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A state-space approach to blind estimation of MIMO wireless channels

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A STATE-SPACE APPROACH TO BLIND ESTIMATION OF MIMO WIRELESS CHANNELS

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To my father, memory of my mother,
my sister Hala,
my wife Radwa, and my son Saifeldin

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Notation and Symbols

$A_{M \times N}$: M-row N-column matrix

A^{-1}: Inverse of A

A^T: Transpose of A

A^*: Complex conjugate transpose of A

I_N: Identity matrix of size $N \times N$

$\mathrm{E}\{.\}$: Expectation

A^+: Pseudo-inverse of A

List of Acronyms

ARE:	Algebraic Riccati equation
BER:	Bit error rate
DRE:	Difference Riccati equation
FIR:	Finite impulse response
LQG:	Linear Quadratic Gaussian
LQR:	Linear Quadratic Regulators
MIMO:	Multi-input, multi-output
MMSE:	Minimum mean squared error
PSD:	Power spectral density
SIMO:	Single-Input, Multi-Output
SNR:	Signal to noise ratio
SVD:	Singular value decomposition
WSS:	Wide-sense stationary
ZF:	Zero-forcing

Abstract

This dissertation focuses on blind channel estimation in wireless communications such that the estimated channel admits the minimum phase property. It assumes only the second order statistics of the transmitted signal at the receive side. Our proposed approach is based on the generalized spectral factorization because of the deficient normal rank for the power spectral density (PSD) function of the received signal. We will show the relationship between the generalized spectral factorization and inner-outer factorizations where the inner is square with smaller size. The inner-outer factorization is in turn related to the generalized Kalman filtering in which the dimension of the input noise processes is greater than the dimension of the output measurement and thus the covariance matrix is always singular. A dual problem is the generalized LQR control in which the dimension of the control input is smaller than the dimension of the controlled output and thus the weighting matrix on control signal is always singular. Iterative algorithms are proposed to obtain stabilizing solutions to algebraic Riccati equations (ARE) associated with the generalized Kalman filtering, LQR control, and spectral factorization. We will show the convergence of the proposed iterative algorithm that provides an effective algorithm for blind channel estimation.

Examples are worked out to illustrate our proposed spectral factorization approach to blind channel estimation with comparisons to the existing method in the literature.

Chapter 1

Introduction

This dissertation continues the existing research in blind channel estimation that is dominated by the subspace method [6, 24, 36, 34]. There is an increased research in blind techniques due to their potential applications to wireless communications which are experiencing fast growth in the past few decades. The classical channel estimation methods require knowledge of the transmitted signals which are known as training signals or pilot tones. The disadvantage of transmitting training signals is that it consumes the precious bandwidth up to 50% [34]. For this reason, blind channel estimation is preferred in which the receiver has no knowledge of the input signal. Specifically in blind estimation problems we aim to estimate the channel information based on the received data with the least information about the source signal such as its statistical characteristics. Indeed there is an extensive work done dealing with blind channel estimation based on the second order statistics (SOS) of the received signals that are noisy in general. For *multi-input/multi-output* (MIMO) channels , the second order statistics are the covariance matrix of the received vector signal, that contains the channel information. It is known that [23, 24] the Toeplitz matrix consisting of the covariance data comprises the signal component and the noise component. Under some reasonable assumptions, the signal component is dependent on the channel information only, and it is separable from the noise component. This is the basic reason why the subspace method is applicable to blind channel estimation. However due to the lack of channel information, none of the transmission zeros of the MIMO channel outside the unit circle, including those at infinity, is identifiable in blind channel estimation. That is, we can only identify channels that are minimum phase without transmission delays in order to obtain a unique channel model. It should be pointed out that the existing estimation algorithms based on the subspace method does not ensure that the identified channel is minimum phase that motivated the research work in this dissertation.

1.1 Dissertation Contributions

Our dissertation work aims to rectify the deficiency of the existing blind channel estimation algorithms which do not ensure the minimum phase or the uniqueness of the channel model. We are motivated by the separability of the signal component from the noise component that implies that the *power spectral density* (PSD) of the signal part is available, which is denoted by $\Phi(z)$. Our approach is to estimate the channel information based on spectral factorization of $\Phi(z)$, due to $\Phi(z) = H(z)\Phi_s(z)H(z)^\sim$ where $\Phi_s(z)$ is the PSD of the transmitted signal that is assumed to be known or white. This approach is different from the existing methods in that the estimated channel is ensured to be minimum phase. However due to the exact separation of the signal component from the noise component in the SOS, $H(z)$ is tall in general. In fact, the taller the better. Hence $\Phi(z)$ does not have full normal rank, giving the rise of difficulty in spectral factorization for $\Phi(z)$. Specifically, the most existing work in spectral factorizations assumes full normal rank for $\Phi(z)$, and there lack effective computational algorithms for spectral factorizations when the PSD matrix does not have full normal rank. Hence in order to approach blind channel estimation using the spectral factorization method, we need first to solve the spectral factorization problem for those PSD matrices having non-full normal rank. In this dissertation work we will show that the aforementioned spectral factorization is closely related to inner-outer factorizations where the inner is square with smaller size, that in turn is related to the generalized Kalman filtering, in which the dimension of the input noise process is smaller than the dimension of the output measurement, and thus the covariance of the observation data is always singular. Therefore we will also study the related generalized Kalman filtering based on which we will develop an efficient iterative algorithm to compute the spectral factors for those PSD matrices with non-full normal ranks. We will prove the convergence of the proposed iterative algorithms.

The contributions of this dissertation are summarized as follows.

- We investigate the generalized Kalman filtering where the dimension of the input process is smaller than the dimension of the output measurement. Using a similar derivation to the standard case, we obtain a similar Riccati equation whose solution provides the Kalman filtering gain. For the stationary Kalman filtering, the associated algebraic Riccati equation (ARE) involves pseudo-inverses, and may contain more than one positive semi-definite solutions with the stabilizing solution the maximum. In fact for stable systems the generalized Kalman filtering problem can be equivalently converted to inner-outer factorizations for non-square stable transfer matrices whose inner is square and has a smaller size that is in turn related to the spectral factorization for PSD matrices whose normal rank is not full. Our approach to generalized Kalman filtering is through tackling the equivalent inner-outer and spectral factorizations, from which we develop an iterative algorithm for computing the stabilizing solution to the ARE associated with the optimization problem. We prove the convergence of the proposed iterative algorithm.

- We study the left spectral factorization problem for PSD matrices that do not have full normal ranks. We show that the spectral factors are outers which are stable and minimum phase. We propose an iterative algorithm to solve the ARE equation associated with the left spectral factor of the PSD matrix $\Phi(z)$. It is known that there may exist more than one positive semi-definite solution to the associated ARE with the minimum solution yielding the left spectral factor. We establish a relationship between the ARE for the generalized Kalman filtering and the ARE for the left spectral factorization. Hence the left spectral factor of the PSD matrix $\Phi(z)$ can be obtained from the inner-outer factorization of the generalized Kalman filter. We will show how to choose the initial conditions for both iterative algorithms that will converge to the required solutions.

- We show the successful application of the left spectral factorization problem to the blind channel estimation and optimal channel equalization. The estimated channel is unique up to factors of unitary matrices. The estimated channel is ensured to be minimum phase without transmission delays that is contrast to the existing subspace method. In addition, these algorithms are numerically efficient compared to the existing algorithms for the channel estimation.

- Because of the duality between the generalized Kalman filtering and the generalized LQR control, we are able to tackle the LQR control in which the dimension of the control input is greater than the dimension of the controlled output, and the weighting matrix on the control signal is singular. Based on the results for the generalized Kalman filtering, we propose a similar solution approach to the generalized LQR control. We develop an iterative algorithm for computing the stabilizing solutions to the generalized LQR control ARE. Also we propose an iterative algorithm for the right spectral factorization problem for PSD matrices that have non-full normal ranks. The duality implies that both algorithms are convergent.

1.2 Background Materials

In this section, we will review some related mathematical materials, and introduce some mathematical notations that are needed in this dissertation.

1.2.1 Basic Notions in Linear Algebra

Denote the set of real/complex numbers by $\mathbf{F} = \mathbf{R}/\mathbf{C}$. A vector space over $\mathbf{F}$ is a nonempty set for which addition and scalar multiplications are closed. An example of such vector space is $\mathbf{F}^n$, which is collection of all n-dimensional column vectors whose entries are in $\mathbf{F}$. A matrix A of size $n \times m$ over $\mathbf{F}$ can be viewed as a linear map from $\mathbf{F}^m$ to $\mathbf{F}^n$. The notions of range space and null space are defined as follows.

Definition 1.1 *Let $A \in \mathbf{F}^{n\times m}$ and $\underline{a}_k$ be its kth column. The range space of A denoted by $\mathcal{R}(A)$ is defined as the subspace spanned by all linear combinations of the columns of A, i.e.*

$$\mathcal{R}(A) = \left\{ \underline{y} \in \mathbf{F}^n | \underline{y} = \sum_{k=1}^{m} \alpha_k \underline{a}_k \right\} \tag{1.1}$$

where $\{\alpha_k\}$ range over $\mathbf{F}$.

By definition, $\mathcal{R}(A)$ is a vector space spanned by the linearly independent column vectors of A as the basis, which is a closed subspace of $\mathbf{F}^n$

Definition 1.2 *Let $A \in \mathbf{F}^{n\times m}$. The null space of A denoted by $\mathcal{N}(A)$ is defined as*

$$\mathcal{N}(A) = \{\underline{x} \in \mathbf{F}^m | A\underline{x} = 0\} \tag{1.2}$$

Similarly, $\mathcal{N}(A)$ is a vector space composed of all vectors $\underline{x} \in \mathbf{F}^m$ such that $A\underline{x} = 0$, which is a closed subspace of $\mathbf{F}^m$. Based on the above definitions, we can examine the existence of the solution to the system of equations $\underline{y} = A\underline{x}$, where $\underline{y} \in \mathbf{F}^n$ and $\underline{x} \in \mathbf{F}^m$. It can be easily shown that there exists a solution $\underline{x} \in \mathbf{F}^m$, if and only if $\underline{y} \in \mathcal{R}(A)$; and the solution is unique, if and only if $\mathcal{N}(\mathcal{A}) = \emptyset$.

A widely used matrix decomposition is singular value decomposition (SVD). For any matrix $A \in \mathbf{F}^{n\times m}$, there exists a SVD $A = USV^*$ where $U \in \mathbf{F}^{n\times n}$ and $V \in \mathbf{F}^{m\times m}$ are unitary matrices, and $S \in \mathbf{F}^{n\times m}$ contains nonzero elements $\{\sigma_k\}_{k=1}^{\rho}$ on the diagonal in descending order where $\rho = min\,\{n,m\}$. Another important decomposition is Schur decompositions. For any square matrix A, there exists Schur decomposition $A = QR_AQ^*$ where Q is a unitary matrix, and R_A is an upper triangular matrix with eigenvalues of A on the diagonal. The next lemma (cf. [46], page 37) gives Schur decomposition for positive semi-definite matrices.

Lemma 1.1 *Let $\Theta_1 = \Theta_1^* \geq 0$ and $\Theta_3 = \Theta_3^* \geq 0$, which may not have the same dimension. If*

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{bmatrix} \geq 0,$$

then $\nabla_1 = \Theta_1 - \Theta_2^*\Theta_3^+\Theta_2 \geq 0$, $\nabla_3 = \Theta_3 - \Theta_2\Theta_1^+\Theta_2^* \geq 0$, and

$$\mathcal{R}(\Theta_2^*) \subseteq \mathcal{R}(\Theta_1), \quad \mathcal{R}(\Theta_2) \subseteq \mathcal{R}(\Theta_3), \tag{1.3}$$

There hold Schur decompositions

$$\Theta = \begin{bmatrix} I & \Theta_2^*\Theta_3^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} \nabla_1 & 0 \\ 0 & \Theta_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ \Theta_3^+\Theta_2 & I \end{bmatrix} \tag{1.4}$$

$$= \begin{bmatrix} I & 0 \\ \Theta_2\Theta_1^{+} & I \end{bmatrix} \begin{bmatrix} \Theta_1 & 0 \\ 0 & \nabla_3 \end{bmatrix} \begin{bmatrix} I & \Theta_1^{+}\Theta_2^{*} \\ 0 & I \end{bmatrix}. \tag{1.5}$$

1.3 Second Order Statistics

In this dissertation, we focus on blind channel estimation based on *second order statistics* (SOS) of the received signal that is a random process in general. Roughly speaking a random process is a sequence of random variables with index being time. The random variable possesses statistical properties, such as mean, variance, and moments. A random process is said to be *stationary*, if all its statistical properties do not change with time. Other processes are called non-stationary. Stationary random processes require very stringent conditions, which often do not hold in practice. A class of random processes satisfying less stringent conditions than stationary is known as wide-sense stationary (WSS) [26, 27]. Assume that the discrete time signal $y(t)$ is a complex sequence of random variables with zero mean. Its mean is assumed to satisfy

$$\mathrm{E}\{y(t)\} = 0, \quad t = 0, \pm 1, \pm 2, \ldots \tag{1.6}$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator. Its *autocovariance sequence* (ACS) is given by

$$r_y(k;t) = \mathrm{E}\{y(t)y^*(t-k)\}, \quad t = 0, \pm 1, \pm 2, \ldots \tag{1.7}$$

where $(\cdot)^*$ is the conjugate of a sequence and conjugate transpose of a vector or matrix. If the above expression is independent of n, i.e., $r_y(k;t) = r_y(k)$, then the random process $\{y(t)\}$ is called WSS.

In the MIMO WSS case, the received signal is given by $\underline{y}(t)$ and its covariance data is given by

$$R_y(k) = \mathrm{E}\left\{\underline{y}(t)\underline{y}^*(t-k)\right\}, \quad t = 0, \pm 1, \pm 2, \ldots \tag{1.8}$$

which is a matrix. There holds $R_y(-k) = R_y^*(k)$. Define the Toeplitz matrix as

$$T_y(m) = \mathrm{E}\left\{\begin{bmatrix} \underline{y}(t-1) \\ \underline{y}(t-2) \\ \vdots \\ \vdots \\ \underline{y}(t-m) \end{bmatrix} \begin{bmatrix} \underline{y}^*(t-1) & \underline{y}^*(t-2) & \ldots & \underline{y}^*(t-m) \end{bmatrix}\right\} \tag{1.9}$$

$$= \begin{bmatrix} R(0) & R(1) & R(2) & \ldots & R(m-1) \\ R(-1) & R(0) & R(1) & \ldots & R(m-2) \\ R(-2) & R(-1) & R(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & R(0) & R(1) \\ R(-(m-1)) & \ldots & R(-2) & R(-1) & R(0) \end{bmatrix} \tag{1.10}$$

It is seen that Toeplitz matrix has the same block element along each block diagonal with size the same as $\underline{y}(t)$.

We assume the following. First, the source signal sequence $\{\underline{s}(t)\}$ is a random WSS process and its second order statistics is known. It has zero mean and its autocorrelation matrix is assumed to be full rank. Second, The noise sequence $\{v(t)\}$

is independent of the source signal. It has zero mean, is white, and has the covariance matrix $\sigma_v^2 I$.

1.3.1 Power Spectral Density

The *power spectral density* (PSD) of a random signal $\underline{y}(n)$ is the discrete time Fourier transform (DTFT) of its ACS

$$\Phi_y(\omega) = \sum_{k=-\infty}^{\infty} R_y(k) e^{-j\omega k} \tag{1.11}$$

In practice, it is difficult to compute the PSD using equation (1.11) because it requires infinite ACS terms to be calculated which is not possible. Instead we can use an approximate PSD formula as follow:

$$\widehat{\Phi}_y^{(N)}(\omega) = \mathrm{E}\left\{ \frac{1}{N} \left(\sum_{t=0}^{N-1} \underline{y}(t) e^{-j\omega t} \right) \left(\sum_{t=0}^{N-1} \underline{y}(t) e^{-j\omega t} \right)^* \right\} \tag{1.12}$$

In the above equation we use only N signal samples which is possible. The convergence of $\Phi_y^{(N)}(\omega)$ to $\Phi_y(\omega)$ as $N \to \infty$ is given in the next theorem [31].

Theorem 1.1 *Suppose that the random signal $\underline{y}(t)$ has a finite averaged power. Then it admits the PSD as defined in (1.11). Let $\Phi_y(\omega)$ be continuous over $[0, 2\pi]$ and $\Phi_y(0) = \phi_y(2\pi)$. Define $\widehat{\Phi}_y^{(N)}(\omega)$ as in (1.12), then*

$$\lim_{N\to\infty} \widehat{\Phi}_y^{(N)}(\omega) = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \left(\sum_{t=0}^{N-1} \underline{y}(t) e^{-j\omega t} \right) \left(\sum_{t=0}^{N-1} \underline{y}(t) e^{-j\omega t} \right)^* \right\} = \Phi_y(\omega) \tag{1.13}$$

for all $[0, 2\pi]$ which means that $\widehat{\Phi}_y^{(N)}(\omega)$ converges uniformly to $\Phi_y(\omega)$.

Theorem (1.1) shows an important property of the PSD:

$$\Phi_y(\omega) \geq 0 \;\; \forall \omega \tag{1.14}$$

That is, PSDs are positive real functions of frequency, even though $\Phi_y(\omega) = \Phi_y(-\omega)$ may not hold in the case of complex signals while it is true in case of the real signals.

1.3.2 State Space Description

Assume that G is a finite-dimensional discrete-time linear time-invariant system. Then G admits a state space description as follow:

$$\underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t), \quad x(0) = x_0 \tag{1.15}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t), \quad t \geq 0 \tag{1.16}$$

where $\underline{x}(t) \in \mathbf{F}^{n\times 1}$ is the *system state variable*, $\underline{x}(0)$ is the *initial condition*, $\underline{u}(t) \in \mathbf{F}^{m\times 1}$ is the *system input*, and $\underline{y}(t) \in \mathbf{F}^{p\times 1}$ is the *system output.* Hence $A \in \mathbf{F}^{n\times n}$, $B \in \mathbf{F}^{n\times m}$, $C \in \mathbf{F}^{p\times n}$ and $D \in \mathbf{F}^{p\times m}$. The transfer matrix from $\underline{u}(n)$ to $\underline{y}(n)$ is defined as

$$Y(z) \;\;=\;\; G\,(z)\,U\,(z)$$

where $U(z)$ and $Y(z)$ are the $\mathcal{Z}$-transform of $\underline{u}(n)$ and $\underline{y}(n)$ respectively with zero initial conditions. Then $G(z)$ admits a state-space realization

$$G(z) = D + C(zI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1.17}$$

The four-tuple matrix (A, B, C, D) is called a realization of $G(z)$.

1.3.3 Important Concepts in Linear System Theory

In this section, we will review some important concepts and notions in linear system theory [7, 45].

Definition 1.3 *The state space system as in (1.15) or the pair* (A, B) *is reachable if for some* $\ell > 0$*, the initial state* $\underline{x}_0$*, and the final state* $\underline{x}_m$*, there exists a bounded control input* $\{\underline{u}(t)\}_{t=0}^{\ell-1}$ *such that* $\underline{x}(\ell) = \underline{x}_m$*.*

Reachability is a system property. It can be shown that (A, B) is reachable, if and only if the eigenvalues of $\lambda_i(A + BF)$ can be arbitrarily assigned through suitable design of the state feedback gain F.

Definition 1.4 *The state space system as described in (1.15) and (1.16) or simply* (C, A) *is observable if for some* $\ell > 0$*, the initial state* x_0 *can be reconstructed uniquely for some* $\ell > 0$ *based on the time history of the input* $\{\underline{u}(t)\}_{t=0}^{\ell-1}$ *and the output* $\left\{\underline{y}(t)\right\}_{t=0}^{\ell-1}$*.*

Also the observability is a system property. It can be shown that (C, A) is observable, if and only if the eigenvalues $\lambda_i(A + LC)$ can be arbitrarily assigned through suitable design of the state estimation gain L.

Reachability and observability of a state-space system depend on its realization. There is a relation between reachability, observability, and minimality of the realization.

Definition 1.5 *A state-space realization* (A, B, C, D) *is said to be minimal if the state vector has the smallest possible dimension among all the realizations for the same system.*

Theorem 1.2 *A state-space realization* (A, B, C, D) *is said to be minimal, if and only if* (A, B) *is reachable and* (C, A) *is observable.*

Another important concept is the stability. We will give its definition as below.

Definition 1.6 *An unforced dynamic system* $\underline{x}(t+1) = A\underline{x}(t)$ *is said to be stable if the solution* $\underline{x}(t)$ *satisfies* $\underline{x}(t) \to 0$ *as* $t \to \infty$ *for an arbitrary initial condition* $\underline{x}_0$.

It follows that stability is equivalent to the condition that all eigenvalues of A are strictly inside the unit circle. Clearly, if A is a stability matrix, then $G(z)$ as in (1.17) is stable. Next, we will give a definition for the stabilizability and detectability which are parallel notations to reachability and observability.

Definition 1.7 *The state space system as described in (1.15) or the pair* (A, B) *is stabilizable if there exists a state feedback law* $\underline{u}(t) = F\underline{x}(t)$ *such that the overall system is stable, i.e.,* $A + BF$ *is a stability matrix. The pair* (C, A) *is said to be detectable if there exists an output injection* $\underline{v}(t) = L\underline{y}(t)$ *such that the overall system is stable, i.e.,* $A + LC$ *is a stability matrix.*

Stabilizability and detectability can be tested by using the Lyapunov method as shown in the next theorem.

Theorem 1.3 *Suppose that* (A, B) *is stabilizable. Then a matrix* A *is a stability matrix, if and only if there exists a unique positive semi-definite hermitian matrix* P *that satisfies the Lyapunov equation*

$$P = APA^* + BB^* \tag{1.18}$$

Suppose that (C, A) is detectable. Then A is a stability matrix, if and only if there exists a unique positive semi-definite hermitian matrix Q that satisfies the Lyapunov equation

$$Q = A^* Q A + C^* C. \tag{1.19}$$

Finally we introduce the notion of strictly minimum phase. The transfer matrix $G(z)$ is strict minimum phase, if and only if

$$\operatorname{rank}\left\{\left[\begin{array}{c|c} A - zI & B \\ \hline C & D \end{array}\right]\right\} = n + \min\{p, m\} \ \ \forall \ |z| \geq 1. \tag{1.20}$$

1.4 Problem Formulation

We consider a communication system that has P transmitter and M receiver antennas. Let $s^{(m)}(t)$ be the symbol from the transmitter m at time tT, where T is the symbol duration and $m = 1, \cdots, P$. The information sequences are assumed to be white and its variance or power is known. We assume that $s^{(m)}(t)$ and $s^{(\ell)}(t)$ are independent whenever $m \neq \ell$.

As shown in Fig. 1.1, the received signal at the ith antenna is

$$y_i(t) = \sum_{m=1}^{P} \sum_{n=-\infty}^{\infty} s^m(n) h_i^{(m)}(t - nT) + w_i(t) \tag{1.21}$$

where $w_i(t)$ is white noise and $h_i^{(m)}(t)$ is the composite channel response between the mth transmitter and the ith receiver antennas which includes the effects of the emission filter, reception filter, and the channel response. There are two assumptions. The first one is that the channel can be considered FIR with length L. The second

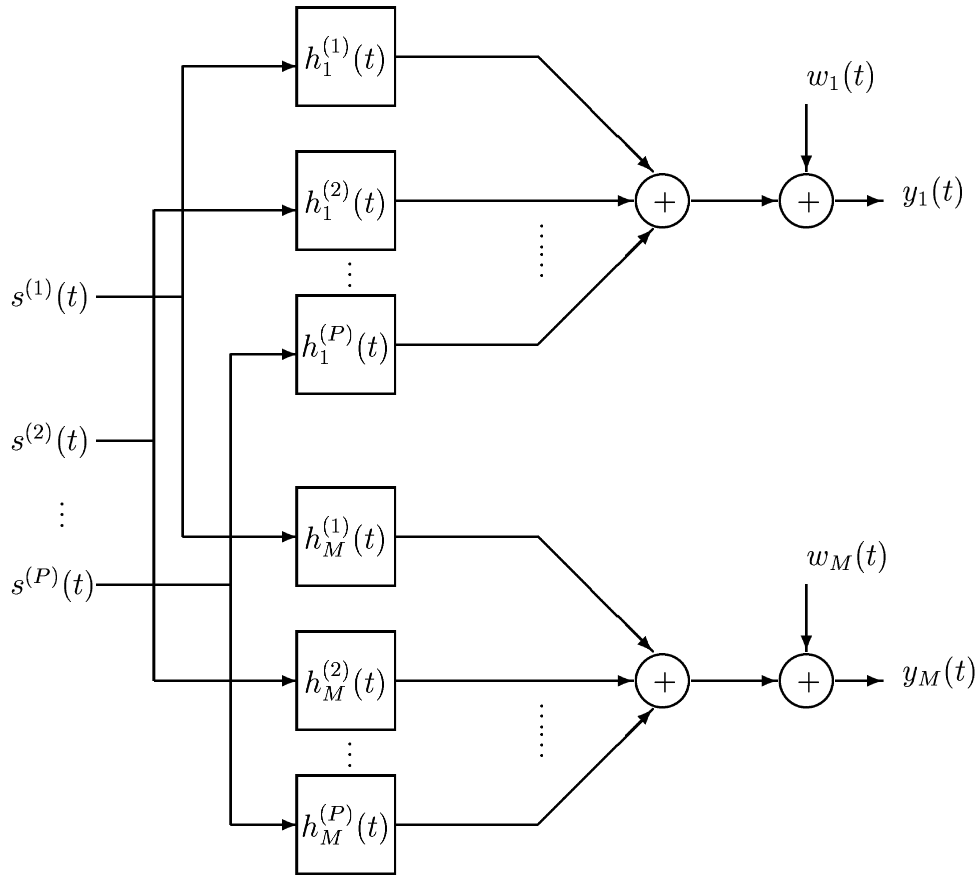

Figure 1.1: Communications system model

one is that the number of receivers M is greater than the number of transmitters P. In addition to the MIMO communication systems which are the ones we are going to investigate throughout this dissertation, we will also consider the one that oversamples the received signal on which most of the existing work in the literature is based.

