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Controller reduction for linear systems

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CONTROLLER REDUCTION FOR LINEAR SYSTEMS

A Dissertation

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To My Family

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Notation and Symbols

$\mathscr{R}$ and $\mathscr{C}$: Fields of real and complex numbers

$\in$: Belong to

$\subset$: Subset

$\cup$: Union

$:=$: Defined as

$\gg$ and $\ll$: Much greater and less than

$|\alpha|$: Absolute value of $\alpha \in \mathscr{C}$

I: Identity matrix

$A \Longleftrightarrow B$: A if and only if B

$A \Longleftarrow B$: If B then A

A^{-1}: Inverse of matrix A

A^T: Transpose of matrix A

A^*: Complex conjugate transpose of matrix A

$\underline{\sigma}(A)$: Smallest singular value of matrix A

$\bar{\sigma}(A)$: Largest singular value of matrix A

$\lambda_i(A)$: The ith eigenvalue of A

$\lambda_{max}(A)$: Eigenvalue of A with maximum modulus

$\sigma_i[G(s)]$: The ith (decreasingly ordered) Hankel singular value of the stable part of $G(s)$

$\rho(A)$: Spectral radius of A

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: Abbreviation of transfer matrix $D + C(sI - A)^{-1}B$

$G^{\sim}(s)$: Shorthand for $G^T(-s)$

$\mathscr{L}_2(-\infty, \infty)$: Time domain square integrable functions

$\mathscr{L}_2(j\mathscr{R})$: Square integrable functions on $\mathscr{C}_0$ including at ∞

$\mathscr{H}_2$: Subspace of $\mathscr{L}_2(j\mathscr{R})$ with functions analytic in $Re(s) > 0$

$\mathscr{L}_\infty(j\mathscr{R})$: Functions bounded on Re(s)=0 including at ∞

$\mathscr{H}_\infty$: The set of $\mathscr{L}_\infty(j\mathscr{R})$ functions analytic in $Re(s) > 0$

$\mathscr{H}_\infty^-$: The set of $\mathscr{L}_\infty(j\mathscr{R})$ functions analytic in $Re(s) < 0$

$\mathscr{R}\mathscr{H}_\infty$: The set of stable proper rational transfer matrices.

$\mathscr{R}\mathscr{H}_\infty^-$: The set of antistable proper rational transfer matrices.

$\|G\|_\infty$: $\mathscr{H}_\infty$ norm of system G

$\|G\|_2$: $\mathscr{H}_2$ norm of system G

$\|G\|_H$: Hankel norm of system G

$[G]_+$: The stable part of G

$\mathcal{F}_\ell(M,Q)$: Lower LFT

$\mathcal{F}_u(M,Q)$: Upper LFT

$\mathcal{S}(M,N)$: Star product

$\mathscr{R}_p(s)$: Rational proper transfer matrics

$\mathrm{Ric}(H)$: The stabilizing solution of an ARE

dom(Ric): The domain of Ric

$\mathrm{diag}(a_1,\ldots,a_n)$: An $n \times n$ diagonal matrix with its ith diagonal element a_i

List of Acronyms

ARE	algebraic Riccati equation
iff	if and only if
lcf	left coprime factorization
rcf	right coprime factorization
MIMO	multi-input and multi-output
SISO	single-input and single-output
LFT	linear fractional transformation
LQG	linear quadratic gaussian
UWA	unweighted additive reduction
UWRCF	unweighted right coprime factor reduction
UWLCF	unweighted left coprime factor reduction
SWA	stability weighted additive reduction
SWRCF	stability weighted right coprime factor reduction
SWLCF	stability weighted left coprime factor reduction
PWA	performance weighted additive reduction
PWRCF	performance weighted right coprime factor reduction
PWLCF	performance weighted left coprime factor reduction

Abstract

This dissertation proposes some $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance preserving controller reduction methods for linear systems. The proposed methods can guarantee robust stability and performance for the closed-loop system with the reduced order controllers.

Several $\mathscr{H}_\infty$ stability and performance preserving controller reduction methods are proposed in this dissertation. It is shown that the weighting functions used in the proposed controller reduction methods can be directly obtained from the parametrization of the $\mathscr{H}_\infty$ controllers. Hence, comparing with the most existing controller reduction approaches, the proposed controller reduction methods require less computation and are easy to apply. At the same time, several algorithms are proposed to simplify some existing controller reduction algorithms. Examples are explored to demonstrate the advantages of the proposed controller reduction methods.

The parallel problems are also discussed for $\mathscr{H}_2$ performance preserving controller reductions. Furthermore, some parallel controller reduction methods are presented to reduce controllers for preserving the closed-loop system stability and performance. Similarly, relevant simplified algorithms are also proposed for those existing $\mathscr{H}_2$ performance preserving controller reduction algorithms. One example is explored to demonstrate those controller reduction methods.

Another $\mathscr{H}_\infty$ controller reduction method is introduced for SISO system to maintain the closed-loop system stability and performance. This approach provides upper bound on the controller weighting function for general SISO $\mathscr{H}_\infty$ control problem, and then a lower order controller is provided using frequency weighted model reduction method, which preserves stability and performance for the closed-loop system.

Finally, some possible future work are outlined.

Chapter 1

Introduction

1.1 Overview of Controller Reduction

It is well-known that the $\mathscr{H}_\infty$ control theory and μ synthesis can be used to design robust performance controllers for highly complex uncertain systems [13], [18], [62], [63]. However, since many physical systems are modeled as high order dynamical models, the controllers designed via many popular methodologies have very high orders (much higher than the plant orders) because of the performance weighting functions and the model uncertainty weighting functions. Generally, high order controllers are not preferred in control system designs because of their complexity. As a result, low order controllers are desirable in real applications for some obvious reasons:

- They are less complex so that they are easy to understand and need shorter time to process;
- They also have high reliability and are easy to implement since there are less chances to get into hardware and software troubles.

Therefore, a low order controller should always be sought first in any control system design if the resulting performance degradation is within an acceptable and reasonable range. Of course, it is critical to reduce the controller order in such a way so that the performance degradation is minimized and it should be clearly noted that the absolute error between the full order controller and the reduced order is not critical. What is the most important is that the error in some critical range should be small [4], [14], [19], [22], [24], [27], [38]-[41], [44], [45], [56], [60].

In general, there are three ways to follow in order to design low order controllers for a high order system model:

- The first possible way is to design low order controllers based on the high order systems directly. The disadvantage of the direct controller reduction is that the existing relevant techniques are insufficient and there are abundant open research problems;

- The second approach is to reduce the high order model first by using model reduction methodologies, and then design a low order controller for the reduced model. However, this method does not guarantee that the low order controller designed on the reduced order model performs well with the high order system. Even worse, it might not stabilize the full order plant because the error information between the full order system and the reduced order system is disregarded during the controller design;

- The third approach to arrive at low order controllers for the full order model is to seek a high order controller based on the high order system, and then reduce the derived high order controller. Our work will mainly focus on the third path, i.e., proposing

several controller reduction methods to reduce high order robust controllers with the objective of preserving robust stability and performance of the closed-loop system.

Figure 1.1 shows the three ways for controller reductions.

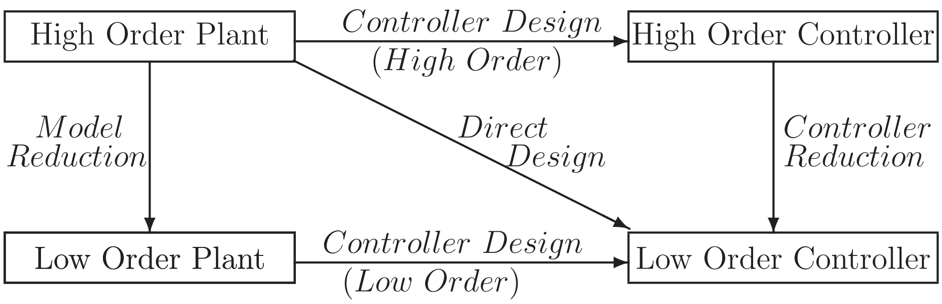


Figure 1.1: Three Ways for Controller Reduction

1.2 Contribution of the Dissertation

Based on full order stability and performance preserving controllers, we explore some new controller reduction methods which can preserve $\mathscr{H}_\infty$ or $\mathscr{H}_2$ stability and performance of closed-loop system for linear systems.

We propose several $\mathscr{H}_\infty$ controller reduction methods that guarantee robust stability and performance for the closed-loop system. One of the advantages of the proposed methods is that the weighting functions for controller reduction is easy to compute and is readily available from standard $\mathscr{H}_\infty$ control design software. The frequency weighted balanced reduction method is used to solve the proposed controller reductions, and computational issue is considered so that the input weighted gramin P and the output weighted gramin Q are simplified. As a result, some algorithms are proposed to simply some existing controller reduction algorithms. The four disk example and the HIMAT example demonstrate that

our proposed methods and algorithms are effective, at least as effective as the best method available in the literature.

Besides, two simple but not easily noticed conclusions are stated:

- The frequency-weighted balanced realization is independent of the particular realizations of G, W_i and W_o;
- Let W and G be scalar transfer functions. Then the input weighted balanced realization of G with input weighting W is the same as the output weighted balanced realization of G with output weighting W.

Then, a new perspective on $\mathscr{H}_2$ controller reductions is presented based on proposed $\mathscr{H}_\infty$ controller reduction methods in this dissertation. Some parallel $\mathscr{H}_2$ controller reduction methods are introduced to preserve closed-loop system stability and performance. Also, the four disk example is explored to illustrate the effectiveness of our proposed $\mathscr{H}_2$ controller reduction methods.

Finally, another $\mathscr{H}_\infty$ controller reduction method is presented to stabilize the closed-loop system and preserve performance. This approach, which includes additive controller reduction and coprime factor controller reduction, aims to obtain upper bound on controller weighting function for general SISO $\mathscr{H}_\infty$ control problem. A reduced controller is derived by applying frequency weighted model reduction method, which guarantees the closed-loop system stability and performance.

1.3 Outline of the Dissertation

The purpose of this dissertation is to provide several controller reduction methods, which can guarantee the closed-loop system stability and robust performance.

This work consists of nine chapters: some basic material and general background are introduced in Chapter 2, which will be used in later chapters. Chapter 3 discusses some model reduction methods and Chapter 4 presents some controller reduction methods. Several $\mathscr{H}_\infty$ controller reduction methods are proposed in Chapter 5. In order to illustrate the proposed methods, two examples are discussed and some simulation results are shown in Chapter 6. Similarly, some parallel $\mathscr{H}_2$ controller reduction approaches are derived in Chapter 7. Another controller reduction method for general SISO $\mathscr{H}_\infty$ controller problem is presented in Chapter 8,which calculates upper bound on the controller weighting function. The further potentially significant work is listed in Chapter 9, followed by conclusions in Chapter 10.

Chapter 2

Preliminaries

In this chapter we review some basic concepts and introduce some general background material needed in later chapters. In Section 2.1, the standard $\mathscr{H}_\infty$ control problem is presented. The standard $\mathscr{H}_2$ or LQG problem is contained in Section 2.2. Small gain theorem, coprime factorization and inner-outer factorization are then discussed in Section 2.3, Section 2.4 and Section 2.5, respectively. Also, the Hankel norm is defined in Section 2.6 and the star product is introduced in Section 2.7.

2.1 $\mathscr{H}_\infty$ Control Problem

This section introduces the linear fractional transformations (LFT) which many interesting control problems can be formulated in such forms:

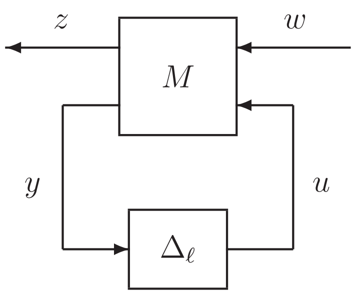


Figure 2.1: The Lower LFT Diagram $\mathcal{F}_\ell(M, \Delta_\ell)$

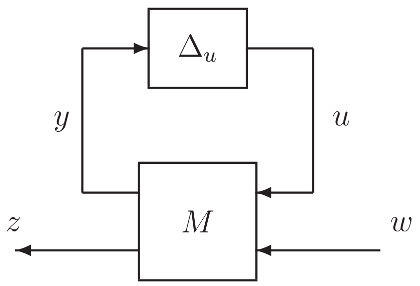


Figure 2.2: The Upper LFT Diagram $\mathcal{F}_u(M, \Delta_u)$

Definition 1 *([63]) Let M be a complex matrix partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathscr{C}^{(p_1+p_2)\times(q_1+q_2)},$$

and let $\Delta_\ell \in \mathscr{C}^{q_2\times p_2}$ and $\Delta_u \in \mathscr{C}^{q_1\times p_1}$ be two other complex matrices. Then we can formally define a lower LFT with respect to Δ_ℓ as the map

$$\mathcal{F}_\ell(M, \bullet) : \mathscr{C}^{q_2\times p_2} \longmapsto \mathscr{C}^{p_1\times q_1}$$

with

$$\mathcal{F}_\ell(M, \Delta_\ell) := M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}$$

provided that the inverse $(I - M_{22}\Delta_\ell)^{-1}$ exists. We can also define an upper LFT with respect to Δ_u as the map

$$\mathcal{F}_u(M, \bullet) : \mathscr{C}^{q_1\times p_1} \longmapsto \mathscr{C}^{p_2\times q_2}$$

with

$$\mathcal{F}_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

provided that the inverse $(I - M_{11}\Delta_u)^{-1}$ exists.

Definition 2 *([63]) Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. Then an algebraic Riccati equation (ARE) is the following matrix equation:*

$$A^*X + XA + XRX + Q = 0. \tag{2.1}$$

Associated with this Riccati equation is a $2n \times 2n$ matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}. \tag{2.2}$$

A matrix of this form is called a Hamiltonian matrix.

Definition 3 *([63]) If the solution X in equation (2.1) is obtained by using H in (2.2) (see details in [63]), define that $Ric : H \longmapsto X$. The domain of Ric is denoted as $dom(Ric)$. Thus, $X = Ric(H)$ and*

$$Ric : dom(Ric) \subset \mathscr{R}^{2n\times 2n} \longmapsto \mathscr{R}^{n\times n}.$$

Many interesting control problems can be put in a linear fractional transformation diagram as in Figure 2.3 and therefore can be treated by using the same techniques.

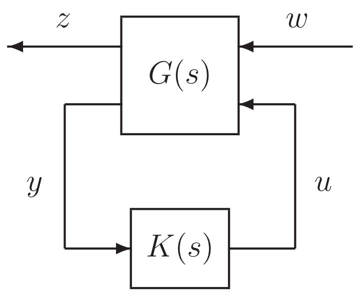


Figure 2.3: The Lower LFT Diagram for Closed-loop System

Suppose K is an m-th order controller which stabilizes the closed-loop system and the n-th order generalized plant G is given by

$$G = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

The following assumptions are made:

(1) $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ is detectable;

(2) D_{12} has full column rank and D_{21} has full row rank;

(3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

T_{zw} is the transfer function from w to z, and

$$\|T_{zw}\|_\infty = \|\mathcal{F}_\ell(G,K)\|_\infty = \left\|G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\right\|_\infty. \tag{2.3}$$

Then all rational internally stabilizing controllers $K(s)$ satisfying $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$ are given by $K = \mathcal{F}_\ell(M_\infty, Q)$ for arbitrary $Q \in \mathscr{RH}_\infty$ such that $\|Q\|_\infty < \gamma$ where

$$M_\infty = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right].$$

All relevant parameters $(\hat{A}, \hat{B}_1, \hat{B}_2, \hat{C}_1, \hat{C}_2, \hat{D}_{11}, \hat{D}_{12}, \hat{D}_{21}, \hat{D}_{22})$ can be found in [62, 63].

For a simple case,

where

$$G = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following assumptions are made:

(1) $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ is detectable;

(2) $(A,\ B_1)$ is controllable and $(C_1,\ A)$ is observable;

(3) $D_{12}^*\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(4) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

The $\mathscr{H}_\infty$ solution involves the following two Hamiltonian matrices:

$$\mathscr{H}_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^* - B_2B_2^* \\ -C_1^*C_1 & -A^* \end{bmatrix}, \mathscr{J}_\infty := \begin{bmatrix} A^* & \gamma^{-2}C_1^*C_1 - C_2^*C_2 \\ -B_1B_1^* & -A \end{bmatrix}.$$

Theorem 1 *[18, 62] There exists an admissible controller such that* $\|T_{zw}\|_\infty < \gamma$ *iff the following three conditions hold:*

(i) $\mathscr{H}_\infty \in dom(Ric)$ *and* $X_\infty := Ric(\mathscr{H}_\infty) > 0$*;*

(ii) $\mathscr{J}_\infty \in dom(Ric)$ *and* $Y_\infty := Ric(\mathscr{J}_\infty) > 0$*;*

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

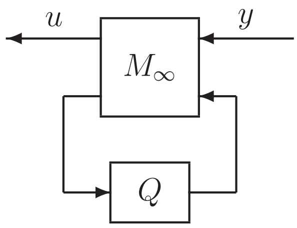
Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\begin{aligned} \hat{A}_\infty &= A + \gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty + Z_\infty L_\infty C_2; \\ F_\infty &= -B_2^*X_\infty; \\ L_\infty &= -Y_\infty C_2^*; \\ Z_\infty &= (I - \gamma^{-2}Y_\infty X_\infty)^{-1}. \end{aligned}$$

Furthermore, the set of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$ *equals the set of all transfer matrices from* y *to* u *in*



$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

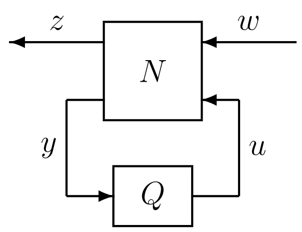
That is,

$$K = \mathcal{F}_\ell(M_\infty,\ Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

where $Q \in \mathscr{RH}_\infty$, $\|Q\|_\infty < \gamma$.

Generally speaking, Q can be chosen to satisfy additional performance objectives. However, how to find such a Q is a challenging problem and continues to be a research topic. In most cases, $Q = 0$ is chosen resulting in a so-called central $\mathscr{H}_\infty$ controller $K_c = \mathcal{F}_\ell(M_\infty,\ 0) = M_{11}$. The problem to be considered here is to find a controller $\hat{K}$ with a minimal possible order such that the $\mathscr{H}_\infty$ performance requirement $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ is satisfied. This is clearly equivalent to finding a Q so that it satisfies the above constraint and the order of $\hat{K}$ is minimized. However, directly finding such a Q has proven to be very difficult. The following lemma is useful in the subsequent development [63].

