of identical mass. These results and the stability of the Roche and Darwin ellipsoids are treated in great detail in Chandrasekhar (1969).

The results of Roche and Darwin were extended to polytropic fluids in a series of papers by Lai, Rasio, and Shapiro. Their work was based on a variational principle and was flexible enough to treat components of differing mass and with arbitrary spins, although the tidal potential was again truncated at the second order (Lai et al., 1993a,b, 1994a,b). The goal of these investigations was to study the onset of the tidal instability. Briefly, if we consider a sequence of synchronous binary stars that each have the same mass but vary in their orbital separation this sequence will reach a minimum in the total energy and angular momentum at some critical separation. If some mechanism, such as the emission of gravitational radiation drives a given system to smaller separation it will reach the minimum and the binary can no longer remain synchronous. This is termed the secular tidal instability. An orbiting body can also suffer a dynamical tidal instability which is analogous to the instability of orbits in the Schwarzschild metric and results from the potential becoming too steep to support circular orbits (this is the case if the potential behaves as $\Phi \propto r^a$ for $a \leq -4$). A more detailed discussion of these instabilities can be found in New (1996).

A numerical approach to studying the equilibrium structure of self-gravitating fluids was introduced by Ostriker and Mark in 1967. They developed the Self-Consistent Field (SCF) method, for generating the equilibrium structure of single rotating polytropes by iteratively solving the equations of hydrostatic equilibrium. (Ostriker & Mark, 1968). The original SCF method was greatly improved by Hachisu (1986a) and in Hachisu’s formulation the SCF technique could be applied to very rapidly rotating figures. The SCF technique was extended to binary systems in Hachisu et al. (1986b). The binary SCF method presented here is similar to the original method of Hachisu et al. It differs in the choice of technique for solving Poisson’s equation, which Hachisu performed through a multipole expansion, the use of spherical coordinates, and the choice of equation of state. To date, our binary SCF code has only been applied to polytropes while the original work of Hachisu et al. treated only the zero-temperature white dwarf equation of state.
We use a cylindrical coordinate grid to describe the model and $R$, $z$ and $\phi$ represent the radial, vertical and azimuthal coordinates respectively. The $x$-axis ($\phi = 0$), is taken to lie along the line of centers for the binary model and the axis of rotation is always taken to be parallel to the $z$-axis. The vector $\mathbf{r}$ refers to an arbitrary point in space and the vector $\mathbf{R}$ is the cylindrical radius vector. Unless otherwise noted we use units where the gravitational constant is unity, the radial extent of the computational grid is unity and the mass scale is set by the maximum density appearing in a given binary model. The density scale is not necessarily the same between different models. We consider only polytropic equations of state in this work, although extending the method to use the zero temperature white dwarf equation of state or even a tabular equation of state presents no fundamental complication. By way of notation we characterize a polytrope by its index $n$, exponent $\gamma = 1 + \frac{1}{n}$, and constant $\kappa$.

The equations governing steady flow for a self-gravitating fluid are as follows (c.f., Frank et al., 1992):

\begin{equation}
\nabla \cdot (\rho \mathbf{v}) = 0, \quad (3.1)
\end{equation}

\begin{equation}
(\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \quad (3.2)
\end{equation}

\begin{equation}
\nabla \cdot \left[ \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \epsilon + p \right) \mathbf{v} \right] = -\nabla \Phi \cdot \mathbf{v}, \quad (3.3)
\end{equation}

where $\rho$ is the mass density, $\mathbf{v}$ is the velocity field, $p$ is the pressure of the fluid, $\Phi$ is the gravitational potential and $\epsilon$ is the internal energy of the fluid per unit mass.

If we consider only binaries that rotate synchronously we can solve eqs. (3.1) – (3.3) in a rotating frame of reference that renders the fluid stationary. In the corotating reference frame both eq. (3.1) and eq. (3.3) are solved trivially and Euler’s equation appears as

\begin{equation}
-\frac{1}{\rho} \nabla p - \nabla \Phi - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{R} = 0, \quad (3.4)
\end{equation}

where $\mathbf{\Omega}$ is the angular frequency of the reference frame. Since we are constraining $\mathbf{v}$ to be zero explicitly, the Coriolis force does not appear in eq. (3.4). The centrifugal force can be written as the gradient of a potential and we can introduce the enthalpy, which is given to within an arbitrary constant by

\begin{equation}
H \equiv \int \frac{dp}{\rho} \quad (3.5)
\end{equation}

to simplify eq. (3.4) to the form

\begin{equation}
\nabla \left( H + \Phi - \frac{1}{2} \mathbf{\Omega}^2 R^2 \right) = 0. \quad (3.6)
\end{equation}
This equation can be trivially integrated to yield

\[ H + \Phi - \frac{1}{2} \Omega^2 R^2 = C_i, \]  

where \( C_i \) are constants of integration and the index \( i \) simply labels the two components. The set of partial differential equations given by (3.1) – (3.3) have been reduced to a single scalar equation for each star. Because of this transformation, the SCF method is sometimes referred to as an integral method.

In the derivation of eq. (3.7) we have assumed that the system center of mass resides at the origin of the coordinate system. As we shall see, this is not sufficiently general. Instead we reformulate (3.7) as

\[ H + \Phi - \frac{1}{2} \Omega^2 |R - R_{\text{com}}|^2 = C_i, \]  

where \( R_{\text{com}} \) is the cylindrical radius vector to the system’s center of mass so that \( |R - R_{\text{com}}| \) is the distance from a grid location at \( R \) to the rotation axis.

An iterative scheme can be based on the master equation (3.8) as follows. An initial guess at the density distribution is formed. Poisson’s equation is then solved to obtain the gravitational potential arising from the mass distribution. This is, by far, the most computationally intensive part of the algorithm. For our work, we have utilized subroutines from the FISHPACK Fortran subroutine set for the solution of elliptic partial differential equations (Schwarztrauber & Sweet, 1975; Schwarztrauber et al., 2000). These subroutines implement a Fourier analysis - cyclic reduction algorithm (Press et al., 1992). The solution of Poisson’s equation for an isolated mass distribution in a finite domain requires that the potential (or its gradient) be specified on a boundary enclosing the mass distribution. We calculate the boundary potential from a spherical harmonic moment expansion of the density distribution utilizing moments through \( \ell = 10 \).

The center of mass of the density distribution is then calculated. With the gravitational potential and coordinates of the system’s center of mass in hand we can use algebraic relations at three boundary points where we force the density field to vanish to set the two integration constants \( C_1 \) and \( C_2 \), and the angular velocity \( \Omega \). The boundary points all lie along the line of centers and correspond to the inner and outer boundary points for one star and the inner boundary point for its companion as illustrated in Fig. 3.1. The value of the gravitational potential at the three boundary points, \( r_A \), \( r_B \), and \( r_C \) are used to solve for \( \Omega \), \( C_1 \), and \( C_2 \) as follows:

\[ \Omega^2 = \frac{\Phi (r_A) - \Phi (r_B)}{\frac{1}{2} \left( |R_A - R_{\text{com}}|^2 - |R_B - R_{\text{com}}|^2 \right)}, \]  

\[ C_1 = \Phi (r_B) - \frac{1}{2} \Omega^2 |R_B - R_{\text{com}}|^2. \]  

(3.9)  

(3.10)
Figure 3.1: Position of the three boundary points in the equatorial plane for a SCF binary model. At the three boundary points (denoted by asterisks and labeled A, B, and C), the density is forced to vanish. The contours represent density levels for the converged model.

\[ C_2 = \Phi (r_C) - \frac{1}{2} \Omega^2 |r_C - R_{com}|^2 \]  \hspace{1cm} (3.11)

All quantities appearing eq. (3.8) are now known for the current density distribution, save for the enthalpy, \( H \) which can be constructed throughout the computational domain. From the enthalpy, an improved density distribution can be constructed using the relation,

\[ \rho = \rho_i^{\text{max}} \left( \frac{H}{H_i^{\text{max}}} \right)^n. \]  \hspace{1cm} (3.12)

As Hachisu (1986a) has explained, it is best to hold the values of \( \rho_i^{\text{max}} \) fixed throughout the iterations.
Table 3.1: Equilibrium Binary Models

<table>
<thead>
<tr>
<th>Model</th>
<th>q</th>
<th>$\rho_1^{\text{max}}$</th>
<th>$R_1$</th>
<th>$P_1^{\text{RL}}$</th>
<th>$\rho_2^{\text{max}}$</th>
<th>$R_2$</th>
<th>$P_2^{\text{RL}}$</th>
<th>VE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>1.00</td>
<td>0.3720</td>
<td>0.3723</td>
<td>1.00</td>
<td>0.3720</td>
<td>0.3723</td>
<td>$1.5 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>1.2111</td>
<td>0.60</td>
<td>0.3893</td>
<td>0.3915</td>
<td>1.00</td>
<td>0.3056</td>
<td>0.3580</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>0.4801</td>
<td>1.00</td>
<td>0.3126</td>
<td>0.3129</td>
<td>1.20</td>
<td>0.3727</td>
<td>0.4401</td>
<td>$3.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>0.1999</td>
<td>0.77</td>
<td>0.2476</td>
<td>0.2478</td>
<td>1.00</td>
<td>0.3817</td>
<td>$&gt; 0.5194$</td>
<td>$2.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>1.0000</td>
<td>1.00</td>
<td>0.2984</td>
<td>0.2778</td>
<td>1.00</td>
<td>0.2984</td>
<td>0.3778</td>
<td>$2.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>6</td>
<td>0.5436</td>
<td>1.00</td>
<td>0.3200</td>
<td>0.3620</td>
<td>1.20</td>
<td>0.3180</td>
<td>0.3919</td>
<td>$2.2 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

The iteration cycle is then repeated using the improved density distribution and the iterations are continued until the relative change from iteration to iteration in $C_1$, $C_2$, $\Omega$, $H_1^{\text{max}}$, and $H_2^{\text{max}}$ are all smaller than some prescribed convergence tolerance, $\delta$. For a grid resolution of 128 radial points by 128 vertical points by 256 points in azimuth, we typically use a tolerance of $\delta = 1 \times 10^{-4}$.

Unfortunately, the self-consistent field method does not allow one to specify physically meaningful parameters such as the binary mass ratio or separation 
*a priori*. Instead, as already described, it is best to specify the locations of the three boundary points and the maximum density for each body. Nevertheless, the method described above remains, to our knowledge, the most effective means of generating fully self-consistent models of synchronously rotating, equilibrium binaries.

The SCF method is insensitive to the functional form of the initial guess of the density distribution that is used to start the iteration. Uniform density spheres and Gaussian density profiles both yield converged models that agree to single precision accuracy (approximately seven digits). We also note that in some instances it is advantageous to use a potential that is a mixture of the previous and current potential in the construction of the enthalpy from eq. (3.8). This is the case for models with soft equations of state (*e.g.*, $n \geq \frac{3}{2}$). For stiffer equations of state there is a tighter coupling between the solution near the boundary points and the global solution (because of the greater amount of material at the edge of the stars) and the method converges quite rapidly. When attempting to generate a model where one component is close to contact, so that a solution will no longer exist for small changes in the input parameters, it can be helpful to begin the iteration with the density field of a previously calculated SCF binary that is close in parameter space.

We gauge the quality of a converged model by the degree to which it satisfies the scalar virial equation. Specifically, we define the following dimensionless quantity to be the virial error,

$$VE \equiv \frac{2K + W}{W} + 3\Pi,$$  

(3.13)
Table 3.2: Convergence for SCF Method

<table>
<thead>
<tr>
<th>R</th>
<th>Z</th>
<th>φ</th>
<th>δ</th>
<th>VE</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>64</td>
<td>128</td>
<td>1.0×10^{-3}</td>
<td>1.0×10^{-3}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-4}</td>
<td>6.0×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-5}</td>
<td>5.5×10^{-4}</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-6}</td>
<td>5.5×10^{-4}</td>
</tr>
<tr>
<td>128</td>
<td>128</td>
<td>256</td>
<td>1.0×10^{-3}</td>
<td>6.9×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-4}</td>
<td>2.0×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-5}</td>
<td>1.5×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-6}</td>
<td>1.4×10^{-4}</td>
</tr>
<tr>
<td>256</td>
<td>256</td>
<td>512</td>
<td>1.0×10^{-3}</td>
<td>6.3×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-4}</td>
<td>1.0×10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-5}</td>
<td>5.2×10^{-5}</td>
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<td></td>
<td></td>
<td></td>
<td>1.0×10^{-6}</td>
<td>4.7×10^{-5}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.0×10^{-7}</td>
<td>4.7×10^{-5}</td>
</tr>
</tbody>
</table>

where the terms appearing in eq. (3.13) are defined by the following integral quantities:

\[ K = \frac{1}{2} \int \rho \mathbf{v} \cdot \mathbf{v} \, dV, \tag{3.14} \]

\[ W = \int \rho \Phi \, dV, \tag{3.15} \]

\[ \Pi = \int p \, dV, \tag{3.16} \]

where \( \mathbf{v} \) is the velocity field. For the present context the motion of the fluid is entirely accounted for by the rotation of the frame and

\[ \mathbf{v} = \Omega \times (\mathbf{R} - \mathbf{R}_{\text{com}}). \tag{3.17} \]

In Fig. 3.2 we plot density contours in the meridional plane for one contact binary system, three semi-detached systems and two detached binaries that were constructed using the SCF technique described here. Fig. 3.3 shows contours in the equatorial plane for the same six systems. The solid lines are at mass density levels of 10^{-4}, 10^{-3}, 10^{-2}, and 10^{-1}, where the density has been normalized to the maximum density for each model, and the dashed lines follow the self-consistently determined critical Roche surface for the system. The more massive component is always shown on the left-hand side of the plots. The binaries all have a polytropic index of \( n = \frac{3}{2} \) and other key parameters for the models are listed in Table 3.1. For detached binary systems we refer to the star that is closest to filling its Roche lobe.
as the primary (denoted by a subscript “1”). Its companion is termed the secondary (denoted by a subscript “2”). This same nomenclature applies also to semi-detached binaries, though for dynamical evolutions we prefer to call the star filling its Roche lobe the donor and its companion the receiver. The listed mass ratios are the ratio of the mass of the primary to the mass of the secondary and the stellar radii ($R_1$ and $R_2$) and Roche lobe radii ($R_{1RL}$ and $R_{2RL}$) have all been normalized to the orbital separation. The radii for these non-spherical bodies are effective radii, that is the radius of a sphere that has a volume equal to the star or Roche lobe. We also list the maximum density for each component and the virial error for the converged model. For Model 4, the Roche lobe of the secondary extends beyond the computational domain so the effective radius in this case is a lower limit. All models were constructed on a grid containing 128 radial and vertical zones by 256 vertical zones.

