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Synthesis of Multiple Feedback Active-Filters.

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ABSTRACT

Two methods of multiple-feedback filter synthesis are introduced. The first method employs second-order resonators, a summing amplifier and a feedback circuit of resistors. It is shown that in the case of bandpass and band-reject filters, it is possible to obtain a general design using identical resonators, with the feedback network dependent only on the type of filter (Chebyshev, Butterworth, etc.).

The second method is a modification of the Sallen and Key cascaded structure, and is applied to the general synthesis of low-pass and high-pass filters. The design combines the bare minimum number of elements of the Sallen and Key structure with the advantages of the multiple-feedback network.

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CHAPTER I

INTRODUCTION

A. Active Filter Synthesis.

Active filter synthesis is the development of an RC active network which realizes a prescribed rational transfer function

$$H(s) = \frac{N(s)}{D(s)} = \frac{V_{out}}{V_{in}} \tag{1.1}$$

The conventional method of synthesis, until quite recently, has been as follows. See for example the work by Tow [1].

(a) The overall transfer function H(s) is factored into biquadratic functions of the form

$$H_i(s) = \frac{a_i s^2 + b_i s + c_i}{d_i s^2 + e_i s + f_i}, \tag{1.2}$$

so that, with the order of H(s) being 2n, we have

$$H(s) = \prod_{i=1}^{n} H_i(s) \tag{1.3}$$

In the case of odd-ordered transfer functions, a bilinear factor is also required.

(b) Subnetworks, or resonators, are developed, using

resistors, capacitors, and operational amplifiers (op amps), to realize each function $H_i(s)$.

(c) The resonators are connected in cascade to realize the overall transfer function H(s) in accordance with (1.3). Proper isolation of the subnetworks to permit cascading is provided by the op amps.

An obvious advantage of the cascade approach is its simplicity of synthesizing a complicated function H(s) by synthesizing a number of relatively simple functions $H_i(s)$, $i = 1, 2, 3, \ldots, n$. A disadvantage of the method is that all the resonators are different and hence the manufacturing process is complicated.

A more serious disadvantage of the cascaded structure, and one which has led many writers in recent years to seek alternate methods of synthesis, is its sensitivity properties. For example, the sensitivity of H(s) with respect to a resonator function $H_i(s)$ is given [2] by

$$S_{H_i}^{H} = \frac{H_i}{H} \frac{\partial H}{\partial H_i} \tag{1.4}$$

which by (1.3) is

$$S_{H_i}^{H} = 1; \quad i = 1,2,\ldots,n. \tag{1.5}$$

That is, a relative change in H_i yields exactly the same relative change in H. This disadvantage of the cascaded method can be overcome by the use of other structures, for example, multiple feedback configurations such as those considered by Laker and Ghausi [3] and by Szentirmai [4].

Our objective in this dissertation is to develop two methods of synthesizing multiple feedback active filters. The first method, considered in Chapters 2 and 3, synthesizes bandpass and band-reject filters by employing a structure made up of a number of identical second-order resonators and a feedback network, along the lines suggested by Hurtig [5]. The resonators are connected as in cascade except that their individual outputs are fed back and, with the filter input, summed to form the input of the first resonator.

The second method of synthesis, which is considered in Chapters IV and V, uses a variation of the cascaded connection of Sallen and Key resonators [6]. The feedback elements are connected to the output of the filter instead of to the output of each resonator. This gives a multiple-feedback filter of the same type as the cascaded structure (that is low-pass, high-pass, etc.) and retains the bare minimum number of elements of the Sallen and Key configuration. This method is applied to the synthesis of low-pass and high-pass filters.

B. Filter Types and Transformations.

A normalized low-pass filter (passband $0 \leq \omega \leq 1$) with transfer function

$$\frac{V_{out}}{V_{in}} = \frac{K}{S^n + b_1 S^{n-1} + \ldots + b_{n-1} S + b_n} \tag{1.6}$$

may be used as a prototype for obtaining other filter types, such as denormalized low-pass, high-pass, bandpass, and band-reject filters. The filter described by (1.6) is of the all-pole type and, depending on the coefficients b_i, may be a Butterworth filter, Chebyshev filter, etc.

Equation (1.6) can be transformed to a denormalized low-pass filter transfer function, with cut-off frequency ω_c, by the transformation

$$S = s/\omega_c \tag{1.7}$$

which is effected by dividing the capacitances of the normalized low-pass filter by ω_c [7]. The low-pass to high-pass transformation is given by

$$S = \omega_c/s \tag{1.8}$$

which is effected by replacing, in the low-pass prototype, the

conductances by capacitances and the capacitances by conductances of equal value, and subsequently dividing the capacitances by ω_c [2]. Transfer functions of bandpass and band-reject filters are obtained by using (1.6) and making the substitutions

$$S = \frac{s^2 + \omega_o^2}{Bs} \tag{1.9}$$

and

$$S = \frac{Bs}{s^2 + \omega_o^2} \tag{1.10}$$

respectively [7]. In both cases ω_o is the center frequency and B is the bandwidth of the band passed in the case of bandpass filters and of the band eliminated in the case of band-reject filters. Instead of B and ω_o, one may specify Q and ω_o, where

$$Q = \frac{\omega_o}{B} \tag{1.11}$$

is the quality factor of the filter. Evidently a high Q implies a relatively narrow band which is passed (or eliminated). A normalized bandpass function ($\omega_o = 1$) with a specified Q is obtained from (1.6) by the transformation

$$S = \frac{Q\ (s^2 + 1)}{s} \tag{1.12}$$

and a normalized band-reject function ensures if

$$S = \frac{s}{Q\ (s^2 + 1)} \tag{1.13}$$

C. <u>Resonators</u>.

The resonators, or subnetworks, which we consider are those with normalized transfer functions with quadratic denominators, having the form in the low-pass case,

$$H(s) = \frac{K}{s^2 + as + b} \tag{1.14}$$

in the high-pass case,

$$H(s) = \frac{Ks^2}{s^2 + as + b} \tag{1.15}$$

in the bandpass case,

$$H(s) = \frac{Ks}{s^2 + \alpha s + 1} \quad , \quad \alpha = 1/Q \tag{1.16}$$

and in the band-reject case,

$$H(s) = \frac{K\ (s^2 + 1)}{s^2 + \alpha s + 1} \quad , \quad \alpha = 1/Q \tag{1.17}$$

In every case $H(s) = {}^{V}\text{out}/V_{\text{in}}$.

The low-pass Sallen and Key resonator [6] which we shall modify to obtain multiple feedback low-pass filters is shown in Fig. 1.1. It may be used to realize (1.14) if we have

$$K = \frac{\mu}{R_1 R_2 C C_1}$$

$$a = \frac{1}{R_2 C_1}(1-\mu) + \frac{1}{R_1 C} + \frac{1}{R_2 C} \tag{1.18}$$

$$b = \frac{1}{R_1 R_2 C C_1}$$

where

$$\mu = 1 + R_4/R_3 \tag{1.19}$$

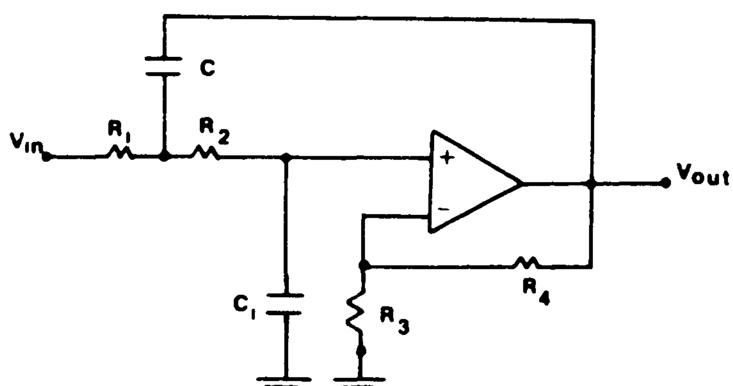

Fig. 1.1. A low-pass Sallen and Key resonator.

Interchanging the resistances and capacitances in an appropriate manner results in a Sallen and Key high-pass resonator.

The combination of the op amp and resistors R_3 and R_4 of Fig. 1.1 constitutes a voltage-controlled voltage source (VCVS). The VCVS may be symbolized as shown in Fig. 1.2, where

$$I = 0, \quad V_2 = \mu V_1 \tag{1.20}$$

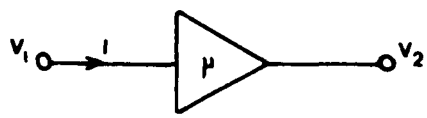

Fig. 1.2. A voltage-controlled voltage source.

Another resonator, which is excellent for realizing bandpass filters with high Q requirements, is obtained by considering the bandpass function

$$\frac{V_{out}}{V_{in}} = \frac{K_1 s}{s^2 + \alpha s + 1} \tag{1.21}$$

where the center frequency is $\omega_o = 1$, and $Q = 1/\alpha$. Rewriting (1.21) we have

$$V_{out}\left[s + \alpha + \frac{1}{s}\right] = K_1 V_{in}$$

Defining V_1 and V_2 by

$$V_1 = -\frac{1}{s} V_{out} \tag{1.22}$$

$$V_{out} = -\frac{1}{s + \alpha} V_2$$

we have

$$V_2 = -K_1 V_{in} - V_1 \tag{1.23}$$

A signal flow graph of (1.22) and (1.23) is shown in Fig. 1.3, from which we obtain the resonator of Fig. 1.4, which realizes (1.21). Note that the units in all figures are ohms and farads.

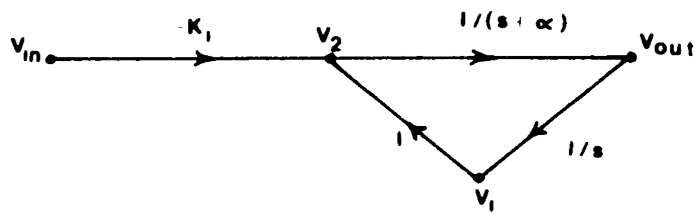

Fig. 1.3. A signal flow graph of a bandpass resonator.