Lemma 1 *Consider a feedback system shown below*



where N is a suitably partitioned transfer matrix

$$N(s) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

Then, the closed-loop transfer matrix from w to z is given by

$$T_{zw} = \mathcal{F}_\ell(N, Q) = N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21}.$$

Assume that the feedback loop is well-posed, i.e., $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$, *and either $N_{21}(j\omega)$ has full row rank for all $\omega \in \mathscr{R} \cup \infty$ or $N_{12}(j\omega)$ has full column rank for all $\omega \in \mathscr{R} \cup \infty$ and $\|N\|_\infty \leq 1$ then $\|\mathcal{F}_\ell(N, Q)\|_\infty < 1$ if $\|Q\|_\infty < 1$.*

Definition 4 *([63]) The real rational subspace of $\mathscr{H}_\infty^-$ is denoted by $\mathscr{R}\mathscr{H}_\infty^-$, which consists of all proper, real rational, antistable transfer matrices (i.e., functions with all poles in the open right-half plane).*

2.2 Standard $\mathscr{H}_2$ Problem

We consider a closed-loop system shown in Figure 2.3 where the n-th order generalized plant G is given by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following standard assumptions are made:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) D_{12} has full column rank and D_{21} has full row rank;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Denote

$$R_1 = D_{12}^* D_{12} > 0, \quad R_2 = D_{21} D_{21}^* > 0$$

and let $X_2 \geq 0$ and $Y_2 \geq 0$ be stabilizing solutions to

$$X_2(A - B_2 R_1^{-1} D_{12}^* C_1) + (A - B_2 R_1^{-1} D_{12}^* C_1)^* X_2 - X_2 B_2 R_1^{-1} B_2^* X_2 + C_1^*(I - D_{12} R_1^{-1} D_{12}^*) C_1 = 0$$

and

$$Y_2(A - B_1 D_{21}^* R_2^{-1} C_2)^* + (A - B_1 D_{21}^* R_2^{-1} C_2) Y_2 - Y_2 C_2^* R_2^{-1} C_2 Y_2 + B_1(I - D_{21}^* R_2^{-1} D_{21}) B_1^* = 0.$$

Define

$$F_2 := -R_1^{-1}(B_2^* X_2 + D_{12}^* C_1), \quad L_2 := -(Y_2 C_2^* + B_1 D_{21}^*) R_2^{-1}.$$

From [13, 62], all controllers that stabilize the system G can be parameterized as

$$K(s) = \mathcal{F}_\ell(H, Q), \quad Q \in \mathscr{RH}_\infty \tag{2.4}$$

with

$$H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $\hat{A}_2 := A + B_2F_2 + L_2C_2$. Furthermore

$$\|T_{zw}\|_2^2 = \min \|T_{zw}\|_2^2 + \left\| R_1^{1/2} Q R_2^{1/2} \right\|_2^2$$

and the optimal controller is ($Q = 0$)

$$K_{opt}(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array} \right] = H_{11}.$$

2.3 Small Gain Theorem

The small gain theorem is an important tool to analyze stability property in control system.

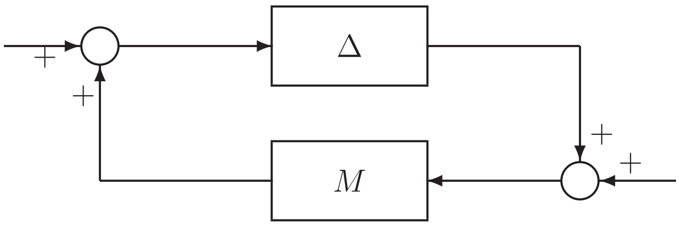


Figure 2.4: $M - \Delta$ Loop for Stability Analysis

Theorem 2 *(Small Gain Theorem)([62, 63]) Suppose $M \in \mathscr{RH}_\infty$ and let $\gamma > 0$. Then the interconnected system shown in Figure 2.4 is well-posed and internally stable for all $\Delta(s) \in \mathscr{RH}_\infty$ with $\|\Delta\|_\infty \leq 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$.*

2.4 Coprime Factorization

Definition 5 *([62, 63]) Two matrices M and N in $\mathscr{RH}_\infty$ are right comprime over $\mathscr{RH}_\infty$ if they have the same number of columns and if there exist matrices X_r and Y_r in $\mathscr{RH}_\infty$ such that*

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_rM + Y_rN = I.$$

Similarly, two matrices $\tilde{M}$ *and* $\tilde{N}$ *in* $\mathscr{RH}_\infty$ *are left comprime over* $\mathscr{RH}_\infty$ *if they have the same number of rows and if there exist matrices* X_l *and* Y_l *in* $\mathscr{RH}_\infty$ *such that*

$$\begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix} \begin{bmatrix} X_l & Y_l \end{bmatrix} = \tilde{M}X_l + \tilde{N}Y_l = I.$$

Let P be a proper real rational matrix. A right coprime factorization (rcf) of P is a factorization $P = NM^{-1}$, where N and M are right coprime over $\mathscr{RH}_\infty$. Similarly, a left coprime factorization (lcf) of P is a factorization $P = \tilde{M}^{-1}\tilde{N}$, where $\tilde{N}$ and $\tilde{M}$ are left coprime over $\mathscr{RH}_\infty$.

2.5 Inner-Outer Factorization

Theorem 3 *([63]) Let* $G \in \mathscr{R}_p$ *be a* $p \times m$ *transfer matrix. Assume* $p \geq m$*. Then there exists a right coprime factorization* $G = NM^{-1}$ *such that* N *is an inner if and only if* $G^{\sim}G > 0$ *on the* $j\omega$*-axis, including at* ∞*. This factorization is unique up to a constant unitary multiple. Furthermore, assume that the realization of* $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is stabilizable and that* $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ *has full column rank for all* $\omega \in \mathscr{R}$*. Then a particular realization of the desired coprime factorization is*

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A + BF & BR^{-1/2} \\ \hline F & R^{-1/2} \\ C + DF & DR^{-1/2} \end{array}\right] \in \mathscr{RH}_\infty$$

where

$$\begin{aligned} R &= D^*D > 0 \\ F &= -R^{-1}(B^*X + D^*C) \end{aligned}$$

and

$$X = Ric \begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I - DR^{-1}D^*)C & -(A - BR^{-1}D^*C)^* \end{bmatrix}.$$

Corollary 1 *([63]) Suppose $G \in \mathscr{RH}_\infty$; then the matrix M in Theorem 3 is an outer. Hence, the factorization $G = NM^{-1}$ given in Theorem 3 is an inner-outer factorization.*

The dual left coprime factorizations is omitted here.

2.6 Hankel Norm

Consider a transfer matrix

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and P and Q are observability and controllability Gramians that can be obtained from the following Lyapunov equations:

$$AP + PA^* + BB^* = 0$$

$$A^*Q + QA + C^*C = 0.$$

Definition 6 *([17]) Let $G(s) = C(sI - A)^{-1}B$ with $Re(\lambda_i(A)) < 0$ for any i, then the Hankel norm of G(s) is defined as*

$$\|G(s)\|_H := \lambda_{max}^{1/2}(PQ).$$

2.7 Star Product

Definition 7 *([63]) Suppose that P and K are compatibly partitioned matrices*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

such that the matrix product $P_{22}K_{11}$ *is well-defined and square, and assume further that* $I - P_{22}K_{11}$ *is invertible. Then the star product of* P *and* K *with respect to this partition is defined as*

$$\mathcal{S}(P, K) = \begin{bmatrix} \mathcal{F}_\ell(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}.$$

Note that this definition is dependent on the partitioning of the matrices P and K.

Chapter 3

Model Reduction Methods

Before discussing controller reduction methods, some model reduction methods and their frequency weighted model reduction methods are introduced in this chapter because they are the basis of modern model approximation. Balanced model reduction and frequency-weighted balanced model reduction are presented in Section 3.1 and 3.2. In Section 3.3 and Section 3.4, Hankel norm approximation and frequency-weighted Hankel norm approximation are introduced, respectively.

3.1 Balanced Model Reduction

The main idea of balanced truncation model reduction method is to eliminate those states which are less controllable and less observable since those system states contribute little to the input and output behavior. Balanced realization was originally introduced by Mullis and Roberts in 1976. In order to overcome the disadvantage that strong controllability (observability) states may have weak observability (controllability), Moore developed the balanced truncation method for model reduction in 1981 [43]. The error bound for the balanced truncation reduction method was obtained by Enns in 1984 [15].

Definition 8 *([62, 63]) Suppose*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{RH}_\infty.$$

Let P and Q denote the controllability gramian and observability gramian, and satisfy $P = Q = \Sigma = diag(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N})$, where $\sigma_1 > \sigma_2 > \cdots > \sigma_N \geq 0$ are called the Hankel singular values *of the system. Then the realization will be referred to as a* balanced realization*.*

Suppose there exists r so that $\sigma_r >> \sigma_{r+1}$ in the above definition, the balanced realization means that, comparing with those states corresponding to $\sigma_1, \ldots, \sigma_r$, those states corresponding to $\sigma_{r+1}, \ldots, \sigma_N$ are less controllable and observable, i.e. they have least contribution for the system frequency response. The information lost for the system, due to truncating those less controllable and less observable states, is small and can be ignored. As a result, it is reasonable to truncate those less controllable and observable states and then the resulted reduced model is acceptable.

Theorem 4 *([62, 63]) Suppose $G(s) \in \mathscr{RH}_\infty$ and*

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

is a balanced realization with Gramian $\Sigma = diag(\Sigma_1, \Sigma_2)$

$$\begin{aligned} \Sigma_1 &= diag(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) \\ \Sigma_2 &= diag(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N}) \end{aligned}$$

and

$$\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_N$$

where σ_i *has multiplicity* $s_i, i = 1, 2, \ldots, N$ *and* $s_1 + s_2 + \cdots + s_N = n$. *Then the truncated system*

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced and asymptotically stable. Therefore, $G_r(s)$ *is one possible approximation to* $G(s)$. *Furthermore,*

$$\|G(s) - G_r(s)\|_\infty \leq 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_N).$$

As we can see, the balanced truncation model reduction method is simple and performs well. However, the disadvantage of this method is that it gives a better approximation for original system at high frequency than low frequency. So it is not desirable in most control system designs which operate at low frequency. A lot of alternative reduction techniques have been researched in order to overcome the drawback of the balanced truncation method, and some improvements have been achieved.

3.2 Frequency-Weighted Balanced Model Reduction

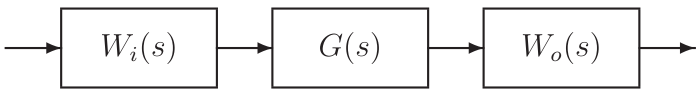


Figure 3.1: Model with Input and Output Weighting Functions

In this section, we introduce the extension of the balanced truncation model reduction method to the frequency-weighted balanced truncation model reduction shown by Enns

in [15]. Consider the model $G \in \mathscr{RH}_\infty$ with the input weighting function $W_i \in \mathscr{RH}_\infty$ and the output weighting function $W_o \in \mathscr{RH}_\infty$, as shown in Figure 3.1. The objective of frequency-weighted balanced truncation method is to find one low order model G_r so that the following norm is as small as possible:

$$\|W_0(G - G_r)W_i\|_\infty .$$

Assume that the state space realizations for G, W_i and W_o are given, respectively, as follows:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right], W_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], W_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right].$$

The weighted input controllability gramian P and the weighted output observability gramian Q can be solved from the following equations:

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}\begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}^* = 0 \tag{3.1}$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}\begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix}^*\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix}\begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix}^* = 0 \tag{3.2}$$

After calculating the weighted input controllability gramian P and the weighted output observability gramian Q, suppose that T is a nonsingular matrix such that

$$TPT^* = (T^{-1})^*QT^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

with $\Sigma_1 = diag(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r})$ and $\Sigma_2 = diag(\sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$, and

$$\left[\begin{array}{c|c} TA_GT^{-1} & TB_G \\ \hline C_GT^{-1} & D_G \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D_G \end{array}\right].$$

One reduced order model G_r will be obtained as:

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D_G \end{array}\right].$$

The detailed procedure has been shown in [62, 63]. It is worth to mention that, unlike the regular (unweighted) case, there is generally no priori error upper bound for the frequency-weighted balanced model reduction approximation. Besides, the reduced order model is usually almost stable for only one-side weighting situation. However, it is not guaranteed to be stable if there are two-side weightings.

In the following, we shall prove two important properties of frequency-weighted balanced realization that are useful in controller order reduction.

Theorem 5 *The frequency-weighted balanced realization is independent of the particular realizations of* G*,* W_i*, and* W_o*.*

Proof It is known that any two minimal realizations of a transfer matrix can be related by a similarity transformation [62]. Hence without loss of generality, we shall assume that any other realizations of G, W_i and W_o are given by

$$G = \left[\begin{array}{c|c} T_gA_GT_g^{-1} & T_gB_G \\ \hline C_GT_g^{-1} & D_G \end{array}\right], \quad W_i = \left[\begin{array}{c|c} T_iA_iT_i^{-1} & T_iB_i \\ \hline C_iT_i^{-1} & D_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} T_oA_oT_o^{-1} & T_oB_o \\ \hline C_oT_o^{-1} & D_o \end{array}\right]$$

for some nonsingular matrices T_g, T_i, and T_o.

Then the input weighted Gramian $\hat{P}$ and the output weighted Gramian $\hat{Q}$ satisfy the following equations:

$$\begin{bmatrix} T_g A_G T_g^{-1} & T_g B_G C_i T_i^{-1} \\ 0 & T_i A_i T_i^{-1} \end{bmatrix} \begin{bmatrix} \hat{P} & \hat{P}_{12} \\ \hat{P}_{12}^* & \hat{P}_{22} \end{bmatrix} + \begin{bmatrix} \hat{P} & \hat{P}_{12} \\ \hat{P}_{12}^* & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} T_g A_G T_g^{-1} & T_g B_G C_i T_i^{-1} \\ 0 & T_i A_i T_i^{-1} \end{bmatrix}^*$$
$$+ \begin{bmatrix} T_g B_G D_i \\ T_i B_i \end{bmatrix} \begin{bmatrix} T_g B_G D_i \\ T_i B_i \end{bmatrix}^* = 0$$

$$\begin{bmatrix} \hat{Q} & \hat{Q}_{12} \\ \hat{Q}_{12}^* & \hat{Q}_{22} \end{bmatrix} \begin{bmatrix} T_g A_G T_g^{-1} & 0 \\ T_o B_o C_G T_g^{-1} & T_o A_o T_o^{-1} \end{bmatrix} + \begin{bmatrix} T_g A_G T_g^{-1} & 0 \\ T_o B_o C_G T_g^{-1} & T_o A_o T_o^{-1} \end{bmatrix}^* \begin{bmatrix} \hat{Q} & \hat{Q}_{12} \\ \hat{Q}_{12}^* & \hat{Q}_{22} \end{bmatrix}$$
$$+ \begin{bmatrix} (C_G T_g^{-1})^* D_o^* \\ (C_o T_o)^* \end{bmatrix} \begin{bmatrix} (C_G T_g^{-1})^* D_o^* \\ (C_o T_o)^* \end{bmatrix}^* = 0$$

These two equations can be simplified to

$$\begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix}^* + \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} T_g^{-1}\hat{P}(T_g^{-1})^* & T_g^{-1}\hat{P}_{12}(T_i^{-1})^* \\ T_i^{-1}\hat{P}_{12}^*(T_g^{-1})^* & T_i^{-1}\hat{P}_{22}(T_i^{-1})^* \end{bmatrix}$$
$$+ \begin{bmatrix} T_g^{-1}\hat{P}(T_g^{-1})^* & T_g^{-1}\hat{P}_{12}(T_i^{-1})^* \\ T_i^{-1}\hat{P}_{12}^*(T_g^{-1})^* & T_i^{-1}\hat{P}_{22}(T_i^{-1})^* \end{bmatrix} \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}^* = 0$$

$$\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* + \begin{bmatrix} T_g^*\hat{Q}T_g & T_g^*\hat{Q}_{12}T_o \\ T_o^*\hat{Q}_{12}^*T_g & \hat{Q}_{22} \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}$$
$$+ \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} T_g^*\hat{Q}T_g & T_g^*\hat{Q}_{12}T_o \\ T_o^*\hat{Q}_{12}^*T_g & \hat{Q}_{22} \end{bmatrix} = 0$$

These equations imply that

$$P = T_g^{-1}\hat{P}(T_g^{-1})^*, \quad Q = T_g^*\hat{Q}T_g$$

and

$$PQ = T_g^{-1}\hat{P}\hat{Q}T_g.$$

Hence the weighted balanced realization will not depend on the particular realizations of G, W_i, and W_o. $\square$

Theorem 6 *Let W and G be scalar transfer functions. Then the input weighted balanced realization of G with input weighting W is the same as the output weighted balanced realization of G with output weighting W.*

Proof Assume that W and G have the following state space realizations:

$$W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right], \quad G(s) = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right].$$

Then

$$W^T(s) = \left[\begin{array}{c|c} A_w^T & C_w^T \\ \hline B_w^T & D_w^T \end{array}\right], \quad G^T(s) = \left[\begin{array}{c|c} A_G^T & C_G^T \\ \hline B_G^T & D_G^T \end{array}\right]$$

are also state space realizations of $W(s)$ and $G(s)$.

Note that the input weighted balanced realization of G with input weighting function W is the same as the output weighted balanced realization of G^T with output weighted function W^T since $(GW)^T = W^T G^T$. Hence by Theorem 5, the input weighted balanced realization of G with input weighting W is the same as the output weighted balanced realization of $G^T(s)$ with output weighting function $W^T(s)$. Then the conclusion follows by noting that $G^T(s)W^T(s) = (W(s)G(s))^T = W(s)G(s)$. $\square$

However, it should be noted that the above conclusion from Theorem 6 does not hold in general for matrix cases. Furthermore, the weighted balanced realization with two-sided weighting functions can be quite tricky as demonstrated in the following example.

Example 1 *Let W_i and W_o be given by*

$$W_i = \frac{s+2}{s+1}, \quad W_o = \frac{1}{s+2}$$

and

$$W = W_i W_o = \frac{1}{s+1}.$$

Let

$$G_1 = \frac{2s+7}{(s+2)(s+5)}.$$

Then the 1st order weighted balanced approximation with input weighting function W_i and output weighting function W_o is given by

$$\hat{G}_1 = \frac{1.79}{s+2.5783}$$

and the 1st order weighted balanced approximation with (input or output) one-sided weighting function W is given by

$$\tilde{G}_1 = \frac{1.82}{s+2.62}.$$

Moreover,

$$\left\| W_o(G_1 - \hat{G}_1)W_i \right\|_\infty = \left\| W(G_1 - \hat{G}_1) \right\|_\infty = 0.0093 < \left\| W(G_1 - \tilde{G}_1) \right\|_\infty = 0.0011.$$

Next, let

$$G_2 = \frac{2(s+1)}{(s+2)(s+5)}.$$

Then the 1st order weighted balanced approximation with input weighting function W_i and output weighting function W_o is given by

$$\hat{G}_2 = \frac{1.5556}{s+5.7037}$$

and the 1st order weighted balanced approximation with (input or output) one-sided weighting function W is given by

$$\tilde{G}_2 = \frac{1.53}{s + 6.097}.$$

Moreover,

$$\left\| W_o(G_2 - \hat{G}_2)W_i \right\|_\infty = \left\| W(G_2 - \hat{G}_2) \right\|_\infty = 0.0727 > \left\| W(G_2 - \tilde{G}_2) \right\|_\infty = 0.0517.$$

This example shows that it is not clear if one-sided weighted method will do better than two-sided weighted method since they produce different results for different problems.

3.3 Hankel Norm Approximation

Another important and powerful state-spaced based model reduction method is optimal Hankel norm approximation. The Hankel norm approximation method was originally developed by V.M.Adamjan, D.Z.Arov and M.G.Krein [1]. Glover [17] made some critical contributions for the method and obtained the unweighted optimal Hankel norm approximation together with an $\mathscr{L}_\infty$ error bound.