Table 3.2 lists the resulting virial error for the contact binary (Model 1 from Table 3.1) constructed on grids of differing resolution up to the highest resolution possible for the largest memory computer to which we have access. As the convergence tolerance, $\delta$, is decreased the number of required iterations increases. For fixed resolution, the overall quality of the solution does not significantly improve beyond some limiting value of $\delta$, regardless of the number of iterations taken. As the resolution is increased, the virial error decreases roughly in proportion to the square root of the number of grid points.

Due to the symmetry of the SCF models about the equatorial plane, we only calculate the models in the half space of $z \geq 0$. Assuming that the line of centers coincides with the $x$ axis, the tidal distortion of each star must be symmetric about the $y = 0$ plane. Further computational efficiency could be obtained with the SCF method by limiting the computational grid to only extend from 0 to $\pi$ in azimuth. To date, we have not enforced this additional symmetry constraint, although in practice the converged models display this property.
Figure 3.2: Slice through the meridional plane for six example SCF binaries. The solid contours are spaced logarithmically at normalized density levels of $10^{-4}$, $10^{-3}$, $10^{-2}$, and $10^{-1}$. The dashed curve traces the critical surface of the self-consistent Roche potential.
Figure 3.3: Same as Fig. 3.2 but slices taken in the equatorial plane.
4. Other Applications of the Self-Consistent Field Technique

In this chapter we present two theoretical investigations that make use of the SCF technique in addition to the construction of equilibrium binaries presented in Chapter 3. The first of these involves the measurement of the volume of the donor’s self-consistent Roche lobe volume in equilibrium systems that approximate cataclysmic variables. The second exercise generalizes the treatment of the fluid as a polytrope with uniform entropy in an attempt to generate polytropes that satisfy the main sequence mass-radius relation for solar type stars.

4.1 Minimum Period for Cataclysmic Variables

A cataclysmic variable (CV) is a binary system consisting of a white dwarf and low mass, main sequence star that is transferring material to the white dwarf through RLOF. The accreted material forms an accretion disk (unless the magnetic field of the white dwarf is sufficiently large to disrupt the disk) that channels the material onto the white dwarf. The accretion disk makes a significant contribution to the light curve from these systems. It is possible, depending on the details of the particular system in question, for the accreted material to accumulate and ignite suddenly (giving rise to a nova) or to burn steadily (giving rise to the super-soft X-ray sources). See Warner (1995) for a thorough discussion of CV systems.

The description of the evolutionary sequence of CVs is a complicated subject which we will largely avoid. It is important to note that in CVs, $q = \frac{M_d}{M_r} < q_{\text{stable}}$ so that mass transfer is stable in the sense described in §2.1. Therefore, mass transfer only occurs because a driving mechanism removes angular momentum from the system and keeps the donor in contact with its Roche lobe. The driving mechanism may be gravitational radiation or magnetic braking for example. The important point for the current discussion is that throughout the time when the CV is visible (transferring matter), there is some mechanism that drives the binary to smaller separations and shorter orbital periods. If this were not the case, the donor would come out of contact with its Roche lobe and the CV would disappear from view.

Of particular interest to our work here is the distribution of orbital periods for the approximately 400 CVs. The period distribution shows two interesting features. First, there is a lack of systems at orbital periods from 2 to 3 hours (termed the period gap). Also, there is an abrupt cutoff in the distribution of CVs at periods shorter than about 80 minutes (the period minimum, hereafter referred to as $P_{\text{min}}$).

Before proceeding further we will explain why the orbital period is an important physical quantity in CVs. Recall Paczyński’s approximation for
Table 4.1: Effective Roche Lobe Radii

<table>
<thead>
<tr>
<th>Mass Ratio</th>
<th>$R_{RL}^{\text{pt. mass}}$</th>
<th>$R_{RL}^{\text{3}/2}$</th>
<th>$R_{RL}^{\text{1}/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.3799</td>
<td>0.3760</td>
<td>0.3623</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3706</td>
<td>0.3668</td>
<td>0.3535</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3604</td>
<td>0.3567</td>
<td>0.3438</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3489</td>
<td>0.3453</td>
<td>0.3327</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3358</td>
<td>0.3323</td>
<td>0.3200</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3207</td>
<td>0.3174</td>
<td>0.3055</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3026</td>
<td>0.2995</td>
<td>0.2882</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2803</td>
<td>0.2774</td>
<td>0.2667</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2506</td>
<td>0.2481</td>
<td>0.2381</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2054</td>
<td>0.2032</td>
<td>0.1945</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1670</td>
<td>0.1652</td>
<td>0.1576</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1012</td>
<td>0.1001</td>
<td>0.0948</td>
</tr>
</tbody>
</table>

Since the donor star is in contact with its Roche lobe this also approximately gives the radius of the donor star as well. If we rearrange Paczyński's relation as,

$$\frac{R_{RL}^{\text{pt. mass}}}{a} = \frac{2}{3^{4/3}} \left( \frac{M_d}{M} \right)^{\frac{4}{3}}. \quad (4.1)$$

we obtain the following relation for the average density of the donor star

$$\frac{3M_d}{4\pi R_{RL}^{3}} = \bar{\rho} = \frac{3^5 \pi}{8GP^2} = 110 P_{\text{hr}}^{-2} \text{ g cm}^{-3}. \quad (4.4)$$

Where $P_{\text{hr}}$ is the orbital period measured in hours. The measurement of the orbital period therefore immediately reveals the average density of the donor star in a CV.

The period minimum is believed to be caused by the “period bounce” that occurs when the donor has lost enough mass for its nuclear energy generation to be extinguished and the star becomes a brown dwarf. The transition from main sequence star to brown dwarf occurs at a mass of about $0.08 M_\odot$. The
brown dwarf has a different adiabatic response to mass loss. In fact, as the donor continues to transfer mass after the system has evolved to the period bounce the mean density of the brown dwarf will decrease and from eq. (4.4) this means that the orbital period will increase. The physical mechanism of the period bounce is thus naturally associated with the observed lack of systems below $P_{\text{min}}$. At the period bounce, the donor is driven out of contact by any further mass loss and RLOF ceases.

Unfortunately, stellar evolutionary calculations predict a period at the bounce of about 70 minutes, about 10% smaller than the observed value of $P_{\text{min}}$ (Kolb & Baraffe, 1999). It is possible that this discrepancy is caused by an overestimate of the effective Roche lobe volume for the donor star. If the self consistent Roche surface were smaller than the value predicted by the point mass Roche model, the period bounce would be reached at a larger separation and at a longer period. From looking at variations in the terms in (4.4) we can see that a reduction of the effective radius of the critical volume by about 7% can entirely explain the difference between the calculated value of the period bounce and the observed value of $P_{\text{min}}$.

To investigate the possibility that the discrepancy between the period bounce and $P_{\text{min}}$ is caused by the inconsistent use of the point mass Roche
potential we have constructed sequences of semi-detached models and measured the volume of the Roche lobe of the donor. Since the white dwarf component of a CV is so much smaller than the donor star (typical white dwarf radii being of order of the Earth’s radius or 6,000 km), we treat the white dwarf as a point mass. In this modified SCF scheme there is one extended star constrained by two boundary points. The location of the point mass is fixed throughout the iteration as is the mass ratio between the white dwarf and fluid star.

The volume of the donor’s Roche lobe is calculated by measuring the value of the Roche potential at the $L_1$ point and summing the volume of all grid cells that have a value of the Roche potential equal to or lower than the value at $L_1$.

We constructed sequences with polytropic indices, $n = \frac{1}{2}, \frac{3}{2}$ at a grid resolution of 128 radial and vertical points by 256 azimuthal points for a range of mass ratios from 0.01 to 1.0. The $n = \frac{3}{2}$ models should be a good approximations to the density distribution of low mass stars. The $n = \frac{1}{2}$ sequence was calculated to examine the size of the difference for a donor star with a stiffer equation of state. The models presented here all have the $L_1$ point coincident with the inner boundary point for the fluid star and the volume of the star is, within two parts in $10^4$, equivalent to the self-
consistent Roche lobe volume. The Roche lobe volume calculated with the SCF technique is then compared with the value obtained from an integration of the point-mass Roche model for the same mass ratio and separation. The integrations of the point mass Roche lobe volume were performed with a program kindly provided by Stefan Mochnaki (Mochnaki, 1984). The effective radii from the SCF models are listed in Table 4.1. All radii have been normalized to the orbital separation. The relative difference between the self-consistent and point mass Roche lobe radii are plotted for the $n = \frac{3}{2}$ sequence in Fig. 4.1 and for the $n = \frac{1}{2}$ sequence in Fig. 4.2.

For both sequences, the self-consistent Roche lobe radius is smaller than the value obtained assuming the gravitational field arises from point masses. For the $n = \frac{3}{2}$ case the self-consistent Roche lobe radius is smaller by only about a percent. The discrepancy is larger for the $n = \frac{1}{2}$ sequence as expected. It appears then that the use of a point-mass model does lead to a prediction of the value of $P_{\text{bounce}}$ that is too small but the effect is not large enough to explain the discrepancy between $P_{\text{min}}$ and $P_{\text{bounce}}$.

Similar calculations have been performed by Uryū & Eriguchi (1999) with a different SCF code for a point mass and extended fluid star in an investigation of the results of Lai et al. when the full tidal potential is used. The models that are common to our work and that of Uryū and Eriguchi are listed in Tables 4.2 and 4.3 for $n = \frac{3}{2}$ and $n = \frac{1}{2}$ respectively. As before, the Roche lobe radii have been normalized by the separation. For the models with $n = \frac{3}{2}$ we find the Roche lobe radii to be slightly larger and for $n = \frac{1}{2}$

### Table 4.2: Comparison of Self-Consistent Roche Lobe Radii, $n = \frac{3}{2}$

<table>
<thead>
<tr>
<th>Mass Ratio</th>
<th>$R_{RL}^{\text{d}}$ Motl</th>
<th>$R_{RL}^{\text{d}}$ Uryū</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.3760</td>
<td>0.3750</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3174</td>
<td>0.3162</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2481</td>
<td>0.2464</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2032</td>
<td>0.2026</td>
</tr>
</tbody>
</table>

### Table 4.3: Comparison of Self-Consistent Roche Lobe Radii, $n = \frac{1}{2}$

<table>
<thead>
<tr>
<th>Mass Ratio</th>
<th>$R_{RL}^{\text{d}}$ Motl</th>
<th>$R_{RL}^{\text{d}}$ Uryū</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
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</tr>
<tr>
<td>0.5</td>
<td>0.3055</td>
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<td>0.2</td>
<td>0.2381</td>
<td>0.2392</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1945</td>
<td>0.1954</td>
</tr>
</tbody>
</table>
slightly smaller than Uyru and Eriguchi have reported though the level of disagreement is typically less than about half a percent.

4.2 Modeling Solar Type Stars

As was noted in §2.1, solar type stars have a radius that is roughly proportional to the mass of the star. The adiabatic response of the radius of a star to a change in its mass can be characterized as

\[ \xi_S \equiv \left. \left( \frac{\partial \ln R}{\partial \ln M} \right) \right|_S, \quad (4.5) \]

where the subscript “S” denotes that the derivative is to be taken with the entropy being held fixed, that is to say the system is adiabatic. For solar type stars, \( \xi_S \approx 1 \) and for a homentropic polytrope, \( \xi_S = \left( \frac{1-n}{3-n} \right) \), so that \( \xi_S = -\frac{1}{3} \) when \( n = \frac{3}{2} \) for example. It is therefore not possible to choose a polytropic index to approximate the adiabatic response of solar type stars with homentropic polytropes. However, it is possible to construct such a model by joining two polytropic fluids, a so-called bipolytrope model (c.f., McCormick & Frank, 1998). Solar type stars can also be modeled by relaxing the restriction that the fluid is homentropic. The structure of a spherical polytrope that is in hydrostatic equilibrium, in the sense that the density is a solution of the Lane-Emden equation for index \( n \), is governed by the structural polytropic exponent, \( \gamma_{\text{struct}} = 1 + \frac{1}{n} \). The pressure within the polytrope is given by

\[ p = \kappa_0 \rho^{\gamma_{\text{struct}}}, \quad (4.6) \]

where \( \kappa_0 \) is a constant whose value can be fixed by assigning the total mass and radius of the polytrope. However, the polytropic exponent \( \gamma_{\text{struct}} \) need not correspond to the exponent that governs the adiabatic change in the fluid’s pressure caused by a change in its density,

\[ \gamma = \frac{c_p}{c_V}, \quad (4.7) \]

where \( c_p \) and \( c_V \) are the specific heats at constant pressure and volume respectively. If we allow the adiabatic equation of state describing the fluid to have the form,

\[ p = \kappa(\rho) \rho^\gamma, \quad (4.8) \]

and we set \( \gamma = \gamma_{\text{struct}} + \alpha \), for some constant \( \alpha \), we can see that the adiabatic equation of state can be consistent with the structure of the fluid if the polytropic “constant” is given by

\[ \kappa(\rho) = \kappa_0 \rho^{-\gamma_{\text{struct}}-\gamma} = \kappa_0 \rho^{-\alpha} \quad (4.9) \]

If \( \alpha > 0 \) the system is stable to convection; if we were to allow \( \alpha < 0 \), the entropy of the fluid would decrease with radius and the star would be
unstable to convection (Hansen & Kawler, 1994). We consider only the case where $\alpha > 0$.

We have applied the SCF technique to construct spherical polytropes whose entropy distribution follows from eq. (4.9) and calculated $\xi_S$ explicitly. We first construct the equilibrium model for the chosen value of the polytropic index $n$ with $\gamma_{\text{struct}}$. From this model we calculate $\kappa (\rho)$ from eq. (4.9) and then remove successive layers of material from the star to construct a sequence of equilibrium models that range from the original star down to a star with only an eighth the mass of the original star.