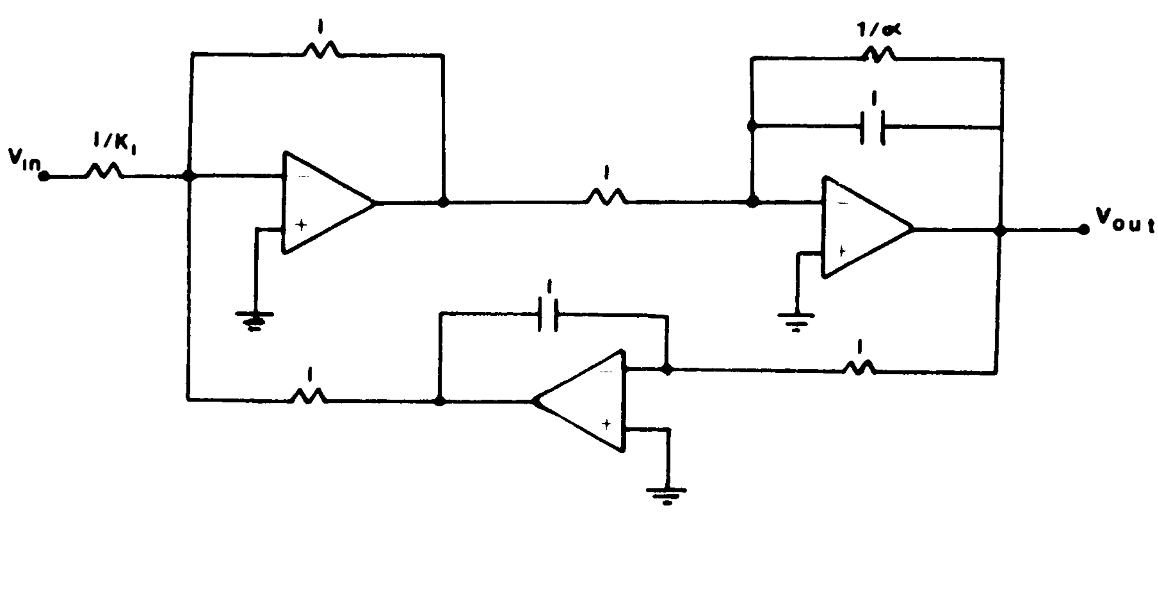

Fig. 1.4. A bandpass resonator.

CHAPTER II

BANDPASS FILTERS USING IDENTICAL RESONATORS

A. The Multiple Feedback Structure.

To obtain the multiple feedback circuit which we shall use to synthesize bandpass and band-reject filters, we consider the signal flow graph of Fig. 2.1. Applying Mason's rule yields

$$\frac{V_{out}}{V_{in}} = \frac{-a_o H_1 H_2 \ldots H_n}{1 + a_1 H_1 + a_2 H_1 H_2 + a_3 H_1 H_2 H_3 + \ldots + a_n H_1 H_2 \ldots H_n} \tag{2.1}$$

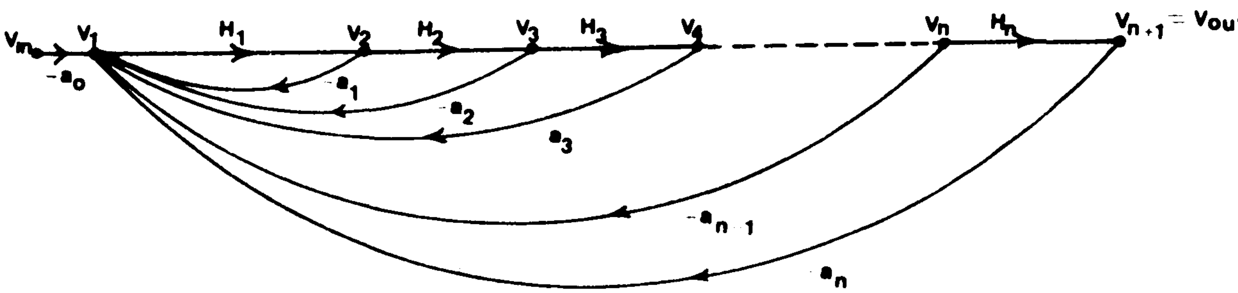

Figure 2.1. The signal flow graph of the filter.

A circuit which realizes (2.1) may be seen from Fig. 2.1 to be a cascade of resonators N_i, i = 1, 2,..., n, having transfer functions $H_i = V_{i+1}/V_i$. Each of the output voltages V_{i+1} is fed back and a weighted sum formed, which is equal to signal V_1, given by

$$V_1 = -(a_o V_{in} + a_1 V_2 + a_2 V_3 + \ldots + a_{n-1} V_n + a_n V_{out}) \tag{2.2}$$

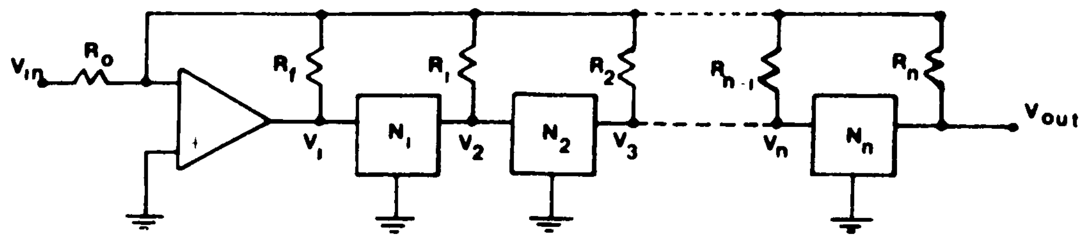

Figure 2.2. The multiple feedback filter

The summing process can be accomplished by a summing amplifier, as shown in Fig. 2.2, from which it may be seen that

$$V_1 = -R_f\left(\frac{V_{in}}{R_o} + \frac{V_2}{R_1} + \frac{V_3}{R_2} + \ldots + \frac{V_n}{R_{n-1}} + \frac{V_{out}}{R_n}\right)$$

Therefore, comparing this result with (2.2) we see that if

$$a_i = \frac{R_f}{R_i}, \quad i = 0, 1, 2, \ldots, n \tag{2.3}$$

then the circuit shown in Fig. 2.2 realizes (2.1).

The filter of Fig. 2.2 is the same type as its resonators. That is, if all the resonators are bandpass sections, then the structure is a bandpass structure. Similar statements hold for low-pass, high-pass, and band-reject filters. As we shall see, in the case of bandpass and band-reject sections, it is possible to have identical resonators.

B. Bandpass Filters.

The network of Fig. 2.2 becomes a bandpass filter when the resonators are bandpass resonators. It is also possible, in this case, to have identical resonators, which is highly desirable from a manufacturing standpoint. To see this, consider the case where each H_i is the bandpass function

$$H_i = \frac{K_1 s}{s^2 + \alpha s + 1} \tag{2.4}$$

of a normalized bandpass resonator with center frequency $\omega_o = 1$ and quality factor $Q_1 = 1/\alpha$. A resonator which realizes (2.4) was given previously in Fig. 1.4.

Consider next the transfer function of the normalized low-pass filter given previously by (1.6). Applying the low-pass to bandpass transformation

$$S = \frac{Q(s^2 + 1)}{s} \tag{2.5}$$

yields

$$\frac{V_{out}}{V_{in}} = \frac{Ks^n/Q^n}{D_1} \tag{2.6}$$

where

$$D_1 = (s^2 + 1)^n + \frac{b_1}{Q} s(s^2 + 1)^{n-1} + \frac{b_2}{Q^2} s^2(s^2 + 1)^{n-2}$$
$$+ \dots + \frac{b_{n-1}}{Q^{n-1}} s^{n-1}(s^2 + 1) + \frac{b_n}{Q^n} s^n \tag{2.7}$$

Equation (2.6) represents a 2n-th order bandpass filter with center frequency 1 rad/sec and a specified Q. A network realizing

(2.6) may be transformed to one with any center frequency $\omega_o = 2\pi f_o$ by proper impedance and frequency scaling.

Substituting (2.4) into (2.1) yields

$$\frac{V_{out}}{V_{in}} = \frac{-a_o K_1^n\ S^n}{D_2} \qquad (2.8)$$

where

$$D_2 = (s^2 + \alpha s + 1)^n + a_1K_1s(s^2 + \alpha s + 1)^{n-1}$$

$$+ a_2K_1^2s^2(s^2 + \alpha s + 1)^{n-2} + \dots + a_{n-1}K_1^{n-1}\ s^{n-1}$$

$$(s^2 + \alpha s + 1) + a_nK_1^ns^n \qquad (2.9)$$

Matching coefficients of like powers in (2.5) and (2.8) leads to n + 1 equations in the n + 3 unknowns, a_0, a_1, ..., a_n, α, and K_1. Therefore, two unknowns may be assigned arbitrarily and the others obtained for a given set of the parameters b_1, b_2, ..., b_n, which characterize the type of filter (Butterworth, Chebyshev, etc.), K, and Q. This will be considered in the following section.

We shall see that the feedback resistors in the realized network depend only on the type of filter desired, that is, on the b_i of (1.6).