1.4.1 MIMO Data Communication Systems

Due to the first assumption, $h_i(t)$ in (1.21) is zero outside $[0, (L-1)T]$. Therefore we have

$$y_i(t) = \sum_{m=1}^{P} \sum_{k=0}^{L-1} s^{(m)}(t-k) h_i^{(m)}(k) + w_i(t) \tag{1.22}$$

Let L be the successive samples of the received signals represented by the vector of size L:

$$\underline{y}_i(t) = \begin{bmatrix} y_i(t) & y_i(t-1) & \cdots & y_i(t-L+1) \end{bmatrix}^T \tag{1.23}$$

Let $\underline{s}^{(m)}(t) = \begin{bmatrix} s^{(m)}(t) & s^{(m)}(t-1) & \cdots & s^{(m)}(t-2L+2) \end{bmatrix}^T$ be a vector with dimension $(2L-1) \times 1$ and $\underline{w}_i(t) = \begin{bmatrix} w_i(t) & w_i(t-1) & \cdots & w_i(t-L+1) \end{bmatrix}^T$ be a vector of size L. Let the channel coefficients $\underline{h}_i^{(m)}$ be a vector defined by

$$\underline{h}_i^{(m)} = \begin{bmatrix} h_i(0) & h_i(1) & \cdots & h_i(L-1) \end{bmatrix}^T. \tag{1.24}$$

Therefore, we can write $\underline{y}_i(t)$ as

$$\underline{y}_i(t) = \sum_{m=1}^{P} H_i^{(m)} \underline{s}^{(m)}(t) + \underline{w}_i^{(m)}(t) \tag{1.25}$$

where $H_i^{(m)}$ has dimension $L \times (2L-1)$ given by

$$H_i^{(m)} = \begin{pmatrix} h_i^{(m)}(0) & \cdots & h_i^{(m)}(L-1) & 0 & \cdots & 0 \\ 0 & h_i^{(m)}(0) & \cdots & h_i^{(m)}(L-1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & h_i^{(m)}(0) & \cdots & h_i^{(m)}(L-1) \end{pmatrix} \tag{1.26}$$

Note that $H_i^{(m)}$ is a wide Toeplitz matrix called *filtering matrix* associated with the ith antenna and mth user. Packing the received signal as follows:

$$\underline{y}(t) = \begin{bmatrix} \underline{y}_1(t)^T & \underline{y}_2(t)^T & \cdots & \underline{y}_M(t)^T \end{bmatrix}^T. \tag{1.27}$$

Similarly we can form an overall noise vector as follows:

$$\underline{w}(t) = \left[\begin{array}{cccc} \underline{w}_1(t)^T & \underline{w}_2(t)^T & \cdots & \underline{w}_M(t)^T \end{array} \right]^T. \tag{1.28}$$

Hence the overall received signal $\underline{y}(t)$ from all the transmitter antennas can be written as

$$\underline{y}(t) = H\underline{s}(t) + \underline{w}(t) \tag{1.29}$$

where $\underline{s}(t)$ has dimension $P(2L-1) \times 1$ given by

$$\underline{s}(t) = \left[\begin{array}{cccc} \underline{s}^{(1)}(t)^T & \underline{s}^{(2)}(t)^T & \cdots & \underline{s}^{(P)}(t)^T \end{array} \right]^T \tag{1.30}$$

and H is the channel matrix that contains multiple copies of the coefficients we wish to estimate. Its dimension is $LM \times P(2L-1)$ and it is given by

$$H = \left[\begin{array}{cccc} H_1^{(1)} & H_1^{(2)} & \cdots & H_1^{(P)} \\ H_2^{(1)} & H_2^{(2)} & \cdots & H_2^{(P)} \\ \vdots & \vdots & \vdots & \vdots \\ H_M^{(1)} & H_M^{(2)} & \cdots & H_M^{(P)} \end{array} \right] \tag{1.31}$$

1.4.2 Oversampling The Received Signals

Most of the work reported in the existing literature is based on oversampling the signal received on each antenna by an integer J assuming several measurements can be performed during the symbol duration period T. At the ith receiver antenna, we form the J signals $x_{ij}(n), 1 \leq j \leq J$, as

$$y_{ij} = y_i\left(tT + \frac{(j-1)}{J}T\right). \tag{1.32}$$

Hence, we obtain MJ channels where each receiver antenna has J virtual channels. Due to the first assumption, $h_i(t)$ in (1.21) is zeros outside $[0, (L-1)T]$. Therefore we have

$$y_{ij}(t) = \sum_{m=1}^{P} \sum_{k=0}^{L-1} s^{(m)}(t-k) h_{ij}^{(m)}(k) + w_{ij}(t) \tag{1.33}$$

where

$$\begin{aligned} h_{ij}^{(m)} &= h_i^m \left(kT + \frac{(j-1)}{J} T \right) && (1.34) \\ w_{ij}^{(m)} &= w_i^m \left(kT + \frac{(j-1)}{J} T \right) && (1.35) \end{aligned}$$

Let L successive samples of the received signal be represented by vector of size L:

$$\underline{y}_{ij}(t) = \begin{bmatrix} y_{ij}(t) & y_{ij}(t-1) & \cdots & y_{ij}(t-L+1) \end{bmatrix}^T \tag{1.36}$$

Let $\underline{s}^{(m)}(t) = \begin{bmatrix} s^{(m)}(t) & s^{(m)}(t-1) & \cdots & s^{(m)}(t-2L+2) \end{bmatrix}^T$ be a vector with dimension $(2L-1) \times 1$ and $\underline{w}_{ij}(t) = \begin{bmatrix} w_{ij}(t) & w_{ij}(t-1) & \cdots & w_{ij}(t-L+1) \end{bmatrix}^T$ be a vector with dimension $(L-1) \times 1$. Let the channel coefficients $\underline{h}_{ij}^{(m)}$ be a vector with dimension $L \times 1$:

$$\underline{h}_{ij}^{(m)} = \begin{bmatrix} [h_{ij}(0) & h_{ij}(1) & \cdots & h_{ij}(L-1) \end{bmatrix}^T. \tag{1.37}$$

Therefore, we can write $\underline{y}_{ij}(n)$ as follow

$$\underline{y}_{ij}(t) = \sum_{m=1}^{P} H_{ij}^{(m)} \underline{s}^{(m)}(t) + \underline{w}_{ij}^{(m)}(t) \tag{1.38}$$

where

$$H_{ij}^{(m)} = \begin{pmatrix} h_{ij}^{(m)}(0) & \cdots & h_{ij}^{(m)}(L-1) & 0 & \cdots & 0 \\ 0 & h_{ij}^{(m)}(0) & \cdots & h_{ij}^{(m)}(L-1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & h_{ij}^{(m)}(0) & \cdots & h_{ij}^{(m)}(L-1) \end{pmatrix} \tag{1.39}$$

has dimension $L \times (2L-1)$. Note that $H_{ij}^{(m)}$ is a wide Toeplitz matrix called *filtering matrix* associated with the ith receiver antenna and jth polyphase component with input from the mth transmitter antenna. Packing all the samples of all the received signals at time index t together results in an $LMJ \times 1$ vector as

$$\underline{y}(t) = \begin{bmatrix} \underline{y}_{11}(t)^T & \underline{y}_{12}(t)^T & \cdots & \underline{y}_{MJ}(t)^T \end{bmatrix}^T . \tag{1.40}$$

Similarly we can form an $LMJ \times 1$ overall noise vector as follow

$$\underline{w}(t) = \begin{bmatrix} \underline{w}_{11}(t)^T & \underline{w}_{12}(t)^T & \cdots & \underline{w}_{MJ}(t)^T \end{bmatrix}^T . \tag{1.41}$$

Hence the overall received signal $\underline{y}(t)$ can be written as

$$\underline{y}(t) = H\underline{s}(t) + \underline{w}(t) \tag{1.42}$$

where $\underline{s}(t)$ is given by

$$\underline{s}(t) = \begin{bmatrix} \underline{s}^{(1)}(t)^T & \underline{s}^{(2)}(t)^T & \cdots & \underline{s}^{(P)}(t)^T \end{bmatrix}^T \tag{1.43}$$

that is a vector of size $P(2L-1)$. Similarly H is the channel matrix that contains multiple copies of the coefficients we wish to estimate. Its dimension is $LMJ \times P(2L-1)$ and it is given by

$$H = \begin{pmatrix} H_{11}^{(1)} & \cdots & H_{11}^{(P)} \\ H_{12}^{(1)} & \cdots & H_{12}^{(P)} \\ \vdots & \vdots & \vdots \\ H_{1J}^{(1)} & \cdots & H_{1J}^{(P)} \\ H_{21}^{(1)} & \cdots & H_{21}^{(P)} \\ \vdots & \vdots & \vdots \\ H_{MJ}^{(1)} & \cdots & H_{MJ}^{(P)} \end{pmatrix} \tag{1.44}$$

As a result we have a virtual MIMO system.

1.4.3 Blind Channel Estimation

A goal for blind channel estimation of multiuser communication systems is to estimate the MPL unknown channel coefficients from the sole observations of $\underline{y}(t)$ given in (1.27). However, the goal of blind channel estimation for the oversampling formulation is to estimate the $MPJL$ unknown channels coefficients from the sole observations of $\underline{y}(t)$ given in (1.40). It has been shown in [8] that the number of unknown parameters for identification using non-integer fractional oversampling is larger than that using integer fractional sampling. Therefore the non-integer fractional sampling has no reduction in identification cost. As in the methods given in [20, 37, 38] for the SIMO case and in [8, 9, 19] for the MIMO case, the estimation is based on $LMJ \times LMJ$ (oversampling case) or $LM \times LM$ (multiuser case). In either case the autocorrelation matrix $R_{\underline{y}}$ of the measurement vector $\underline{y}(t)$ is employed which is given by

$$R_y = \mathrm{E}\{\underline{y}(t)\underline{y}(t)^*\} \tag{1.45}$$

The additive measurement noise is assumed to be independent of the transmitted sequences. Hence, from (1.40) and (1.27) we can express R_y as follows:

$$R_y = HR_sH^* + R_w \tag{1.46}$$

where $R_s = E(\underline{s}(t)\underline{s}(t)^*) = \sigma_s^2 I$ and $R_w = E(\underline{w}(t)\underline{w}(t)^*) = \sigma_w^2 I$ denoting respectively the autocorrelation matrices of the transmitted symbol vector $\underline{s}(t)$ and the autocorrelation matrices of the measurement noise vector $\underline{w}(t)$. Blind channel estimation based on SOS has received considerable attention since the publication of [37]. It has been studied for both SIMO and MIMO cases. In the next section we will summarize blind SIMO channel identification and blind MIMO channel identification methods.

1.5 Overview for Blind Channel Estimation

Blind channel estimation has received great interest due to their potential applications to wireless communications. Blind channel estimation using higher order statistics can be found in [5, 15, 28, 32]. The approach based on the second order statistics has received considerable attention since the publication of [37] that considers a SIMO communication system. The main idea in [37] makes the use of the fact that the channel vector is in a one dimensional subspace of the received signal that yields very accurate channel estimation results.

The general subspace method can handle MIMO channels. It belongs to the family of the second order statistics and it exploits the subspace structure of the received signals. The subspace algorithms have the advantage that the channel estimates can often be obtained in a closed form from optimizing a quadratic cost function [34]. However the subspace method has some disadvantages. The first disadvantage is that it depend on the property that the channel lies in a subspace, and thus it may not be robust against modeling errors especially when the channel matrix H is close to be non-full in its column rank. Another disadvantage is that it is often more computationally expensive.

Based on the idea of the spectral redundancy in a cyclostationary signal [11, 12, 13], a second-order statistical property can be used to identify a nonminimum phase channel. However it requires oversampling, i.e., more than one samples are needed in a single symbol period. Therefore, if the received signal is sampled at the baud rate, only minimum-phase channels can be identified from the second-order statistics of the received signal. On the other hand, if the sampling rate is higher than the baud rate, the resulting sequence is wide sense cyclostationary and the nonminimum phase channels can be identified. Most of the literature work is based on sampling the received signal with sampling frequency higher than the baud rate which requires extra cost and complexity. It was shown in the previous section that the number of the estimated parameters in the oversampling case is the number of the estimated parameters in the multiuser case multiplied by the oversampling parameter which increase the complexity.

It was shown in [36, 39] that the channel can be uniquely identified up to a constant factor from the autocorrelation function of the received data if and only if subchannels are coprime. The classical subspace method which based on the noise subspace of the received data space-time correlation matrix was shown in [23, 24]. The basic idea is to force the signal space to have the block Toeplitz form. This method deals with the SIMO case. We can summarize this method as follows [23, 24, 34]: Let $\underline{h}_i = \begin{bmatrix} h_i^{(1)} & \cdots & h_i^{(M)} \end{bmatrix}^T$ and $\underline{h}^T = \begin{bmatrix} \underline{h}_0^T & \cdots & \underline{h}_L^T \end{bmatrix}$. Suppose that $\underline{V} = \begin{bmatrix} v_1 & \cdots & v_{ML} \end{bmatrix}^T$ is in the orthogonal complement of the range space of $T_L(\underline{h})$, i.e.

$$\begin{bmatrix} \underline{V}_1^* & \cdots & \underline{V}_L^* \end{bmatrix} \begin{pmatrix} \underline{h}_0 & \cdots & \underline{h}_L & & \\ & \ddots & & \ddots & \\ & & \underline{h}_0 & \cdots & \underline{h}_L \end{pmatrix} = 0. \tag{1.47}$$

where $\underline{V}_k$ is the kth subvector of $\underline{V}$, $\underline{V}_k = \begin{bmatrix} v_{(k-1)M+1} & \cdots & v_{Mk} \end{bmatrix}^T$. Equation (1.47) can be written as a linear equation with respect to the channel parameter $\underline{h}$ as follow,

$$\begin{bmatrix} \underline{h}_0^* & \cdots & \underline{h}_L^* \end{bmatrix} \cdot \begin{pmatrix} \underline{V}_1 & \cdots & \underline{V}_L & & \\ & \ddots & & \ddots & \\ & & \underline{V}_1 & \cdots & \underline{V}_L \end{pmatrix} = \underline{h}^* T_{(L+1)}(\underline{V}) = 0. \tag{1.48}$$

Equation (1.48) can be used to identify the channel vector $\underline{h}$ provided that it has a unique solution. Authors of [23, 24] gave the following theorem.

Theorem 1.4 *Let $\mathcal{N}(\underline{h}) = span\{\underline{V}_1, \cdots, \underline{V}_L\}$ be the orthogonal complement of the column space of $T_L(\underline{h})$. For any $\underline{h}$ and $\underline{h}_1$ satisfying that subchannels are coprime, $\mathcal{N}(\underline{h}) = \mathcal{N}(\underline{h}_1)$ if and only if $\underline{h} = \alpha \underline{h}_1$. In addition, for all i, $\underline{h}$ satisfies the following equation*

$$T^*_{(L+1)}(\underline{V}_i)\underline{h} = 0. \tag{1.49}$$

Having the estimated basis $\hat{V}_i$ of the orthogonal complement of $T_L(\underline{h})$, channel parameters identification can be achieved by the following minimization

$$\hat{\underline{h}} = arg \min \sum_i \underline{h}^* T_{(L+1)} \left(\underline{V}_i\right) T^*_{(L+1)} \left(\underline{V}_i\right) \underline{h}. \tag{1.50}$$

From the above theorem, the channel parameters estimation can be achieved by estimating the orthogonal complement of the $T_L(\underline{h})$ first. This can be accomplished using the signal-noise space decomposition approach. We have

$$\hat{R}_y = T_L(\underline{h}) \hat{R}_s T_L^*(\underline{h}) + \sigma^2 I \tag{1.51}$$

The SVD of $\hat{R}_y$ can be given in the following form

$$\hat{R}_y = \hat{U} \mathtt{diag} \left\{ \lambda_1^2 + \sigma^2, \cdots, \lambda_{2L}^2 + \sigma^2, \sigma^2, \cdots, \sigma^2 \right\} \hat{U}^* \tag{1.52}$$

where λ_i are the singular values of $T_L(\underline{h})$. If the subchannels are coprime, the orthogonal complement of the range space of $T_L(\underline{h})$ is given by the singular vectors $\{\hat{u}_i\}_{i=2L+1}^{ML}$ of $\hat{R}_y$ associated with the singular value σ^2. The advantage of this method is its capability to provide the exact channel when there is little noise, even for a relatively small number of symbols. However, it suffers when the channel matrix becomes ill-conditioned due to common zeros of the different channels. Another disadvantage of the subspace methods is that it requires the knowledge of channel model orders. Therefore, it is sensitive to errors in channel order estimates. The method for SIMO channels given in [23] is generalized to the identification of MIMO FIR channels in [19, 41]. Also, the necessary and sufficient condition for MIMO FIR systems to be identified is established up to an ambiguity unitary matrix using second-order statistics.

A liner prediction based approach (LPA) was first given in [29, 30] and was later generalized in [2]. It is found that the linear prediction approach is robust when the channel order is over-estimated. The outer-product decomposition algorithm (OPDA) was presented in [8] to solve the over estimated channel order issue and generates superior identification results than the LPA algorithm. However, this algorithm suffers from high computational complexity.

Chapter 2

Generalized Kalman Filtering and Inner-Outer Factorizations

2.1 Introduction

In this chapter we investigate two seemingly unrelated problems which are generalized Kalman filtering, and inner-outer factorizations. As we recall that blind channel estimation is dominated by the subspace method. For MIMO channels, the second order statistics are the covariance matrices of the MIMO output signal, that contains the channel information. It is known that the Toeplitz matrix consisting of the covariance data comprises the signal component and the noise component. Under reasonable assumptions, the signal component is dependent on the channel information only, and it is separable from the noise component. The separability of the signal component from the noise component implies that the PSD of the signal part $\Phi(z)$ is available. Because of $\Phi(z) = H(z)\Phi_s(z)H(z)^\sim$, we can estimate the channel information based on spectral factorization of $\Phi(z)$. However ambiguities exist. We can only estimate the channel with all poles/zeros inside the unit circle based on PSD of the signal. For simplicity it is assumed that the signal is white, i.e. $\Phi_s(z) = \sigma_s^2 I \ \ \forall z$. Although there is an extensive work done dealing with blind channel estimation based on the second order statistics of the received signals, none of the existing work guarantees that the estimated channel is minimum phase. Our approach ensures that the estimated channel is minimum phase, that is in contrast to the existing work in the field.

Let the transfer matrix $\Phi(z)$ of size $p \times p$ has the form given in (1.11). It follows that $\Phi(z)$ is a hermitian matrix for any z on the unit circle. If in addition $\Phi(z) \geq 0$ $\forall \ |z| = 1$, then $\Phi(z)$ qualifies a PSD. Let $\{R_k\}$ be the *inverse discrete-time Fourier transform* (IDTFT) of $\Phi(z)$. Then $\{R_k\}$ are the covariance sequence. Let the normal rank of $\Phi(z)$ be $r < p$. We are interested in the spectral factorization

$$\Phi(z) = W_L(z) W_L(z)^{\sim} \tag{2.1}$$

where $W_L(z)$ has size $p \times r$, and more importantly it is causal, stable, and strict minimum phase. In other words, all poles and zeros of $W_L(z)$ are strictly inside the unit circle. In this case $W_L(z)$ is called the left spectral factor of $\Phi(z)$. Extensions can be made for spectral factors to include poles and zeros on the unit circle. But for the application to channel estimation, we shall not do so in this dissertation. Instead we assume that $\Phi(z)$ is a bounded hermitian positive matrix with rank r for all z on the unit circle, which excludes the possibilities of poles and zeros on the unit circle for the spectral factors. It is worth to pointing out that most of the existing work on spectral factorizations assume that $r = p$, and there lack effective computational algorithms for spectral factorizations in the case of $0 < r < p$. For applications to blind channel estimation, $W_L(z)$ determines the estimated channel model. Specifically once the spectral factor $W_L(z)$ is available, $H(z) = W_L(z)/\sigma_s$, which is the channel to be estimated. In practice only estimated PSD $\widehat{\Phi}(z)$ is available, and thus $\widehat{H}(z) = W_L(z)/\sigma_s$, is the estimated channel.

For multipath MIMO channels, the channel transfer matrix $H(z)$ is a matrix polynomial of z^{-1}. Hence it admits a state-space realization (A, B, C, D) or equivalently

$$H(z) = D + C(zI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.2}$$

It is assumed that $H(z)$ has size $p \times m$ with $m < p$. Hence m is the normal rank of the PSD: $\Phi(z) = H(z)H(z)^{\sim}$. In this chapter we will also consider inner-outer factorization $H(z) = H_{\mathrm{o}}(z)H_{\mathrm{i}}(z)$ where $H_{\mathrm{i}}(z)$ is inner of size $m \times m$, and $H_{\mathrm{o}}(z)$ is outer having no pole and zero outside the unit circle, including at infinity. This problem is different from the existing results in that the inners are square of smaller size. We note, however, that $H(z)$ may have zeros strictly outside the unit circle, and its realization is subject to the constraint (due to the transmission delays)

$$0 < \mathrm{rank}\{D\} \le \min\{m, p\} \tag{2.3}$$

where $D \in \mathbf{F}^{p\times m}$. The reason for investigating the inner-outer factorization is that an inner factor of the channel represents ambiguities that do not show up in the PSD of the received signal. Consequently we can only estimate the outer factor of the channel. On the other hand many existing blind channel estimation algorithms do not guarantee minimum phase of the estimated channel. Thus how to compute the outer part of a given transfer matrix $H(z)$ (that can be the estimated channel model using some existing blind channel estimation algorithm) becomes an important issue in blind channel estimation in order to eliminate the ambiguity part, or the inner factor, of the channel model.

It will be shown that the inner-outer factorization in this chapter has a close relation to the generalized Kalman filtering. In conventional standard Kalman filtering, the dimension of the input noise process is no smaller than the dimension of the output measurement. However in generalized Kalman filtering, the dimension of the input noise process is strictly smaller than the dimension of the output measurement. Thus the covariance matrix of the observation noise is always singular. The inner-outer factorization in this chapter cannot be solved without solving the generalized Kalman filtering. We will thus investigate the generalized Kalman filtering first in the next section. Similar to the standard Kalman filtering, an algebraic Riccati equation (ARE) needs to be solved in order to obtain that optimal linear state estimator. Due to the singularity of the covariance matrix of the observation noise, the existing ARE solvers cannot be applied. Hence an iterative algorithm is proposed for computing the positive semi-definite solution to the corresponding ARE. In Section 2.3 we investigate the problem of the inner-outer factorization. The stabilizing solution to the ARE associated with the generalized Kalman filtering will be used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. In Section 2.4, several numerical examples are employed to illustrate the results obtained in this chapter for computing the outer part of the given channel. The numerical examples also point out the convergence issues in the proposed iterative algorithm for solving the required ARE, which will be tackled together with the spectral factorization in Chapter 4. Section 2.5 concludes the chapter.

2.2 Generalized Kalman Filtering

In this section, we study the generalized Kalman filtering, which involves the notion of pseudo-inverses. For a matrix $M \neq 0$, its pseudo-inverse, denoted by M^+, satisfies $MM^+M = M$. In general there are more than one pseudo-inverses. Let $M = USV^*$ be the SVD. One of the pseudo-inverses of M is $M^+ = VS^+U^*$ where S^+ computes inverses of the diagonal elements for only those non-zero singular values.

Given a random process governed by the state-space model

$$\underline{x}(t+1) = A\underline{x}(t) + B\underline{v}(t), \quad \underline{y}(t) = C\underline{x}(t) + D\underline{v}(t), \tag{2.4}$$

where $\underline{x}(0)$ has mean $\underline{0}$ and covariance Σ_0, and $\underline{v}(t)$ is a WSS random process satisfying

$$\mathtt{E}[\underline{v}(t)] = 0, \quad \mathtt{E}[\underline{v}(t+k)\underline{v}^*(t)] = \delta(k)I, \quad \delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases} \tag{2.5}$$

Kalman filtering aims to estimate $\underline{x}(t+1)$ based on observation $\{\underline{y}(k)\}_{k=0}^{t}$ that minimizes the estimation error variance. In our problem setting, the dimension of the input noise $\{\underline{v}(t)\}$ is m, that is smaller than the dimension of the output measurement $\{\underline{y}(t)\}$ which is p. Since $p > m$, the covariance of the observation noise $D\underline{v}(t)$ is always singular which is different from the standard Kalman filtering. The standard Kalman filtering deals with the case when D is "fat" and has full row rank. However, we have a "tall" D, which may not have a full column rank: $\text{rank}\{D\} \leq m$.

The following lemma [3] is important to derive the generalized Kalman filter equations.

Lemma 2.1 *Let $\underline{X}$ and $\underline{Y}$ be two jointly distributed random vectors with*

$$\mathrm{E}\left\{\begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}\right\} = \begin{bmatrix} \underline{m}_x \\ \underline{m}_y \end{bmatrix}, \ \mathrm{cov}\left\{\begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}\right\} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \tag{2.6}$$

and Σ_{yy} is singular. Then the linear MMSE estimator of $\underline{X}$ in terms of $\underline{Y}$ is given by

$$\hat{\underline{X}} = \underline{m}_x + \Sigma_{xy}\Sigma_{yy}^{+}(\underline{Y} - \underline{m}_y) \tag{2.7}$$

The error covariance associated with $\hat{\underline{X}}$ is unconditioned and given by

$$\mathrm{E}\left\{(\underline{X} - \hat{\underline{X}})(\underline{X} - \hat{\underline{X}})^*\right\} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{+}\Sigma_{yx}. \tag{2.8}$$

If $\underline{X}$ and $\underline{Y}$ are jointly Gaussian, then (2.7) is also the MMSE estimate $\hat{\underline{X}} = E\{\underline{X}|\underline{Y}\}$ that is optimal among all (linear and nonlinear) estimators [3].

The following theorem introduces the solution to the generalized Kalman filter.

Theorem 2.1 *Consider linear time-varying state-space process:*

$$\underline{x}(t+1) = A_t\underline{x}(t) + B_t\underline{v}(t), \quad \underline{y}(t) = C_t\underline{x}(t) + D_t\underline{v}(t).$$

Let $\Sigma_0 = \mathrm{E}\{\underline{x}(0)\underline{x}(0)^*\}$ and define $\{\Sigma_t\}, t > 0$ as the solution to the DRE:

$$\begin{aligned} \Sigma_{t+1} &= A_t\Sigma_tA_t^* - (A_t\Sigma_tC_t^* + B_tD_t^*)(D_tD_t^* + C_t\Sigma_tC_t^*)^{+}(A_t\Sigma_tC_t^* + B_tD_t^*)^* + B_tB_t^*, \\ L_t &= -(A_t\Sigma_tC_t^* + B_tD_t^*)(D_tD_t^* + C_t\Sigma_tC_t^*)^{+}, \quad \Sigma_0 \geq 0. \end{aligned} \tag{2.9}$$

Then the optimal linear estimator, or the generalized Kalman filter is given by

$$\underline{\hat{x}}(t+1) = (A_t + L_t C_t)\underline{\hat{x}}(t) - L_t \underline{y}(t), \quad t \geq 0. \tag{2.10}$$

The matrix gain L_t as above is termed as Kalman gain.

Proof: We derive the optimal linear estimator using the first principle similar to [3]. To shorten the notation, the time index is moved to as subscript. We adopt the notation $\Sigma_{0/-1} = \Sigma_0$. The random variable $\begin{bmatrix} \underline{x}_0 \\ \underline{y}_0 \end{bmatrix}$ has mean $\begin{bmatrix} \underline{\bar{x}}_0 \\ C_0 \underline{\bar{x}}_0 \end{bmatrix}$. The covariance matrices can be computed directly as follows:

$$\begin{aligned}
\Sigma_{\underline{x}_0 \underline{x}_0} &= \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][\underline{x}_0 - \underline{\bar{x}}_0]^*\} = \Sigma_{0/-1}, & (2.11)\\
\Sigma_{\underline{x}_0 \underline{y}_0} &= \mathrm{cov}[\underline{x}_0, \underline{y}_0] = \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][\underline{y}_0 - \underline{\bar{y}}_0]^*\} \\
&= \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][C_0 \underline{x}_0 - C_0 \underline{\bar{x}}_0 + D_0 \underline{v}_0]^*\} \\
&= \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][C_0(\underline{x}_0 - \underline{\bar{x}}_0) + D_0 \underline{v}_0]^* \\
&= \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][\underline{x}_0 - \underline{\bar{x}}_0]^*\} C_0^* = \Sigma_{0/-1} C_0^*, & (2.12)\\
\Sigma_{\underline{y}_0 \underline{y}_0} &= \mathrm{E}\{[\underline{y}_0 - \underline{\bar{y}}_0][\underline{y}_0 - \underline{\bar{y}}_0]^*\} \\
&= \mathrm{E}\{[C_0 \underline{x}_0 + D_0 \underline{v}_0 - C_0 \underline{\bar{x}}_0][C_0 \underline{x}_0 + D_0 \underline{v}_0 - C_0 \underline{\bar{x}}_0]^*\} \\
&= \mathrm{E}\{[C_0(\underline{x}_0 - \underline{\bar{x}}_0) + D_0 \underline{v}_0][C_0(\underline{x}_0 - \underline{\bar{x}}_0) + D_0 \underline{v}_0]^*\} \\
&= C_0 \mathrm{E}\{[\underline{x}_0 - \underline{\bar{x}}_0][\underline{x}_0 - \underline{\bar{x}}_0)]^*\} C_0^* + D_0 E\{\underline{v}_0 \underline{v}_0^*\} D_0^* \\
&= C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^*, & (2.13)
\end{aligned}$$

So, the covariance of $\begin{bmatrix} \underline{x}_0 \\ \underline{y}_0 \end{bmatrix}$ is $\begin{bmatrix} \Sigma_{0/-1} & \Sigma_{0/-1} C_0^* \\ C_0 \Sigma_{0/-1} & C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^* \end{bmatrix}$. Applying Lemma

2.1, the optimal linear estimate of $\underline{x}_0$ conditioned on $\underline{y}_0$ has mean and covariance

$$\hat{\underline{x}}_{0/0} = \bar{\underline{x}}_0 + \Sigma_{0/-1} C_0^* (C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^*)^+ (\underline{y}_0 - C_0 \bar{\underline{x}}_0), \tag{2.14}$$

$$\Sigma_{0/0} = \Sigma_{0/-1} - \Sigma_{0/-1} C_0^* (C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^*)^+ C_0 \Sigma_{0/-1}, \tag{2.15}$$

respectively. From (2.4), it follows that $\underline{x}_1$ conditioned on $\underline{y}_0$ is gaussian. Hence by

$$\begin{aligned} \underline{x}_1 &= A_0 \underline{x}_0 + B_0 \underline{v}_0, \\ \underline{y}_0 &= C_0 \underline{x}_0 + D_0 \underline{v}_0, \end{aligned} \tag{2.16}$$

$\bar{\underline{x}}_1 = A_0 \bar{\underline{x}}_0$ and $\bar{\underline{y}}_0 = C_0 \bar{\underline{x}}_0$, and thus

$$\begin{aligned} \Sigma_{\underline{x}_1 \underline{x}_1} &= \mathrm{E}\{[A_0 \underline{x}_0 + B_0 \underline{v}_0 - A_0 \bar{\underline{x}}_0][A_0 \underline{x}_0 + B_0 \underline{v}_0 - A_0 \bar{\underline{x}}_0]^*\} \\ &= A_0 \Sigma_{0/-1} A_0^* + B_0 B_0^* \end{aligned} \tag{2.17}$$

$$\begin{aligned} \Sigma_{\underline{x}_1 \underline{y}_0} &= \mathrm{E}\{[\underline{x}_1 - A_0 \bar{\underline{x}}_0][\underline{y}_0 - C_0 \bar{\underline{x}}_0]^*\} \\ &= [A_0 \underline{x}_0 + B_0 \underline{v}_0 - A_0 \bar{\underline{x}}_0][C_0 \underline{x}_0 + D_0 \underline{v}_0 - C_0 \bar{\underline{x}}_0]^* \\ &= A_0 \Sigma_{0/-1} C_0^* + B_0 D_0^* \end{aligned} \tag{2.18}$$

$$\Sigma_{\underline{y}_0 \underline{x}_1} = C_0 \Sigma_{0/-1} A_0^* + D_0 B_0^* \tag{2.19}$$

$$\Sigma_{\underline{y}_0 \underline{y}_0} = C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^* \tag{2.20}$$