The independent coordinate in this one-dimensional SCF code is the total mass enclosed by a sphere of radius $r$,

$$M (r) = 4\pi \int_0^r r^2 \rho \, dr'. \quad (4.10)$$

The grid in this case consists of shells of matter that each contain a constant amount of mass.
The gravitational potential follows simply from

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}.$$  \hspace{1cm} (4.11)

The enthalpy is calculated as

$$H = \Phi_{\text{surface}} - \Phi,$$  \hspace{1cm} (4.12)

where

$$\Phi_{\text{surface}} = -\frac{GM_*}{R_*}.$$  \hspace{1cm} (4.13)

is the value of the potential at the surface of the star. We use a scaling relation to give the density of the fluid in terms of the enthalpy for the initial equilibrium structure

$$\rho = \rho_{\text{max}}^{\text{max}} \left( \frac{H}{H_{\text{max}}} \right)^n,$$  \hspace{1cm} (4.14)

and the density is given by

$$\rho = \left[ \frac{nH}{\kappa (\gamma_{\text{struct}} + \alpha)} \right]^{1/(\gamma_{\text{struct}} + \alpha - 1)},$$  \hspace{1cm} (4.15)
Figure 4.5: The adiabatic response in the radius of the nonhomentropic polynomials to a change in mass as a function of the total mass of the star. The stars have the structure of an $n = \frac{3}{2}$ polytrope and the different curves correspond to polytropes that have an adiabatic exponent given by $\gamma_{\text{struct}} + \alpha = \frac{5}{3} + \alpha$ for the indicated value of $\alpha$.

when we are removing layers of material and need to account for the non-uniform entropy content of the star.

From the density and the known amount of mass in each shell we can then solve for the radius of each layer. The iteration for each model continues until the relative change in each radial location of the mass shells is smaller than $10^{-5}$.

The maximum density for the original model is set by the relation between the central density for a polytrope and its mean density

$$\rho_{\text{max}} = \frac{M_* \zeta_n^3}{4\pi R_*^3 \left( -\zeta_n^2 \frac{d\Theta_n}{d\xi} \right)}$$

where $M_*$ and $R_*$ are the mass and radius of the star respectively. $\zeta_n$ and $\left( \zeta_n^2 \frac{d\Theta_n}{d\xi} \right)$ depend only on the polytropic index and are tabulated in Chandrasekhar (1939). The value of the central (maximum) density for the sequence of models experiencing mass loss can not be found from the previous
expression. As layers of material are removed we no longer know the radius of the star but the mass loss is assumed to be adiabatic allowing us to eliminate the radius from the previous expression using a relation between $R_*$ and $\kappa_0$ yielding

$$\rho_{\text{max}} = \left[ (M - N\Delta M) \left( \frac{4\pi G}{(n + 1) \kappa_0} \right)^{\frac{3}{2}} \left( \frac{1}{4\pi \left( -\kappa_0^2 \frac{d\kappa_0}{d\kappa_n} \right)} \right)^{\frac{2n}{3-2n}} \right], \quad (4.17)$$

where $M - N\Delta M$ is the mass of the star after $N$ shells (each containing a mass $\Delta M$) have been removed. The relations derived and presented in this section rely heavily on the results presented in Chandrasekhar (1939).

We have calculated the adiabatic response to mass loss of a spherical, non-homentropic star with a polytropic index $n = \frac{3}{2}$, structural polytropic exponent of $\gamma_{\text{struct}} = \frac{5}{3}$ for $\alpha = 0.0, 0.1, 0.5, 1.0, 1.5$. The star was taken to have the mass and radius of the Sun initially and 175 of the total 200 mass layers were removed from the star to form a sequence of models. The value of $\kappa$ for the fluid in the initial model is plotted as a function of the mass enclosed in Fig. 4.3. As can be seen, the entropy content of the fluid rises rapidly at the edge of the star. The resulting mass-radius relation is shown in Fig. 4.4 for the five sequences corresponding to the different values of $\alpha$ considered.

When $\alpha = 0$ the star follows the correct behavior for the homentropic mass-radius relation, namely, $R \propto M^{-\frac{1}{3}}$ and for $\alpha \geq 0.5$ the mass-radius relation has been reversed meaning that the star contracts upon mass loss. This can be seen more clearly in Fig. 4.5 where we plot the adiabatic response to mass loss, $\xi_S$ as defined by eq. (4.5). For the homentropic case, $\xi_S$ is constant and as $\alpha$ is increased, $\xi_S$ increases and exhibits a successively steeper drop for the first few mass shells. A fluid system that evolves along an adiabat characterized by $\gamma_{\text{struct}} = \frac{5}{3}$ and $\alpha = 0.5$, for example, should provide an adequate approximation to Solar type stars for the purpose of simulating a radiative donor star that is undergoing RLOF.
5. Hydrodynamics Description

In this chapter we describe in detail the gravitational hydrodynamics code that we have implemented to conduct the evolutions presented in this dissertation. It is hoped that enough information is presented to make the numerical techniques comprehensible to other researchers who will use and further develop the current algorithms.

We begin by distinguishing two different ways to view, and compute, the flow of a fluid. In the first, the Lagrangian picture, the coordinates that describe the fluid are carried along with the fluid itself. This approach has the advantage that the equations describing the fluid are expressed in terms of the total time derivative of the fluid variables. In the other view, the fluid flows through a coordinate system (though that coordinate system need not be stationary in time). This is termed the Eulerian view and the time rate of change of a variable now has two components, one due to the explicit time dependence of the variable and the other due to its flow through space. This can be expressed as the equivalence of the following two operators:

\[
\left( \frac{d}{dt} \right)_{\text{Lagrangian}} \equiv \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)_{\text{Eulerian}}. \tag{5.1}
\]

Despite the simplicity of time evolution in the Lagrangian view, this approach is impractical for multi-dimensional flows due to the complexity of the coordinate system necessary to describe all but the simplest flows. We therefore proceed with the Eulerian view in which the evolution of a fluid state is given by the flow of the fluid through space (advection) and the action of sources and sinks on the fluid state (which, because they are the same in both views are qualified as Lagrangian source terms).

5.1 Continuum Mechanics Formalism

We have developed an explicit, conservative, finite-volume, Eulerian hydrodynamics code that is second-order accurate in both time and space to evolve the equilibrium binaries in time. The program is similar to the ZEUS code developed by Stone & Norman (1992). The integration scheme is designed to evolve five primary variables that are densities of conserved quantities: the mass density, \( \rho \), the angular momentum density, \( \mathbf{A} \), the radial momentum density, \( S \), the vertical momentum density, \( T \) and an entropy tracer, \( \tau \). The entropy tracer is defined as,

\[
\tau \equiv (\epsilon \rho)^{\frac{1}{\gamma}}, \tag{5.2}
\]

where \( \epsilon \) is the internal energy per unit mass and \( \gamma \) is the selected ratio of specific heats of the fluid. For the evolutions presented here we have set
\( \gamma = 1 + \frac{1}{n} \). The entropy tracer is related to the specific entropy of the fluid through the relation
\[
s = c_p \ln \frac{\tau}{\rho}. \tag{5.3}\]

Using the entropy tracer in lieu of the internal energy per unit mass or the total energy density allows us to avoid the finite difference representation of the divergence of the velocity field that must otherwise be used to express the work done by pressure on the fluid.

The set of differential equations that we solve is based on the conservation laws for these five conserved densities. Mass conservation is governed by the continuity equation,
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{5.4}
\]
where \( \mathbf{v} \) is the velocity field. The velocity vector, \( \mathbf{v} \), is expressed in terms of its components in a cylindrical coordinate system as \( \mathbf{v} = u \hat{e}_R + v \hat{e}_\phi + w \hat{e}_z \). The three components of Euler’s equation govern changes in the momentum densities. We express these equations in a frame of reference rotating with a constant angular velocity, \( \Omega \), as follows:
\[
\frac{\partial S}{\partial t} + \nabla \cdot (S \mathbf{v}) = -\rho \frac{\partial \Phi_{\text{eff}}}{\partial R} + \frac{A^2}{\rho R^3} + 2\Omega \frac{A}{R}, \tag{5.5}
\]
\[
\frac{\partial T}{\partial t} + \nabla \cdot (T \mathbf{v}) = -\rho \frac{\partial \Phi_{\text{eff}}}{\partial z}, \tag{5.6}
\]
\[
\frac{\partial A}{\partial t} + \nabla \cdot (A \mathbf{v}) = -\rho \frac{\partial \Phi_{\text{eff}}}{\partial \phi} - 2\Omega SR, \tag{5.7}
\]
where,
\[
\Phi_{\text{eff}} \equiv H + \Phi - \frac{1}{2} \Omega^2 R^2. \tag{5.8}
\]
The second and third terms appearing on the right-hand side of eq. (5.5) represent the curvature of cylindrical coordinates and the radial component of the Coriolis force, respectively. Likewise, the last term appearing on the right hand side of eq. (5.7) represents the azimuthal component of the Coriolis force.

From the first law of thermodynamics we know that in the most general case, the entropy tracer obeys the expression,
\[
\frac{\partial \tau}{\partial t} + \nabla \cdot (\tau \mathbf{v}) = \frac{\tau}{c_p} \frac{ds}{dt}. \tag{5.9}
\]
Here we will be considering only adiabatic flows in which case \( \frac{ds}{dt} = 0 \) and the entropy tracer obeys an advection equation of precisely the same form as the continuity equation, namely,
\[
\frac{\partial \tau}{\partial t} + \nabla \cdot (\tau \mathbf{v}) = 0. \tag{5.10}
\]
Even though we are performing adiabatic evolutions we can not simply use an adiabatic equation of state \( p = \kappa \rho^\gamma \) and disregard the first law of thermodynamics because the polytropic constant is, in general, different for each binary component.

Finally, we solve Poisson’s equation once every integration timestep in order to calculate the force of gravity arising from the instantaneous mass distribution,

\[
\nabla^2 \Phi = 4\pi G \rho; \quad (5.11)
\]

and we use the ideal gas law as the equation of state to close the system of equations

\[
p = (\gamma - 1) \tau = (\gamma - 1) \rho \epsilon. \quad (5.12)
\]

It may be argued that our treatment of the thermodynamics of the system as the purely adiabatic flow of an ideal fluid is overly simplified. However, we believe that the self-consistent treatment of both binary components in the presence of the full nonlinear tidal forces is sufficiently complex and novel to warrant the use of a simple equation of state at the present time. This will allow us to establish the qualitative behavior of systems in this limiting case before additional complications leading to nonadiabatic heat transport are introduced into the simulations.

5.2 Finite Volume Representation

Before proceeding with the discussion of the hydrodynamics algorithm that we have implemented to solve the equations presented in §5.1 we first describe the discretization that has been used to represent the exact partial differential equations when they are expressed as approximate algebraic relations between distinct points in the computational grid. As in the ZEUS code, all scalar variables and the diagonal components of tensors are defined at cell centers. The components of vectors are defined at the corresponding faces of the cell. A volume element and the relative positions of the variables within each cell is illustrated in Fig. 5.1. The cell extends from \( R_i \) to \( R_{i+1} \) in radius, from \( z_j \) to \( z_{j+1} \) in the vertical coordinate, and from \( \phi_k \) to \( \phi_{k+1} \) in the azimuthal coordinate. We represent the staggered variables in the computational mesh with a half-index notation; the coordinates of the center of a grid cell are given by \( R_{i+\frac{1}{2}}, z_{j+\frac{1}{2}}, \phi_{k+\frac{1}{2}} \) for example. A complete listing of the variables and their centering is given in Table 5.1.

5.3 Treatment of Advection Terms

Through the method of operator splitting (or Strang splitting as it is sometimes termed), one can construct a numerical scheme that groups terms of the same physical character together (Strang, 1968). Again, following along the lines of the ZEUS code we implement a splitting scheme that separates treatment of the Eulerian transport (advection) terms from the
Figure 5.1: Volume element for a cell-centered quantity (defined at the open circle labeled A). Radial, vertical and azimuthal face-centered quantities are defined at points B, C, and respectively. Table 5.1 lists the variables used in the hydrodynamics code and their centering.
Table 5.1: Hydrodynamic Variables and their Centering

<table>
<thead>
<tr>
<th>Centering</th>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$R_{i+\frac{1}{2}}$</td>
<td>Cylindrical Radius Coordinate</td>
</tr>
<tr>
<td></td>
<td>$z_{j+\frac{1}{2}}$</td>
<td>Vertical Coordinate</td>
</tr>
<tr>
<td></td>
<td>$\phi_{k+\frac{1}{2}}$</td>
<td>Azimuthal Coordinate</td>
</tr>
<tr>
<td></td>
<td>$\rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Mass Density</td>
</tr>
<tr>
<td></td>
<td>$\tau_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Entropy Tracer</td>
</tr>
<tr>
<td></td>
<td>$p_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Pressure</td>
</tr>
<tr>
<td></td>
<td>$H_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Enthalpy</td>
</tr>
<tr>
<td></td>
<td>$\Phi_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Gravitational Potential</td>
</tr>
<tr>
<td></td>
<td>$Q_{D_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}$</td>
<td>Diagonal Components of Artificial Viscosity</td>
</tr>
<tr>
<td>B</td>
<td>$S_{i,j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Radial Momentum Density</td>
</tr>
<tr>
<td></td>
<td>$u_{i,j+\frac{1}{2},k+\frac{1}{2}}$</td>
<td>Radial Velocity</td>
</tr>
<tr>
<td>C</td>
<td>$T_{i+\frac{1}{2},j,k+\frac{1}{2}}$</td>
<td>Vertical Momentum Density</td>
</tr>
<tr>
<td></td>
<td>$w_{i+\frac{1}{2},j,k+\frac{1}{2}}$</td>
<td>Vertical Velocity</td>
</tr>
<tr>
<td>D</td>
<td>$A_{i+\frac{1}{2},j+\frac{1}{2},k}$</td>
<td>Angular Momentum Density</td>
</tr>
<tr>
<td></td>
<td>$v_{i+\frac{1}{2},j+\frac{1}{2},k}$</td>
<td>Azimuthal Velocity</td>
</tr>
</tbody>
</table>

treatment of source terms. In this section we describe our algorithm for the advection terms.