C. Determination of the Feedback Circuit .

The feedback circuit of Fig. 2.2., for a given set of resonators, is determined by calculating the values of the resistors R_o, R_f, R_1,..., R_n required to match the coefficients

of (2.6) and (2.8). The work is facilitated somewhat by applying the binomial theorem to D_2 in (2.9), resulting in

$$D_2 = \sum_{i=0}^{n} \binom{n}{i} (s^2+1)^{n-i} (\alpha s)^i +$$

$$a_1 K_1 s \sum_{i=0}^{n-1} \binom{n-1}{i} (s^2+1)^{n-1-i} (\alpha s)^i +$$

$$a_2 K_1^2 s^2 \sum_{i=0}^{n-2} \binom{n-2}{i} (s^2+1)^{n-2-i} (\alpha s)^i + \dots +$$

$$a_{n-1} K_1^{n-1} s^{n-1} (s^2+1) + a_{n-1} K_1^{n-1} s^{n-1} (\alpha s) + a_n K_1^n s^n$$

Equating coefficients of $s^k(s^2+1)^{n-k}$, $k = 0,1,2,\dots,n$, in D_2 and D_1 given by (2.7), we have

$$\begin{aligned}
k = 0&: \quad 1 = 1 \\
k = 1&: \quad \binom{n}{1}\alpha + a_1 K_1 \binom{n-1}{0} = \frac{b_1}{Q} \\
k = 2&: \quad \binom{n}{2}\alpha^2 + a_1 K_1 \binom{n-1}{1}\alpha + a_2 K_1^2 \binom{n-2}{0} = \frac{b_2}{Q^2} \qquad (2.10) \\
&\ \vdots \\
k = n&: \quad \alpha^n + a_1 K_1 \alpha^{n-1} + a_2 K_1^2 \alpha^{n-2} + \dots + a_{n-1} K_1^{n-1} \alpha \\
&\qquad + a_n K_1^n = \frac{b_n}{Q^n}
\end{aligned}$$

with the general case being

$$\binom{n}{k}\alpha^k + \sum_{i=1}^{k} a_i K_1^i \binom{n-i}{k-i}\alpha^{k-i} = \frac{b_k}{Q^k}, \quad k = 1,2,\dots,n \qquad (2.11)$$

In addition, we must match the numerators of (2.6) and (2.8), resulting in

$$\frac{K}{Q^n} = -a_0 K_1^n \tag{2.12}$$

One approach in solving (2.11) and (2.12) is to make the assignments

$$K_1 = \frac{1}{Q}, \quad \alpha = \frac{\beta}{Q} \tag{2.13}$$

where β is to be assigned, if possible, to make the a_i nonnegative. We note from (2.12) that this choice of K_1 yields

$$a_0 = -K \tag{2.14}$$

and hence we must have $K<0$. That is, the filter must have an inverting gain.

Substituting (2.13) into (2.11), simplifying, and solving for a_k in terms of a_1, a_2, ..., a_{k-1}, results in

$$a_1 = b_1 - n\beta$$

$$a_k = b_k - \binom{n}{k}\beta^k - \sum_{i=1}^{k-1} \binom{n-i}{k-i} a_i \beta^{k-i}; \quad k = 2,3, \ldots, n \tag{2.15}$$

Thus, for a chosen value of β, we may obtain successively a_1, a_2, ..., a_n. Also we may solve (2.15) explicitly for the a_k in terms of β, n, k, and the b_i. The pattern that emerges as each successive a_k is obtained is given by

$$a_k = (-1)^k \sum_{i=0}^{k} (-1)^i \binom{n-i}{k-i} \beta^{k-i} b_i, \quad k = 1,2, \ldots, n \tag{2.16}$$

where we note that $b_o = 1$. That (2.16) is the solution of (2.15) may be readily shown by substitution and verifying that the coefficients of the various powers of β vanish.

The range of β for nonnegative a_k depends on the low-pass prototype coefficients b_i. It is interesting to note from (2.16) that

$$a_n = P_n(-\beta) \tag{2.17}$$

where

$$P_n(s) = s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n$$

is the denominator of the transfer function of the low-pass prototype filter, given by (1.6). Since $P_n(s)$ is strictly Hurwitz, all its zeros occur in the left half of the s-plane. Also $P_n(0)>0$. Therefore, denoting the real zero of $P_n(s)$ which is nearest the origin by $-\sigma(\sigma>0)$, we note that any value of β on $0<\beta<\sigma$ results in $a_n>0$. Of course, by the first of (2.15), a_1 is nonnegative if

$$0 < \beta \le b_1/n \tag{2.18}$$

Other bounds on β are obtained, for the various cases, by considering the other equations of (2.15).

The gain H_o of a bandpass filter is defined to be the value of the transfer function H(s) at the center frequency ω_o. By (2.6) and (2.7) we see that the gain is given by

$$H_o = \frac{V_{out}(j1)}{V_{in}(j1)} = \frac{K}{b_n} \tag{2.19}$$

Therefore specifying the gain, for a given filter type, determines K. The gain, as we have noted, must be negative.

As an example, suppose we want a 6th order normalized Butterworth bandpass filter for a given gain H_o and Q. This requires a 3rd order low-pass Butterworth prototype, for which $b_1 = b_2 = 2$ and $b_3 = 1$. By (2.19) we have $K = H_o$, and by (2.14) we have $a_o = -H_o$. By (2.16) we have

$$\begin{aligned} a_1 &= 2 - 3\beta \\ a_2 &= 2 - 4\beta + 3\beta^2 \\ a_3 &= 1 - 2\beta + 2\beta^2 - \beta^3 \end{aligned} \qquad (2.20)$$

All the a_i are nonnegative if we have $0<\beta\leq 2/3$. Therefore, if we choose $\beta = 1/3$, we have $a_1 = a_2 = 1$ and $a_3 = 14/27$, and by (2.13), we have $\alpha = 1/3Q$ and $K_1 = 1/Q$. These values determine the resonator of Fig. 1.4, and we may then construct the bandpass filter of Fig. 2.2 with three such resonators, and a feedback network of R_o, R_f, R_1, R_2, and R_3. These latter values are determined by (2.3) for a selected value of R_f. Choosing $R_f = 1$, we have $R_o = -1/H_o$, $R_1 = R_2 = 1$, and $R_3 = 27/14$.

We note that the foregoing design results in a normalized bandpass filter ($\omega_o = 1$). If we wish a filter with a given center frequency f_o, we select standard values C of the capacitors in the resonators, and multiply the normalized values of the resistances of both the feedback circuit and the resonators by $1/2\pi f_o C$. The result is a bandpass Butterworth filter with center

frequency f_o Hz, a gain H_o, and a quality factor Q.

We observe that in the above example, $\alpha = 1/3Q$, which by (1.21) is equivalent to $1/3Q = 1/Q_1$, or $Q = Q_1/3$, where Q_1 is the quality factor of the resonator. Thus Q of the filter is less than Q of the resonator, as is always the case when $\beta<1$, since in general $Q = \beta Q_1$. Therefore, for a high Q filter we must have resonators capable of producing very high Q. The resonator of Fig. 1.4 was chosen for that reason. .

As a final note, if β is chosen so that one of the a_i is zero, then this eliminates one of the feedback resistors. For example, if $\beta = b_1/n$, we see from (2.15) that $a_1 = 0$ and hence R_1 is infinite (open circuit). For the eighth order Butterworth and chebyshev (various dB ripples) bandpass filters, we chose β such that R_1 became infinite. The results for R_f = 1. and a gain of H_o are shown in table 1.

Type	R_o	R_2	R_3	R_4
Butterworth	$1.000/H_o$	1.172	2.613	4.911
Chebyshev (1/2 dB)	$2.639/H_o$	0.848	4.722	4.951
Chebyshev (1 dB)	$3.623/H_o$	0.898	6.315	5.819
Chebyshev (2 dB)	$4.854/H_o$	0.940	8.831	6.639
Chebyshev (3 dB)	$5.650/H_o$	0.959	11.121	7.067

Table 1. Feedback Circuit Values

D. Sensitivity of the Multiple Feedback Filter.

As observed in Section A of the first chapter, the sensitivity of the transfer function H(s) of a cascaded structure with respect to a resonator function $H_i(s)$ is given by

$$S^{H}_{H_i} = \frac{H_i}{H}\frac{\partial H}{\partial H_i} = 1 \tag{2.21}$$

This means, as indicated previously, that a relative change in H_i yields exactly the same relative change in H.

The sensitivity $S^{H}_{H_i}$ of the multiple feedback filter may be obtained from (2.1) . To facilitate the work we write (2.1) in the form

$$H = \frac{N}{D}$$

where

$$N = -a_o \prod_{i=1}^{n} H_i \tag{2.22}$$

and

$$D = 1 + \sum_{i=1}^{n} a_i \prod_{j=1}^{i} H_j \tag{2.23}$$

We also define the quantity

$$\begin{aligned} D_k &= 1 + a_1H_1 + a_2H_1H_2 + \ldots + a_{k-1}H_1H_2 \cdots H_{k-1} \\ &= 1 + \sum_{i=1}^{k} a_i \prod_{j=1}^{i} H_j \quad ; k = 1,2,\ldots,n+1 \end{aligned} \tag{2.24}$$

from which we note that $D = D_{n+1}$.

Applying (1.4) to H we have

$$S^{H}_{H_i} = \frac{H_i}{H} \left[\frac{DN - N(a_i H_1 H_2 \ldots H_i + \ldots + a_n H_1 H_2 \ldots H_n)}{H_i D^2} \right]$$

or

$$S^{H}_{H_i} = \frac{D_i}{D} \qquad (2.25)$$

Thus for $i = n$, we have $S^{H}_{H_i} = 1$, as in the case of the cascaded structure. However, for $0 < i < n$, Eq. (2.25) is complex for general values of s, and meaningful comparisons cannot be made unless we restrict s to real values or else modify the definition of sensitivity. This may be done by considering absolute values of $S^{H}_{H_i}$ or by comparing some real quantity associated with $S^{H}_{H_i}$ with its counterpart in the cascaded structure, as is done in [4]. It is not our purpose here to consider sensitivity in depth, but it should be clear from (2.25) that for $i < n$, D_i has fewer terms than D, and it is plausible that by some suitable definition of sensitivity, the multiple feedback filter should compare favorably with the cascaded filter.

