Thermomechanical Interaction Analysis of Bodies Subjected to Oscillatory Heat Source

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THERMOMECHANICAL INTERACTION ANALYSIS OF BODIES SUBJECTED TO OSCILLATORY HEAT SOURCE

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mechanical Engineering

by

Jun Wen
B.S., China University of Petroleum, 1995
M.S., China University of Petroleum, 1998
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Dedication

To my mother, my wife and my son
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Abstract

Surface heating by application of a moving heat source is a common problem in most manufacturing processes as well as in many tribological applications. The existing literature on studies of moving heat source problems primarily concentrates on unidirectional moving heat sources. Yet, a more complete analysis requires understanding of thermal behavior of bodies in relative sliding contacts subjected to oscillatory heat sources and thermomechanical interaction.

In this dissertation, two analytical models are developed for rapid evaluation of the transient temperature in solids undergoing oscillatory heat source. In the first model, an analytical technique is presented to treat oscillatory heat source problems by using the Duhamel theorem. An explicit analytical solution for temperature variation in a rectangular domain subjected to oscillatory heat flux is developed. In the second model, the solutions for distributed heat sources undergoing oscillatory motion on the surface of a semi-infinite body are developed. The appropriate governing equations for different heat sources are derived and an efficient algorithm is developed to solve them. Finally, analytical expressions are provided for predicting maximum surface temperature.

Two computationally-efficient algorithms are developed to handle complex transient moving heat source problems based on transfer matrix method combined with the dual reciprocity boundary element and finite element methods. The time integration is processed by an iteration transfer matrix method, making both models numerically stable. For the model associated with boundary element method, the domain integrals due to transient heat transfer are converted to a boundary one by the dual reciprocity method. The influence matrices are evaluated by an adaptive precise time integration method.

In the analysis of oscillatory contacts, it is further necessary to take into consideration the thermomechanical interaction between the contact bodies. To this end, thermomechanical models are developed that enable one to efficiently treat such computationally demanding problems. The thermomechanical coupling process involves a transient solution scheme where the frictional heat is automatically partitioned between contact surfaces. The coupling between thermal and mechanical interaction is treated by an iteration method. In addition, a three dimensional computational model is presented. The numerical results are verified by the experimental measurements.
1 Introduction

1.1 Thermal and Thermomechanical Behavior Involved with Moving Heat Sources

Moving heat source problems occur in a wide variety of industrial applications, such as manufacturing processes that involve cutting, grinding, welding, and surface heat treatment using laser irradiation along the workpiece. A common feature in each of these applications is that the temperature distribution in the heated zone at or near the surface can play an important role in the metallurgical microstructure, thermal shrinkage, thermal cracking, residual stress, and many other performance parameters of the component.

There are also many tribological applications involved with the moving heat source when the component surfaces are in relative sliding motion. Examples include the interaction of the rolling elements and races in roller bearings, shafts in journal bearings, and steel rolls in rolling mills, where the interacting bodies undergo one or more moving heat sources due to rubbing. Relative to the heat source, the stationary body is subjected to a fixed heat source. The sliding body is periodically heated over a small contact surface area while cooled over all or part of surface area by convention. As the frictional heat flows into the contacting bodies, the contact area changes due to thermal expansion, resulting in the change of the contact pressure, which in turn further intensifies the heat generation and temperature. The cycle of frictional heating leading to thermal expansion and higher contact pressures and temperature often becomes unstable and causes component failure. Thermoelastic instability, hot spotting, seizure, cracking, thermal blistering, and scuffing are all in effect a manifestation of unstable thermal runaway brought about as a result of thermomechanical interaction of contacting bodies.

Therefore, the knowledge of temperature in dry or lubricated sliding contact plays a vital role in the design of machine elements. Since direct temperature measurement is often impractical, efficient models are needed at the design stage to predict the friction-induced temperatures rise under sliding conditions. The early pioneering work on the moving heat source problems was reported by Block [1] and Jaeger [2] on the treatment of moving heat source acting on a semi-infinite solid. Since that, many notable publications have devoted significant research toward the prediction of the temperature in sliding contact. For example, Tian & Kennedy [3] analyzed the maximum and average surface temperature rise for a semi-infinite body due to different moving heat sources for the entire Peclet number range. Hou and Komanduri [4] derived the general solutions for moving plane heat source. Hirano & Yoshida [5] analyzed the surface temperature of a semi-infinite body subjected to a rectangular heat source with reciprocating motion. Greenwood & Alliston-Greiner [6] presented an analysis of surface temperature in fretting contact by separating the sinusoidal heat source into a constant and a periodic heat input using the Fourier transformation. Tian & Kennedy studied the problem of frictional heating in oscillatory sliding to predict the surface temperature rise in Ref. [7], and developed a contact surface temperature model for finite bodies in dry and lubricated sliding systems in Ref. [8]. Also, significant efforts have been devoted by numerical techniques to study the moving heat source problems. Kennedy et. al. [9-11] performed a

In this dissertation, the following contributions to the analyses of moving heat source problems have been made.

1) Analytical solutions for transient temperature involving oscillatory heat source.

Many analytical analyses of the surface temperature in the sliding contact have been based on the solution for temperature distribution due to an instantaneous point heat source by Carslaw and Jaeger [14] to develop the solution for a distributed heat flux within a contact patch on the surface of half space. See for example Refs. [3-7]. Analytical solution provides an efficiently prediction of temperature in sliding contact. However, with the exception of a few papers [5-7], most of analytical studies dealing with this aspect heave focused their attention to unidirectional, moving heat source. In this dissertation, two analytical models for the transient temperature distribution in the domain subjected to oscillatory heat sources are developed for rapid evaluation of the temperature rise in the oscillatory contact. The first model is developed by using the Duhamel’s theorem to treat a rectangular domain experiencing oscillating heat flux on its boundary. The second one presents the solutions for the different heat sources (circular, rectangular and parabolic) undergoing oscillatory motion on a semi-infinite body. Simple expressions for prediction of the maximum surface temperature for different heat sources are also provided by a surface-fitting method based on an extensive number of simulations.

2) Numerical modes for moving heat source problems based on the combination of transfer matrix method with dual reciprocity boundary element method or finite element method.