The optimal Hankel norm approximation problem can be stated as: given a transfer function $G(s)$ of McMillan degree n, find $\hat{G}(s)$ of McMillan degree $k(k < n)$ such that the Hankel norm error $\left\| G(s) - \hat{G}(s) \right\|_H$ is minimized [62]. It is known from [1, 17] that

$$\inf \left\| \mathrm{G(s)} - \hat{\mathrm{G}}\mathrm{(s)} \right\|_{\mathrm{H}} = \sigma_{\mathrm{k+1}}.$$

Furthermore, there exists an upper bound on the $\mathscr{L}_\infty$ norm of the error.

Theorem 7 *([17, 19]) Suppose that $G(s) \in \mathscr{RH}_\infty$ is a strictly proper transfer function, $\hat{G}(s) \in \mathscr{RH}_\infty$ is a strictly proper r-th order optimal Hankel norm approximation to $G(s)$ and $\sigma_1 > \sigma_2 > \cdots > \sigma_N \geq 0$ are the* Hankel singular values *of $G(s)$. Then*

$$\left\| G(s) - \hat{G}(s) \right\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_N)$$

and there exists a constant D_0 such that

$$\left\| G(s) - \hat{G}(s) - D_0 \right\|_\infty \leq (\sigma_{r+1} + \cdots + \sigma_N).$$

3.4 Frequency-Weighted Hankel Norm Approximation

Just as is stated at the beginning of this chapter, it is highly desirable for engineers to have different requirements on model reduction approximation error at different frequency ranges. The extension to frequency-weighted optimal Hankel norm approximation has been derived by Latham and Anderson in 1985 [37]. The error bounds have been obtained by Anderson for one side weighted case in 1986 [3]. However, there is not any satisfactory $\mathscr{L}_\infty$ error bound for the general two-side weighting situation, except some special cases such as the first order weighting function [31], and with a multivariable version in Zhou [59]. Zhou also gave the complete solution to the frequency-weighted Hankel norm approximation with antistable weightings [61].

Some main results of frequency-weighted optimal Hankel norm approximation will be summarized in this part.

Theorem 8 *([61]) Let $G(s) \in \mathscr{RH}_\infty$, $W_1(s) \in \mathscr{RH}_\infty^-$ and $W_2(s) \in \mathscr{RH}_\infty^-$ with minimal*

state-space realizations

$$W_1(s) = \left[\begin{array}{c|c} A_{1w} & B_{1w} \\ \hline C_{1w} & D_{1w} \end{array}\right], \quad W_2(s) = \left[\begin{array}{c|c} A_{2w} & B_{2w} \\ \hline C_{2w} & D_{2w} \end{array}\right].$$

Suppose that

$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \in \mathscr{RH}_\infty$$

is a r-th order optimal Hankel norm approximation of $[W_1 G W_2]_+$*, i.e.,*

$$G_1(s) = arg \inf_{degQ \le r} \|[W_1 G W_2]_+ - Q\|_H$$

and assume

$$\begin{bmatrix} A_{1w} - \lambda I & B_{1w} \\ C_{1w} & D_{1w} \end{bmatrix}, \quad \begin{bmatrix} A_{2w} - \lambda I & B_{2w} \\ C_{2w} & D_{2w} \end{bmatrix}$$

have, respectively, full row rank and full column rank for all $\lambda = \lambda_i(A_r), i = 1, \dots, r$*. Then there exist matrices* X, Y, Q *and* Z *such that*

$$\begin{aligned} A_{1w}X - XA_1 + B_{1w}Y &= 0 \\ C_{1w}X + D_{1w}Y &= C_1 \\ QA_{2w} - A_1 Q + ZC_{2w} &= 0 \\ QB_{2w} + ZD_{2w} &= B_1. \end{aligned}$$

Furthermore

$$G_r := \left[\begin{array}{c|c} A_1 & Z \\ \hline Y & 0 \end{array}\right]$$

is the frequency-weighted optimal Hankel norm approximation, i.e.,

$$\inf_{deg\hat{G} \le r} \left\| W_1 (G - \hat{G}) W_2 \right\|_H = \|W_1 (G - G_r) W_2\|_H = \sigma_{r+1}([W_1 G W_2]_+).$$

Note that $Y = C_1$ if $W_1 = I$ and $Z = B_r$ if $W_2 = I$.

Chapter 4

Controller Reduction Methods

Low order controller is normally preferred over high order controller in control system design. The main idea of controller reduction is to reduce the full order controller K to a lower order controller $\hat{K}$ which can preserve the system stability and minimize the performance degradation of the closed-loop system with new reduced order controller. The techniques listed in Chapter 3, including (frequency-weighted) balanced truncation and (frequency-weighted) optimal Hankel norm approximation, can be used in controller reduction. Additional controller reduction methods will be reviewed in Section 4.1 and Section 4.2, respectively. Additive controller reduction method for stability and performance preservation are shown in Section 4.1.1 and Section 4.2.1, respectively. Coprime factor controller reduction method for stability and performance preservation are presented in Section 4.1.2 and Section 4.2.2, respectively. Nagado and Usui [45] provide a $\mathscr{H}_\infty$ controller reduction method in Section 4.2.3, and relevant performance preserving issue is discussed by Hadian and Yazdanpanzh [28] in Section 4.2.4.

4.1 Stability Preservation Controller Reduction

The purpose of this part is to introduce some controller reduction methods which guarantee the closed-loop stability because the property of stability is the most basic and important requirement for control system. Consider the closed-loop system shown in Figure 2.3 where the generalized plant G is given by

$$G = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$. Suppose that K is a full order controller which can stabilize the closed-loop system, it is desirable to find a reduced order controller $\hat{K}$ to stabilize the same system and guarantee the performance.

One trivial approach to obtain the reduced controller is to minimize $\left\|K - \hat{K}\right\|_\infty$ by using model reduction methods. Unfortunately, this approach can be very conservative because it makes $\left\|K - \hat{K}\right\|_\infty$ as small as possible over all frequencies. On the other hand, a reduced order stabilizing controller only needs to minimize the approximation error over those selected frequency ranges that affect the closed-loop stability and performance.

4.1.1 Additive Controller Reduction

Lemma 2 *([62]) Suppose that the full order stabilizing controller K and the reduced order controller $\hat{K}$ have the same number of right half plane poles. Define*

$$\Delta := K - \hat{K}, \quad W_a := (I - G_{22}K)^{-1}G_{22}.$$

Then the closed-loop system with $\hat{K}$ is also stable if either

$$\|W_a\Delta\|_\infty < 1$$

or

$$\|\Delta W_a\|_\infty < 1.$$

In order to get a reduced order stabilizing controller $\hat{K}$ for the feedback system, some model reduction methods can be used here such as frequency-weighted balance model reduction, frequency-weighted Hankel norm model reduction or other model reduction methods. If K is unstable, then K is generally separated into two parts: stable part K_s and unstable part K_u, i.e., $K = K_s + K_u$. To make sure that K and $\hat{K}$ have the same number of right half plane poles, model reduction is applied to the stable part K_s to obtain $\hat{K}_s$, then the final reduced controller is given by: $\hat{K} = \hat{K}_s + K_u$.

Remark 1 *([62]) The stability condition in Lemma 2 is only sufficient. That means even if $\|W_a\Delta\|_\infty \geq 1$ or $\|\Delta W_a\|_\infty \geq 1$, then it is still possible that $\hat{K}$ is a stabilizing reduced controller.*

4.1.2 Coprime Factor Controller Reduction

Note that the additive controller reduction method only performs order reduction on the stable part of the full order controller. If the closed-loop system is totally unstable, this method cannot be used to reduce the controller order. In this section, coprime factor controller reduction method will be shown to solve this kind of problem.

Suppose that G_{22} and K have the right and left coprime factorizations, respectively

$$G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad K = UV^{-1} = \tilde{V}^{-1}\tilde{U}.$$

Define

$$\begin{bmatrix} \tilde{N}_n & \tilde{M}_n \end{bmatrix} := (\tilde{M}V - \tilde{N}U)^{-1} \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = V^{-1}(I - G_{22}K)^{-1} \begin{bmatrix} G_{22} & I \end{bmatrix}$$

and

$$\begin{bmatrix} N_n \\ M_n \end{bmatrix} := \begin{bmatrix} N \\ M \end{bmatrix} (\tilde{V}M - \tilde{V}N)^{-1} = \begin{bmatrix} G_{22} \\ I \end{bmatrix} (I - KG_{22})^{-1}\tilde{V}^{-1}.$$

Lemma 3 *([62]) Let $\hat{U}, \hat{V} \in \mathscr{RH}_\infty$ be the reduced order right coprime factors of U and V. Then $\hat{K} = \hat{U}\hat{V}^{-1}$ stabilizes the system if*

$$\left\| \begin{bmatrix} -\tilde{N}_n & \tilde{M}_n \end{bmatrix} \left(\begin{bmatrix} U \\ V \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1.$$

Similarly, Let $\hat{\tilde{U}}, \hat{\tilde{V}} \in \mathscr{RH}_\infty$ be the reduced order left coprime factors of $\tilde{U}$ and $\tilde{V}$. Then $\hat{K} = \hat{\tilde{V}}^{-1}\hat{\tilde{U}}$ stabilizes the system if

$$\left\| \left(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \begin{bmatrix} -N_n \\ M_n \end{bmatrix} \right\|_\infty < 1.$$

Remark 2 *This method is significantly different from the addition controller reduction method so that there is no restriction on unstable poles of the controller in coprime factor reduction method. K and $\hat{K}$ may have different number of right half plane poles, and even different positions of those right half plane poles.*

Remark 3 *The stability condition in Lemma 3 is also only sufficient.*

4.2 $\mathscr{H}_\infty$ Controller Reduction

Several controller reduction methods that can guarantee closed-loop system stability and performance are introduced here.

Consider a closed-loop system shown in Figure 2.3 and assume that those four assumptions in Section 2.1 are satisfied. From Equation (2.3), all stabilizing controllers which guarantee the closed-loop system performance $\|T_{zw}\|_\infty$ satisfy:

$$\begin{aligned}\|T_{zw}\|_\infty &= \|\mathcal{F}_\ell(G,K)\|_\infty \\ &= \left\|G_{11} + G_{12}K(I - G_{22}K^{-1})G_{21}\right\|_\infty < \gamma.\end{aligned}$$

Here we aim to find a controller $\hat{K}$ with a minimal possible order to satisfy the performance requirement $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$.

4.2.1 Additive Controller Reduction by Goddard and Glover

Suppose K_0 is a nominal and high order controller such that the $\mathscr{H}_\infty$ performance bound $\|\mathcal{F}_\ell(G,\ K_0)\|_\infty < \gamma$ is satisfied. Let us consider a class of reduced order controller in the form:

$$\hat{K} = K_0 + W_2 \Delta W_1$$

where Δ is a stable perturbation with stable, minimum phase and invertible weighting functions W_1 and W_2. The motivation here is to figure out if a reduced order controller $\hat{K}$ can be obtained so that $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$. Theorem 1 shows that if there is a such controller, then there must exist a Q so that $\hat{K} = \mathcal{F}_\ell(M_\infty, Q)$, and it can be calculated that

$$Q = \mathcal{F}_\ell(K_a^{-1}, \hat{K})$$

where

$$K_a^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M_\infty^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Furthermore,

$$\begin{aligned} \|Q\|_\infty < \gamma &\iff \left\|\mathcal{F}_\ell(K_a^{-1}, \hat{K})\right\|_\infty < \gamma \\ &\iff \left\|\mathcal{F}_\ell(K_a^{-1}, K_0 + W_2 \Delta W_1)\right\|_\infty < \gamma \\ &\iff \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1 \end{aligned}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix},$$

and R is given by star product

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(K_a^{-1}, \begin{bmatrix} K_0 & I \\ I & 0 \end{bmatrix}).$$

Theorem 9 *([19–22, 60, 63]) Suppose W_1 and W_2 are stable, minimum phase and invertible transfer matrices such that $\tilde{R}$ is a contraction. If K_0 is a stabilizing controller which satisfies $\|\mathcal{F}_\ell(G,\ K_0)\|_\infty < \gamma$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$ if*

$$\|\Delta\|_\infty = \left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1.$$

Remark 4 *There is a limitation for the set of feasible reduced order controllers because the form of $\hat{K}$ decides that $\hat{K}$ and K_0 must have the same unstable poles.*

Remark 5 *In Theorem 9, there are infinite choices for W_1, W_2 such that $\tilde{R}$ is contractive. However, the "largest" W_1 and W_2 [20] in some sense are preferred in order to make $\left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1$.*

One procedure summarized in the following algorithm [62] can be used to obtain a reduced order controller $\hat{K}$ which preserves the $\mathcal{H}_\infty$ performance bound $\left\| \mathcal{F}_\ell(G,\ \hat{K}) \right\|_\infty < \gamma$.

Algorithm 1

- K_0 is a full-order controller satisfying $\|\mathcal{F}_\ell(G,\ K_0)\|_\infty < \gamma$;
- Compute W_1 and W_2 so that $\tilde{R}$ is a contraction;
- Use frequency-weighted model reduction method to find a reduced controller $\hat{K}$ so that $\left\| W_2^{-1}(\hat{K} - K_0)W_1^{-1} \right\|_\infty < 1$.

4.2.2 Coprime Factor Controller Reduction by Goddard and Glover

The coprime factor controller reduction for stability preservation has been introduced in Section 4.1.2. The $\mathscr{H}_\infty$ performance preservation problem is also considered in coprime factor framework here [21, 22].

Lemma 4 *All stabilizing controllers that satisfy the $\mathscr{H}_\infty$ performance bound $\|T_{zw}\|_\infty < \gamma$ can also be parameterized as:*

$$
\begin{aligned}
K = \mathcal{F}_\ell(M_\infty, Q) &= (\Theta_{11}Q + \Theta_{12})(\Theta_{21}Q + \Theta_{22})^{-1} = UV^{-1} \\
&= (Q\tilde{\Theta}_{12} + \tilde{\Theta}_{22})^{-1}(Q\tilde{\Theta}_{11} + \tilde{\Theta}_{21}) = \tilde{V}^{-1}\tilde{U}
\end{aligned}
$$

where $Q \in \mathscr{RH}_\infty$, $\|Q\|_\infty < \gamma$, and UV^{-1} and $\tilde{V}^{-1}\tilde{U}$ are right and left coprime factorizations over $\mathscr{RH}_\infty$, respectively, and

$$
\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{11}\hat{D}_{21}^{-1} \\ -\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{21}^{-1} \end{array}\right]
$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 & \hat{B}_1 - \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11} & -\hat{B}_2 \hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{21} - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{D}_{11} & -\hat{D}_{22} \hat{D}_{12}^{-1} \\ \hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{12}^{-1} \hat{D}_{11} & \hat{D}_{12}^{-1} \end{array}\right].$$

Theorem 10 *Let $K_0 = \Theta_{12}\Theta_{22}^{-1}$ be the central $\mathscr{H}_\infty$ controller ($Q = 0$) such that $\|\mathcal{F}_\ell(G,\ K_0)\|_\infty < \gamma$ and let $\hat{U}, \hat{V} \in \mathscr{RH}_\infty$ with $\det\hat{V}(\infty) \neq 0$ be such that*

$$\left\| \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{U}\hat{V}^{-1}$ is also a stabilizing controller so that $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$.

Theorem 11 *Let $K_0 = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$ be the central $\mathscr{H}_\infty$ controller ($Q = 0$) such that $\|\mathcal{F}_\ell(G,\ K_0)\|_\infty < \gamma$ and let $\hat{\tilde{U}}, \hat{\tilde{V}} \in \mathscr{RH}_\infty$ with $\det\hat{\tilde{V}}(\infty) \neq 0$ be such that*

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{\tilde{V}}^{-1}\hat{\tilde{U}}$ is also a stabilizing controller so that $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$.

Algorithm 2

- Let $K = \Theta_{12}\Theta_{22}^{-1}$ (or $K = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$) be the central $\mathscr{H}_\infty$ controller ($Q = 0$) such that $\|T_{zw}\|_\infty < \gamma$.

- Use model reduction method to find a reduced order controller $\hat{K} = \hat{U}\hat{V}^{-1}$ (or $K = \hat{\tilde{V}}^{-1}\hat{\tilde{U}}$) such that the following inequality holds

$$\left\| \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}$$

or

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

- Then $\hat{K}$ can stablilize the system and maintain the performance requirement $\left\|\mathcal{F}_\ell(G,\ \hat{K})\right\|_\infty < \gamma$.

4.2.3 Nagado and Usui's Controller Reduction Method

It is now well accepted that the most effective controller reduction methods use the information of closed-loop system in reduction algorithms, and in many cases, controller reduction problems can be (conservatively) reduced to some frequency weighted model reduction problems in certain way. Many controller reduction methods [22, 38, 52] were proposed over the years and use the information from the closed-loop system in frequency weightings. However, the $\mathscr{H}_\infty$ performance cannot be guaranteed in some methods, and in some other methods, high-order frequency weighting functions or complex computation are involved even if those methods can guarantee the $\mathscr{H}_\infty$ performance of the closed-loop system. Nagado and Usui [45] proposed a new $\mathscr{H}_\infty$ controller reduction method which considers the stability property of the closed-loop system using frequency weightings found with an standard representation of the solution for an $\mathscr{H}_\infty$ controller problem. In this part, the stability preserving method will be introduced first.

Note that $K_0(s) := M_{11}(s)$ is the central controller that satisfies $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Now suppose $\hat{K}$ is a reduced order controller that also satisfies $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$. Then $\hat{K}$ can be represented as

$$\hat{K} = \mathcal{F}_\ell(M_\infty, Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}.$$

Let

$$\Delta := \hat{K} - K_0 \tag{4.1}$$

then

$$\Delta = M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

and Q can be expressed in Δ as

$$\begin{aligned} Q &= (I + M_{12}^{-1}\Delta M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta M_{21}^{-1} \\ &= M_{12}^{-1}(I + \Delta M_{21}^{-1}M_{22}M_{12}^{-1})^{-1}\Delta M_{21}^{-1}. \end{aligned}$$

Theorem 12 *Suppose that* $\Delta := \hat{K} - K_0$ *is stable. Then* $Q = M_{12}^{-1}(I + \Delta M_{21}^{-1}M_{22}M_{12}^{-1})^{-1}\Delta M_{21}^{-1} \in \mathscr{H}_\infty$ *if*

$$\left\|\Delta M_{21}^{-1}M_{22}M_{12}^{-1}\right\|_\infty < 1$$

or

$$\left\|M_{21}^{-1}M_{22}M_{12}^{-1}\Delta\right\|_\infty < 1$$

where

$$\begin{aligned} M_{12}^{-1} &= -\hat{D}_{12}^{-1}\hat{C}_1(sI - \hat{A} + \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1)^{-1}\hat{B}_2\hat{D}_{12}^{-1} + \hat{D}_{12}^{-1} \\ M_{21}^{-1} &= -\hat{D}_{21}^{-1}\hat{C}_2(sI - \hat{A} + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2)^{-1}\hat{B}_1\hat{D}_{21}^{-1} + \hat{D}_{21}^{-1} \\ M_{22} &= \hat{C}_2(sI - \hat{A})^{-1}\hat{B}_2 + \hat{D}_{22}. \end{aligned}$$

Nagado and Usui [45] used Enn's frequency-weighted balanced realization truncation method [15] to reduce the controller K_0 and obtained a reduced controller $\hat{K}$ which has the high possibility to guarantee the stability of the closed-loop system.

4.2.4 Hadian and Yazdanpanzh's Controller Reduction Method

The performance preserving issue is discussed by Hadian and Yazdanpanzh in [28]. Note again that $Q = (I + M_{12}^{-1}\Delta M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta M_{21}^{-1}$.