Given the density $\lambda$ of any conserved quantity $\Lambda$ that satisfies a generic conservation law of the form,

$$ \frac{\partial \lambda}{\partial t} + \nabla \cdot (\lambda \mathbf{v}) = 0, \quad (5.13) $$

we can replace the differential equation (5.13) with an equivalent integral equation,

$$ \frac{\partial}{\partial t} \int_V \lambda dV = - \int_V \nabla \cdot (\lambda \mathbf{v}) \, dV = - \int_{S(V)} \lambda \mathbf{v} \cdot d\mathbf{S}. \quad (5.14) $$

Equation (5.14) must hold for any volume. In particular, it must hold for every volume element within the computational grid. The exact integral relation is then expressible in the following finite volume form for each grid cell:

$$ \frac{\lambda^{(n+\text{advection})} - \lambda^{(n)}}{\Delta t} = - \frac{1}{\Delta V} \sum_{i=1}^{6} \lambda_i^{(n)} \mathbf{v} \cdot \Delta \mathbf{S}_i, \quad (5.15) $$
where the summation is over all six faces on the surface of the three dimensional cell. The notation of “$n + \text{advection}$” is meant as the variable $\lambda$ updated from its value at timestep $n$ by the action of the advection operator. We use a similar notation for other terms that update the state of the fluid. The surface elements, $\Delta S_i$, are naturally face-centered with respect to the control volume in question, so averages must be taken to obtain the advection velocity components necessary to perform the dot product for the momentum densities as shown in eq. (5.15). We use second-order accurate, linear averages to construct the advection velocities in this case. The amount of $\Lambda$ advected through each face is given by an upwind biased, linear interpolation of the distribution of $\lambda$ to give $\lambda^*$ as described by van Leer (1979). By construction, the amount of $\Lambda$ that is transported out of one cell immediately flows into the neighboring cell; thus ensuring the conservative nature of the advection scheme.

Unlike the ZEUS code, we do not use operator splitting along the three separate dimensions during the advection step. Instead, we perform the updates due to advection in all three dimensions simultaneously. We thus avoid concerns about bias that may be introduced by using an unsymmetrized ordering of the advection sweeps. A discussion of how we obtain second-order accuracy in time for the advection step through time centering of the terms appearing in eq. (5.15) is presented in §5.6.

Our advection scheme automatically reverts to a first-order accurate (upwind) scheme at local extrema in the primary fluid variables. In addition, it is necessary to introduce an artificial viscosity to stabilize the scheme in the presence of shocks. The artificial viscosity prescription we have implemented is detailed in §5.5.

### 5.4 Treatment of Source Terms

The Lagrangian source terms for the momenta that are shown on the right-hand sides of eqs. (5.5)-(5.7) arise from the forces of pressure and gravity, as well as from the differentiation of the curvilinear basis vectors and the rotation of the reference frame. We have found it advantageous to combine the pressure gradient with the gradient of the gravitational potential, which results in a gradient of the sum of $H$ and $\Phi$. Since the centrifugal force can also be expressed as the gradient of a potential, it is included as well to form an effective potential as defined in eq. (5.8). As explained in §3, our initial models have the property that $\Phi^{\text{eff}} = \text{constant}$ everywhere, hence to reasonably high precision $\nabla \Phi^{\text{eff}} = 0$ throughout both stars initially.

The expressions we have used for the source term updates of the momentum densities are given here by expressions (5.16)–(5.18). As with the advection step, we do not use an operator splitting technique to evaluate the source terms along the three separate coordinate dimensions; instead, at each cell location, all updates due to Lagrangian source terms are performed
at once.

\[
\frac{S_{i,j+\frac{1}{2},k+\frac{1}{2}}^{(n+\text{source})} - S_{i,j+\frac{1}{2},k+\frac{1}{2}}^{(n)}}{\Delta t} = \]

\[
- \frac{\hat{\rho}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}{\Delta R} \left[ \Phi_{\text{eff}}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \Phi_{\text{eff}}_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \right] + \frac{\left( \hat{A}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \right)^2}{\rho_{i,j+\frac{1}{2},k+\frac{1}{2}} R_i^3} + \frac{2\Omega \hat{A}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}{R_i};
\]

\[
\frac{T_{i+\frac{1}{2},j,k+\frac{1}{2}}^{(n+\text{source})} - T_{i+\frac{1}{2},j,k+\frac{1}{2}}^{(n)}}{\Delta t} = \]

\[
- \frac{\hat{\rho}_{i+\frac{1}{2},j,k+\frac{1}{2}}}{\Delta z} \left[ \Phi_{\text{eff}}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \Phi_{\text{eff}}_{i+\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}} \right] ;
\]

\[
\frac{A_{i+\frac{1}{2},j+\frac{1}{2},k}^{(n+\text{source})} - A_{i+\frac{1}{2},j+\frac{1}{2},k}^{(n)}}{\Delta t} = \]

\[
- \frac{\hat{\rho}_{i+\frac{1}{2},j+\frac{1}{2},k}}{\Delta \phi} \left[ \Phi_{\text{eff}}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \Phi_{\text{eff}}_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} \right] - 2\Omega \hat{S}_{i+\frac{1}{2},j+\frac{1}{2},k} R_{i+\frac{1}{2}}.
\]

Note that a caret identifies a variable whose value has been interpolated to a spatial location that is different from the variable’s primary definition point as shown in Fig. 5.1. These variables are given by volume-weighted averages as follows:

\[
\hat{A}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{4R_i} \left[ \left( A_{i+\frac{1}{2},j+\frac{1}{2},k} + A_{i+\frac{1}{2},j+\frac{1}{2},k+1} \right) \left( R_i + \frac{1}{4} \Delta R \right) \right.
\]

\[
+ \left( A_{i-\frac{1}{2},j+\frac{1}{2},k} + A_{i-\frac{1}{2},j+\frac{1}{2},k+1} \right) \left( R_i - \frac{1}{4} \Delta R \right) \right],
\]

\[
\hat{S}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{1}{4R_i+\frac{1}{2}} \left[ \left( S_{i+1,j+\frac{1}{2},k+\frac{1}{2}} + S_{i+1,j+\frac{1}{2},k-\frac{1}{2}} \right) \left( R_i+\frac{1}{2} + \frac{1}{4} \Delta R_{i+\frac{1}{2}} \right) \right.
\]

\[
+ \left( S_{i,j+\frac{1}{2},k+\frac{1}{2}} + S_{i,j+\frac{1}{2},k-\frac{1}{2}} \right) \left( R_i+\frac{1}{2} - \frac{1}{4} \Delta R \right) \right],
\]
\[ \hat{\rho}_{i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2R_i} \left[ \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \left( R_i + \frac{1}{4} \Delta R \right) + \rho_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \left( R_i - \frac{1}{4} \Delta R \right) \right], \quad (5.21) \]

\[ \hat{\rho}_{i+\frac{1}{2},j,k+\frac{1}{2}} = \frac{1}{2} \left( \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \rho_{i+\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}} \right), \quad (5.22) \]

\[ \hat{\rho}_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{1}{2} \left( \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \rho_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} \right). \quad (5.23) \]

### 5.5 Artificial Viscosity

To stabilize the scheme, we employ a planar, von Neumann artificial viscosity that is active only for zones that are undergoing compression. (See Stone & Norman 1992 or Bowers & Wilson 1991, p. 311 for more detailed discussions of artificial viscosity in Eulerian hydrodynamics.) The momentum densities are updated from the following finite difference equations,

\[
\frac{S_{i,j+\frac{1}{2},k+\frac{1}{2}}^{(n+\text{viscosity})} - S_{i,j+\frac{1}{2},k+\frac{1}{2}}^{(n)}}{\Delta t} = \frac{1}{\Delta R} \left( Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{RR} - Q_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{RR} \right), \quad (5.24) 
\]

\[
\frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{(n+\text{viscosity})} - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{(n)}}{\Delta t} = \frac{1}{\Delta z} \left( Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{zz} - Q_{i+\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2}}^{zz} \right), \quad (5.25) 
\]

\[
\frac{A_{i+\frac{1}{2},j+\frac{1}{2},k}^{(n+\text{viscosity})} - A_{i+\frac{1}{2},j+\frac{1}{2},k}^{(n)}}{\Delta t} = \frac{1}{\Delta \phi} \left( Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{\phi\phi} - Q_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}^{\phi\phi} \right), \quad (5.26) 
\]

where the diagonal components of the artificial viscosity are given by,

\[
Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{RR} = \nu \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \left( u_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}} \right)^2, \quad (5.27) 
\]

\[
Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{zz} = \nu \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \left( w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - w_{i+\frac{1}{2},j,k+\frac{1}{2}} \right)^2, \quad (5.28) 
\]

\[
Q_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{\phi\phi} = \nu \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \left( v_{i+\frac{1}{2},j+\frac{1}{2},k+1} - v_{i+\frac{1}{2},j+\frac{1}{2},k} \right)^2, \quad (5.29) 
\]

if the velocity difference is negative; otherwise the components of \( Q \) are zero. Note that we neglect the shear components of viscosity. The factor \( \nu \) is a parameter that roughly dictates the number of zones across which shock structures will be spread. A value of \( \nu = 2 \) is typically sufficient. In keeping
with our overall adiabatic treatment of the flow (see §5.1), we neglect the generation of entropy by shock compression.

To illustrate the numerical instability that we are guarding against with the introduction of artificial viscosity, we show plots of the radial and azimuthal velocities through the center of the same fluid star at the same point in time from a simulation with artificial viscosity (solid line) and without artificial viscosity (plotted as crosses). Without the dissipation associated with the artificial viscosity term the solution will develop a numerical instability evidenced here as the sawtooth compression and expansion pattern in the star. We note that this is not a fault in the implementation of the code itself but is instead a consequence of Godunov’s theorem which states that any advection scheme, more accurate than first order, that depends linearly on the solution itself will be numerically unstable (introduce new extrema) at discontinuities in the solution (Zalsek, 1997).
5.6 Time Centering

The timestep cycle is split between the application of source, advection and viscosity operators. First, the source terms are applied for one half of a timestep. Next, all updates due to advection are performed for a full timestep and the viscosity updates are applied to the momentum densities. Finally, the second half of the source operators are applied. The source and advection steps are thereby staggered in time when viewed over several iteration cycles for a constant value of the timestep.

The advection is time-centered by first performing half a timestep of fictitious advection in order to obtain “time-centered” velocities for constructing the face-centered advection velocity components that appear in eq. (5.15). The full timestep of advection is then performed. The components of the viscosity tensor are constructed from the velocity and density estimates at the midpoint of the timestep as well.

Since the momentum densities themselves also appear in the source terms of eqs. (5.5) and (5.7), similar care must be taken with their centering in time. The source operators are applied in a fictitious source step to obtain the angular and radial momentum densities at a point half a timestep in the future. These values are then used to update the momentum densities through a full timestep. As the timestep value changes from iteration to iteration, this algorithm for time centering the source terms is not formally accurate to second order. However, in real computations the character of the flow and, hence, the maximal signal velocity do not change rapidly over the course of a timestep cycle so that one may expect the resulting inaccuracies in the time centering of the source terms to be small. The other terms that appear in the source operators, including the gravitational potential, are all calculated at the approximate midpoint in time between the source steps.

5.7 Timestep Formulation and Boundary Conditions

Since we explicitly integrate the fluid equations in time, the timestep is limited in size by the familiar Courant-Friedrichs-Lewy (CFL) stability criterion which limits the time increment to be small enough so that no characteristic can cross a cell in a single timestep. Specifically,

$$\Delta t = \min \left[ \frac{\Delta R}{c + |u|}, \frac{\Delta Z}{c + |w|}, \frac{R \Delta \phi}{c + |v|} \right]$$

where $c$ is the speed of sound. In practice we limit the timestep to a half the CFL time. Also, since we have introduced the diffusion terms associated with artificial viscosity, the timestep must also satisfy the condition, (p. 270 of Bowers & Wilson 1991)

$$\Delta t \leq \frac{1}{4} \min \left[ \frac{\rho \Delta R}{Q^{RR}}, \frac{\rho \Delta Z}{Q^{ZZ}}, \frac{\rho R \Delta \phi}{Q^{\phi\phi}} \right]^{1/2}$$
The boundary conditions for the fluid variables at the external boundaries are to allow the fluid to flow freely off the grid but to not allow material to flow back from the outermost layer of boundary cells. The central annulus of cells that has an inner radius at the coordinate axis is treated as a single azimuthally averaged cell for each layer in the vertical direction.

The hydrodynamics code is not capable of resolving arbitrarily small densities, in part simply because one must divide the momentum densities by the mass density to form the velocities used in the advection of the fluid. A small limiting density, \( \rho_{\text{min}} \) is always maintained in each cell. For the simulations presented here the value of the limiting density was \( 10^{-7} \) for early simulations and was lowered to \( 10^{-10} \) for later runs. The maximum density in all simulations is of order one. A value of the limiting density that is too high can produce unphysical results while a value too low implies that errors in the source terms will be magnified into correspondingly larger velocities which in turn limit the timestep as described above.

5.8 Parallelization of Hydrodynamics Code

As it is our intention to perform high resolution simulations, it is imperative that the work load within the simulation be distributed amongst many processors so that the simulations may be conducted in a reasonable amount of time and not exceed the available memory of a single computer. The fluid dynamics equations, being hyperbolic partial differential equations, are ideally suited to a simple domain decomposition or single program multiple data (SPMD) parallelization model. Each computational task performs the same operations on their own block of the global data arrays with communication only being necessary between nearest neighbor tasks that share a boundary of ghost zones that is one-cell thick (this ghost zone thickness is dictated by the order of our advection and finite-difference operators). We have written the program in Fortran90 with explicit message passing being performed with MPI (Message Passing Interface) subroutine calls. The resulting parallel code performance scales linearly with the number of processors for 4 to 128 processors on the Cray T3E. Similar behavior is also seen on the IBM SP platform.

5.9 Solution of Poisson’s Equation

We are seeking to solve Poisson’s equation for an isolated distribution of mass. The correct boundary condition in this instance is that the potential goes to zero at infinity. As we only construct the solution on a finite domain we must specify the gravitational potential (or its gradient) on some boundary that encloses all the mass in the simulation. We construct the boundary potential using a novel technique based on a compact representation of the cylindrical Greens function in terms of half-integer degree Legendre functions of the second kind as described by Cohl & Tohline (1999). The boundary potential is then simply given by the convolution of the appropriate Greens
function with the density distribution. This method is capable of generating the exact solution for a discretized mass distribution and has the attractive feature that it can be applied to very flattened bodies without suffering penalties in either performance or accuracy.

In order to obtain the interior solution for the gravitational potential, Poisson's equation is first Fourier transformed in the azimuthal direction and then the resulting set of two-dimensional partial differential equations (Helmholtz equations) for the decoupled Fourier amplitudes are solved using an alternating direction implicit (ADI) scheme (c.f., Peaceman & Rachford, 1955; Black & Bodenheimer, 1975). The solution is then transformed back to real space.