Numerical method such as the Finite Element Method (FEM) can be applied to handle complex moving heat source problems, but it often suffers from numerical oscillation at high sliding velocities, so that accurate prediction of temperature generally requires extremely fine mesh, which is computationally expensive [9]. This is particularly crucial when dealing with three-dimensional oscillatory heat source problems where a large number of cycles are needed for the system to reach a steady state [13]. The Transfer Matrix Method (TMM) provides an efficient solution, which can reduce the amount of computer storage and computing time. This method is widely used in structure mechanics [15, 16]. An application of transfer matrix method in heat transfer analysis of periodic heating problems and its efficiency is reported by Fan & Barber [17] where they presented a very efficient method for the analysis of linear periodic heating problems, and studied the temperature profile in a two dimensional media undergoing periodic variable boundary temperatures. In this dissertation, a computationally efficient method is developed for simulating the transient temperature distribution in a body subjected to
unidirectional sliding or oscillatory heat source along its boundary by utilizing the combination of finite element technique and transfer matrix method. A more computational efficient method is developed for transient heat conduction in moving heat source problems by a combination of the boundary element method and the transfer matrix method. By limiting the discretization of the model to its boundary rather than the whole volume, the Boundary Element Method (BEM) reduces the dimensionality of the problem. The Dual Reciprocity Boundary Element Method (DRBEM) is used for converting the domain integral into a boundary one. Both developed models can efficiently handle complicated moving heat source problems in manufacturing or tribological applications.

3) Thermomechanical effects on transient temperature in non-conformal contacts experiencing reciprocating sliding motion.

In sliding contacts, the frictional heat generated in the contact interface results in the changes of the contact area due to thermal expansion, affecting the contact pressure distribution, which in turn influences the heat generation and temperature. Therefore, accurate prediction of the temperature in sliding contact must treat the thermomechanical interaction simultaneously. A great majority of published literature deal with the unidirectional moving heat sources or sliding rough surfaces [18-22]. Nevertheless, a comprehensive study of transient temperature variation for an oscillatory sliding contact with the thermomechanical effects is still lacking. In this dissertation, a thermomechanical model is developed to study the interaction of a semi-infinite elastic solid in contact with a rigid adiabatic sphere subjected to oscillatory sliding motion. The effects of transient heat transfer and thermomechanical loads on the temperature variation in the solid are investigated.

4) Thermomechanical coupling in oscillatory system with application to journal bearing seizure

The seizure is one of the common failures of journal bearings. Due to the structural distortion caused by frictional heating, the bearing may lose its designed clearance relative to the shaft, resulting in a multiple contact to the extent that a complete loss of clearance occurs with catastrophic seizure. It has been found that large temperature rise in journal bearings is responsible for the failure initiation, which changes the contact state between the shaft and the bushing. Most of the literatures [12, 23-26] are concentrated on the temperature and thermomechanical behavior in continuous rotating journal bearings in which the load is constant and the velocity is unidirectional. Yet there are many applications in which the load varies dynamically and the velocity changes directions in an oscillatory fashion, such as pin-bushing assemblies in heavy construction machinery, the bearings in reciprocating components of internal combustion engines and in reciprocating compressors. Therefore, a comprehensive study of thermomechanical behavior of oscillatory journal bearings is required. In this dissertation, a model for thermomechanical coupling in oscillatory system is developed. The model is applied to the seizure analysis of a journal bearing. The thermomechanical interaction between the
shaft and bushing is employed. The interdependent thermal and mechanical interaction is handled by an iteration method. The heat division between the contacting bodies is fully accounted for.

5) Transient heat transfer in an actual journal bearing.

Most of the literatures studying the thermal or thermomechanical behavior of journal bearings such as Refs. [12, 23-26] assumed two dimensional (2-D) journal bearings. In this dissertation, a three dimensional (3-D) computational model for thermal analysis of an actual journal bearing with experimental verification is presented. The actual distribution of the frictional heat is considered. The model can predict the maximum contact temperature and the transient temperature field in oscillatory journal bearings.

1.2 Outline of the Dissertation

This dissertation focuses on the thermal and thermomechanical analysis on the problem involving moving heat sources. Seven topics are presented, each of which is treated as a separate chapter and written in the form of a journal paper. Two of these papers [27-28] have been published. Another two of them, related to Chapters 4 and 5, are being processed for acceptance with minor changes.

Chapter 2 and Chapter 3 deal with the analytical models for oscillatory heat source problems. Simple and effective methods are developed in these two chapters. Chapter 2 presents an analytical technique for treating heat conduction problems involving a body experiencing oscillatory heat flux on its boundary by using Duhamel’s theorem. To verify the results, numerical solution is compared with the analytical solution. Chapter 3 develops an analytical solution for the transient temperature profile involving circular, rectangular and parabolic heat sources undergoing oscillatory motion on a semi-infinite body. The method is verified by the published literature. Application of the method to fretting contact is presented. Based on an extensive number of simulations, more simple expressions for predicting the maximum surface temperature for different heat sources is developed. Both analytical solutions in Chapter 2 and Chapter 3 are used to verify the numerical models developed in the following chapters.

Numerical models are useful for treating the complicated moving heat source problems. By utilizing the combination of transfer matrix method with the boundary element or finite element method, two efficient numerical models for simulating the transient temperature distribution in a body subjected to unidirectional sliding or oscillatory heat sources along its boundary are developed in Chapter 4 and Chapter 5, respectively. The model combines the advantages of both transfer matrix method and finite element or boundary element method, resulting in much savings in computer time. The efficiency and application of the models in different moving heat source problems are presented in Chapter 4 and Chapter 5.

The computational efficient thermal model developed in Chapters 4 or 5 makes it practical to handle the problems involving thermomechanical interaction in oscillatory
contacts. Chapter 6 and Chapter 7 present the thermomechanical models for the oscillatory contact by treating the thermal and mechanical parts of the contact solution independently and then combined together through a numerical iteration scheme. Chapter 6 deals with thermomechanical effects on transient temperature in non-conformal contacts experiencing reciprocating sliding motion. A comprehensive study of transient temperature variation for an oscillatory sliding contact with the thermomechanical interaction is presented in Chapter 6. Chapter 7 investigates the thermomechanical interaction behavior of journal bearings with oscillatory motion undergoing thermally induced seizure. A wide range of operating parameters is simulated. The significance of applied load, contact clearance, friction coefficient, oscillation parameters, convective heat transfer condition and varied load directions in the thermally induced seizure are studied in Chapter 7.

Chapter 8 extends the 2-D analyses of Chapters 7. In this chapter, heat transfer analysis of an actual journal bearing is performed with consideration of the non-uniform distribution of the frictional heat along the axial direction. The computational model can predict the maximum contact temperature and the temperature distribution in the journal bearing. The comparison of the simulation results along with the experiment results is presented.