Theorem 13 *Suppose that $\Delta := \hat{K} - K_0$ is stable. Then Q is stable and $\|Q\|_\infty < \gamma$ if $\left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty \le \frac{\gamma}{1+\gamma\alpha}$, where $\alpha = \|M_{22}\|_\infty$.*

Proof

$$\begin{aligned}
\|Q\|_\infty &= \left\|(I + M_{12}^{-1}\Delta M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty \\
&\le \frac{\left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty}{1 - \left\|M_{12}^{-1}\Delta M_{21}^{-1}M_{22}\right\|_\infty} \\
&\le \frac{\left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty}{1 - \left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty \|M_{22}\|_\infty}.
\end{aligned}$$

Suppose $\alpha = \|M_{22}\|_\infty$ and $\beta = \left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty$, then

$$\|Q\|_\infty \le \frac{\beta}{1 - \beta\alpha}.$$

Hence $\|Q\|_\infty < \gamma$ if

$$\beta < \frac{\gamma}{1 + \gamma\alpha}.$$

So the performance maintaining problem ($\|Q\|_\infty < \gamma$) has been transferred to find one reduced controller to satisfy the $\mathscr{H}_\infty$ norm requirement $\left\|M_{12}^{-1}\Delta M_{21}^{-1}\right\|_\infty \le \frac{\gamma}{1+\gamma\alpha}$. □

The advantage of the two methods from Section 4.2.3 and Section 4.2.4 is that weighting functions can be derived from the representation for the $\mathscr{H}_\infty$ controller problem, which reduces computation complexity comparing with many other methods in [22, 38, 52]. Some numerical examples are given in [28, 45].

Chapter 5

$\mathscr{H}_\infty$ Performance Preserving Controller Reduction

In this chapter, several controller reduction methods are introduced with the objective that the closed-loop stability is guaranteed and the closed-loop performance degradation is made as small as possible with the reduced controller. First, some sufficient conditions for system stability guarantee and controller reduction algorithms are given in Section 5.1. Section 5.2 presents $\mathscr{H}_\infty$ controller reduction methods which can guarantee the closed-loop stability and the performance as well. Computational issues are discussed in Section 5.3, where the details for obtaining reduced controllers in frequency-weighted balanced truncation are exhibited.

5.1 Proposed Stability Preserving Controller Reduction

As shown in Chapter 2, all stabilizing controllers that satisfy the performance $\|T_{zw}\|_\infty < \gamma$ can be parameterized as:

$$K = \mathcal{F}_\ell(M_\infty,\ Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}.$$

Note that $K_0(s) := M_{11}(s)$ is the central controller that satisfies $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Suppose that $\hat{K}$ is a reduced order controller that also satisfies $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$. Then there exists Q so that $\hat{K}$ can be represented as

$$\hat{K} = \mathcal{F}_\ell(M_\infty,\ Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

where

$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right].$$

Let

$$\Delta_K := \hat{K} - K_0.$$

Then

$$\Delta_K = M_{12}Q(I - M_{22}Q)^{-1}M_{21}$$

and Q can be expressed in Δ_K as

$$Q = (I + M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta_K M_{21}^{-1}.$$

Hence finding a reduced order controller $\hat{K}$ such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ is reduced to find a $\hat{K}$ such that $Q \in \mathscr{H}_\infty$ and $\|Q\|_\infty < \gamma$.

Here, it is noted that $M_{12}^{-1} \in \mathscr{H}_\infty$, $M_{21}^{-1} \in \mathscr{H}_\infty$, and it can also be verified that $M_{21}^{-1}M_{22} \in \mathscr{H}_\infty$ and $M_{22}M_{12}^{-1} \in \mathscr{H}_\infty$.

Furthermore,

$$
\begin{aligned}
M_{12}^{-1} &= \left[\begin{array}{c|c} \hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1} \end{array}\right] \\
M_{21}^{-1} &= \left[\begin{array}{c|c} \hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1} \end{array}\right] \\
M_{21}^{-1}M_{22} &= \left[\begin{array}{cc|c} \hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ 0 & \hat{A} & \hat{B}_2 \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1}\hat{D}_{22} \end{array}\right] \\
M_{22}M_{12}^{-1} &= \left[\begin{array}{cc|c} \hat{A} & \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} \\ 0 & \hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2 & \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{22}\hat{D}_{12}^{-1} \end{array}\right].
\end{aligned}
$$

After removing the uncontrollable and unobservable part of $M_{21}^{-1}M_{22}$ and $M_{22}M_{12}^{-1}$:

$$
\begin{aligned}
M_{21}^{-1}M_{22} &= \left[\begin{array}{c|c} \hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1}\hat{D}_{22} \end{array}\right] \\
M_{22}M_{12}^{-1} &= \left[\begin{array}{c|c} \hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2-\hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{22}\hat{D}_{12}^{-1} \end{array}\right].
\end{aligned}
$$

Lemma 5 *Suppose that $\Delta_K := \hat{K} - K_0$ is stable. Then*

$$Q = (I + M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22})^{-1}M_{12}^{-1}\Delta_K M_{21}^{-1} \in \mathscr{H}_\infty$$

if one of the following conditions holds

(a) $\left\|\Delta_K M_{21}^{-1}M_{22}M_{12}^{-1}\right\|_\infty < 1$;

(b) $\left\|M_{21}^{-1}M_{22}M_{12}^{-1}\Delta_K\right\|_\infty < 1$;

(c) $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}\right\|_\infty < 1$;

(d) $\left\| M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty < 1$*;*

(e) $\left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty < 1$ *for some square* L *such that* $L, L^{-1} \in \mathscr{H}_\infty$*;*

(f) $\left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty < 1$ *for some square* J *such that* $J, J^{-1} \in \mathscr{H}_\infty$*.*

Proof Note that

$$\begin{aligned} Q &= (I + M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22})^{-1} M_{12}^{-1} \Delta_K M_{21}^{-1} \\ &= M_{12}^{-1} \Delta_K M_{21}^{-1} (I + M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1})^{-1} \\ &= M_{12}^{-1} (I + \Delta_K M_{21}^{-1} M_{22} M_{12}^{-1})^{-1} \Delta_K M_{21}^{-1} \\ &= M_{12}^{-1} \Delta_K (I + M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K)^{-1} M_{21}^{-1}. \end{aligned}$$

Then by small gain theorem, Q is stable if Δ_K is stable and either one of the following conditions is true:

(a) $\left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{-1} \right\|_\infty < 1$;

(b) $\left\| M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K \right\|_\infty < 1$;

(c) $\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} \right\|_\infty < 1$;

(d) $\left\| M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty < 1$.

For any square transfer matrices L and J such that $L, L^{-1}, J, J^{-1} \in \mathscr{H}_\infty$, we have

$$\begin{aligned} Q &= (I + M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22})^{-1} M_{12}^{-1} \Delta_K M_{21}^{-1} \\ &= L^{-1} (I + L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1})^{-1} L M_{12}^{-1} \Delta_K M_{21}^{-1} \\ &= M_{12}^{-1} \Delta_K M_{21}^{-1} J (I + J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J)^{-1} J^{-1}. \end{aligned}$$

Again by small gain theorem, Q is stable if Δ_K is stable and either

$$\left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty < 1,$$

or

$$\left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty < 1.$$

$\square$

Conditions (a) and (b) were used in [45] to obtain reduced order controllers with impressive results.

Lemma 6 *Let L and J be square transfer matrices such that $L, L^{-1}, J, J^{-1} \in \mathscr{H}_\infty$. Then*

$$\min_{L, L^{-1} \in \mathscr{H}_\infty} \left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty \leq \left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{-1} \right\|_\infty$$

and

$$\min_{J, J^{-1} \in \mathscr{H}_\infty} \left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty \leq \left\| M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K \right\|_\infty .$$

Proof Let an inner-outer factorization of M_{12}^{-1} be given by

$$M_{12}^{-1} = M_{12}^o M_{12}^i$$

such that $M_{12}^o, (M_{12}^o)^{-1}, M_{12}^i \in \mathscr{H}_\infty$ and

$$M_{12}^i(s)(M_{12}^i(-s))^T = I.$$

Now let $L_0 = (M_{12}^o)^{-1}$. Then

$$\min_{L, L^{-1} \in \mathscr{H}_\infty} \left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty$$

$$
\begin{aligned}
&\leq \left\| L_0 M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L_0^{-1} \right\|_\infty \\
&= \left\| L_0 M_{12}^{o} M_{12}^{i} \Delta_K M_{21}^{-1} M_{22} L_0^{-1} \right\|_\infty \\
&= \left\| M_{12}^{i} \Delta_K M_{21}^{-1} M_{22} M_{12}^{o} \right\|_\infty \\
&= \left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{o} \right\|_\infty \\
&= \left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{o} M_{12}^{i} \right\|_\infty \\
&= \left\| \Delta_K M_{21}^{-1} M_{22} M_{12}^{-1} \right\|_\infty .
\end{aligned}
$$

The other inequality can be shown using the same technique.

□

This lemma shows that the least conservative stability conditions are (e) and (f). Hence we shall propose the following improved controller reduction procedure.

Algorithm 3

- Let $L = I$.
- Repeat:
  - Find a reduced order controller $\hat{K}$ using the following criterion

    $$\left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty .$$

  - Find an L such that $L, L^{-1} \in \mathscr{H}_\infty$ and

    $$\min_{L, L^{-1} \in \mathscr{H}_\infty} \left\| L M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} L^{-1} \right\|_\infty .$$

Algorithm 4

- Let $J = I$.
- Repeat:
  - Find a reduced order controller $\hat{K}$ using the following criterion
$$\left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty .$$
  - Find a J such that $J, J^{-1} \in \mathscr{H}_\infty$ and
$$\min_{J, J^{-1} \in \mathscr{H}_\infty} \left\| J^{-1} M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} J \right\|_\infty .$$

Note that Lemma 5 in this section gives some sufficient conditions to guarantee the stability of Q which in turn may result in the closed-loop system stability. The reduced order controllers derived from Algorithm 3 and Algorithm 4 can only satisfy the stability requirement of the system. In other words, there is no guarantee that $\|Q\|_\infty < \gamma$ will be satisfied even if any of the above conditions is satisfied. Hence the reduced order controller is not guaranteed to satisfy $\left\| \mathcal{F}_\ell(G, \hat{K}) \right\|_\infty < \gamma$ and this condition has to be verified for each reduced controller. Besides, how to find the optimal L and J is still an open question.

5.2 Proposed $\mathscr{H}_\infty$ Performance Preserving Controller Reduction

Hadian and Yazdanpannah proposed a controller reduction approach in [28] which considers the error $M_{12}^{-1} \Delta_K M_{21}^{-1}$. According to Theorem 13, $\left\| \mathcal{F}_\ell(G, \hat{K}) \right\|_\infty < \gamma$ is guaranteed if the weighted approximation error $\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty$ is sufficiently small.

Algorithm 5

- Find a reduced order controller $\hat{K}$ using the following criterion

$$\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty .$$

Nevertheless, this method may still be conservative. We shall propose some other methods below.

Theorem 14 *Let $K_0 = M_{11}$ be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and $\varepsilon > 0$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ if*

$$\left\| M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}.$$

Proof Let

$$\tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}_1 & \tilde{\Delta}_2 \end{bmatrix} := M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix}.$$

Then

$$\begin{aligned} Q &= (I + \frac{\tilde{\Delta}_1}{\gamma\varepsilon})^{-1}\tilde{\Delta}_2 = \left(I + \frac{\tilde{\Delta}}{\gamma\varepsilon} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \tilde{\Delta} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \left(I + \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \begin{bmatrix} \frac{1}{\sqrt{1+\varepsilon^2}} I \\ 0 \end{bmatrix} \right)^{-1} \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \begin{bmatrix} 0 \\ I \end{bmatrix} \gamma \\ &= \mathcal{F}_\ell \left(N, \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \right) \gamma \end{aligned}$$

where

$$N = \begin{bmatrix} 0 & \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} I \\ \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{1+\varepsilon^2}} I \\ 0 \end{bmatrix} \end{bmatrix}$$

and $N'N = I$. By Lemma 1, $\|Q\|_\infty < \gamma$ if

$$\left\| \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} \frac{\tilde{\Delta}}{\gamma} \right\|_\infty < 1$$

or equivalent

$$\left\| \tilde{\Delta} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}.$$

□

Similarly, we have the following dual result.

Theorem 15 *Let $K_0 = M_{11}$ be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and $\varepsilon > 0$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ if*

$$\left\| \begin{bmatrix} \varepsilon\gamma M_{22} \\ I \end{bmatrix} M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty < \frac{\varepsilon\gamma}{\sqrt{1+\varepsilon^2}}.$$

Remark 6 *Note that $\varepsilon > 0$ should be used as a design parameter. One may start from $\varepsilon = 0$ and in this case the above controller reduction methods are reduced to*

$$\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty.$$

However, in this case, there is no guarantee if the reduced controller will satisfy the $\mathscr{H}_\infty$ performance and the exact $\mathscr{H}_\infty$ performance has to be verified for each reduced order controller. The reason is that the performance guarantee condition is only sufficient, which means it is possible to find one reduced controller to satisfy performance requirement even if $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}\right\|_\infty \geq \frac{\gamma}{1+\gamma\|M_{22}\|_\infty}$. On the other hand, when ε is very large, the method is equivalent to

$$\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}M_{22} \right\|_\infty$$

or

$$\left\| M_{22} M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

Again the exact $\mathscr{H}_\infty$ performance resulting from these criteria has to be verified for each reduced order controller.

Algorithm 6

- Find a reduced order controller $\hat{K}$ using the following criterion

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \left[\begin{array}{cc} \varepsilon\gamma M_{22} & I \end{array} \right] \right\|_\infty .$$

The related state space realization of the relevant transfer matrix is given by

$$M_{21}^{-1} \left[\begin{array}{cc} \varepsilon\gamma M_{22} & I \end{array} \right] \; = \; \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 & \varepsilon\gamma \left(\hat{B}_2 - \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22} \right) & -\hat{B}_1 \hat{D}_{21}^{-1} \\ \hline \hat{D}_{21}^{-1} \hat{C}_2 & \varepsilon\gamma \hat{D}_{21}^{-1} \hat{D}_{22} & \hat{D}_{21}^{-1} \end{array} \right].$$

Algorithm 7

- Find a reduced order controller $\hat{K}$ using the following criterion

$$\left\| \left[\begin{array}{c} \varepsilon\gamma M_{22} \\ I \end{array} \right] M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

The related state space realization of the relevant transfer matrix is given by

$$\left[\begin{array}{c} \varepsilon\gamma M_{22} \\ I \end{array} \right] M_{12}^{-1} \; = \; \left[\begin{array}{c|c} \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{B}_2 \hat{D}_{12}^{-1} \\ \hline \varepsilon\gamma \left(\hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1 \right) & \varepsilon\gamma \hat{D}_{22} \hat{D}_{12}^{-1} \\ -\hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{12}^{-1} \end{array} \right].$$

5.3 Computational Issues in Frequency-Weighted Balanced Controller Reduction

In this section, we shall look at how the frequency-weighted balanced model reduction method in Chapter 3 can be used to solve the controller reductions in the last two sections. Define

$$G = K_0 = M_{11} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right].$$

The aim is to find G_r, i.e. $\hat{K}$, with lower order such that the following norm is as small as possible:

$$\left\| W_o(\hat{K} - K_0)W_i \right\|_\infty .$$

5.3.1 Computational Issues for Stability Preserving Controller Reduction

Lemma 5 gives several sufficient conditions which guarantee the stability of Q. However, those reduced order controllers are not guaranteed to satisfy $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$ and we have to verify if the closed-loop system performance is preserved by using these controller reduction approaches.

Since

$$\begin{aligned} M_{21}^{-1}M_{22} &= \left[\begin{array}{c|c} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1}\hat{D}_{22} \end{array}\right] \\ M_{22}M_{12}^{-1} &= \left[\begin{array}{c|c} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{22}\hat{D}_{12}^{-1} \end{array}\right]. \end{aligned}$$

Hence

$$M_{21}^{-1}M_{22}M_{12}^{-1} = \left[\begin{array}{cc|c} \hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & (\hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1}\hat{C}_1 & (\hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1} \\ 0 & \hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1} \end{array}\right].$$

Theorem 16 *Suppose that $\hat{A}$ is stable. Then*

(a) Corresponding to the controller reduction method in Lemma 5(a), where $W_o = I$ and $W_i = M_{21}^{-1}M_{22}M_{12}^{-1}$, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}\begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 \\ 0 & \hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & (\hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1}\hat{C}_1 \\ 0 & 0 & \hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 \end{bmatrix}$$

$$\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix} = \begin{bmatrix} \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1} \\ (\hat{B}_2-\hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1}\hat{C}_1 \\ -\hat{B}_2\hat{D}_{12}^{-1} \end{bmatrix}$$

$$Q\hat{A} + \hat{A}^*Q + \hat{C}_1^*\hat{C}_1 = 0.$$

(b) Corresponding to the controller reduction method in Lemma 5(b), where $W_i = I$ and $W_o = M_{21}^{-1}M_{22}M_{12}^{-1}$, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

$$P\hat{A}^* + \hat{A}P + \hat{B}_1\hat{B}_1^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}\begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix}^*\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix}\begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix}$$

$$= \begin{bmatrix} \hat{A} & 0 & 0 \\ (\hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1}\hat{C}_1 & \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & (\hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22})\hat{D}_{12}^{-1}\hat{C}_1 \\ -\hat{B}_2\hat{D}_{21}^{-1}\hat{C}_1 & 0 & \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 \end{bmatrix}$$

$$\begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} \hat{C}_1^*(\hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1})^* \\ (\hat{D}_{21}^{-1}\hat{C}_2)^* \\ (\hat{D}_{21}^{-1}\hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1)^* \end{bmatrix}.$$

(c) *Corresponding to the controller reduction method in Lemma 5(c), where* $W_o = M_{12}^{-1}$ *and* $W_i = M_{21}^{-1}M_{22}$, *the input weighted gramian* P *and the output weighted gramian* Q *can be computed from the following Lyapunov equations*

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_GD_i \\ B_i \end{bmatrix} \begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \\ 0 & \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \end{bmatrix}$$

$$\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix} = \begin{bmatrix} \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \end{bmatrix}$$

$$Q(\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1) + (\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1)^*Q + (\hat{D}_{12}^{-1}\hat{C}_1)^*\hat{D}_{12}^{-1}\hat{C}_1 = 0.$$

(d) *Corresponding to the controller reduction method in Lemma 5(d), where* $W_i = M_{21}^{-1}$ *and* $W_o = M_{22}M_{12}^{-1}$, *the input weighted gramian* P *and the output weighted gramian* Q *can be computed from the following Lyapunov equations*

$$(\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2)P + P(\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2)^* + \hat{B}_1\hat{D}_{21}^{-1}(\hat{B}_1\hat{D}_{21}^{-1})^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 \end{bmatrix}$$

$$\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} (\hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1)^* \\ (\hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1)^* \end{bmatrix}.$$

5.3.2 Computational Issues for Performance Preserving Controller Reduction

In Section 5.2, some controller reduction methods are proposed to guarantee the closed-loop system performance, that is, $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$. Similarly, we shall look at how the frequency weighted balanced model reduction method can be used to derive reduced order controllers which satisfy system performance requirements. We shall start with the simple case

$$\left\| M_{12}^{-1} (\hat{K} - K_0) M_{21}^{-1} \right\|_\infty .$$

Define $W_i = M_{21}^{-1}$ and $W_o = M_{12}^{-1}$.