CHAPTER III

BAND-REJECT FILTERS USING IDENTICAL RESONATORS

A. The Resonator.

It was noted in Section A of Chapter II that the multiple feedback filter of Fig. 2.2 may be designed with identical resonators in the case of bandpass and band-reject filters. This was shown in the bandpass case in the previous chapter, and we shall show it also to be the case for band-reject filters in this chapter. First, however, we consider a band-reject resonator which may be used.

The resonator we need is a second-order band-reject filter whose transfer function is given generally, by (1.6) and (1.10). For n = 2, we have

$$\frac{V_{out}}{V_{in}} = \frac{K\,(s^2 + \omega_o^2)}{s^2 + B\,s + \omega_o^2} \tag{3.1}$$

The center frequency of the rejected band is ω_o and $B = \omega_0/Q$ is the width of the rejected band. The quantity Q, as in the bandpass filter, is the quality factor which characterizes the narrowness of the rejecting band of the filter.

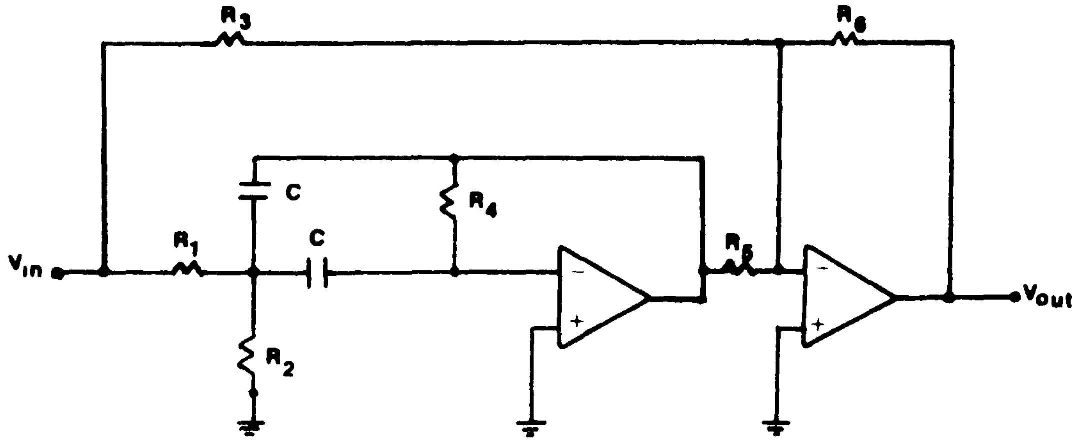

Fig. 3.1 A second-order bandreject resonator.

One resonator which realizes (3.1) is due to Huelsman [8], and is shown in Fig. 3.1. Analysis of the circuit yields, for

$$R_3 R_4 = 2 R_1 R_5 \tag{3.2}$$

the center frequency and bandwidth

$$\omega_o^2 = \frac{1}{R_4 C^2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{3.3}$$

$$B = 2/R_4 C$$

and an inverting gain

$$H_o = -R_6/R_3 \tag{3.4}$$

The gain is defined as $H_o = H(0)$, which is given by K in (3.1).

In the normalized case, $\omega_o = 1$ and $B = 1/Q$, in which case (3.1) becomes

$$\frac{V_{out}}{V_{in}} = \frac{K_1 (s^2 + 1)}{s^2 + \alpha s + 1} \tag{3.5}$$

where $\alpha = 1/Q_1$ and K_1 and Q_1 are the gain and quality factor of the resonator.

B. The Feedback Circuit.

To develop a 2n-th order multiple feedback band-reject filter of the type shown in Fig. 2.2, we shall use n identical second-order resonators N_i, having transfer functions given by (3.5). The overall filter function, by Eq. (2.1), then will be

$$\frac{V_{out}}{V_{in}} = \frac{- a_o K_1^n (s^2 + 1)^n}{D_1} \tag{3.6}$$

where

$$D_1 = (s^2 + \alpha s + 1)^n + a_1 K_1 (s^2 + 1)(s^2 + \alpha s + 1)^{n-1} + a_2 K_1^2 (s^2 + 1)^2 (s^2 + \alpha s + 1)^{n-2} + \ldots + a_{n-1} K_1^{n-1} (s^2 + 1)^{n-1} (s^2 + \alpha s + 1) + a_n K_1^n (s^2 + 1)^n$$

or

$$D_1 = \sum_{i=o}^{n} \binom{n}{i} (\alpha s)^{n-i} (s^2 + 1)^i + a_1 K_1 (s^2 + 1) \sum_{i=o}^{n-1} \binom{n-1}{i} (\alpha s)^{n-i-1} (s^2 + 1)^i + a_2 K_1^2 (s^2 + 1)^2 \sum_{i=o}^{n-2} \binom{n-2}{i} (\alpha s)^{n-i-2} (s^2 + 1)^i + \ldots + a_n K_1^n (s^2 + 1)^n \tag{3.7}$$

This result will be matched with the general band-reject transfer function given in Section B of Chapter I by

$$\frac{V_{out}}{V_{in}} = \left.\frac{K}{S^n + b_1 S^{n-1} + b_2 S^{n-2} + \ldots + b_{n-1} S + b_n}\right|_{S = \frac{s}{Q(s^2+1)}}$$

or

$$\frac{V_{out}}{V_{in}} = \frac{KQ^n (s^2 + 1)^n}{D_2} \tag{3.8}$$

where

$$D_2 = s^n + b_1 Q(s^2+1)s^{n-1} + b_2 Q^2 (s^2+1)^2\, s^{n-2} + \ldots +$$

$$b_n Q^n (s^2 + 1)^n \tag{3.9}$$

Equation (3.8) represents a band-reject filter with center frequency $\omega_o = 1$ and quality factor Q. Matching the numerators of (3.6) and (3.8) results in

$$-a_o K_1^n = KQ^n \tag{3.10}$$

and equating coefficients of $s^{n-k}(s^2 + 1)^k$, $k = 0,1,\ldots, n$ in (3.7) and (3.9) yields

$$\alpha^{n} = 1$$

$$\alpha^{n-1}\left[\binom{n}{1} + a_1K_1\binom{n-1}{0}\right] = b_1Q$$

$$\alpha^{n-2}\left[\binom{n}{2} + a_1K_1\binom{n-1}{1} + a_2K_1^2\binom{n-2}{0}\right] = b_2Q^2$$

$$\vdots \tag{3.11}$$

$$\alpha^{n-k}\left[\binom{n}{k} + a_1K_1\binom{n-1}{k-1} + a_2K_1^2\binom{n-2}{k-2} + \ldots + a_kK_1^k\binom{n-k}{0}\right] = b_kQ^k$$

$$1 + a_1K_1 + a_2K_1^2 + \ldots + a_nK_1^n = b_nQ^n$$

One way to proceed is to assign K_1 the value

$$K_1 = \beta Q \tag{3.12}$$

where $\beta>0$ is an arbitrary constant, resulting in

$$a_o = -K/\beta^n \tag{3.13}$$

Therefore, as in the bandpass case of the previous chapter, the gain of the filter must be an inverting gain. Also by (3.11) we have

$$\alpha = 1 \tag{3.14}$$

and

$$\binom{n}{k} + \sum_{i=1}^{k} a_i\beta^iQ^i\binom{n-i}{k-i} = b_kQ^k;\ k = 1,2,\ldots, n \tag{3.15}$$

The latter represents n equations in the n+1 unknowns a_1, a_2, ..., a_n, β and may be solved quite readily for a_1, a_2,..., a_n in that order, in terms of β. Hopefully β can be assigned so that the a_i are nonnegative.

The pattern that emerges as each consecutive a_i is found in terms of β, Q, and the b_j from (3.15) is given by

$$a_k = \frac{1}{\beta^k Q^k} \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} b_i Q^i ; \; k = 1,2,\ldots,n \qquad (3.16)$$

This may be proved quite readily by substituting a_k from (3.16) into (3.15) and observing the vanishing of the coefficients of Q^j, $j = 1,2,\ldots, k$.

We note from (3.16) that a_1 is given by

$$a_1 = (b_1 Q - n)/\beta Q$$

and hence is nonnegative if

$$Q \geq n/b_1 \qquad (3.17)$$

A lower bound on Q may be found, for specific cases of n, by considering (3.17) and the other equations of (3.16), for k = 2,3,..., n. It is interesting to note from the case k = n, that

$$a_n = P_n(-1/Q)/\beta^n \qquad (3.18)$$

where $P_n(s)$ is the denominator of the transfer function of the lowpass prototype. Therefore by an argument identical to that given in Section C of Chapter II for the bandpass filter, there is always a positive lower bound on Q for which $a_n \geq 0$.

We may also obtain by differentiating (3.16),

$$\frac{da_k}{dQ} = \frac{(n-k+1)\,a_{k-1}}{\beta^k Q^{k+1}} \quad ; \quad k = 2,3,\ldots, n$$

$$\frac{da_1}{dQ} = n/\beta Q^2 \tag{3.19}$$

Therefore, if $a_1 \geq 0$, then $\frac{da_2}{dQ} > 0$, and we should expect to find, for some $Q > 0$, a nonnegative a_2. The argument may be repeated for a_3, a_4, etc., and as we have seen, is certainly valid for a_n.

