Projectives and Injectives in a Setting of Axiomatic-Exactness.

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Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

William Alan Vekovius
B.A., Wayne State University, 1965
M.A., Wayne State University, 1967
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ABSTRACT

Let \( m \) be a commutative monoid with identity, 0, and let \( \mathcal{A} \) be a subset of \( m^5 \) consisting of elements which are of the form \( (c,a,b,c,0) \), such that \( \mathcal{A} \) satisfies the following five axioms prescribed by Jack E. Ohm in his paper "An Axiomatic Approach to Homological Dimension" (to appear).

1. \( (0,0,0,0,0) \in \mathcal{A} \)
2. \( (0,a,b,c,0) \in \mathcal{A} \), and
   \[ m \in m \Rightarrow (0,a+m,b+m,c,0) \in \mathcal{A} \quad \text{and} \quad (0,a+b+m,c+m,0) \in \mathcal{A} \]
3. i) \( (0,m,m',0,0) \in \mathcal{A} \Rightarrow m = m' \)
   ii) \( (0,0,m,m',0) \in \mathcal{A} \Rightarrow m = m' \)
   iii) \( (0,m,0,m',0) \in \mathcal{A} \Rightarrow m = m' = 0 \)
4. \( (0,a,b,c,0) \in \mathcal{A} \) and \( (0,k,m,b,0) \in \mathcal{A} \Rightarrow \) there exists \( m' \in m \) such that
   \( (0,m',m,c,0) \in \mathcal{A} \) and \( (0,k,m',a,0) \in \mathcal{A} \)
5. (Existence of pullbacks) \((0, A, B, C, 0) \in \mathcal{S}\) and \((0, A', B', C, 0) \in \mathcal{S}\) there exists \(M \in \mathcal{M}\) such that \((0, A, M, B', 0) \in \mathcal{S}\) and \((0, A', M, B, 0) \in \mathcal{S}\).

Ohm calls a pair \((\mathcal{M}, \mathcal{S})\) a monoid with exact sequences. A tuple in \(\mathcal{S}\) is called exact.

In the same paper Ohm defines a generalization of a projective (or dually, injective) module. An element \(P \in \mathcal{M}\) is called jective (relative to \(\mathcal{S}\)) if for any \(K, K', P'\) in \(\mathcal{M}\) such that \((0, K', P', A, 0)\) and \((0, K, P, C, 0)\) are exact, there exists \(M \in \mathcal{M}\) such that \((0, K', M, K, 0)\) and \((0, M, P' + P, B, 0)\) are exact.

In Chapter II, jectives are characterized as follows:

Theorem. An element \(M \in \mathcal{M}\) is jective if and only if every sequence of the form \((0, A, B, M, 0)\) splits. That is, \(B = A + M\).

In Chapter III constructions are given for sets \((\mathcal{M}, \mathcal{S})\) and used to answer, negatively, several questions raised by Professor Ohm during his seminar at Louisiana State University, 1972-73.

(1) Is the sum of two exact sequences always exact?

(2) Is a summand of a jective necessarily jective?
(3) If $P$ is injective and $A + P$ is injective, is $A$ injective?

(4) If the order of the 5-tuples in the elements of a set of short exact sequences is reversed, is the new set of 5-tuples a set of short exact sequences?
CHAPTER I


In [10] Ohm provides an axiomatic approach to homological dimension. He remarks that "In defining homological dimension, morphisms never really enter into the picture. All that matters is the existence of certain exact sequences." His approach unifies the theories of projective and injective dimension, and his axioms are strong enough to provide the dimension theorem (see [11], pp. 37).
2. Example. Ohm's axioms on exactness are modeled after the properties of exact sequences of \(R\)-isomorphism classes of \(R\)-modules, where \(R\) is a ring. We now take a look at this example, especially with respect to the properties axiomatized by Ohm.

Let \((\mathcal{M},+)\) denote the \(R\)-isomorphism classes of finitely generated \(R\)-modules over a noetherian ring \(R\).

Then \(\mathcal{M}\) is a set and is closed under \(+\), defined by \(\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{M}_1 \oplus \mathcal{M}_2\). (For each \(n \geq 0\), let \(F_n = \bigoplus_{i=1}^{n} R\).)

Every finitely generated \(R\)-module is isomorphic to a module defined by a congruence on \(F_n\). (See \(\mathcal{M}\).)

A congruence is a subset of \(F_n \times F_n\), and all congruences on \(F_n\) is a subset of \(2^{F_n \times F_n}\). Hence \(\bigcup_{n \geq 0} \) (congruences on \(F_n\)) is a set of the same cardinality as \(\mathcal{M}\).)

1.1 Definition. Let \(\mathcal{S} \subseteq \mathcal{M}^5\) be such that each element of \(\mathcal{S}\) is of the form \((\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})\) and such that \((\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})\) is in \(\mathcal{S}\) if there exist \(R\)-module homomorphisms \(\alpha\) and \(\beta\) such that

\[
\begin{align*}
\alpha &\quad \beta \\
0 &\to A &\to R &\to C &\to 0
\end{align*}
\]

is an exact sequence of \(R\)-modules. An element of \(\mathcal{S}\) is called a short exact sequence or simply exact.

Given a sequence \((\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})\) in \(\mathcal{S}\), there exists, by definition, an exact sequence
of R-modules. The homomorphism \( \alpha \) and \( \beta \) are not, in general, unique as the following example shows:

1.2 Example. Let \( R = \mathbb{Z} \); the integers, so that our R-modules are now abelian groups. Let

\[
A = \mathbb{Z}, \quad B = \mathbb{Z} \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}_2), \quad \text{and} \quad C = \mathbb{Z}_2 \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}_2),
\]

and let \( \alpha : A \to B \) be defined to be

\[
\alpha(x) = (2x, 0, 0, \ldots, 0, \ldots),
\]

and let \( \beta : B \to C \) be defined to be

\[
\beta(x, y_1, y_2, \ldots, y_n, \ldots) = (x, y_1, y_2, \ldots, y_n, \ldots).
\]

Then the sequence \( 0 \to A \to B \to C \to 0 \) does not split. For if it did, there would exist \( \gamma : C \to B \) such that

\[
\beta \circ \gamma = l_C.
\]

Then \( l_C \) restricted to the first summand of \( C \) would yield \( l_{\mathbb{Z}_2} \), and would factor through \( \mathbb{Z} \).

There is, however, no \( \mathbb{Z} \)-homomorphism from \( \mathbb{Z}_2 \) into \( \mathbb{Z} \) except for the zero-homomorphism. Hence this sequence cannot be a split.
Since $B = A \oplus C$, there is a pair of maps, $i$ and $p$, the canonical injection and projection respectively, which do make the sequence split.

Thus $\alpha$ and $\beta$ need not be unique.

1.3 Definition. An element $(\overline{0}, A, B, C, \overline{0})$ of $\mathcal{S}$ is called a split exact sequence if $B = \overline{A} + \overline{C}$.

1.4 Example. Let $R$ be a noetherian ring and let $(\mathfrak{m}, +)$ be the $R$-isomorphism classes of $R$-modules under direct sum, and let $\mathcal{S}$ be the subset of $\mathfrak{m}^5$ consisting of all exact sequences. Then $\mathcal{S}$ has the following five properties:

Property 1. $(0, 0, 0, 0, 0) \in \mathcal{S}$.

Proof. The sequence $0 \to 0 \to 0 \to 0 \to 0$ is an exact sequence of $R$-modules.

Property 2. $\mathcal{S}$ contains the split exact sequences, and given any sequence $(\overline{0}, A, B, C, \overline{0}) \in \mathcal{S}$, and any split exact sequence $(\overline{0}, D, D+E, E, \overline{0}) \in \mathfrak{m}^5$, their sum

$$(\overline{0}, A + D, B + D + E, E, \overline{0})$$

is in $\mathcal{S}$.
Proof. If \( C \) and \( D \) are in \( \mathfrak{m} \), then the sequence of \( R \)-modules, \( 0 \rightarrow C \xrightarrow{i} C \oplus D \xrightarrow{p} D \rightarrow 0 \), where \( i \) and \( p \) are the canonical injection and projection respectively, is always exact, \( C \oplus D \) is finitely generated, and therefore

\[
(\mathfrak{m}, C, C+D, D, 0) \in \mathfrak{s}.
\]

The second statement of property 2 follows from the more general property enjoyed by this example which follows:

**Property 2'.** Let \((\mathfrak{m}, A_1, B_1, C_1, 0)\) and \((\mathfrak{m}, A_2, B_2, C_2, 0)\) be in \( \mathfrak{s} \). Then their coordinate-wise sum

\[
(\mathfrak{m}, A_1+A_2, B_1+B_2, C_1+C_2, 0)
\]

is in \( \mathfrak{s} \).

**Proof.** There exist \( R \)-module homomorphisms \( \alpha_1, \beta_1, \alpha_2, \beta_2 \), such that

\[
0 \rightarrow A_1 \xrightarrow{\alpha_1} B_1 \xrightarrow{\beta_1} C_1 \rightarrow 0
\]

\[
0 \rightarrow A_2 \xrightarrow{\alpha_2} B_2 \xrightarrow{\beta_2} C_2 \rightarrow 0
\]
are exact sequences of finitely generated $R$-modules.

Then their sum

$$0 \to A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \to 0$$

is also exact, and $A_1 \oplus A_2$, $B_1 \oplus B_2$, and $C_1 \oplus C_2$ are finitely generated.

**Property 3.**

1) If $(\overline{0},M,M',\overline{0},\overline{0}) \in \mathcal{S}$, then $M = M'$.

2) If $(\overline{0},\overline{0},M',M,\overline{0}) \in \mathcal{S}$, then $M = M'$.

3) If $(\overline{0},\overline{M},\overline{0},M',\overline{0}) \in \mathcal{S}$, then $M = M' = \overline{0}$.

**Proof.**

1) $(\overline{0},\overline{M},\overline{M}',\overline{0},\overline{0}) \in \mathcal{S} \Rightarrow \exists \alpha, \beta$ such that

$$0 \to M \to M' \to 0 \to 0$$

is exact. Thus $\alpha$ is an isomorphism and $M = M'$.

The proof of 2) is similar to the proof of 1).

3) $(\overline{0},\overline{M},\overline{0},M',\overline{0}) \in \mathcal{S} \Rightarrow \exists \alpha, \beta$ such that

$$0 \to \overline{M} \to \overline{0} \to M' \to \overline{0}$$

is an exact sequence of $R$-modules. Therefore,

$$M = \overline{0} = M' \text{ or } \overline{M} = \overline{0} = \overline{M}'$$

**Property 4.** (Pullback) If $(\overline{0},\overline{A},\overline{B},\overline{C},\overline{0})$ and $(\overline{0},\overline{K},\overline{M},\overline{B},\overline{0})$ are exact, then there exists an $M' \in \mathcal{M}$ such that $(\overline{0},\overline{M}',\overline{M},\overline{C},\overline{0})$ and $(\overline{0},\overline{K},\overline{M}',\overline{A},\overline{0})$ are exact.
Proof. There exist pairs \((\alpha, \beta)\) and \((\gamma, \delta)\) of \(R\)-module homomorphisms such that
\[ 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0 \]
and \[ 0 \to K \overset{\gamma}{\to} M \overset{\delta}{\to} B \to 0 \] are exact sequences of \(R\)-modules.