Finally the results of all the work are summarized in the conclusion section in Chapter 9. Possible future research topics are also discussed in Chapter 9.
2 Analytical Formulation for the Temperature Profile by Duhamel’s Theorem in Bodies Subjected to an Oscillatory Heat Source

This chapter presents an analytical technique for treating heat conduction problems involving a body experiencing oscillating heat flux on its boundary. The boundary heat flux is treated as a combination of many point heat sources, each of which emits heat intermittently based on the motion of the flux. The working function of the intermittent heat source with respect to time is evaluated by using the Fourier series and temperature profile of each point heat source is derived by using the Duhamel’s theorem. Finally, by superposition of the temperature fields over all the point heat sources, the temperature profile due to the original moving heat flux is determined. Prediction results and verification using finite element method are presented for an oscillatory heat flux in a rectangular domain.

2.1 Introduction

Many heat conduction problems are concerned with a moving heat source traversing along over one of the boundaries, for example, as a result of a solid sliding back and forth on another body. The heat generated within the contact region is due to friction, whose magnitude is dependent on the friction coefficient, sliding velocity and pressure in the contact area. Depending on the thermomechanical properties of the bodies, part of the interfacial heat transfers to the sliding body and the rest conducts into the stationary solid. Thus relative to the respective coordinate systems attached to each body, the sliding body is subject to a fixed heat source, and the stationary body is subject to a moving heat source. The temperature rise at the interface has a significant effect on the tribological behavior of the contact materials, causing the materials to distort, which in turn affects the contact geometry, pressure distribution and the temperature. In some applications, depending on the operating conditions, a positive feedback loop develops where the contact pressures and temperature become exceedingly high leading to gross surface damage and ultimately failure of the system. Thus, an efficient methodology for prediction of the temperature field as a function of time is needed at the design stage.

Pioneering work on the moving heat source problems was reported by Blok [1] with particular interest in the meshing of gear teeth giving rise to the concept of flash temperature. The work was later extended by Jaeger [2] who expressed the surface temperature of each solid in terms of heat flux. Tian & Kennedy [3] analyzed the surface temperature rise for a semi-infinite body due to different moving heat sources for the entire range of Peclet number using a Green’s function method. Ju & Farris [29], Gao & Lee [30] developed a transient temperature model based on the fast Fourier transform method, respectively. Qiu & Cheng [31] did a numerical simulation of the temperature rise for a three-dimensional rough surface sliding against a smooth surface in mixed

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lubricated contact by the moving grid method. Hirano & Yoshida [5] analyzed the surface temperature of semi-infinite body subjected to a rectangular heat source with reciprocating motion. Greenwood & Greiner [6] presented an analysis of surface temperature in fretting contact by assuming the source remains stationary. Additional references about the models for flash temperature can be found in Ref. [32], where the difference of the predicted temperature among the models is investigated. With the exception of Ref. [5, 6], most of the papers focused their attention to unidirectional, moving heat source.

In this chapter the Duhamel’s theorem is used to analytically determine the temperature in a rectangular domain subjected to oscillating heat source on its boundary, extending the semi-analytical treatment of the problem as developed by Krishnamurthy & Khonsari [33].

The approach in this chapter is as follows. The moving heat flux is considered to consist of many intermittent point heat sources and their working function \( f(\xi, t) \) with respect to time is evaluated by using the Fourier series. The temperature field \( T_\xi(x, y, t) \) of each point heat is derived by using Duhamel’s theorem. Then, the superposition of \( T_\xi(x, y, t) \) over all the heat sources yields the desired solution for \( T(x, y, t) \) of the original problem. The mathematical formulation is illustrated in §2.2.

2.2 Mathematical Formulation

Referring to Fig. 2.1, consider a rectangular domain of width \( L \) and height \( h \) on which an oscillating heat flux of magnitude \( q'' \) is acting over a contact width \( l \) \((l < L/2)\). The oscillation velocity is \( \nu \). The other three sides are at constant temperature \( T_0 \), and the initial temperature of the domain is \( T_i \). Along the top surface, the local coordinate \( \xi \) denotes the location of a point heat source. The heat flux extending over the width \( l \) is treated as a combination of many point heat sources. Without the loss of generality, it is assumed that the oscillation starts from the left side.
2.2.1 Working Function $f(\xi,t)$ of a Time-dependent Point Heat Source

Each point heat source on the top surface emits heat intermittently. Referring to Fig. 2.2, the working function $f(\xi,t)$ for a point heat source at $x = \xi$ ($y = h$) within one cycle is a periodic square wave function with magnitude of 1 (on) or 0 (off).

![Fig. 2.2 Periodic square wave of working function of the point heat source](image)

The oscillation period and the angular frequency are $T = \frac{2(L-l)}{v}$ and $\omega = \frac{2\pi}{T} = \frac{\pi v}{L-l}$, respectively. The function $f(\xi,t)$ can be treated as an even function. Its Fourier series is given by

$$f(\xi,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega t) \quad (2.1a)$$

where

$$a_0 = \frac{2}{T} \int_{0}^{T/2} f(\xi,t)dt \quad (2.1b)$$

$$a_n = \frac{4}{T} \int_{0}^{T/2} f(\xi,t) \cos(n \omega t)dt \quad (2.1c)$$

Then for the interval $0 < \xi < L$, the Fourier series of $f(\xi,t)$ is

$$f(\xi,t) = \begin{cases} f_1(\xi,t) & \text{if } 0 < \xi \leq l \\ f_2(\xi,t) & \text{if } l < \xi \leq L-l \\ f_3(\xi,t) & \text{if } L-l < \xi \leq L \end{cases} \quad (2.2a)$$

where

$$f_1(\xi,t) = \frac{\xi}{L-l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n \pi \xi}{L-l} \right) \cos \left( \frac{n \pi \omega t}{L-l} \right) \quad (2.2b)$$

$$f_2(\xi,t) = \frac{l}{L-l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \sin \left( \frac{n \pi \xi}{L-l} \right) - \sin \left( \frac{n \pi (\xi-l)}{L-l} \right) \right] \cos \left( \frac{n \pi \omega t}{L-l} \right) \quad (2.2c)$$
\[ f_\xi(\xi, t) = \frac{L - \xi}{L - l} - \frac{2}{\pi } \sum_{n=1}^{\infty} \sin \left( \frac{n\pi(\xi - l)}{L - l} \right) \cos \left( \frac{n\pi vt}{L - l} \right) \] 

(2.2d)

2.2.2 Temperature Profile \( T_\xi(x, y, t) \) of a Time-dependent Point Heat Source

The governing equation for time-dependent heat conduction is

\[
\frac{\partial^2 T_\xi(x, y, t)}{\partial x^2} + \frac{\partial^2 T_\xi(x, y, t)}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T_\xi(x, y, t)}{\partial t} \quad \text{in} \quad 0 < x < L, \ 0 < y < h, \ t > 0 
\]

(2.3a)

where \( T_\xi(x, y, t) \) is the transient temperature field of a time-dependent point heat source at location \((\xi, h)\). \( k \) is the thermal conductivity of the material and \( \alpha \) represents the thermal diffusivity.