C. The Band-reject Filter.

By the previous sections of this chapter we have seen that a band-reject filter of order 2n, of the type shown in Fig. 2.2 may be designed with realizable (nonnegative) feedback resistances. For the normalized case ($\omega_o = 1$), the resonators are all identical and have transfer functions given by (3.5) where $\alpha = 1/Q_1$, and K_1 and Q_1 are the gain and quality factor of each resonator. By (3.12), (3.13), and (3.14), we have

$$\begin{aligned} K_1 &= \beta Q \\ a_o &= -K/\beta^n \\ \alpha &= 1 \end{aligned} \tag{3.20}$$

where Q is the desired quality factor of the band-reject filter, and K is the numerator of the low-pass prototype function. Since the gain H_o of the band-reject filter is its transfer function at s = 0, we have from (3.8) and (3.9),

$$H_o = K/b_n \tag{3.21}$$

By (3.20) we see that each resonator must have a noninverting gain of βQ and a quality factor of $Q_1 = 1$ ($\alpha = 1/Q = 1$). The resonator of Fig. 3.1 has an inverting gain and therefore if it is to be used we must invert its output. Adding an inverter to the output and adjusting the elements to obtain a center frequency $\omega_o = 1$, a gain of βQ, and a quality factor of 1, we obtain the result shown in Fig. 3.2. The transfer function is given by

$$\frac{V_{out}}{V_{in}} = \frac{\beta Q(s^2 + 1)}{s^2 + s + 1} \tag{3.22}$$

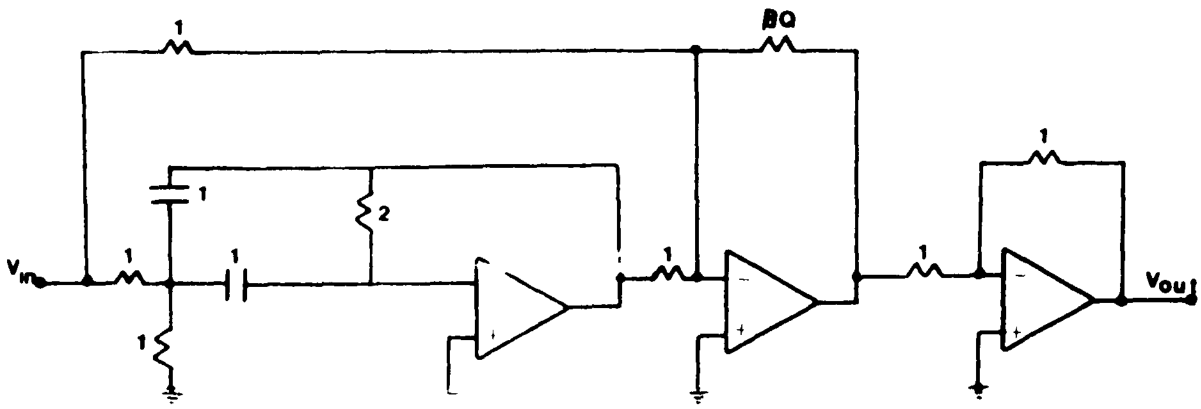

Fig. 3.2 A resonator with noninverting gain.

In summary, suppose we want a 2n-th order band-reject filter, of a certain type (Butterworth or Chebyshev), with a center frequency f_o (Hz), an inverting gain of H_o (negative), and a quality factor Q.

We first obtain a normalized filter (center frequency $\omega_o = 1$ rad/s), noting that the type of filter desired determines the coefficients $b_1, b_2, \ldots, b_n$. The n identical resonators will be of the type shown in Fig. 3.2. By (3.20) and (3.21) we have

$$a_o = - b_n H_o/\beta^n \tag{3.23}$$

and the remaining feedback constants a_k, $k = 1,2,3,\ldots,n$, are determined by (3.16). The feedback network resistors are then obtained from

$$R_i = \frac{R_f}{a_i} \quad , \; i = 0,1,\ldots, n \tag{3.24}$$

The quantities R_f and β are arbitrary positive numbers and may be chosen for convenience. We should note that the given Q must be sufficiently large for all the a_i to be nonnegative.

Finally the filter is obtained from the normalized circuit by replacing the 1 F capacitors by convenient standard value C and multiplying all the resistances by the scale factor $K = 1/2\pi f_o C$.

The advantages of the method are that H_o, Q, and f_o may all be specified, the resonators are all identical, and the arbitrariness of R_f and β allows us to have some control over the range of values of the resistances. This is important from a sensitivity standpoint.

As an example, suppose we want a 6th order Butterworth band-

reject filter with $\omega_o = 1$, $Q = 10$, and $H_o = -10$. Then we have $b_1 = b_2 = 2$, and $b_3 = 1$. By (3.23) we have

$$a_o = 10/\beta^3$$

and by (3.16) we have

$$a_1 = (2Q - 3)/\beta Q$$

$$a_2 = (2Q^2 - 4Q + 3)/\beta^2 Q^2$$

$$a_3 = (Q^3 - 2Q^2 + 2Q - 1)/\beta^3 Q^3$$

Evidently $Q > n/b_1 = 3/2$ results in positive a_i and for $Q = 10$, we have

$$a_1 = 1.7/\beta,\ a_2 = 1.63/\beta^2,\ a_3 = 0.819/\beta^3 \qquad (3.25)$$

We then have, for the resistances,

$$R_o = R_f\beta^3/10,$$

$$R_1 = R_f\beta/1.7$$

$$R_2 = R_f\beta^2/1.63$$

$$R_3 = R_f\beta^3/1.63$$

where β and R_f are arbitrary positive numbers.

CHAPTER IV

AN ALTERNATE METHOD OF MULTIPLE-FEEDBACK SYNTHESIS

A. Low-pass Filters using a Variation of Sallen and Key Resonators.

In a brief discussion in Chapter I, the use of Sallen and Key resonators in synthesizing filters was discussed. The resonator network was shown in Fig. 1.1, and its transfer function was described by (1.14) and (1.18).

In this chapter, we consider the synthesis of all-pole low-pass filters using a variation of the Sallen and Key resonators in the following manner. The filter is simply a cascaded connection of low-pass Sallen and Key resonators with the exception that the feedback elements are connected to the output of the filter rather than the output of each resonator. Fig. 4.1 shows a general configuration of a network of this kind which realizes a low-pass filter of even order. In the case where the order of the filter is odd, a first-order RC section is cascaded at the input port of the network. An example showing this latter case is the network of Fig. 4.2, where the order of the transfer function is n = 3.

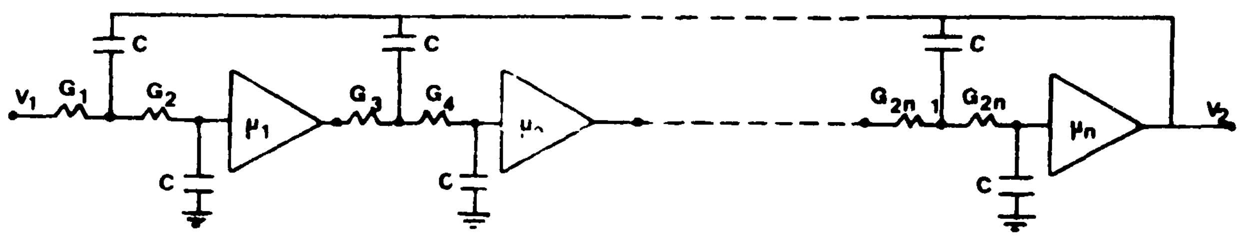

Fig. 4.1. A low-pass filter of order 2n.

Some features of this approach to synthesis are worth noting.

(a) The network retains the bare minimum number of elements of the Sallen and Key cascaded configuration.

(b) All the values of the capacitances in the circuit can be preselected, using standard values. In the work to follow we shall have $C = C_1$; however, in some cases of high gain, we may take $C_1 = 2C$ to keep the resistances within a proper range.

(c) The synthesis is exact with respect to the gain, contrary to the cascaded connection, where a change in the prescribed gain is necessary in many cases for obtaining a stable output.

The analysis shows that the transfer function related to Fig. 4.1 is of the form

$$\frac{V_2}{V_1} = \frac{H_o A_o}{s^n + A_{n-1}s^{n-1} + \ldots + A_1 s + A_o} \tag{4.1}$$

where, with the capacitor values preselected, H_o is the product of the gains of the amplifiers and A_i, $i = 0,1,\ldots,$ $n-1$, is a function of conductances and gains of the amplifiers in the network.

B. Low-pass Filter Transfer Functions.

The synthesis of all-pole low-pass filters (such as Butterworth, Chebyshev, etc.) using this method will be discussed in Chapter V. There we shall need the transfer function of (4.1) with specified order. For this reason the transfer functions for $n = 3$ through $n = 8$ of Fig. 4.1, where the values of all

capacitances are one farad (normalized case), are tabulated with the respective networks, which are shown in Figures 4.2 through 4.7.