Theorem 17 *Suppose $\hat{A}$ is stable. Then the input weighted gramian P and the output weighted gramian Q in Algorithm 5 can be computed from the following Lyapunov equations*

$$(\hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2) P + P (\hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2)^* + \hat{B}_1 \hat{D}_{21}^{-1} (\hat{B}_1 \hat{D}_{21}^{-1})^* = 0$$

$$Q (\hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1) + (\hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1)^* Q + (\hat{D}_{12}^{-1} \hat{C}_1)^* \hat{D}_{12}^{-1} \hat{C}_1 = 0.$$

Proof Note that in equations (3.1) and (3.2),

$$\begin{aligned}
\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} &= \begin{bmatrix} \hat{A} & \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 \\ 0 & \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 \end{bmatrix} \\
\begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} &= \begin{bmatrix} \hat{B}_1 \hat{D}_{21}^{-1} \\ -\hat{B}_1 \hat{D}_{21}^{-1} \end{bmatrix} \\
\begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} &= \begin{bmatrix} \hat{A} & 0 \\ -\hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 \end{bmatrix} \\
\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} &= \begin{bmatrix} \hat{C}_1^* (\hat{D}_{12}^{-1}) \\ \hat{C}_1^* (\hat{D}_{12}^{-1})^* \end{bmatrix}.
\end{aligned}$$

Then

$$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} + \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix}^* = 0$$

$$\begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0.$$

□

Similarly, we have the following results.

Theorem 18 *Suppose $\hat{A}$ is stable. Then*

(a) The input weighted gramian P and the output weighted gramian Q in Algorithm 6 can be computed from the following Lyapunov equations

$$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 \\ 0 & \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 \end{bmatrix}$$

$$\begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} = \begin{bmatrix} \varepsilon\gamma \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22} & \hat{B}_1 \hat{D}_{21}^{-1} \\ \varepsilon\gamma \left(\hat{B}_2 - \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22}\right) & -\hat{B}_1 \hat{D}_{21}^{-1} \end{bmatrix}$$

$$Q(\hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1) + (\hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1)^* Q + (\hat{D}_{12}^{-1} \hat{C}_1)^* \hat{D}_{12}^{-1} \hat{C}_1 = 0.$$

(b) *The input weighted gramian P and the output weighted gramian Q in Algorithm 7 can be computed from the following Lyapunov equations*

$$(\hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2) P + P(\hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2)^* + \hat{B}_1 \hat{D}_{21}^{-1} (\hat{B}_1 \hat{D}_{21}^{-1})^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 \end{bmatrix}$$

$$\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} \varepsilon\gamma (\hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1)^* & (\hat{D}_{12}^{-1} \hat{C}_1)^* \\ \varepsilon\gamma \left(\hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1\right)^* & -(\hat{D}_{12}^{-1} \hat{C}_1)^* \end{bmatrix}.$$

Now let T be a nonsingular matrix such that

$$TPT^* = (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) with $\Sigma_1 = \operatorname{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r})$ and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$ and partition full order controller accordingly as

$$K_0 = \left[\begin{array}{c|c} T\hat{A}T^{-1} & T\hat{B}_1 \\ \hline \hat{C}_1 T^{-1} & \hat{D}_{11} \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ A_{21} & A_{22} & B_{12} \\ \hline C_{11} & C_{12} & 0 \end{array}\right].$$

Then a reduced-order controller $\hat{K}$ is obtained as

$$\hat{K} = \left[\begin{array}{c|c} A_{11} & B_{11} \\ \hline C_{11} & D_{11} \end{array}\right].$$

It is important to verify if this reduced order controller will satisfy the $\mathscr{H}_\infty$ performance.

Note that if $\hat{A}$ is not stable, then we need to write

$$K_0 = K_{0s} + K_{0u}$$

such that K_{0s} is stable and K_{0u} is antistable. Now let the reduced order controller be

$$\hat{K} = \hat{K}_{0s} + K_{0u}$$

such that $\hat{K}_{0s}$ is a stable approximation of K_{0s} obtained using any algorithm proposed above.

Chapter 6

Examples

The four disk example and the HIMAT example are explored in this chapter to illustrate the effectiveness of those proposed controller reduction methods.

6.1 Four Disk Example

We shall consider the four disk control system studied by Enns [1984] in this chapter, and then compare the simulation results derived from our methods to those recent reduced controller methods shown in [62]. Note that (frequency weighted) balance reduction method is applied in this section. We shall set up the dynamical system in the standard linear fractional transformation form

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= \begin{bmatrix} \sqrt{q_1} H \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\
y &= C_2 x + \begin{bmatrix} 0 & I \end{bmatrix} w
\end{aligned}$$

where $q_1 = 1 \times 10^{-6}, q_2 = 1$ and

$$B_1 = \begin{bmatrix} \sqrt{q_2} B_2 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 6.4432 \times 10^{-3} & 2.3196 \times 10^{-3} & 7.1252 \times 10^{-2} & 1.0002 & 0.10455 & 0.99551 \end{bmatrix}$$

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -3.982 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The optimal $\mathscr{H}_\infty$ norm for T_{zw} is $\gamma_{opt} = 1.1272$ and one 8th order suboptimal controller is derived. Here, $\gamma = 1.2$ is chosen to design an $\mathscr{H}_\infty$ controller for the system and the resulting controller is an 8^{th} order controller K_0 which satisfies $\|\mathcal{F}_\ell(G, K_0)\|_\infty < 1.2$. Let the coprime factorizations of K_0 be $K_0 = \Theta_{12}\Theta_{22}^{-1} = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$ as obtained from Lemma 4. We can also calculate the matrix M_∞ for this system when $\gamma = 1.2$:

$$M_\infty = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right].$$

The controller K_0 is reduced using several methods and the comparison of results is listed in Table 6.1 and where some abbreviations are made:

UWA Unweighted additive reduction

$$\left\| K_0 - \hat{K} \right\|_\infty$$

UWRCF Unweighted right coprime factor reduction

$$\left\| \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right\|_\infty$$

UWLCF Unweighted left coprime factor reduction

$$\left\| \begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right\|_\infty$$

SWA Stability weighted additive reduction

$$\left\| W_a(K_0 - \hat{K}) \right\|_\infty$$

SWRCF Stability weighted right coprime factor reduction

$$\left\| \begin{bmatrix} -\tilde{N}_n & \tilde{M}_n \end{bmatrix} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty$$

SWLCF Stability weighted left coprime factor reduction

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \begin{bmatrix} -N_n \\ M_n \end{bmatrix} \right\|_\infty$$

PWA Performance weighted additive reduction

$$\left\| W_2^{-1}(K_0 - \hat{K}) W_1^{-1} \right\|_\infty$$

PWRCF Performance weighted right coprime factor reduction

$$\left\| \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty$$

PWLCF Performance weighted left coprime factor reduction

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty$$

Some tables show the comparision results between our proposed controller reduction methods and some other effective order reduction methods. Several facts are worth noting:

Order of $\hat{K}$	7	6	5	4	3	2	1	0
PWA	1.196	1.196	1.199	1.197	U	4.99	U	U
PWRCF	1.2	1.196	1.207	1.195	2.98	1.674	U	U
PWLCF	1.197	1.196	U	1.197	U	U	U	U
UWA	U	1.321	U	U	U	U	U	U
UWRCF	1.198	1.196	1.199	1.196	U	U	U	U
UWLCF	1.985	1.258	27.04	5.059	U	U	U	U
SWA	1.327	1.199	2.27	1.47	23.5	U	U	U
SWRCF	1.236	1.197	1.251	1.201	13.91	1.415	U	U
SWLCF	1.417	1.217	48.04	3.031	U	U	U	U
$\left\Vert\Delta_K M_{21}^{-1} M_{22} M_{12}^{-1}\right\Vert_\infty$	1.1966	1.1964	1.1997	1.1963	U	2.9775	U	U
$\left\Vert M_{21}^{-1} M_{22} M_{12}^{-1}\Delta_K\right\Vert_\infty$	1.1966	1.1964	1.1997	1.1963	U	2.9775	U	U
$\left\Vert M_{12}^{-1}\Delta_K M_{21}^{-1} M_{22}\right\Vert_\infty$	U	1.1964	U	1.1968	U	U	U	U
$\left\Vert M_{22} M_{12}^{-1}\Delta_K M_{21}^{-1}\right\Vert_\infty$	U	1.1964	U	1.1969	U	U	U	U
$\left\Vert M_{12}^{-1}\Delta_K M_{21}^{-1}\right\Vert_\infty$	U	1.1964	U	1.1967	U	U	U	U

Table 6.1: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller: U–closed-loop system is unstable

Remark 7 *In all our proposed methods, it is obvious that the performance degradation resulting from order reduction is very small compared to the other methods, except the cases where the closed-loop system is unstable.*

Remark 8 *PWA and PWRCF (PWRLF) are usually preferred methods to obtain reduced controllers in order to guarantee the stability and performance. Table 6.1 shows that the proposed controller reduction methods also work as well as PWA and PWRCF (PWRLF). However, the advantages of our proposed methods include that they require less computation*

and resulting weighting functions can be easily derived from the representation for the $\mathscr{H}_\infty$ controller problem, i.e., M_∞.

Remark 9 *The controller reduction methods in Lemma 5(a) and Lemma 5(b) were used in [45] to obtain reduced order controllers. Their simulation results are verified here in the cases of $\left\|\Delta_K M_{21}^{-1} M_{22} M_{12}^{-1}\right\|_\infty$ and $\left\|M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K\right\|_\infty$. The result shows that these controller reduction methods work almost as well as performance weighted additive reduction (PWA). They can guarantee the closed-loop system stability and performance as well for this example. They also support Theorem 6 that the weighted balanced realization of scalar G with one-side scalar weighting function W does not change whether W is input weighting function or W is output weighting function. Hence, the weighted approximations of $\left\|\Delta_K M_{21}^{-1} M_{22} M_{12}^{-1}\right\|_\infty$ and $\left\|M_{21}^{-1} M_{22} M_{12}^{-1} \Delta_K\right\|_\infty$ are equal for the same reduced order.*

Remark 10 *The controller reduction methods resulting from Lemma 5(c) and Lemma 5(d) can also stablilize the closed-loop system and satisfy the performance requirement when the order of controller is reduced to the lower order. Although the closed-loop system is unstable with some reduced order controller, the result from these two controller reduction approaches are still impressive.*

Remark 11 *Similarly, the controller reduction approach from Algorithm 5 is also effective when the controller is reduced to the sixth and the fourth orders. Here, we put weighting functions to one side and get the results listed in Table 6.2. But it is emphasized that the last two rows in Table 6.2 are included here only to show that applying balanced reduction methods*

to $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}\right\|_\infty$, $\left\|M_{21}^{-1}M_{12}^{-1}\Delta_K\right\|_\infty$, $\left\|\Delta_K M_{21}^{-1}M_{12}^{-1}\right\|_\infty$ *might not lead to the same result even for SISO system, which has been shown in Example 1.*

Order of $\hat{K}$	7	6	5	4	3	2	1	0
$\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}\right\|_\infty$	U	1.1964	U	1.1967	U	U	U	U
$\left\|\Delta_K M_{21}^{-1}M_{12}^{-1}\right\|_\infty$	1.1970	1.1964	1.1995	1.1963	U	3.1116	U	U
$\left\|M_{21}^{-1}M_{12}^{-1}\Delta_K\right\|_\infty$	1.1970	1.1964	1.1995	1.1963	U	3.1116	U	U

Table 6.2: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller: U–closed-loop system is unstable

Remark 12 *Actually, the result in Table 6.1 is not perfect as we have hoped. For example, the closed-loop system is unstable by using method* $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}\right\|_\infty$ *when the controller is reduced to the fifth order. One possible reason is that the frequency weighted balanced model reduction method is not optimal; another reason might be due to the skewed problem of weighting functions since the maximal Hankel singular value of* $M_{21}^{-1}M_{22}$ *is 217.8523 and the minimal Hankel singular value is 0.6559. This ill-conditioned transfer function might lead to unsatisfactory results.*

In order to solve the ill-conditioned problem, we truncate the weighting functions to lower orders according to the Hankel singular values. For example, for $\left\|M_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}\right\|_\infty$ *case, when truncating both the input and output weighting functions to the fourth order, it is found that the closed-loop system is stable with the fifth order controller and the performance value* $\gamma = 1.1980 < 1.2$ *as shown in Table 6.4. However, we cannot conclude that better or worse results will be obtained after truncating weighting function because it depends each specific case. Some truncation results are shown as follows:*

Order of $\hat{K}$	7	6	5	4	3	2	1	0
No truncating	1.1966	1.1964	1.1997	1.1963	U	2.9775	U	U
Truncate W_i to order 10	1.1966	1.1964	1.1998	1.1963	U	2.9655	U	U
Truncate W_i to order 6	1.1964	1.1964	**1.2002**	1.1963	U	2.9516	U	U
Truncate W_i to order 4	1.1967	1.1964	1.1990	1.1964	U	2.9449	U	U
Truncate W_i to order 2	1.1971	1.1964	1.1999	1.1963	U	8.9027	U	U

Table 6.3: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller and with truncated weighting functions: U–closed-loop system is unstable: $W_i = M_{21}^{-1} M_{22} M_{12}^{-1}$: $W_o = I$

(a) *$W_i = M_{21}^{-1} M_{22} M_{12}^{-1}$ (Table 6.3). The Hankel singular values of W_i are: 854.5507; 230.2799; 16.0836; 15.5794; 1.5845; 1.5621; 0.9230; 0.8540; 0.6553; 0.5075; 0.3203; 0.3189; 0.0909; 0.0884; 0.0207; 0.0198. Table 6.3 shows that the performance does not change too much after truncating the input weighting function. If the input weighting function W_i is truncated to 6th order, the result becomes even worse ($\gamma = 1.2002 > 1.2$) when the controller is reduced to the 5th order. It shows that worsening results might be obtained with truncating weighting functions. As a result, it is not concluded that truncating weighting functions can bring better or worse results. We have to verify for each specific case. The case of $W_o = M_{21}^{-1} M_{22} M_{12}^{-1}$ is omitted here because of the similar structure.*

(b) *$W_i = M_{21}^{-1} M_{22}$ and $W_o = M_{12}^{-1}$ (Table 6.4, 6.5, 6.6). The Hankel singular values of W_i are: 217.8523; 55.1882; 6.9652; 6.7540; 1.5931; 1.5385; 0.6824; 0.6559. The Hankel singular values of W_o are: 1.7753; 0.5515; 0.3527; 0.3101; 0.2446; 0.1301; 0.1269; 0.0880. Table 6.4 shows that the system is stable with the 5th order controller when W_i*

and W_o are truncated to the fourth order. And no matter how to truncate weighting function in our list, the closed-loop system is not stable with the first order controller.

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1963	1.1964	1.2135	1.1966	U	U	U	U
Truncate W_i to order 4	1.1964	1.1964	**1.1980**	1.1964	U	U	U	U
Truncate W_i to order 2	1.1964	1.1964	1.1997	1.1965	U	1.9621	U	U
Truncate W_i to order 1	1.1979	1.1970	U	1.1975	U	U	U	U

Table 6.4: $W_o = M_{12}^{-1}$ is truncated to 4 order : $W_i = M_{21}^{-1} M_{22}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1963	1.1964	1.2124	1.1966	U	U	U	U
Truncate W_i to order 4	1.1964	1.1964	1.1979	1.1964	U	U	U	U
Truncate W_i to order 2	1.1964	1.1964	1.1995	1.1965	2.6227	2.0055	U	U
Truncate W_i to order 1	1.1978	1.1967	U	1.1975	U	U	U	U

Table 6.5: $W_o = M_{12}^{-1}$ is truncated to 2 order : $W_i = M_{21}^{-1} M_{22}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1964	1.1964	1.2138	1.1965	16.4578	U	U	U
Truncate W_i to order 4	1.1965	1.1964	1.1978	1.1964	15.2623	U	U	U
Truncate W_i to order 2	1.1965	1.1964	U	1.1964	13.8309	1.8294	U	U
Truncate W_i to order 1	U	1.1970	U	1.1973	U	U	U	U

Table 6.6: $W_o = M_{12}^{-1}$ is truncated to 1 order : $W_i = M_{21}^{-1} M_{22}$

(c) $W_o = M_{22} M_{12}^{-1}$ and $W_i = M_{21}^{-1}$ (Table 6.7, 6.8, 6.9). The Hankel singular values of W_i are: 253.1167; 61.6390; 7.4828; 7.4466; 1.0619; 1.0297; 0.4553; 0.4365. The Hankel singular values of W_o are: 2.1450; 0.8473; 0.5902; 0.5811; 0.4909; 0.3267; 0.2627;

0.1101. As shown in Table 6.1, the system is unstable with the 7th order controller. But from the tables below, it is shown that the system is stabilized with one 7th order controller if weighting functions are truncated.

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1965	1.1964	U	1.1967	U	U	U	U
Truncate W_i to order 4	1.1966	1.1964	U	1.1965	U	U	U	U
Truncate W_i to order 2	1.1968	1.1965	U	1.1966	U	2.3753	U	U
Truncate W_i to order 1	1.2022	1.1971	U	1.1979	U	U	U	U

Table 6.7: $W_o = M_{22}M_{12}^{-1}$ is truncated to 6 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1963	1.1964	U	1.1967	U	U	U	U
Truncate W_i to order 4	1.1963	1.1964	U	1.1965	U	U	U	U
Truncate W_i to order 2	1.1963	1.1964	U	1.1966	U	2.1639	U	U
Truncate W_i to order 1	1.1978	1.1968	U	1.1982	U	U	U	U

Table 6.8: $W_o = M_{22}M_{12}^{-1}$ is truncated to 4 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1965	1.1964	U	1.1966	U	U	U	U
Truncate W_i to order 4	1.1966	1.1964	U	1.1964	U	U	U	U
Truncate W_i to order 2	1.1968	1.1964	U	1.1965	2.1696	3.4787	U	U
Truncate W_i to order 1	1.2015	1.1967	U	1.1974	U	U	U	U

Table 6.9: $W_o = M_{22}M_{12}^{-1}$ is truncated to 1 order : $W_i = M_{21}^{-1}$

(d) *$W_o = M_{12}^{-1}$ and $W_i = M_{21}^{-1}$ (Table 6.10, 6.11, 6.12, 6.13). The Hankel singular values of W_i are: 253.1167; 61.6390; 7.4828; 7.4466; 1.0619; 1.0297; 0.4553; 0.4365. The*

Hankel singular values of W_o are: 1.7753; 0.5515; 0.3527; 0.3101; 0.2446; 0.1301; 0.1269; 0.0880. The tables here show that the closed-loop system can be stable with seventh order controller for some truncated W_i and W_o.