According to Lemma 2.1, $\underline{x}_1$ conditioned on $\underline{y}_0$ has mean and covariance as follows:

$$\begin{aligned} \hat{\underline{x}}_{1/0} &= \bar{\underline{x}}_1 + \Sigma_{\underline{x}_1 \underline{y}_0} \Sigma^+_{\underline{y}_0 \underline{y}_0} (\underline{y}_0 - \bar{\underline{y}}_0) \\ &= A_0 \bar{\underline{x}}_0 + (A_0 \Sigma_{0/-1} C_0^* + B_0 D_0)(C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^*)^+ (\underline{y}_0 - C_0 \bar{\underline{x}}_0) \\ &= A_0 \hat{\underline{x}}_{0/0} + B_0 D_0^* (C_0 \Sigma_{0/-1} C_0^* + D_0 D_0^*)^+ (\underline{y}_0 - C_0 \bar{\underline{x}}_0) \end{aligned} \tag{2.21}$$

Denote $\Sigma_1 = \Sigma_{1/0}$ and recall $\Sigma_0 = \Sigma_{0/-1}$. Its associated estimation error covariance is

$$\Sigma_1 \;=\; (A_0\Sigma_0 A_0^* + B_0 B_0^*) - (A_0\Sigma_0 C_0^* + B_0 D_0)(C_0\Sigma_0 C_0^* + D_0 D_0^*)^+(A_0\Sigma_0 C_0^* + B_0 D_0)^*$$

The above concludes the optimality of the generalized Kalman filter (2.10) for $t = 0$. From (2.4) it follows that $\underline{y}_1$ conditioned on $\underline{y}_0$ is gaussian and has mean and covariance

$$\begin{aligned}
\underline{y}_1 &= C_1\underline{x}_1 + D_1\underline{v}_1 \\
\mathrm{E}\{\underline{y}_1|\underline{y}_0\} &= \mathrm{E}\{C_1\underline{x}_1 + D_1\underline{v}_1|\underline{y}_0\} \\
&= \mathrm{E}\{C_1\underline{x}_1|\underline{y}_0\} + \mathrm{E}\{D_1\underline{v}_1|\underline{y}_0\} = C_1\mathrm{E}\{\underline{x}_1|\underline{y}_0\} = C_1\hat{\underline{x}}_{1/0}, \\
\mathrm{E}\{[\underline{y}_1 - \hat{\underline{y}}_{1/0}][\underline{y}_1 - \hat{\underline{y}}_{1/0}]^*|\underline{y}_0\} &= C_1\mathrm{E}\{[\underline{x}_1 - \hat{\underline{x}}_{1/0}][\underline{x}_1 - \hat{\underline{x}}_{1/0}]^*\}C_1^* + D_1 D_1^* \\
&= C_1\Sigma_{1/0}C_1^* + D_1 D_1^*, \qquad (2.22)
\end{aligned}$$

respectively. Also, we can state that

$$\mathrm{E}\{[\underline{x}_1 - \hat{\underline{x}}_{1/0}][\underline{y}_1 - \hat{\underline{y}}_{1/0}]^* \mid \underline{y}_0\} = \mathrm{E}\{[\underline{x}_1 - \hat{\underline{x}}_{1/0}][C_1(\underline{x}_1 - \hat{\underline{x}}_{1/0}) + D_1\underline{v}_1]^*\} = \Sigma_{1/0}C_1^* \qquad (2.23)$$

Hence the random variable $\begin{bmatrix} \underline{x}_1 \\ \underline{y}_1 \end{bmatrix}$ conditioned on $\underline{y}_0$ has mean and covariance as follow:

$$\begin{bmatrix} \hat{\underline{x}}_{1/0} \\ C_1\hat{\underline{x}}_{1/0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Sigma_{1/0} & \Sigma_{1/0}C_1^* \\ C_1\Sigma_{1/0} & C_1\Sigma_{1/0}C_1^* + D_1 D_1^* \end{bmatrix} \qquad (2.24)$$

Applying Lemma 2.1, we conclude that $\underline{x}_1$ conditioned on $\underline{y}_0$ and $\underline{y}_1$ has

$$\hat{\underline{x}}_{1/1} = \hat{\underline{x}}_{1/0} + \Sigma_{1/0}C_1^*(C_1\Sigma_{1/0}C_1^* + D_1 D_1^*)^+(\underline{y}_1 - C_1\hat{\underline{x}}_{1/0}) \qquad (2.25)$$

and covariance

$$\Sigma_{1/1} = \Sigma_{1/0} - \Sigma_{1/0}C_1^*(C_1\Sigma_{1/0}C_1^* + D_1D_1^*)^+C_1\Sigma_{1/0}. \tag{2.26}$$

With updating of time indices, we can find

$$\underline{\hat{x}}_{2/1} = A_1\underline{\hat{x}}_{1/1} + B_1D_1^*(C_1\Sigma_{1/0}C_1^* + D_1D_1^*)^+(\underline{y}_1 - C_1\underline{\hat{x}}_{1/1}) \tag{2.27}$$

$$\Sigma_{2/1} = A_1\Sigma_{1/1}A_1^* + B_1B_1^* - B_1D_1^*(C_1\Sigma_{1/0}C_1^* + D_1D_1^*)^+D_1B_1^* \tag{2.28}$$

$$\underline{\hat{x}}_{2/2} = \underline{\hat{x}}_{2/1} + \Sigma_{2/1}C_2^*(C_2\Sigma_{2/1}C_2^* + D_2D_2^*)^+(\underline{y}_2 - C_2\underline{\hat{x}}_{2/1}) \tag{2.29}$$

$$\Sigma_{2/2} = \Sigma_{2/1} - \Sigma_{2/1}C_2^*(C_2\Sigma_{2/1}C_2^* + D_2D_2^*)^+C_2\Sigma_{2/1} \tag{2.30}$$

More generally, repetition of these steps yields

$$\underline{\hat{x}}_{t/t} = \underline{\hat{x}}_{t/t-1} + \Sigma_{t/t-1}C_t^*(C_t\Sigma_{t/t-1}C_t^* + D_tD_t^*)^+(\underline{y}_t - C_t\underline{\hat{x}}_{t/t-1}) \tag{2.31}$$

$$\underline{\hat{x}}_{t+1/t} = A_t\underline{\hat{x}}_{t/t} + B_tD_t^*(C_t\Sigma_{t/t-1}C_t^* + D_tD_t^*)^+(\underline{y}_t - C_t\underline{\hat{x}}_{t/t}) \tag{2.32}$$

$$\Sigma_{t/t} = \Sigma_{t/t-1} - \Sigma_{t/t-1}C_t^*(C_t\Sigma_{t/t-1}C_t^* + D_tD_t^*)^+C_t\Sigma_{t/t-1} \tag{2.33}$$

$$\Sigma_{t+1/t} = A_t\Sigma_{t/t}A_t^* + B_tB_t^* - B_tD_t^*(C_t\Sigma_{t/t-1}C_t^* + D_tD_t^*)^+D_tB_t^* \tag{2.34}$$

Hence we obtain linear optimal estimate for x_{t+1} based on observation up to t as

$$\underline{\hat{x}}_{t+1} = A_t\underline{\hat{x}}_t + (A_t\Sigma_tC_t^* + B_tD_t^*)(C_t\Sigma_tC_t^* + D_tD_t^*)^+(\underline{y}_t - C_t\underline{\hat{x}}_t) \tag{2.35}$$

$$\begin{aligned} &= [A_t - (A_t\Sigma_tC_t^* + B_tD_t^*)(C_t\Sigma_tC_t^* + D_tD_t^*)^+C_t]\underline{\hat{x}}_t + \\ &\quad (A_t\Sigma_tC_t^* + B_tD_t^*)(C_t\Sigma_tC_t^* + D_tD_t^*)^+\underline{y}_t \end{aligned} \tag{2.36}$$

$$= (A_t + L_tC_t)\underline{\hat{x}}_t - L_t\underline{y}_t \tag{2.37}$$

So substituting (2.33) into (2.34) yields the DRE and Kalman gain

$$\Sigma_{t+1} = A_t\Sigma_tA_t^* - (A_t\Sigma_tC_t^* + B_tD_t^*)(D_tD_t^* + C_t\Sigma_tC_t^*)^+(A_t\Sigma_tC_t^* + B_tD_t^*)^* + B_tB_t^*,$$

$$L_t = -(A_t\Sigma_t C_t^* + B_t D_t^*)(D_t D_t^* + C_t\Sigma_t C_t^*)^+$$

that completes the proof. ∎

Our problem for generalized Kalman filter assumes that the state-space process is WSS, which specializes to the following.

Theorem 2.2 *Consider the state-space process in (2.4) with $\{\underline{v}(t)\}$ WSS described as in (2.5). If (C, A) is detectable, then as $t \to \infty$, the solution to the DRE*

$$\Sigma_{t+1} = A\Sigma_t A^* - (A\Sigma_t C^* + BD^*)(DD^* + C\Sigma_t C^*)^+(A\Sigma_t C^* + BD^*)^* + BB^* \quad (2.38)$$

converges to some $\Sigma \geq 0$, and the DRE converges the ARE

$$\Sigma = A\Sigma A^* - (A\Sigma C^* + BD^*)(DD^* + C\Sigma C^*)^+(A\Sigma C^* + BD^*)^* + BB^*. \quad (2.39)$$

If in addition that the limit Σ is stabilizing, then

$$L = -(A\Sigma C^* + BD^*)(DD^* + C\Sigma C^*)^+, \quad (2.40)$$

is the stationary Kalman gain, and the generalized Kalman filter is given by

$$\underline{\hat{x}}(t+1) = (A + LC)\underline{\hat{x}}(t) - L\underline{y}(t). \quad (2.41)$$

Proof: We note that for finite time horizon, the Kalman filter is

$$\underline{\hat{x}}(t+1) = (A + L_t C)\underline{\hat{x}}(t) - L_t\underline{y}(t), \quad L_t = -(A\Sigma_t C^* + BD^*)(DD^* + C\Sigma_t C^*)^+$$

in light of Theorem 2.1. Taking the difference of the above Kalman filter from the state-space process in (2.4) yields

$$\underline{e}(t+1) = (A + L_t C)\underline{e}(t) + (B + L_t D)\underline{v}(t)$$

where $\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$. By the independence of $\underline{e}(t)$ and $\underline{v}(t)$, we obtain the error covariance as

$$\Sigma_{t+1} = (A + L_tC)\Sigma_t(A + L_tC)^* + (B + L_tD)(B + L_tD)^*$$

which is the Lyapunov form of the DRE (2.38), from which we conclude that $\Sigma_t \geq 0$ $\forall t$, in light of $\Sigma_0 \geq 0$. In the limiting case of $t \to \infty$, the limit $\Sigma \geq 0$ clearly holds, but we need show that Σ is bounded. By the detectability of (C, A), stabilizing L_s exists such that $(A + L_sC)$ is a stability matrix. In this case the state-space model can be written as

$$\underline{x}(t+1) = (A + L_sC)\underline{x}(t) + (B + L_sD)\underline{v}(t) - L_s\underline{y}(t).$$

That is, we have a new state-space model with A replaced by $(A + L_sC)$ and B by $(B + L_sD)$. Since Kalman filter is optimal, we obtain that the limit Σ satisfies $\Sigma \leq \Sigma_s$ where Σ_s is the unique solution to the Lyapunov equation

$$\Sigma_s = (A + L_sC)\Sigma_s(A + L_sC)^* + (B + L_sD)(B + L_sD)^*$$

by stability of $(A + L_sC)$. Hence the limit Σ is bounded. If in addition Σ is a stabilizing solution to the ARE (2.39), then the stationary optimal linear estimator is indeed given by (2.41). ∎

The proof of Theorem 2.1 and Theorem 2.2 show that a positive semi-definite solution $\Sigma \geq 0$ can be obtained iteratively: For $k = 0, 1, \cdots$, with $\Sigma_0 \geq 0$, do the following:

$$\begin{aligned} L_k &= -(A\Sigma_k C^* + BD^*)(DD^* + C\Sigma_k C^*)^+, \\ \Sigma_{k+1} &= (A + L_k C)\Sigma_k (A + LC_k)^* + (B + L_k D)(B + L_k D)^*. \end{aligned} \tag{2.42}$$

In practice the algorithm is terminated when $\|\Sigma_N - \Sigma_{N+1}\|$ satisfies some pre-specified tolerance bound.

Remark 2.1 We would like to make the following remarks:

(a) Different from the standard stationary Kalman filtering, we can not conclude stability of $A + LC$ despite the fact that (C, A) is detectable. That is, the estimation error system

$$\underline{e}(t+1) = (A + LC)\underline{e}(t) + (B + LD)\underline{v}(t), \quad \delta\underline{y}(t) = C\underline{e}(t) + D\underline{v}(t) \tag{2.43}$$

with L in (2.40), may not be internally stable, even though the error covariance is bounded. A careful reflection concludes that any unstable modes of $(A + LC)$ are unreachable based on the control input matrix $(B+LD)$. That is, the unstable modes of $(A + LC)$ are also unreachable modes of $(A + LC, B + LD)$.

(b) The proposed iterative algorithm (2.42) is basically finite horizon non-stationary Kalman filter algorithm. Hence the iterative algorithm is convergent for any initial condition $\Sigma_0 \geq 0$.

(c) The ARE (2.39) may admit more than one positive semi-definite solutions. Each one can be viewed as the equilibrium to the DRE in (2.38). However there is a unique maximal solution $\Sigma_{\max}$, and a unique minimal solution $\Sigma_{\min}$ such that any other positive semi-definite solution Σ to the ARE (2.39) satisfies

$$0 \leq \Sigma_{\min} \leq \Sigma \leq \Sigma_{\max}.$$

If the initial value $\Sigma_0 = 0$ for the iterative algorithm in (2.42), then Σ_k is likely to converge to $\Sigma_{\min}$ as $k \to \infty$. This is intuitively true based on the optimality of Kalman filter. On the other hand, if $\Sigma_0 = \rho I$ with $\rho > 0$ sufficiently large, then Σ_k is likely to converge to $\Sigma_{\max}$ as $k \to \infty$. In particular if $\Sigma_0 \geq 0$ is close to some $\Sigma \geq 0$ satisfying the ARE (2.39), then Σ_k is likely to be trapped to the same Σ in a few iterations.

(d) For the problem of inner-outer factorization, A is assumed to be a stability matrix. If Σ_0 is chosen as the solution to the Lyapunov equation

$$\Sigma_0 = A\Sigma_0 A^* + BB^*, \tag{2.44}$$

then $\Sigma_0 \geq 0$. Moreover taking the difference between the above Lyapunov equation and the ARE (2.39) yields

$$(\Sigma_0 - \Sigma) = A(\Sigma_0 - \Sigma)A^* + (A\Sigma C^* + BD^*)(DD^* + C\Sigma C^*)^+(A\Sigma C^* + BD^*)^*.$$

Stability of A implies that $\Sigma_0 \geq \Sigma$ for any positive semi-definite solution to the ARE (2.39). Hence the maximal solution to the ARE (2.39) is likely to be obtained with the iterative algorithm (2.42) with the initial value Σ_0 satisfying (2.44).

(e) A solution $\Sigma \geq 0$ to the ARE (2.39) is said to be a stabilizing solution, if $(A + LC)$ is a stability matrix where L has the expression in (2.40). It can be argued as in the standard Kalman filter that the stabilizing solution to the ARE (2.39) is maximal among all positive semi-definite solutions to (2.39), and thus is $\Sigma_{\rm max}$, if it exists. The existence of the stabilizing solution $\Sigma_{\rm max}$ is hinged to the condition (which is similar to the standard Kalman filtering):

$$\text{rank}\left\{\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n + m \quad \forall\, \theta \in \mathbf{R}, \tag{2.45}$$

in addition to the detectability of (C, A). It will be shown later that $\Sigma = \Sigma_{\rm max}$ is what needed for computing the inner-outer factorization. How to obtain $\Sigma = \Sigma_{\rm max}$ will be answered in a later chapter. ∎

The above remarks indicate that the limiting optimal solution Σ to the generalized Kalman filtering is dependent on the initial condition Σ_0. The resultant estimator can not be implemented in practice, unless $(A + LC)$ is a stability matrix, in which case $\Sigma = \Sigma_{\rm max}$. For ease of the reference, we denote $L_{\rm m}$ as the optimal estimation gain associated with $\Sigma_{\rm max}$ as follows:

$$L_{\rm m} = -(A\Sigma_{\rm max}C^* + BD^*)(DD^* + C\Sigma_{\rm max}C^*)^+. \tag{2.46}$$

The next section will examine the inner-outer factorization problem associated with generalized Kalman filtering. The development of the construction algorithm for the inner and outer factors require the stabilizing solution $\Sigma_{\rm max}$ and the stabilizing Kalman gain $L_{\rm m}$.

2.3 Inner-Outer Factorization

In this section we investigate inner-outer factorizations for the following case:

$$m < p: \quad H(z) = H_{\rm o}(z)H_{\rm i}(z), \tag{2.47}$$

where $H_{\rm i}(z)$ is a square inner of the smaller size, and $H_{\rm o}(z)$ is an outer. A square transfer matrix $H_{\rm i}(z)$ is called inner, if it is stable, and $H_{\rm i}(e^{j\omega})$ is a unitary matrix for all $\omega \in \mathbf{R}$. In other words, $H_{\rm i}(z)H_{\rm i}(z)^{\sim} = I$. A non-square transfer matrix $H_{\rm o}(z)$ is called outer, if it is both stable, and minimum phase. A moment of reflection reveals that all zeros of $H_{\rm i}(z)$ are strictly outside the unit circle, and are thus unstable. On the other hand, zeros of $H_{\rm o}(z)$ are all in the unit disc, including the unit circle. This inner-outer factorizations are intimately related to spectral factorizations. In fact, $H_{\rm o}(z)$ is the left spectral factor of $\Phi(z) = H(z)H(z)^{\sim}$. Hence inners are ambiguities that can not be estimated based on the PSD $\Phi(z)$. The results in this section can be used to eliminate the ambiguity part of the channel estimated using the existing methods that can not guarantee the minimum phase of the estimated channels. It will be shown that the results on generalized Kalman filtering is crucial to tackle the inner-outer factorization in this section. The assumption that $D \neq 0$ has no loss of generality, because any causal transfer matrix $H(z)$ can be written as $H(z) = z^{-k}\tilde{H}(z)$ for some $k \geq 0$ and causal transfer matrix $\tilde{H}(z)$ such that $\tilde{D} = \tilde{H}(\infty) \neq 0$. Thus inner-outer factorizations of $\tilde{H}(z)$ can then be studied with z^{-k} subsumed into the inner.

We introduce the following two lemmas to help in finding a solution for the inner-outer factorization in (2.47).

Lemma 2.2 *Suppose that* $\operatorname{rank}\{D\} \leq m < p$ *with* $p \times m$ *the dimensions of* $H(z)$ *as in* (2.2)*. Let* $\Sigma \geq 0$ *be a solution to the ARE* (2.39) *and* L *as in* (2.40)*. Denote* $\Pi = DD^* + C\Sigma C^*$*. Then there hold* $L\Pi = -(A\Sigma C^* + BD^*)$*, and*

$$H(z)H(z)^{\sim} = \left[I - C(zI-A)^{-1}L\right](DD^* + C\Sigma C^*)\left[I - C(zI-A)^{-1}L\right]^{\sim}. \quad (2.48)$$

Moreover all unstable models of $A+LC$ *are unreachable modes of* $(A+LC, B+LD)$*.*

Proof: The above lemma shows that the rank of $\Pi = DD^* + C\Sigma C^*$ is the same as the normal rank of $H(z)$ in (2.2). If Π is invertible, then Π^{-1} exists, and $L\Pi = -(A\Sigma C^* + BD^*)$ is true in light of (2.40). Assume that $\operatorname{rank}\{\Pi\} = m_o < p$. Then there exists a Cholesky factorization $\Pi = \Omega\Omega^*$ with $p \times m_o$ the size of Ω. Thus there exists $\Omega_\perp$ of size $p \times (p - m_o)$ such that

$$\Omega_\perp \Omega_\perp^+ = I - \Omega\Omega^+, \quad \det\left(\begin{bmatrix} \Omega & \Omega_\perp \end{bmatrix}\right) \neq 0. \quad (2.49)$$

Since $\Omega^+\Omega = I$, and $\Omega_\perp^+\Omega_\perp = I$, the above implies

$$\begin{bmatrix} \Omega^+ \\ \Omega_\perp^+ \end{bmatrix} \begin{bmatrix} \Omega & \Omega_\perp \end{bmatrix} = I, \quad \Omega_\perp^+\Omega = 0. \quad (2.50)$$

With the above notation, $L = -(A\Sigma C^* + BD^*)(\Omega^+)^*\Omega^+$. Let the rank of Σ be $r > 0$. Then

$$\Sigma = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = U_1\Sigma_1 U_1^*, \quad \Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r) > 0, \quad (2.51)$$

by the SVD of Σ where $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ is a unitary matrix. It follows that

$$\Pi = \begin{bmatrix} CU_1\Sigma_1^{1/2} & D \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2}U_1^*C^* \\ D^* \end{bmatrix} = \Omega\Omega^* \quad \Longrightarrow \quad \begin{bmatrix} CU_1\Sigma_1^{1/2} & D \end{bmatrix} = \Omega V \quad (2.52)$$

for some orthogonal matrix V satisfying $VV^* = I$. By $\Omega^+\Omega = I$, the above yields

$$\begin{aligned} L\Pi &= -(A\Sigma C^* + BD^*)(\Omega^+)^*\Omega^+\Omega\Omega^* = -\begin{bmatrix} AU_1\Sigma_1^{1/2} & B \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2}U_1^*C^* \\ D^* \end{bmatrix} (\Omega^+)^*\Omega^+ \\ &= -\begin{bmatrix} AU_1\Sigma_1^{1/2} & B \end{bmatrix} (\Omega V)^*(\Omega^+)^*\Omega^* = -\begin{bmatrix} AU_1\Sigma_1^{1/2} & B \end{bmatrix} (\Omega V)^* \\ &= -(A\Sigma C^* + BD^*) \end{aligned}$$

which concludes that $L\Pi = -(A\Sigma C^* + BD^*)$. The next step of the proof is to show that the rank of $\Pi = DD^* + C\Sigma C^*$ is the same as the normal rank of $H(z)$ in (2.2). Noticing that

$$\Sigma - A\Sigma A^* = (zI - A)\Sigma(z^{-1}I - A^*) + (zI - A)\Sigma A^* + A\Sigma(z^{-1}I - A^*) \qquad (2.53)$$

By the expression of the state estimation gain L as in (2.40), ARE (2.39),

$$\Sigma - A\Sigma A^* = L(A\Sigma C^* + BD^*)^* + BB^* = -L(DD^* + C\Sigma C^*)L^* + BB^*. \qquad (2.54)$$

It follows from the above, (2.53) and $\Pi = DD^* + C\Sigma C^*$

$$\begin{aligned} BB^* &= \Sigma - A\Sigma A^* + L(DD^* + C\Sigma C^*)L^* \\ &= (zI - A)\Sigma(z^{-1}I - A^*) + (zI - A)\Sigma A^* + A\Sigma(z^{-1}I - A^*) + L\Pi L^* \end{aligned}$$

Denote $\Theta(z) = C(zI - A)^{-1}$. Multiply $\Theta(z)$ from left and $\Theta(z)^\sim$ from right, then

$$\Theta(z)BB^*\Theta(z)^\sim = C\Sigma C^* + \Theta(z)L\Pi L^*\Theta(z)^\sim + C\Sigma A^*\Theta(z)^\sim + \Theta A\Sigma C^* \qquad (2.55)$$

Since $H(z)H(z)^{\sim} = DD^* + \Theta(z)BD^* + DB^*\Theta(z)^{\sim} + \Theta(z)BB^*\Theta(z)^{\sim}$,

$$\begin{aligned} H(z)H(z)^{\sim} &= DD^* + \Theta(z)BD^* + DB^*\Theta(z)^{\sim} + C\Sigma C^* \\ &\quad + \Theta(z)L\Pi L^*\Theta(z)^{\sim} + C\Sigma A^*\Theta(z)^{\sim} + \Theta A\Sigma C^* \\ &= DD^* + \Theta(z)(BD^* + A\Sigma C^*) + (DB^* + C\Sigma A^*)\Theta(z)^{\sim} \\ &\quad + C\Sigma C^* + \Theta(z)L\Pi L^*\Theta(z)^{\sim} \\ &= \Theta(z)(BD^* + A\Sigma C^*) + (DB^* + C\Sigma A^*)\Theta(z)^{\sim} + \Theta(z)L\Pi L^*\Theta(z)^{\sim} + \Pi \end{aligned} \tag{2.56}$$

Using $L\Pi = -(A\Sigma C^* + BD^*)$, we obtain

$$H(z)H(z)^{\sim} = \Theta(z)L\Pi L^*\Theta(z)^{\sim} - \Theta(z)L\Pi - \Pi L^*\Theta(z)^{\sim} + \Pi \tag{2.57}$$

$$= [I - \Theta(z)L](DD^* + C\Sigma C^*)[I - \Theta(z)L]^{\sim} \tag{2.58}$$

$$= [I - \Theta(z)L]\Pi[I - \Theta(z)L]^{\sim} \tag{2.59}$$

which concludes the proof. ∎

It should be now clear that the rank of $\Pi = DD^* + C\Sigma C^*$ is the same as the normal rank of $H(z)$, by the fact that $[I - \Theta(z)L]$ has the full normal rank.

Lemma 2.3 *Suppose that* $\operatorname{rank}\{D\} \le m < p$ *with* $p \times m$ *the dimension of* $H(z)$ *as in* (2.2). *Let* $\Sigma \ge 0$ *be a solution to the ARE* (2.39), *and the state estimation gain* L *be as in* (2.40). *If* $H(z)$ *as in* (2.2) *has normal rank* $m < p$, *then* $\Pi = DD^* + C\Sigma C^*$ *has rank* m. *Let* $\Pi = \Omega\Omega^*$ *be the Cholesky factorization with* $p \times m$ *the size of* Ω. *Then*

$$G(z) = (I - \Omega\Omega^+)C(zI - A - LC)^{-1}(B + LD) = 0. \tag{2.60}$$

Proof: The identity (2.48) shows that the rank of $\Pi = DD^* + C\Sigma C^*$ is the same as the normal rank of $H(z)$, which is m. Hence there exist Ω of size $p \times m$ and $\Omega_\perp$ of size $p \times (p-m)$ such that $\Pi = \Omega\Omega^*$, and (2.49) and (2.50) hold. With the above notation, the state estimation gain L has the expression $L = -(A\Sigma C^* + BD^*)(\Omega^+)^*\Omega^+$. It follows that (using $\Omega^+\Omega_\perp = 0$)

$$L(I - \Omega\Omega^+) = -(A\Sigma C^* + BD^*)(\Omega^+)^*\Omega^+\Omega_\perp\Omega_\perp^+ = 0 \tag{2.61}$$

By the SVD in (2.51)

$$\Pi = \Omega\Omega^* = DD^* + C\Sigma C^* = \begin{bmatrix} CU_1\Sigma_1^{1/2} & D \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2}U_1^*C^* \\ D^* \end{bmatrix} \tag{2.62}$$

Thus $\Omega_\perp^+\Omega = 0$ implies that $\Omega_\perp^+\Pi = \Omega_\perp^+\Omega\Omega^* = 0$, $\Omega_\perp^+CU_1 = 0$ and $\Omega_\perp^+D = 0$, yielding

$$\Omega_\perp\Omega_\perp^+D = (I - \Omega\Omega^+)D = 0, \qquad \Omega_\perp\Omega_\perp^+CU_1 = (I - \Omega\Omega^+)CU_1 = 0 \tag{2.63}$$

Rewrite the ARE (2.39) in the form of Lyapunov equation

$$\Sigma = (A + LC)\Sigma(A + LC)^* + (B + LD)(B + LD)^* \tag{2.64}$$

where $\Sigma \geq 0$ by the hypothesis. Clearly all unstable poles of $G(z)$ as in (2.60) are unreachable modes of $(A + LC, B + LD)$. Applying the similarity transform $S = U = [U_1 \quad U_2]$ to the realization of $G(z)$ gives

$$\begin{aligned}
U^*(A + LC)U &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} (A + LC) \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
U^*(B + LD) &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} (B + LD) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
(I - \Omega\Omega^+)CU &= (I - \Omega\Omega^+)C \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.
\end{aligned}$$

It follows from (2.63) that $C_1 = 0$. Multiplying (2.64) by U^* from left, and U from right, and using the above partitions and the SVD of Σ in (2.51) yield

$$
\begin{aligned}
U^*\Sigma U &= U^*(A+LC)\Sigma(A+LC)^*U + U^*(B+LD)(B+LD)^*U \quad \text{or} \\
\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \Sigma_1 \begin{bmatrix} A_{11}^* & A_{21}^* \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}
\end{aligned}
$$

The fact that $\Sigma_1 > 0$ implies that $A_{21} = 0$ and $B_2 = 0$ in light of the above equation, which coupled with $C_1 = 0$ concludes that (2.60) is true. ∎

The next result presents the solution to the inner-outer factorization in (2.47).