The boundary conditions are:

\[
T_\xi = T_0 \quad \text{at} \quad x = 0, \ x = L, \ t > 0
\]

(2.3b)

\[
T_\xi = T_0 \quad \text{at} \quad y = 0, \ t > 0
\]

(2.3c)

\[
k \frac{\partial T_\xi}{\partial y} = q^" \delta(x - \xi) f(\xi, t) \quad \text{at} \quad y = h, \ t > 0
\]

(2.3d)

where \( \delta(x - \xi) \) represents the Dirac delta function defined as

\[
\delta(x - \xi) = \begin{cases} \infty & \text{if} \ x = \xi \\ 0 & \text{if} \ x \neq \xi \end{cases}
\]

(2.3e)

with

\[
\int_{-\infty}^{\infty} \delta(x - \xi) dx = 1
\]

(2.3f)

for \( \varepsilon > 0 \), and the following property for any function \( g(x) \)

\[
\int \delta(x - \xi) g(x) dx = g(\xi)
\]

(2.3g)

The initial condition is:

\[
T_\xi = T_i \quad \text{at} \quad t = 0, \ \text{in} \ 0 \leq x \leq L, \ 0 \leq y \leq h
\]

(2.3h)

Let

\[
T_\xi(x, y, t) = \phi_1(x, y, t) + \phi_2(x, y, t)
\]

(2.4)

the solution of problem (2.3) becomes the superposition of the following two problems.

\[
\begin{align*}
\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} & = \frac{1}{\alpha} \frac{\partial \phi_1}{\partial t} \quad \text{in} \quad 0 < x < L, \ 0 < y < h, \ t > 0 \\
\phi_1 & = T_0 \quad \text{at} \quad x = 0, \ x = L, \ t > 0 \\
\phi_1 & = T_0 \quad \text{at} \quad y = 0, \ t > 0 \\
\frac{\partial \phi_1}{\partial y} & = 0 \quad \text{at} \quad y = h, \ t > 0 \\
\phi_1 & = T_i \quad \text{at} \quad t = 0
\end{align*}
\]

(2.5)
and
\[
\begin{align*}
\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} &= \frac{1}{\alpha} \frac{\partial \phi_z}{\partial t} \quad \text{in} \quad 0 < x < L, \quad 0 < y < h, \quad t > 0 \\
\phi_z &= 0 \quad \text{at} \quad x = 0, \quad x = L, \quad t > 0 \\
\phi_z &= 0 \quad \text{at} \quad y = 0, \quad t > 0 \tag{2.6}
\end{align*}
\]

\[
k \frac{\partial \phi_z}{\partial y} = q^* \delta(x - \xi) f(\xi, t) \quad \text{at} \quad y = h, \quad t > 0
\]

\[
\phi_z = 0 \quad \text{at} \quad t = 0
\]

It can be shown that the Green’s function for the problem is

\[
G(x, y, t \mid x', y', \tau) = \frac{4}{L h} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)} \sin(\beta_m x) \sin(\eta_p y) \sin(\beta_m x') \sin(\eta_p y') \tag{2.7}
\]

Then the solution for the problem (2.5) can be obtained by use of the Green’s function as

\[
\phi = T_0 + \frac{4}{L h} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{1 - (-1)^m}{\beta_m \eta_p} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \sin(\beta_m x) \sin(\eta_p y) \tag{2.8a}
\]

where

\[
\beta_m = \frac{m \pi}{L} \quad (m = 1, 2, \ldots, \infty). \tag{2.8b}
\]

\[
\eta_p = \frac{(2p - 1) \pi}{2h} \quad (p = 1, 2, \ldots, \infty) \tag{2.8c}
\]

To solve the problem (2.6) using the Duhamel’s theorem, the auxiliary problem is defined as follows.

\[
\begin{align*}
\frac{\partial^2 \theta_z}{\partial x^2} + \frac{\partial^2 \theta_z}{\partial y^2} &= \frac{1}{\alpha} \frac{\partial \theta_z}{\partial t} \quad \text{in} \quad 0 < x < L, \quad 0 < y < h, \quad t > 0 \\
\theta_z &= 0 \quad \text{at} \quad x = 0, \quad x = L, \quad t > 0 \\
\theta_z &= 0 \quad \text{at} \quad y = 0, \quad t > 0 \tag{2.9}
\end{align*}
\]

\[
k \frac{\partial \theta_z}{\partial y} = \delta(x - \xi) \quad \text{at} \quad y = h, \quad t > 0
\]

\[
\theta_z = 0 \quad \text{at} \quad t = 0
\]

Then by Duhamel’s theorem the solution of the problem (2.6) is written as

\[
\phi_z(x, y, t) = \int_{t=0}^{t} q^* f(\xi, \tau) \frac{\partial \theta_z(x, y, t - \tau)}{\partial t} d\tau \tag{2.10}
\]

Application of the same Green’s function as that in Equation (2.7) yields the solution for the auxiliary problem (2.9)

\[
\theta_z(x, y, t) = \frac{4}{L h k} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^{p+1} \sin(\beta_m \xi)}{\beta_m^2 + \eta_p^2} \left(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t}\right) \sin(\beta_m x) \sin(\eta_p y) \tag{2.11}
\]
where $\beta_m$, $\eta_p$ are the same as those in Equation (2.8b) and (2.8c), respectively.