The solution of Poisson's equation requires special care in the context of parallel computing because the solution necessarily involves global communication as the character of the underlying physical law is action at a distance. The algorithms we have implemented for computing the gravitational potential are well suited to a cylindrical geometry and very efficient in a distributed computing environment. Parallel communications are used to transpose the data so that all the data in a given dimension are in local memory at one time. When operations are to be performed along a different dimension, the data is transposed again. This allows us to send a relatively few number of large messages. Further details regarding our solution of Poisson's equation in a parallel computing environment can be found in Cohl et al. (1997).

5.10 Diagnostic Calculations

In order to measure parameters of interest, the timestep cycle is interrupted after a set number of timesteps to perform diagnostic calculations. An immediate problem that one must confront is how to decide what fluid belongs to which star. We use two algorithms to accomplish this task. In the first, we identify the layer of grid cells that mark the boundary of each star (those that are at a transition between density levels above and below \( \rho_{\text{boundary}} \) where we have chosen \( \rho_{\text{boundary}} \) to be \( 10^{-5} \)). We then calculate the average energy of these cells and, depending on which side of the grid a cell lies on, it “belongs” to a star if it is more tightly bound than the layer of cells at \( \rho_{\text{boundary}} \).

To complement this identification technique, and to allow us to visualize the fluid that has been accreted by the receiving star, we have implemented a multi-fluid algorithm within the hydrodynamics code. For every timestep we track two mass fractions that correspond to the portion of the total mass in each cell of material that was initially part of one star or the other. This is done by updating the mass fractions as well as the mass density when eq. (5.15) is computed. We assume that at all times the fluid components are evenly mixed within the grid cells.

Parameters that are calculated include the mass of each component and the mass contained in the envelope; the total \( z \) component of angular momentum and the \( z \) component of spin angular momentum for each star; the total kinetic, potential and thermal energies; the center of mass of the sys-
tem and of each component separately; the volumes of each star at different density levels and of the Roche lobes; and the components of the second time derivative of the moment of inertia tensor that are required to calculate the strain tensor (in the transverse-traceless gauge) of gravitational radiation (Finn & Evans, 1990; Rasio & Shapiro, 1992).
6. Test Simulations

Here we present results from three different types of test cases that we have used to evaluate and quantify the accuracy of our computational tools. This is especially important as the algorithms have been written in a new parallel form. In all tests we compare a known, although not necessarily analytical, solution with the calculated numerical solution.

6.1 Sod’s Shock Tube Test

As a check of the stability of our code in the presence of, ideally, discontinuous jumps in the fluid variables we have solved Sod’s shock tube problem (Sod, 1978). Sod’s shock tube problem presents a useful hydrodynamic test because the solution is known analytically and contains the three simple waves that can occur in ideal fluid flow. Of these simple waves, it is the shock wave that concerns us most. Our goal is not to resolve the details of the shock structure but rather to ensure that our algorithm is well behaved (numerically stable and yields an acceptable solution) in the presence of shocks.

In a binary system that is undergoing mass transfer, the accretion stream will necessarily undergo a shock transition as it is decelerated upon impact with the receiving star or when it intersects itself if the stream has sufficient angular momentum to orbit the receiving star. Furthermore, there may be weak but stationary shock fronts (stationary in a corotating reference frame and weak in the sense that the pressure jump is small) along the surface of both stars if the mass transfer event is nonconservative and material from a common envelope falls back onto the stars.

The initial conditions and the solution as a function of time are listed in Table 6.1. The solution given in Table 6.1 is obtained by matching the shock jump conditions (c.f., Courant & Fredricks, 1948) to the self-similar solution for the rarefaction wave (c.f., Landau & Lifshitz, 1959). The solution is divided into five regions (here labeled 1 to 5 from the bottom to the top of the grid) and involves a shock wave traveling supersonically into the less dense gas followed by a contact discontinuity trailing the shock front and a rarefaction (expansion) wave traveling into the undisturbed, high density, gas. The boundary between regions 1 (undisturbed gas) and 2 (expanding gas) is the head of the rarefaction wave and travels as

\[ z_{\text{head}}(t) = z_0 - c_1 t. \]  \hspace{1cm} (6.1)

The boundary between regions 2 and 3 is the tail of the rarefaction wave which is given by

\[ z_{\text{tail}}(t) = z_0 + (w_3 - c_3) t. \]  \hspace{1cm} (6.2)
Table 6.1: Solution to Sod’s Shock Tube Problem

<table>
<thead>
<tr>
<th>Region</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
<td>$w$</td>
<td>$\frac{2}{\gamma+1} \left( c_1 + \frac{z-z_0}{t} \right)$</td>
<td>0.9274</td>
<td>0.9274</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>$p_1 \left( 1 - \frac{\gamma-1}{2} \frac{w^2}{c_1} \right)^{\frac{2\gamma}{\gamma-1}}$</td>
<td>0.3031</td>
<td>0.3031</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>$\rho_1 \left( 1 - \frac{\gamma-2}{2} \frac{w^2}{c_1} \right)^{\frac{2}{\gamma-1}}$</td>
<td>0.4263</td>
<td>0.2656</td>
<td>0.125</td>
</tr>
<tr>
<td></td>
<td>$\epsilon$</td>
<td>$\frac{p_2}{(\gamma-1)\rho}$</td>
<td>1.778</td>
<td>2.853</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>$\tau$</td>
<td>$(\rho\epsilon)^{\frac{1}{\gamma}}$</td>
<td>0.8203</td>
<td>0.8203</td>
<td>0.3715</td>
</tr>
<tr>
<td></td>
<td>$\frac{\tau}{\rho}$</td>
<td>$\frac{\tau}{\rho}$</td>
<td>1.924</td>
<td>3.088</td>
<td>2.972</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$\sqrt{\gamma (\gamma - 1) \epsilon}$</td>
<td>0.9978</td>
<td>1.264</td>
<td>1.058</td>
</tr>
</tbody>
</table>

The contact discontinuity marks the location of the initial discontinuity as a function of time and travels as

$$z_{\text{contact}}(t) = z_0 + w_4t.$$  \hspace{1cm} (6.3)

Finally, the shockwave advances supersonically as

$$z_{\text{shock}}(t) = z_0 + Wt,$$ \hspace{1cm} (6.4)

where $W$ is the shock speed which is given by

$$W = c_5 \left[ 1 + \frac{\gamma + 1}{2\gamma} \frac{p_4 - p_5}{p_5} \right]^{\frac{1}{2}}.$$ \hspace{1cm} (6.5)

The computed solution for the vertical velocity, $w$, pressure, $p$, mass density $\rho$, and the ratio $\frac{\tau}{\rho}$ which is proportional to the adiabatic constant, $\kappa$ of the fluid is plotted along with the analytic solution at time $t = 0.247$ in Fig. 6.1. The computed points are not average values but are instead the values for a random column of cells at constant radius and azimuth within the grid. The calculation was performed with a coefficient for the artificial viscosity of $\nu = 2.0$ and with 130 vertical zones. The initial discontinuity
Figure 6.1: Clockwise from the upper left panel, the vertical velocity, pressure, the ratio of the entropy tracer to the mass density (which is proportional to the adiabatic constant) and the mass density at time $t = 0.247$ for Sod’s shock tube problem. The initial discontinuity (see Table 6.1) was placed at a plane of constant $z$ and the computed solution for a column of cells at constant radius and azimuth is plotted as crosses. The solid curve is the analytical solution.

was placed at $z_0 = -0.1$ and the grid extended from $-\frac{1}{2}$ to $\frac{1}{2}$ in the vertical direction.

The results from this simulation compare favorably to the results produced by other second-order accurate, Eulerian hydrodynamics programs with artificial viscosity (c.f., Hawley et al., 1984; Stone & Norman, 1992; Lufkin & Hawley, 1993). The shock front is spread out over approximately three zones, and there is no indication of numerical instability in the solution for the shocked gas. The contact discontinuity is likewise spread out over about three zones due to the numerical diffusion inherent in a second-order accurate Eulerian scheme. There is some disagreement between the computed and analytical solution at the tail of the rarefaction wave. This phenomenon has been investigated by Norman & Winkler (1983) and results from an inconsistent representation of the analytic viscous equations in finite difference form. Finally, there is a difference in the computed and analytical solution for the ratio of $\tau$ to $\rho$ in the shocked gas. This is the result of
our performing an adiabatic evolution while the proper solution for a shock entails the generation of entropy by shock compression of the gas.

We note that we have used the gradient of the pressure as opposed to the density times the gradient of the enthalpy for the solution of Sod’s problem. Due to the pathological nature of the discontinuous initial conditions, a correct solution cannot be obtained if the enthalpy term is used with our chosen centering of the fluid variables.

To compare the quality of the solution from our hydrodynamics code with those of other numerical techniques we have performed the Sod shock tube calculation with a one-dimensional, Lagrangian piecewise-parabolic method (PPM) scheme (Colella & Woodward, 1984). The PPM scheme utilizes parabolic distributions of the fluid variables and interface values between cells are calculated from an approximate solution to the nonlinear Riemann problem. The scheme represents the highest quality numerical technique in current use in computational astrophysics and the computational expense is warranted when shock heating or similar effects are crucial to describing the underlying physics. The solution, shown in Fig. 6.2 was computed with a grid of 200 mass elements using the artificial viscosity described in Colella & Woodward (1984) with the initial discontinuity placed at $x = \frac{1}{2}$. Many of the numerical defects discussed previously are absent in the PPM solu-
tion though the sharpness of the contact discontinuity is due to the use of Lagrangian coordinates and not entirely due to the higher accuracy of the PPM scheme. As stated previously, we are not interested in the details of shocks in our current simulations and for our present purposes the solution plotted in Fig. 6.1 is adequate.

6.2 Test of Poisson Solver

Cohl & Tohline (1999) have published exhaustive tests showing the accuracy with which we are able to evaluate the gravitational potential on the boundary of our cylindrical coordinate grid. In order to ascertain the accuracy with which we are able to determine the force of gravity arising from the fluid everywhere inside the grid, we have calculated the potential and its derivatives for a uniform-density sphere of radius \( R^*_s \) and density \( \rho^*_s \), centered at an arbitrary position on the grid, \( r_0 \). The analytical potential is

\[
\Phi (r) = \begin{cases} 
-2\pi G \rho_s \left( R^2 - \frac{d^2}{3} \right) & d < R^*_s \\
-\frac{4\pi}{3} G \rho_s \frac{R^3}{d} & d > R^*_s 
\end{cases},
\]

where \( d = |r - r_0| \). The corresponding derivatives appearing in the gravitational force are:

\[
\frac{\partial \Phi}{\partial R} = \frac{4\pi G \rho_s}{3} \left[ (R \cos \phi - x_0) \cos \phi + (R \sin \phi - y_0) \sin \phi \right],
\]

\[
\frac{\partial \Phi}{\partial z} = \frac{4\pi G \rho_s}{3} (z - z_0),
\]

\[
\frac{\partial \Phi}{\partial \phi} = \frac{4\pi G \rho_s}{3} \left[ -R (R \cos \phi - x_0) \sin \phi + R (R \sin \phi - y_0) \cos \phi \right],
\]

for \( d < R^*_s \), and

\[
\frac{\partial \Phi}{\partial R} = \frac{4\pi G \rho_s}{3} \frac{R^3}{d^3} \left[ (R \cos \phi - x_0) \cos \phi + (R \sin \phi - y_0) \sin \phi \right],
\]

\[
\frac{\partial \Phi}{\partial Z} = \frac{4\pi G \rho_s}{3} \frac{R^3}{d^3} (z - z_0),
\]

\[
\frac{\partial \Phi}{\partial \phi} = \frac{4\pi G \rho_s}{3} \frac{R^3}{d^3} \left[ -R (R \cos \phi - x_0) \sin \phi + R (R \sin \phi - y_0) \cos \phi \right],
\]

for \( d > R^*_s \).

In Table 6.2 we present the average relative error in the potential and the average absolute error in the three derivatives for a uniform density sphere (\( \rho^*_s = 1 \)) of radius \( R^*_s = \frac{1}{3} \) placed at the origin and at \( x_0 = \frac{1}{2} \) for a representative set of grid resolutions. The grid extends from 0 to 1 in radius and
from $-\frac{1}{2}$ to $\frac{1}{2}$ in the vertical direction. Similarly, in Table 6.3 we present the maximum errors for the same quantities. For the derivatives of $\Phi$ we report the absolute error because the solution has stationary points where the inverse of the gradient of the potential is ill-defined.

For the axisymmetric density distribution the resulting potential is axisymmetric to machine accuracy. The average relative error in the potential and the average absolute error in the radial and vertical derivatives all decrease by a factor of about three as the radial and vertical resolutions are doubled. As expected for an axisymmetric mass distribution the quality of the solution is independent of the number of azimuthal zones. The maximum values of the relative error in the potential decreases by a factor of about three as well and the maximum value in the absolute error of the radial and vertical derivatives has been cut in half as the number of radial and vertical zones doubles.

Table 6.2: Average Error for Gravitational Potential and Derivatives

<table>
<thead>
<tr>
<th>Origin</th>
<th>$R$</th>
<th>$z$</th>
<th>$\phi$</th>
<th>$\Phi$</th>
<th>$\partial_R \Phi$</th>
<th>$\partial_z \Phi$</th>
<th>$\partial_\phi \Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>66</td>
<td>66</td>
<td>64</td>
<td>$1.0 \times 10^{-2}$</td>
<td>$3.4 \times 10^{-3}$</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$4.0 \times 10^{-18}$</td>
</tr>
<tr>
<td></td>
<td>66</td>
<td>66</td>
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<td>0.5x</td>
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<td>66</td>
<td>64</td>
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<td>$7.1 \times 10^{-5}$</td>
<td>$3.6 \times 10^{-5}$</td>
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Table 6.3: Maximum Error for Gravitational Potential and Derivatives

<table>
<thead>
<tr>
<th>Origin</th>
<th>$R$</th>
<th>$z$</th>
<th>$\phi$</th>
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<td>$2.1 \times 10^{-2}$</td>
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<td>$1.8 \times 10^{-2}$</td>
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</table>
Figure 6.3: Contours of the absolute value of the relative error in the gravitational potential of a uniform density sphere centered at the origin (left column) and centered at $x = 0.5$ (right column). The top row shows contours in the equatorial plane while the bottom row shows contours in the plane $y = 0$. The grid has resolution of 130 radial and vertical zones by 256 azimuthal zones. Ten equally spaced contours between the indicated minimum and maximum values are shown. For both centerings the error occurs predominantly at the surface of the sphere.
When the sphere is placed off axis the convergence pattern is much more difficult to recognize. For the off-axis test at the highest resolution (the same radial and azimuthal resolution that we currently use for binary evolutions), we are able to obtain a solution that is accurate to one part in $10^4$, on average for the potential. Similarly, the finite-difference and analytical components of the derivatives of the potential agree to better than 4 decimal places on average.