Consider the following diagram of \(R\)-modules constructed from these two sequences:

\[
\begin{array}{c}
\text{O} \\
\downarrow \\
\text{R} \\
\downarrow \gamma \\
\text{M} \\
\downarrow \alpha \downarrow \delta \\
A \overset{\beta}{\to} B \to C \to 0.
\end{array}
\]

This diagram can be completed with \(M'\), the pullback of the diagram

\[
\begin{array}{c}
A \\
\downarrow \gamma \downarrow \delta \\
B
\end{array}
\]

as follows:
The pullback $M'$ is a submodule of $M$, which is a finitely generated module over a noetherian ring, and is therefore finitely generated (see [6], p. 11). Thus $(\mathcal{O}, M', M, \mathcal{O})$ and $(\mathcal{O}, K, M', A, \mathcal{O})$ are in $\mathcal{S}$.

**Property 5. (Pullback)** If $(\mathcal{O}, A, B, C, \mathcal{O})$ and $(\mathcal{O}, A', B', C, \mathcal{O})$ are in $\mathcal{S}$, then there exists an $M \in \mathfrak{m}$ such that $(\mathcal{O}, A, M, B', \mathcal{O})$ and $(\mathcal{O}, A', M, B, \mathcal{O})$ are in $\mathcal{S}$.

**Proof.** There exist pairs $(\alpha, \beta)$ and $(\gamma, \delta)$ of $R$-module homomorphisms making $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ and $0 \rightarrow A' \xrightarrow{\gamma} B' \xrightarrow{\delta} C \rightarrow 0$ exact sequences of $R$-modules. The diagram
can be completed with the pullback, \( M \), of the diagram

\[
\begin{array}{c}
\text{O} \\
\vdots \\
\alpha \downarrow \\
\begin{array}{ccc}
A & \rightarrow & B \\
\beta & \rightarrow & C \\
\delta & \rightarrow & O
\end{array}
\end{array}
\]

in the following manner:

\[
\begin{array}{c}
\text{O} \\
\vdots \\
\cdots \rightarrow A \rightarrow \cdots \\
\downarrow \\
M \\
\downarrow \\
\vdots \\
\cdots \rightarrow B \rightarrow \cdots \\
\downarrow \\
\text{O}
\end{array}
\]

(see [8], I, 20.3). Since \( M \subseteq B \oplus B \), and \( R \) is noetherian, we have \( M \) finitely generated.

3. Injectives. The concept of a projective (or dually, injective) \( R \)-module is one of the most important in homological algebra.
**Definition.** Let $R$ be a ring, and let $P$ be an $R$-module. We say that $P$ is **projective** if whenever there are $R$-modules $A$ and $B$, $R$-module homomorphism $\alpha: A \rightarrow B$ and $R$-module epimorphism $\beta: A \rightarrow B \rightarrow 0$, then there exists $\gamma: P \rightarrow B$ such that $\beta \gamma = \alpha$.

\[
\begin{array}{c}
A \\
\downarrow \beta \\
B \\
\downarrow 0 \\
P \\
\downarrow \gamma \\
A \\
\downarrow \alpha \\
B \\
\downarrow 0 \\
\end{array}
\]

The following well-known characterization of projective $R$-modules will be used. Although proofs are found in most every book on homological algebra, the proofs will also be included here because the theorems will be considered again in the more general setting of Ohm's axioms.

**1.5 Theorem.** Let $R$ be a ring and let $P$ be an $R$-module. Then the following statements are equivalent:

(a) $P$ is projective.

(b) Every short exact sequence

\[
O \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} P
\]

of $R$-modules splits.

(c) $P$ is isomorphic to a direct summand of a free module over $R$.
Proof. (a) ⇒ (b). Assume $R$ is projective and consider the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\beta} & P \\
\downarrow l_p & & \downarrow l_p \\
B & \xrightarrow{\gamma} & P
\end{array}
$$

with $l_p: P \rightarrow P$, the identity. By definition there exists $\gamma: P \rightarrow B$ such that

$$\beta \gamma = l_p.$$

Hence,

$$B \cong \alpha(A) \oplus \gamma(P) \cong A \oplus P.$$

(b) ⇒ (c) Let $F$ be the free module on the set $P$. Then the identity

$$1: P \rightarrow P$$

can be extended to a homomorphism

$$J: F \rightarrow P.$$

Letting $K$ be the kernel of $J$, we have

$$0 \rightarrow K \rightarrow F \rightarrow P \rightarrow 0$$

is exact, and as it splits, we conclude from (b),
\[ K \oplus P \cong F. \]

(c) = (a). Assume \( K \oplus P \cong F \), the free module on the set \( S \). Consider the diagram

\[
\begin{array}{c}
\alpha \\
\downarrow \\
A \\[-> B \\
\beta \\
\end{array} \]

Extend \( \beta : P \to B \) to \( \beta' : F \to B \) by

\[ \beta'(f) = \beta'(x, y) = \beta(x) \]

\( f \in F, f = (x, y), x \in P, y \in K. \)

There exists \( \Gamma : F \to A \) such that

\[ \alpha \Gamma = \beta \]

as we see by sending \( s \in S \) to any element of

\[ \alpha^{-1} (\beta'(s)). \]

Then if \( i \) is the inclusion \( i : P \to P \oplus K \),

\[ \gamma = \Gamma i \]

is the required homomorphism.

Q.E.D.

1.6 Definition. Let \( \mathcal{M} \) be the monoid of \( R \)-isomorphism classes of \( R \)-modules. An element \( \overline{P} \in \mathcal{M} \) is called
jective if given any pair of exact sequences
\((\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})\) and \((\overline{0}, \overline{K}', \overline{P}, \overline{C}, \overline{0})\) in \(\mathcal{B}\), and an element \(\overline{K} \in \mathcal{M}\) such that \((\overline{0}, \overline{K}, \overline{P}, \overline{C}, \overline{0})\) is in \(\mathcal{B}\), then there exists \(\overline{M} \in \mathcal{M}\) such that \((\overline{0}, \overline{K}', \overline{M}, \overline{K}, \overline{0})\) and \((\overline{0}, \overline{M}, \overline{P}', \overline{P}, \overline{0})\) are exact.

1.7 Theorem. An element \(\overline{P}\) in \(\mathcal{M}\) isjective if and only if \(\overline{P}\) is a projective \(R\)-module.

Proof. Suppose \(\overline{P}\) is a projective \(R\)-module, and let \((\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0}), (\overline{0}, \overline{K}', \overline{P}', \overline{A}, \overline{0})\), and \((\overline{0}, \overline{K}, \overline{P}, \overline{A}, \overline{0})\) be given exact sequences. Then there exist \(R\)-module homomorphisms \(a, \beta, \gamma, \delta, \gamma', \delta'\), such that the following are exact sequences of \(R\)-modules:

\[
\begin{align*}
0 & \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \overset{0}{\to} \\
0 & \to K \overset{\gamma}{\to} P \overset{\delta}{\to} A \overset{0}{\to} \\
0 & \to K' \overset{\gamma'}{\to} P' \overset{\delta'}{\to} A' \overset{0}{\to} 
\end{align*}
\]

Consider the following diagram:
Let $\lambda : P \to B$ be the $R$-module homomorphism guaranteed by the projectivity of $P$ such that

$$\delta = \beta \lambda .$$

Define $\delta'': P' \oplus P \to B$ by

$$\delta''(p', p) = \alpha \delta'(p') + \lambda(p)$$

and let $M = \ker \delta''$, and $\alpha'' : M \to P \oplus P'$ be the inclusion. Now define $i : P' \to P \oplus P'$, and $p : P \oplus P' \to P$ to be the injection and projection maps, respectively.

Then all sequences are exact, the diagram commutes, and the top row can therefore be completed by the Nine Lemma (see [8], I, 16.1). Hence the sequences $(\overline{0}, \overline{M}, \overline{F} \oplus F, \overline{R}, \overline{O})$ and $(\overline{0}, \overline{K}', \overline{M}, \overline{R}, \overline{O})$ are in $\mathcal{A}$.

Conversely, suppose that $\overline{P}$ is a projective element of $\mathcal{M}$. Let $P$ be a finitely generated free $R$-module.
which resolves $P$. (Let $\{x_\alpha\}$ be a finite set of
generators for $P$. Let $F\{x_\alpha\}$ be the free module on
the set $\{x_\alpha\}$. Then inclusion: $\{x_\alpha\} \to P$ can be
extended to a unique homomorphism $F\{x_\alpha\} \to P$ which
is onto since the $\{x_\alpha\}$ generate $P$.}) Let $K$ be the
kernel. Then $K$ is finitely generated since $F$ is
noetherian, and we have the exact sequence of $R$-modules:

$$0 \to K \to F \to P \to 0.$$ 

Hence $(\overline{0}, K, F, P, \overline{0})$ is in $\mathcal{S}$, and since $P$ is jective,
and $(\overline{0}, \overline{0}, A, A, \overline{0})$ and $(\overline{0}, \overline{0}, P, P, \overline{0})$ are in $\mathcal{S}$, we can
find $M \in \mathfrak{m}$ such that $(\overline{0}, M, A+P, F, \overline{0})$ and $(\overline{0}, \overline{0}, M, \overline{0}, \overline{0})$
are in $\mathcal{S}$. Then $\overline{M} = \overline{0}$ and $\overline{F} = A + \overline{P}$ by property 3.

Hence $P$ is isomorphic to a direct summand of a free
module, and is therefore projective by theorem 1.5.

Thus, the jective elements $P$ of $\mathcal{S}$ are those
for which $P$ is projective, and have the other usual
properties of projectives holding for these jectives,
namely:

i) If $P$ is jective, and $P' + \overline{Q} \cong P$, then $P'$
is jective (a summand of a jective is necessarily
jective).

Proof. Suppose $P$ is jective and $P' + \overline{Q} \cong P$. Then
$P$ is a projective $R$-module and $P' \oplus Q \not\cong P$. 

Thus $P'$ is a direct summand of $P$, and therefore must be projective. Hence $P'$ is projective by theorem 1.7.

ii) If $P$ and $Q$ are projective, then $P+Q$ is projective.

**Proof.** By theorem 1.7, $P$ and $Q$ are projective, therefore $P \oplus Q$ is projective. Hence $P \oplus Q = P + Q$ is projective by theorem 1.7.

iii) Given $(\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})$ and $(\overline{0}, \overline{A'}, \overline{B'}, \overline{C'}, \overline{0})$ exact, then their sum

$$(\overline{0}, \overline{A+A'}, \overline{B+B'}, \overline{C+C'}, \overline{0})$$

is also exact.

**Proof.** Since $(\overline{0}, \overline{A}, \overline{B}, \overline{C}, \overline{0})$ is in $\mathcal{A}$ and there exist R-module homomorphisms $\alpha$ and $\beta$ such that

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is an exact sequence of R-modules, and there exist $\alpha'$ and $\beta'$ such that

$$0 \to A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \to 0$$

is an exact sequence of R-modules.
Their sum

\[
\begin{align*}
a + a' &= \beta + \beta' \\
0 &\to A + A' \to B + B' \to C + C' \to 0
\end{align*}
\]

is then easily seen to be exact. Thus

\[
(\overline{0}, \overline{A + A'}, \overline{B + B'}, \overline{C + C'}, \overline{0}) \in \mathfrak{S}.
\]
CHAPTER II

1. The Axioms. Let \((m,+\rangle\) be a commutative monoid with identity \(0\). Let \(\mathcal{S}\) be a subset of \(m^5\) of elements of the form \((0,A,B,C,0)\) which satisfy the following axioms.