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	U	1.1964	U	1.1967	U	U	U	U
Truncate W_i to order 4	U	1.1964	1.1981	1.1965	U	U	U	U
Truncate W_i to order 2	U	1.1964	U	1.1966	U	1.8134	U	U
Truncate W_i to order 1	U	1.1969	U	1.1979	U	U	U	U

Table 6.10: $W_o = M_{12}^{-1}$ is truncated to 7 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1964	1.1964	1.2067	1.1966	U	U	U	U
Truncate W_i to order 4	1.1964	1.1964	1.1974	1.1964	U	U	U	U
Truncate W_i to order 2	1.1965	1.1964	1.1993	1.1965	U	1.8757	U	U
Truncate W_i to order 1	U	1.1970	U	1.1975	U	U	U	U

Table 6.11: $W_o = M_{12}^{-1}$ is truncated to 4 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1964	1.1964	1.2062	1.1966	U	U	U	U
Truncate W_i to order 4	1.1964	1.1964	1.1973	1.1964	U	U	U	U
Truncate W_i to order 2	1.1965	1.1964	1.1991	1.1965	2.2091	1.8750	U	U
Truncate W_i to order 1	U	1.1970	U	1.1975	U	U	U	U

Table 6.12: $W_o = M_{12}^{-1}$ is truncated to 2 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
Truncate W_i to order 6	1.1965	1.1964	1.2060	1.1965	37.6697	U	U	U
Truncate W_i to order 4	1.1965	1.1964	1.1970	1.1964	36.4878	U	U	U
Truncate W_i to order 2	U	1.1964	1.1986	1.1965	U	2.3034	U	U
Truncate W_i to order 1	U	1.1970	U	1.1972	U	U	U	U

Table 6.13: $W_o = M_{12}^{-1}$ is truncated to 1 order : $W_i = M_{21}^{-1}$

Order of $\hat{K}$	7	6	5	4	3	2	1	0
$\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty$	U	1.1964	U	1.1967	U	U	U	U
ε=0	U	1.1964	U	1.1967	U	U	U	U
ε=0.1	U	1.1964	U	1.1967	U	U	U	U
ε=1	U	1.1964	U	1.1968	U	U	U	U
ε=10	U	1.1964	U	1.1968	U	U	U	U
ε=100	U	1.1964	U	1.1968	U	U	U	U
$\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} M_{22} \right\|_\infty$	U	1.1964	U	1.1968	U	U	U	U

Table 6.14: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller: U– closed-loop system is unstable

Order of $\hat{K}$	7	6	5	4	3	2	1	0
$\left\| M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty$	U	1.1964	U	1.1967	U	U	U	U
ε=0	U	1.1964	U	1.1967	U	U	U	U
ε=0.1	U	1.1964	U	1.1967	U	U	U	U
ε=1	U	1.1964	U	1.1968	U	U	U	U
ε=10	U	1.1964	U	1.1969	U	U	U	U
ε=100	U	1.1964	U	1.1969	U	U	U	U
$\left\| M_{22} M_{12}^{-1} \Delta_K M_{21}^{-1} \right\|_\infty$	U	1.1964	U	1.1969	U	U	U	U

Table 6.15: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller: U– closed-loop system is unstable

Remark 13 *The results in Table 6.14 and Table 6.15 support those controller reduction approaches explored in Algorithm 6 and Algorithm 7.*

Note that $\varepsilon > 0$ *from Algorithm 6 and Algorithm 7 should be used as a design parameter. One may start from* $\varepsilon = 0$ *and in this case the controller reduction methods* $\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \begin{bmatrix} \varepsilon\gamma M_{22} & I \end{bmatrix} \right\|_\infty$ *and* $\left\| \begin{bmatrix} \varepsilon\gamma M_{22} \\ I \end{bmatrix} M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty$ *are reduced to* $\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty$. *However, in this case, there is no guarantee if the reduced controller will satisfy the* $\mathscr{H}_\infty$ *performance and the exact* $\mathscr{H}_\infty$ *performance has to be verified for each reduced order controller. One the other hand, when* ε *is very large, the methods are equivalent to* $\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}M_{22} \right\|_\infty$ *and* $\left\| M_{22}M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty$. *Again the exact* $\mathscr{H}_\infty$ *performance resulting from these criteria has to be verified for each reduced order controller.*

Order of $\hat{K}$		7	6	5	4	3	2	1
$\left\| M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty$	B	U	1.1964	U	1.1967	U	U	U
	H	1.1960	1.2018	1.1973	1.2111	2.8532	1.4127	U

Table 6.16: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller: U–closed-loop system is unstable. B: Balance reduction with or without weighting. H: Hankel reduction with or without weighting.

Remark 14 *We have looked at how the frequency weighted balanced model reduction technique can be used to solve the controller reductions in above sections. Since Hankel norm approximation is also one important and powerful reduction method, we are interested in exploring the results from this two different model reduction methods. The result for* $\left\| M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty$ *is listed in the Table 6.16. The table shows that, for the performance weighted controller reduction method* $\left\| M_{12}^{-1}\Delta_K M_{21}^{-1} \right\|_\infty$, *the Hankel reduction methods works better than balance*

reduction in most cases. It can stabilize system with most reduced controllers, and also satisfy the performance requirement with the 7th and the 5th order controllers.

6.2 HIMAT Example

HIMAT control problem [7, 63] is considered to design reduced order robust controllers in this chapter. The system diagram is shown in Figure 6.1 where

$$W_{del} = \begin{bmatrix} \frac{50(s+100)}{s+10000} & 0 \\ 0 & \frac{50(s+100)}{s+10000} \end{bmatrix}, W_p = \begin{bmatrix} \frac{0.5(s+0.018)}{s+0.018} & 0 \\ 0 & \frac{0.5(s+0.018)}{s+0.018} \end{bmatrix}, W_n = \begin{bmatrix} \frac{2(s+1.28)}{s+320} & 0 \\ 0 & \frac{2(s+1.28)}{s+320} \end{bmatrix}$$

$$G_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right]$$

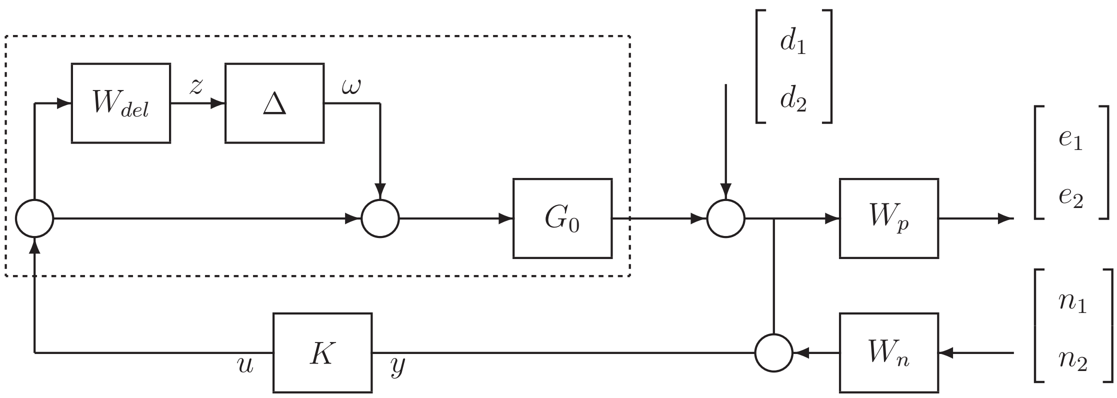


Figure 6.1: HIMAT Closed Loop Interconnection.

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Let $T(s)$ denote the transfer function from disturbances $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and noises $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ to errors $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. The robust performance objective is to design a controller $K(s)$ so that

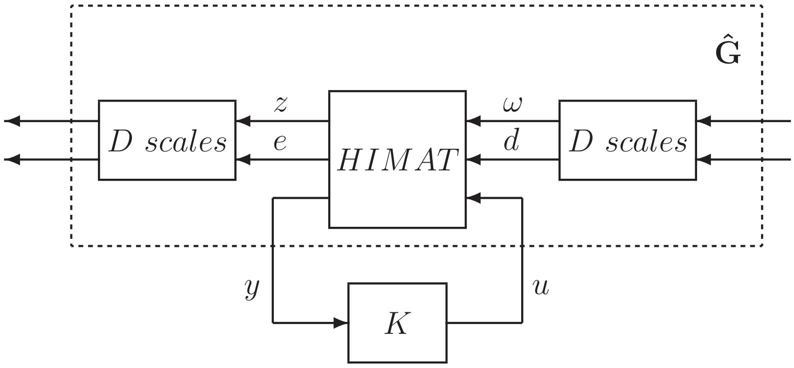

Figure 6.2: HIMAT LFT Interconnection.

$\|T(s)\|_\infty < 1$ for all $\|\Delta\|_\infty < 1$. First of all, the HIMAT closed-loop interconnection of Figure 6.1 is converted to LFT interconnection [19] shown in Figure 6.2 where $e = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$, $d = \begin{bmatrix} d_1 & d_2 & n_1 & n_2 \end{bmatrix}^T$ and $D\ scales$ are from D-K iterations. One 30 states controller is designed in [7] using D-K iteration to achieve the robust performance objective $\gamma = 0.97$. The aim of this chapter is to design one reduced order robust controller which preserves robust performance, i.e., $\left\|\mathcal{F}_\ell(\hat{G}, K)\right\|_\infty < 1$. Controller reduction methods proposed in this thesis are used in reducing robust controller order. Some of our findings are presented and the following representations are used in Tables:

Criterion A: Controller reduction via criterion $\left\|M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}\right\|_\infty$;

Criterion B: Controller reduction via criterion $\left\|M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}M_{22}\right\|_\infty$;

Criterion C: Controller reduction via criterion $\left\|M_{22}M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}\right\|_\infty$;

Criterion D: Controller reduction via criterion $\left\|(\hat{K} - K_0)M_{21}^{-1}M_{22}M_{12}^{-1}\right\|_\infty$;

Criterion E: Controller reduction via criterion $\left\|M_{21}^{-1}M_{22}M_{12}^{-1}(\hat{K} - K_0)\right\|_\infty$.

Remark 15 *The bold type numbers in Table 6.18 represent the performance value, where the HIMAT system preserves robust performance with the lowest order robust controller for*

different controller reduction methods. For example, one 11th order controller can be obtained in Table 6.18 to achieve performance objective $\gamma = 0.9998 < 1$ *via controller reduction criterion* $\left\| M_{22}M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty$.

Orders of $\hat{K}$	Criterion A	Criterion B	Criterion C	Criterion D	Criterion E
29	0.9998	0.9998	0.9998	0.9998	0.9998
28	U	0.9998	U	0.9998	0.9998
27	0.9998	0.9998	0.9998	0.9998	0.9998
26	0.9998	0.9998	0.9998	0.9998	0.9998
25	0.9998	0.9998	0.9998	0.9998	0.9998
24	0.9998	0.9998	0.9998	0.9998	0.9998
23	0.9998	0.9998	0.9998	0.9998	0.9998
22	0.9998	0.9998	0.9998	0.9998	0.9998
21	0.9998	0.9998	0.9998	0.9998	0.9998
20	0.9998	0.9998	0.9998	0.9998	0.9998
19	0.9998	0.9998	0.9998	0.9998	0.9998
18	0.9998	U	0.9998	0.9998	0.9998
17	0.9998	0.9998	0.9998	0.9998	0.9998
16	0.9998	0.9998	0.9998	0.9998	0.9998
15	0.9998	0.9998	0.9998	0.9998	0.9998
14	0.9998	0.9998	0.9998	**0.9998**	0.9998
13	**0.9998**	**0.9998**	0.9998	1.0000	**0.9999**
12	1.0503	1.0010	0.9999	1.0191	1.0167
11	1.0579	1.0012	**0.9998**	1.0260	1.0254
10	1.0099	1.0005	1.0013	1.0328	1.0458
9	1.0468	1.0018	1.0159	1.1566	1.1641
8	1.0054	1.0479	1.0150	1.1691	1.3650

7	1.0238	1.0158	1.0144	1.2433	1.3271
6	1.0242	1.0163	1.0138	1.2374	1.3231
5	U	U	U	3.9824	4.0933
4	U	U	U	U	U
3	U	U	U	U	U
2	U	U	U	U	U
1	U	U	U	U	U
0	6557.088	6557.088	6557.088	6557.088	6557.088

Table 6.17: $\mathcal{F}_\ell(G,\hat{K})$ with reduced order controller:U– closed-loop system is unstable

Remark 16 *Similar to the four disk example, Table 6.18 almost supports the facts:*

$$\left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\left[\begin{array}{cc}\varepsilon\gamma M_{22} & I\end{array}\right]\right\|_\infty \xrightarrow{\varepsilon\to\infty} \left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}M_{22}\right\|_\infty;$$
$$\left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\left[\begin{array}{cc}\varepsilon\gamma M_{22} & I\end{array}\right]\right\|_\infty \xrightarrow{\varepsilon\to 0} \left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty;$$
$$\left\|\left[\begin{array}{c}\varepsilon\gamma M_{22}\\ I\end{array}\right]M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty \xrightarrow{\varepsilon\to\infty} \left\|M_{22}M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty;$$
$$\left\|\left[\begin{array}{c}\varepsilon\gamma M_{22}\\ I\end{array}\right]M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty \xrightarrow{\varepsilon\to 0} \left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty.$$

$\hat{K}$	**Cri. B**	ε=10	ε=1	ε=0.1	**Cri. A**	ε=0.1	ε=1	ε=10	**Cri. C**
29	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
28	**0.9998**	0.9998	0.9998	0.9998	**U**	U	U	U	**U**
27	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
26	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
25	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
24	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
23	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**

22	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
21	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
20	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
19	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
18	**U**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
17	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
16	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
15	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
14	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
13	**0.9998**	0.9998	0.9998	0.9998	**0.9998**	0.9998	0.9998	0.9998	**0.9998**
12	**1.0010**	1.0011	1.0190	1.0504	**1.0503**	1.0492	1.0112	0.9998	**0.9998**
11	**1.0012**	1.0011	1.0044	1.0662	**1.0579**	1.0544	1.0013	0.9998	**0.9998**
10	**1.0005**	1.0005	1.0024	1.0098	**1.0099**	1.0096	1.0028	1.0014	**1.0013**
9	**1.0018**	1.0017	1.0222	1.0461	**1.0468**	1.0465	1.0372	1.0163	**1.0159**
8	**1.0479**	1.0473	1.0199	1.0128	**1.0054**	1.0221	1.0155	1.0151	**1.0150**
7	**1.0158**	1.0159	1.0197	1.0238	**1.0238**	1.0237	1.0188	1.0145	**1.0144**
6	**1.0157**	1.0164	1.0201	1.0241	**1.0242**	1.0241	1.0192	1.0139	**1.0138**
5	**U**	U	U	U	**U**	U	U	U	**U**

Table 6.18: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller:U–closed-loop system is unstable

Remark 17 *It has been known from the four disk example, truncating on weighting functions sometimes can impact controller reduction results. However, it is hard to conclude whether truncated weighting will arrive better result or not. In HIMAT example, we shall continue to explore how truncating weighting can affect controller reduction. For example, the criterion* $\left\| M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1} \right\|_\infty$ *is used to reduce controller and guarantee robust performance. The results in Table 6.19 display that slightly higher order controller can be gained by*

truncating M_{12}^{-1} and M_{21}^{-1} to the 2nd order, but low order weighting functions can contribute greatly to simplify calculation.

Orders of $\hat{K}$	$\left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty$ No truncating	$\left\|M_{12}^{-1}(\hat{K}-K_0)M_{21}^{-1}\right\|_\infty$ $W_o : 2$; $W_i : 2$
15	0.9998	0.9998
14	0.9998	**0.9998**
13	**0.9998**	1.0015
12	1.0503	1.0017
11	1.0579	1.0019

Table 6.19: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller:U– closed-loop system is unstable

Remark 18 *After obtaining reduced order controller via proposed controller reduction algorithms, it is still possible to get better results by optimizing constant term and other parameters of the reduced controller. Here, we only optimize the constant term D and parameter B in $\hat{K} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. The resulting performance is listed in Table 6.20, where the reduced controller is optimized at the parameters D and B. We can see that the controller order is reduced greatly after optimization for Criterion A, Criterion B and Criterion C.*

Orders of	Criterion A		Criterion B		Criterion C	
$\hat{K}$	opt	no opt	opt	no opt	opt	no opt
15	0.9988	0.9998	0.9970	0.9998	0.9966	0.9998
14	0.9990	0.9998	0.9980	0.9998	0.9965	0.9998
13	0.9957	**0.9998**	0.9971	**0.9998**	0.9987	0.9998
12	0.9965	1.0503	0.9968	1.0010	0.9982	0.9999
11	0.9966	1.0579	0.9964	1.0012	0.9976	**0.9998**

10	0.9960	1.0099	0.9990	1.0005	0.9988	1.0013
9	0.9985	1.0468	**0.9985**	1.0018	0.9969	1.0159
8	**0.9992**	0.9997	1.0157	1.0479	0.9980	1.0150
7	1.0031	1.0238	1.0018	1.0158	0.9985	1.0144
6	1.0052	1.0242	1.0014	1.0163	**0.9988**	1.0138

Table 6.20: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller:U– closed-loop system is unstable

Remark 19 *Since the frequency weighted balance realization does not depend on the particular realization of K_0, balancing K_0 before reducing usually can arrive better result in controller reduction. For instance, in Criterion A, one 28th controller can stabilize system and grantee performance if balancing K_0 before reducing.*

Orders of $\hat{K}$	$\left\|M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}\right\|_\infty$ No balancing	$\left\|M_{12}^{-1}(\hat{K} - K_0)M_{21}^{-1}\right\|_\infty$ Balancing K_0
29	0.9998	0.9998
28	**U**	**0.9998**
27	0.9998	0.9998

Table 6.21: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller:U– closed-loop system is unstable

Remark 20 *In order to compare our proposed methods with those in [22], one modified HIMAT closed-loop interconnection without noise weighting W_n is presented in Figure 6.3. By using the proposed controller reduction methods and frequency weighted balance reduction, the corresponding performance results are shown in Table 6.22. It indicates that part of our*

proposed approaches (Criterion C) can get to lower order (11th order) performance preserving controllers than methods mentioned in [22] (which is 14th order) prior to optimization.

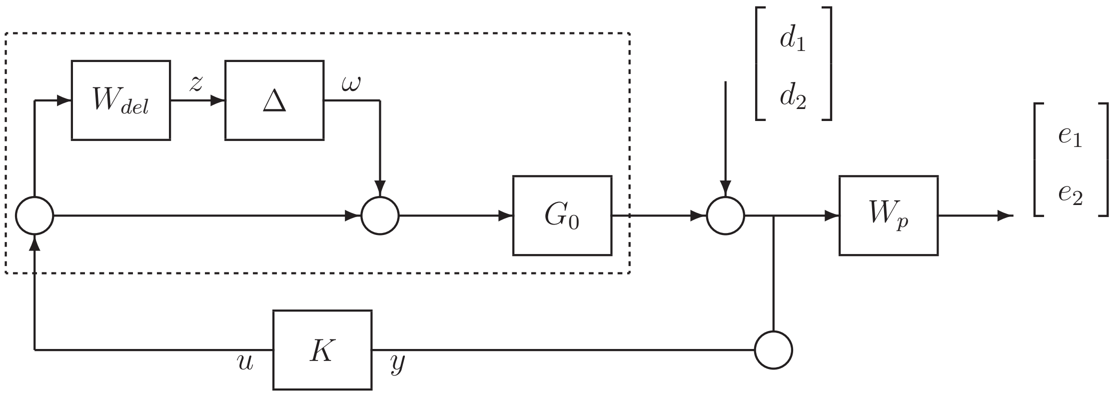

Figure 6.3: Modified HIMAT Closed Loop Interconnection.