Introducing Equation (2.11) to (2.10), the solution of problem (2.6) is obtained as

$$
\phi_\xi(x, y, t) = q^* \frac{4\alpha}{Lhk} \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} (-1)^{p+1} \sin(\beta_m x) \sin(\eta_p y) \sin(\beta_m \xi) \int_{0}^{t} f(\xi, \tau) e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)} d\tau \tag{2.12}
$$

Substituting Equation (2.8a) and (2.12) into (2.4), temperature profile of a time-dependent point heat source at location $(\xi, h)$ with working function $f(\xi, t)$ is obtained as

$$
T_\xi(x, y, t) = T_0 + \frac{4(T_i - T_0)}{Lh} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{1 - (-1)^m}{\beta_m \eta_p} e^{-\alpha(\beta_m^2 + \eta_p^2)\tau} \sin(\beta_m x) \sin(\eta_p y) \sin(\beta_m \xi) \int_{0}^{t} f(\xi, \tau) e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)} d\tau + q^* \frac{4\alpha}{Lhk} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} (-1)^{p+1} \sin(\beta_m x) \sin(\eta_p y) \sin(\beta_m \xi) \int_{0}^{t} f(\xi, \tau) e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)} d\tau \tag{2.13}
$$

2.2.3 Transient Temperature Profile $T(x, y, t)$ for the Entire Domain

In Eqn. (2.13) the last term, i.e., $\phi_\xi(x, y, t)$, is the only term that accounts for the contribution of a point heat source at location $(\xi, h)$ to the entire domain temperature. The temperature profile $T(x, y, t)$ due to all the point heat sources is determined by integrating the last term in Equation (2.13) over the top surface of the domain as

$$
T(x, y, t) = \phi_1(x, y, t) + \int_{\xi = 0}^{L} \phi_\xi(x, y, t) d\xi \tag{2.14}
$$

Substituting Equation (2.8a) and (2.12) into (2.14) and performing the indicated operations yields following solution for the transient field temperature of the entire domain.

$$
T(x, y, t) = T_0 + \frac{4(T_i - T_0)}{Lh} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{1 - (-1)^m}{\beta_m \eta_p} e^{-\lambda_{np}t} \sin(\beta_m x) \sin(\eta_p y)
+ q^* \frac{4\alpha}{Lhk} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} (-1)^{p+1} \sin(\beta_m x) \sin(\eta_p y) \int_{0}^{t} f(\xi, \tau) e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)} d\tau \tag{2.15a}
$$

$$
= \left\{ \frac{1 - (-1)^m}{\lambda_{np} \beta_m l} \sin(\beta_m l) \left(1 - e^{-\lambda_{np}l}\right) + \sum_{n=1}^{\infty} S_n \frac{\mu_n \sin(\mu_n t) + \lambda_{np} \cos(\mu_n t) - \lambda_{np} e^{-\lambda_{np}t}}{\mu_n^2 + \lambda_{np}^2} \right\}
$$

where

$$\mu_n = \frac{n\pi \nu}{L - l} \tag{2.15b}
$$

$$\lambda_{np} = \alpha(\beta_m^2 + \eta_p^2) \tag{2.15c}$$
\[
S_n = \begin{cases} 
\frac{1 - (-1)^{n+m}}{n\pi} \sin(\beta_n l) \left( \frac{1}{\beta_n - \mu_n / \nu} - \frac{1}{\beta_n + \mu_n / \nu} \right) & \text{if } \beta_n - \mu_n / \nu \neq 0 \\
\frac{1 - (-1)^{n+m}}{n\pi} \sin(\beta_n l) \left( -\frac{1}{\beta_n + \mu_n / \nu} \right) + \frac{L-l}{n\pi} \left[ 1 - \cos \left( \frac{n\pi l}{L-l} \right) \right] & \text{if } \beta_n - \mu_n / \nu = 0
\end{cases}
\]

(2.15d)

\(\beta_m, \eta_p\) are determined from Equation (2.8b) and (2.8c), respectively.

2.3 Results and Discussion

In this section, simulations are done using equation (2.15). Table 2.1 shows the input parameters. In the simulations, The truncation error is set to be \(10^{-5}\), and thus the maximum numbers of the terms for the infinite series in Equation (2.15) are \(m = 50\), \(p = 48\) and \(n = 7\). A finite element model, shown in Fig. 2.3, was also developed to verify the results. The top surface is divided into 30 elements. The flux traverses from the left to the right side after 20 steps and then oscillates back to the left. The step time \(\Delta t = L / 30 / \nu = 0.00833\) seconds. The user-defined subroutine DFLUX is used to apply the thermal load of the oscillatory flux. The simulations are done using the ABAQUS on a 3.2GHz Pentium 4 computer. It takes about eleven hours to perform 2000 seconds of simulations for the oscillatory heat flux. In contrast, the computations of the analytical solution take only a few minutes on the same computer. The temperature contour at the steady state is shown in Fig. 2.4.

<table>
<thead>
<tr>
<th>Table 2.1 Parameters used in the simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width of the domain (L) (m)</td>
</tr>
<tr>
<td>Height of the domain (h) (m)</td>
</tr>
<tr>
<td>Length of heat flux (l) (m)</td>
</tr>
<tr>
<td>Heat Flux (q'') (W/m²)</td>
</tr>
<tr>
<td>Velocity of oscillation (\nu) (m/s)</td>
</tr>
<tr>
<td>Thermal conductivity (k) (W/m·K)</td>
</tr>
<tr>
<td>Thermal diffusivity (\alpha) (m²/s)</td>
</tr>
<tr>
<td>Boundary Temperature (T_0) (°C)</td>
</tr>
<tr>
<td>Initial Temperature (T_i) (°C)</td>
</tr>
</tbody>
</table>

Figure 2.5 shows a comparison between the predicted analytical results and ABAQUS simulations for the temperature rise at locations \((L/2, h)\), \((L/6, h)\), \((5L/6, h)\) and \((L/2, h/2)\) specified in Fig. 2.3. The analytical and numerical results are in excellent agreement. In the analytical solution, the temperature rise at \((L/6, h)\) is a slightly higher than that at \((5L/6, h)\). The reason is that although the geometry of the domain is symmetric along the
center line in the y-direction, the motion of the flux is not exactly symmetric because of our assumption that the flux begins to oscillate from the left. If assuming the beginning location at the center of the top surface, the difference will disappear. The analytical solution provides a direct and efficient methodology for problems involving oscillatory heat flux.