\(\mathcal{S}_1\). \((0,0,0,0,0)\in \mathcal{S}\)

\(\mathcal{S}_2\). \(\mathcal{S} + \mathrm{Sp} \subseteq \mathcal{S}\)

\(\mathcal{S}_3\). i) \((0,M,M',0,0)\in \mathcal{S} \Rightarrow M = M'\)

ii) \((0,0,M,M',0)\in \mathcal{S} \Rightarrow M = M'\)

iii) \((0,M,0,M',0)\in \mathcal{S} \Rightarrow M = M' = 0\)

\(\mathcal{S}_4\). (Pullback) Given \((0,A,B,C,0)\in \mathcal{S}\) and \((0,K,M,B,0)\in \mathcal{S}\) there exists \(M'\in m\) such that \((0,M',M,C,0)\in \mathcal{S}\) and \((0,K,M',A,0)\in \mathcal{S}\)

\(\mathcal{S}_5\). (Pullback) Given \((0,A,B,C,0)\in \mathcal{S}\) and \((0,A',B',C,0)\in \mathcal{S}\). There exists \(M\in m\) such that \((0,A,M,B',0)\in \mathcal{S}\) and \((0,A',M,B,0)\in \mathcal{S}\).

A pair \((m,\mathcal{S})\) is called a monoid with exact sequences, and an element of \(\mathcal{S}\) will be called a set of short exact sequences.
As was verified in example 1.4, the monoid of isomorphism classes of R-modules under direct sum satisfies these axioms. Other examples will be constructed in Chapter III.

Two diagrams will be considered throughout this paper in order to illustrate axioms δ4 and δ5.

For the pullbacks in δ4 and δ5, one should have the following diagrams in mind:

\[
\begin{array}{c}
\begin{array}{c}
\text{0} \\
\text{0} \\
\text{K} \\
\text{0} \\
\text{0}
\end{array}
\quad \begin{array}{c}
\text{0} \\
\text{0} \\
\text{A} \\
\text{0} \\
\text{0}
\end{array}

\begin{array}{c}
\text{M'} \\
\text{M} \\
\text{C} \\
\text{0}
\end{array}
\quad \begin{array}{c}
\text{A'} \\
\text{A} \\
\text{B} \\
\text{0}
\end{array}

\begin{array}{c}
\text{0} \\
\text{0}
\end{array}
\end{array}
\]\n
\[\delta^4 \quad \delta^5\]

The sequences \((0, M', M, C, 0)\) and \((0, K, M', A, 0)\) in the \(\delta^4\) diagram, and \((0, A, M', P, 0)\) and \((0, A, M', B', 0)\) in the \(\delta^5\) diagram are called the completions of the diagrams. An \(\delta^4\) or \(\delta^5\) diagram together with its completion is called a completed diagram.

3. Jectives. Let \((\mathfrak{m}, \delta)\) be a set of short exact sequences. An element \(P \in \mathfrak{m}\) is called jective if for
any \((0, A, B, C, 0) \in \mathfrak{R}\), \((0, K', P', A, 0) \in \mathfrak{R}\), and \((0, K, P, C, 0) \in \mathfrak{R}\) in \(\mathfrak{R}\), there exists \(M \in m\) such that there exist \((0, K', M, K, 0) \in \mathfrak{R}\) and \((0, K, P'+P, B, 0) \in \mathfrak{R}\).

The following diagram is helpful:

```
0   0   0
|   |   |
|   |   |
|   |   |
0   K'   M   K   0
|   |   |
|   |   |
0   P   P+P   P
|   |   |
|   |   |
A   B   C   0
|   |   |
|   |   |
0   0   0
```

We have seen in Chapter I that when \(m\) is the monoid of isomorphism classes of \(R\)-modules, that an element \(P\) in \(m\) is jective if and only if \(P\) is a projective \(R\)-module. Projective \(R\)-modules are also characterized as those modules \(P\) which split every exact sequence \((0, A, B, P, 0)\). We now prove an analogous result for an arbitrary monoid with exact sequences \((m, \mathfrak{R})\).

2.1 Theorem. Let \((m, \mathfrak{R})\) be a monoid with exact sequences and let \(P \in m\). Then \(P\) is jective if and only if \(P\) splits every short exact sequence \((0, A, B, P, 0) \in \mathfrak{R}\) (or \(A \oplus P\)).
Proof. Suppose \( P \) is jective, and let \((0,A,I,P,0)\)
be exact. Then the diagram

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\vdots & & \\
0\cdots0\cdots0\cdots0\cdots0 \\
\vdots & & \\
A & A+P & P \\
\vdots & & \\
0\longrightarrow A\longrightarrow B\longrightarrow P\longrightarrow 0 \\
\vdots & & \\
0 & 0 & \\
\end{array}
\]

can be completed as shown. Hence \( A+P \) is by axiom \( 8^+ \).

Conversely, suppose \( P \in \mathcal{M} \), and every sequence
\((0,D,E,P,0)\) in \( \mathcal{S} \) splits, and let

\[
\begin{array}{cccc}
0 & 0 \\
\uparrow & & \\
K' & K \\
\uparrow & & \\
P & P \\
\uparrow & & \\
0\longrightarrow A\longrightarrow K\longrightarrow C\longrightarrow 0 \\
\vdots & & \\
0 & 0 \\
\end{array}
\]

be a jective diagram which we wish to complete.

By axiom \( 8^+ \), there exists \( N \in \mathcal{M} \) such that
can be completed (as shown), resulting in sequences 
(0,A,N,P,0) and (0,K,N,B,0) in 3. Since P splits 
every such sequence, A + P = N. Hence, substituting 
A + P into the second sequence for N, we have 
(C,K,A+P,B,0) ∈ 3.

By axiom 35, the diagram

\[
\begin{array}{c}
\vdots \\
K' \\
\vdots \\
O \cdots A \cdots N \cdots P \cdots 0 \\
\vdots \\
O \longrightarrow A + P \longrightarrow B \longrightarrow 0 \\
\vdots \\
O \\
\end{array}
\]

can be completed.

The completion of this diagram also completes the 
rective diagram with which we started.

Q.E.D.
Another result analogous to the case of R-modules follows.

2.2 Corollary. Let \((\mathcal{M}, \delta)\) be a monoid with exact sequences. If \(P\) and \(Q\) are elements of \(\mathcal{M}\) which are injective, then \(P + Q\) is also injective.

Proof. By the theorem we only need to show that any sequence in \(\delta\) of the form \((0, A, B, P+Q, 0)\) splits. We construct the following \(\delta\) diagram:

\[
\begin{array}{c}
0 & 0 \\
\vdots \\
A & A \\
\vdots \\
0 & \cdots & M & \cdots & B & \cdots & Q & \cdots & 0 \\
\vdots \\
0 & \cdots & P & \cdots & P+Q & \cdots & Q & \cdots & 0 \\
\vdots \\
0 & 0 \\
\end{array}
\]

There exists \(M \in \mathcal{M}\) which completes the diagram as shown with \((0, A, M, P, 0) \in \delta\), and \((0, M, B, Q, 0) \in \delta\). Now, since \(P\) is injective, the sequence \((0, A, M, P, 0)\) splits and hence \(A + P = M\). Since \(Q\) is injective we have \(M + Q = B\). Substituting for \(M\),
\[ A + P + Q = B. \]

Hence \((0,A,R,P+Q,0)\) splits.

Q.E.D.

Another property of a projective module in the category of \(R\)-modules is that a summand of a projective module is projective. This property is false for general \((\mathfrak{T},\mathfrak{S})\), as we shall see in Chapter III. There is a result, however, which holds for cancellative jectives in a monoid \(\mathfrak{M}\) with exact sequences \(\mathfrak{S}\).

**2.3 Definition.** Let \((\mathfrak{M},\mathfrak{T})\) be a monoid and \(x \in \mathfrak{M}\). The element \(x\) is called cancellative if for all \(y, y' \in \mathfrak{M}\)

\[ x + y = x + y' \Rightarrow y = y'. \]

**2.4 Corollary.** Let \((\mathfrak{M},\mathfrak{S})\) be a set of short exact sequences, and let \(Q\) be a cancellative element in \(\mathfrak{M}\). Then if \(P + Q\) is jective, so is \(P\).

**Proof.** Let \((0,A,R,P,0)\) be in \(\mathfrak{S}\). We wish to see that this sequence splits. By axiom \(\mathfrak{S}2\), the sequence \((0,A,R+Q,P+Q,0)\) is in \(\mathfrak{S}\). Since \(P + Q\) is jective, the theorem gives us that
\[ A + P + Q = B + Q. \]

Now cancelling \( Q \), we have

\[ A + P = B. \]

Q.E.D.

**4. The Addition of Exact Sequences.** In the category of \( R \)-modules, two short exact sequences can be added and the sum remains exact. In Chapter III we will see an example \((\mathfrak{m}, \mathfrak{g})\) where there exists a sequence which cannot be added to itself. With more hypothesis, something positive can be said.

**2.5 Theorem.** Let \((\mathfrak{m}, \mathfrak{g})\) be a monoid with exact sequences, and let \((0, A, B, C, 0)\) and \((0, A', B', C', 0)\) be elements of \( \mathfrak{g} \). If \( A \) (or \( A' \)) is ejective, then \((0, A+A', B+B', C+C', 0)\) is in \( \mathfrak{g} \).

**Proof.** Since \((0, A, B, C, 0) \in \mathfrak{g}\), we have \((0, A, B+C', C+C', 0) \in \mathfrak{g}\) by axiom \( \mathfrak{g}2 \). Similarly, \((0, A', B'+B, C'+B, 0) \in \mathfrak{g}\). Now using these elements we construct an \( \mathfrak{g}4 \) diagram.
which can be completed with a pullback $M \in \mathcal{M}$. Hence $(0, A', M, A, 0) \in \delta$ and $(0, M, B+B' C+C', 0) \in \delta$. Since $A$ is injective, $(0, A', M, A, 0)$ splits, and $M \cdot A = A'$. Substituting for $M$, we have $(0, A+A', B+C', C+C', 0) \in \delta$.

Q.E.D.
1. **Diagrams.** Let $\mathcal{D}$ be an $\mathcal{D}^{1}$ diagram constructed from the elements $(0, A, B, C, 0)$ and $(0, E, F, B, 0)$ in $\mathcal{D}$ and let $\mathcal{D}'$ be an $\mathcal{D}^{5}$ diagram constructed from the elements $(0, M, N, S, 0)$ and $(0, U, T, S, 0)$ in $\mathcal{D}$ as shown below:

\[
\begin{array}{ccc}
& & O \\
& E & \\
& F & \\
0 & A & B & C & 0 \\
& O & \\
\end{array}
\quad
\begin{array}{ccc}
& & O \\
& U & \\
& T & \\
0 & M & N & S & 0 \\
& O & \\
\end{array}
\]

$\mathcal{D}$

$\mathcal{D}'$

Suppose there exist sequences

\[(0, A_{i}, B_{i}, C_{i}, 0) \quad \text{and} \quad (0, E_{i}, F_{i}, B_{i}, 0) \in \mathcal{D}, \quad i = 1, 2,\]

\[(0, M_{i}, N_{i}, S_{i}, 0) \quad \text{and} \quad (0, U_{i}, T_{i}, S_{i}, 0) \in \mathcal{D}, \quad i = 1, 2.\]
such that

\[(0, A, B, C, 0) = (0, A_1 + A_2, B_1 + B_2, C_1 + C_2, 0)\]

\[(0, E, F, R, 0) = (0, E_1 + E_2, F_1 + F_2, R_1 + R_2, 0)\]

and

\[(0, M, N, S, 0) = (0, M_1 + M_2, N_1 + N_2, S_1 + S_2, 0)\]

\[(0, U, T, S, 0) = (0, U_1 + U_2, T_1 + T_2, S_1 + S_2, 0)\].