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Orders of $\hat{K}$	Criterion A	Criterion B	Criterion C	Criterion D	Criterion E
17	0.99997	0.99997	0.99997	0.99997	0.99997
16	0.99997	0.99997	0.99997	0.99997	**0.99997**
15	0.99997	0.99997	0.99997	0.99997	1.00016
14	0.99997	0.99997	0.99997	**0.99999**	1.00050
13	0.99997	0.99997	0.99997	1.02075	1.00619
12	**0.99998**	**0.99998**	0.99997	1.01066	1.00122
11	1.01493	1.00349	**0.99997**	1.01436	1.04479
10	1.01678	1.00663	1.01573	1.04332	1.25062
9	1.108731	1.20381	1.14818	1.01800	1.18222

Table 6.22: $\mathcal{F}_\ell(G, \hat{K})$ with reduced order controller:U–closed-loop system is unstable

Chapter 7

A New Perspective on $\mathscr{H}_2$ Controller Reduction

Some recent results on $\mathscr{H}_\infty$ controller reduction [28, 45] and discussion in Chapter 5 have motivated us to look at the $\mathscr{H}_2$ controller reduction from a different perspective in this chapter. This chapter considers $\mathscr{H}_2$ controller reduction from all (suboptimal) stabilizing $\mathscr{H}_2$ controller parametrization. It is shown that the closed-loop stability is guaranteed and performance degradation is limited if certain weighted controller reduction errors are small. The proposed $\mathscr{H}_2$ controller reduction is introduced in Section 7.1. In Section 7.2, one example is shown to support the proposed controller reduction methods.

7.1 $\mathscr{H}_2$ Controller Reduction

Recall from Chapter 2 that all controllers that satisfy $\|T_{zw}\|_2 < \gamma$ are given by $K = \mathcal{F}_\ell(H, Q)$ $\left(\left\|R_1^{1/2} Q R_1^{1/2}\right\|_2^2 < \gamma^2 - min\|T_{zw}\|_2^2\right)$. Let $\hat{K}$ be a reduced order controller. Then there must be a Q such that

$$\hat{K}(s) = \mathcal{F}_\ell(H, Q) = H_{11} + H_{12}Q(I - H_{22}Q)^{-1}H_{21}.$$

We can also write Q in terms of

$$\Delta_{K2} := \hat{K} - K_{opt} \tag{7.1}$$

as

$$\Delta_{K2} = H_{12}Q(I - H_{22}Q)^{-1}H_{21} \tag{7.2}$$

$$Q = (I + H_{12}^{-1}\Delta_{K2}H_{21}^{-1}H_{22})^{-1}H_{12}^{-1}\Delta_{K2}H_{21}^{-1}$$

where

$$\begin{aligned}
H_{21}^{-1} &= \left[\begin{array}{c|c} A + B_2F_2 & L_2 \\ \hline -C_2 & I \end{array}\right] \\
H_{12}^{-1} &= \left[\begin{array}{c|c} A + L_2C_2 & -B_2 \\ \hline F_2 & I \end{array}\right] \\
H_{21}^{-1}H_{22} &= \left[\begin{array}{c|c} A + B_2F_2 & B_2 \\ \hline -C_2 & 0 \end{array}\right] \\
H_{22}H_{12}^{-1} &= \left[\begin{array}{c|c} A + L_2C_2 & -B_2 \\ \hline C_2 & 0 \end{array}\right]
\end{aligned}$$

are all stable.

Theorem 19 *Suppose that $\Delta_{K2} = \hat{K} - K_{opt}$ is stable. Then*

$$Q = (I + H_{12}^{-1}\Delta_{K2}H_{21}^{-1}H_{22})^{-1}H_{12}^{-1}\Delta_{K2}H_{21}^{-1}$$

is also stable if one of the following is true

$$\left\|H_{12}^{-1}\Delta_{K2}H_{21}^{-1}H_{22}\right\|_\infty < 1;$$

$$\left\|\Delta_{K2}H_{21}^{-1}H_{22}H_{12}^{-1}\right\|_\infty < 1;$$

$$\left\|H_{21}^{-1}H_{22}H_{12}^{-1}\Delta_{K2}\right\|_\infty < 1;$$

$$\left\| H_{22} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} \right\|_\infty < 1.$$

Furthermore, $\hat{K}$ is a stabilizing controller.

Proof Note that

$$\begin{aligned} Q &= (I + H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22})^{-1} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} \\ &= H_{12}^{-1} (I + \Delta_{K2} H_{21}^{-1} H_{22} H_{12}^{-1})^{-1} \Delta_{K2} H_{21}^{-1} \\ &= H_{12}^{-1} \Delta_{K2} (I + H_{21}^{-1} H_{22} H_{12}^{-1} \Delta_{K2})^{-1} H_{21}^{-1} \\ &= H_{12}^{-1} \Delta_{K2} H_{21}^{-1} (I + H_{22} H_{12}^{-1} \Delta_{K2} H_{21}^{-1})^{-1} \end{aligned}$$

and $H_{12}^{-1}, H_{21}^{-1}, H_{21}^{-1} H_{22}, H_{22} H_{12}^{-1}$ are all stable. Since Δ_{K2} is also assumed to be stable, by the small gain theorem, Q will be stable if one of the following is true

$$\left\| H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22} \right\|_\infty < 1;$$

$$\left\| \Delta_{K2} H_{21}^{-1} H_{22} H_{12}^{-1} \right\|_\infty < 1;$$

$$\left\| H_{21}^{-1} H_{22} H_{12}^{-1} \Delta_{K2} \right\|_\infty < 1;$$

$$\left\| H_{22} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} \right\|_\infty < 1.$$

The stability of Q implies that

$$\hat{K}(s) = \mathcal{F}_\ell(H, Q) = H_{11} + H_{12} Q (I - H_{22} Q)^{-1} H_{21}$$

is a stabilizing controller for the system. $\square$

Remark 21 *If K_{opt} itself is stable, then Δ_{K2} is stable as long as the reduced order controller $\hat{K}$ is stable. On the other hand, if K_{opt} itself is not stable, then we need to write*

$$K_{opt} = K_{0s} + K_{0u}$$

such that K_{0s} is stable and K_{0u} is antistable. Now let the reduced order controller be

$$\hat{K} = \hat{K}_{0s} + K_{0u}$$

such that $\hat{K}_{0s}$ is a stable approximation of K_{0s}. Then Δ_{K2} is also stable.

Hence one way to reduce the order of the controller is to use the above stability conditions as controller approximation criteria.

Algorithm 8: Stability Weighted Controller Reduction

- Let $K_{opt} = K_{0s} + K_{0u}$ such that K_{0s} is stable and K_{0u} is antistable.
- Let $\Delta_{K2} = \hat{K}_{0s} - K_{0s}$ and find a stable lower order $\hat{K}_{0s}$ such that one of the following four conditions is true:
  - (a)

    $$\left\| H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22} \right\|_\infty < 1.$$

    In this case when K_{opt} is stable, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

    $$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & L_2 C_2 \\ 0 & A + B_2 F_2 \end{bmatrix}$$

$$\begin{bmatrix} BD_i \\ B_i \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$Q(A + L_2 C_2) + (A + L_2 C_2)^* Q + F_2^* F_2 = 0.$$

– (b)

$$\left\| \Delta_{K2} H_{21}^{-1} H_{22} H_{12}^{-1} \right\|_\infty < 1.$$

In this case when K_{opt} is stable, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

$$\begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} \begin{bmatrix} B_G D_i \\ B_i \end{bmatrix}^* + \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & B_G C_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & L_2 C_2 & 0 \\ 0 & A + B_2 F_2 & B_2 F_2 \\ 0 & 0 & A + L_2 C_2 \end{bmatrix}$$

$$\begin{bmatrix} B_G D_i \\ B_i \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \\ -B_2 \end{bmatrix}$$

$$Q\hat{A}_2 + \hat{A}_2^* Q + F_2^* F_2 = 0.$$

– (c)

$$\left\| H_{21}^{-1} H_{22} H_{12}^{-1} \Delta_{K2} \right\|_\infty < 1.$$

In this case when K_{opt} is stable, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

$$P\hat{A}_2^* + \hat{A}_2 P + L_2 L_2^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & 0 & 0 \\ B_2 F_2 & A + B_2 F_2 & B_2 F_2 \\ -B_2 F_2 & 0 & A + L_2 C_2 \end{bmatrix}$$

$$\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} 0 \\ -C_2^* \\ 0 \end{bmatrix}.$$

– (d)

$$\left\| H_{22} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} \right\|_\infty < 1.$$

In this case when K_{opt} is stable, the input weighted gramian P and the output weighted gramian Q can be computed from the following Lyapunov equations

$$(A + B_2 F_2) P + P (A + B_2 F_2)^* + L_2 L_2^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0$$

with

$$\begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & 0 \\ -B_2 F_2 & A + L_2 C_2 \end{bmatrix}$$
$$\begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} 0 \\ C_2^* \end{bmatrix}.$$

- The reduced order controller is given by

$$\hat{K} = \hat{K}_{0s} + K_{0u}.$$

To find the optimal reduced order controller, it is necessary to find a $\hat{K}$ such that $\left\| R_1^{1/2} Q R_2^{1/2} \right\|_2^2$ is as small as possible. Since

$$\begin{aligned}
\left\| R_1^{1/2} Q R_2^{1/2} \right\|_2 &= \left\| R_1^{1/2} (I + H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22})^{-1} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_2 \\
&\leq \frac{\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_2}{1 - \left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22} R_1^{-1/2} \right\|_\infty} \\
&\leq \frac{\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_2}{1 - \left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_\infty \left\| R_2^{-1/2} H_{22} R_1^{-1/2} \right\|_\infty}.
\end{aligned}$$

We propose to approximate the controller by performing the frequency weighted balanced reduction on either one of the following errors

$$\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22} R_1^{-1/2} \right\|_\infty$$

or

$$\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_2.$$

Theorem 20 *Suppose that K_{opt} is stable. Then the input weighted gramian P and the output weighted gramian Q in*

$$\left\| R_1^{1/2} H_{12}^{-1} (\hat{K} - K_{opt}) H_{21}^{-1} R_2^{1/2} \right\|_2$$

can be computed from the following Lyapunov equations

$$(A + B_2F_2)P + P(A + B_2F_2)^* + L_2R_2L_2^* = 0$$

$$Q(A + L_2C_2) + (A + L_2C_2)^*Q + F_2^*R_1F_2 = 0.$$

Proof Denote

$$\begin{aligned}
W_i &:= H_{21}^{-1}R_2^{1/2} = \left[\begin{array}{c|c} A + B_2F_2 & L_2R_2^{1/2} \\ \hline -C_2 & R_2^{1/2} \end{array}\right] \\
W_o &:= R_1^{1/2}H_{12}^{-1} = \left[\begin{array}{c|c} A + L_2C_2 & -B_2 \\ \hline R_1^{1/2}F_2 & R_1^{1/2} \end{array}\right] \\
G &= K_{opt} = \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right] = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right]
\end{aligned}$$

Then

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & L_2C_2 \\ 0 & A + B_2F_2 \end{bmatrix},$$

$$\begin{bmatrix} A_G & 0 \\ B_oC_G & A_o \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & 0 \\ -B_2F_2 & A + L_2C_2 \end{bmatrix},$$

$$\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix} = \begin{bmatrix} -L_2R_2^{1/2} \\ L_2R_2^{1/2} \end{bmatrix},$$

$$\begin{bmatrix} C_G^*D_o^* \\ C_o^* \end{bmatrix} = \begin{bmatrix} F_2^*R_1^{1/2} \\ F_2^*R_1^{1/2} \end{bmatrix}.$$

Now it is easy to verify that

$$\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}\begin{bmatrix} P & -P \\ -P & P \end{bmatrix} + \begin{bmatrix} P & -P \\ -P & P \end{bmatrix}\begin{bmatrix} A_G & B_GC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}\begin{bmatrix} B_GD_i \\ B_i \end{bmatrix}^* = 0 \tag{7.3}$$

$$\begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix} + \begin{bmatrix} A_G & 0 \\ B_o C_G & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} + \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C_G^* D_o^* \\ C_o^* \end{bmatrix}^* = 0. \tag{7.4}$$

Hence P and Q satisfy the following equations:

$$(A + B_2 F_2) P + P (A + B_2 F_2)^* + L_2 R_2 L_2^* = 0$$

$$Q (A + L_2 C_2) + (A + L_2 C_2)^* Q + F_2^* R_1 F_2 = 0.$$

□

Remark 22 *It is noted that equations (7.3) and (7.4) are still satisfied by the matrices P and Q even if K_{opt} is not stable. However, in this case the P and Q are not weighted gramians for K_{0s}.*

Algorithm 9: Performance Weighted Controller Reduction

- Suppose that K_{opt} is stable and let $\Delta_{K2} = \hat{K} - K_{opt}$.
- Using weighted balanced reduction method to find a stable lower order $\hat{K}$ such that

  $$\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} R_2^{1/2} \right\|_2$$

  is as small as possible. The weighted gramians P and Q can be obtained from

  $$(A + B_2 F_2) P + P (A + B_2 F_2)^* + L_2 R_2 L_2^* = 0$$

  $$Q (A + L_2 C_2) + (A + L_2 C_2)^* Q + F_2^* R_1 F_2 = 0.$$

It is noted that this controller reduction method is exactly the same as the BCRAM method in [57] although it was motivated from a completely different perspective.

Algorithm 10 Another Stability Weighted Controller Reduction

- Suppose that K_{opt} is stable and let $\Delta_{K2} = \hat{K} - K_{opt}$.

- Using weighted balanced reduction method to find a stable lower order $\hat{K}$ such that

$$\left\| R_1^{1/2} H_{12}^{-1} \Delta_{K2} H_{21}^{-1} H_{22} R_1^{-1/2} \right\|_\infty < 1.$$

  The weighted gramians P and Q can be obtained from

$$\begin{bmatrix} \hat{A}_2 & L_2C_2 \\ 0 & A+B_2F_2 \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} \hat{A}_2 & L_2C_2 \\ 0 & A+B_2F_2 \end{bmatrix}^*$$
$$+ \begin{bmatrix} 0 \\ B_2 \end{bmatrix} R_1^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}^* = 0$$

$$Q(A+L_2C_2) + (A+L_2C_2)^*Q + F_2^* R_1 F_2 = 0.$$

In either Algorithm 9 or Algorithm 10, let T be a nonsingular matrix such that

$$TPT^* = (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) with $\Sigma_1 = \mathrm{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r})$ and $\Sigma_2 = \mathrm{diag}(\sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$ and partition full order controller accordingly as

$$\begin{aligned} K_{opt} &= \left[\begin{array}{c|c} T\hat{A}_2T^{-1} & -TL_2 \\ \hline F_2T^{-1} & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ A_{21} & A_{22} & B_{12} \\ \hline C_{11} & C_{12} & 0 \end{array}\right]. \end{aligned}$$

Then a reduced-order controller $\hat{K}$ is obtained as

$$\hat{K} = \left[\begin{array}{c|c} A_{11} & B_{11} \\ \hline C_{11} & 0 \end{array}\right].$$

7.2 An Example

We also consider a four disk control system studied in [15, 62] and in Chapter 6. Then $R_1 = R_2 = 1$ and the optimal controller is an 8th order controller and the optimal cost is

$$\min \|T_{zw}\|_2 = 0.3689.$$

The optimal controller is reduced using various controller reduction criteria with frequency weighted balanced truncation method. The results are shown in Table 7.1 with the first column indicating the model reduction criteria used in the frequency weighted balanced truncation method.

Methods\Order of $\hat{K}$	7	6	5	4	3	2	1
$H_{21}^{-1}H_{22}H_{12}^{-1}\Delta_{K2}$	0.3690	0.3690	0.3699	0.3697	0.4110	0.3746	U
$\Delta_{K2}H_{21}^{-1}H_{22}H_{12}^{-1}$	0.3690	0.3690	0.3699	0.3697	0.4110	0.3746	U
$H_{12}^{-1}\Delta_{K2}H_{21}^{-1}H_{22}$	0.3690	0.3690	0.3702	0.3697	U	0.3777	U
$H_{22}H_{12}^{-1}\Delta_{K2}H_{21}^{-1}$	U	0.3690	0.3715	0.3698	U	0.3825	U
$H_{12}^{-1}\Delta_{K2}H_{21}^{-1}$	0.3690	0.3690	0.3699	0.3697	U	0.3741	U
$H_{21}^{-1}H_{12}^{-1}\Delta_{K2}$	0.3690	0.3690	0.3698	0.3697	0.3774	0.3743	U
$\Delta_{K2}H_{21}^{-1}H_{12}^{-1}$	0.3690	0.3690	0.3698	0.3697	0.3774	0.3743	U

Table 7.1: $\|T_{zw}\|_2$ with reduced order controllers: U–closed-loop system is unstable

Since this is a SISO system, it is clear that

$$H_{21}^{-1}H_{22}H_{12}^{-1}\Delta_{K2} = \Delta_{K2}H_{21}^{-1}H_{22}H_{12}^{-1} = H_{12}^{-1}\Delta_{K2}H_{21}^{-1}H_{22} = H_{22}H_{12}^{-1}\Delta_{K2}H_{21}^{-1}$$

and

$$H_{12}^{-1}\Delta_{K2}H_{21}^{-1} = H_{21}^{-1}H_{12}^{-1}\Delta_{K2} = \Delta_{K2}H_{21}^{-1}H_{12}^{-1}.$$

Hence one might expect that the first four methods should produce the same results and the last three methods should also produce the same results. However, since the frequency weighted balanced model reduction is not optimal, the resulting reduced order controllers from these criteria are different as shown in the table. In particular, it is obvious that the reduction is poor whenever two-sided weighting functions are used in frequency weighted balanced reduction. This also shows that the method proposed in [57] may not work well since that method is equivalent to a two-sided weighted balanced reduction by Algorithm 9.

It should be pointed out that the last two criteria in the table do not make sense for MIMO systems. It is included here only to show that the weighted balanced model reduction does not work well with two-sided weighting functions.

Chapter 8

Another Controller Order Reduction

In addition to those $\mathscr{H}_\infty$ performance controller reduction methods introduced in Chapter 5, we will propose another robust controller reduction method for SISO system with two different ways in this chapter: one is additive controller reduction and another is coprime factor controller reduction. This method can provide upper bound on the controller weighting function, which results in a reduced controller using weighted model reduction method, and the closed-loop system maintains robust stability and performance with the derived low order controller. The proposed additive controller reduction is shown in Section 8.1. The proposed coprime factor controller reduction method is presented in Section 8.2.

8.1 Proposed Additive Controller Reduction

Consider the general closed-loop control system framework in Figure 2.3. Suppose a n-th order generalized plant G is given by

$$G = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and $K = K_0 + \Delta_K$ is an m-th order controller which stabilizes the closed-loop system, where the transfer function $K_0(s)$ represents the nominal controller and it is subject to additive perturbation, hence

$$K(s) = K_0(s) + \Delta_K(s) = K_0(s) + W_K(s)\hat{\Delta}_K(s), \hat{\Delta}_K(s) \in \mathscr{H}_\infty, \left\|\hat{\Delta}_K(s)\right\|_\infty < 1.$$

Here $W_K(s)$ is weighting function bounding the uncertainties.