![Fig. 2.3 Model of finite element method](image)

![Fig. 2.4 Temperature contour at steady state](image)

![Fig. 2.5 Comparison of temperature rise obtained analytically and by Finite Element Method at locations (L/2, h), (L/6, h), (5L/6, h) and (L/2, h/2)](image)
2.4 Concluding Remarks

In this chapter, an analytical method for treating heat conduction problems involving a body subjected to oscillating heat flux on one of its boundaries is developed. The method can be easily applied to the problem where the working function and fundamental solution are easily determined. Otherwise, the semi-analytical treatment of the problem reported in Ref. [33] can be used. Prediction results by the analytical solution and verification using finite element method are also presented. The analysis demonstrates that the result can efficiently predict the field temperature of such moving boundary problems or be used to verify the solution from other methods. The method can be readily extended to three-dimensional problems as well as problems involving other types of periodic heat flux on a boundary.
This chapter presents an analytical approach for treating problems involving oscillatory heat source. The transient temperature profile involving circular, rectangular and parabolic heat sources undergoing oscillatory motion on a semi-infinite body is determined by integrating the instantaneous solution for a point heat source throughout the area where the heat source acts with an assumption that the body takes all the heat. An efficient algorithm for solving the governing equations is developed. The results of a series simulations are presented, covering a wide range of operating parameters including a new dimensionless frequency $\tilde{\omega} = \omega l^2 / 4\alpha$ and the dimensionless oscillation amplitude $\tilde{A} = A/l$, whose product can be interpreted as the Peclet number involving oscillatory heat source, $Pe = \tilde{\omega}\tilde{A}$. Application of the present method to fretting contact is presented. The predicted temperature is in good agreement with published literature. Furthermore, analytical expressions for predicting the maximum surface temperature for different heat sources are provided by a surface-fitting method based on an extensive number of simulations.

3.1 Introduction

The frictional heat in sliding systems such as bearings, gears, and the like is a function of friction coefficient, sliding velocity and pressure in the contact area. The heat causes high flash temperatures on the contacting bodies and a relatively steep temperature gradient in the substrate. The temperature field in the contact zone can have a significant effect on the material properties and the thermal contact stresses. Thus, the knowledge of temperature in dry or lubricated sliding contact plays a vital role in the design of machine elements. Since direct temperature measurement is often impractical, efficient models are needed at the design stage to predict the friction-induced temperatures rise under sliding conditions.

Surface temperature at a sliding contact interface has long been of interest. Pioneering work on the moving heat source problems was reported by Blok [1] with particular interest in the meshing of gear teeth, which gave birth to the concept of flash temperature. The work was later extended by Jaeger [2] who expressed the surface temperature of each solid in terms of heat flux. Francis [38] gave an analytical expression for the steady state interfacial temperature distribution in a sliding circular Hertzian contact. Kuhlmann-Wilsdorf [39] developed expressions for the evaluation of flash temperature at circular and elliptical contact spots. Tian & Kennedy [3] analyzed the maximum and average surface temperature rise for a semi-infinite body due to different moving heat sources for the entire Peclet number range. Bos and Moes [40] presented a numerical algorithm to solve the steady state heat partition relationship and the associated flash temperatures for arbitrary shaped contacts by matching the surface temperatures inside the contact area.

Greenwood [41] repeated one of cases treated by Bos and Moes using a different method and shed some light on the concept of a heat partitioning factor. All of these studies have concentrated primarily on the steady state condition.

The transient analyses of the flash temperature were included in references [5-7, 9, 10, 30, 31, 42, 43]. Hirano & Yoshida [5] analyzed the surface temperature of a semi-infinite body subjected to a rectangular heat source with reciprocating motion. Gecim & Winer [42] derived an analytical expression for a two-dimensional transient temperature distribution in the vicinity of a small stationary circular heat source. Greenwood & Alliston-Greiner [6] presented an analysis of surface temperature in fretting contact by separating the sinusoidal heat source into a constant and a periodic heat input using the Fourier transformation, assuming that the heat source remains stationary. Ovaert & Talmage [43] investigated the temperature in an anisotropic half-space under a sliding rectangular heat source. Qiu & Cheng [31], Gao & Lee [30] developed a transient temperature model based on the fast Fourier transform method and the moving grid method, respectively. Kennedy et al. [9, 10] performed a finite element analysis of sliding surface temperatures. Tian & Kennedy [7] studied the problem of frictional heating in oscillatory sliding, and applied the Green’s function method for finite contacting bodies derived in the parallel study [8] to predict the surface temperature rise. Recently, Mansouri & Khonsari [13] developed a model to predict the surface temperature in a pin-bushing system with relative oscillatory motion. Additional references on published research on flash temperature can be found in [32], where the differences between the predictions of various models are also discussed.

A survey of published literature reveals that with an exception of a few papers [5-10, 13], most studies dealing with this aspect have focused their attention to unidirectional, moving heat source. In contrast, the surface temperature solution in oscillatory sliding differs from that in unidirectional sliding: the heat source is time varying and the sliding motion is periodic. In this study, the transient temperature profile for different oscillating heat sources on a semi-infinite body is derived by integrating the solution for an instantaneous point heat source throughout the area where the heat source acts. Results of a parametric study for a wide range of dimensionless operating parameters are presented. Analytical expressions for maximum surface temperature for different types of heat sources are provided by a surface-fitting method based on an extensive number of simulations.

### 3.2 Analytical Development

Consider a system consisting of a stationary body A in contact with a sliding body B undergoing oscillating motion. If all the energy dissipated in the frictional contact is converted into heat, the heat flux density on the body A is given by

\[
q = \mu p |v|
\]  

(3.1)

where \( \mu \) is the coefficient of friction, \( p \) is the contact pressure, and \( v(t) \) is the oscillatory velocity. It is assumed that slider B takes no heat.
In most applications the oscillatory heat source acts over a relatively small portion of body A so that body A can be treated as a semi-infinite geometry. Referring to Fig. 3.1, a semi-infinite body is subjected to heat source \( q(x', y', z', t) \) that oscillates with velocity \( \nu(t) \) along the \( x \) direction and acts on the area \( \Omega(t) \) at time \( t \). \( o'\xi\eta \) is a local coordinate system parallel to coordinate system \( oxy \) with origin \( o' \) at the center of the flux and moving with it. \( s(t) \) is the displacement of the origin \( o' \) relative to \( o \). Thus, the coordinates \((x', y', z')\) of points on the surface can be given in the coordinate system \( o'\xi\eta \) as

\[
\begin{align*}
  x'(\xi, t) &= s(t) + \xi \\
  y'(\eta) &= \eta \\
  z' &= 0
\end{align*}
\]

(3.2a)

where

\[
s(t) = \int_0^t \nu(\tau) d\tau
\]

(3.2b)

and the heat source can be rewritten as

\[
q = \begin{cases} 
  q(x', y', z', t) & \text{if } (x', y', z') \in \Omega(t) \\
  0 & \text{if } (x', y', z') \notin \Omega(t)
\end{cases}
\]

(3.3)

Fig. 3.1 Semi-infinite body subjected to an oscillating heat source

The solution for the temperature in the half space is based on the solution for an instantaneous point heat source at location \((x', y', z')\), which is given by Carslaw & Jaeger [14] as

\[
\varphi(x, y, z, t \mid x', y', z') = \frac{1}{4\rho\epsilon(\pi\alpha)^{3/2}} \int_0^t q(x', y', z', t') dt' \times \exp \left\{ -\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4\alpha(t-t')} \right\} \frac{dt'}{(t-t')^{3/2}}
\]

(3.4)
where $\phi(x,y,z,t | x',y',z')$ represents the temperature rise at $(x,y,z)$ at time $t$ in a semi-infinite body subjected to a point heat source at $(x',y',z')$ up to time $t$. The parametric $\rho$ denotes the density of the material, $c$ is the specific heat, $\alpha$ denotes the thermal diffusivity, and $q(x',y',z',t')$ represents the point flux.