We show contour plots of the absolute value of the relative error in the gravitational potential in the equatorial plane and the plane $y = 0$ in Fig. 6.3. The left column corresponds to the axisymmetric sphere while the right column shows the absolute value of the relative error in the potential for the off axis sphere. We show ten contours that are equally spaced between the indicated minimum and maximum relative errors. In all cases the largest error occurs at the edge of the sphere where the density drops discontinuously to zero. The edge of the rigid sphere is represented only approximately in terms of cylindrical volume elements. The errors in the potential presented here probably over estimate the errors arising in the evolution of self-gravitating compressible fluid bodies where the density distribution falls to zero more smoothly.

### 6.3 Test of Hydrostatic Equilibrium

A stringent test of our coupled solution of Poisson’s equation and the fluid dynamical equations - and one that may seem trivial at first mention - is how well we are able to maintain hydrostatic equilibrium for a simple system such as a spherical polytrope that is placed off axis in the grid. While our hydrodynamics implementation is conservative with respect to the advection of the fluid, there is no guarantee that the total momentum is conserved once the action of the Lagrangian source terms are included. Throughout a mass-transfer simulation, the bulk of the fluid should remain near hydrostatic equilibrium and the correct response of both components to their changing mass can be limited by the accuracy to which force balance is maintained.

To perform this test, we have placed a spherical, $n = \frac{3}{2}$ polytrope of radius $R = 0.38$ in a cylindrical grid of total radius 1.0 but with a variety of different resolutions. The polytrope is centered at $x \approx 0.58$. In each case, the initial density distribution was generated with our SCF code (with only one star present and no frame rotation), and the initial velocities were zero everywhere. Using our full gravitational hydrodynamics code, we then permitted the fluid system to evolve in time.

Over the course of the evolutions the stars drift outwards as if acted on by a constant force. This drift is shown in Fig. 6.4 where we have plotted the location of the center of mass of the star as a function of time for grids of varying resolution. We have normalized the evolution time to the dynamical time as given by eq. (2.1). The size and rate of the drift decrease as the azimuthal resolution increases.

As another measure of the quality of the solution we present the virial error as defined by eq. (3.13). The logarithm of the separate components
Figure 6.4: The center of mass as a function of time (measured in dynamical times) for the spherical polytrope described in §6.3. The different curves correspond to calculations performed at the indicated resolution in terms of the number of radial by vertical by azimuthal zones. The curves representing the simulations at resolutions of 66 x 66 x 128 and 130 x 130 x 128 lie on top of one another.

appearing in eq. (3.13) are plotted separately for the highest resolution simulation (computed with 130 radial and vertical zones and 256 azimuthal zones) as a function of the dynamical time in Fig. 6.5. The kinetic energy term, $T$ is much smaller than either the gravitational or thermal terms and is dominated by noise. This noise arises from discrete events where a small region at the outer boundary of the star acquires an outward acceleration. The density in the fluid in this region is small but the resulting velocity can be large. Because of the noise in $T$ we have not included it in the calculation of the virial error for the simulations as a function of time that is shown in Fig. 6.6. Overall, the virial error decreases by a factor of approximately 6 from the lowest to the highest resolution simulation.

There is no significant improvement in the drift of the system center of mass when only the radial and vertical resolutions are increased (compare the curves for the simulations at resolutions of 66 radial, 66 vertical by 128 azimuthal zones and 130 radial, 130 vertical by 128 azimuthal zones) though there is an improvement in the virial error between these two simu-
Figure 6.5: The logarithm of the three components to the virial error as given in eqs. (3.14)–(3.16) for the spherical polytrope evolved in a grid of resolution 130 radial and vertical zones by 256 azimuthal zones as a function of the dynamical time.

Figure 6.6: The virial error, as defined in eq. (3.13) as a function of the dynamical time for the spherical polytrope described in §6.3. The meaning of the different curves is as in Fig. 6.4.
lations. This suggests that there are (at least) two limiting numerical effects in play. One dictates the resolution of the equilibrium state itself and the other causes a constant displacement of that equilibrium state. The former effect converges with the finite-difference size isotropically while the latter depends only on the azimuthal resolution. When trying to resolve a highly nonaxisymmetric object such as an off-axis sphere within a uniform cylindrical coordinate grid, different parts of the star are resolved to varying degrees and it is not surprising that the convergence of the numerical solution is not simply describable. Further tests of hydrostatic equilibrium will be performed in the context of equilibrium binary systems in Chapter 7.
7. Detached Binary Simulations

In this section we present results from two benchmark simulations of detached binaries that we have performed to ascertain the precision which we can expect to attain in simulations of semi-detached binary systems (systems undergoing mass transfer), and gauge the computational workload of such simulations. One binary is an equal mass system with identical components (see Model 5 in Table 3.1 and Figs. 3.2 and 3.3; hereafter referred to as the EB system) and the other binary has a mass ratio $q = 0.8436$ (see Model 6 in Table 3.1 and Figs. 3.2 and 3.3; hereafter referred to as unequal binary or UB system). The EB system was constructed to resemble the single star used for the test of hydrostatic equilibrium in §6.3 to enable us to compare the systematic errors in the case of a binary system given the errors observed when only star, initially at rest, was evolved. Each component of the EB system differs from the isolated, spherical star in that each is flattened by rotation and tidally distorted by its companion, but the components have a comparable size, in terms of grid cells, and the same central density and polytropic index as the isolated sphere.

Previous simulations of equal-mass, barotropic binary systems have shown that it is important to conduct the evolutions in a frame of reference that renders the binary as close to static as possible in order to minimize the effects of numerical diffusion arising from the finite accuracy of Eulerian advection schemes (New & Tohline, 1997). With this in mind, our EB and UB simulations have been conducted in a frame of reference rotating with the orbital angular velocity of the system as obtained by our SCF technique. In dealing with unequal-mass systems we have discovered another subtle, but important issue that should be addressed with care when “transporting” an initial hydrostatic model from the grid of the SCF code into the grid of the hydrodynamics code. During each SCF iteration, the system’s center of mass is not fixed to any particular location beyond the fact that, by symmetry, it must lie along the line of centers. In general, then, we must translate the density field as we introduce it into the hydrocode so that the system center of mass coincides with the $z$-axis, which is taken to be the rotation axis for the hydrodynamic evolution. The translation of the initial model is performed with a bicubic interpolation scheme (Press et al., 1992). If the translation could be performed exactly, the initial velocities would be identically zero relative to the hydrodynamic reference frame. Because of the inherent symmetry of an equal-mass binary system, this was in fact the case for our EB system by construction. For our UB system, however, the center of mass of the translated binary model was displaced by a small distance from the rotation axis. Specifically, $\mathbf{R}_{\text{com}} = 2.842 \times 10^{-6}\mathbf{x}$ (the other two components of $\mathbf{R}_{\text{com}}$ being of the same order as the floating point precision using 64 bit words). This displacement corresponded to only $4 \times 10^{-4} \Delta R$, where $\Delta R$ is the radial extent of each grid cell. As we introduced the SCF model into the hydrodynamical grid, we therefore also ascribed
nonzero velocities as initial conditions according to the relation,
\[ \mathbf{v} = -\Omega \times \mathbf{R}_{\text{com}}. \]  

Because the displacement \( \mathbf{R}_{\text{com}} \) was quite small for our UB system, the initial velocities prescribed through eq. (7.1) were also very small. Nevertheless, it was necessary to include them in order to achieve the best possible steady-state configuration corresponding to the stars following circular orbits. However, this does imply a net velocity for the system center of mass which in turn limits the number of orbits the evolution of an unequal mass system can be followed before there is significant motion of the two stars relative to the reference frame of the calculation. If the uniform motion of the system were preserved by the hydrodynamics code, the unequal binary could be evolved for approximately 400 orbits before the system center of mass had moved more than one grid cell away from the coordinate axis.

We summarize the EB and UB evolutions in Table 7.1. We list the length of the evolution, \( t \), in units of the orbital period, \( P \); the number of timesteps calculated in the simulation; the relative change in the total mass for each component, \( \frac{\Delta M_i}{M_i} \), as well as for the binary system as a whole, \( \frac{\Delta M}{M} \); the relative change in the \( z \) component of the total angular momentum of the system, \( \frac{\Delta J_z}{J_z} \); the relative change in the orbital separation, \( \frac{\Delta a}{a} \); and the computing platform, number of processors and the wall clock time (total execution time) for the simulations. The EB system was run on 64 processors of the Cray T3E 600 at the San Diego Supercomputing Center and the UB evolution was performed on 8 dual-processor nodes of Louisiana State University’s IBM SP3. Both simulations were performed at a resolution of 130 radial by 98 vertical by 256 azimuthal zones for a total of about three and a quarter million cells.

For both systems, the components are largely static and remain well within their Roche lobes throughout the simulations. In Fig. 7.1 we plot the computed Roche lobe volume and the volumes occupied by material more dense than \( 10^{-1}, 10^{-2}, 10^{-3}, \) and \( 10^{-4} \) for one component of the EB system. (For reference, the initial SCF density fields have values of a few times \( 10^{-5} \) at the edge of the stars.) The same quantities are plotted for the primary and secondary components of the UB system in Figs. 7.2 and 7.3.

Unlike in the single star case presented in §6.3, there is no evidence of a systematic outwards force in either the EB or UB system despite the fact that the EB (UB) system has evolved for the equivalent of approximately 90 (75) dynamical times. It appears as though the introduction of a rotating frame of reference and the associated centrifugal potential and Coriolis force have provided a feedback mechanism that acts to correct the systematic imbalance discussed previously. The equatorial plane trajectory of the system center of mass and the center of mass for each binary component during the EB and UB evolutions is shown in Figs. 7.4 and 7.5. The linear scale of the plots is the same for all the frames in each figure and in the left-hand frame of each we have indicated the resolution of the computational grid, \( \Delta R \) is \( 7.87 \times 10^{-3} \). In both evolutions, the binary system as a whole is moving with a small, nearly constant velocity which, when viewed in the rotating frame of reference, appears as a spiral trajectory. In the UB case, the drift velocity
Table 7.1: Quantities of Interest for Benchmark Simulations

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Equal Mass Binary (EB)</th>
<th>Unequal Mass Binary (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{t}{P}$</td>
<td>5.314</td>
<td>5.178</td>
</tr>
<tr>
<td>timesteps</td>
<td>165,000</td>
<td>150,000</td>
</tr>
<tr>
<td>$\frac{\Delta M_1}{M_1}$</td>
<td>$-1.025 \times 10^{-4}$</td>
<td>$-1.233 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\frac{\Delta M_2}{M_2}$</td>
<td>$-1.128 \times 10^{-4}$</td>
<td>$-3.394 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\frac{\Delta M}{M}$</td>
<td>$-1.022 \times 10^{-4}$</td>
<td>$-7.120 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\frac{\Delta J_z}{J_z}$</td>
<td>$6.056 \times 10^{-4}$</td>
<td>$7.7908 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\frac{\Delta a}{a}$</td>
<td>$-1.545 \times 10^{-3}$</td>
<td>$-9.278 \times 10^{-4}$</td>
</tr>
<tr>
<td>Machine</td>
<td>Cray T3E 600</td>
<td>IBM SP3</td>
</tr>
<tr>
<td>Processors</td>
<td>64</td>
<td>16</td>
</tr>
<tr>
<td>$T_{WallClock}$</td>
<td>173 hours</td>
<td>265 hours</td>
</tr>
</tbody>
</table>
Figure 7.1: The Roche volume (dashed curve) and volume occupied by material more dense than $10^{-1}$, $10^{-2}$, $10^{-3}$, and $10^{-4}$ (solid curves from bottom to top) as a function of the number of orbits for one component of the EB system.

of the system is on average about a factor of 15 larger than the velocity prescribed as an initial condition by eq. (7.1). Nonetheless, an evolution of the UB system through 20 orbits should leave the system center of mass within one grid spacing of the origin, provided that the drift of the center of mass proceeds at a constant rate. At this level the motion of the components resulting from the motion of the system center of mass should be insignificant. The initial conditions produced with the SCF code describe circular orbits for the binary components to a remarkable degree with only a slight amount of epicyclic motion apparent in the EB system and an oscillation in the orbital separation that is of amplitude less than one part in $10^3$. As with the center of mass of the binary systems as a whole, the stars individually show only a small drift over the course of the evolution. In the EB case the drift is more ordered than in the UB system; we suggest that this is due to the lower accuracy in the initial orbital parameters as determined by the SCF technique.

The orbits for both systems exhibit a slow decay as can be seen in Figs. 7.6 and 7.7. The decay will cause both components to lead the rotating frame of reference as is the case for both the EB, and later in its evolution the UB
system, (in all simulations presented here we use a right handed coordinate system and the positive sense of rotation of the frame is counter-clockwise - from the x axis into the y direction). The orbital separation decays at a rate of $\Delta a/a \approx 1.8 \times 10^{-4} (2.8 \times 10^{-4})$ of its initial value per orbit in the UB (EB) system. This is most likely caused by the dissipation that accompanies the fluid simulation due to the introduction of artificial viscosity to mediate shocks. While these values are large compared to some physical effects that one may wish to model, such as the decay of a close, compact binary’s orbit due to the emission of gravitational radiation, they are small compared to the changes that are expected to result from dynamical mass-transfer events such as the ones discussed in §2.1.

As summarized in Table 7.1, the total mass is conserved to about one part in $10^{-4}$. The mass loss observed is not due to an error in the code but is instead caused by the flow of low density material from the atmosphere of the stars off of the grid. The rate at which mass is lost due to this numerical “wind” does impose a limit on the mass transfer rate that we can measure from simulations of RLOF. The steady-state outflow from the stars is shown in Figs. 7.8 and 7.9. The flow pattern throughout the simulation is similar to...
to the two snapshots presented here. There is a loss of low density ($\approx 10^{-5}$ to $10^{-6}$) material from the outward facing edge of each component (along the outwards directed, radial cell faces). This outflow is then in turn shaped by the Coriolis force into the asymmetric plumes that trail the stars. Noting the steep gradient in the density at the edge of the star we can see that the wind results from a lack of resolution there. In both simulations presented in this chapter the central region of the grid was inactive and was in essence a hole. This was done to shorten the time taken to complete these two test simulations. The “hole” is especially apparent in Fig. 7.9.