Theorem 2.3 *Suppose that $H(z)$ as in* (2.2) *has normal rank $m < p$, satisfies the condition* (2.45)*, and A is a stability matrix. Let $\Sigma = \Sigma_{\rm max} \geq 0$ be the maximal solution to* (2.42)*, and $L_{\rm m}$ be as in* (2.46)*. Then there holds the inner-outer factorization $H(z) = H_{\rm o}(z)H_{\rm i}(z)$ where, with $\Omega_{\rm m}\Omega_{\rm m}^* = \Pi = DD^* + C\Sigma_{\rm max}C^*$, the inner and outer factors of $H(z)$ are given respectively by*

$$
H_{\rm i}(z) = \Omega_{\rm m}^+ \left[\begin{array}{c|c} A + L_{\rm m}C & B + L_{\rm m}D \\ \hline C & D \end{array}\right], \quad H_{\rm o}(z) = \left[\begin{array}{c|c} A & -L_{\rm m} \\ \hline C & I \end{array}\right] \Omega_{\rm m}. \tag{2.65}
$$

Proof: Using $\Theta(z) = C(zI - A)$,

$$
T(z) = [I - \Theta L]^{-1} H(z) = \left[\begin{array}{c|c} A + L_{\rm m}C & B + L_{\rm m}D \\ \hline C & D \end{array}\right] \tag{2.66}
$$

satisfies $T(z)T(z)^\sim = DD^* + C\Sigma_{\rm max}C^* = \Omega_{\rm m}\Omega_{\rm m}^*$. That is, $\Omega_{\rm m}$ has the same column rank as the normal rank of $H(z)$, which is m, and $H_{\rm i}(z) = \Omega_{\rm m}^+ T(z)$ is square and satisfies $H_{\rm i}(z)H_{\rm i}(z)^\sim = I_m$. Because the unstable modes of $(A+L_{\rm m}C)$ are unreachable modes of $(A+L_{\rm m}C, B+L_{\rm m}D)$, they can be eliminated through Kalman decomposition.

Hence $H_{\rm i}(z)$ with minimal realization is stable, which is indeed an inner. To verify the expression of $H_{\rm o}(z)$ as in (2.65), we have

$$
\begin{aligned}
H_{\rm o}(z)H_{\rm i}(z) &= \left[\begin{array}{c|c} A & -L_{\rm m} \\ \hline C & I \end{array}\right] \Omega_{\rm m}\Omega_{\rm m}^{+} \left[\begin{array}{c|c} A+L_{\rm m}C & B+L_{\rm m}D \\ \hline C & D \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & -L_{\rm m}\Omega_{\rm m}\Omega_{\rm m}^{+}C & -L_{\rm m}\Omega_{\rm m}\Omega_{\rm m}^{+}D \\ 0 & A+L_{\rm m}C & B+L_{\rm m}D \\ \hline C & \Omega_{\rm m}\Omega_{\rm m}^{+}C & \Omega_{\rm m}\Omega_{\rm m}^{+}D \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & -L_{\rm m}C & -L_{\rm m}D \\ 0 & A+L_{\rm m}C & B+L_{\rm m}D \\ \hline C & \Omega_{\rm m}\Omega_{\rm m}^{+}C & D \end{array}\right], \qquad (2.67)
\end{aligned}
$$

in light of (2.61) and (2.49). Using $T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ and applying similarity transformation to (2.67), we obtain

$$
H_{\rm o}(z)H_{\rm i}(z) = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A+L_{\rm m}C & B+L_{\rm m}D \\ \hline C & -(I-\Omega_{\rm m}\Omega_{\rm m}^{+})C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & B \end{array}\right] = H(z), \qquad (2.68)
$$

in light of lemma 2.3. Hence the Theorem is true. ∎

Theorem 2.3 shows that the stabilizing solution to the ARE in (2.39) is crucial to the construction of the inner-outer factors. Although an iterative algorithm is proposed for computing the stabilizing solution in the previous section, it is unclear how to choose the initial condition $\Sigma_0 \geq 0$ for the iterative algorithm in (2.42), that will ensure their convergence to the required stabilizing solutions. It turns out that such an issue has to be resolved together with that for spectral factorizations, which will be investigated in Chapter 4. Before concluding this section we would like to indicate that if $H(z)$ is minimum phase with D full column rank, then the ARE (2.42) admits only one nonnegative solution that is $\Sigma_{\max} = 0$. On the other hand if $H(z)$ is minimum phase but D does not full column rank, then again the ARE (2.42) admits a unique nonnegative solution that is $\Sigma_{\min}$, and in this case $\Sigma_{\max} = \Sigma_{\min}$.

2.4 Illustrative Examples

In this section we present examples to demonstrate the proposed iterative algorithm to solve the ARE associated with the generalized Kalman filtering and its application to channel equalization for wireless data communications.

Example 2.1 This example is motivated from multiuser wireless data communications. Because of the multipath phenomena, the discretized wireless channels is represented by

$$H(z) = H_0 + H_1 z^{-1} + \cdots + H_\ell z^{-\ell}, \qquad H_k \in \mathbf{C}^{p \times m}. \tag{2.69}$$

Thus the received signal data are convolution of the discretized channel with the transmitted digital data, plus the observation noise. As such it causes the problem of inter-symbol-interference (ISI) that poses the difficulty in symbol detection. A conventional approach to eliminate the ISI is through channel equalization. Because redundancies are often introduced purposely in communications, the channel transfer matrix $H(z)$ is tall, i.e., $p > m$, and can be assumed to be strictly minimum phase.

A simple state-space realization for $H(z)$ is

$$A = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} H_1 & H_2 & \cdots & H_\ell \end{bmatrix}, \quad D = H_0. \tag{2.70}$$

In [14], it is assumed that $D = H_0$ has rank m, and has derived optimal channel equalizer, which is the causal and stable left inverse of $H(z)$, that has the smallest $\mathcal{H}_2$-norm. However due to the existence of the transmission delay, full rank assumption on H_0 is not realistic. Hence we apply the algorithm in Section 2.3 to compute inner-outer factorization $H(z) = H_{\rm o}(z)H_{\rm i}(z)$, where $H_{\rm i}(z)$ is FIR, and has all its zeros at infinity, and $H_{\rm o}(\infty)$ has the full column rank. Let d be the minimum integer such that $z^{-d}H_{\rm i}(z)^\sim$ is causal. Then the optimal linear equalizer can be obtained as

$$G(z) = z^{-d}H_{\rm i}(z)^\sim H_{\rm o}^{(\rm inv)}(z) \quad \Longrightarrow \quad G(z)H(z) = z^{-d}I, \tag{2.71}$$

where $H_{\rm o}^{(\rm inv)}(z)$ is the causal and stable left inverse of $H_{\rm o}(z)$, that has the smallest $\mathcal{H}_2$-norm. How to compute such as an optimal channel equalizer has been solved in [14].

In the following we consider a numerical example with $H(z)$ of size 3×2, specified by

$$C = \begin{bmatrix} -0.1199 & -0.5955 & -0.0793 & -1.3474 & 0.0359 & 0.5529 \\ -0.0653 & -0.1497 & 1.5352 & 0.4694 & -0.6275 & -0.2037 \\ 0.4853 & -0.4348 & -0.6065 & -0.9036 & 0.5354 & -2.0543 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0.7233 & 1.0172 \\ 0.2104 & 0.2959 \\ -0.5664 & -0.7965 \end{bmatrix}$$

where $\ell = 3$. Recall realization as in (2.70). Each element in D and C is generated as a normal random variable with zero mean, and unit variance, and the two columns of $D = H_0$ are linearly dependent. It can be verified that $H(z)$ is strictly minimum phase.

It is noted that in applying the iterative algorithm in (2.42), we do not need begin with initial condition $\Sigma_0 = A\Sigma_0 A^* + BB^*$, with (A, B, C, D) a realization of $H(z)$, due to the assumption that $H(z)$ is strictly minimum phase. In fact with $\Sigma_0 = 0$, the iterative algorithm in (2.42) often converges faster, based on our numerical experience. For the given numerical example with the error tolerance 10^{-10}, the number of iterations is only 2, if $\Sigma_0 = 0$; The number of iterations is 6, if $\Sigma_0 = A\Sigma_0 A^* + BB^*$. Both give the same $\Sigma_{\max}$. With $\Sigma_{\max}$ obtained, the inner and outer factors are computed according to Theorem 2.3. It turns out that the reachability gramian of $H_{\rm i}(z)$ has rank 1, implying that the minimal realization of $H_{\rm i}(z)$ has an order at most 1. Since $H(z)$ has only one zero outside the unit circle at infinity, $H_{\rm i}(z)$ has McMillan degree exactly 1. After eliminating the unreachable modes of $H_{\rm i}(z)$ we obtain

$$H_{\rm i}(z) = \begin{bmatrix} -0.5738 & -0.8070 \\ -0.0811 & -0.1140 \end{bmatrix} + \begin{bmatrix} -0.1399 \\ 0.9902 \end{bmatrix} \begin{bmatrix} -0.8150 & 0.5795 \end{bmatrix} z^{-1},$$

which is indeed an inner. The outer factor is specified by

$$\Omega_{\mathrm{m}} = \begin{bmatrix} -1.2013 & -0.4196 \\ -0.3548 & -0.0840 \\ 1.0584 & -0.5044 \end{bmatrix}, \qquad L_{\mathrm{m}} = \begin{bmatrix} -0.3341 & -0.0913 & 0.1324 \\ -0.4698 & -0.1284 & 0.1862 \\ -0.7230 & -0.1400 & -0.9752 \\ 0.5141 & 0.0995 & 0.6935 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The iterative algorithm works quite efficiently in terms of the computational complexity. ∎

The next example consider linear receiver design using the generalized Kalman filtering.

Example 2.2 Again this example is motivated from multiuser wireless data communications. Consider the symbol detection in multiuser wireless data communications. The multipath channel is described by

$$\underline{r}(t) = \sum_{k=0}^{l} H_k(t)\underline{s}(t-k) \tag{2.72}$$

Assume the channel impulse response $\{H_k(t)\}$ is known at time t. Our goal is to design linear receivers that detect the symbol $\underline{s}(t)$ with the minimum error variance. Assume $\{\underline{s}(t)\}$ is WSS with zero mean and covariance $\sigma_s^2 I$.

The channel model can be associated with a realization with the state vector

$$\underline{x}(t) = \begin{bmatrix} \underline{s}(t-1)^T & \underline{s}(t-2)^T & \cdots & \underline{s}(t-l)^T \end{bmatrix}^T$$

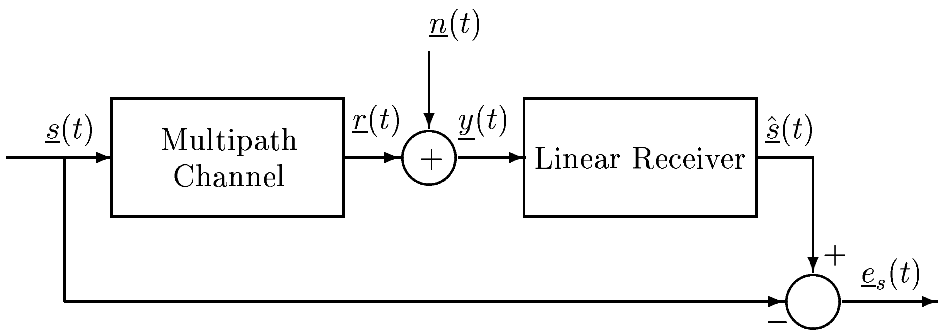

Figure 2.1: Detection of the symbol inputs

Denote $\underline{v}(t) = \begin{bmatrix} \underline{s}(t)^T & \underline{n}(t)^T \end{bmatrix}^T$. Assume the dimension of the symbol vector $\underline{s}(t)$ is μ and the dimension of the noise vector $\underline{n}(t)$ is ν. The observed signal $\underline{y}(t)$ at the receiver site can be described the state-space model in (2.4) with

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{0}_{\mu\times\mu} \\ I_{\mu\times(l-1)} & \mathbf{0} \end{bmatrix}, \quad B = \begin{bmatrix} I_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{\mu(l-1)\times\nu} \end{bmatrix},$$
$$C = \begin{bmatrix} H_1 & H_2 & \cdots & H_\ell \end{bmatrix}, \quad D = [H_0 \quad \sigma_n I].$$

where σ_n^2 is the noise covariance which assumed to be white and WSS. Assuming the covariance of the observation noise $D\underline{v}(t)$, is singular because H_0 is not fill rank, in this case we should apply the generalized Kalman filtering. Hence the optimal linear estimator of $\underline{s}(t-l)$ for all $0 \le k < l$ based on observations $\left\{\underline{y}(k)\right\}_{k=0}^{t}$ is given by

$$\hat{\underline{x}}(t+1) = (A+LC)\hat{\underline{x}}(t) - L\underline{y}(t), \quad L = -(A\Sigma C^* + BD^*)(DD^* + C\Sigma C^*)^+,$$
$$\Sigma = A\Sigma A^* - (A\Sigma C^* + BD^*)(DD^* + C\Sigma C^*)^+(A\Sigma C^* + BD^*)^* + BB^*.$$

Hence the signal to be estimated is given by

$$\hat{\underline{s}}(t-l) \;\; = \;\; \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & I_\mu \end{bmatrix} \hat{\underline{x}}(t)$$

The quantization can be applied to $\{\hat{\underline{s}}(t-l)\}$ to obtain the detected symbol. ∎

2.5 Conclusion

In this chapter we investigated the generalized Kalman filtering and inner-outer factorization problems. We proposed an iterative algorithm for computing the positive semi-definite solution to the corresponding ARE that was employed to solve the inner-outer factorization problem considered in this chapter. The stabilizing solution to the ARE associated with the generalized Kalman filtering was used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. Examples are used to illustrate the proposed solution algorithms. While we can assume stability for inner-outer factorization, we can not assume the stability of A for the generalized Kalman filtering in general. A simple way to bypass the stability issue is to set the Kalman gain $L = L_1 + L_2$ where $(A+L_1C)$ is a stability matrix, and then consider the generalized Kalman filter for

$$\underline{x}(t+1) = (A+L_1C)\underline{x}(t) + (B+L_1D)\underline{v}(t) - L_1\underline{y}(t), \quad \underline{y}(t) = C\underline{x}(t) + D\underline{v}(t) \quad (2.73)$$

The above leads to the following modified algorithm:

$$\begin{aligned} L_k &= -[(A+L_1C)\Sigma_k C^* + (B+L_1D)D^*](DD^* + C\Sigma_k C^*)^+ + L_1, \\ \Sigma_{k+1} &= (A+L_kC)\Sigma_k(A+LC_k)^* + (B+L_kD)(B+L_kD)^*, \end{aligned} \quad (2.74)$$

Although iterative algorithm is derived for computing a solution to the ARE associated with generalized Kalman filtering, it is unclear how to choose the boundary condition $\Sigma_0 \geq 0$ that will ensure its convergence to the required stabilizing solution.

It turns out that such an issue has to be resolved together with that for spectral factorization which will be discussed in Chapter 4.

Chapter 3

Generalized LQR Control and Inner-Outer Factorizations

3.1 Introduction

LQR control stands for linear quadratic regulator, which is a well studied control problem in late 60s and early 70s [3, 17]. We consider generalized LQR control that is dual problem to the generalized Kalman filtering, in which the dimension of the control input signal is strictly greater than the dimension of the controlled output signal, and thus the penalty weighting matrix for the control input energy is always singular. The duality between the generalize Kalman filtering and the generalized LQR control implies that the results in Chapter 2 can be translated to those in this chapter. However it is noted that the generalized Kalman filtering deals with stochastic processes, while LQR control is deterministic in nature. Hence a new derivation for LQR control is needed. We will develop an iterative algorithm for computing the stabilizing solutions to the ARE associated with the generalized

LQR control. In addition we will develop similar formulas to compute inner-outer factorizations dual to that in the previous chapter.

At the first glance, the generalized LQR control seems unrelated to the scope of this dissertation. We will see in the next chapter that the spectral factorization problem in this dissertation is also related to the generalized LQR control. Recall that the spectral factorization is our proposed approach to blind channel estimation for MIMO wireless channels. Specifically let the transfer matrix $\Phi(z)$ of size $p \times p$ has the form given in (1.11). It follows that $\Phi(z)$ is a hermitian matrix for any z on the unit circle. If in addition $\Phi(z) \geq 0 \ \forall \ |z| = 1$, then $\Phi(z)$ qualifies a PSD with $\{R_k\}$ the covariance sequence. Let the normal rank of $\Phi(z)$ be $r < p$. We are interested in the spectral factorization

$$\Phi(z) = W_R(z)^{\sim} W_R(z) \tag{3.1}$$

where $W_R(z)$ has size $r \times p$, and more importantly it is causal, stable, and strict minimum phase. In other words, all poles and zeros of $W_R(z)$ are strictly inside the unit circle. In this case $W_R(z)$ is called the right spectral factor of $\Phi(z)$. We assume that $\Phi(z)$ is a bounded hermitian positive matrix with rank r for all z on the unit circle, which excludes the possibilities of poles and zeros on the unit circle for the spectral factors. It is worth to pointing out that most of the existing work on spectral factorizations assume that $r = p$, and there lack effective computational algorithms for spectral factorizations in the case of $0 < r < p$. Hence the results in this chapter complement those in the previous chapter that help construct a coherent spectral factorization theory applicable to blind channel estimation.

Assume that the underlying system is finite-dimensional with transfer matrix $H(z)$. Then it admits a state-space realization (A, B, C, D) or equivalently

$$H(z) = D + C(zI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3.2}$$

It is assumed that $H(z)$ has size $p \times m$ with $m > p$, which is opposite to the scenario in the previous chapter. Hence m is the normal rank of the PSD $H(z)^{\sim}H(z)$. Clearly $H(z)$ involves ambiguities that do not show up in the PSD $H(z)^{\sim}H(z)$, leading to the problem of inner-outer factorization $H(z) = H_{\rm i}(z)H_{\rm o}(z)$ where $H_{\rm i}(z)$ is inner of size $p \times p$, and $H_{\rm o}(z)$ is outer having no pole and zero outside the unit circle, including at infinity. This problem is dual to the inner-outer factorization in the previous chapter, and thus different from the existing results. Recall that the inners are square of smaller size. We note, however, that $H(z)$ may have zeros strictly outside the unit circle, and its realization is subject to the constraint (due to the transmission delays)

$$0 < \text{rank}\{D\} \le \min\{m, p\} \tag{3.3}$$

where $D \in \mathbf{F}^{p\times m}$.

Similar to the previous chapter, it will be shown that the inner-outer factorization in this chapter has a close relation to the generalized LQR control. In fact the inner-outer factorization in this chapter can not be solved without solving the generalized LQR control. We will thus investigate the generalized LQR control first in the next section. Similar to the standard LQR control, an algebraic Riccati equation (ARE) needs to be solved in order to obtain the optimal control law. An iterative algorithm is proposed for computing the positive semi-definite solution to the corresponding ARE. In Section 3.3 we investigate the problem of the inner-outer factorization. The stabilizing solution to the ARE associated with the generalized LQR control will be used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. In Section 3.4, we will investigate the generalized Linear-Quadratic-Gaussian (LQG) optimal control problem. LQG control is different from LQR in that the state variables are not measurable that have to be estimated based on measurements of the plant input and the output. In Section 3.5, several numerical examples are employed to illustrate the results obtained in this chapter for computing the outer part of the given channel. The numerical examples also point out the convergence issues in the proposed iterative algorithm for solving the required ARE which will be tackled together with the spectral factorization in Chapter 4. The chapter is concluded in Section 3.6.

3.2 Generalized LQR Control

In this section, we study the generalized LQR control. The generalized LQR control assumes the state-space model

$$\underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t), \quad \underline{z}(t) = C\underline{x}(t) + D\underline{u}(t), \quad \underline{x}(0) = \underline{x}_0 \neq 0, \tag{3.4}$$

and searches for the control input $\underline{u}(t)$ to minimize the quadratic performance index

$$J = \sum_{t=0}^{\infty} \|\underline{z}(t)\|^2 = \sum_{t=0}^{\infty} \underline{z}^*(t)\underline{z}(t) = \sum_{t=0}^{\infty} \begin{bmatrix} \underline{x}^*(t) & \underline{u}^*(t) \end{bmatrix} \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) \end{bmatrix}. \tag{3.5}$$

We assume that the control input $\underline{u}(t)$ has size m, the controlled output $\underline{z}(t)$ has size p, and $m > p$. Stability of A is not assumed for the generalized LQR control, and $\text{rank}\{D\} \leq p$.

This problem differs from the standard LQR control in that D is a "fat" matrix by $m > p$, and its rank can be strictly smaller than p. That is, the penalty weighting matrix on the control signal is always singular. The conventional approach is to consider optimal control over the finite time horizon, and then take the limit to the infinity time horizon. So we now allow time-dependence of the realization, and modify (3.4), and (3.5) to

$$\begin{bmatrix} \underline{x}(t+1) \\ \underline{z}(t) \end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) \end{bmatrix}, \quad \underline{x}(0) = \underline{x}_0 \neq 0, \tag{3.6}$$

$$J(0,T) = \underline{x}^*(T)X_T\underline{x}(T) + \sum_{t=0}^{T-1} \|\underline{z}(t)\|^2, \tag{3.7}$$

respectively, with $X_T \geq 0$ the penalty for the terminal state at time $t = T$. Then (3.6) implies

$$\begin{aligned}
\rho(t) &:= \underline{x}^*(t+1)X_{t+1}\underline{x}(t+1) + \underline{z}^*(t)\underline{z}(t) \\
&= \begin{bmatrix} \underline{x}^*(t+1) & \underline{z}^*(t) \end{bmatrix} \begin{bmatrix} X_{t+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \underline{x}(t+1) \\ \underline{z}(t) \end{bmatrix} \\
&= \begin{bmatrix} \underline{x}^*(t) & \underline{u}^*(t) \end{bmatrix} \begin{bmatrix} A_t^* & C_t^* \\ B_t^* & D_t^* \end{bmatrix} \begin{bmatrix} X_{t+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \underline{x}^*(t) & \underline{u}^*(t) \end{bmatrix} \begin{bmatrix} A_t^* X_{t+1} A_t + C_t^* C_t & A_t^* X_{t+1} B_t + C_t^* D_t \\ B_t^* X_{t+1} A_t + D_t^* C_t & D_t^* D_t + B_t^* X_{t+1} B_t \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) \end{bmatrix} \tag{3.8}$$

Denoting $\Theta_1 = A_t^* X_{t+1} A_t + C_t^* C_t$, $\Theta_2 = B_t^* X_{t+1} A_t + D_t^* C_t$, $\Theta_3 = D_t^* D_t + B_t^* X_{t+1} B_t$, and applying the Schur decomposition in (1.4) to the middle matrix of $\rho(t)$ in (3.8) yield

$$\rho(t) = \begin{bmatrix} \underline{x}^*(t) & \underline{u}^*(t) + \underline{x}^*(t)\Theta_2^*\Theta_3^+ \end{bmatrix} \begin{bmatrix} X_t & 0 \\ 0 & \Theta_3 \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{u}(t) + \Theta_3^+ \Theta_2 \underline{x}(t) \end{bmatrix} \tag{3.9}$$

where $X_t = \nabla_1$ is the Schur complement, satisfying the difference Riccati equation (DRE)

$$X_t = A_t^* X_{t+1} A_t + C_t^* C_t - (A_t^* X_{t+1} B_t + C_t^* D_t)(D_t^* D_t + B_t^* X_{t+1} B_t)^+ (B_t^* X_{t+1} A_t + D_t^* C_t), \tag{3.10}$$

by $\nabla_1 = \Theta_1 - \Theta_2^* \Theta_3^+ \Theta_2$. Consequently

$$\rho(t) = \underline{x}^*(t+1) X_{t+1} \underline{x}(t+1) + \underline{z}^*(t)\underline{z}(t) = \underline{x}^*(t) X_t \underline{x}(t) + \underline{\mu}^*(t)[D_t^* D_t + B_t^* X_{t+1} B_t]\underline{\mu}(t),$$

where $\underline{\mu}(t) = \underline{u}(t) - F_t \underline{x}(t)$, and

$$F_t = -(D_t^* D_t + B_t^* X_{t+1} B_t)^+ (B_t^* X_{t+1} A_t + D_t^* C_t). \tag{3.11}$$

Since $X_{t+1} \geq 0$, $\rho(t) \geq \underline{x}^*(t) X_t \underline{x}(t)$. Hence if $X_t \geq 0$ for $0 \leq t \leq T$, we have

$$\begin{aligned} J(0,T) &= \underline{x}^*(T) X_T \underline{x}(T) + \sum_{k=0}^{T-1} \|\underline{z}(t)\|^2 = \rho(T-1) + \sum_{k=0}^{T-2} \|\underline{z}(t)\|^2 \\ &\geq \underline{x}^*(T-1) X_{T-1} \underline{x}(T-1) + \sum_{k=0}^{T-2} \|\underline{z}(t)\|^2 = \rho(T-2) + \sum_{k=0}^{T-3} \|\underline{z}(t)\|^2 \\ &\geq \underline{x}^*(T-2) X_{T-2} \underline{x}(T-2) + \sum_{k=0}^{T-3} \|\underline{z}(t)\|^2 \geq \cdots \geq \underline{x}^*(0) X_0 \underline{x}(0), \end{aligned} \tag{3.12}$$

where the boundary condition $X_T \geq 0$ is assumed. The above derivation is similar to that for the standard LQR control, and is summarized in the following result.

Theorem 3.1 *Let the m-input/p-output system be given as in* (3.6)*, with* $m > p \geq \text{rank}\{D_t\}$*. Let* $\{X_t\}_{t=0}^{T-1}$ *be the solutions to the DRE* (3.10) *with the boundary condition* $X_T \geq 0$*. Then* $X_t \geq 0$ *for* $0 \leq t < T$*, and* $J(0,T) \geq J_{\min}(0,T) = \underline{x}_0^* X_0 \underline{x}_0$*, which is achieved by taking the optimal control law* $\underline{u}_{\text{opt}}(t) = F_t \underline{x}(t)$ *with* F_t *as in* (3.11)*. If the time horizon* $T \to \infty$*, and the system in* (3.4) *is time-invariant, then* $X_t \to X \geq 0$ *for each finite* $t \geq 0$*, satisfying the ARE*

$$X = A^* X A + C^* C - (A^* X B + C^* D)(D^* D + B^* X B)^+ (B^* X A + D^* C), \qquad (3.13)$$

provided that (A,B) *is stabilizable. Moreover the optimal performance index* $J_{\min}(0,T)$ *approaches to* $J_{\min} = \underline{x}_0^* X \underline{x}_0$*, and the optimal control law approaches to* $\underline{u}_{\text{opt}}(t) = F\underline{x}(t)$ *as* $T \to \infty$ *with*

$$F = -(D^* D + B^* X B)^+ (B^* X A + D^* C). \qquad (3.14)$$

Proof: We notice that the DRE (3.10) can be written as

$$X_t = (A_t + B_t F_t)^* X_{t+1} (A_t + B_t F_t) + (C_t + D_t F_t)^* (C_t + D_t F_t)$$

from which we conclude that $X_t \geq 0$, provided that $X_{t+1} \geq 0$. Hence in light of (3.12), $J(0,T) \geq \underline{x}^*(0) X_0 \underline{x}(0)$. The minimum value of $J(0,T)$ is achievable by taking $\underline{u}(t) = \underline{u}_{\text{opt}}(t) = F_t \underline{u}(t)$, which is in the form of state feedback. If the state-space model is time-invariant, then the stabilizability condition of (A,B) implies the existence of F_s such that $(A + BF_s)$ is a stability matrix. Hence with the control input $\underline{u}(t) = F_s\underline{x}(t)$, the closed-loop system is given by

$$\underline{x}(t+1) = (A + BF_s)\underline{x}(t), \quad \underline{z}(t) = (C + DF_s)\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0 \neq 0.$$

The corresponding quadratic performance index is

$$J_{F_s} = \sum_{t=0}^{\infty} \|\underline{z}(t)\|^2 = \underline{x}_0^* P \underline{x}_0$$

where $P \geq 0$ is the solution to the Lyapunov equation

$$P = (A + BF_s)^* P (A + BF_s) + (C + DF_s)^*(C + DF_s).$$

It can be verified that the above P is finite, and given by

$$P = \sum_{k=0}^{\infty} [(A + BF_s)^k]^* (C + DF_s)^* (C + DF_s)(A + BF_s)^k.$$

On the other hand, denote $X_\infty(t)$ as the solution to the DRE (3.10) for the limiting case $T \to \infty$. Then the optimality of the control law

$$\underline{u}_{\text{opt}}(t) = F_t \underline{x}(t), \qquad F_t = -(D^*D + B^*X_\infty(t)B)^+(B^*X_\infty(t)A + D^*C),$$

implies that $X_\infty(t) \geq 0$ for $t \geq 0$. If the initial time is $t_0 \geq 0$ with the initial condition $\underline{x}(t_0) = \underline{x}_0$, an application of the control law $\underline{u}_{\text{opt}}(t) = F_t\underline{x}(t)$ leads to

$$J_{\min}(t_0, \infty) = \sum_{t=t_0}^{\infty} \|\underline{z}(t)\|^2 = \underline{x}_0^* X_\infty(t) \underline{x}_0 \leq \underline{x}_0^* P \underline{x}_0 \quad \forall\, \underline{x}_0 \neq 0.$$

Hence $0 \leq X_\infty(t_0) \leq P$ for all finite $t_0 \geq 0$. Since the system is time-invariant, $J_{\min}(t_0, \infty)$ depends only on $\underline{x}(t_0) = \underline{x}_0$, rather than t_0 by the fact that the time horizon of the quadratic performance index is infinity, yielding the conclusion that $X_\infty(t_0) = X$, independent of time t_0 as well. The standard argument in the existing LQR theory concludes that $\underline{u}_{\rm opt}(t) = F\underline{x}(t)$ with F as in (3.14) is the optimal control law which achieves the minimum value $J_{\min} = \underline{x}_0^* X \underline{x}_0$. ∎

Theorem 3.1 provides an algorithm to compute a positive semi-definite solution $X \geq 0$: For $k = 0, 1, \cdots$, with $X^{(0)} \geq 0$ given, do the following:

$$\begin{aligned} F^{(k)} &= -(D^*D + B^*X^{(k)}B)^+(B^*X^{(k)}A + D^*C), \\ X^{(k+1)} &= (A + BF^{(k)})^*X^{(k)}(A + BF^{(k)}) + (C + DF^{(k)})^*(C + DF^{(k)}). \end{aligned} \tag{3.15}$$

The algorithm can be terminated if $\|X^{(N)} - X^{(N+1)}\|$ is smaller than some pre-specified tolerance bound.