Introducing Eq. (3.3) into (3.4) and integrating the result throughout the surface $\Omega$ where the heat source acts results in the following expression for the temperature field in the body.

$$T(x,y,z,t) = \frac{1}{4\rho c \pi \alpha^{3/2}} \int_0^t dt' \int_{\Omega} q(x',y',z',t) \times \exp\left(-\frac{\left[(x-x'(\xi,t' \rightarrow t))^2 + (y-y'(\eta))^2 + (z-z')^2\right]}{4\alpha(t-t')}\right) d\xi d\eta$$

(3.5)

where $x'(\xi,t' \rightarrow t) = s(t' \rightarrow t) + \xi$, and $s(t' \rightarrow t)$ is the displacement of the heat source from time $t'$ to $t$.

Letting $\tau = t-t'$, Eq. (3.5) becomes

$$T(x,y,z,t) = \frac{1}{4\rho c \pi \alpha^{3/2}} \int_0^\tau d\tau \int_{\Omega} q(\xi,\eta,t-\tau) \times \exp\left(-\frac{\left[(x-s(t-\tau \rightarrow t)-\xi)^2 + (y-\eta)^2 + z^2\right]}{4\alpha \tau}\right) d\xi d\eta$$

(3.6)

Equation (3.6) represents the temperature field for a moving heat source. In the following sections we treat three common categories of heat source configurations shown in Fig. 3.2: circular, rectangular, and elliptical. The heat source is assumed to oscillate sinusoidally according to

$$\nu(t) = A \omega \sin(\omega t)$$

(3.7a)

where $A$ is the oscillation amplitude, $\omega$ is the angular frequency. The corresponding displacement is given by:

$$s(t) = A [1 - \cos(\omega t)]$$

(3.7b)

It is noted that Eq. (3.6) gives the temperature variation at a point stationary relative to the center of the oscillatory heat source. If the neutral position of oscillation is fixed on the origin of the coordinate system $\text{ox}$, the temperature variation at a fixed point on the surface of a semi-infinite body can be obtained by substituting $x$ in Eq. (3.6) by $x + s(t) - A = -A \cos(\omega t)$, where $-A$ is the initial $x$ coordinate of the center of the heat source in the fixed coordinate system.

The following dimensionless parameters are used in the derivation of the solutions.

$$\overline{X} = \frac{x}{l}, \quad \overline{Y} = \frac{y}{l}, \quad \overline{Z} = \frac{z}{l}, \quad \overline{A} = \frac{A}{l}$$

$$\overline{\omega} = \frac{\omega l^2}{4\alpha}, \quad \Phi = \omega t, \quad \phi = \omega \tau$$

(3.8)
where \( l \) is half of the contact length along the sliding direction. For circular heat source \( l = R \), rectangular heat source \( l = l_x \), and elliptical heat source \( l = a \). The product of \( \bar{\omega} \) and \( \bar{A} \) can be interpreted as the Peclet number for the oscillation case (i.e. \( Pe = \bar{\omega} \bar{A} \)).

![Diagram of heat source configurations]

Fig. 3.2 Heat source configurations: (a) Circular heat source. (b) Rectangular heat source. (c) Elliptical heat source.

### 3.2.1 Circular Configuration

**Case I. Uniform heat source**

Referring to Fig. 3.2(a), for the case of a uniform heat source, the heat distribution is

\[
q(t) = q_0 |\sin(\omega t)|
\]

where \( q_0 = \mu p_0 A \omega \), and \( p_0 \) is the average contact pressure.

From Eq. (3.6), the transient temperature in the dimensionless form (see Eq. (3.8) for the dimensionless parameters) is

\[
T(X, Y, Z, \Phi) = \frac{\sqrt{\bar{\omega}}}{4\pi^{3/2}} \int_0^\Phi |\sin(\Phi - \phi)| d\phi \times \int_0^{2\pi} \int_{r=0} r \left[ \frac{[\bar{X} + \bar{A} \cos(\Phi) - \bar{A} \cos(\Phi - \phi) - \bar{r} \cos(\theta)]^2 + [\bar{Y} - \bar{r} \sin(\Phi)]^2 + \bar{Z}^2}{\bar{\omega}} \right] r dr d\theta
\]

where \( T(X, Y, Z, \Phi) = T_k / 2Rq_0 \), \( R \) is the radius of the heat source.

**Case II. Parabolic heat source**

For the case of a parabolic heat source
\[
q(r,t) = q_m \left(1 - r^2 / R^2 \right)^{1/2} \left|\sin(\omega t)\right|
\]
where \( q_m = \mu \rho_m A \omega \), and \( p_m \) is the maximum contact pressure, the dimensionless transient temperature is

\[
\bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = \frac{\sqrt{\phi}}{4\pi^{3/2}} \int_0^\phi \sin(\Phi - \phi) d\phi \int_0^{2\pi} \int_{\eta - 1}^{1} \left(1 - \bar{r}^2\right)^{1/2} \exp \left\{ - \frac{\left[\bar{X} - \bar{A} \cos(\Phi) - \bar{A} \cos(\Phi - \phi) - \bar{X}\right]^2 + \left[\bar{Y} - \bar{r} \sin(\theta)\right]^2 + \bar{Z}^2}{\phi / \phi^3} \right\} d\bar{r} d\bar{r} d\theta
\]

where \( \bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = Tk / 2Rq_m \).

It should be noted that the Eqs. (3.10) and (3.12) are singular at \( \phi = 0 \). As described in section 3.3, these integrations are evaluated numerically.