We have tried different centering schemes in an effort to minimize the outflow from the components and thereby increase the dynamic range of mass transfer rates that can be computed accurately. The different schemes have affected the character of the errors at the edge of the stars but have not significantly changed the amplitude of the outflow. While the problem can be made less significant by increasing the resolution of the simulations, a preferable solution would be the use of adaptive mesh refinement techniques where regions of significant error in the solution are dynamically assigned a larger number of smaller grid cells to enhance the resolution. In any event,
Figure 7.4: Trajectory of the center of mass of the left star, the system center of mass, and the right star in the equatorial plane for the EB system through approximately 5 orbits. We have subtracted off the initial coordinates and have, for reference, indicated the size of a grid cell.
Figure 7.5: Same as Fig. 7.4 for (from left to right) the secondary center of mass, system center of mass, and primary center of mass for the UB system evolved through approximately 5 orbits.
Figure 7.6: The orbital separation, normalized to its initial value, as a function of the number of orbits for the EB system.

Figure 7.7: The orbital separation, normalized to its initial value, as a function of the number of orbits for the UB system.
the current hydrodynamics code is convergent which implies that given sufficient computing power all the effects summarized in Table 7.1 can be made arbitrarily small.

From these benchmark simulations we observe that, while computationally expensive, it is practical to evolve two distinct, and fully self-consistent fluid bodies with an explicit, Eulerian hydrodynamics code. For 64 processors of the Cray T3E 600 the calculation of one orbit costs about 28 hours of computing time. Using 16 processors of the newer IBM SP-3, the evolution of one orbit costs 51 hours (which on a per-processor basis represents a speedup of about a factor of two). The computational workload of a mass transfer simulations is - even at higher resolution than performed here - within the reach of current parallel computers, given the linear scaling of our gravitational hydrodynamics code with the number of processors. We emphasize that the simulation of mass transfer events with the techniques of three-dimensional numerical hydrodynamics for a self-gravitating fluid has not previously been attempted by other researchers.
Figure 7.8: Contours of the density for the UB system in the equatorial plane at one orbit. The contours are evenly spaced in the logarithm of the density from $10^{-1}$ to $10^{-9}$. Note the low density “wind” extending predominantly from the outwards trailing edge of each star. The density level in this outflow is $\approx 10^{-5}$ to $10^{-6}$. 
Figure 7.9: Contours of the density for the UB system in the equatorial plane at two orbits. The contours are evenly spaced in the logarithm of the density from $10^{-1}$ to $10^{-9}$. The flow is nearly identical to that shown in Fig. 7.8 indicating that the “numerical wind” outflow from the stars is in steady state.
8. Mass Transfer Simulations

In this chapter we present results from three simulations of mass transfer in semi-detached binary systems. We will discuss each of these simulations in the sequence in which they were performed as each simulation has given us new insights into how to improve the calculations and thus have influenced subsequent work. The first simulation we will consider is a binary whose mass ratio is smaller than one, that is to say, the donor is the less massive star. We will refer to this simulation as the QLT1 (for $q < 1$) evolution. We have also performed evolutions of a binary with the more massive component undergoing RLOF and present two simulations of this system. In one of these we have introduced a term that removes angular momentum from the system so that the donor is driven into deeper contact. The necessity for this will be explained below. The system with the more massive star undergoing RLOF is referred to hereafter as the QGT1 (for $q > 1$) evolution and the case with the driving term in place will be termed the QGT1D ($q > 1$ with “driving”) evolution.

The initial states for the simulations are listed for reference in Table 8.1. Both the QGT1 and QGT1D simulations proceeded from the same SCF binary model. The SCF models are homentropic polytropes of index $n = \frac{3}{2}$ and exponent, $\gamma = 1 + \frac{1}{n} = \frac{5}{3}$. As we are simulating polytropes, there is no intrinsic length or mass scale ascribed to these models. The SCF binaries could represent - to varying degrees of validity - stars, or stellar remnants (white dwarfs and neutron stars). We use a dimensionless system of units in which the gravitational constant, maximum density within a model, and the edge of the computational grid are all set to unity. For example, in the chosen units, a test mass in circular orbit at the edge of the grid about a point mass of unit strength at the origin will have an orbital period of $2\pi$ and an orbital frequency, $\Omega = 1$. All evolutions were conducted at a resolution of 130 radial zones by 98 vertical zones by 256 azimuthal zones.

In Table 8.1 we list the mass ratio of the binary (as in §2.1, $q \equiv \frac{M_d}{M_r}$); the orbital period, $P$; separation, $a$; and angular frequency, $\Omega$; the $z$ component of the total angular momentum about the center of mass, $J_z$; the virial error as defined in eq. (3.13); the initial displacement of the system center of mass from the rotation axis, $R_{\text{com}}$; and the magnitude of the initial velocity of the system arising from this displacement as given by eq. (7.1); the minimum density allowed in the simulation, $\rho_{\text{min}}$ as discussed in §5.7; and for each component we list the maximum density, $\rho_{\text{max}}^i$, the (uniform) polytropic constant, $\kappa_i$; the mass, $M_i$; effective radius $R_i$ and effective Roche lobe radius $R^{RL}_i$. The effective radii of the stars and their Roche lobes have been normalized to the orbital separation. Images of isodensity surfaces for the QLT1 and QGT1 models are reproduced in Figs. 8.14 and 8.15.

We note that the three evolutions presented in this chapter were performed with a version of the hydrodynamics code that was similar, though not identical, to the version described in Chapter 5. In particular, these sim-
Table 8.1: Initial Semi-Detached Models

<table>
<thead>
<tr>
<th>Quantity</th>
<th>QLT1</th>
<th>QGT1</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>0.8697</td>
<td>1.323</td>
</tr>
<tr>
<td>P</td>
<td>29.03</td>
<td>29.73</td>
</tr>
<tr>
<td>a</td>
<td>0.8750</td>
<td>0.8882</td>
</tr>
<tr>
<td>Ω</td>
<td>0.2164</td>
<td>0.2113</td>
</tr>
<tr>
<td>Jz</td>
<td>1.404 × 10⁻³</td>
<td>1.400 × 10⁻³</td>
</tr>
<tr>
<td>VE</td>
<td>-7.3 × 10⁻⁴</td>
<td>-7.7 × 10⁻⁴</td>
</tr>
<tr>
<td></td>
<td>Rcom</td>
<td>8.083 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td>vcom</td>
<td>1.749 × 10⁻⁵</td>
</tr>
<tr>
<td>ρmin</td>
<td>1.0 × 10⁻⁷</td>
<td>1.0 × 10⁻¹⁰</td>
</tr>
<tr>
<td>ρdmax</td>
<td>0.68</td>
<td>0.60</td>
</tr>
<tr>
<td>κd</td>
<td>3.130 × 10⁻²</td>
<td>3.722 × 10⁻²</td>
</tr>
<tr>
<td>Md</td>
<td>1.438 × 10⁻²</td>
<td>1.755 × 10⁻²</td>
</tr>
<tr>
<td>Rd</td>
<td>0.3621</td>
<td>0.3509</td>
</tr>
<tr>
<td>RRLd</td>
<td>0.3624</td>
<td>0.3547</td>
</tr>
<tr>
<td>ρrmax</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>κr</td>
<td>3.054 × 10⁻²</td>
<td>2.636 × 10⁻²</td>
</tr>
<tr>
<td>Mr</td>
<td>1.654 × 10⁻²</td>
<td>1.326 × 10⁻²</td>
</tr>
<tr>
<td>Rr</td>
<td>0.3331</td>
<td>0.2672</td>
</tr>
<tr>
<td>RRLr</td>
<td>0.3870</td>
<td>0.3111</td>
</tr>
</tbody>
</table>

Simulations did not include artificial viscosity (see §5.5) and the centering for the momentum densities and velocities was different as well. For the computations presented here, the radial and vertical components of the momentum and velocities are centered at the vertex of a cell and the angular momentum density is cell-centered (see Fig. 5.1 and the related discussion in §5.2).

The initial velocities of the systems are substantially higher in the simulations presented in this chapter as compared to those of Chapter 7. Chronologically, the detached binary simulations discussed in Chapter 7 were performed after the results detailed here and represent the highest quality results obtained, to date, in this research project. The significant factor differentiating these two sets of simulations is the use of 64 bit arithmetic for the generation of SCF models and for the translation of the model. We also note that semi-detached binary systems are more difficult to obtain as they represent an
end point along a sequence of models. For small changes in the parameters, equilibrium binary solutions no longer exist.

The final states for all three binary evolutions are summarized in Table 8.2. We list the length of the evolutions in units of the orbital period, $t_P$; the total number of timesteps in the integration; the fractional change in the mass of the donor, receiver and the total system ($\Delta M_d$, $\Delta M_r$, and $\Delta M$ respectively); the relative change in the $z$ component of total angular momentum about the center of mass, $\Delta J_z$; the relative change in the orbital separation, $\Delta a$; the average mass loss rate from the donor per orbit in units of solar masses per year, $M_d$ (assuming the donor has the same mass and radius as the Sun); and the total time to compute the evolution. The QGT1 and QGT1D evolutions were calculated on 64 processors of the Cray T3E 600 at the San Diego Supercomputing Center and the QLT1 evolution was performed on 64 processors of the Cray T3E 900 at the Stennis Major Shared Resource Center.

For comparison with the mass transfer rates reported in Table 8.2, the Sun is losing mass in the form of a stellar wind at an average rate of about $10^{-14}M_\odot$ per year and stable mass transferring binaries have rates of between $10^{-9}$ to $10^{-10}M_\odot$ per year. $\beta$ Lyrae is, for example, commonly considered to be undergoing very rapid and complex mass exchange and has a mass transfer rate of perhaps $10^{-5}M_\odot$ per year. From these comparisons we can conclude that the mass-loss rates listed in Table 8.2 are very rapid indeed and would lead to very short lived interactions. If we take the donor to be a solar type star (with the solar mass and radius) the orbital period would be only about 8 hours for the models presented here. If, instead, we consider a white dwarf binary, as a precursor to a Type Ia supernova, the orbital period would be only about 30 seconds. At a constant mass transfer rate of $3 \times 10^{-5}M_\odot$ per orbit (taken to be characteristic of the first two simulations presented here) the mass transfer events considered here last for a negligible time compared to the lifetime of the system in its inactive state. For higher mass transfer rates, the probability of observing a binary in a phase of dynamically unstable mass transfer grows even smaller.

8.1 QLT1 Evolution

We have evolved the QLT1 ($q = 0.87$) system through almost 8 orbits. From eq. (2.21) we see that this mass ratio exceeds the stability boundary for an $n = \frac{3}{2}$ polytrope, so we expect the mass transfer rate to grow with time as the donor star expands. The chosen mass ratio also slightly exceeds the limit for the Roche lobe to contract upon mass loss, $q = \frac{5}{6}$, as derived from eq. (2.15).

For the QLT1 binary evolution we used the Roche lobes to identify the fluid that belonged to one star or the other. As discussed in §1.2, this is strictly valid only when the stars are stationary in the corotating frame. In Fig. 8.1 we show the change in total mass for the donor and the receiver. To-
Figure 8.1: The change in mass of donor (bottom curve) and receiver (top curve) throughout the evolution of the QLT1 system. Due to accretion of background material the receiver gains mass by an amount that is greater than the mass lost from the donor. Note that in the last few orbits of the evolution, the identification of material in the donor by its location relative to its Roche lobe fails.

Towards the end of the evolution there is significant motion of both stars relative to the reference frame and, consequently, the surface $\Phi_{\text{Roche}} = \Phi_{\text{Roche}}(L_1)$ no longer corresponds to the actual critical equipotential for the stars. This effect is most severe for the donor star, which is in contact with its Roche lobe.

We expect that, at least initially when the mass transfer rate is small, $\Delta M_r = -\Delta M_d$ or that all the mass that leaves the donor arrives at the receiver. From the asymmetry in Fig. 8.1 we can also see that, in addition to the mass transfer from the donor to the receiver, both stars are gaining mass. From Table 8.2, the QLT1 system as a whole has gained about 0.05% of its initial mass over the course of the simulation. This mass is “created” in grid cells where the density has fallen below the value $\rho_{\text{min}}$ and been reset to $\rho_{\text{min}}$ in order to maintain an acceptable timestep increment (see §5.7 for a discussion of the necessity of $\rho_{\text{min}}$). In subsequent simulations of semi-detached systems, the value of $\rho_{\text{min}}$ was lowered to $10^{-10}$. We further note
Figure 8.2: For the donor (top plot) and receiver (bottom plot) in the QLT1 evolution, the volume of the Roche lobe (dashed curve) and the volume occupied by material more dense than $10^{-5}$, $10^{-4}$, $10^{-3}$, $10^{-2}$, and $10^{-1}$ (solid curves from top to bottom).
that, while the system is exhibiting a net gain in total mass, there is material flowing off of the grid from the wind discussed in §7.

We plot the volume of the donor’s and receiver’s Roche lobe in Fig. 8.2 along with the volume occupied by material more dense than $10^{-5}$, $10^{-4}$, $10^{-3}$, $10^{-2}$, and $10^{-1}$. The Roche lobe volume oscillates with an amplitude of about 2%. Consequently, a steady mass transfer stream is not established during the simulation as the density of the donor material at the Roche surface is changing with time. The deviation from a circular orbit is demonstrated in Fig. 8.4 where we plot the orbital separation as a function of the number of orbits. The separation has been normalized to its initial value.

Unlike the other two simulations presented in this chapter, the initial conditions for the QLT1 simulation were generated by translating the binary’s center of mass to the coordinate axis as described in §7. This translated density distribution was then used as the starting point for another set of SCF iterations. The purpose of this procedure was to correct errors in the self-consistent density field that may have been created by the interpolation. While this did improve the virial error somewhat it had no significant benefit for the orbit as can be seen in Fig. 8.5 where we show the trajectories of the center of mass of the donor, system, and receiver. The motion of the binary is significantly worse than in the detached simulations show in Figs. 7.4 and

Figure 8.3: The $z$ component of the total angular momentum normalized by its initial value for the QLT1 evolution.
7.5. In subsequent simulations we only translate the density distribution in the preparation of the SCF model for evolution in the hydrodynamics code. In Fig. 8.3 we show the $z$ component of the total angular momentum. This should be a conserved quantity with the exception that this is not a strictly closed system. As material flows off the grid it carries angular momentum with it. In addition, the procedure of maintaining $\rho^{\text{min}}$ must also cause a violation of the conservation of angular momentum. We have not disentangled these effects from numerical inaccuracies that may occur in the treatment of the source terms for the angular momentum density as discussed in §5.4 though preliminary checks indicate that the flow of material off of the grid and generation of material are small effects compared to the 0.25% loss of angular momentum shown in Fig. 8.3.