Remark 3.1 We would like to make the following remarks:

(a) Different from the standard LQR theory, we can not conclude stability of $A+BF$ despite the fact that (A, B) is stabilizable. That is, the optimal feedback system

$$\underline{x}(t+1) = (A + BF)\underline{x}(t), \quad \underline{z}(t) = (C + DF)\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0 \neq 0, \tag{3.16}$$

with F in (3.14), may not be internally stable, even though the energy of the controlled output

$$J_{\min} = \|\underline{z}\|_2^2 = \sum_{t=0}^{\infty} \|\underline{z}(t)\|^2 = \underline{x}_0^* X \underline{x}_0 < \infty$$

is bounded. A careful reflection concludes that any unstable modes of $(A+BF)$ are unobservable based on the controlled output $\underline{z}(t) = (C+DF)\underline{x}(t)$. That is, the unstable modes of $(A+BF)$ are also unobservable modes of $(C+DF, A+BF)$.

(b) It is noted that $X^{(k)} = X_{T-k}$ is the solution to the DRE in (3.10) at time $t = T-k$ with $X_T = X^{(0)} \geq 0$ for the time-invariant system. If $t = T-k \geq 0$ is finite and fixed as $T, k \to \infty$, then $X^{(k)} \to X_\infty(t) = X$ in light of Theorem 3.1. Hence the iterative algorithm in (3.15) is convergent for any initial condition $X^{(0)} \geq 0$.

(c) The ARE (3.13) may admit more than one positive semi-definite solutions. Each one can be viewed as the equilibrium to the DRE

$$X_t = A^*X_{t+1}A+C^*C-(A^*X_{t+1}B+C^*D)(D^*D+B^*X_{t+1}B)^+(B^*X_{t+1}A+D^*C).$$

However there is a unique maximal solution $X_{\max}$, and a unique minimal solution $X_{\min}$ such that any other positive semi-definite solution X to the ARE (3.13) satisfies

$$0 \leq X_{\min} \leq X \leq X_{\max}.$$

If the initial value $X^{(0)} = 0$ for the iterative algorithm in (3.15), then $X^{(k)}$ is likely to converge to $X_{\min}$ as $k \to \infty$. This is intuitively true based on the optimality of the feedback control gain in (3.11) corresponding to $X^{(t)} = X_{T-t}$ for each $t \geq 0$. On the other hand, if $X^{(0)} = \rho I$ with $\rho > 0$ sufficiently large, then $X^{(k)}$ is likely to converge to $X_{\max}$ as $k \to \infty$. In particular if $X^{(0)} \geq 0$ is close to some $X \geq 0$ satisfying the ARE (3.13), then $X^{(k)}$ is likely to be trapped to the same X in a few iterations.

(d) For the problem of inner-outer factorization in Case (i) of (3.20), A is assumed to be a stability matrix. If $X^{(0)} = W$ is chosen as the solution to the Lyapunov equation

$$W = A^* W A + C^* C, \tag{3.17}$$

then $W \geq 0$. Moreover taking the difference between the above Lyapunov equation and the ARE (3.13) yields

$$(W - X) = A^*(W - X)A + (A^* X_{t+1} B + C^* D)(D^* D + B^* X_{t+1} B)^+ (B^* X_{t+1} A + D^* C).$$

Stability of A implies that $W \geq X$ for any positive semi-definite solution to the ARE (3.13). Hence the maximal solution to the ARE (3.13) is likely to be obtained with the iterative algorithm (3.15) with the initial value $X^{(0)} = W$.

(e) A solution $X \geq 0$ to the ARE (3.13) is said to be a stabilizing solution, if $(A + BF)$ is a stability matrix where F has the expression in (3.14). It can be argued as in the standard LQR control that the stabilizing solution to the ARE (3.13) is maximal among all positive semi-definite solutions to (3.13), and thus is $X_{\max}$, if it exists. The existence of the stabilizing solution $X_{\max}$ is hinged to the condition (which is similar to the standard LQR control):

$$\text{rank}\left\{ \begin{bmatrix} A - e^{j\theta} I & B \\ C & D \end{bmatrix} \right\} = n + p \quad \forall\, \theta \in \mathbf{R}, \tag{3.18}$$

in addition to the stabilizability of (A, B). It will be shown later that $X = X_{\max}$ is what needed for computing the inner-outer factorization for Case (i) in (3.20). How to obtain $X = X_{\max}$ using the iterative algorithm (3.15) will be answered in the next section. ∎

The above remarks are dual to those in the previous chapter, and indicate that the limiting optimal solution X to the generalized LQR control is dependent on the boundary condition $X^{(0)}$. The resultant control law can not be implemented in practice, unless $(A + BF)$ is a stability matrix, in which case $X = X_{\max}$. For ease of the reference, we denote F_{m} as the optimal feedback gain associated with $X_{\max}$ as follows:

$$F_{\mathrm{m}} = -(D^*D + B^*X_{\max}B)^+(B^*X_{\max}A + D^*C). \tag{3.19}$$

3.3 Inner-Outer Factorization

In this section we investigate inner-outer factorizations for the following case:

$$m > p: \quad H(z) = H_{\mathrm{i}}(z)H_{\mathrm{o}}(z), \tag{3.20}$$

where $H_{\mathrm{i}}(z)$ is a square inner of the smaller size, and $H_{\mathrm{o}}(z)$ is an outer. In fact, $H_{\mathrm{o}}(z)$ is the right spectral factor of $\Phi(z) = H(z)^{\sim}H(z)$. It will be shown that the results on generalized LQR control is crucial to tackle the inner-outer factorization in this section. The assumption that $D \neq 0$ has no loss of generality, because any causal transfer matrix $H(z)$ can be written as $H(z) = z^{-k}\tilde{H}(z)$ for some $k \geq 0$ and causal transfer matrix $\tilde{H}(z)$ such that $\tilde{D} = \tilde{H}(\infty) \neq 0$. Thus inner-outer factorizations of $\tilde{H}(z)$ can then be studied with z^{-k} subsumed into the inner.

We introduce the following lemmas to help in finding a solution for the inner-outer factorization in (3.20).

Lemma 3.1 *Suppose that* $0 < \text{rank}\{D\} \leq p < m$ *with* $p \times m$ *the dimension of* $H(z)$ *as in* (3.2). *Let* $X \geq 0$ *be a solution to the ARE* (3.13), *and the state feedback gain* F *be as in* (3.14). *Denote* $\Pi = D^*D + B^*XB$. *Then* $\Pi F = -(B^*XA + D^*C)$.

Proof: If Π is invertible, then Π^{-1} exists, and the lemma is clearly true in light of (3.14). Assume that $\text{rank}\{\Pi\} = p_o < m$. Then there exists a Cholesky factorization $\Pi = \Omega^*\Omega$ with $p_o \times m$ the size of Ω. Thus there exists $\Omega_\perp$ of size $(m - p_o) \times m$ such that

$$\Omega_\perp^+\Omega_\perp = I - \Omega^+\Omega, \quad \det\left(\begin{bmatrix} \Omega \\ \Omega_\perp \end{bmatrix}\right) \neq 0. \tag{3.21}$$

Since $\Omega\Omega^+ = I$, and $\Omega_\perp\Omega_\perp^+ = I$, the above implies

$$\begin{bmatrix} \Omega \\ \Omega_\perp \end{bmatrix} \begin{bmatrix} \Omega^+ & \Omega_\perp^+ \end{bmatrix} = I, \quad \Omega_\perp\Omega^+ = 0. \tag{3.22}$$

With the above notation, $F = -\Omega^+(\Omega^+)^*(B^*XA + D^*C)$. Let the rank of X be $r > 0$. Then

$$X = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = U_1\Sigma_1U_1^*, \quad \Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r) > 0, \tag{3.23}$$

by the SVD of X where $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ is a unitary matrix. It follows that

$$\Pi = \begin{bmatrix} B^* U_1 \Sigma_1^{1/2} & D^* \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2} U_1^* B \\ D \end{bmatrix} = \Omega^* \Omega \quad \Longrightarrow \quad \begin{bmatrix} \Sigma_1^{1/2} U_1^* B \\ D \end{bmatrix} = V\Omega \tag{3.24}$$

for some orthogonal matrix V satisfying $V^*V = I$. By $\Omega\Omega^+ = I$, the above yields

$$\begin{aligned} \Pi F &= -\Omega^* \Omega \Omega^+ (\Omega^+)^* (B^* X A + D^* C) = -\Omega^* (\Omega^+)^* \begin{bmatrix} B^* U_1 \Sigma_1^{1/2} & D^* \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2} U_1^* A \\ C \end{bmatrix} \\ &= -\Omega^* (\Omega^+)^* \Omega^* V^* \begin{bmatrix} \Sigma_1^{1/2} U_1^* A \\ C \end{bmatrix} = -(V\Omega)^* \begin{bmatrix} \Sigma_1^{1/2} U_1^* A \\ C \end{bmatrix} = -(B^* X A + D^* C), \end{aligned}$$

which concludes the proof. ∎

The next lemma shows that $\Pi = D^*D + B^*XB$ has the same rank as the normal rank of $H(z)$, which is an important property.

Lemma 3.2 *Suppose that* $\operatorname{rank}\{D\} \leq p < m$ *with* $p \times m$ *the dimension of* $H(z)$ *as in* (3.2)*. Let* $X \geq 0$ *be a solution to the ARE* (3.13)*, and the state feedback gain* F *be as in* (3.14)*. Then there holds*

$$H(z)^{\sim} H(z) = \left[I - F(zI - A)^{-1} B \right]^{\sim} (D^*D + B^*XB) \left[I - F(zI - A)^{-1} B \right]. \tag{3.25}$$

Proof: To show (3.25), we notice that

$$X - A^*XA = (z^{-1}I - A^*)X(zI - A) + (z^{-1}I - A^*)XA + A^*X(zI - A). \tag{3.26}$$

By the expression of the state feedback gain F as in (3.14), the ARE (3.13) can be written as

$$X - A^*XA = -F^*(D^*D + B^*XB)F + C^*C.$$

It follows from the above and (3.26) that (recall that $\Pi = D^*D + B^*XB$)

$$\begin{aligned} C^*C &= X - A^*XA + F^*(D^*D + B^*XB)F \\ &= (z^{-1}I - A^*)X(zI - A) + (z^{-1}I - A^*)XA + A^*X(zI - A) + F^*\Pi F. \end{aligned}$$

Denote $\Theta(z) = (zI - A)^{-1}B$. Then multiplying $\Theta(z)$ from right and $\Theta(z)^\sim$ from left yields

$$\Theta(z)^\sim C^*C\Theta(z) = B^*XB + \Theta(z)^\sim F^*\Pi F\Theta(z) + B^*XA\Theta(z) + \Theta(z)^\sim A^*XB.$$

Since $H(z)^\sim H(z) = D^*D + D^*C\Theta(z) + \Theta(z)^\sim C^*D + \Theta(z)^\sim C^*C\Theta(z)$,

$$H(z)^\sim H(z) = \Theta(z)^\sim F^*\Pi F\Theta(z) + (D^*C + B^*XA)\Theta(z) + \Theta(z)^\sim(A^*XB + C^*D) + \Pi.$$

By Lemma 3.1, $(D^*C + B^*XA) = -\Pi F$, we thus obtain

$$\begin{aligned} H(z)^\sim H(z) &= \Theta(z)^\sim F^*\Pi F\Theta(z) - \Pi F\Theta(z) - \Theta(z)^\sim F^*\Pi + \Pi \\ &= [I - F\Theta(z)]^\sim(D^*D + B^*XB)[I - F\Theta(z)] \end{aligned}$$

which concludes the proof. ∎

It should be now clear that the rank of $\Pi = (D^*D + B^*XB)$ is the same as the normal rank of $H(z)$, by the fact that $[I - F\Theta(z)]$ has the full normal rank. The following observation is also important.

Lemma 3.3 *Suppose that* $\operatorname{rank}\{D\} \le p < m$ *with* $p \times m$ *the dimensions of* $H(z)$ *as in* (3.2). *Let* $X \ge 0$ *be a solution to the ARE* (3.13), *and the state feedback gain* F *be as in* (3.14), *If* $H(z)$ *as in* (3.2) *has normal rank* $p < m$. *then* $\Pi = D^*D + B^*XB$

has rank p. Let $\Pi = \Omega^*\Omega$ be its Cholesky factorization with $p \times m$ the size of Ω, and Ω^+ satisfy (3.22). *Then*

$$G(z) = (C + DF)(zI - A - BF)^{-1}B(I - \Omega^+\Omega) = 0. \tag{3.27}$$

Proof: The identity (3.25) shows that the rank of $\Pi = D^*D + B^*XB$ is the same as the normal rank of $H(z)$, which is p. Hence there exist Ω of size $p \times m$ and $\Omega_\perp$ of size $(m-p) \times m$ such that $\Pi = \Omega^*\Omega$, and (3.21) and (3.22) hold. With the above notation, the feedback gain in (3.14) has the expression $F = -\Omega^+(\Omega^+)^*(B^*XA + D^*C)$. It follows that

$$(I - \Omega^+\Omega)F = -\Omega_\perp^+\Omega_\perp\Omega^+(\Omega^+)^*(B^*XA + D^*C) = 0, \tag{3.28}$$

by $\Omega_\perp\Omega^+ = 0$ as in (3.22). By the SVD in (3.23),

$$\Pi = \Omega^*\Omega = B^*XB + D^*D = \begin{bmatrix} B^*U_1\Sigma^{1/2} & D^* \end{bmatrix} \begin{bmatrix} \Sigma_1^{1/2}U_1^*B \\ D \end{bmatrix}.$$

Thus $\Omega\Omega_\perp^+ = 0$ implies that $D\Omega_\perp^+ = 0$, and $U_1^*B\Omega_\perp^+ = 0$, yielding

$$D\Omega_\perp^+\Omega_\perp = D(I - \Omega^+\Omega) = 0, \quad U_1^*B\Omega_\perp^+\Omega_\perp = U_1^*B(I - \Omega^+\Omega) = 0. \tag{3.29}$$

Rewrite the ARE (3.13) in the form of Lyapunov equation:

$$X = (A + BF)^*X(A + BF) + (C + DF)^*(C + DF) \tag{3.30}$$

where $X \geq 0$ by the hypothesis. Clearly all unstable poles of $G(z)$ as in (3.27) are unobservable modes of $(C + DF, A + BF)$. Applying the similarity transform

$S = U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ to the realization of $G(z)$ gives

$$\begin{aligned} U^*(A+BF)U &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} (A+BF) \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ U^*B(I-\Omega^+\Omega) &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} B(I-\Omega^+\Omega) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ (C+DF)U &= (C+DF) \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \end{aligned}$$

with compatible partitions. It follows from (3.29) that $B_1 = 0$. Multiplying (3.30) by U^* from left, and U from right, and using the above partitions and the SVD of X in (3.23) yield

$$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \Sigma_1 \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} + \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

The fact that $\Sigma_1 > 0$ implies that $A_{12} = 0$ and $C_2 = 0$ in light of the above equation, which coupled with $B_1 = 0$ concludes that (3.27) is true. ∎

We are now ready to present our result on the inner-outer factorization as in (3.20).

Theorem 3.2 *Suppose that $H(z)$ as in (3.2) has normal rank $p < m$ satisfies the condition (3.18), and A is a stability matrix. Let $X_{\mathrm{max}} \geq 0$ be the maximal solution to (3.15), and F_{m} be as in (3.19). Then there holds the inner-outer factorization $H(z) = H_{\mathrm{i}}(z)H_{\mathrm{o}}(z)$ where, with $\Omega_{\mathrm{m}}^*\Omega_{\mathrm{m}} = \Pi = D^*D + B^*X_{\mathrm{max}}B$, the inner and outer are given respectively by*

$$H_{\mathrm{i}}(z) = \left[\begin{array}{c|c} A+BF_{\mathrm{m}} & B \\ \hline C+DF_{\mathrm{m}} & D \end{array}\right] \Omega_{\mathrm{m}}^+, \quad H_{\mathrm{o}}(z) = \Omega_{\mathrm{m}} \left[\begin{array}{c|c} A & B \\ \hline -F_{\mathrm{m}} & I \end{array}\right]. \tag{3.31}$$

Proof: We note that the previous three lemmas hold for any solution $X \geq 0$ to the ARE (3.13). Thus by the proof of Lemma 3.2 and $\Theta(z) = (zI - A)^{-1}B$,

$$T(z) = H(z)[I - F_{\rm m}\Theta(z)]^{-1} = \left[\begin{array}{c|c} A + BF_{\rm m} & B \\ \hline C + DF_{\rm m} & D \end{array}\right]$$

satisfies $T(z)^{\sim}T(z) = D^*D + B^*X_{\rm max}B = \Omega_{\rm m}^*\Omega_{\rm m}$. That is, $\Omega_{\rm m}$ has the same column rank as the normal rank of $H(z)$, which is p, and $H_{\rm i}(z) = T(z)\Omega_{\rm m}^+$ is square and satisfies $H_{\rm i}(z)^{\sim}H_{\rm i}(z) = I_p$. Because the unstable modes of $(A+BF_{\rm m})$ are unobservable modes of $(C+DF_{\rm m}, A+BF_{\rm m})$, they can be eliminated through Kalman decomposition. Hence $H_{\rm i}(z)$ with minimal realization is stable, which is indeed an inner. To verify the expression of $H_{\rm o}(z)$ as in (3.31), we have

$$\begin{aligned} H_{\rm i}(z)H_{\rm o}(z) &= \left[\begin{array}{c|c} A + BF_{\rm m} & B \\ \hline C + DF_{\rm m} & D \end{array}\right] \Omega_{\rm m}^+\Omega_{\rm m} \left[\begin{array}{c|c} A & B \\ \hline -F_{\rm m} & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ -BF_{\rm m} & A + BF_{\rm m} & B\Omega_{\rm m}^+\Omega_{\rm m} \\ \hline -DF_{\rm m} & C + DF_{\rm m} & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A + BF_{\rm m} & -B(I - \Omega_{\rm m}^+\Omega_{\rm m}) \\ \hline C & C + DF_{\rm m} & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = H(z), \end{aligned}$$

in light of Lemma 3.3. Stability of A implies stability of $H_{\rm o}(z)$. Moreover the hypotheses of the theorem imply that the maximal solution $X_{\rm max}$ is stabilizing, or $(A+BF_{\rm m})$ is a stability matrix. Consequently $H_{\rm o}(z)$ admits a right inverse given by

$$H_{\rm o}^+(z) = \left[\begin{array}{c|c} A + BF_{\rm m} & B \\ \hline F_{\rm m} & I \end{array}\right] \Omega_{\rm m}^+$$

which is stable. Hence $H_{\rm o}(z)$ is strict minimum phase, and an outer. ∎

We comment that the outer factor $H_{\rm o}(z)$ has no transmission zeros at $z = \infty$, due to full rank of $\Omega_{\rm m}$ which has size $p \times m$, and the same rank as the normal rank of $H(z)$. The possible transmission zeros of $H(z)$ at $z = \infty$ are now transmission zeros of the inner factor $H_{\rm i}(z)$, which is evident by its expression in (3.31). In the case when $G(z)$ is strict minimum phase, i.e.,

$$\text{rank}\left\{\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}\right\} = n + p \quad \forall\, |z| \geq 1,$$

there is a unique positive semi-definite solution $X \geq 0$. In fact $X = 0$, if $\text{rank}\{D\} = p$.

3.4 Generalized LQG

In this section we study the generalized Linear-Quadratic-Gaussian (LQG) optimal control. As in the standard LQG problem we consider the disturbance rejection problem for

$$\underline{x}(t+1) = A\underline{x}(t) + B_1\underline{d}(t) + B_2\underline{u}(t), \quad \underline{x}(0) = 0 \tag{3.32}$$

$$\underline{w}(t) = C_1\underline{x}(t) + D_{11}\underline{d}(t) + D_{12}\underline{u}(t) \tag{3.33}$$

where $\underline{d}(t)$ is a white disturbance satisfying $\mathrm{E}\left[\underline{d}(t)\right] = 0$ and $\mathrm{E}\left[\underline{d}(t+\tau)\underline{d}^*(t)\right] = I\delta(\tau)$, and $\underline{w}(t)$ is to be controlled. However we do not have measurements of $\underline{x}(t)$ and $\underline{d}(t)$. The only measurements we have is the output measurements

$$\underline{y}(t) = C_2\underline{x}(t) + D_{21}\underline{d}(t) + D_{22}\underline{u}(t), \quad D_{22} = 0 \tag{3.34}$$

The case $D_{22} \neq 0$ will be studied later. Assume that $\underline{d}(t)$ has size m_1, $\underline{u}(t)$ has size m_2, $\underline{w}(t)$ has size p_1, and $\underline{y}(t)$ size is p_2. Different from the standard LQG, the generalized LQG the dimension of the control input $\underline{u}(t)$ is greater than the dimension of the controlled output $\underline{w}(t)$, i.e. $m_2 > p_1$, and the dimension of the input noise $\underline{d}(t)$ is smaller than the dimension of the output measurements $\underline{y}(t)$, i.e. $p_2 > m_1$. Consequently $\det(D_{12}^* D_{12}) = 0$ and $\det(D_{21} D_{21}^*) = 0$ that is contrast to the standard LQG control. However using the same derivation as in the standard LQG case, we can obtain the state-space formulae for design of generalized LQG controller. The strategy is to first design a full information controller, assuming that $\underline{x}(t)$, and $\underline{d}(t)$ are available, and then design an output estimator to estimate linear combination of $\underline{x}(t)$ and $\underline{d}(t)$, based on output measurement $\underline{y}(t)$.

As in Remark 3.1, the control ARE is modified as follows:

$$X = A^*XA + C_1^*C_1 - (A^*XB_2 + C_1^*D_{12})(D_{12}^*D_{12} + B_2^*XB_2)^+(A^*XB_2 + C_1^*D_{12})^* \quad (3.35)$$

The existence of the stabilizing solution to the ARE (3.35) is hinged to the condition

$$\operatorname{rank}\left\{\begin{bmatrix} A - zI & B_2 \\ C_1 & D_{12} \end{bmatrix}\right\} = n + p_1 \quad \forall\, |z| = 1 \quad (3.36)$$

in addition to the stabilizability of (A, B_2). The following theorem is applied for the first step of the design strategy.

Theorem 3.3 *Let $X_{\max}$ be the stabilizing solution to the control ARE* (3.35). *Suppose that the full information, i.e., $x(t)$ and $d(t)$ are available. Then the unique optimal full information control is given by $\underline{u}(t) = F_{\mathrm{m}}x(t) + F_0 d(t)$ with*

$$F_{\mathrm{m}} = -(D_{12}^*D_{12} + B_2^*X_{\max}B_2)^+(B_2^*X_{\max}A + D_{12}^*C_1), \quad (3.37)$$

$$F_0 = -(D_{12}^*D_{12} + B_2^*X_{\max}B_2)^+(B_2^*X_{\max}B_1 + D_{12}^*D_{11}), \quad (3.38)$$

Proof: A simple way to prove the theorem is to convert the full information control into state feedback control by augmenting the state vector into $\underline{x}_a(t) = \underline{x}(t) \oplus \underline{d}(t)$. Hence the state-space model in (3.32) is now augmented into

$$\begin{aligned}
\underline{x}_a(t+1) &= \begin{bmatrix} \underline{x}(t+1) \\ \underline{d}(t+1) \end{bmatrix} = \begin{bmatrix} A & B_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{d}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{d}(t+1) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \underline{u}(t) \\
&= \tilde{A}\underline{x}_a(t) + \tilde{B}_2\underline{u}(t), \qquad \underline{\tilde{d}}(t) = \underline{d}(t+1), \\
\underline{w}(t) &= \begin{bmatrix} C_1 & D_{11} \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{d}(t) \end{bmatrix} + D_{12}\underline{u}(t) = \tilde{C}_1\underline{x}_a(t) + D_{12}\underline{u}(t)
\end{aligned}$$

Now applying the result on generalized LQR control we obtain the optimal control law as

$$\underline{u}(t) = \tilde{F}_{\mathrm{m}}\underline{x}_a(t), \qquad \tilde{F}_{\mathrm{m}} = -(D_{12}^*D_{12} + \tilde{B}_2^*\tilde{X}_{\mathrm{m}}B_2)^+(\tilde{B}_2^*\tilde{X}_{\mathrm{m}}\tilde{A} + D_{12}^*\tilde{C}_1),$$

$$\tilde{X}_{\mathrm{m}} = \tilde{A}^*\tilde{X}_{\mathrm{m}}\tilde{A} + \tilde{C}_1^*\tilde{C}_1 - (\tilde{A}^*\tilde{X}_{\mathrm{m}}\tilde{B}_2 + \tilde{C}_1^*D_{12})(D_{12}^*D_{12} + \tilde{B}_2^*\tilde{X}_{\mathrm{m}}\tilde{B}_2)^+(\tilde{A}^*\tilde{X}_{\mathrm{m}}\tilde{B}_2 + \tilde{C}_1^*D_{12})^*$$

with $\tilde{X}_{\mathrm{m}}$ as the stabilizing solution. Partition $\tilde{X}_{\mathrm{m}}$ as

$$\tilde{X}_{\mathrm{m}} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix}, \qquad \tilde{X}_{11} = X_{\max},$$

where $\tilde{X}_{11}$ has the same size as A. Then the $(1,1)$ position of the augmented ARE is identical to ARE (3.35) as follows:

$$X_{\max} = A^*X_{\mathrm{m}}A + C_1^*C_1 - (A^*X_{\max}B_2 + C_1^*D_{12})(D_{12}^*D_{12} + B_2^*X_{\max}B_2)^+(A^*X_{\mathrm{m}}B_2 + C_1^*D_{12})^* \tag{3.39}$$

Moreover the augmented optimal state feedback gain is given by

$$\tilde{F}_{\mathrm{m}} = \begin{bmatrix} F_{\mathrm{m}} & F_0 \end{bmatrix} = -(D_{12}^*D_{12} + B_2^*X_{\max}B_2)^+(B_2^*X_{\max}\begin{bmatrix} A & B_1 \end{bmatrix} + D_{12}^*\begin{bmatrix} C_1 & D_{11} \end{bmatrix})$$

That is, $\tilde{F}_{\rm m}$ is related to $X_{\max}$ only that verifies the expressions in (3.37) and (3.38).