Then we can write $\mathcal{B}$ as the sum of the two diagrams $\mathcal{B}_1$ and $\mathcal{B}_2$ shown below:

\[\begin{array}{c}
0 \\
E_1 \\
F_1 \\
A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0
\end{array} \hspace{1cm} \begin{array}{c}
0 \\
E_2 \\
F_2 \\
A_2 \longrightarrow B_2 \longrightarrow C_2 \longrightarrow 0
\end{array}\]

$\mathcal{B}_1$ $\mathcal{B}_2$

and $\mathcal{B}'$ may be written as the sum of the two diagrams $\mathcal{B}_1'$ and $\mathcal{B}_2'$ shown below:
We say that $\mathcal{B}_1, \mathcal{B}_2$ ($\mathcal{B}_1', \mathcal{B}_2'$) are summands of $\mathcal{B}$ ($\mathcal{B}'$) and that $\mathcal{B}$ ($\mathcal{B}'$) is the sum of the diagrams $\mathcal{B}_1$ and $\mathcal{B}_2$ ($\mathcal{B}_1'$ and $\mathcal{B}_2'$).

The following lemmas aid in completing diagrams.

In the following lemmas we will assume that $\mathcal{M}$ is a commutative monoid and $\mathcal{S} \subseteq \mathcal{M}^5$ is a set of elements of the form $(0, A, B, C, 0)$.

3.1 Lemma. Suppose $\mathcal{B}$ is a diagram of type $\mathcal{S}_4$ or $\mathcal{S}_5$, constructed from the elements of $\mathcal{S}$, and suppose that $\mathcal{B}$ can be written as the sum of two diagrams $\mathcal{B}_1$ and $\mathcal{B}_2$ (of the same type as $\mathcal{B}$). If both $\mathcal{B}_1$ and $\mathcal{B}_2$ can be completed with elements of $\mathcal{S}$ and if the sums of the completions of $\mathcal{B}_1$ and $\mathcal{B}_2$ result...
in rows and columns which are elements of $\mathcal{S}$, then the sum of the completed $S_1$ and $S_2$ completes $S$ in $\mathcal{S}$.

Proof. Add them.

3.2 Lemma. Suppose $A \subseteq \mathcal{S}$, and $A + A \subseteq \mathcal{S}$. Then

i) An $S^4$ diagram with column in $A$ can be completed in $\mathcal{S}$.

ii) An $S^5$ diagram with either column or row in $A$ can be completed in $\mathcal{S}$.

iii) An $S^5$ diagram with both column and row in $A$ can be completed in $A$.

Proof. i) Let $S$ be an $S^4$ diagram constructed with row $(0,A,B,C,0) \in \mathcal{S}$, and $(0,D,D+B,A,B,0)$ in $A$. Then $S$ can be completed as follows:

```
  0   0
  .   .
  D   D
  .   .
0\ldots D+A\ldots D+B\ldots C\ldots 0
  .   .
  0--A--B--C--\ldots 0
  .   .
  0   0
```

(S completed) .
ii) Let $\mathcal{B}$ be an $A_5$ diagram with row $(0,A,B,C,0) \in \mathcal{B}$ and column $(0,D,D+C,D,0) \in \mathcal{A}$. Then $\mathcal{B}$ can be completed with elements of $\mathcal{B}$ as follows:

$$
\begin{array}{cccccc}
0 & 0 & & & & \\
\vdots & & & & & \\
D & D & & & & \\
\vdots & & & & & \\
0 & \cdots & A & \cdots & D+B & \cdots & D+C & \cdots & 0 \\
\vdots & & & & & \\
0 & \cdots & A & \cdots & A+B & \cdots & D+B & \cdots & 0 \\
\vdots & & & & & \\
0 & \cdots & A & \cdots & A+B & \cdots & B & \cdots & 0 \\
\vdots & & & & & \\
0 & 0 & & & & \\
\end{array}
$$

$\mathcal{B}$ Completed.

iii) Let $\mathcal{B}$ be an $A_5$ diagram with row $(0,A,A+B,B,0) \in \mathcal{A}$ and column $(0,D,D+B,B,0) \in \mathcal{A}$. Then $\mathcal{B}$ can be completed with elements of $\mathcal{A}$ as follows:

$$
\begin{array}{cccccc}
0 & 0 & & & & \\
\vdots & & & & & \\
D & D & & & & \\
\vdots & & & & & \\
0 & \cdots & A & \cdots & A+D+B & \cdots & D+B & \cdots & 0 \\
\vdots & & & & & \\
0 & \cdots & A & \cdots & A+B & \cdots & B & \cdots & 0 \\
\vdots & & & & & \\
0 & 0 & & & & \\
\end{array}
$$
3.3 Lemma. Suppose \( \mathcal{D} \subseteq \mathcal{S} \), and let \( \mathcal{B} \) be an \( \mathcal{S} \)-diagram. If the horizontal row is either of the form \((0, C, C, 0, 0)\) or \((0, 0, C, C, 0)\), then the diagram can be completed in \( \mathcal{S} \).

Proof. If the diagram is constructed with the two sequences i) \((0, A, B, C, 0)\) and \((0, C, C, 0, 0)\) or ii) \((0, A, B, C, 0)\) and \((0, 0, C, C, 0)\), then it can be completed as shown below:

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\vdots & | & & | & & \\
A & A & & & & \\
\vdots & | & & | & & \\
0\cdots B\cdots B\cdots 0\cdots 0 & | & & | & & \\
\vdots & | & & | & & \\
0\cdots C\cdots C\cdots 0 & | & & | & & \\
\vdots & | & & | & & \\
0 & 0 & & & & \\
\end{array}
\]

i)  

ii) .

Q.E.D.

3.4 Lemma. Suppose \( \mathcal{D} \subseteq \mathcal{S} \), and let \((0, A, B, C, 0)\) \( \in \mathcal{S} \). The \( \mathcal{S} \)-diagram constructed with \((0, A, B, C, 0)\) and \((0, A, B, C, 0)\) can be completed in \( \mathcal{S} \) (actually in \( \mathcal{D} \)).
Proof.

\[
\begin{array}{c}
0 & 0 \\
\vdots & \\
A & A \\
\vdots & \\
0 \cdots A + B \cdots B \cdots 0 \\
\vdots & \\
0 & A & B & C & 0 \\
\vdots & \\
0 & 0
\end{array}
\]

\[\text{Q.E.D.}\]

We now utilize the preceding lemmas on diagram completion to show how to construct a simple example of a set of short exact sequences.

3.5 Definition. Let \( X \) be a set. The free commutative monoid on \( X \), \( \mathcal{M}(X) \), is defined to be all functions from \( X \) into the non-negative integers with the following operation:

\[
(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in X.
\]

\[
0(x) = 0 \quad \text{for all } x \in X.
\]

A congruence on \( \mathcal{M}(X) \) is an equivalence relation on \( \mathcal{M}(X) \) which satisfies one additional axiom, additivity. So a congruence is a subset of \( \mathcal{M}(X) \times \mathcal{M}(X) \).
which satisfies

i) (Reflexivity) \( (a,a) \in C \quad \forall a \in \mathcal{M}(X) \).

ii) (Symmetry) \( (a,b) \in C \Rightarrow (b,a) \in C \quad \forall a,b \in \mathcal{M}(X) \).

iii) (Transitivity) \( (a,b) \in C \) and \( (b,c) \in C \)

\[ \Rightarrow (a,c) \in C \quad \forall a,b,c \in \mathcal{M}(X) \).

iv) (Additivity) \( (a,b) \in C \) and \( (c,d) \in C \)

\[ \Rightarrow (a+c,b+d) \in C \quad \forall a,b,c,d \in \mathcal{M}(X) \).

Notice that the fourth condition guarantees that the set of equivalence classes partitioned by this equivalence relation will have a binary operation induced by the binary operation on \( \mathcal{M}(X) \), which is independent of the representative of the equivalence class.

Let \( S \) be a subset of \( \mathcal{M}(X) \times \mathcal{M}(X) \). Since the intersection of any family of congruences is a congruence, and \( \mathcal{M}(X) \times \mathcal{M}(X) \) is itself a congruence which contains \( S \), there is a smallest congruence which contains \( S \).

3.6 Definition. Let \( \mathcal{M}(X) \) be the free commutative monoid on a set \( X \). Let \( S \) be a subset of \( \mathcal{M}(X) \times \mathcal{M}(X) \). Then we define the commutative monoid with relations \( S \) and generators \( X \) to be the monoid of equivalence classes defined by the smallest congruence in \( \mathcal{M}(X) \times \mathcal{M}(X) \) which contains \( S \).
Instead of the notation \((a,b) \in \mathcal{C}\) we will use the notation \(a \sim b\). (**a** is equivalent to **b**.)

3.7 Theorem. Let \(\mathcal{M}\) be any commutative monoid such that \(M + M' = 0 \Rightarrow M, M' = 0\). Let \(\delta \in \mathcal{D}\), the split exact sequences. Then \(\delta\) is a set of short exact sequences.

**Proof.** 1. \(\delta\) satisfies axiom \(\delta_1\).

**Proof.** The element \((0,0,0,0,0)\) is a split exact sequence as \(0 + 0 = 0\).

2. \(\delta\) satisfies axiom \(\delta_2\).

**Proof.** The sum of two splits is a split.

3. \(\delta\) satisfies axiom \(\delta_3\).

**Proof.** If i) \((0,M,M',0,0) \in \delta\) or if ii) \((0,0,M',M,0) \in \delta\), then \(M + 0 = M'\), since either splits. If iii) \((0,M,0,M',0) \in \delta\), then \(M + M' = 0\), and by definition of \(\mathcal{M}\), \(M, M' = 0\).

4. \(\delta\) satisfies axioms \(\delta_4\) and \(\delta_5\).

**Proof.** Since the column is a split, the diagram can be completed by lemma 3.2.

Q.E.D.
The following example shows that the sums of two short exact sequence need not be exact.

2.5 Example. Let \( \mathcal{E} \subseteq \mathbb{F}^5 \) be the set of all elements
\[
\mathcal{E} = s(0,A,A,A,0) + m(0,A,A,0,0) + n(0,0,A,A,0)
\]
where \( s = 0 \) or \( 1 \); \( m,n \geq 0 \), so each element of \( \mathcal{E} \) can be written in the form
\[
(s, (s+m)A, (s+m+n)A, (s+n)A, 0).
\]

Then \( \mathcal{E} \) is a set of short exact sequences which is not a submonoid of \( \mathbb{F}^5 \). \((0,A,A,A,0) + (0,A,A,0,0) \in \mathcal{E} \).