On balance, the QLT1 simulation was not illuminating due to the accretion of background material that was of the same magnitude as the mass transfer rate. The results of the QLT1 system have been presented here as an example of early attempts to simulate mass transfer in binaries and to point out the lessons learned. Any comparisons of the results from the QLT1 evolution with real binaries or with other simulations should be made only with great caution.

Figure 8.4: The orbital separation normalized by its initial value for the QLT1 evolution.
Figure 8.5: The center of mass trajectories for (from left to right) the donor, system center of mass and receiver for the QLT1 binary through 7.8 orbits.
8.2 QGT1 Evolution

From the QLT1 evolution we saw that it was necessary to lower the value of $\rho_{\text{min}}$ and also to improve the methods used to ascertain which fluid elements belonged to which binary component. We implemented the multiple fluid tracking scheme discussed in §5.10 as well as a definition of the stellar’ material that depends on the average total energy of a fluid layer on the boundary of the star and was thus not limited by the assumption of circular orbits as is the case in the Roche approximation. These new techniques are sufficiently general to operate accurately for fluid bodies moving on eccentric orbits or even on unbound trajectories through the grid.

We also decided to focus our attention on a more extreme system, one for which it was expected that the mass transfer rate would grow more quickly and rise above the noise inherent in the finite accuracy of our hydrodynamics code. For a binary system with mass ratio $q > 1$, the orbital separation should shrink until $q$ reaches equality. This happens in addition to the donor star (an $n = \frac{3}{2}$ polytrope) expanding due to mass loss as discussed in §2.1.

We constructed an equilibrium SCF model that is described in Table 8.1 with a mass ratio of $q = 1.3$. The problem of mass creation has, to the precision that we can measure, been eliminated in the QGT1 evolution as

Figure 8.6: The change in mass of the donor (bottom curve) and receiver (top curve) throughout the QGT1 evolution.
Figure 8.7: For the donor (top plot) and receiver (bottom plot), the volume of the Roche lobe (dashed curve) and the volume occupied by material more dense than $10^{-4}$, $10^{-3}$, $10^{-2}$, and $10^{-1}$ (solid curves from top to bottom) for the QGT1 evolution.
Figure 8.8: The $z$ component of the total angular momentum about the center of mass normalized by its initial value for the QGT1 evolution.

can be seen in Table 8.2. The net increase in the total mass found in the QLT1 simulation has become a net mass loss from the system due to winds advecting material off of the grid as discussed previously.

The rate of mass transfer oscillates with the orbital period as the variation in the separation alternately brings the donor closer and further from contact. The donor is driven well out of contact towards the end of the simulation. In Fig. 8.6, the change in mass flattens out and the receiver even begins to lose mass after about 5.5 orbits. The donor's retreat from contact can be seen in the volume plot for the donor in Fig. 8.7. The increase in the effective volume of the donor's Roche lobe is consistent with increase in the orbital separation which is shown in Fig. 8.9. From Table 8.2 we can see that the orbital separation grows by about 2.5% over the course of the evolution. To further complicate matters, conservation of angular momentum is violated to a more significant degree in the QLT1 evolution; Fig. 8.8 shows that the binary has gained angular momentum in the amount of 0.55% of the total over the course of the evolution (If the change in angular momentum were due to material being lost from the simulation by flowing off the grid, we would expect the total angular momentum to have decayed, not increased.)

As seen in Fig. 8.6, the mass transfer event proceeds at an approximately constant rate of mass loss (gain) for the donor (receiver). There is again, an offset between the amount the donor loses and the amount that the receiver
Figure 8.9: The orbital separation normalized by its initial value for the QGT1 evolution.

gains. This time, however, the offset is due to the loss of matter from the boundary region of the stars and eventually from the system.

Although small in absolute terms, the violation of conservation of angular momentum signals an error. The change in the total angular momentum is, nevertheless, almost a factor of ten too small to account for the change in the orbital separation that one would expect from considering variations in the relation between the orbital angular momentum and separation,

\[ J_{\text{orb}} = \mu \Omega a^3 = \mu M^\frac{1}{2} a^\frac{3}{2}. \]  

We suggest instead that, in part at least, the increase in separation is physical and results from the transfer of momentum in the flow through the inner Lagrange point. Unfortunately, this simulation is not capable of providing more than a hint at the possibility that very rapid mass transfer events may be self-regulating in that a high mass transfer rate may ultimately drive the donor out of contact. Further cause for caution on this interpretation arises from the results obtained in the QGT1D evolution which is described next.
8.3 Driven QGT1 Evolution

In attempt to force the mass transfer rate to a high enough amplitude to dominate the evolution (as compared to spurious numerical effects in the conservation of angular momentum, for example) we introduced a driving term in the evolution of the QGT1 binary. Specifically, we removed angular momentum from the orbit at a constant rate of two percent per orbit. The goal was to initially drive the system to a very high mass transfer rate and then, going back to a saved state, resume the evolution with the driving term turned off. A still image of the resulting mass transfer stream is shown in Fig. 8.16.

For the QGT1D evolution, we again present plots of the change in mass for the donor and receiver (Fig. 8.10), the volume of the stars as compared to their Roche lobe volume (Fig. 8.11), and the response of the system’s angular momentum and orbital separation (Figs. 8.12 and 8.13).

The mass transfer rate does indeed reach a high amplitude, the peak change in mass being about a factor of twenty higher than in the QGT1 simulation. However, at slightly past two orbits the mass transfer ceases. At this point in the evolution, the separation increases rapidly (by roughly five percent) and the donor is essentially driven back out of contact with its Roche
Figure 8.11: For the donor (top plot) and receiver (bottom plot), the volume of the Roche lobe (dashed curve) and the volume occupied by material more dense than $10^{-4}$, $10^{-3}$, $10^{-2}$, and $10^{-1}$ (solid curves from top to bottom) for the QGT1D evolution. Note the large degree of over-contact for the donor star from between 0.5 and 2.5 orbits.
lobe. The separation increases despite the fact that angular momentum is still being drained from the system. Furthermore, as shown in Fig. 8.12, the total angular momentum of the binary begins increasing as in the QGT1 evolution, despite the fact that angular momentum is being removed at a constant rate.

As the angular momentum change is correspondingly higher in the QGT1D evolution as compared to the QGT1 simulation it is plausible that the numerical defect is itself dependent on the mass transfer rate. Of the terms appearing in the equations of motion (see §5.1 and in particular eqs. 5.5 - 5.7), it is the coupling in the Coriolis force that is most likely corrupting the computations. The Coriolis terms are themselves dependent on the velocity field which is substantial on the surface of the star where material is flowing towards the inner Lagrange point. The velocity of this flow is expected to be of order the local sound speed. This view is further supported by the fact that the detached binaries (where there is no substantial flow as compared to the RLOF simulations) do not suffer from this problem even though the they are acted on by the same source terms for gravity, pressure and the centrifugal potential. We note that the curvature term is also small for the
Figure 8.13: The orbital separation normalized by its initial value for the QGT1D evolution. The dashed line shows the expected behavior of the separation given the constant angular momentum drain acting on the system, assuming for simplicity a point mass binary.

detached binary simulations as the bulk of the fluid is relatively far from the axis.

Without a rigorous understanding of the unexpected behavior observed in the QGT1 and QGT1D evolutions, little can be said about the relevance of these simulations to any realistic binary system. By pushing our modeling efforts to this stage, however, a great deal has been learned about the degree to which present simulation tools can be utilized to study the structure, stability and dynamical evolution of close binary systems that are susceptible to mass transfer events. We have also uncovered a previously unknown numerical instability that may be present in similar, finite volume, Eulerian hydrodynamic codes that employ a rotating reference frame.

To proceed beyond the QGT1 and QGT1D calculations we must understand the precise nature of this numerical instability and recast our algorithm so as to avoid it.
Figure 8.14: A side view of the SCF model for the QLT1 evolution. The shells correspond to density levels of 0.75 (innermost surface), $10^{-1}$, $10^{-2}$, and $10^{-3}$ (outermost surface). The donor star (the star in contact with its Roche lobe) is at bottom in this image.
Figure 8.15: Rendered view from the side of the SCF model for the QGT1 evolution. The colored shells correspond to density levels of 0.75 (innermost surface), $10^{-1}$, $10^{-2}$, and $10^{-3}$ (outermost surface). The donor is the bottom star in this view.
Figure 8.16: The QGT1D system viewed from the top down at a time of 0.83 orbits when the donor is vigorously overflowing its Roche lobe. Density surfaces are at $0.3$, $5 \times 10^{-2}$, $2 \times 10^{-3}$, and $1.0 \times 10^{-4}$. 
Table 8.2: Evolution Summaries

<table>
<thead>
<tr>
<th>Quantity</th>
<th>QLT1</th>
<th>QGT1</th>
<th>QGT1D</th>
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<tr>
<td>$\frac{t}{P}$</td>
<td>7.773</td>
<td>5.810</td>
<td>3.095</td>
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<td>timesteps</td>
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<td>250,000</td>
<td>120,000</td>
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<td>$\frac{\Delta M_d}{M_d}$</td>
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<td>$-1.867 \times 10^{-4}$</td>
<td>$-2.335 \times 10^{-3}$</td>
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<td>$\frac{\Delta M_r}{M_r}$</td>
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<td>$1.566 \times 10^{-4}$</td>
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</tr>
<tr>
<td>$\Delta M$</td>
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<td>$-1.623 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\frac{\Delta J_z}{J_z}$</td>
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<td>$\frac{\Delta a}{a}$</td>
<td>$2.300 \times 10^{-3}$</td>
<td>$2.572 \times 10^{-2}$</td>
<td>$-2.539 \times 10^{-2}$</td>
</tr>
<tr>
<td>$-\dot{M}<em>d \ (M</em>{\odot} \text{ year}^{-1})$</td>
<td>$4.0 \times 10^{-2}$</td>
<td>$3.6 \times 10^{-2}$</td>
<td>$8.4 \times 10^{-1}$</td>
</tr>
<tr>
<td>$T_{\text{WallClock}}$</td>
<td>154 hours</td>
<td>286 hours</td>
<td>150 hours</td>
</tr>
</tbody>
</table>
9. Conclusions

As mentioned in the introduction, the calculations presented in this dissertation represent the first self-consistent, three-dimensional treatment of Roche lobe overflow with the techniques of Eulerian hydrodynamics for self-gravitating fluids. From these simulations, we are able to conclude that the mass transfer rates can be orders of magnitude in excess of those found in system that can be readily observed. While the results presented in §8 for the QGT1, driven QGT1, and QLT1 evolutions do not match the expectations outlined in the discussion of stability in binaries undergoing mass exchange, this may in part be due to the simplified nature of the arguments used. Clearly more work is required to untangle the numerical effects from those that have a basis in the physical laws governing these systems. Once this has been accomplished, the techniques applied in the evolution of semi-detached systems here could be used to study the merger and fate of double white dwarf binaries, the detailed simulation of the flow along the surface of the donor to the inner Lagrange point, the evolution of dynamically unstable mass-transferring systems, and the self-consistent calculation of the formation of an accretion disk.

With the experience of the simulations conducted to date we can with some confidence set limits on the length of time an unequal binary can be evolved given the quality of the initial conditions. We have also developed techniques, such as the ability to track multiple fluid components throughout the evolution, that will prove of utility in future work examining the accretion of material onto a compact object and the mixing of material from the donor into the accretor's envelope.

From the simulation of detached binaries in §7 we can see that, at current resolution, stable equilibrium binaries can be maintained with a high degree of accuracy for timescales of a few orbits (which is the equivalent of approximately 100 dynamical times for the binary components themselves) with relative ease. The numerical defects that are present in the detached binary simulations, such as the loss of material form both components in a spurious wind, stem from the lack of resolution at the edge of the stars. These defects can be further minimized by using a more intelligent, adaptive resolution scheme or by simply running the existing code on the current generation of high performance computing platforms at higher resolution.

In the context of both detached and interacting binaries there is interesting research work that can be done with present tools and techniques to investigate the spectra of normal modes of oscillation for the binary components and study their coupling to the orbital frequency. The phenomena of apsidal motion - caused by the extended mass distributions - can also be studied in greater detail than previously possible with the fully three-dimensional binary models developed here.

We have demonstrated a simple technique for extending the range of adiabatic response to mass loss for polytropic fluids to include the behavior of solar type stars by allowing the bodies to have nonuniform entropy distri-
butions. This technique should ultimately allow us to model systems with more moderate mass transfer rates. For such systems, the results from simulations could be compared directly with observations of binaries such as W Serpentis and β Lyrae to gauge the relative importance of different physical mechanisms that are believed to be operating.

In addition to generating the equilibrium binaries that we evolve in time with the hydrodynamics code described here, we have seen that the SCF technique can be of use in examining the impact of deviations from the point-mass Roche model that may be of importance to some areas of binary research. The discrepancy between the point-mass Roche model and self-consistent Roche model predictions for the surface area of binary components could be tabulated for a range of mass ratios and filling factors for example. These could then be used as correction factors in the reduction of data from eclipsing binary systems to form model “images” of the light emitting surface.

We have specifically examined the role of a self-consistent Roche lobe radius in determining the minimum period of Cataclysmic Variables. While our results, as well as the more recent results of Uryū et al., indicate that the deviations from the Roche model are not of sufficient strength to account for the entire discrepancy between the predicted and observed values for the period gap, they can account for part of the discrepancy.
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Vita

Patrick Michael Motl was born in Beaverdam, Wisconsin, on June 4, 1971. He was raised in Waterloo, Wisconsin, until the age of five when he and his family moved to Grandview, Indiana. Patrick graduated from South Spencer High School in 1989. From 1989 to 1992 he attended Indiana University in Bloomington, Indiana, and in 1993 he studied abroad at the University of Kent in Canterbury in the United Kingdom, through a student exchange program. He received his bachelor of science degree from Indiana University in physics in 1993. From 1993 to the year 2000 he was a graduate student at Louisiana State University in Baton Rouge. From November of 2000 to the present he has worked as a postdoctoral researcher in astrophysics at the University of Missouri - Columbia. Patrick expects to receive his doctorate degree in December of 2001.