$\blacksquare$

For the general case of LQG control, the full information is unavailable. We introduce

$$\underline{u}(t) = \underline{v}(t) + F_{\rm m}\underline{x}(t) + F_0\underline{d}(t) \tag{3.40}$$

that is commonly known as variable substitution. Substituting (3.40) into (3.32) and (3.33) shows that

$$\underline{x}(t+1) = (A + B_2F_{\rm m})\underline{x}(t) + (B_1 + B_2F_0)\underline{d}(t) + B_2\underline{v}(t), \tag{3.41}$$

$$\underline{w}(t) = (C_1 + D_{12}F_{\rm m})\underline{x}(t) + (D_{11} + D_{12}F_0)\underline{d}(t) + D_{12}\underline{v}(t). \tag{3.42}$$

It follows that $\underline{w}(t) = H(q)\underline{d}(t) + G(q)\underline{v}(t)$ with q^{-1} the delay operator and

$$H(z) = \left[\begin{array}{c|c} A + B_2F_{\rm m} & B_1 + B_2F_0 \\ \hline C_1 + D_{12}F_{\rm m} & D_{11} + D_{12}F_0 \end{array}\right] =: \left[\begin{array}{c|c} A_F & B_H \\ \hline C_F & D_H \end{array}\right], \tag{3.43}$$

$$G(z) = \left[\begin{array}{c|c} A + B_2F_{\rm m} & B_2 \\ \hline C_1 + D_{12}F_{\rm m} & D_{12} \end{array}\right] =: \left[\begin{array}{c|c} A_F & B_G \\ \hline C_F & D_G \end{array}\right]. \tag{3.44}$$

With the above notation the Lyaponov form of the ARE in (3.39) is equivalent to

$$X_{\max} = A_F^* X_{\max} + C_F^* C_F$$

Recall the identity in (3.26), the above equation leads to

$$\begin{aligned} C_F^*C_F &= X_{\max} - A_F^* X_{\max} \\ &= (z^{-1}I - A_F^*)X_{\max}(zI - A_F) + (z^{-1}I - A_F^*)X_{\max}A_F + A_F^*X_{\rm m}(zI - A_F) \end{aligned}$$

With a similar procedure as in the previous sections, we have

$$\begin{aligned} B_G^*(z^{-1}I - A_F^*)^{-1}C_F^*C_F(zI - A_F)^{-1}B_H &= B_G^*(z^{-1}I - A_F^*)^{-1}A_F^*X_{\max}B_H \\ &+ B_G^*X_{\max}A_F(zI - A_F)^{-1}B_H \\ &+ B_G^*X_{\max}B_H \end{aligned}$$

Direct computation with the above relation leads to

$$\begin{aligned} G(z)^{\sim}H(z) &= D_G^*D_H + B_G^*X_{\max}B_H + (D_G^*C_F + B_G^*X_{\max}A_F)(zI - A_F)^{-1}B_H \\ &\quad + B_G^*(z^{-1}I - A_F^*)^{-1}(D_H^*C_F + B_H^*X_{\mathrm{m}}A_F)^* \end{aligned}$$

By the expressions of A_F, C_F, B_G, and D_G,

$$\begin{aligned} D_G^*D_H + B_G^*X_{\max}B_H &= (D_{12}^*D_{11} + B_2^*X_{\max}B_1) + (D_{12}^*D_{12} + B_2^*X_{\max}B_2)F_0 = 0, \\ D_G^*C_F + B_G^*X_{\max}A_F &= (D_{12}^*C_1 + B_2^*X_{\max}A) + (D_{12}^*D_{12} + B_2^*X_{\max}B_2)F_{\mathrm{m}} = 0 \end{aligned}$$

in light of Lemma 3.1 and Theorem 3.3. As a result

$$G(z)^{\sim}H(z) = B_G^*(z^{-1}I - A_F^*)^{-1}(D_H^*C_F + B_H^*X_{\mathrm{m}}A_F)^* = \sum_{k=1}^{\infty} S_k z^k \tag{3.45}$$

for some $\{S_k\}$ that is strictly anticausal. Because the disturbance is the only external input, ultimately $\underline{v}(t) = T_v(q)\underline{d}(t)$. Hence

$$\underline{w}(t) = H(q)\underline{d}(t) + G(q)\underline{v}(t) = H(q)\underline{d}(t) + G(q)T_v(q)\underline{d}(t).$$

Recall that $\underline{d}(t)$ is white with zero mean and identity covariance. We thus obtain

$$\|\underline{w}\|_{\mathcal{P}}^2 = \|H + GT_v\|_2^2 = \|H\|_2^2 + \|GT_v\|_2^2 + 2J_s$$

where the cross term is given by

$$J_s = \mathrm{Tr}\left\{\frac{1}{j2\pi}\oint_{|z|=1} G(z)^{\sim} H(z) z^{-1}\ dz\right\} = \sum_{k=1}^{\infty}\left\{\frac{1}{j2\pi}\oint_{|z|=1} S_k z^{k-1}\ dz\right\} = 0$$

by Cauchy integral Theorem. Therefore it is concluded that

$$\|\underline{w}\|_{\mathcal{P}}^2 = \|H + GT_v\|_2^2 = \|H\|_2^2 + \|GT_v\|_2^2. \tag{3.46}$$

Furthermore $G(z)\Omega_{\mathrm{m}}^{+}$ is an inner by the result in the previous section. We also recall the result in Lemma 3.3 and equation (3.29) that yield the conclusion

$$G(z)\Omega_{\perp}^{+}\Omega_{\perp} = 0, \qquad \begin{bmatrix} \Omega_{\mathrm{m}}^{+} & \Omega_{\perp}^{+} \end{bmatrix}\begin{bmatrix} \Omega_{\mathrm{m}} \\ \Omega_{\perp} \end{bmatrix} = I$$

We thus obtain the following equality:

$$G(q)\underline{v}(t) = G(q)\begin{bmatrix} \Omega_{\mathrm{m}}^{+} & \Omega_{\perp}^{+} \end{bmatrix}\begin{bmatrix} \Omega_{\mathrm{m}} \\ \Omega_{\perp} \end{bmatrix}\underline{v}(t) = G(q)\Omega_{\mathrm{m}}^{+}\Omega_{\mathrm{m}}\underline{v}(t)$$

where $\Omega_{\mathrm{m}}^{*}\Omega_{\mathrm{m}} = (D_{12}^{*}D_{12} + B_2^{*}XB_2)$. The above relation now leads to

$$\|\underline{w}\|_{\mathcal{P}}^2 \ = \ \|H\|_2^2 + \|Gv\|_2^2 = \|H\|_2^2 + \|\Omega_{\mathrm{m}}v\|_2^2 \tag{3.47}$$

Hence the controller design amounts to design of $\underline{u}(t)$ such that it minimizes

$$\|\Omega_{\mathrm{m}}[\underline{u}(t) - F_{\mathrm{m}}\underline{x}(t) - F_0\underline{d}(t)]\|_2^2 \tag{3.48}$$

which is an output estimation problem. Thus the second step of the design strategy is to design an output estimator to estimate the linear combination of $\underline{x}(t)$ and $\underline{d}(t)$ based on the output measurement $\underline{y}(t)$. We now have

$$\underline{x}(t+1) \ = \ A\underline{x}(t) + B_1\underline{d}(t) + B_2\underline{u}(t), \tag{3.49}$$

$$\underline{y}(t) \ = \ C_2\underline{x}(t) + D_{21}\underline{d}(t). \tag{3.50}$$

We need to estimate $\underline{\mu}(t) = F_{\mathrm{m}}\underline{x}(t) + F_0\underline{d}(t)$, such that

$$\underline{z}(t) = \Omega_{\mathrm{m}}\left[\underline{u}(t) - F_{\mathrm{m}}\underline{x}(t) - F_0\underline{d}(t)\right]. \tag{3.51}$$

has the smallest variance. We assume that the filtering ARE

$$\Sigma = A\Sigma A^* - (A\Sigma C_2^* + B_1D_{21}^*)(D_{21}D_{21}^* + C_2\Sigma C_2^*)^+(A\Sigma C_2^* + B_1D_{21}^*)^* + B_1B_1^* \tag{3.52}$$

has a stabilizing solution $\Sigma_{\max} \geq 0$. The existence of this stabilizing solution is hinged to the condition

$$\operatorname{rank}\left\{\begin{bmatrix} A - zI & B_1 \\ C_2 & D_{21} \end{bmatrix}\right\} = n + m_1 \quad \forall\, |z| = 1 \tag{3.53}$$

in addition to the detectabilty of (C_2, A).

Theorem 3.4 *Let $\Sigma_{\max}$ be the stabilizing solution to the ARE* (3.52). *Then the optimal output estimator to that described in* (3.49) – (3.51) *is given by*

$$\begin{aligned} \hat{\underline{x}}(t+1) &= (A + L_{\mathrm{m}}C_2)\hat{\underline{x}}(t) - L_{\mathrm{m}}\underline{y}(t) + B_2\underline{u}(t), & (3.54) \\ \begin{bmatrix} L_{\mathrm{m}} \\ L_0 \end{bmatrix} &= -\begin{bmatrix} (A\Sigma_{\max}C_2^* + B_1D_{21}^*) \\ (F_{\mathrm{m}}\Sigma_{\max}C_2^* + F_0D_{21}^*) \end{bmatrix}(D_{21}D_{21}^* + C_2\Sigma_{\max}C_2^*)^+. & (3.55) \end{aligned}$$

Substituting $\underline{u}(t) = (F_{\mathrm{m}} + L_0C_2)\hat{\underline{x}}(t) - L_0\underline{y}(t)$ yields LQG controller

$$\begin{aligned} \hat{\underline{x}}(t+1) &= (A + B_2F_{\mathrm{m}} + L_{\mathrm{m}}C_2 + B_2L_0C_2)\hat{\underline{x}}(t) - (B_2L_0 + L_{\mathrm{m}})\underline{y}(t), & (3.56) \\ \underline{u}(t) &= (F_{\mathrm{m}} + L_0C_2)\hat{\underline{x}}(t) - L_0\underline{y}(t). & (3.57) \end{aligned}$$

Although the LQG controller given in the above theorem is derived for the case $D_{22} = 0$, it can be easily adapted to the $D_{22} \neq 0$ case as follows. Let $K_0(z) =$

$\hat{D} + \hat{C}(zI - \hat{A})^{-1}\hat{B}$ where $\hat{D} = -L_0$, $\hat{C} = F_{\rm m} + L_0 C_2$, and

$$\hat{A} = (A + B_2 F_{\rm m} + L_{\rm m} C_2 + B_2 L_0 C_2), \qquad \hat{B} = -(B_2 L_0 + L_{\rm m})$$

that is the LQG controller for the case $D_{22} = 0$. From (3.34) we have $\underline{y}(t) - D_{22}\underline{u}(t) = C_2\underline{x}(t) + D_{21}\underline{d}(t)$, leading to

$$\underline{u}(t) = K_0(q)\left[\underline{y}(t) - D_{22}\underline{u}(t)\right] \tag{3.58}$$

That is, we have

$$\begin{aligned} \hat{\underline{x}}(t+1) &= \hat{A}\hat{\underline{x}}(t) + \hat{B}[\underline{y}(t) - D_{22}\underline{u}(t)], & (3.59)\\ \underline{u}(t) &= \hat{C}\hat{\underline{x}}(t) + \hat{D}[\underline{y}(t) - D_{22}\underline{u}(t)]. & (3.60) \end{aligned}$$

The second equation in the above leads to

$$\underline{u}(t) = (I + \hat{D}D_{22})^{+}[\hat{C}\hat{\underline{x}}(t) + \hat{D}\underline{y}(t)], \tag{3.61}$$

which yields to the feedback controller

$$\begin{aligned} \hat{\underline{x}}(t+1) &= [\hat{A} - \hat{B}D_{22}(I + \hat{D}D_{22})^{+}\hat{C}]\hat{\underline{x}}(t) + \hat{B}(I + D_{22}\hat{D})^{+}\underline{y}(t) & (3.62)\\ \underline{u}(t) &= (I + \hat{D}D_{22})^{+}\hat{C}\hat{\underline{x}}(t) + (I + \hat{D}D_{22})^{+}\hat{D}\underline{y}(t) & (3.63) \end{aligned}$$

This concludes the generalized LQG control.

3.5 Illustrative Examples

In this section we present examples to demonstrate the proposed iterative algorithm to solve the ARE associated with the generalized LQR control and its applications.

Example 3.1 This example is minimization of the mean-squared error for tracking in feedback control system design shown in Figure 3.1 where the control input $\underline{u}(t)$ has dimension m, and the controlled signal $\underline{y}(t)$ has dimension $p < m$. The plant model $P(z)$ is assumed to be stable, and the command signal $\underline{r}(t)$ is white. The goal is to design controller $K(z)$ such that the closed-loop system maintains its stability, and the mean-squared error for tracking is minimized.

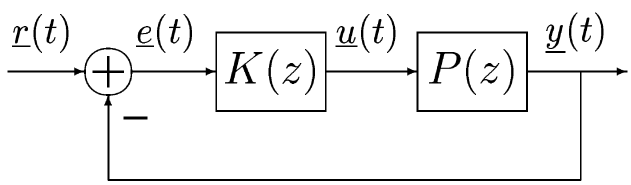

Figure 3.1: Block diagram for tracking feedback control

Because $P(z)$ is stable, all controllers that maintain stability of the closed-loop system are parameterized by $K(z) = [I - Q(z)P(z)]^{-1}Q(z)$ where $Q(z)$ is causal and stable, which is termed as Q-parameterization [44]. Let q^{-1} be the unit delay operator. The tracking error is now given by

$$\underline{e}(t) = \underline{r}(t) - \underline{y}(t) = [I - P(q)Q(q)]\,\underline{r}(t). \tag{3.64}$$

In light of the Parseval's theorem, the mean-squared error for tracking has the expression

$$E[\|\underline{e}(t)\|^2] = \|I - PQ\|_2^2 := \text{Trace}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[I - P(e^{j\omega})Q(e^{j\omega})\right]\left[I - P(e^{j\omega})Q(e^{j\omega})\right]^* d\omega\right\}. \tag{3.65}$$

Let (A, B, C, D) be a realization of $P(z)$ with A a stability matrix. Then we have

$$\underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t), \qquad \underline{e}(t) = \underline{r}(t) - \underline{y}(t) = -C\underline{x}(t) - D\underline{u}(t) + \underline{r}(t), \tag{3.66}$$

where $\underline{u}(t) = Q(q)\underline{r}(t)$. The minimization of the tracking error is over all possible causal and stable $Q(z)$. Since $\underline{r}(t)$ is command input, it is known to the controller. Setting $\underline{\tilde{x}}(t) = \underline{x}(t) \oplus \underline{r}(t)$ and $\underline{d}(t) = \underline{r}(t+1)$ yields the augmented state-space equation

$$\underline{\tilde{x}}(t+1) = \tilde{A}\underline{\tilde{x}}(t) + \tilde{B}_1\underline{d}(t) + \tilde{B}_2\underline{u}(t), \qquad \underline{e}(t) = \tilde{C}\underline{\tilde{x}}(t) + \tilde{D}\underline{u}(t) \tag{3.67}$$

where $\tilde{C} = \begin{bmatrix} -C & I \end{bmatrix}$, $\tilde{D} = -D$, and

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_1 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \tilde{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Due to the fact that $\underline{d}(t)$ in (3.67) represents the dummy equation for $\underline{r}(\cdot) = \underline{r}(\cdot)$, it has no effect on the tracking performance. Thus we have an equivalent $\mathcal{H}_2$ optimal control problem: design control input $\underline{u}(t)$ such that the $\mathcal{H}_2$-norm of the closed-loop system is minimized. Such an optimal control problem is usually decomposed into two separate problems with the first one an equivalent LQR control, and the second an optimal estimation. However we note that $\tilde{D} = -D$ has size $p \times m$ with $p < m$, and may not have rank p. Thus regular LQR solution is not applicable here.

If A has size $n \times n$, then an application of the results in Section 3 leads us to the ARE of size $(n+p) \times (n+p)$. Due to the special structure of $\tilde{A}$ and $\tilde{B}_2$, we need solve only the following ARE:

$$X = A^*XA + C^*C - (A^*XB + C^*D)(D^*D + B^*XB)^+(B^*XA + D^*C)$$

which is identical to (3.13). Hence the stabilizing solution $X_{\max}$ can be obtained by using the iterative algorithm in (3.15) with initial value W, solved from (3.17). The optimal control gain can then be obtained as

$$\tilde{F}_m = \begin{bmatrix} F_m & J_m \end{bmatrix} = -(D^*D + B^*X_{\max}B)^{-1} \begin{bmatrix} B^*X_{\max}A + D^*C & -D^* \end{bmatrix}. \quad (3.68)$$

Finally substituting the optimal control input $\underline{u}_{\text{opt}}(t) = \tilde{F}_m\underline{\tilde{x}}(t)$ into (3.66) yields

$$\underline{x}(t+1) = (A + BF_m)\underline{x}(t) + BJ_m\underline{r}(t), \qquad \underline{u}(t) = \underline{u}_{\text{opt}}(t) = F_m\underline{x}(t) + J_m\underline{r}(t), \quad (3.69)$$

that is the realization for $Q(z)$. Recall that $Q(z) = K(z)[I + P(z)K(z)]^{-1}$ is the transfer matrix from $\underline{r}(t)$ to $\underline{u}(t)$. The computation of $K(z)$ based on $Q(z)$ is omitted.

■

We would like to comment that although we have derived the solution procedure for the optimal tracking problem, it may not lead to good performance, due to the lack of other performance parameters in design. In order to achieve robust performance, model uncertainty may have to be addressed, and the penalty on the control signal $\underline{u}(t)$ needs be taken into account as well. On the other hand the above example does indicate that generalized LQR control has applications to feedback control system design.

Example 3.2 This example is motivated from multiuser wireless data communications. Because of the multipath phenomena, the discretized wireless channels is represented by

$$H(z) = H_0 + H_1z^{-1} + \cdots + H_\ell z^{-\ell}, \qquad H_k \in \mathbf{C}^{p\times m}. \quad (3.70)$$

Thus the received signal data are convolution of the discretized channel with the transmitted digital data, plus the observation noise. As such it causes the problem of inter-symbol-interference (ISI) and inter-channel interference (ICI) that poses the difficulty in symbol detection. A conventional approach to eliminate such effects is through channel equalization. A precoding filter (precoder) at the transmitter side and is applied prior to transmission to equalize the signal at the output of the receive filter. This is especially needed in the downlink mobile radio channel where we aim to keep the receiver units as simple as possible. Precoders are used in transmitters to compensate for distortion caused by channel response, and to reduce the effects of ISI and ICI. We deal with the case when $p < m$ for the precoder design which is a commonly used assumption for the downlink model, where the transmitter side can have plentiful resources. Therefore the channel transfer matrix $H(z)$ is fat and can be assumed to be strictly minimum phase. A simple state-space realization for $H(z)$ is

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ \vdots \\ H_\ell \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \quad D = H_0. \tag{3.71}$$

In [21], it is assumed that $D = H_0$ has rank p, and has derived optimal ZF precoder, which is the causal and stable right inverse of $H(z)$, that has the smallest $\mathcal{H}_2$-norm. However due to the existence of the transmission delay, full rank assumption on H_0

is not realistic. Hence we apply the algorithm in Section 3 to compute inner-outer factorization $H(z) = H_{\rm i}(z)H_{\rm o}(z)$, where $H_{\rm i}(z)$ is FIR, and has all its zeros at infinity, and $H_{\rm o}(\infty)$ has the full column rank. Let d be the minimum integer such that $z^{-d}H_{\rm i}(z)^{\sim}$ is causal. Then the optimal precoder can be obtained as

$$F(z) = z^{-d}H_{\rm o}(z)^{(\rm inv)}H_{\rm i}^{\sim}(z) \quad \Longrightarrow \quad H(z)F(z) = z^{-d}I, \tag{3.72}$$

where $H_{\rm o}^{(\rm inv)}(z)$ is the causal and stable right inverse of $H_{\rm o}(z)$, that has the smallest $\mathcal{H}_2$-norm. How to compute such as an optimal channel equalizer has been solved in [21].

In the following we consider a numerical example with $H(z)$ of size 2×3, specified by

$$D = \begin{bmatrix} -1.4410 & 0.5711 & -0.3999 \\ -2.0265 & 0.8032 & -0.5624 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.1867 & 1.0668 & 0.7143 \\ 0.7258 & 0.0593 & 1.6236 \\ -0.5883 & -0.0956 & -0.6918 \\ 2.1832 & -0.8323 & 0.8580 \\ -0.1364 & 0.2944 & 1.2540 \\ 0.1139 & -1.3362 & -1.5937 \end{bmatrix}$$

where $\ell = 3$. Recall realization as in (3.71). Each element in D and B is generated as a normal random variable with zero mean, and unit variance, and the two rows of $D = H_0$ are linearly dependent. It can be verified that $H(z)$ is strictly minimum phase. It is noted that in applying the iterative algorithm in (3.15), we do not need begin with initial condition $X^{(0)} = A^* X^{(0)} A + C^* C$, with (A, B, C, D) a realization of $H(z)$, due to the assumption that $H(z)$ is strictly minimum phase. In fact with $X^{(0)} = 0$, the iterative algorithm in (3.15) often converges faster, based on our numerical experience. For the given numerical example with the error tolerance 10^{-10}, the number of iterations is only 2, if $X^{(0)} = 0$; The number of iterations is 6, if $X^{(0)} = A^* X^{(0)} A + C^* C$. Both give the same $X_{\max}$. With $X_{\max}$ obtained, the inner and outer factors are computed according to Theorem 3.2. It turns out that the reachability gramian of $H_{\mathrm{i}}(z)$ has rank 1, implying that the minimal realization of $H_{\mathrm{i}}(z)$ has an order at most 1. Since $H(z)$ has only one zero outside the unit circle at infinity, $H_{\mathrm{i}}(z)$ has McMillan degree exactly 1. After eliminating the unreachable modes of $H_{\mathrm{i}}(z)$ we obtain

$$H_{\mathrm{i}}(z) = \begin{bmatrix} 0.5486 & 0.1869 \\ 0.7715 & 0.2627 \end{bmatrix} + \begin{bmatrix} 0.815 \\ -0.5795 \end{bmatrix} \begin{bmatrix} 0.3224 & -0.9466 \end{bmatrix} z^{-1},$$

which is indeed an inner. The outer factor is specified by

$$\Omega_{\mathrm{m}} = \begin{bmatrix} -2.5385 & 1.2021 & -0.7689 \\ -0.2594 & -0.4727 & 0.1172 \end{bmatrix},$$

$$F_{\mathrm{m}} = \begin{bmatrix} 0.3234 & 0.4547 & -0.5785 & 0.4113 & -0.0000 & 0.0000 \\ 0.2124 & 0.2985 & -1.2349 & 0.8782 & -0.0001 & 0.0000 \\ -0.0222 & -0.0312 & 0.3207 & -0.2281 & 0.0000 & -0.0000 \end{bmatrix}.$$

The iterative algorithm works quite efficiently in terms of the computational complexity. ∎

3.6 Conclusion

In this chapter we investigated the generalized LQR control and inner-outer factorization problems. We proposed an iterative algorithm for computing the positive semi-definite solution to the corresponding ARE that was employed to solve the inner-outer factorization problem considered in this chapter. The stabilizing solution to the ARE associated with the generalized LQR control was used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. Examples are used to illustrate the proposed solution algorithms. In addition to the above, we investigated the generalized LQG optimal controller problem. While we can assume stability for inner-outer factorization, we can not assume the stability of A for the generalized LQR control in general. A simple way to bypass the instability issue is to set the control gain $F = F_1 + F_2$ where $(A + BF_1)$ is a stability matrix, and then consider the generalized LQR control for

$$x(t+1) = (A + BF_1)x(t) + Bu(t), \quad z(t) = (C + DF_1)x(t) + Du(t). \tag{3.73}$$

The above leads to the following modified algorithm

$$\begin{aligned} F^{(k)} &= -(D^*D + B^*X^{(k)}B)^+[B^*X^{(k)}(A + BF_1) + D^*(C + DF_1)] + F_1, \\ X^{(k+1)} &= (A + BF^{(k)})^*X^{(k)}(A + BF^{(k)}) + (C + DF^{(k)})^*(C + DF^{(k)}), \end{aligned} \tag{3.74}$$

Although iterative algorithm is derived for computing a solution to the ARE associated with generalized Kalman filtering, it is unclear how to choose the boundary condition $X^(0) \geq 0$ that will ensure its convergence to the required stabilizing solution. It turns out that such an issue has to be resolved together with that for spectral factorization which will be discussed in Chapter 4.

Chapter 4

Spectral Factorization and Convergence Results

4.1 Introduction

In this chapter we investigate the spectral factorization problem, entailed by blind channel estimation. The results in this chapter are also applicable to generalized Kalman filtering, generalized LQR control, and to inner-outer factorizations for non-square transfer function matrices where the inner factor is square and has a smaller size. Recall that in the previous two chapters, recursive algorithms are proposed to iteratively solve the algebraic Riccati equations (AREs) associated with the generalized Kalman filtering and generalized LQR control. However there is no guarantee for the convergence of the two iterative algorithms in solving the stabilizing solution to the corresponding AREs. In addition the inner-outer factorizations are not feasible without the stabilizing solution to the same AREs for the generalized Kalman filtering and generalized LQR control.

As discussed in Chapter 1, the PSD matrix considered in this dissertation has the normal rank strictly smaller than the size of the PSD matrix, that is motivated by blind channel estimation based on the second order statistics of the received signal. Such a spectral factorization problem is considerably harder than the one in the case of full normal rank. Most of the existing work on spectral factorizations assume the full normal rank for the PSD matrix. There lack effective computational algorithms for spectral factorizations where the PSD matrix does not admit full normal rank. We again propose an iterative algorithm for computing the required spectral factorization entailed by the blind channel estimation. More importantly we reveal an intrinsic relation between the required spectral factorizations and the inner-outer factorizations studied in the previous two chapters characterized by their corresponding AREs. Such a relation helps to choose the correct initial or boundary conditions for the corresponding AREs for which the convergence of the recursive algorithms to the stabilizing solutions can be established.

The contents of this chapter are organized as follows. After the introduction section, the spectral factorization problem is formulated in Section 4.2 together with some background materials, which motivate the iterative algorithm for spectral factorizations. In Section 4.3, we establish the relation between the inner-outer factorization and the spectral factorization, based on which we establish the convergence of the proposed iterative algorithm. Such a convergence result also provides the initial or boundary condition for the iterative algorithm in solving the stabilizing solutions to the AREs associated with the generalized Kalman filtering and the generalized LQR control. Section 4.4 contains several numerical examples that illustrate the results in this chapter. The chapter is concluded in Section 4.5.

4.2 Iterative Algorithm for Spectral Factorization

Consider the $q \times q$ para-hermitian matrix $\Phi(z)$ which is positive semi-definite on the unit circle. Such a para-hermitian matrix has the form

$$\Phi(z) = \sum_{k=-\infty}^{\infty} R_k z^{-k} = R_0 + C_\Phi (zI - A)^{-1} B_\Phi + B_\Phi^* (z^{-1} I - A^*)^{-1} C_\Phi^*, \tag{4.1}$$

where A is a stability matrix, and the normal rank of $\Phi(z)$ is $\rho < q$. This problem is much harder than the case of full normal rank. Since $\Phi(z)$ is positive semi-definite on the unit circle, there exist minimal degree factorizations [1]

$$\Phi(z) = W_G(z)^{\sim} W_G(z) = W_K(z) W_K(z)^{\sim}, \tag{4.2}$$

where $W_G(z)$ of size $\rho \times q$ and $W_K(z)$ of size $q \times \rho$ are both stable, given by

$$W_G(z) = \left[\begin{array}{c|c} A & B_\Phi \\ \hline G & D_G \end{array}\right], \quad W_K(z) = \left[\begin{array}{c|c} A & K \\ \hline C_\Phi & D_K \end{array}\right], \tag{4.3}$$

for some (G, D_G) and (K, D_K). The following result is modified from [1]. For the sake of completion, we include a proof for the lemma.

Lemma 4.1 *Let the positive para-hermitian matrix $\Phi(z)$ be given in (4.1) where A is a stability matrix. There exist minimal degree factorizations as in (4.2) for some $W_G(z)$ and $W_K(z)$ in the form of (4.3), if and only if*

$$P = A^*PA + G^*G, \qquad C_\Phi = D_G^* G + B_\Phi^* P A, \qquad R_0 = D_G^* D_G + B_\Phi^* P B_\Phi, \tag{4.4}$$

$$Q = AQA^* + KK^*, \qquad B_\Phi^* = D_K K^* + C_\Phi Q A^*, \qquad R_0 = D_K D_K^* + C_\Phi Q C_\Phi^*, \tag{4.5}$$

admit solutions (P, G, D_G), *and* (Q, K, D_K) *respectively.*

Proof: Suppose that (4.4) and (4.5) admit solutions (P, G, D_G), and (Q, K, D_K) respectively. Then stability of A implies that $P \geq 0$, and $Q \geq 0$ that satisfy the Lyapunov equations in (4.4) and (4.5), respectively. By direct calculation we have

$$G^*G = P - A^*PA = (z^{-1}I - A^*)P(zI - A) + (z^{-1}I - A^*)PA + A^*P(zI - A), \quad (4.6)$$

$$KK^* = Q - AQA^* = (zI - A)Q(z^{-1}I - A^*) + (zI - A)QA^* + AQ(z^{-1}I - A^*). \quad (4.7)$$

Denote

$$\begin{aligned} \Phi_G(z) &= B_\Phi^*(z^{-1}I - A^*)^{-1}G^*G(zI - A)^{-1}B_\Phi, \\ \Phi_K(z) &= C_\Phi(zI - A)^{-1}KK^*(z^{-1}I - A^*)^{-1}C_\Phi^*. \end{aligned} \quad (4.8)$$

Then multiplying (4.6) by $B_\Phi^*(z^{-1}I - A^*)^{-1}$ from left and by $(zI - A)^{-1}B_\Phi$ from the right, and multiplying (4.7) by $C_\Phi(zI - A)^{-1}$ from left and by $(z^{-1}I - A^*)^{-1}C_\Phi^*$ from right yield

$$\Phi_G(z) = B_\Phi^*PB_\Phi + B_\Phi^*PA(zI - A)^{-1}B_\Phi + B_\Phi^*(z^{-1}I - A^*)^{-1}A^*PB_\Phi, \quad (4.9)$$

$$\Phi_K(z) = C_\Phi QC_\Phi^* + C_\Phi(zI - A)^{-1}AQC_\Phi^* + C_\Phi QA^*(z^{-1}I - A^*)^{-1}C_\Phi^*. \quad (4.10)$$

The later two equalities in (4.4) and the expression in (4.9) imply that the para-hermitian matrix in (4.1) can be written as

$$\begin{aligned} \Phi(z) &= D_G^*D_G + B_\Phi^*PB_\Phi + [D_G^*G + B_\Phi^*PA](zI - A)^{-1}B_\Phi \\ &\qquad + B_\Phi^*\left(z^{-1}I - A^*\right)^{-1}[D_G^*G + B_\Phi^*PA]^* \\ &= D_G^*D_G + \Phi_G(z) + D_G^*G(zI - A)^{-1}B_\Phi + B_\Phi^*(z^{-1}I - A^*)^{-1}D_G^*G \\ &= \left[D_G^* + B_\Phi^*(z^{-1}I - A^*)^{-1}G^*\right]\left[D_G + G(zI - A)^{-1}B_\Phi\right] = W_G(z)^\sim W_G(z). \end{aligned}$$

Similarly the later two equalities in (4.5) and the expression in (4.10) imply that the para-hermitian matrix in (4.1) can be written as

$$\begin{aligned}\Phi(z) &= D_K D_K^* + C_\Phi Q C_\Phi^* + C_\Phi (zI-A)^{-1}[D_K K^* + C_\Phi Q A^*]^* \\ &\qquad + [D_K K^* + C_\Phi Q A^*](z^{-1}I - A^*)^{-1} C_\Phi^* \\ &= D_K D_K^* + \Phi_K(z) + C_\Phi (zI-A)^{-1} K D_K^* + D_K K^* (z^{-1}I - A^*)^{-1} C_\Phi^* \\ &= \left[D_K + C_\Phi (zI-A)^{-1} K\right]\left[D_K^* + K^*(z^{-1}I - A^*)^{-1} C_\Phi^*\right] = W_K(z) W_K(z)^\sim.\end{aligned}$$

Hence the existence of solutions (P, G, D_G) and (Q, K, D_K) to (4.4) and (4.5), respectively is sufficient to the existence of minimal degree factors in (4.3).