Proof. Let \( \mathcal{E} \) be an \( \mathcal{E}^4 \) diagram constructed from two elements \((0,E,F,G,0)\) and \((0,L,M,F,0)\) of \( \mathcal{E} \):

\[
\begin{array}{c}
\mathcal{E}^4 \\
\begin{array}{c}
| \\
| \\
| \\
E \rightarrow F \rightarrow G \rightarrow 0
\end{array}
\end{array}
\]

Since \((0,E,F,G,0)\) is in \( \mathcal{E} \), there exist \( s = 0 \) or \( 1 \); \( m,n \geq 0 \) such that
\[(0, F, F, G, 0) : \]
\[(0, (s+m)A, (s+m+n)A, (s+n)A, 0) \]
and there exist \(s' \leq 0\) or \(l, m', n' \geq 0\) such that
\[(0, L, M, F, 0) = (0, (s'+m')A, (s'+m'+n')A, (s'+n')A, 0) \]
and, as \(\mathbb{F} = (s+m+n)A = (s'+n')A\) and \(M\) being free gives
\[s + m + n = s' + n'. \]

If \(s' = 0\), the column \((0, L, M, F, 0)\) splits, and the diagram can be completed by lemma 3.2.

If \(s' = 1\), then \((0, L, M, F, 0)\) can be written
\[(0, A, A, A, 0) + (0, m' A, (m'+n')A, n' A, 0) \].

Now, with the assumption \(s' = 1\), if we also have \(s = 1\), then
\[l + m + n = l + n' \]
and then
\[m + n = n'.\]
The diagram can be written as the sum of the two subdiagrams $\mathcal{D}_1$ and $\mathcal{D}_2$

\[
\begin{array}{cc}
0 & 0 \\
\downarrow & \downarrow \\
A & m'A \\
\downarrow & \downarrow \\
A & (m'm')A \\
0 -- A -- A -- A -- 0 & 0 -- nA -- (n+m)A -- m'A -- 0 \\
\end{array}
\]

The first of these, $\mathcal{D}_1$, can be completed as shown below:

\[
\begin{array}{cc}
0 & 0 \\
\vdots & \\
A & A \\
\vdots & \\
0 \cdots A \cdots A \cdots (i) & \\
\vdots & \\
(i -- A -- A -- A -- i) & \\
\vdots & \\
0 & 0 \\
\end{array}
\]

($\mathcal{D}_1$ completed.)

The second, $\mathcal{D}_2$, has a split column and row and can be completed by lemma 3.2, with splits.
The sum of $B_1$ completed and $B_2$ completed is an $\&$-diagram.

If $s' = l$ and $s = o$, then

$$m + n = l + n'.$$

The diagram $B$ is now

$$\begin{array}{c}
0 \\
| \quad \quad \quad (l+m')A \\
| \quad \quad \quad (l+m'+n')A \\
0 \quad mA \quad (m+n)A \quad nA \quad 0 \\
\end{array}$$

which if $m \neq o$ can be written as the sum of the two subdiagrams $B_1$ and $B_2$

$$\begin{array}{c}
0 \\
| A \\
| A \\
0 \quad A \quad A \quad 0 \quad 0 \\
\end{array} \quad \begin{array}{c}
0 \\
| n'A \\
| (m'+n')A \\
0 \quad (m-1)A \quad [(m-1)n]A \quad nA \quad 0 \\
\end{array}$$
Diagram $\mathcal{D}_1$ can be completed by lemma 3.2. Diagram $\mathcal{D}_2$ can be completed by lemma 3.2, as the column and row split. The last case, $m \to o$, can be completed by lemma 3.3. The sum of diagram $\mathcal{D}_1$ completed and $\mathcal{D}_2$ completed is in $\mathcal{S}$ as $\mathcal{D}_2$ can be completed with split.

Q.E.D.

$\mathcal{S}$ satisfies axiom $\mathcal{S}_5$. Let $\mathcal{D}$ be an $\mathcal{S}_5$ diagram constructed with two elements $(0,E,F,G,0)$ and $(L,M,G,o)$ of $\mathcal{S}$. 

\[
\begin{array}{c}
\circ \\
L \\
M \\
E \rightarrow F \rightarrow G \\
\circ
\end{array}
\]

Since $(0,E,F,G,0)$ is in $\mathcal{S}$, there exist $s', o$ or $1$, and $m,n \geq o$ such that

\[(0,E,F,G,0) \rightarrow (0,(s+m)A,(s+m+n)A,(s+n)A,o)\]

and since $(0,L,M,G,o)$ is in $\mathcal{S}$ there exist $s', o$ or $1$, and $m,n \geq o$ such that
Equating coordinates, we have

\[(0, L, M, G, 0) = (0, (s'+m')A, (s'+m'+n')A, (s'+m')A, 0)\].

and, as \(m\) is free on \(A\),

\[s + n = s' + n' \quad (1)\].

Suppose that either \(s = 0\) or \(s' = 0\). If \(s = 0\), then

\[(0, E, F, G, 0) = (0, mA, (m+n)A, nA, 0)\]

which is a split.

If \(s' = 0\), then

\[(0, L, M, G, 0) = (0, m'A, (m'+n')A, n'A, 0)\]

a split.

In either case, the diagram can be completed in \(\mathcal{S}\).

be a lemma 3.2.

If \(s = s' = 1\), then from equation (1)

\[1 + m = 1 + n' \quad \text{or} \quad n = m'.\]

The diagram may be written as the sum of two subdiagrams \(\mathcal{B}_1\) and \(\mathcal{B}_2\).
The first of these, $\mathcal{D}_1$, can be completed in $\mathcal{S}$ as follows:

Diagram $\mathcal{D}_2$ has a split row and split column and may therefore be completed with split exact sequences by lemma 3.2.

The sum of the completed subdiagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ consists of sequences in $\mathcal{S}$ because $\mathcal{D}_1$ consists of
elements of \( \mathcal{S} \), and \( B_2 \) consists entirely of splits. Their sum consists of elements of \( \mathcal{S} \) by axiom \( \mathcal{S} \).

Q.E.D.

3.9 Remark. In example 3.8 the element \((0, A, A, A, 0)\) is in \( \mathcal{S} \). We now see that the element

\[(0, A, A, A, 0) + (0, A, A, A, 0) = (0, 2A, 2A, 2A, 0)\]

which is the sum of an element of \( \mathcal{S} \) with itself is an element of \( \mathcal{M} \) which is not in \( \mathcal{S} \).

Proof. Suppose \((0, 2A, 2A, 2A, 0)\) is in \( \mathcal{S} \). Then there exist \( s = 0 \) or \( 1 \), and \( m,n \geq 0 \) such that

\[(0, 2A, 2A, 2A, 0) = (0, (s+m)A, (s+m+n)A, (s+n)A, 0) .\]

Equating coordinates we have

\[ 2A = (s+m)A \]
\[ 2A = (s+m+n)A \]
\[ 2A = (s+n)A . \]

By freeness of \( \mathcal{M} \) we have the following three equations:
Equations (1) and (2) imply $n = o$, which together with equation (3) gives

$$s = 2.$$ 

Thus as $s = o$ or 1 for elements of $\mathcal{E}$, $(0,2A,2A,2A,0)$ is not in $\mathcal{E}$.

Q.E.D.

3.10 Example. Let $\mathcal{M}$ be the commutative monoid generated by the two elements $A$ and $B$ with relations:

1) $B + B \sim B$
2) $A + B$.

We denote the equivalence class of $A$ by $\overline{A}$, and that of $B$ by $\overline{B}$.

Let $\mathcal{E}$ be the submonoid of the product monoid $\mathcal{M}$, which consists of all elements of the form

$$n(\overline{0},\overline{A},\overline{A},\overline{A},\overline{0}) + (\overline{0},n_1\overline{A}+m_1\overline{B},(n_1+n_2)\overline{A}+(m_1+m_2)\overline{B},n_2\overline{A}+m_2\overline{B},\overline{0})$$

$$= (\overline{0},(n+n_1)\overline{A}+m_1\overline{B},(n+n_1+n_2)\overline{A}+(m_1+m_2)\overline{B},(n+n_2)\overline{A}+m_2\overline{B},\overline{0})$$
where $n, n', n'', m', m'' \geq 0$.

Note that $(\overline{0}, n_1 A + m_1 B, (n_1 + n_2) A + (m_1 + m_2) B, n_1 A + m_2 B, 0)$ is an arbitrary split in $\mathcal{M}_1$.

The set $\mathcal{S}$ is a set of short exact sequences such that $(\mathcal{M}, \mathcal{S})$ has the property that the element $\overline{B}$ is injective, the element $\overline{A}$ satisfies $\overline{A} + \overline{B} = \overline{B}$, but $\overline{A}$ is not injective.

Before verifying that $\mathcal{S}$ satisfies axiom $\mathcal{S}_1 - \mathcal{S}_4$, we will determine exactly all relations defined by the smallest congruence $\mathcal{C}$ on $\mathcal{M}(A, B)$ for which $B \ + \ B \sim B$ and $A + B \sim B$ hold.

3.11 Lemma. Let $\mathcal{C}$ be the smallest congruence on $\mathcal{M}(A, B)$ for which

1) $B + B \sim B$

and

2) $A + B \sim B$,

then the relations defined by $\mathcal{C}$ are exactly

$\mathcal{R}: nA + mB \sim n'A + m'B \quad n, m, n', m' \geq 0$

with the condition that

$m \sim 0 \sim m', \quad$ and then $m \sim n'$. 
Proof. We first show that the above relation $R$ does
form a congruence on $\mathbb{M}(A,B)$.

1. $R$ is reflexive.

Proof. Let $nA + mB$ be in $\mathbb{M}(A,B)$. Then

$$nA + mB \sim nA + mB$$

holds since $m \sim m = 0$, and $n = n$.

2. $R$ is symmetric.

Proof. Let $nA + mB$ and $n'A + m'B$ be in $\mathbb{M}(A,B)$.

If $mA + nB \sim n'A + m'B$, then

$$m \sim m' = 0$$

and then $n = n'$

is a symmetric statement of $m$ and $m'$, $n$ and $n'$.

3. $R$ is transitive.

Proof. Let $nA + mB$, $n'A + m'B$, and $n''A + m''B$ be
in $\mathbb{M}(A,B)$ and suppose that

$$(1) \quad nA + mB \sim n'A + m'B$$

and

$$(2) \quad n'A + m'B \sim n''A + m''B.$$ 

Then if $m \sim 0$, $(1)$ implies that $m' \sim 0$ and $n \sim n'$.

Then $(2)$ implies $m'' \sim 0$, and $n' \sim n''$. Therefore,
By symmetry,

m'' \circ m = o, \text{ and } n'' \circ n.

Hence \( \mathcal{R} \) is transitive.

\( h. \) \( \mathcal{R} \) is additive.

Proof. Let \( n_1A + m_1B, n_1'A + m_1'B, m_2'A + m_2'B, m_2''A + m_2''B \) be elements of \( \mathcal{M}(A, \mathcal{R}) \).

Suppose that

\[ n_1A + m_1B \sim n_1'A + m_1'B, \]

and

\[ n_2'A + m_2'B \sim n_2''A + m_2''B. \]

We wish to see that

\[ (n_1 + n_2)A + (m_1 + m_2)B \sim (n_1' + n_2')A + (m_1' + m_2')B. \]

If \( m_1 + m_2 = o \), then \( m_1 - m_2 = o \), by their non-negativity, thus,

\[ m_1 \circ m = m_1' \circ m, \text{ and } n_1 = n_1'. \]

and

\[ m_2 \circ m = m_2' \circ m, \text{ and } n_2 = n_2'. \]

m = o \Rightarrow m'' = o \text{ and } n = n'.