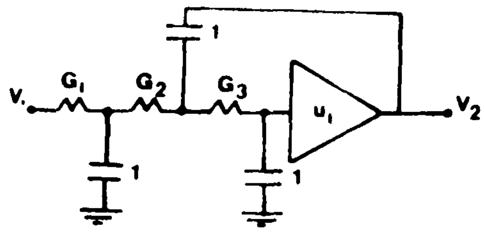

FIG.4.2 3RD. ORDER LOWPASS NETWORK

```
V2/V1=H0*A0/(A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1
A0=G1*G2*G3
A1=(G2+2.*G3-U1*G3)*(G1+G2)+G2*G3-G2**2
A2=G1+2.*G2+2.*G3-U1*G3
A3=1.
```

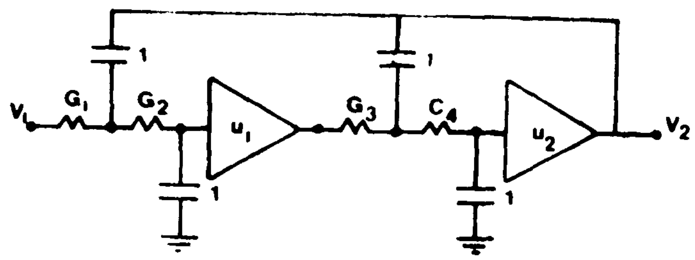

FIG.4.3 4TH. ORDER LOWPASS NETWORK

```
V2/V1=H0*A0/(A4*S**4+A3*S**3+A2*S**2+A1*S+A0)


WHERE THE COEFFICIENTS ARE

H0=U1*U2
A0=G1*G2*G3*G4
A1=G1*G2*(G3+2.*G4-U2*G4)+G3*G4*(G1+2.*G2)
    -U1*U2*G2*G3*G4
A2=G1*G2+G3*G4+(G1+2.*G2)*(G3+2.*G4-U2*G4)
A3=G1+2.*G2+G3+2.*G4-U2*G4
A4=1.
```

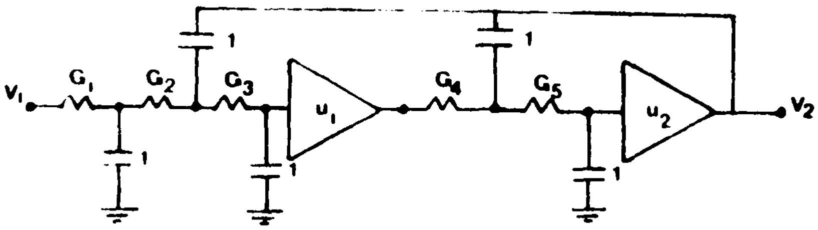

FIG.4.4 5TH. ORDER LOWPASS NETWORK

```
V2/V1=H0*A0/A5*S**5+A4*S**4+A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1*U2
A0=G1*G2*G3*G4*G5
A1=B*G4*G5+A*C-G2*G2*G4*G5-G2*G2*G3*C
    -U1*U2*G3*G4*G5*(G1+G2)
A2=G4*G5*D+B*C+A-G2*G2*C-G2*G2*G3-U1*U2* G3*G4*G5
A3=G4*G5+D*C+B-G2*G2
A4=C+D
A5=1.
```

WITH

```
A=G1*G2*G3+G2*G2*G3
B=G2*G3+(G1+G2)*(G2+2.*G3)
C=G4+2.*G5-U2*G5
D=G1+2.*G2+2.*G3
```

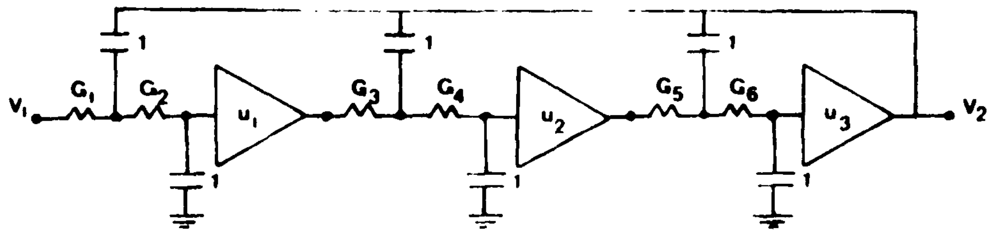

FIG.4.5 6TH. ORDER LOWPASS NETWORK

```
V2/V1=H0*A0/(A6*S**6+A5*S**5+A4*S**4
        +A3*S**3+A2*S**2+A1*S+A0)

WHERE THE COEFFICIENTS ARE

H0=U1*U2*U3
A0=G1*G2*G3*G4*G5*G6
A1=G5*G6*D+G1*G2*G3*G4*B-U2*U3*G1*G2*G4*G5*G6
    -U1*U2*U3*G2*G3*G4*G5*G6
A2=G5*G6*C+C1*G2*G3*G4+B*D-U2*U3*G4*G5*G6
    *(G1+2.*G2)
A3=G5*G6*A+B*C+D-U2*U3*G4*G5*G6
A4=G5*G6+A*B+C
A5=B+A
A6=1.

WITH

A=G1+2.*G2+G3+2.*G4
B=G5+2.*G6-U3*G6
C=G1*G2+G3*G4+(G1+2.*G2)*(G3+2.*G4)
D=G1*G2*(G3+2.*G4)+G3*G4*(G1+2 *G2)
```

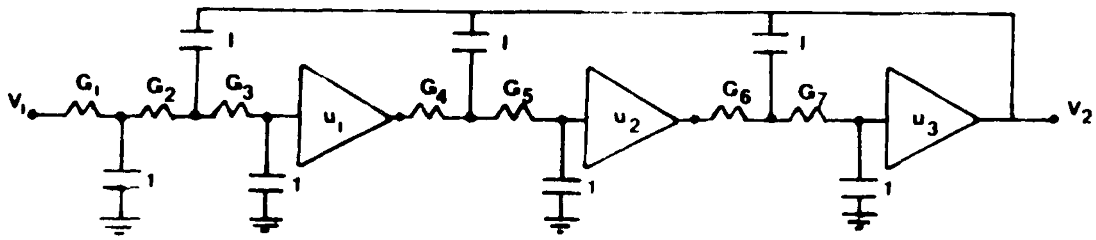

FIG.4.6 SEVENTH ORDER LOW-PASS NETWORK

```
V2/V1 H0*A0/(A7*S**7+A6*S**6+A5*S**5+A4*S**4
      +A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1*U2*U3
A0=G1*G2*G3*G4*G5*G6*G7
A1=G4*G5*G6*G7*E+G2*G3*(G1+G2)*F-U1*U2*U3*G3*G4*
        G5*G6*G7*(G1+G2)-G2*G2*G4*G5*G6*G7-G2*G2*G3*F
A2=G4*G5*G6*G7*B+E*F+G2*G3*(G1+G2)*D-G2*G2*G3*D-
   U1*U2*U3*G3*G4*G5*G6*G7-G2*G2*F
A3=G4*G5*G6*G7+B*F+E*D+G2*G3*(G1+G2)*A-G2*G2*D-
   G2*G2*G3*A
A4=F+B*D+E*A+G2*G3*(G1+G2)-G2*G2*A-G2*G2*G3
A5=D+B*A+E-G2*G2
A6=A+B
A7=1.
```

WITH

```
A=G4+2.*G5+G6+2.*G7-U3*G7
B=G1+2.*G2+2.*G3
C=G6+2.*G7-U3*G7
D=G6*G7+(G4+2.*G5)*C+G4*G5
E=G2*G3+(G1+G2)*(G2+2.*G3)
F=G6*G7*(G4+2.*G5)+G4*G5*C-U2*U3*G5*G6*G7
```

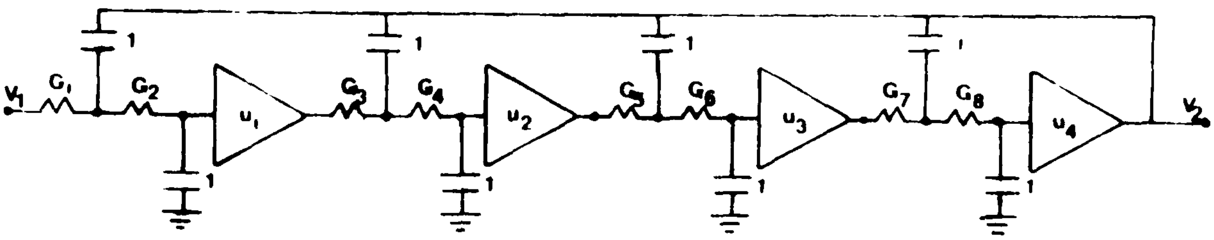

FIG.4.7 EIGHTH ORDER LOW-PASS NETWORK

```
V2/V1=H0*A0/(A8*S**8+A7*S**7+A6*S**6+A5*S**5+A4*S**4
          +A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1*U2*U3*U4
A0=G1*G2*G3*G4*G5*G6*G7*G8
A1=G1*G2*G3*G4*H+G5*G6*G7*G8*F
   -U2*U3*U4*G1*G2*G4*G5*G6*G7*G8
   -U3*U4*G1*G2*G3*G4*G6*G7*G8-P
A2=G1*G2*G3*G4*E+G5*G6*G7*G8*D+H*F
   -U2*U3*U4*G4*G5*G6*G7*G8*(G1+2.*G2)-U3*U4*G6*G7*G8*F
A3=G5*G6*G7*G8*B+G1*G2*G3*G4*C+D*H+E*F
   -U2*U3*U4*G4*G5*G6*G7*G8-U3*U4*G6*G7*G8*D
A4=G1*G2*G3*G4+G5*G6*G7*G8+B*H+C*F+D*E
   -U3*U4*G6*G7*G8*B
A5=H+F+B*E+C*D-U3*U4*G6*G7*G8
A6=D+E+B*C
A7=B+C
A8=1.
```

WITH

```
A=G7+2.*G8-U4*G8
B=G1+2.*G2+G3+2.*G4
C=G5+2.*G6+G7+2.*G8-U4*G8
D=G1*G2+G3*G4+(G1+2.*G2)*(G3+2.*G4)
E=G5*G6+G7*G8+(G5+2.*G6)*A
F=G1*G2*(G3+2.*G4)+G3*G4*(G1+2.*G2)
H=G5*G6*A+G7*G8*(G5+2.*G6)
P=U1*U2*U3*U4*G2*G3*G4*G5*G6*G7*G8
```

C. High-pass Filter Transfer Functions.

In this section we consider the analysis of high-pass filters using a modification of the cascaded connection of the high-pass Sallen and Key resonators. The low-pass to high-pass transformation is discussed in Part B of Chapter I. This transformation converts the network of Fig. 4.1 to the network of Fig. 4.8. Obviously the networks are similar with the exception that the capacitances and the conductances have changed places. The values of the capacitors in the network are preselected and are all identical. In this case, as in the case of low-pass filters, each capacitor is set to be one farad.