Conversely, assume that the minimal degree factors in (4.3) exist such that (4.2) holds true. That is, $\Phi(z) = W_G(z)^\sim W_G(z) = W_K(z) W_K(z)^\sim$ where $W_G(z)$ and $W_K(z)$ are given as in (4.3) for some (D_G, G_K) and (G, K). Then $\Phi(z) = W_G(z)^\sim W_G(z)$ implies that

$$\Phi(z) = D_G^* D_G + D_G^* G (zI - A)^{-1} B_\Phi + B_\Phi^* (z^{-1}I - A^*)^{-1} G^* + \Phi_G(z). \qquad (4.11)$$

Stability of A implies that $P = A^* P A + G^* G$ has a unique solution $P \geq 0$, and thus (4.6) and (4.9) hold true. It follows that

$$\begin{aligned}\Phi(z) &= D_G^* D_G + B_\Phi^* P B_\Phi + [D_G^* G + B_\Phi^* P A]\,(zI-A)^{-1} B_\Phi \\ &\qquad + B_\Phi^* \left(z^{-1}I - A^*\right)^{-1} [D_G^* G + B_\Phi^* P A]^*\end{aligned}$$

and hence $R_0 = D_G^* D_G + B_\Phi^* P B_\Phi$ and $C_\Phi = D_G^* G + B_\Phi^* P A$ in light of the expression of $\Phi(z)$ in (4.1), implying that (4.4) has some solution (P, G, D_G). Similarly $\Phi(z) =$

$W_K(z)W_K(z)^{\sim}$ yields

$$\Phi(z) = D_K D_K^* + D_K K^*(z^{-1}I - A^*)^{-1}C_\Phi^* + C_\Phi(zI - A)^{-1}KD_K^* + \Phi_K(z). \quad (4.12)$$

Stability of A implies that $Q = AQA + KK^*$ has a unique solution $Q \geq 0$, and thus (4.7) and (4.10) hold true. It follows that

$$\begin{aligned} \Phi(z) \;\; &= \;\; D_K D_K^* + C_\Phi Q C_\Phi^* + C_\Phi(zI - A)^{-1}(KD_K^* + AQC_\Phi^*) \\ &\qquad\qquad + (D_K K^* + C_\Phi Q A^*)(z^{-1}I - A^*)^{-1}C_\Phi^*. \end{aligned}$$

Hence $R_0 = D_K D_K^* + C_\Phi Q C_\Phi^*$ and $B_\Phi^* = D_K K^* + C_\Phi Q A^*$ hold true in light of the expression of $\Phi(z)$. The proof is now complete. ∎

Lemma 4.1 and its proof show that in order to obtain the minimal degree factors $W_G(z)$ and $W_K(z)$ in (4.3), i.e., (G, D_G) and (K, D_K), we need first solve for P and Q in (4.4) and (4.5), respectively, which are two Lyapunov equations. Since A is stable, $P \geq 0$, and $Q \geq 0$, if they exist. In fact (4.4), and (4.5) have solutions (P, G, D_G), and (Q, K, D_K), respectively, if and only if $\Phi(z) \geq 0$ for all $|z| = 1$, in light of Lemma 4.1. However more than one set of such solutions (P, G, D_G), or (Q, K, D_K) exist, implying that more than one pair of minimal degree factors exist. Nevertheless there are unique sets of solutions (P, G, D_G), and (Q, K, D_K) such that both $W_G(z)$ and $W_K(z)$ as in (4.3) are outer functions, i.e.,

$$\text{rank } \{ W_G(z) \} = \text{rank} \{ W_K(z) \} = \rho \;\; \forall \; |z| \geq 1. \quad (4.13)$$

Such $W_K(z)$ and $W_G(z)$ are exactly the left and right spectral factors, respectively.

The spectral factorization problem in this section corresponds one set of the minimal degree factors, and the spectral factors are unique upto a factor of unitary matrices. Because not every set of solutions to (4.4) or to (4.5) yields spectral factors of $\Phi(z)$, our goal is to obtain the right sets of solutions such that the resultant $W_G(z)$, and $W_K(z)$ are spectral factors of $\Phi(z)$, and satisfy (4.2). For this purpose the results from the previous chapters play the pivotal role.

Let us examine further the equalities

$$C_\Phi = D_G^* G + B_\Phi^* P A, \qquad B_\Phi^* = D_K K^* + C_\Phi Q A^* \tag{4.14}$$

from (4.4) and (4.5), respectively. The above equations imply

$$\mathcal{R}(C_\Phi - B_\Phi^* P A) \subseteq \mathcal{R}(D_G^*), \quad \mathcal{R}(B_\Phi^* - C_\Phi Q A^*) \subseteq \mathcal{R}(D_K). \tag{4.15}$$

As a result, $G = (D_G^+)^*(C_\Phi - B_\Phi^* P A)$, $K = (B_\Phi - AQC_\Phi^*)(D_K^+)^*$, and hence

$$\begin{aligned} G^*G &= (C_\Phi - B_\Phi^* PA)^*(D_G^* D_G)^+(C_\Phi - B_\Phi^* PA) \qquad (4.16) \\ &= (C_\Phi - B_\Phi^* PA)^*(R_0 - B_\Phi^* P B_\Phi)^+(C_\Phi - B_\Phi^* PA), \\ KK^* &= (B_\Phi - AQC_\Phi^*)(D_K D_K^*)^+(B_\Phi - AQC_\Phi^*)^* \qquad (4.17) \\ &= (B_\Phi - AQC_\Phi^*)(R_0 - C_\Phi Q C_\Phi^*)^+(B_\Phi - AQC_\Phi^*)^*. \end{aligned}$$

Consequently the two Lyapunov equations in (4.4) and (4.5) now have the form of AREs:

$$P = A^*PA + (C_\Phi - B_\Phi^* PA)^*(R_0 - B_\Phi^* P B_\Phi)^+(C_\Phi - B_\Phi^* PA), \tag{4.18}$$

$$Q = AQA^* + (B_\Phi - AQC_\Phi^*)(R_0 - C_\Phi Q C_\Phi^*)^+(B_\Phi - AQC_\Phi^*)^*, \tag{4.19}$$

respectively. The following result is again modified from [1].

Lemma 4.2 *Suppose that* $\Phi(z) \geq 0$ *for all* $|z| = 1$*. Then all solutions* P *and* Q *to* (4.18)*, and* (4.19) *respectively are non-negative definite. There exist maximal solutions* $P_{\max}$*,* $Q_{\max}$*, and minimal solutions* $P_{\min}$*,* $Q_{\min}$ *to* (4.18)*, and* (4.19)*, respectively. All other solutions* P*, and* Q *to* (4.18)*, and* (4.19)*, respectively satisfy* $P_{\min} \leq P \leq P_{\max}$ *and* $Q_{\min} \leq Q \leq Q_{\max}$*.*

The solution sets corresponding to $P_{\min}$, and $Q_{\min}$ are associated with right, and left spectral factors of $\Phi(z)$, respectively, while $P_{\max}$, and $Q_{\max}$ are associated with those factors $W_G(z)$, and $W_K(z)$, whose transmission zeros are all outside the unit circle, respectively. Any other solutions P, and Q being neither minimal, nor maximal correspond to those factors $W_G(z)$, and $W_K(z)$ which contain some non-minimum phase zeros. The computation of $P_{\min}$, and $Q_{\min}$ is the main focus of this section, which yield the minimal degree spectral factors of $\Phi(z)$ in (4.1). We propose the following iterative algorithm.

- Set initial values $P_0 = 0$, and $Q_0 = 0$.
- For $k = 0, 1, \cdots$, compute

$$\begin{aligned} P_{k+1} &= A^* P_k A + (C_\Phi - B_\Phi^* P_k A)^* (R_0 - B_\Phi^* P_k B_\Phi)^+ (C_\Phi - B_\Phi^* P_k A), \quad (4.20) \\ Q_{k+1} &= A Q_k A^* + (B_\Phi - A Q_k C_\Phi^*)(R_0 - C_\Phi Q_k C_\Phi^*)^+ (B_\Phi - A Q_k C_\Phi^*)^* \quad (4.21) \end{aligned}$$

- If $\|P_N - P_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{P_k\}$; If $\|Q_N - Q_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{Q_k\}$.

The above algorithm in computing the minimal solutions $P_{\min}$ and $Q_{\min}$ is similar to the iterative algorithms proposed in the previous two chapters. The difference lies in how the initial values are specified. Recall that the iterative algorithms in the previous two chapters have not fixed the initial or boundary values in their respective iterative algorithms. In the next section we will reveal an intrinsic relation between the AREs in the previous two chapters and the AREs in this chapter that will help choose the correct initial or boundary values for the two iterative algorithms in the previous two chapters that lead to the stabilizing solutions or the maximal solutions.

4.3 Convergence of the Iterative Algorithm

In this section we will show that the spectral factorization algorithm proposed in the previous section is convergent with limit $P_{\min}$, and $Q_{\min}$. For this purpose define $D_{G_{\mathrm{m}}}$ and $D_{K_{\mathrm{m}}}$ as the minimum Cholesky factors via

$$D_{G_{\mathrm{m}}}^* D_{G_{\mathrm{m}}} = R_0 - B_\Phi^* P_{\min} B_\Phi, \quad D_{K_{\mathrm{m}}} D_{K_{\mathrm{m}}}^* = R_0 - C_\Phi Q_{\min} C_\Phi^*. \tag{4.22}$$

Similarly define G_{m} and K_{m} as

$$G_{\mathrm{m}} = (D_{G_{\mathrm{m}}}^+)^* (C_\Phi - B_\Phi^* P_{\min} A), \quad K_{\mathrm{m}} = (B_\Phi - A Q_{\min} C_\Phi^*)(D_{K_{\mathrm{m}}}^+)^*. \tag{4.23}$$

Then $(A, K_{\mathrm{m}}, C_\Phi, D_{K_{\mathrm{m}}})$, and $(A, B_\Phi, G_{\mathrm{m}}, D_{G_{\mathrm{m}}})$ are realizations associated with left, and right spectral factors of $\Phi(z)$, respectively. That is,

$$W_{K_{\mathrm{m}}}(z) = \left[\begin{array}{c|c} A & K_{\mathrm{m}} \\ \hline C_\Phi & D_{K_{\mathrm{m}}} \end{array}\right], \quad W_{G_{\mathrm{m}}}(z) = \left[\begin{array}{c|c} A & B_\Phi \\ \hline G_{\mathrm{m}} & D_{G_{\mathrm{m}}} \end{array}\right] \tag{4.24}$$

are the left, and right spectral factors of $\Phi(z)$, respectively, and are thus outers. In light of Lemma 3.2 and Theorem 3.2, $D_{G_{\mathrm{m}}}$ has rank ρ, and in light of Lemma 2.2 and Theorem 2.3, $D_{K_{\mathrm{m}}}$ also has rank ρ. As a result, $D_{G_{\mathrm{m}}}$ and $D_{K_{\mathrm{m}}}$ have dimensions $\rho \times q$ and $q \times \rho$, respectively, and thus have the full rank. Recall that ρ is the normal rank of $\Phi(z)$. However for any other minimal degree factors $W_K(z)$ and $W_G(z)$ as in (4.3) which are not spectral factors of $\Phi(z)$, the associated D_G and D_K may have ranks strictly smaller than ρ. It is crucial to observe that the right spectral factor of $\Phi(z)$ can be obtained from the inner-outer factorization of $H(z) = W_G(z)$ as in (3.20), and the left spectral factor of $\Phi(z)$ can be obtained from the inner-outer factorization of $H(z) = W_K(z)$ as in (2.47). Hence the following result is true.

Theorem 4.1 *Consider minimal degree factors $W_G(z)$ of size $\rho \times q$, and $W_K(z)$ of size $q \times \rho$ as in* (4.3)*, which are not spectral factors of $\Phi(z)$, but satisfy* (4.2) *with $\rho < q$, where $\Phi(z) \geq 0$ for all $|z| = 1$. Then for any $X^{(0)} \geq 0$, and $\Sigma_0 \geq 0$ with $T > 0$, the following DREs*

$$X^{(k+1)} = A^* X^{(k)} A + G^* G - (A^* X^{(k)} B_\Phi + G^* D_G)\Gamma^+ (A^* X^{(k)} B_\Phi + G^* D_G)^*, \tag{4.25}$$

$$\begin{aligned} \Sigma_{k+1} &= A\Sigma_k A^* + KK^* - (A\Sigma_k C_\Phi^* + KD_K^*)\Lambda^+ (A\Sigma_k C_\Phi^* + KD_K^*)^*, \qquad (4.26) \\ \Gamma &= D_G^* D_G + B_\Phi^* X^{(k)} B_\Phi, \qquad \Lambda = D_K D_K^* + C_\Phi \Sigma_k C_\Phi^*, \end{aligned}$$

have solutions $\{X^{(k)}\}_{k=1}^{T}$, *and* $\{\Sigma_k\}_{k=1}^{T}$, *respectively, which are non-negative definite. Suppose that* $X^{(0)} \geq 0$ *and* $\Sigma_0 \geq 0$ *are chosen such that* $X^{(T)}$ *converges to* $X_{\max} \geq 0$, *and* Σ_T *converges* $\Sigma_{\max} \geq 0$, *respectively, as* $T \to \infty$, *satisfying the AREs*

$$X_{\max} = A^* X_{\max} A + G^* G - (A^* X_{\max} B_\Phi + G^* D_G)\Gamma_{\mathrm{m}}^{+}(B_\Phi^* X_{\max} A + D_G^* G), \quad (4.27)$$

$$\Sigma_{\max} = A\Sigma_{\max} A^* + KK^* - (A\Sigma_{\max} C_\Phi^* + KD_K^*)\Lambda_{\mathrm{m}}^{+}(C_\Phi \Sigma_{\max} A^* + D_K K^*), \quad (4.28)$$

$$\Gamma_{\mathrm{m}} = D_G^* D_G + B_\Phi^* X_{\max} B_\Phi, \qquad \Lambda_{\mathrm{m}} = D_K D_K^* + C_\Phi \Sigma_{\max} C_\Phi^*.$$

In this case, realizations of the left and right spectral factors in (4.24) *are uniquely specified* (*up to a factor of unitary matrices*) *respectively by*

$$D_{G_{\mathrm{m}}}^* D_{G_{\mathrm{m}}} = D_G^* D_G + B_\Phi^* X_{\max} B_\Phi, \qquad G_{\mathrm{m}} = (D_{G_{\mathrm{m}}}^{+})^*(B_\Phi^* X_{\max} A + D_G^* G), \quad (4.29)$$

$$D_{K_{\mathrm{m}}} D_{K_{\mathrm{m}}}^* = D_K D_K^* + C_\Phi \Sigma_{\max} C_\Phi^*, \qquad K_{\mathrm{m}} = (A\Sigma_{\max} C_\Phi^* + KD_K^*)(D_{K_{\mathrm{m}}}^{+})^*, \quad (4.30)$$

where $D_{G_{\mathrm{m}}}$ *and* $D_{K_{\mathrm{m}}}$ *are the minimum rank Cholesky factors.*

Proof: For factorizations in (4.2) with $W_G(z)$ given in (4.3), the inner-outer factorization of $H(z) = W_G(z)$ as in (3.20) can be applied to obtain the right spectral factor $W_{G_{\mathrm{m}}}(z) = H_{\mathrm{o}}(z)$. Hence the DRE in (4.25) is obtained using the iterative algorithm (3.15) with $D = D_G$, $C = G$, and $B = B_\Phi$, leading to the limit ARE in (4.27) as $T \to \infty$ under the hypothesis that $X_k \to X_{\max}$. It follows that $D_{G_{\mathrm{m}}} = \Omega_{\mathrm{m}}$ and thus in light of Theorem 3.2,

$$G_{\mathrm{m}} = -D_{G_{\mathrm{m}}} F_{\mathrm{m}} = D_{G_{\mathrm{m}}} D_{G_{\mathrm{m}}}^{+} (D_{G_{\mathrm{m}}}^{+})^*(B_\Phi^* X_{\max} A + D_G^* G) = (D_{G_{\mathrm{m}}}^{+})^*(B_\Phi^* X_{\max} A + D_G^* G).$$

Thus (4.29) holds. A similar argument can be used to prove its dual in (4.30) as follows. For factorizations in (4.2) with $W_K(z)$ given in (4.3), the inner-outer factorization of $H(z) = W_K(z)$ as in (2.47) can be applied to obtain the left spectral factor $W_{K_{\rm m}}(z) = H_{\rm o}(z)$. Hence the DRE in (4.26) is obtained using the iterative algorithm (2.42) with $D = D_K$, $C = C_\phi$, and $B = K_{\rm m}$, leading to the limit ARE in (4.28) as $T \to \infty$ under the hypothesis that $\Sigma_k \to \Sigma_{\max}$. It follows that $D_{K_{\rm m}} = \Omega_{\rm m}$ and thus in light of Theorem 2.3.

$$K_{\rm m} = -L_{\rm m} D_{K_{\rm m}} = (A\Sigma_{\max} C_\Phi^* + KD_K^*)(D_{K_{\rm m}}^+)^* D_{K_{\rm m}}^+ D_{K_{\rm m}} = (A\Sigma_{\max} C_\Phi^* + KD_K^*)(D_{K_{\rm m}}^+)^*.$$

Therefore (4.30) holds true. ∎

Theorem 4.1 shows that the minimal solutions $P_{\min} \geq 0$, and $Q_{\min} \geq 0$ to the AREs (4.18) and (4.19) can be computed from

$$P_{\min} = A^* P_{\min} A + G_{\rm m}^* G_{\rm m}, \quad Q_{\min} = A Q_{\min} A^* + K_{\rm m} K_{\rm m}^*, \tag{4.31}$$

respectively, which are basically the special cases of (4.4), and (4.5). It also indicates that

$$\begin{aligned} P_{\min} &= A^* P_{\min} A + (A^* X_{\max} B_\Phi + G^* D_G)\Gamma_{\rm m}^+ (B_\Phi^* X_{\max} A + D_G^* G), && (4.32)\\ Q_{\min} &= A Q_{\min} A^* + (A\Sigma_{\max} C_\Phi^* + KD_K^*)\Lambda_{\rm m}^+ (C_\Phi \Sigma_{\max} A^* + D_K K^*), && (4.33) \end{aligned}$$

in light of (4.29) and (4.30) where $\Gamma_{\rm m} = D_G^* D_G + B_\Phi^* X_{\max} B_\Phi$ and $\Lambda_{\rm m} = D_K D_K^* + C_\Phi \Sigma_{\max} C_\Phi^*$. Adding (4.32) to (4.27), and (4.33) to (4.28), respectively yield

$$\begin{aligned} (X_{\max} + P_{\min}) &= A^*(X_{\max} + P_{\min})A + G^*G, && (4.34)\\ (\Sigma_{\max} + Q_{\min}) &= A^*(\Sigma_{\max} + Q_{\min})A + KK^*. && (4.35) \end{aligned}$$

Comparing the above two Lyapunov equations with those in (4.4) and (4.5), respectively concludes that $P = X_{\max} + P_{\min}$, and $Q = \Sigma_{\max} + Q_{\min}$. Note that $X_{\max}$ is dependent on G, while $\Sigma_{\max}$ is dependent on K, but $P_{\min}$ and $Q_{\min}$ are not. Hence we may switch to the notations

$$X_{\max} = X_{\max}(G), \quad P = P(G), \quad \Sigma_{\max} = \Sigma_{\max}(K), \quad Q = Q(K),$$

respectively. The aforementioned analysis leads to the relation

$$P(G) = X_{\max}(G) + P_{\min}, \quad Q(K) = \Sigma_{\max}(K) + Q_{\min}, \tag{4.36}$$

which are associated with $W_G(z)$ and $W_K(z)$ in (4.3), respectively. It follows that

$$\begin{aligned} \Phi(z) &= D_G^* D_G + B_\Phi^* P(G) B_\Phi + [D_G^* G + B_\Phi^* P(G) A](zI - A)^{-1} B_\Phi && (4.37) \\ &\quad + B_\Phi^* (z^{-1} I - A^*)^{-1} [D_G^* G + B_\Phi^* Q(G) A]^*, \\ &= D_K D_K^* + C_\Phi Q(K) C_\Phi^* + C_\Phi (zI - A)^{-1} [K D_K^* + A^* Q(K) C_\Phi^*] && (4.38) \\ &\quad + [G D_K^* + A^* Q(K) C_\Phi^*]^* (z^{-1} I - A^*)^{-1} C_\Phi^*, \end{aligned}$$

where $P(G)$ and $Q(K)$ are given as in (4.36). We are now ready for the main result of this chapter.

Theorem 4.2 *Let $\Phi(z)$ of size $q \times q$ as in* (4.1) *have normal rank $\rho < q$. Suppose that A is a stability matrix, and $\Phi(z) \geq 0$ and* $\operatorname{rank}\{\Phi(z)\} = \rho$ *for all $|z| = 1$. Then the iterative formulas* (4.20) *and* (4.21) *in the proposed algorithm are convergent with limits $P_{\min}$, and $Q_{\min}$, which are the minimum solutions to the AREs* (4.18) *and* (4.19)*, respectively.*

Proof: We first prove that the limiting solution to (4.20) is $P_{\min}$. By (4.37) and $\Phi(z)$ in (4.1),

$$R_0 = D_G^* D_G + B_\Phi^* P(G) B_\Phi, \quad C_\Phi = D_G^* G + B_\Phi^* P(G) A$$

for some G and D_G with $P(G) = P_{\min} + X_{\max}(G)$. Denote $\Delta_k = P(G) - P_k$. Substituting the above into (4.20) yields

$$P_{k+1} = A^* P_k A + (A^* \Delta_k B_\Phi + G^* D_G)(D_G^* D_G + B_\Phi^* \Delta_k B_\Phi)^+ (B_\Phi^* \Delta_k A + D_G^* G). \quad (4.39)$$

Because $P(G) = A^* P(G) A + G^* G$ by (4.4), the above equation leads to

$$\Delta_{k+1} = A^* \Delta_k A + G^* G - (A^* \Delta_k B_\Phi + G^* D_G)(D_G D_G^* + B_\Phi^* \Delta_k B_\Phi)^+ (B_\Phi^* \Delta_k A + D_G^* G)$$

which is identical to (4.25) with $\Delta_i = X^{(i)}$ for $i = k$ and $k+1$. Since $P_0 = 0$, $\Delta_0 = P(G) - P_0 \geq X \geq 0$ for any positive semi-definite solution to ARE

$$X = A^* X A + G^* G - (A^* X B_\Phi + G^* D_G)(D_G^* D_G + B_\Phi^* X B_\Phi)^+ (B_\Phi^* X A + D_G^* G) \quad (4.40)$$

which is the same ARE as in (4.27). In light of (d) in Remark 3.1, and the results in Chapter 3, the iterative algorithm (3.15) is convergent with $X^{(k)} \to X_{\max}$ most likely. Moreover the iterative algorithm in (4.20) is in fact independent of D_G and G. In other words for every possible pair of $(D_G,\, G)$ the iterative solutions P_k in (4.20) satisfies

$$\Delta_k = P(G) - P_k = X^{(k)}$$

with $X^{(k)}$ the iterative solutions to (4.25). We may thus choose D_G and G such that $W_G(z)$ is strictly minimum phase. Such D_G and G clearly exist by the hypothesis that $\Phi(z)$ has the rank ρ for all z on the unit circle. As such the positive semi-definite solution to the ARE in (4.40) is unique, which is $X_{\max}(G)$. As a result $\Delta_k = X^{(k)}$ is convergent, implying that $P_k = P(G) - \Delta_k$ is convergent to $P_{\min} = P(G) - X_{\max}(G)$ by (4.36). The proof for the limiting solution of (4.21) to $Q_{\min}$ can be shown similarly, which is omitted. ∎

It is noted that the convergence of the proposed spectral factorization algorithm embodied in (4.20) and (4.21) is established under the zero initial condition $P_0 = Q_0 = 0$. If $P_0 \geq 0$ and $Q_0 \geq 0$ are arbitrary, then the convergence of the DREs in (4.20) and (4.21) remains unknown, that is very different from the inner-outer factorization algorithms in the previous section.

Remark 4.1 Theorem 4.2 not only shows the convergence of the proposed algorithm for spectral factorizations, but also provides the right initial values $X^{(0)}$ and Σ_0 for the iterative algorithms in (3.15), and (2.42), respectively, in order to ensure the limits $X_{\max}$ and $\Sigma_{\max}$, respectively. That is, the initial condition $X^{(0)} = W$ with W the solution to the Lyapunov equation (3.17) can ensure that the iterative algorithm in (3.15) admits limit $X_{\max}$, as required for the inner-outer factorization in (3.20); Similarly if Σ_0 satisfies $\Sigma_0 = A\Sigma_0 A^* + BB^*$ is chosen, then the iterative algorithm (2.42) admits the limit $\Sigma_{\max}$, as required for the inner-outer factorization in (2.47). ∎

4.4 Illustrative Examples

In this section we present an example to demonstrate the proposed spectral factorization iterative algorithm.

Example 4.1 This example examines the spectral factorization with the PSD modified from [43]:

$$\Phi(z) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} - vv^* + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -0.5 & 0 \end{bmatrix} z \quad (4.41)$$

where $v^* = \begin{bmatrix} 0.1086 & 0.4052 & 0.9732 \end{bmatrix}$. It can be verified that $\Phi(e^{j\omega})$ has rank 2 approximately with its third eigenvalue no greater than 10^{-5} for all ω. To apply the algorithm in (4.20), we set

$$A_\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_\Phi = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

It takes only two iterations for the algorithm to terminate with error tolerance 10^{-6} in computing $P_{\min}$. The right spectral factor is obtained via computing

$$D_{\mathrm{m}} = \begin{bmatrix} -0.39540 & 2.36795 & 0.00000 \\ -0.78774 & -0.13154 & 0.00003 \end{bmatrix}, G_{\mathrm{m}} = \begin{bmatrix} 0.41085 & -0.41085 & -0.20543 \\ -0.20622 & 0.20622 & 0.10311 \end{bmatrix}$$

as in (4.22) and (4.23), respectively. Since the spectral factors are unique upto a factor of unitary matrices, we have

$$UW_G(z) = \begin{bmatrix} 0.80847 & -0.00000 & -0.00000 \\ -0.35110 & 2.37160 & -0.00000 \end{bmatrix} + \begin{bmatrix} 0.18312 & -0.18312 & -0.09156 \\ 0.42166 & -0.42166 & -0.21083 \end{bmatrix} z^{-1}$$

is also a right spectral factor, which agrees with the example in [43], where

$$U = \begin{bmatrix} -0.05546 & -0.99846 \\ 0.99846 & -0.05546 \end{bmatrix}$$

is a unitary matrix. On the other hand, to apply the algorithm in (4.21), we set

$$A_\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_\Phi = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again, it takes only two iterations for the algorithm to terminate with error tolerance 10^{-6} in computing $Q_{\min}$. The right spectral factor is obtained via computing

$$D_{K_\mathrm{m}} = \begin{bmatrix} -0.994079 & -0.004377 \\ 1.050317 & -0.024639 \\ 0.105392 & 0.204264 \end{bmatrix}, \quad K_\mathrm{m} = \begin{bmatrix} 0.000000 & 0.000000 \\ -0.997444 & -1.933173 \\ 0.000000 & 0.000000 \end{bmatrix}$$

as in (4.22) and (4.23), respectively. Since the spectral factors are unique upto a factor of unitary matrices, we have

$$W_K(z) = \begin{bmatrix} -0.994079 & -0.004377 \\ 1.050317 & -0.024639 \\ 0.105392 & 0.204264 \end{bmatrix} + \begin{bmatrix} 0.000000 & 0.000000 \\ -0.997444 & -1.933173 \\ 0.000000 & 0.000000 \end{bmatrix} z^{-1}$$

is also a left spectral factor It is surprising to see the faster convergence in our proposed iterative algorithm. But we need keep in mind that the highest power in this PSD matrix is only 1. Usually the number of iterations increases with respect to the highest power of $\Phi(z)$. ∎

We would like to comment that in practical numerical examples, the normal rank of the PSD matrix is almost always full, as in the above example. For this reason we have to determine the normal rank numerically, which can be difficult in practice. On the other hand, it is possible that the normal rank is known in advance, and its inflation is due to the noise in estimation of the covariance data $\{R_k\}$, which is the case for blind channel estimation in [6, 23, 35]. In our numerical study for spectral factorization, we also observed the slow convergence whenever the spectral factors have zeros on the unit circle, which is consistent with what observed in [43].

4.5 Conclusion

This chapter investigated the spectral factorization problem and its relation to the inner-outer factorization problem. We proposed an iterative algorithm for computing the solution to the spectral factorization problem. An intrinsic relation between the spectral factorization AREs and the inner-outer factorization AREs is revealed. Such a relationship helped us to choose the correct initial or boundary conditions for the corresponding AREs in the generalized Kalman filtering and the generalized LQR control for which the convergence of the recursive algorithms to the stabilizing solutions can be established. Also we proved the convergence of the proposed iterative algorithm to the minimal solutions of the AREs corresponding to the spectral factorization in this chapter. An example is used to illustrate the proposed spectral factorization algorithm that shows its fast convergence.