By symmetry,

m'' \circ o = m = o, \text{ and } n'' \circ n.
Therefore,

\[ m_1 + m_2 = 0 \Rightarrow m_1' + m_2' = 0 \quad \text{and} \quad n_1 + n_2 = n_1' + n_2'. \]

By symmetry,

\[ m_1' + m_2' = 0 \Rightarrow m_1 + m_2 = 0 \quad \text{and} \quad n_1' + n_2' = n_1 + n_2. \]

Thus \( R \) is a congruence.

Now suppose that \( C' \) is any congruence on \( m(A, B) \) for which

\begin{enumerate}
  \item \( A + B \sim B \)
  \item \( A + B \sim B \).
\end{enumerate}

We will show that \( R \subseteq C' \).

1. \( nB \sim mB \) (or strictly positive \( m, n \)).

Proof. Since \( C' \) is reflexive,

\[ B \sim B. \]

Proceed by induction on \( k \) to conclude \( nB \sim B \): Assume that

\[ kB \sim B, \quad k \geq 1. \]

By additivity of \( C' \),

\[ B + kB \sim B + B. \]
This, together with condition 1), and the transitivity of $C'$ implies

$$(k+1)P \sim B.$$ 

Therefore, by induction,

$$nB \sim B \quad \text{for all } n \geq 0.$$ 

Let $m, n$ be strictly positive integers. Then

$$nB \sim B, \quad \text{and } mB \sim B.$$ 

By symmetry,

$$B \sim mB.$$ 

Now transitivity of $C'$ gives

$$nB \sim mB.$$ 

2. $C'$ implies $nA + mB \sim n'A + m'B$ for strictly positive $n, m, n', m'$.

Proof. Let $m, n, m', n' > 0$. Adding ii) to itself $n$ times we have

$$nA + nB \sim nB.$$ 

Now apply i), that $mB \sim nB$, which added to the relation $nA \sim nA$, yields
\( nA + mB \sim C, \quad nA + nB \sim C. \)

By the transitivity of \( C' \),
\[ nA + mB \sim C, \quad nB \sim C, \quad nA + mB \sim C, \]
and since by 1, \( nB \sim C, B \), we have
\[ nA + mB \sim C, \quad n'A + m'B \sim C'. \]

for all \( n,m > 0 \). Hence for \( m',n' \) as well,
\[ n'A + m'B \sim B. \]

Or by symmetry,
\[ nA + mB \sim B \quad \text{and} \quad B \sim n'A + m'B. \]

which with transitivity imply
\[ nA + mB \sim n'A + m'B. \]

when \( n,m,n',m' \) are positive.

3. \( kA \sim C, kA \) for all \( k \geq 0 \) by the
reflectivity of \( C' \).

4. \( nA + mB \sim m'B \) for \( m,m' > 0, n \geq 0 \).

Proof. If \( n = 0 \), this follows from 1. If \( n > 0 \),
then in the proof of 2. we had \( nA + mB \sim C \). Thus
as $B \sim m'B$ from 1., we have $nA + mB \sim m'B$ by transitivity.

Thus by 1., 2., 3., 4., $R \subseteq C'$, and $R$ is the smallest congruence which contains $B + B \sim B$ and $A + B \sim B$.

Q.E.D.

We now return to example 3.10.

1. $\mathcal{B}$ satisfies axiom $\mathcal{B}_1$

Proof. Let $n = n_1 - n_2 = m_1 - m_2$.

2. $\mathcal{B}$ satisfies axiom $\mathcal{B}_2$.

Proof. Let

$$E = n(\overline{D},\overline{A},\overline{A},\overline{A},\overline{O}) + (\overline{D},\overline{D},\overline{A}+\overline{E},\overline{E},\overline{0})$$

be an element of $\mathcal{B}$ and let

$$S = (\overline{D},\overline{D}',\overline{D}+\overline{E}',\overline{E}',\overline{O})$$

be a split exact sequence. Then

$$E + S = n(\overline{D},\overline{A},\overline{A},\overline{A},\overline{O}) + (\overline{D},\overline{D}',(\overline{D}+\overline{D}'),(\overline{E}+\overline{E}'),\overline{E}+\overline{E},\overline{0})$$

which is in $\mathcal{B}$.

3. $\mathcal{B}$ satisfies axiom $\mathcal{B}_3$. 
Proof. i) Let

\((\overline{O}, M, M', \overline{O}, \overline{O})\) be in \(S\).

There exist \(n, n_1, n_2, m_1, m_2 \geq 0\) such that

\((\overline{O}, M, M', \overline{O}, \overline{O}) \sim \)

\((\overline{O}, (n+n_1)\overline{A} + m_1\overline{B}, (n+n_1+n_2)\overline{A} + (m_1+m_2)\overline{B}, (n+n_2)\overline{A} + m_2\overline{B}, \overline{O})\).

Equating coordinates, we get

\[M = (n+n_1)\overline{A} + m_1\overline{B}\]

\[M' = (n+n_1+n_2)\overline{A} + (m_1+m_2)\overline{B}\]

\[o\overline{A} + o\overline{B} = \overline{O} = (n+n_2)\overline{A} + m_2\overline{B}\,.

The last equation implies

\[m_2 = 0\,\text{, and } n + n_2 = 0;\]

hence, since \(n\) and \(n_2\) are \(\geq 0\),

\[n = n_2 = m_2 = 0.\]

Therefore

\[M = n_1\overline{A} + m_1\overline{B}\]

\[M' = n_1\overline{A} + m_1\overline{B}.

Hence \(M = M'\).
Part ii) of axiom $\delta 3$ follows from the symmetry of the elements of $\delta$.

Part iii) Let

\[(\overline{0}, \overline{M}, \overline{O}, \overline{M}', \overline{O})\]

be in $\delta$. There exist $n, n_1, n_2, m_1, m_2 \geq 0$ such that

\[(\overline{0}, \overline{M}, \overline{O}, \overline{M}', \overline{O}) = (\overline{0}, (n+n_1)\overline{A} + m_1\overline{B}, (n+n_1+n_2)\overline{A} + (m_1+m_2)\overline{B}, (n_1+n_2)\overline{A} + m_2\overline{B}, \overline{O}) .\]

Thus, equating coordinates,

\[\overline{M} = (n+n_1)\overline{A} + m_1\overline{B} \]
\[\overline{O} = (n+n_1+n_2)\overline{A} + (m_1+m_2)\overline{B} \]
\[\overline{M}' = (n_1+n_2)\overline{A} + m_2\overline{B} .\]

The second equation implies

\[oA + oB \sim (n+n_1+n_2)A + (m_1+m_2)B .\]

Thus, \(o = m_1 + m_2\), and \(o = n + n_1 + n_2\), and as they are all non-negative,

\[m_1 = m_2 = o = n = n_1 = n_2 .\]
Hence,

\[ M \sim 0 \quad \text{and} \quad M' \sim 0 \]

or

\[ M = M' = \emptyset. \]

The following lemma will be used during the proofs that \( \mathcal{G} \) satisfies axioms \( \mathcal{A}^h \) and \( \mathcal{A}^r \).

### 3.12 Lemma

Let

\[ (0, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, (n+n_2)A+m_2B, 0) \]

be in \( \mathcal{G} \). Then this sequence splits if and only if one of the following three conditions is satisfied:

i) \( n = 0 \)

ii) \( m_1 \neq 0 \)

iii) \( m_2 \neq 0 \).

**Proof.** The above sequence splits if and only if

\[ ((n+n_1)A+m_1B) + ((n+n_2)A+m_2B) = (n+n_1+n_2)A + (m_1+m_2)B \]

or equivalently,

\[ (2n+n_1+n_2)A + (m_1+m_2)B \sim (n+n_1+n_2)A + (m_1+m_2)B \]
which holds if and only if

\[ m_1 + m_2 \neq o \]

or \( m_1 + m_2 = o \) and 

\[ 2n + n_1 + n_2 = n + n_1 + n_2. \]

\( m_1 + m_2 \neq o \) is equivalent to

\[ m_1 \neq o \text{ or } m_2 \neq o, \]

and

\[ m_1 + m_2 = o \]

\[ 2n + n_1 + n_2 = n + n_1 + n_2 \]

hold if and only if \( n = o = m_1 = m_2 \).

Q.E.D.

4. \( \mathcal{S} \) satisfies axiom \( \mathcal{S}^4 \).

Proof. Let \((0, E, F, G, O)\) and \((0, M, N, F, O)\) be elements of \( \mathcal{S} \) which form the \( \mathcal{S}^4 \) diagram.
Since $(\mathcal{O}, E, F, G, O)$ is in $\mathcal{S}$, there exist $n, n_1, m_2, m_1, m_2 \geq 0$ such that

$$(\mathcal{O}, E, F, G, O) = (n, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, (n+n_2)A+m_2B, O)$$

and there exist $n', n_1', m_2', m_1' \geq 0$ such that

$$(\mathcal{O}, M, N, F, O) = (n', (n'+n_1')A+m_1'B, (n'+n_1'+n_2')A+(m_1'+m_2')B, (n'+n_2')A+m_2'B, O).$$

By the lemma, if $m_1' \neq 0$, $m_2' \neq 0$, or $n' = 0$, then the column splits, and the diagram can be completed by Lemma 3.2.

Therefore we will assume that

$$m_1' = 0, m_2' = 0, \text{ and } n' \neq 0.$$ 

Equating coordinates in the expressions for our sequences, we have
\[
F = (n + n_1 + n_2)\bar{A} + (m_1 + m_2)\bar{B} = (n' + n_2')\bar{A} + m_1'\bar{B}
\]
or
\[
(n + n_1 + n_2)\bar{A} + (m_1 + m_2)\bar{B} \sim (n' + n_2')\bar{A} + m_2'\bar{B}.
\]
Now since \(m_2' = 0\), we have \(m_1 + m_2 = 0\) and

\[
n + n_1 + n_2 = n' + n_2',
\]
and since \(m_1\) and \(m_2\) are non-negative, we have

\[
m_1 - m_2 = 0.
\]

Thus the diagram is reduced to

\[
\begin{array}{c}
0 \\
(n' + n_1')\bar{A} \\
(n' + n_1' + n_2')\bar{A} \\
0 \rightarrow (n + n_1)\bar{A} \rightarrow (n + n_1 + n_2)\bar{A} \rightarrow (n + n_2)\bar{A} \rightarrow 0
\end{array}
\]

which we write as the sum of \(n'\) subdiagrams selected from (a), (b), and (c) below,
plus $n'_2$ subdiagrams selected from $(a')$, $(b')$, and $(c')$ below,

\[
\begin{align*}
\begin{array}{ccc}
\bar{0} & \bar{0} & \bar{5} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} & \begin{array}{ccc}
\bar{0} & \bar{0} & \bar{6} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} & \begin{array}{ccc}
\bar{0} & \bar{0} & \bar{6} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} \\
\begin{array}{ccc}
\bar{0} & \bar{0} & \bar{6} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} & \begin{array}{ccc}
\bar{0} & \bar{0} & \bar{6} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} & \begin{array}{ccc}
\bar{0} & \bar{0} & \bar{6} \\
\bar{A} & \bar{A} & \bar{A} \\
\bar{A} & & \\
\end{array} \\
\end{align*}
\]

$(a')$ \hspace{1cm} $(b')$ \hspace{1cm} $(c')$

Such integers exist by the following lemma:

\textbf{2.13 Lemma.} Suppose $n, n_1, n_2, n'$, and $n'_2$ are non-negative integers and 

$$n + n_1 + n_2 = n' + n'_2.$$ 

Then there exist non-negative integers
\[ |(a)|, |(a')|, |(b)|, |(b')|, |(c)|, \text{ and } |(c')| \]
such that
\[ |(a)| + |(a')| = n \]
\[ |(b)| + |(b')| = n_1 \]
\[ |(c)| + |(c')| = n_2 \]
and
\[ |(a)| + |(b)| + |(c)| = n' \]
\[ |(a')| + |(b')| + |(c')| = n_2' . \]

**Proof.** Since \( n' \leq n' + n_2' = n + n_1 + n_2 \), there exists integers \(|(a)|, |(b)|, \text{ and } |(c)|\) such that
\[ |(a)| \leq n, |(b)| \leq n_1, \text{ and } |(c)| \leq n_2 \]
and
\[ |(a)| + |(b)| + |(c)| = n' . \]

Set
\[ |(a')| = n - |(a)| \]
\[ |(b')| = n_1 - |(b)| \]
\[ |(c')| = n_2 - |(c)| . \]
Then, \( |(a')| + |(b')| + |(c')| = \)

\[ (n-|a|) + (n_1-|b|) + (n_2-|c|) \]

\[ = n + n_1 + n_2 - (|a| + |b| + |c|) \]

\[ = (n+n_1+n_2) - n' \]

\[ = n'_2 . \]

Q.E.D.