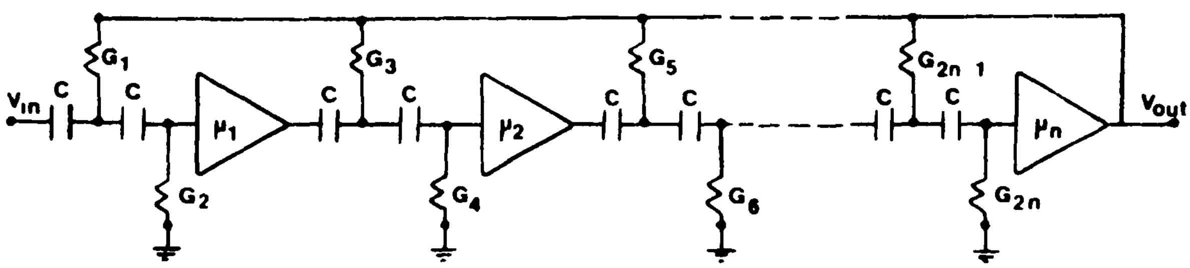

Fig. 4.8 A high-pass filter of order 2n

The general form of the transfer function for the network in Fig. 4.8 is

$$\frac{V_2}{V_1} = \frac{H_o s^n}{s^n + A_{n-1}s^{n-1} + \ldots + A_1 s + A_o} \tag{4.2}$$

where H_o is the product of the gains of the amplifiers, and A_i, i = 0, 1, 2, ..., n-1, is a function of gains and conductances. High-pass networks for n = 3 through n = 8 are shown in Figs. 4.9 through 4.14, respectively. The transfer function for each network is tabulated with the respective figure.

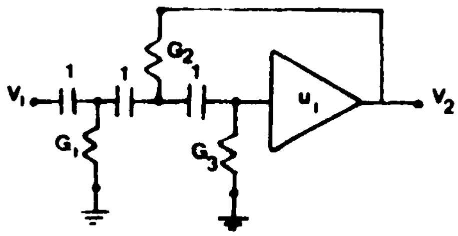

FIG.4.9 3RD. ORDER HIGHPASS NETWORK

```
V2/V1=H0*S**3/(A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1
A0=G1*G2*G3
A1=2.*G2*G3+G1*(G2+2.*G3-U1*G2)
A2=2.*(G2+2.*G3-U1*G2)-G3+G1
A3=1.
```

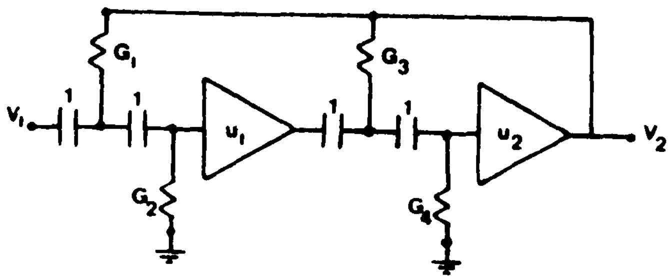

FIG.4.10 4TH. ORDER HIGHPASS NETWORK

V2/V1=HC*S**4/(A4*S**4+A3*S**3+A2*S**2+A1*S+A0)

WHERE THE CCEFFICIENTS ARE

HO=U1*U2
A0=G1*G2*G3*G4
A1=G3*C4*(G1+2.*G2)+C1*G2*(G3+2.*G4-U2*G3)
A2=G1*C2+G3*G4+(G1+2.*G2)*(G3+2.*G4-L2*G3)
A3=G3+2.*C4-U2*G3+G1+2.*G2-U1*L2*G1
A4=1.

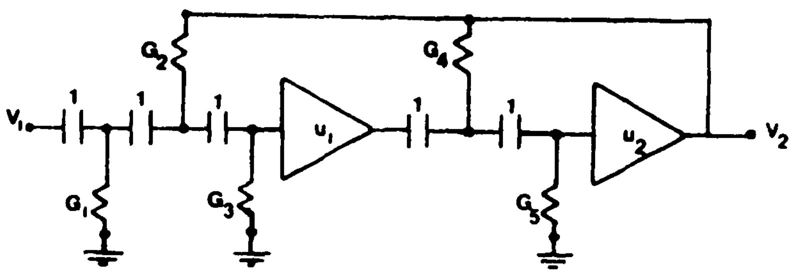

FIG.4.11 5TH. ORDER HIGHPASS NETWORK

```
V2/V1=H0*S**5/A5*S**5+A4*S**4+A3*S**3+A2*S**2+A1*S+A0)

WHERE THE COEFFICIENTS ARE

H0=U1*U2
A0=G1*G2*G3*G4*G5
A1=2.*G2*G3*G4*G5+G1*A
A2=2.*A+G1*B-G3*G4*G5
A3=2.*B+G1*C-G4*G5-G3*(2.*G5+G4-U2*G4)
A4=2.*C+G1-(2.*G5+G4-U2*G4+G3)
A5=1.

WITH

A=G4*G5*(G2+2.*G3)+G2*G3*(2.*G5+G4-U2*G4)
B=G4*G5+(G2+2.*G3)*(2.*G5+G4-U2*G4)+G2*G3
C=2.*G5+G4-U2*G4+G2+2.*G3-U1*U2*G2
D=2.*G5+G4-U2*G4
```

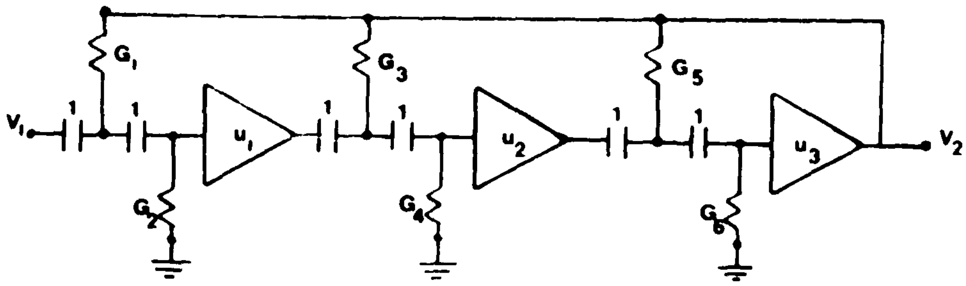

FIG.4.12 6TH. ORDER HIGHPASS NETWORK

```
V2/V1=H0*S**6/(A6*S**6+A5*S**5+A4*S**4+
        A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
H0=U1*U2*U3
A0=G1*G2*G3*G4*G5*G6
A1=(G1+2.*G2)*G3*G4*G5*G6+G1*G2*D
A2=G3*G4*G5*G6+(G1+2.*G2)*D+G1*G2*C
A3=D+(G1+2.*G2)*C+G1*G2*B
A4=G1*G2+C+(2.*G2+G1)*B
A5=B+G1+2.*G2-U1*U2*U3*G1
A6=1.
```

WITH

```
A=2.*G6+G5-U3*G5
B=A+2.*G4+G3-U2*U3*G3
C=G5*G6+(G3+2.*G4)*A+G3*G4
D=G5*G6*(G3+2.*G4)+G3*G4*A
```

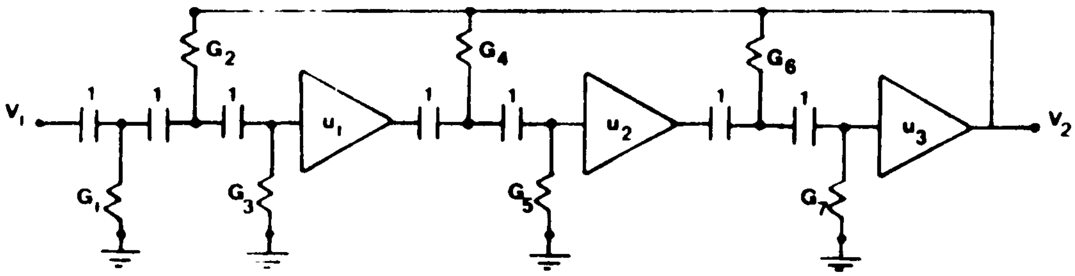

FIG. 4.13 7TH. ORDER HIGHPASS NETWORK

```
V2/V1=H0*S**7/(A7*S**7+A6*S**6+A5*S**5+A4*S**4+
        A3*S**3+A2*S**2+A1*S+A0))


WHERE THE COEFFICIENTS ARE

H0=U1*U2*U3
A0=G1*G2*G3*G4*G5*G6*G7
A1=2.*G2*G3*G4*G5*G6*G7+G1*K
A2=2.*K+G1*J-G3*G4*G5*G6*G7
A3=2.*J+G1*I-G4*G5*G6*G7-G3*E
A4=2.*I+G1*H-G3*D-E
A5=2.*H+G1*F-D-G3*C
A6=2.*F+G1-C-G3
A7=1.

WITH

A=2.*G7+G6-U3*G6
B=G4+2.*G5
C=U2*U3*G4+A+B
D=G4*G5+G6*G7+A*B
E=G6*G7*B+G4*G5*A
L=G2+2.*G3
F=C+L-U1*U2*U3*G2
H=D+L*C+G2*G3
I=E+D*L+G2*G3*C
J=G4*G5*G6*G7+L*E+G2*G3*D
K=G2*G3*E+G4*G5*G6*G7*L
```

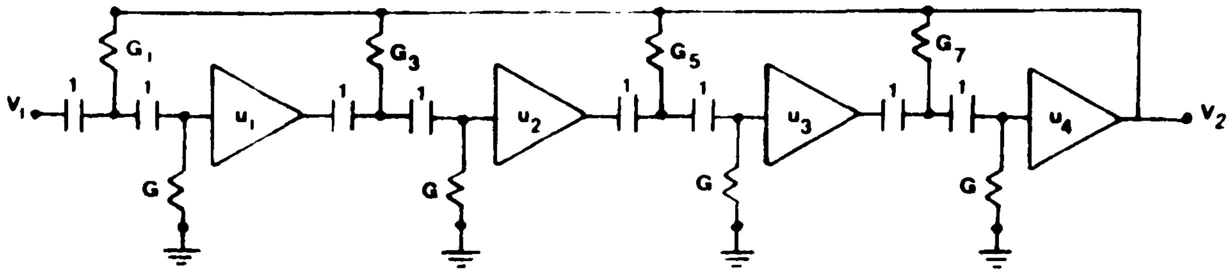

FIG.4.14 8TH. ORDER HIGHPASS NETWORK

```
V2/V1=HO*S**8/(A8*S**8+A7*S**7+A6*S**6+A5*S**5+
      A4*S**4+A3*S**3+A2*S**2+A1*S+A0)
```