Chapter 5

Blind Channel Estimation and Conclusion

In this chapter, we apply our theory on generalized spectral factorization developed in this dissertation to blind channel estimation. As discussed in Chapter 1, blind channel estimation is dominated by the subspace method. Although it is effective especially for SIMO (single-input/multiple-output) channels, it does not ensure the minimum phase property and thus the uniqueness for the estimated channel. Moreover it is difficult to be extended to MIMO (multiple-input/multiple-output) channels. On the other hand the spectral factorization method treats the MIMO and SIMO channels the same way with much lower computational complexity compared to the subspace method. Hence the spectral factorization method is more preferred. We will use numerical examples to simulate blind channel estimation and compare our proposed spectral factorization method with the existing subspace method. The simulation examples show that the two methods have comparable channel estimation results for the SIMO case although the subspace method employs optimization tools and has much higher complexity. We will also present the simulation results on MIMO channels.

This chapter will also conclude our dissertation work with a summary on the contributions of this thesis and discussions on the existing problems for the spectral factorization approach to blind channel estimation. We will outline the direction for the future research in the problem area of blind channel estimation.

5.1 Single-Input/Multi-Output (SIMO) Channels

We consider a communication system that has single transmitter and M receiver antennas. Let $s(t)$ be the emitted symbol from the user time tT, where T is the symbol duration and t is integer valued. The information sequences are assumed white and its variance or power is known.

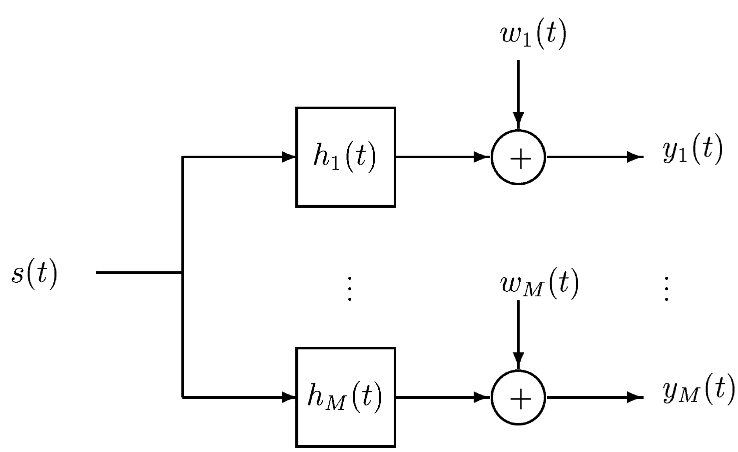

Figure 5.1: SIMO system model

Assume that the channel is FIR with length L. As shown in Figure 5.1 the received signal at the ith antenna is given by

$$y_i(t) = \sum_{k=0}^{L-1} s(t-k)h_i(k) + w_i(t) \tag{5.1}$$

where $w_i(t)$ is white noise and $h_i(t)$ is the composite channel response between the user and the ith antenna which includes the effects of the emission filter, reception filter, channel response. Let the output vector at time t be represented by

$$\underline{y}_a(t) = \left[\begin{array}{cccc} y_1(t) & y_2(t) & \cdots & y_M(t) \end{array} \right]^T \tag{5.2}$$

that is an $M \times 1$ vector and the added white noise vector at time t be given by

$$\underline{w}_a(t) = \left[\begin{array}{cccc} w_1(t) & w_2(t) & \cdots & w_M(t) \end{array} \right]^T \tag{5.3}$$

Normally $M >> 1$ to ensure sufficient spatial diversity. Let L successive signal of the received signals be represented by

$$\underline{y}(t) = \left[\begin{array}{cccc} \underline{y}_a(t) & \underline{y}_a(t-1) & \cdots & \underline{y}_a(t-L+1) \end{array} \right]^T \tag{5.4}$$

that is an $ML \times 1$ vector. Denote $\underline{s}(t) = \left[\begin{array}{cccc} s(t) & s(t-1) & \cdots & s(t-2L+2) \end{array} \right]^T$ as a vector with dimension $(2L-1)\times 1$ and $\underline{w}(t) = \left[\begin{array}{cccc} \underline{w}_a(t) & \underline{w}_a(t-1) & \cdots & \underline{w}_a(t-L+1) \end{array} \right]^T$ as a vector with dimension $ML \times 1$. Let the vector $\underline{h}(t)$ be defined by

$$\underline{h}(t) = \left[\begin{array}{cccc} h_1(t) & h_2(t) & \cdots & h_M(t) \end{array} \right]^T \tag{5.5}$$

that has dimension $M \times 1$. Then we have

$$\underline{y}(t) = H\underline{s}(t) + \underline{w}(t) \tag{5.6}$$

where H is the channel matrix with dimension $ML \times (2L-1)$ given by

$$H = \begin{pmatrix} \underline{h}(0) & \cdots & \underline{h}(L-1) & 0 & \cdots & 0 \\ 0 & \underline{h}(0) & \cdots & \underline{h}(L-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{h}(0) & \cdots & \underline{h}(L-1) \end{pmatrix} \tag{5.7}$$

Because $M > 1$, H has $(2L-1)$ column and has rank at most $(2L-1)$. The autocorrelation matrix $R_{\underline{y}}$ of the measurement vector $\underline{y}(t)$ has the expression:

$$R_y = \mathrm{E}\{\underline{y}(t)\underline{y}(t)^*\} \tag{5.8}$$

The additive measurement noise is assumed to be independent of the transmitted sequences. Hence, R_y can be obtained as follows:

$$R_y = HR_sH^* + R_w \tag{5.9}$$

where $R_s = \mathrm{E}(\underline{s}(t)\underline{s}(t)^*) = \sigma_s^2 I$ and $R_w = \mathrm{E}(\underline{w}(t)\underline{w}(t)^*) = \sigma_w^2 I$ denote respectively the autocorrelation matrices of the transmitted symbol vector $\underline{s}(t)$ and the autocorrelation matrices of the measurement noise vector $\underline{w}(t)$.

In the following we outline the algorithm for blind channel estimation using spectral factorization based on the covariance matrix R_y that have a Toeplitz structure.

1. Compute the Toeplitz matrix R_y of the received data as follows:

$$R_y = \begin{bmatrix} R(0) & R(1) & R(2) & \ldots & R(L-1) \\ R(-1) & R(0) & R(1) & \ldots & R(L-2) \\ R(-2) & R(-1) & R(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & R(0) & R(1) \\ R(-(L-1)) & \ldots & R(-2) & R(-1) & R(0) \end{bmatrix} \tag{5.10}$$

where $R(k) = \mathrm{E}\left\{\underline{y}_a(t)\underline{y}_a^*(t-k)\right\}$.

2. Compute the SVD of the Toeplitz matrix to remove the noise part.

$$\begin{aligned}
\hat{R}_y &= U(:,1:2L-1) * S(1:2L-1, 1:2L-1) * V(:,1:2L-1)^* \quad (5.11)\\
&= \begin{bmatrix} \hat{R}(0) & \hat{R}(1) & \hat{R}(2) & \ldots & \hat{R}(L-1) \\ \hat{R}(-1) & \hat{R}(0) & \hat{R}(1) & \ldots & \hat{R}(L-2) \\ \hat{R}(-2) & \hat{R}(-1) & \hat{R}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{R}(0) & \hat{R}(1) \\ \hat{R}(-(L-1)) & \ldots & \hat{R}(-2) & \hat{R}(-1) & \hat{R}(0) \end{bmatrix} \quad (5.12)
\end{aligned}$$

3. Because $\hat{R}_y$ is positive semi-definite, its PSD can be written as

$$\Phi(z) = \sum_{k=-(L-1)}^{L-1} \hat{R}_k z^{-k} = \hat{R}_0 + C_\Phi(zI - A)^{-1}B_\Phi + B_\Phi^*(z^{-1}I - A^*)^{-1}C_\Phi^*, \quad (5.13)$$

A simple state-space realization for $\Phi(z)$ is

$$A = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \quad B_\Phi = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (5.14)$$

$$C_\Phi = \begin{bmatrix} \hat{R}(1) & \hat{R}(2) & \cdots & \hat{R}(L-1) \end{bmatrix}, \quad D = \frac{\hat{R}(0)}{2}. \quad (5.15)$$

4. Apply the following iterative algorithm to compute $Q_{\min}$: Let $Q_0 = 0$ and for $k = 0, 1, \cdots$, compute

$$Q_{k+1} = AQ_kA^* + (B_\Phi - AQ_kC_\Phi^*)(R_0 - C_\Phi Q_k C_\Phi^*)^+(B_\Phi - AQ_kC_\Phi^*)^*. \quad (5.16)$$

If $\|Q_N - Q_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{Q_k\}$.

5. Compute $D_{K_{\rm m}}$ via finding the minimum Cholesky factors of

$$D_{K_{\rm m}} D_{K_{\rm m}}^* = R_0 - C_\Phi Q_{\rm min} C_\Phi^*. \tag{5.17}$$

Then, Compute $K_{\rm m}$ as

$$K_{\rm m} = (B_\Phi - A Q_{\rm min} C_\Phi^*)(D_{K_{\rm m}}^+)^*. \tag{5.18}$$

The next is a numerical example for blind estimation of a SIMO channel.

Example 5.1 This example was presented in [24]. A Monte Carlo simulation was conducted to evaluate the performance of the proposed algorithms in a digital communication situation. The emitted signal has a binary format. The number of receiver antennas is $M = 4$ and the channel length is $L = 5$. The channel coefficients are complex given by

$$\begin{aligned}
h_1 &= [(-0.049, 0.359), (0.482, -0.569), (-0.556, 0.587), (1.0, 0.0), (-0.171, 0.061)] \\
h_2 &= [(0.443, -0.0364), (1.0, 0.0), (0.921, -0.194), (0.189, -0.208), (-0.087, -0.054)] \\
h_3 &= [(-0.211, -0.322), (-0.199, 0.918), (1.0, 0.0), (-0.284, -0.524), (0.136, -0.19)] \\
h_4 &= [(0.417, 0.030), (1.0, 0.0), (0.873, 0.145), (0.285, 0.309), (-0.049, 0.161)]
\end{aligned}$$

The white Gaussian noise is added to the receiver site and the SNR is 25 dB. The number of symbols used is varied across the experiments: 250, 500, 1000, 2000, 5000. 100 Monte-Carlo runs were executed for each situation. Two error criteria were used in [24] to measure the performance for blind channel estimation which are:

- *The mean bias b*: It is defined as

$$b = \frac{1}{ML}\sum_{i=1}^{ML} |\langle H_i \rangle - H_i|$$

where $\langle H_i \rangle$ denotes the average over the MonteCarlo runs of the estimates of H_i, the *ith* component of the actual coefficient vector.

- *The mean-square error*: It is defined as the mean-square error for estimated channel vector.

We have adopted the same error criteria as above and carried out the numerical simulation for channel estimation based on both the spectral factorization method and the subspace method as in [24]. Figure 5.2 shows the mean bias of the parameters estimates where the solid line is based on the spectral factorization method and the dashed line is based on subspace method proposed in [24]. The simulation shows that the mean bias error for our proposed method is lower than the method in the literature. However the bias error does not seem to decrease much as the number of symbols increases.

Figure 5.3 shows that numerical results for mean-square error. Again the solid and dashed lines represent the errors for our proposed method and the method in the literature, respectively. It is seen that our method results in larger error. The reason lies in the fact that the method in [24] employed minimization algorithm to minimize the mean-square error which suffers from high complexity. Indeed it requires computing the eigenvalue and eigenvectors for 40×40 matrix as well as 20×20 matrix.

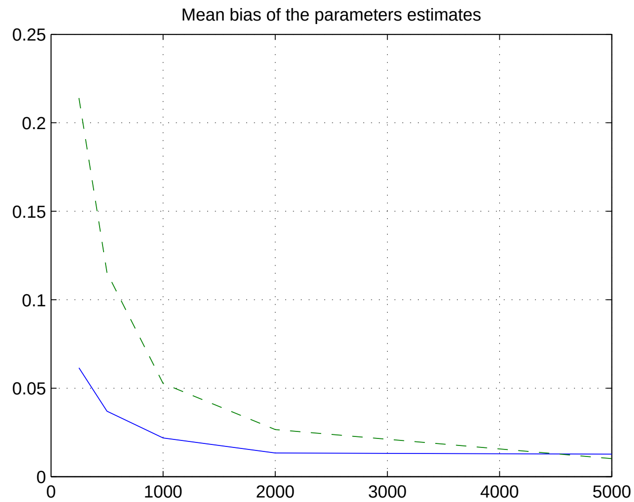

Figure 5.2: Mean bias error in the SIMO with Binary format input case

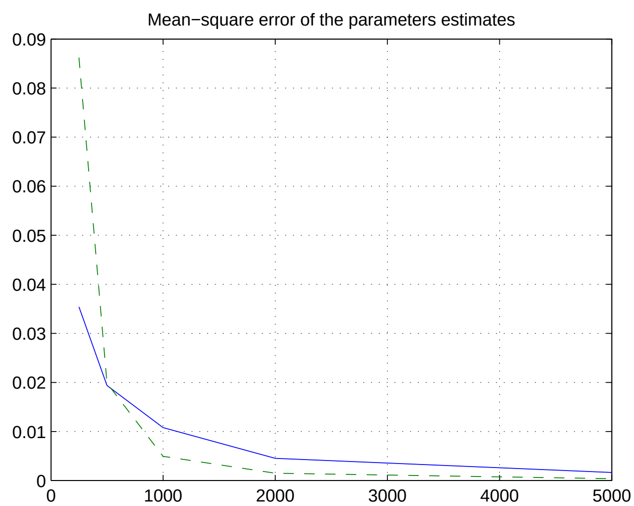

Figure 5.3: Mean-square error in the SIMO with Binary format input case

On other hands, our method requires computing the singular value decomposition for 20×20 matrix once and computing the Pseudo inverse for 4×4 matrix. However our proposed algorithm requires $\hat{R}_y$ to be Toeplitz which may not be after removing the noise. If we forced it to be Toeplitz, then it no longer positive semi-definite. As a result, the iterative algorithm starts diverge so we have to stop it early giving worse mean-squared error. ∎

Example 5.2 This example is the same as Example 5.1 but the emitted symbols have quaternary-QAM format [25] and the SNR is 15 dB. Again we compute the two errors to compare the performance of the two different algorithms. The results are given in Figures 5.4 and 5.5 which are similar to the previous example.

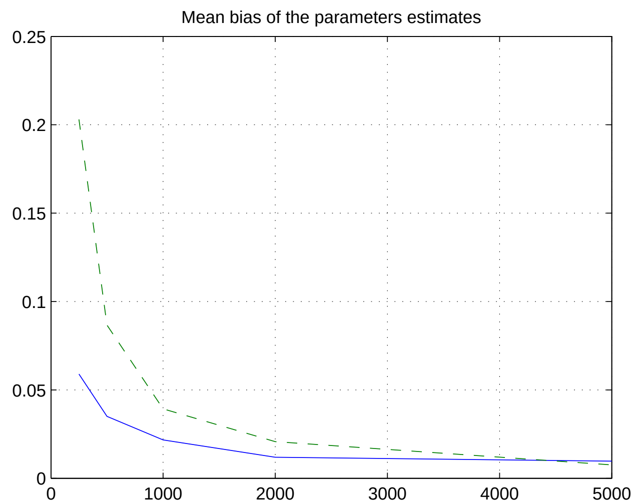

Figure 5.4: Mean bias error in the SIMO with QAM input case

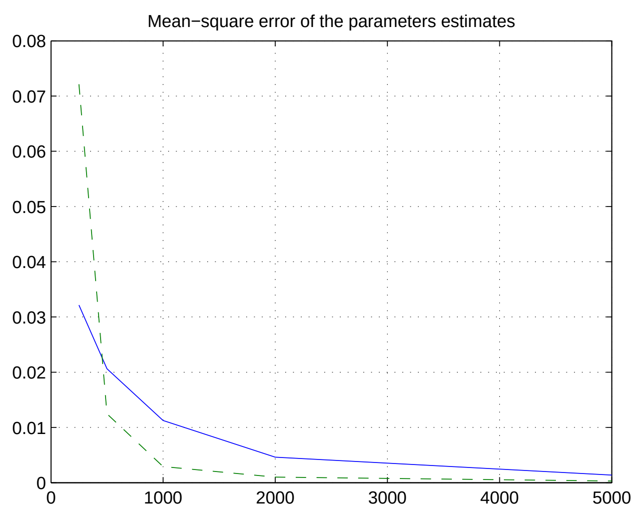

Figure 5.5: Mean-square error in the SIMO with QAM input case

This illustrates the fact that the relative performance of these two algorithms is insensitive to the modulation format. We would like to comment that our proposed algorithm is applicable to blind channel estimation of the true MIMO channel that is contrast to that in [24] to be shown in the next section. ∎

5.2 Multi-Input/Multi-Output (MIMO) Channels

We consider a communication system that has P transmitter and M receiver antennas. Let $s^{(m)}(t)$ be the symbol from the transmitter m at time tT, where T is the symbol duration and $m = 1, \cdots, P$. The information sequences are assumed to be white and its variance or power is known. We assume that $s^{(m)}(t)$ and $s^{(\ell)}(t)$ are independent whenever $m \neq \ell$. The channel is assumed FIR with length L. The received signal at the ith antenna is

$$y_i(t) = \sum_{m=1}^{P} \sum_{k=0}^{L-1} s^{(m)}(t-k) h_i^{(m)}(k) + w_i(t) \tag{5.19}$$

where $w_i(t)$ is white noise and $h_i^{(m)}(t)$ is the composite channel response between the user m and the ith antenna which includes the effects of the emission filter, reception filter, channel response. Let the input vector be

$$\underline{s}_a(t) = \begin{bmatrix} s_1(t) & s_2(t) & \cdots & s_P(t) \end{bmatrix}^T \tag{5.20}$$

and output vector at time t be

$$\underline{y}_a(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_M(t) \end{bmatrix}^T \tag{5.21}$$

The additive white noise vector at time t has the same as size and is given by

$$\underline{w}_a(t) = \begin{bmatrix} w_1(t) & w_2(t) & \cdots & w_M(t) \end{bmatrix}^T \tag{5.22}$$

Let the N successive signal of the received signals be represented by

$$\underline{y}(t) = \begin{bmatrix} \underline{y}_a(t) & \underline{y}_a(t-1) & \cdots & \underline{y}_a(t-N+1) \end{bmatrix}^T \tag{5.23}$$

which is a $MN \times 1$ vector. Denote $\underline{s}(t) = \begin{bmatrix} \underline{s}_a(t) & \underline{s}_a(t-1) & \cdots & \underline{s}_a(t-L-N+2) \end{bmatrix}^T$ as a vector with dimension $P(L+N-1) \times 1$ and $\underline{w}(t) = \begin{bmatrix} \underline{w}_a(t) & \underline{w}_a(t-1) & \cdots & \underline{w}_a(t-N+1) \end{bmatrix}^T$ as a vector with dimension $MN \times 1$. Let $H_a(t)$ be a matrix with dimension $M \times P$:

$$H_a(t) = \begin{bmatrix} h_1^{(1)}(t) & \cdots & h_1^{(P)}(t) \\ h_2^{(1)}(t) & \cdots & h_2^{(P)}(t) \\ \vdots & \vdots & \vdots \\ h_M^{(1)}(t) & \cdots & h_M^{(P)}(t) \end{bmatrix}. \tag{5.24}$$

Then $\underline{y}(t)$ has the following expression:

$$\underline{y}(t) = H\underline{s}(t) + \underline{w}(t) \tag{5.25}$$

where H is the channel matrix that contains multiple copies of the coefficients we wish to estimate and is given by

$$H = \begin{pmatrix} H_a(0) & \cdots & H_a(L-1) & 0 & \cdots & 0 \\ 0 & H_a(0) & \cdots & H_a(L-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_a(0) & \cdots & H_a(L-1) \end{pmatrix} \tag{5.26}$$

Its dimension is $MN \times P(L+N-1)$. We have to choose N such that the H matrix is tall. This condition yields $N > \frac{P(L-1)}{M-P}$. The autocorrelation matrix $R_{\underline{y}}$ of the measurement vector $\underline{y}(t)$ is given by (5.9). For the MIMO case, we apply the same algorithm as in SIMO case because the spectral factorization does not distinguish the two and has the similar computational complexity.

Example 5.3 In this example, we assume $P = 2$ and $M = 4$. The emitted signal has a binary format. The channel length is $L = 5$. The channel coefficient matrices are given by:

$$H_a(0) = \begin{bmatrix} (0.1865, 1.0384) & (1.2368, 0.3558) \\ (1.8060, 0.2430) & (-0.2684, 0.1256) \\ (-0.2381, -0.0582) & (1.1727, 1.2766) \\ (-0.6096, -0.8477) & (0.6332, -0.5178) \end{bmatrix}$$

$$H_a(1) = \begin{bmatrix} (0.5444, 0.0870) & (0.0235, -0.5816) \\ (2.0900, 1.0544) & (-0.0894, 1.0931) \\ (0.4457, 0.1324) & (0.0484, 0.6358) \\ (1.2498, -1.0096) & (0.4379, 1.0780) \end{bmatrix}$$

$$H_a(2) = \begin{bmatrix} (2.0299, -0.3624) & (-0.3108, 0.2658) \\ (0.5749, -0.0781) & (-0.0325, -0.0623) \\ (0.3362, 0.3910) & (0.6246, 0.3273) \\ (-0.3782, -0.6866) & (0.7651, 0.9140) \end{bmatrix}$$

$$H_a(3) = \begin{bmatrix} (0.2005, -0.5491) & (-0.1887, -0.0172) \\ (-0.2926, -0.7939) & (0.2085, -0.4623) \\ (-0.1786, -0.6155) & (-0.1706, 1.1072) \\ (-0.2179, -0.1717) & (0.1249, 0.0122) \end{bmatrix}$$

$$H_a(4) = \begin{bmatrix} (0.1965, -1.0143) & (0.8262, -1.4625) \\ (-0.0921, -0.9955) & (-0.2275, 0.7724) \\ (0.8362, -0.3282) & (-0.4186, 0.7437) \\ (-0.3989, -0.2490) & (-0.1427, 1.2747) \end{bmatrix}$$

White Gaussian noise is added to the output and the SNR is 25 dB. The number of symbols used is varied across the experiments: 250, 500, 1000, 2000, 5000, 10000. 100 Monte-Carlo runs were executed for each situation. We again use the same two error criteria as mentioned in the previous examples to compute the performance of our proposed algorithm. The results are illustrated in Figure 5.6 and Figure 5.7. Because of the lack of the similar algorithm to that in [24] for MIMO channels, only our simulation results are presented.

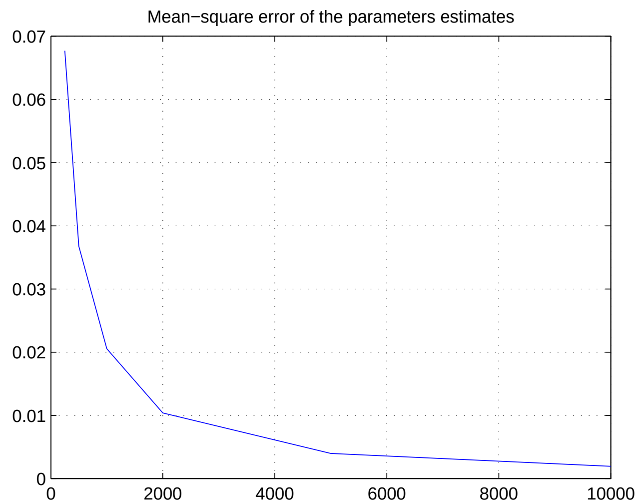

Figure 5.6: Mean-square error in the QAM input case

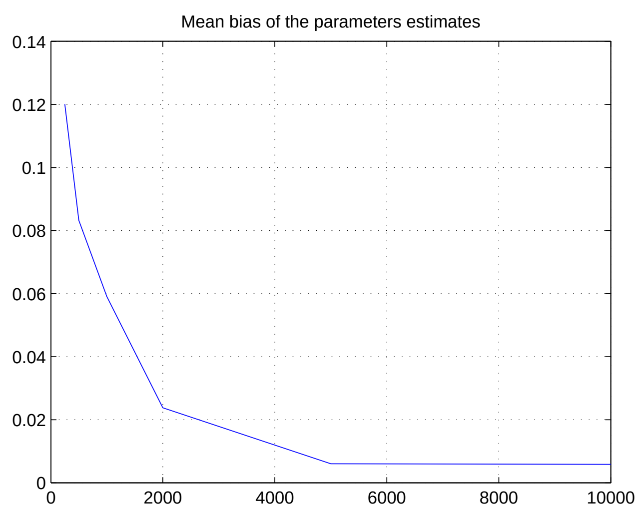

Figure 5.7: Mean bias error in the QAM input case

5.3 Conclusion and Future Research

In this dissertation, we have focused on blind channel estimation such that the estimated channel admits the minimum phase property. Our proposed approach is based on the generalized spectral factorization for PSD of the received signal. This approach is different from the existing methods in that it ensures the minimum phase for the estimated channel. Because the associated PSD does not have full normal rank, it gives the rise of difficulty in the spectral factorization problem. There lack effective computational algorithms for spectral factorizations when the PSD matrix does not have full normal rank. Hence in order to approach blind channel estimation using the spectral factorization method, we first solved the spectral factorization problem for those PSD matrices having deficient normal rank. In this dissertation we showed that the aforementioned spectral factorization is closely related to inner-outer factorizations where the inner is square with smaller size, that in turn is related to the generalized Kalman filtering. Therefore we studied the related generalized Kalman filtering based on which we developed an efficient iterative algorithm to compute the spectral factors for those PSD matrices with deficient normal ranks.

Specifically in Chapter 2, we investigated the generalized Kalman filtering. Using a similar derivation to the standard case, we obtained a similar Riccati equation whose solution provides the Kalman filtering gain. An iterative algorithm was proposed for computing the positive semi-definite solution to the corresponding ARE. The stabilizing solution to the ARE associated with the generalized Kalman filtering was used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. Numerical example was employed to illustrate the results obtained in this chapter for computing the outer part of the given channel. We note that the inner-outer factorization in this chapter has applications to channel equalization.

In Chapter 3, due to the duality between the generalized Kalman filtering and the generalized LQR control, we were able to tackle the LQR control. Based on the results for the generalized Kalman filtering, we proposed a similar solution approach to the generalized LQR control. We developed an iterative algorithm for computing the stabilizing solutions to the generalized LQR control ARE. The stabilizing solution to the ARE associated with the generalized LQR control was used to construct both the inner and the outer factors, that constitutes the algorithm for computing the inner-outer factorization. Also, we investigated the LQG optimal control problem. Numerical examples were employed to illustrate the results obtained for computing the outer part of the given channel.

In Chapter 4, we investigated the spectral factorization problem, entailed by blind channel estimation. We established the relation between the inner-outer factorization and the spectral factorization. We proposed an iterative algorithm for the spectral factorizations. Based on the relation between the inner-outer factorization and the spectral factorization, we established the convergence of the proposed iterative algorithm. Such a convergence result also provided the initial or boundary condition for the iterative algorithm in solving the stabilizing solutions to the AREs associated with the generalized Kalman filtering and the generalized LQR control that derived in Chapter 2 and Chapter 3.

In Chapter 5, we applied the proposed spectral factorization algorithm to the blind channel estimation and worked out numerical examples. First we applied the proposed algorithm to the SIMO channel using a channel model from [24]. With a large number of simulations, we compared our results with the previous work in [24]. Although our mean square error is larger, we achieved smaller mean bias error. Moreover an MIMO model is also employed to demonstrate the effectiveness of our proposed algorithm and the results are analyzed.

While our thesis has contributed to generalized LQR control, Kalman filtering, and LQG control in addition to blind channel estimation, more research problems exist. As analyzed in the previous two sections, the performance of our proposed algorithm is hinged to that the covariance matrix after removing the noise is positive semi-definite with a prescribed rank condition. This does not hold in practice nor in the simulation examples. Hence it is crucial to resolve this particular problem. Other problems include slow convergence for inner-outer factorization when transmission zeros are close to the unit circle and possibility of blind channel equalization without channel estimation. These are discussed in more detail in the following.

- Removing the noise part in the covariance matrix: As discussed in the previous two sections, after removing the noise part via SVD, the resultant matrix has the correct rank but does not have Toeplitz structure. If we make it Toeplitz, then it becomes indefinite. Such a problem causes the divergence of our proposed iterative algorithm for spectral factorization. We believe that a possible future research direction can be characterization of the positive semi-definite Toeplitz matrices with a prescribed rank condition which can be used for removal of the noise component in the covariance matrix.

- The existence of transmission zeros for H(z) close to the unit circle: It has been observed in our simulation examples, the existence of transmission zeros close to the unit circle slows down the convergence in computation of the inner-outer factorizations. This is similar to that for spectral factorizations. How to speed up the convergence rate for our proposed iterative algorithms in the presence of the unit circle zeros is still open problem.

- Direct channel equalization: An interesting problem is blind channel equalization without channel estimation because it will eliminate the unnecessary complications of the blind channel estimation. This problem is similar to direct method for adaptive feedback control [4] and will be much harder to solve.

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