Theorem 21 *For the general closed-loop system shown in Figure 2.3, assume controller K is a scalar transfer function and $K = K_0 + W_K\hat{\Delta}_K(s)$. Then the closed-loop system has robust performance $\|T_{zw}\|_\infty < \gamma$ if the following inequality holds for every frequency ω:*

$$|W_K| \leq \frac{\gamma|1 - G_{22}K_0| - \bar{\sigma}[G_{11} + (G_{12}G_{21} - G_{11}G_{22})K_0]}{\bar{\sigma}(G_{12}G_{21} - G_{11}G_{22}) + \gamma|G_{22}|}$$

Proof From (2.3),

$$\begin{aligned}
T_{zw} &= G_{11} + G_{12}K(1 - G_{22}K)^{-1}G_{21} \\
&= G_{11} + \frac{G_{12}G_{21}K}{(1 - G_{22}K)} \\
&= \frac{G_{11} + (G_{12}G_{21} - G_{11}G_{22})K}{1 - G_{22}K} \\
&= \frac{G_{11} + (G_{12}G_{21} - G_{11}G_{22})(K_0 + W_K\hat{\Delta}_K)}{1 - G_{22}(K_0 + W_K\hat{\Delta}_K)} \\
&= \frac{[G_{11} + (G_{12}G_{21} - G_{11}G_{22})K_0] + (G_{12}G_{21} - G_{11}G_{22})W_K\hat{\Delta}_K}{1 - G_{22}K_0 - G_{22}W_K\hat{\Delta}_K}
\end{aligned}$$

then

$$|W_K| \leq \frac{\gamma|1 - G_{22}K_0| - \bar{\sigma}[G_{11} + (G_{12}G_{21} - G_{11}G_{22})K_0]}{\bar{\sigma}(G_{12}G_{21} - G_{11}G_{22}) + \gamma|G_{22}|}$$

implies that

$$\bar{\sigma}(T_{zw}) \leq \frac{\bar{\sigma}[G_{11} + (G_{12}G_{21} - G_{11}G_{22})K_0] + \bar{\sigma}(G_{12}G_{21} - G_{11}G_{22})|W_K|}{|1 - G_{22}K_0| - |G_{22}||W_K|} \leq \gamma.$$

As an example, we apply the above result to a special SISO case shown in Figure 8.1, where

$$P(s) = P_0(s) + \Delta_P(s) = P_0(s) + W_P(s)\hat{\Delta}_P(s), \hat{\Delta}_P(s) \in \mathscr{H}_\infty, \left\| \hat{\Delta}_P(s) \right\|_\infty < 1,$$

$$K(s) = K_0(s) + \Delta_K(s) = K_0(s) + W_K(s)\hat{\Delta}_K(s), \hat{\Delta}_K(s) \in \mathscr{H}_\infty, \left\| \hat{\Delta}_K(s) \right\|_\infty < 1,$$

and $W_P(s)$ and $W_K(s)$ are weighting functions bounding the uncertainties. Then put the feedback control system shown in Figure 8.1 to the general framework shown in Figure 8.2,

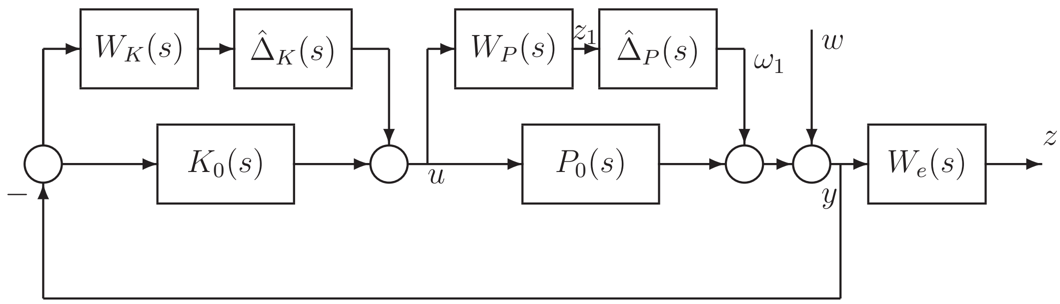

Figure 8.1: Feedback Control System with Additive Uncertainties.

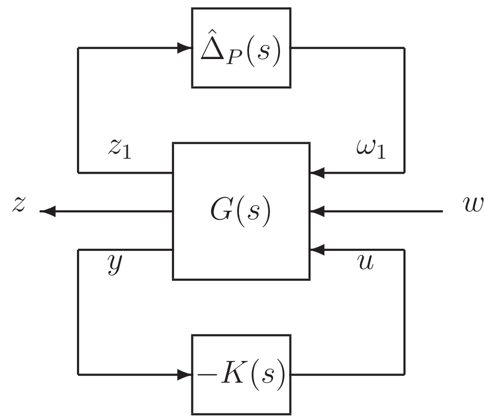

Figure 8.2: General Framework.

where

$$G(s) = \left[\begin{array}{cc|c} 0 & 0 & W_P \\ W_e & W_e & W_e P_0 \\ \hline I & I & P_0 \end{array}\right].$$

Theorem 22 *From Theorem 21, the SISO closed-loop system in Figure 8.2 has robust performance* $\|T_{zw}\|_\infty < \gamma$ *if the following inequality holds for every* ω*:*

$$|W_K| \leq \frac{\gamma|1 + P_0 K_0| - \sqrt{2}\sqrt{|W_e|^2 + |W_P K_0|^2}}{\sqrt{2}|W_P| + \gamma|P_0|}.$$

Proof By Theorem 21, robust performance is satisfied if the following inequality holds:

$$\begin{aligned}
|W_K| \quad &\leq \quad \frac{\gamma|1 + P_0 K_0| - \bar{\sigma}\left(\begin{bmatrix} 0 & 0 \\ W_e & W_e \end{bmatrix} + \left(\begin{bmatrix} W_P \\ W_e P_0 \end{bmatrix}\begin{bmatrix} I & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ W_e & W_e \end{bmatrix} P_0\right)(-K_0)\right)}{\bar{\sigma}\left(\begin{bmatrix} W_P \\ W_e P_0 \end{bmatrix}\begin{bmatrix} I & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ W_e & W_e \end{bmatrix} P_0\right) + \gamma|P_0|} \\
&= \quad \frac{\gamma|1 + P_0 K_0| - \bar{\sigma}\begin{bmatrix} -W_P K_0 & -W_P K_0 \\ W_e & W_e \end{bmatrix}}{\bar{\sigma}\left(\begin{bmatrix} W_P \\ 0 \end{bmatrix}\begin{bmatrix} I & I \end{bmatrix}\right) + \gamma|P_0|} \\
&= \quad \frac{\gamma|1 + P_0 K_0| - \sqrt{2}\sqrt{|W_e|^2 + |W_P K_0|^2}}{\sqrt{2}|W_P| + \gamma|P_0|}
\end{aligned}$$

When $\gamma = 1$,

$$|W_K| \leq \frac{|1 + P_0 K_0| - \sqrt{2}\sqrt{|W_e|^2 + |W_P K_0|^2}}{\sqrt{2}|W_P| + |P_0|}. \tag{8.1}$$

□

At the same time, another sufficient condition for robust performance preserving can be obtained as follow:

Theorem 23 *The SISO closed-loop system in Figure 8.1 has robust performance* $\|T_{zw}\|_\infty < \gamma$ *if the following inequality holds for every* ω*:*

$$|W_K| \leq \frac{|1+P_0K_0| - |W_PK_0| - (1/\gamma)|W_e|}{|P_0| + |W_P|}.$$

Proof Suppose that T_{zw} is the transfer function from input to output of the closed-loop system, then

$$\begin{aligned} T_{zw}(s) &= \frac{W_e(s)}{1+P(s)K(s)} \\ &= \frac{W_e(s)}{1+(P_0(s)+W_P(s)\hat{\Delta}_P(s))(K_0(s)+W_K(s)\hat{\Delta}_K(s))} \\ &= \frac{W_e(s)}{1+P_0(s)K_0(s)+P_0(s)W_K(s)\hat{\Delta}_K(s)+W_P(s)\hat{\Delta}_P(s)K_0(s)+\hat{\Delta}_K(s)\hat{\Delta}_P(s)W_K(s)W_P(s)} \end{aligned}$$

and

$$\begin{aligned} |T_{zw}(s)| &\leq \frac{|W_e(s)|}{|1+P_0(s)K_0(s)| - |P_0(s)W_K(s)| - |W_P(s)K_0(s)| - |W_P(s)W_K(s)|} \\ &= \frac{|W_e(s)|}{|1+P_0(s)K_0(s)| - |W_P(s)K_0(s)| - |W_K(s)|(|P_0(s)| + |W_P(s)|)} \end{aligned}$$

If $|T_{zw}(s)| \leq \gamma$, the robust performance of closed-loop system is satisfied. That is

$$\begin{aligned} |T_{zw}(s)| \leq \gamma &\Longleftarrow (1/\gamma)|W_e(s)| \leq |1+P_0(s)K_0(s)| - |W_P(s)K_0(s)| - |W_K(s)|(|P_0(s)| + |W_P(s)|) \\ &\Longleftrightarrow |W_K(s)| \leq \frac{|1+P_0(s)K_0(s)| - |W_P(s)K_0(s)| - (1/\gamma)|W_e(s)|}{|P_0(s)| + |W_P(s)|} \end{aligned}$$

When $\gamma = 1$,

$$|W_K(s)| \leq \frac{|1+P_0(s)K_0(s)| - |W_P(s)K_0(s)| - |W_e(s)|}{|P_0(s)| + |W_P(s)|}. \tag{8.2}$$

$\square$

Remark 23 *The sufficient condition for performance preserving from Theorem 22 is more conservative because (8.1) implies (8.2):*

$$\frac{|1+P_0K_0| - \sqrt{2}\sqrt{|W_e|^2 + |W_PK_0|^2}}{\sqrt{2}|W_P| + |P_0|} \leq \frac{|1+P_0(s)K_0(s)| - |W_P(s)K_0(s)| - |W_e(s)|}{|P_0(s)| + |W_P(s)|}.$$

Note that the same performance guarantee inequality can be obtained as presented in [12] when $\gamma = 1$.

8.2 Proposed Coprime Factor Controller Reduction

Suppose that K_0 has the right coprime factorization $K_0 = N_0M_0^{-1}$ or the left coprime factorization $K_0 = \tilde{M}_0^{-1}\tilde{N}_0$. Here, we only take the right coprime factorization $K_0 = N_0M_0^{-1}$ and present corresponding results because similar results can be obtained for the left coprime fatorization $K_0 = \tilde{M}_0^{-1}\tilde{N}_0$.

Assume that K has a coprime factorization as

$$K = \frac{N_0 + \Delta_N W_K}{M_0 + \Delta_M W_K} = (N_0 + \Delta_N W_K)(M_0 + \Delta_M W_K)^{-1} = NM^{-1}$$

where $\|\Delta\|_\infty = \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty \leq 1$, then

$$K - K_0 = \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} N - N_0 \\ M - M_0 \end{bmatrix} = \begin{bmatrix} \Delta_N W_K \\ \Delta_M W_K \end{bmatrix} = \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} W_K = \Delta W_K.$$

Theorem 24 *For the general closed-loop system shown in Figure 2.3, assume controller K is a scalar transfer function and $K = (N_0 + \Delta_N W_K)(M_0 + \Delta_M W_K)^{-1}$. Then the closed-loop system has robust performance $\|T_{zw}\|_\infty < \gamma$ if the following inequality holds for every ω:*

$$|W_K| \leq \frac{\gamma|M_0 - G_{22}N_0| - \bar{\sigma}[G_{11}M_0 + (G_{12}G_{21} - G_{11}G_{22})N_0]}{\bar{\sigma}\left(\begin{bmatrix} G_{11} & G_{12}G_{21} - G_{11}G_{22} \end{bmatrix}\right) + \gamma\bar{\sigma}\left(\begin{bmatrix} 1 & -G_{22} \end{bmatrix}\right)}. \tag{8.3}$$

Proof From (2.3),

$$\begin{aligned}
T_{zw} &= G_{11} + G_{12}K(1-G_{22}K)^{-1}G_{21} \\
&= G_{11} + \frac{G_{12}KG_{21}}{1-G_{22}K} \\
&= \frac{G_{11}M_0 + G_{11}\Delta_M W_K - G_{11}G_{22}(N_0+\Delta_N W_K) + G_{12}G_{21}(N_0+\Delta_N W_K)}{M_0 - G_{22}N_0 + (\Delta_M - G_{22}\Delta_N)W_K} \\
&= \frac{G_{11}M_0 + (G_{12}G_{21}-G_{11}G_{22})N_0 + \begin{bmatrix} G_{11} & G_{12}G_{21}-G_{11}G_{22} \end{bmatrix}\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} W_K}{M_0 - G_{22}N_0 + \begin{bmatrix} 1 & -G_{22} \end{bmatrix}\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} W_K}
\end{aligned}$$

so

$$\bar{\sigma}(T_{zw}) \leq \frac{\bar{\sigma}[G_{11}M_0 + (G_{12}G_{21}-G_{11}G_{22})N_0] + \bar{\sigma}\Big(\begin{bmatrix} G_{11} & G_{12}G_{21}-G_{11}G_{22} \end{bmatrix}\Big)W_K}{|M_0 - G_{22}N_0| - \bar{\sigma}\Big(\begin{bmatrix} 1 & -G_{22} \end{bmatrix}\Big)W_K}.$$

Here we can conclude that

$$|W_K| \leq \frac{\gamma|M_0 - G_{22}N_0| - \bar{\sigma}[G_{11}M_0 + (G_{12}G_{21}-G_{11}G_{22})N_0]}{\bar{\sigma}\Big(\begin{bmatrix} G_{11} & G_{12}G_{21}-G_{11}G_{22} \end{bmatrix}\Big) + \gamma\bar{\sigma}\Big(\begin{bmatrix} 1 & -G_{22} \end{bmatrix}\Big)}$$

guarantees $\bar{\sigma}(T_{zw}) \leq \gamma$. □

Also, for special SISO situation, the following result can be derived when K has the right coprime factorization. Here, we suppose that the system model P does not have uncertainties, that is, $P = P_0$.

Theorem 25 *If controller K has the right coprime factorization, the SISO closed-loop system has robust performance $\|T_{zw}\|_\infty < \gamma$ if the following inequality holds for every ω:*

$$|W_K| \leq \frac{\gamma|M_0 + PN_0| - |W_e M_0|}{|W_e| + \gamma(1+|P|^2)^{1/2}}.$$

In particular, the nominal performance requires

$$|W_e M_0| \leq \gamma |M_0 + P N_0|.$$

Proof If we suppose that P does not have uncertainties,

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} W_e & W_e P \\ 1 & P \end{bmatrix}$$

From 8.3,

$$\begin{aligned} |W_K| &\leq \frac{\gamma |M_0 + G_{22} N_0| - \bar{\sigma}[G_{11} M_0 - (G_{12} G_{21} - G_{11} G_{22}) N_0]}{\bar{\sigma}\left(\begin{bmatrix} G_{11} & G_{12} G_{21} - G_{11} G_{22} \end{bmatrix}\right) + \gamma \bar{\sigma}\left(\begin{bmatrix} 1 & -G_{22} \end{bmatrix}\right)} \\ &= \frac{\gamma |M_0 + P N_0| - |W_e M_0|}{\bar{\sigma}\begin{bmatrix} W_e & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & P \end{bmatrix}} \\ &= \frac{\gamma |M_0 + P N_0| - |W_e M_0|}{|W_e| + \gamma (1 + |P|^2)^{1/2}} \end{aligned}$$

□

Now, an upper bound of controller uncertainty has been obtained. Hence, compute the frequency of upper bound of weighting function W_K for each ω, and then use MATLAB command **fitmag** to fit one stable and minimum phase transfer function W as new weighting function. The reduced controller $\hat{K}$ derived from the weighted model reduction method with W will preserve the closed-loop $\mathscr{H}_\infty$ performance, i.e., $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

Chapter 9

Future Work

Several controller reduction approaches and algorithms are introduced in this dissertation for linear system. The four disk example and HIMAT example are explored to show that the proposed controller reduction methods can work well. At least, they can work as effective as the best method available in the literature. The main advantages of the proposed methods include that the weighting functions for controller reduction are easy to compute and are readily available from standard $\mathscr{H}_\infty$ control design software. However, there are still potentially significant work which may complement this dissertation.

We discussed several sufficient conditions and algorithms to guarantee the closed-loop system stability in Chapter 5. One of conditions is that $\left\|LM_{12}^{-1}\Delta_K M_{21}^{-1}M_{22}L^{-1}\right\|_\infty < 1$ for some square L such that $L, L^{-1} \in \mathscr{H}_\infty$, and related algorithm is also stated in Algorithm 3 that one final reduced controller can be found by iterating reduced order controller and parameter L. The similar case is referred to parameter J. However, how to find the optimal L and J is still an open question and worth studying.

As stated in Chapter 6, although the four disk and HIMAT examples have partially demonstrated the advantages of proposed methods over other approaches, the simulation

results are not as perfect as hoped. One of possible reasons is from the special data of the examples, especially HIMAT example. In HIMAT example, the original full order controller is one 30^{th} order controller, and the (input and output) weighting functions are more than 20 orders. So the model itself is considerablly complicated, and computation errors are inevitable during simulation. In addition, both examples have skewed problems, which also probably bring on undesired simulation results. Hence, more general examples will be explored to verify the effectiveness of the proposed methods.

The Table 6.20 shows that the better results could be obtained if the parameters of already reduced controller from proposed methods are optimized. However, we only listed one case where we optimize the constant term D and parameter B. As a result, further work included optimizing more parameters and trying different optimization approaches are needed.

Another $\mathscr{H}_\infty$ performance controller reduction method was introduced for SISO system in Chapter 8. We are interested in extending the approach to MIMO system, and it should be more complicated in that case.

Chapter 10

Conclusion

Controller reduction is an important research topic in control system design since low order controllers are always important and welcome for engineers. This dissertation has proposed some $\mathscr{H}_\infty$ or $\mathscr{H}_2$ controller reduction methods which can guarantee stability and performance for closed-loop linear systems.

The main part of this thesis introduced several $\mathscr{H}_\infty$ performance preserving controller reduction approaches. One advantage of the proposed controller reduction is less computational complexity. Comparing with other current controller reduction methods, another advantage is the simplicity for obtaining weighting function, which can be derived from the representation for the $\mathscr{H}_\infty$ control problem. The four disk example and HIMAT example are explored to demonstrate those proposed controller reduction methods, and simulation results and analysis also support the conclusion and the effectiveness of proposed methods. All simulation results showed the effectiveness of techniques. In addition, two important but easily ignored results on weighted model reduction are also described in this dissertation.

Based on those proposed $\mathscr{H}_\infty$ controller reduction methods, we looked at the $\mathscr{H}_2$ controller reductions from a different perspective in Chapter 7. Several $\mathscr{H}_2$ stability preserving

and performance preserving controller reduction approaches are introduced and related algorithms are also presented. Four disk example is explored to illustrate $\mathscr{H}_2$ controller reduction methods and algorithms.

Another separate $\mathscr{H}_\infty$ stability and performance preserving controller reduction method is derived for SISO linear system in Chapter 8. The core of this proposed method is that, we can get upper bound on the weighting function of controller reduction for general SISO $\mathscr{H}_\infty$ control problem, and then one new weighting function is obtained by fitting upper bound. Hence, one lower order controller can be derived by using frequency weighted model reduction method, which guarantees the closed-loop system stability and performance.

Finally, some further potentially significant work which may complement this dissertation are discussed.

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