Diagram (a) can be completed with \( \bar{A} \)'s, and the others can be completed by lemma 3.2 or 3.3. The sum of their completions completes the diagram.

5. \( \mathfrak{g} \) satisfies axiom \( \mathfrak{g}^5 \)

Proof. Let \((\mathcal{O}, E, F, \mathcal{O}, \mathcal{O})\) and \((\mathcal{O}, M, N, \mathcal{O}, \mathcal{O})\) be elements of \( \mathfrak{g} \) which form the \( \mathfrak{g}^5 \) diagram

\[
\begin{array}{c}
\mathcal{O} \\
\downarrow \\
M \\
\downarrow \\
N \\
\downarrow \\
\mathcal{O} \\
\end{array}
\]

\[
\begin{array}{c}
E \\
\downarrow \\
F \\
\downarrow \\
\mathcal{O} \\
\end{array}
\]
Since \((0, E, F, G, 0)\) is in \(\mathfrak{G}\), there exist
\(n, n_1, n_2, m_1, m_2 \geq 0\) such that
\[
(0, E, F, G, 0) = (0, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, (n+n_2)A+m_2B, 0)
\]
and there exist \(n', n_1', n_2', m_1', m_2' \geq 0\) such that
\[
(0, E, F, G, 0) = (0, (n'+n_1')A+m_1'B, (n'+n_1'+n_2')A+(m_1'+m_2')B, (n'+n_2')A+m_2'B, 0)
\]
If \(m_1 \not\equiv 0\), \(m_2 \not\equiv 0\), or \(n = 0\), then by the lemma the row splits and the diagram can then be completed by lemma 3.2. Similarly if \(m_1' \not\equiv 0\), \(m_2' \not\equiv 0\), or \(n' = 0\), the column splits and the diagram can be completed by lemma 3.2. Hence we assume
\[
m_1 - m_2 = m_1' - m_2' = 0, n \not\equiv 0, \text{ and } n' \not\equiv 0.
\]
Equating expressions for \(G\) in our sequences we have
\[
(n+n_2)A + m_2B = (n'+n_2')A + m_2'B
\]
or
\[
(n+n_2)A + m_2B \sim (n'+n_2')A + m_2'B.
\]
Thus, as $m_2 = m_2' = 0$, we have

$$n + n_2 = n' + n_2' .$$

Our diagram is now reduced to

$$
\begin{array}{c}
0 \\
(n' + n_1')\overline{A} \\
(n' + n_1' + n_2')\overline{A} \\
0 \quad (n+ n_1)\overline{A} \quad (n+ n_1 + n_2)\overline{A} \quad (n+ n_2)\overline{A} \quad 0 \\
0
\end{array}
$$

By symmetry, we may assume that $n \leq n'$. Then

$$n' + n_2' = n + (n' - n) + n_2' ,$$

where

$$n_2 = (n' - n) + n_2' .$$

We can write our diagram as the sum of two subdiagrams $\mathcal{D}_1$ and $\mathcal{D}_2$. 
Diagram $B_2$ has a split row and can therefore be completed in $8$ by lemma 3.2.

Diagram $B_1$ can be completed as shown below

Their sum is composed of sums of elements in $8$, and is therefore completed in $8$. 

Q.E.D.
**Remark.** In example 3.9, we have elements 
\( A, B \in m \) and that \( A + B = B \), with \( B \) jective, but \( A \) not jective.

1) The element \( B \) is jective.

**Proof.** Let \((0, M, N, B, 0)\) be in \( \mathcal{A} \). It suffices by to see that it splits. Since \((0, M, N, B, 0)\) is in \( \mathcal{A} \), there exist \( n, m_1, m_2, n_1, n_2 \geq 0 \) such that

\[
(0, M, N, B, 0) = (0, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, (n+n_2)A+n_2B, 0).
\]

Equating coordinates we have,

1) \( M = (n+n_1)A = m_1B \)

2) \( N = (n+n_1+n_2)A + (m_1+m_2)B \)

3) \( B = (n+n_2)A + m_2B \).

By 3.10 equation (3) implies \( m_2 \neq 0 \), hence the sequence splits by lemma 3.11.

2) The element \( A \) is not jective.

**Proof.** The sequence \((0, A, A, A, 0)\) is not split by lemma 3.11.
We remark also, that $\overline{A}$ is jectively equivalent to
the jective $\overline{B}$ (that is there exist jectives $\overline{P}, \overline{P}'$
such that $\overline{A} + \overline{P} = \overline{B} + \overline{P}'$) but is itself not jective
(see [10]).

3.15 Definition. Let $S$ be a set and let $\mathfrak{S}$ be a
subset of $S^5$. We define

$$\mathfrak{S}' = \{(0, C, B, A, 0) \mid (0, A, B, C, 0) \in \mathfrak{S}\}.$$

The following is an example of a set $\mathfrak{S}$ of short
exact sequences for which $\mathfrak{S}'$ is not a set of short
exact sequences.

3.16 Example. Let $\mathcal{M} = \mathcal{M}(A, B)$ be the free commutative
monoid on the two elements $A$ and $B$, and let $\mathfrak{S}$ be
the subset of $\mathcal{M}^5$ consisting of all elements of the
form

$$n(0, A, A, B, 0) + (0, C, C+D, D, 0)$$

where $n \geq 0$ and $C, D$ are in $\mathcal{M}$. Then $\mathfrak{S}$ is a set
of short exact sequences.

Proof. We observe that since $C$ and $D$ are in $\mathcal{M}$,
there exist $n_1, m_1, n_2, m_2 \geq 0$ such that

$$C = n_1 A + m_1 B, \text{ and } D = n_2 A + m_2 B.$$
Thus an element of $\mathcal{S}$ has form

$$n(0, A, A, B, 0) + (0, n_1 A + m_1 B, (n_1 + n_2) A + (m_1 + m_2) B, n_2 A + (m_1 + m_2) B, 0)$$

which when combined into one sequence results in

$$(0, (n + n_1) A + m_1 B, (n + n_1 + n_2) A + (m_1 + m_2) B, n_2 A + (m_1 + m_2) B, 0).$$

We now proceed to show that $\mathcal{S}$ satisfies axioms $\mathcal{S}1 - \mathcal{S}5$:

1. $\mathcal{S}$ satisfies axiom $\mathcal{S}1$

Proof. Let $n = 0$ and $C = D = 0$.

2. $\mathcal{S}$ satisfies axiom $\mathcal{S}2$

Proof. Let $E = (0, A, A, B, 0)$. Then any element of $\mathcal{S}$ has form

$$nE + S$$

where $S$ is a split.

Let $S'$ be any split in $\mathcal{M}^5$. Then

$$(nE + S) + S' = nE + (S + S')$$

which is again in $\mathcal{S}$ since $S + S'$ is again a split.
3. \( S \) satisfies axiom \( \delta 3 \)

Proof. 1) Let \((0,M,M',0,0)\) be in \( S \). Then there exist \( n,n_1,m_1,n_1',m_1' \geq 0 \) such that

\[
(0,M,M',0,0) = (0,(n+m_1)A+m_1B,(n_1+n_2)A+(m_1+m_2)B,n_2A+(n+m_2)B,0).
\]

Equating coordinates, we have

\[
M = (n+n_1)A + m_1B
\]

\[
M' = (n_1+n_2)A + (m_1+m_2)B
\]

\[
o = n_2A + (n+m_2)B.
\]

Therefore, as \( m \) is free, we conclude by the last equation that

\[
n_2 = (n+m_2) = 0.
\]

Now by non-negativity of \( n \) and \( m_2 \), we have

\[
n_2 = n = m_2 = 0.
\]

Thus,

\[
M = n_1A + m_1B
\]

\[
M' = n_1A + m_1B,
\]

and

\[
M = M'.
\]
Part ii) of axiom $\mathfrak{A}3$ follows by a similar argument.

iii) Suppose an element of the form

$$(0, M, 0, M', 0)$$

is in $\mathfrak{A}$. Then there exist $n, n_1, n_2, m_1, m_2 \geq 0$ such that

$$(0, M, 0, M', 0) = (0, (n+n_1)A + m_1B, (n+n_1+n_2)A + (m_1+m_2)B, n_2A + (n+m_2)B, 0).$$

Equating coordinates we have

$$M = (n+n_1)A + m_1B$$
$$0 = (n+n_1+n_2)A + (m_1+m_2)B$$
$$M' = n_2A + (n+m_2)B.$$

The second equation and $m$ being free imply

$$o = n + n_1 + n_2$$
$$o = m_1 + m_2,$$

and as they are all non-negative,

$$n = n_1 = n_2 = m_1 = m_2 = o.$$

Hence

$$M = 0 - M'.$$
4. $\mathcal{S}$ satisfies axiom $\mathcal{A}_4$

Proof. Consider the $\mathcal{A}_4$ diagram constructed with two elements $(0, E, F, G, 0)$ and $(0, M, N, F, 0)$ in $\mathcal{S}$:

Since $(0, E, F, G, 0)$ and $(0, M, N, F, 0)$ are in $\mathcal{S}$ there exist $n, n_1, n_2, m_1, m_2 \geq 0$, and $n', n_1', n_2', m_1', m_2' \geq 0$ such that

$$(0, E, F, G, 0) = (0, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, n_2A+(n+m_2)B, 0)$$

and

$$(0, M, N, F, 0) = (0, (n'+n_1')A+m_1'B, (n'+n_1'+n_2')A+(m_1'+m_2')B, n_2'A+(n'+m_2')B, 0) .$$

Equating our expressions for $F$, we have

$$F = (n+n_1+n_2)A+m_1B = n_2'A+(n'+m_1')B .$$
Now, since \( \mathfrak{m} \) is free, we have

\[
\begin{align*}
n + n_1 + n_2 &= n_2' \\
\lambda_1 + \lambda_2 &= \lambda' + \lambda_2'.
\end{align*}
\]

We can therefore write the original diagram \( \mathcal{D} \)

\[
\begin{array}{c}
0 \\
\downarrow \\
(n'+n_1')A + \lambda_1'B \\
\downarrow \\
(n'+n_1'+n_2')A + (\lambda_1' + \lambda_2')B \\
\downarrow \\
0 - (n+n_1)A + \lambda_1B - (n+n_1+n_2)A + (\lambda_1 + \lambda_2)B - n_2A + (n+n_2)B - 0 \\
\downarrow \\
0
\end{array}
\]

as the sum of the two subdiagrams \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) shown below:

\[
\begin{array}{c}
0 \\
\downarrow \\
n_1'A + \lambda_1'B \\
\downarrow \\
(n_1'+n_2')A + \lambda_1'B \\
\downarrow \\
0 - (n+n_1)A - (n+n_1+n_2)A - n_2A + nB - 0 \\
\downarrow \\
0
\end{array}
\]

\( \mathcal{D}_1 \)
Subdiagram $\delta_2$ can be completed in $\beta$ by lemma 3.2 because the column

$$(0, n_1'A + m_1'B, (n_1' + n_2')A + m'B, (n + n_1 + n_2)A, 0)$$

is a split by the condition

$$n + n_1 + n_2 = n_2'.$$

Subdiagram $\delta_2$ may be written as a sum of $n'$ subdiagrams, each of which is of type (a) or (b) below:
plus \( m_2' \) subdiagrams, each of which is of type \((a')\) or \((b')\) below:

\[
\begin{array}{cccccc}
0 & \rightarrow & B & \rightarrow & B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
O & \rightarrow & O & \rightarrow & O & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

\((a')\) \hspace{2cm} \((b')\)

These are selected in such a way that if

\[ |(n)| = \text{number of subdiagrams of type } (n), \]
then
\[ |(a)| + |(a')| = m_1 \]
\[ |(b)| + |(b')| = m_2 . \]

Such exist by the following lemma.