WHERE THE COEFFICIENTS ARE

```
HO=U1*U2*U3*U4
A0=G1*G2*G3*G4*G5*G6*G7*G8
A1=G1*G2*(G5*G6*G7*G8*(G3+2.*G4)+G3*G4*E)+
   (G1+2.*G2)*G3*G4*G5*G6*G7*G8
A2=G3*G4*G5*G6*G7*G8+(G1+2.*G2)*(G5*G6*G7*G8*
   G1*G2*(G3+2.*G4)*E+G3*G4*D
   (G3+2.*G4)+G3*G4*E)+G1*G2*G7*G8*G5*G6+
A3=G5*G6*G7*G8*(G3+2.*G4)+G3*G4*E+(G1+2.*G2)*G5*G6*
   G7*G8+(G1+2.*G2)*(G3+2.*G4)*E+(G1+2.*G2)*G3*G4*D+
   G1*G2*E+G1*G2*G3*G4*B+G1*G2*(G3+2.*G4)*D
A4=G5*G6*G7*G8+(G3+2.*G4)*E+G3*G4*D+(G1+2.*G2)*
   (E+G3*G4*B+(G3+2.*G4)*D)+G1*G2*(D+(G3+2.*G4)*B)
   +G1*G2*G3*G4
A5=E+G3*G4*E+(G3+2.*G4)*D+(G1+2.*G2)*(D+B+G3*G4)+
   G1*G2*C
A6=D+(G3+2.*G4)*B+G3*G4+(G1+2.*G2)*C+G1*G2
A7=C+G1+2.*G2-U1*U3*U3*U4*G1
A8=1.
```

WITH

```
A=G7+2.*G8-U4*G7
B=G5+2.*G6+A-U3*U4*G5
C=B+G3+2.*G4-U2*U3*U4*G3
D=G5*G6+A*(G5+2.*G6)+G7*G8
E=G5*G6*A+G7*G8*(G5+2.*G6)
```

CHAPTER V

SYNTHESIS

A. Synthesis Approach.

Before we begin to synthesize the filters of Chapter IV, we notice from Figs. 4.1 and 4.8 and the related transfer functions that the number of conductances in the filters is always equal to n, the order of the transfer function. The following method is applied in synthesizing low-pass transfer functions of the form

$$\frac{V_2}{V_1} = \frac{K_1}{s^n + B_{n-1}s^{n-1} + \ldots + B_1 s + B_o} \tag{5.1}$$

or high-pass transfer functions of the form

$$\frac{V_2}{V_1} = \frac{K_2 s^n}{s^n + B_{n-1}s^{n-1} + \ldots + B_1 s + B_o} \tag{5.2}$$

Let

$$\underline{B} = [B_o, B_1, \ldots, B_{n-1}, B_n]^T \tag{5.3}$$

where $B_n = K_1$ of (5.1) in the case of low-pass filters, and $B_n = K_2$ of (5.2) in the case of high-pass filters.

Also let

$$\underline{G} = [G_1, G_2, \ldots, G_n]^T \tag{5.4}$$

where G_i, i = 1,2,...,n, are the conductances in the respective networks. Finally, let

$$\underline{A} = [A_o, A_1, \ldots, A_{n-1}, A_n]^T \tag{5.5}$$

where $A_n = H_o A_o$ in (4.1) or $A_n = H_o$ in (4.2). Since the elements of $\underline{A}$ are functions of $\underline{G}$ and the gains of the amplifiers in the circuit, we can write

$$\underline{A} = \underline{A}\ (\underline{G}, \underline{U}) \tag{5.6}$$

where $\underline{U}$ is a vector whose elements are the gains of the amplifiers in the circuit. The problem is to find the vectors $\underline{G}$ and $\underline{U}$ to satisfy the system of equations

$$\underline{A} - \underline{B} = \underline{A}(\underline{G}, \underline{U}) - \underline{B} = \underline{0} \tag{5.7}$$

The above system can be solved by a variety of methods.

(a) Using any computer search program with constraints on the elements of $\underline{G}$ and $\underline{U}$ to confine them between some boundaries depending on the specifications. For example,

$$a_i \le G_i \le b_i,\ i = 1,2,\ldots n \tag{5.8}$$

where a_i and b_i are some positive numbers, and

$1 \leq U_j, \ j = 1,2,\ldots, [n/2]$

where $[n/2]$ is the number of amplifiers in the circuit; The number $[n/2]$ is the largest integer less than or equal to $n/2$.

(b) Assigning reasonable practical values to the elements of $\underline{U}$ and reducing the problem to a system of n nonlinear algebraic equations in n unknowns and then solving for $\underline{G}$.

(c) A combination of the above two approaches.

The method we used is (c). For the simpler cases (lower values of n) we assigned values to all the gains U_i except one. This determined the unassigned gain since the product of the U_i is the given gain of the filter. Then we applied the Newton-Raphson method [9] to the remaining n equations in the n conductances. We repeated this procedure for a number of cases (assigning different values to U_i) and selected the one which gave the best range of conductance values.

For the more complex cases (higher values of n), we used the Powell [10] search technique, restricting the conductances to positive values and the gains U_i to numbers greater than or equal to one. The answers obtained were not sufficiently accurate, so we used rounded-off values of the U_i obtained and applied the Newton-Raphson method to find the conductances, taking as initial values those obtained with the search program.

The following examples (and many others which are not included in this paper) were solved as discussed in the previous paragraph. The results were excellent as may be seen from the error column, $B_i - A_i$, in the tables of the following examples.

B. Example 1.

An 8th order low-pass Butterworth is synthesized using the network of Fig. 4.7 and the transfer function related to it. The results are incorporated in Table 2.

G_i Conductances	B_i Given Coeff.	A_i Matching Coeff.	$B_i - A_i$ Error
G_1 = 1.7529662	B_0 = 1.0	A_0 = .99999999	$.01 \times 10^{-6}$
G_2 = .1111362	B_1 = 5.12580013	A_1 = 5.12608286	$-.02 \times 10^{-4}$
G_3 = .8253321	B_2 = 13.13710022	A_2 = 13.13710017	$.5 \times 10^{-6}$
G_4 = .6701567	B_3 = 21.84620667	A_3 = 21.84620660	$.7 \times 10^{-7}$
G_5 = .7834489	B_4 = 25.68840027	A_4 = 25.68840022	$.5 \times 10^{-7}$
G_6 = 1.187908	B_5 = 21.84620667	A_5 = 21.84620665	$.2 \times 10^{-6}$
G_7 = 1.589579	B_6 = 13.13710022	A_6 = 13.13709531	$.491 \times 10^{-5}$
G_8 = 6.2732136	B_7 = 5.12580013	A_7 = 5.12580013	0.0
Given overall gain: $K = H_o = 10$			
Gain of amplifiers: $U_1 = 3.8$; $U_2 = 1$; $U_3 = 1$; $U_4 = 2.63$			

Table 2. Results for Example 1.

8th order low-pass Butterworth

C. Example 2.

A 5th order high-pass Chebyshev (1 dB ripple) is synthesized using a gain H_o = 10. The results are incorporated in Table 3.

G_i Conductances	B_i Given Coeff.	A_i Matching Coeff.	$B_i - A_i$ Error
G_1 = 8.39050750	B_0 = 8.14155259	8.14155237	$.22 \times 10^{-6}$
G_2 = 0.98575527	B_1 = 7.62717003	7.62701531	$.5472 \times 10^{-4}$
G_3 4.79840617	B_2 = 13.74958485	13.74958459	$.26 \times 10^{-6}$
G_4 = 0.23176788	B_3 = 7.93309717	7.93309712	$.5 \times 10^{-7}$
G_5 = 0.88[illegible]11486	B_4 = 4.72644988	4.72644988	0.0
Overall given gain: K = H_o = 10.			
Gain of amplifier: U_1 = 1; U_2 = 10			

Table 3. Results for Example 2.

5th order high-pass Chebyshev

CHAPTER VI

CONCLUSIONS

The first method of synthesis described in Chapters II and III can be used to obtain multiple feedback realizations, using identical resonators, for bandpass and band-reject filters. The realizations may be obtained for given values of center frequency, gain, and Q. This method overcomes the sensitivity difficulties, discussed in Chapter I, associated with cascaded structures, and has the further advantages of identical resonators and feedback resistors which determine, independent of the resonators, the type of filter desired (Chebyshev, Butterworth, etc.).

It is rather easy to extend the first method to low-pass and high- pass filters, but analysis of the general case indicates that the resonators cannot be made identical. For these two cases we develop the second method, described in Chapters IV and V. This second method has the advantages of the multiple feedback circuits, and can be used to obtain realizations using the bare minimum number of elements of the Sallen and Key cascaded structure. This method could also be applied to bandpass filters, but the structures would have the inherently low Q's associated with Sallen and Key networks [2]. However, if low Q is specified, such circuits would be very economical solutions to the filter problem.

Recommendations for further research to be pursued are as follows:

1. Although the method of Chapter III seems to be a very efficient way of synthesizing the band-reject filters, the resonator used as the basic block for this purpose may not be the best one. Further research may develop a better band-reject resonator.
2. In this dissertation we have not concerned ourselves with sensitivity except in a very specialized sense. It would be interesting to conduct a thorough sensitivity study of the multiple-feedback filter as compared with the cascade structure.

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VITA

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Date of Examination:

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