3.17 Lemma. Suppose \( m_1, m_2, n', m_2' \) are non-negative integers and

\[ m_1 + m_2 = n' + m_2' . \]

Then there exist non-negative integers

\[ |(a)|, |(b)|, |(a')|, |(b')| \]

such that

\[ |(a)| + |(a')| = m_1 \]
\[ |(b)| + |(b')| = m_2 \]
\[ |(a)| + |(b)| = n' \]
\[ |(a')| + |(b')| = m_2' . \]

Proof. Since \( n' \leq n' + m_2' = m_1 + m_2 \), there exist

\[ |(a)| \leq m_1, |(b)| \leq m_2 \]

such that

\[ |(a)| + |(b)| = n' . \]
Set

\[ |(a')| = m_1 - |(a)| \]
\[ |(b')| = m_2 - |(b)| . \]

Then

\[ |(a')| + |(b')| = m_1 - |(a)| + m_2 - |(b)| \]
\[ = m_1 + m_2 - n' = n_2' \]

and

\[ |(a)| + |(a')| = |(a)| + m_1 - |(a)| = m_1 \]
\[ |(b)| + |(b')| = |(b)| + m_2 - |(b)| = m_2 . \]

Q.E.D.

Each of the diagrams (a), (b), (a'), and (b') can be completed by lemma 3.3. Their sum completes diagram \( \mathcal{D}_1 \), and since \( \mathcal{D} \) is closed under addition, the original diagram \( \mathcal{D} \) is completed by adding \( \mathcal{D}_1 \) completed and \( \mathcal{D}_2 \) completed.

Q.E.D.
5. $\mathcal{A}$ satisfies axiom $\mathcal{A}_5$

Proof. Let $\mathcal{A}$ be an $\mathcal{A}_5$ diagram constructed from the two elements $(0, E, F, G, 0)$ and $(0, M, N, G, 0)$ of $\mathcal{A}$ as shown below:

```
      O
      |   M
      |    N
      | 0---E---F---G---0
      |    |   |
      O
```

Since $(0, E, F, G, 0)$ is in $\mathcal{A}$, there exist $n, n_1, n_2, m_1, m_2 \geq 0$ such that

$$(0, E, F, G, 0) = (0, (n+n_1)A+m_1B, (n+n_1+n_2)A+(m_1+m_2)B, n_2A+(n+m_2)B, 0)$$

and there exist $n', n_1', n_2', m_1', m_2' \geq 0$ such that

$$(0, M, N, G, 0) = (0, (n'+n_1')A+m_1'B, (n'+n_1'+n_2')A+(m_1'+m_2')B, n_2'A+(n'+m_2')B, 0).$$
Equating the expressions for $G$, we have

$$n_2A + (n+m_2)B = n_2'A + (n'+m_2')B$$

and, as $m$ is free, we conclude that

$$n_2 = n_2'$$
$$n + m_2 = n' + m_2'.$$

The diagram $B$ (shown below) can then be written as the sum of two subdiagrams $B_1$ and $B_2$ as follows:

$$
\begin{align*}
0 & \quad (n' + n_1')A + m_1'B \\
& \quad (n' + n_1' + n_2')A + (m_1' + m_2')B \\
0 & \quad (n + n_1)A + m_1B -(n + n_1 + n_2)A + (m_1 + m_2)B - n_2A + (n + m_2)B - 0
\end{align*}
$$

$B$

$$
\begin{align*}
0 & \quad n_1'A + m_1'B \\
& \quad (n_1' + n_2')A + m_1'B \\
0 & \quad n_1A + m_1B - (n_1 + n_2)A + m_1B - n_2A - (n + m_2)B - 0
\end{align*}
$$

$B_1$
Subdiagram $\mathcal{B}_1$ can be completed in $\mathcal{B}$ by lemma 3.2 since the row and column both split.

Subdiagram $\mathcal{B}_2$ can be completed as follows:

Suppose $n' > n$. Then

$$n' + m' = n + (n' - n) + m'.$$

We break $\mathcal{B}_2$ into two subdiagrams $\mathcal{B}_2'$ and $\mathcal{B}_2''$ as shown below:

$$
\begin{array}{c}
\begin{array}{c}
0 \\
| \\
nA \\
| \\
nA + m_2'B \\
| \\
0 \\
\end{array}
\quad \begin{array}{c}
0 \\
| \\
(n'-n)A \\
| \\
(n'-n)A + m_2'B \\
| \\
0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\
| \\
nA - nA \\
| \\
nA - 0 \\
| \\
0 \\
\end{array}
\quad \begin{array}{c}
0 \\
| \\
0 - m_2'B \\
| \\
0 \\
\end{array}
\end{array}
$$
Diagram $\mathcal{D}_2'$ has a split row and can be completed by lemma 3.2.

Diagram $\mathcal{D}_2'$ can be completed as follows:

$$
\begin{array}{cccc}
0 & 0 \\
\vdots & | \\
\vdots & | \\
0 \cdots nA \cdots nA \cdots & 0 \\
\vdots & | \\
0 \cdots nA \cdots nA \cdots nA \cdots & 0 \\
\vdots & | \\
0 & 0 \\
\end{array}
$$

($\mathcal{D}_2'$ completed).

Their sum completes diagram $\mathcal{D}_2$, and $\mathcal{D}$ is completed by adding the completed $\mathcal{D}_1$ and $\mathcal{D}_2$. (Note that $\mathcal{D}$ is closed under sum.) The case $n' < n$ is symmetric to the case just given.

Q.E.D.

Thus $\mathcal{D}$ is a set of short exact sequences. We will now show that $\mathcal{D}'$ is not a set of short exact sequences.

3.18 Remark. $\mathcal{D}'$ is not a set of short exact sequences.
Proof. Consider the following $\delta'$ diagram constructed from the elements of $\delta'$:

\[
\begin{array}{c}
0 \\
| \\
B \\
| \\
A \\
| \\
0 \rightarrow B \rightarrow A \rightarrow A \rightarrow 0 \\
| \\
0
\end{array}
\]

Suppose there is a completion in $\delta'$. That is, suppose there exists $M \in \mathbb{M}$ such that

\[(0, M, A, A, 0) \in \delta' \quad \text{and} \quad (0, B, M, B, 0) \in \delta'.\]

Equivalently,

\[(0, A, A, M, 0) \in \delta \quad \text{and} \quad (0, B, M, B, 0) \in \delta'.\]

Since $(0, A, A, M, 0)$ is in $\delta$, there exist $n, n_1, n_2, m_1, m_2 \geq 0$ such that

\[(0, A, A, M, 0) = (0, (n + n_1)A + m_1B, (n + n_1 + n_2)A + (m_1 + m_2)B, n_2A + (n + m_2)B, 0).\]
Equating coordinates, we have

\[ A = (n+n_1)A + m_1B \]
\[ A = (n+n_1+n_2)A + (m_1+m_2)B \]
\[ M = n_2A + (n+m_2)B \]

and, since \( m \) is free, these equations give us

\[ l = n + n_1 \quad \text{and} \quad m_1 = 0 \]
\[ l = n + n_1 + n_2 \quad \text{and} \quad m_1 + m_2 = 0. \]

Thus,

\[ m_2 = 0 \quad \text{and} \quad n_2 = 0 \]

and, as \( n \) and \( n_1 \) are non-negative, we have either

\[ n = 0 \quad \text{and} \quad n_1 = 1 \]

or

\[ n = 1 \quad \text{and} \quad n_1 = 0. \]

If the first holds,

\[ n = 0 \quad \text{and} \quad n_1 = 1, \]

then

\[ M = n_2A + (n+m_2)B = 0A + (0+0)B = 0 \]
and hence

\[(0, B, M, B, 0) = (0, B, 0, B, 0) \in \mathcal{S} .\]

contradicting the fact that \( \mathcal{S} \) satisfies axiom \( \mathcal{S}_3 \). Thus, it must be the other case, that

\[n = 1 \quad \text{and} \quad n_1 = 0\]

which holds. Hence

\[M = n_2 A + (n + m_1) B + o A + (1 + o) B - B .\]

Thus,

\[(0, B, M, B, 0) = (0, B, B, B, 0) \in \mathcal{S} .\]

Now, if \((0, B, B, B, 0)\) is to be in \( \mathcal{S} \), there must exist \(n', n_1', n_2', m_1', m_2' \geq 0\) such that

\[(0, B, B, B, 0) = (0, (n' + n_1') A + m_1 B, (n' + n_1' + n_2') A + (m_1' + m_2') B, n_2' A + (n' + m_2') B, 0) .\]

Equating coordinates, we have

\[B = (n' + n_1') A + m_1 B\]

\[B = (n' + n_1' + n_2') A + (m_1' + m_2') B\]

\[B = n_2' A + (n' + m_2') B .\]
Now, using that $\mathcal{M}$ is free we conclude

$$n' + n_1' = 0 \quad \text{and} \quad m_1' = 1$$

$$n' + n_1' + n_2' = 0 \quad \text{and} \quad m_1' + m_2' = 1$$

$$n_2' = 0 \quad \text{and} \quad n' + m_2' = 1.$$ 

Thus, as $n'$, $n_1'$, and $n_2'$ are all non-negative, we have

$$n' = n_1' = n_2' = 0.$$ 

Since $m_1' = 1$ and $m_1' + m_2' = 1$, we have

$$m_2' = 0$$

and $n' + m_2' = 1$, $m_2' = 0$, and $n' = 0$, imply

$$0 = 1$$

a contraction.

Thus $(0,B,B,B,0) \not\in \mathcal{S}$, and there is no pair of elements of $\mathcal{S}'$ which complete the diagram.

Q.E.D.
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VITA

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Title of Thesis: PROJECTIVES AND INJECTIVES IN A SETTING OF AXIOMATIC EXACTNESS

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July 16, 1973