### 3.2.2 Rectangular Configuration

For a uniformly distributed rectangular heat source \( q(t) = q_0 \left|\sin(\omega t)\right| \), and \( q_0 = \mu \rho_n A \omega \), as shown in Fig. 3.2(b). Based on Eq. (3.6) the dimensionless temperature solution (see Eq. (3.8) for the dimensionless parameters) is:

\[
\bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = \frac{\sqrt{\phi}}{4\pi^{3/2}} \int_0^\phi \sin(\Phi - \phi) d\phi \int_{\xi - 1}^{1} \int_{\eta - 1}^{1} \exp \left\{ - \frac{\left[\bar{X} + \bar{A} \cos(\Phi) - \bar{A} \cos(\Phi - \phi) - \bar{X}\right]^2 + \left[\bar{Y} - \bar{r} \sin(\theta)\right]^2 + \bar{Z}^2}{\phi / \phi^3} \right\} d\bar{z} d\bar{y} d\theta
\]

where \( \bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = Tk / 2l_1 q_0 \), and \( \epsilon_r = l_1 / l_x \).

Note that Eq. (3.13) is singular at \( \phi = 0 \). If one is interested in the surface temperature, where the maximum temperature occurs, then it is possible to analytically remove the singularity by separating the integration interval into two segments: a very small one including the singular points and the rest without singularity [5, 31]. When \( z = z_0 \Rightarrow \bar{Z} = 0 \), Eq. (3.13) becomes

\[
\bar{T}(\bar{X}, \bar{Y}, \Phi) = \frac{\sqrt{\phi}}{4\pi^{3/2}} \left(I_1 + I_2\right)
\]

where

\[
I_1 = \int_0^\phi \sin(\Phi - \phi) d\phi \int_{\xi - 1}^{1} \int_{\eta - 1}^{1} \exp \left\{ - \frac{\left[\bar{X} + \bar{A} \cos(\Phi) - \bar{A} \cos(\Phi - \phi) - \bar{X}\right]^2 + \left[\bar{Y} - \bar{r} \sin(\theta)\right]^2 + \bar{Z}^2}{\phi / \phi^3} \right\} d\bar{z} d\bar{y} d\theta
\]

\[
I_2 = \int_{\phi - \epsilon}^{\phi} \sin(\Phi - \phi) d\phi \int_{\xi - 1}^{1} \int_{\eta - 1}^{1} \exp \left\{ - \frac{\left[\bar{X} + \bar{A} \cos(\Phi) - \bar{A} \cos(\Phi - \phi) - \bar{X}\right]^2 + \left[\bar{Y} - \bar{r} \sin(\theta)\right]^2 + \bar{Z}^2}{\phi / \phi^3} \right\} d\bar{z} d\bar{y} d\theta
\]
$I_1 = \int_0^{\Phi^*} \frac{\sin(\Phi - \phi)}{\phi^{3/2}} \, d\phi$

$\times \int_{\xi=-1}^{\xi=1} \int_{\eta=-\epsilon_\xi}^{\eta=\epsilon_\xi} \exp\left( \left[ \frac{X + A \cos(\Phi) - A \cos(\Phi - \phi) - \xi}{\phi} \right]^2 + \left[ \frac{Y - \eta}{\phi} \right]^2 \right) \, d\xi \, d\eta \quad (3.14b)$

$I_2 = \int_0^{\Phi^*} \frac{\sin(\Phi - \phi)}{\phi^{3/2}} \, d\phi$

$\times \int_{\xi=-1}^{\xi=1} \int_{\eta=-\epsilon_\xi}^{\eta=\epsilon_\xi} \exp\left( \left[ \frac{X + A \cos(\Phi) - A \cos(\Phi - \phi) - \xi}{\phi} \right]^2 + \left[ \frac{Y - \eta}{\phi} \right]^2 \right) \, d\xi \, d\eta \quad (3.14c)$

and $\Phi^*$ is a small value in $[0, \Phi]$.

$I_1$ has no singularity and can be integrated directly by an appropriate numerical method. To evaluate $I_2$, we rewrite Eq. (3.14c) as

$I_2 = \int_0^{\Phi^*} \frac{\sin(\Phi - \phi)}{\phi^{3/2}} \, d\phi$

Where

$I_\xi = \int_{\xi=-1}^{\xi=1} \phi^{-1/2} \exp\left( -\frac{\left[ X + A \cos(\Phi) - A \cos(\Phi - \phi) - \xi \right]^2}{\phi} \right) \, d\xi \quad (3.15b)$

$I_\eta = \int_{\eta=-\epsilon_\xi}^{\eta=\epsilon_\xi} \phi^{-1/2} \exp\left( -\frac{\left( \frac{Y - \eta}{\phi} \right)^2}{\epsilon_\xi} \right) \, d\eta = \sqrt{\frac{\pi}{4\epsilon_\xi}} \left[ \text{erf}\left( \frac{1 + X + A \cos(\Phi) - A \cos(\Phi - \phi)}{\phi} \right)^{1/2} + \text{erf}\left( \frac{1 - X - A \cos(\Phi) + A \cos(\Phi - \phi)}{\phi} \right)^{1/2} \right] \quad (3.15c)$

It is noted that $I_\xi$ and $I_\eta$ have the following properties when $\Phi^*$ is small enough.

i) $I_\xi = \sqrt{\frac{\pi}{4\epsilon_{\xi}}}$, when $|X| = 1$ \hspace{1cm} (3.16a)

ii) $I_\xi = \sqrt{\frac{\pi}{\epsilon_{\xi}}}$, when $|X| < 1$ \hspace{1cm} (3.16b)

iii) $I_\xi = 0$, when $|X| > 1$ \hspace{1cm} (3.16c)

and

i) $I_\eta = \sqrt{\frac{\pi}{4\epsilon_{\eta}}}$, when $|Y| = \epsilon_R$ \hspace{1cm} (3.17a)

ii) $I_\eta = \sqrt{\frac{\pi}{\epsilon_{\eta}}}$, when $|Y| < \epsilon_R$ \hspace{1cm} (3.17b)
\[ I_n = 0, \quad \text{when } |\bar{Y}| > \varepsilon_R \quad (3.17c) \]

Generally, \( \Phi^* \) is set such that either \( \sin(\Phi - \phi) \geq 0 \) or \( \sin(\Phi - \phi) < 0 \) for \( \phi \in [0, \Phi^*] \), so that \( I_2 \) can be evaluated by Fresnel integrals as

\[
I_2 = I_\xi I_\eta \int_0^{\Phi^*} \frac{\sin(\Phi - \phi)}{\phi^{1/2}} d\phi
\]

\[
= \pm I_\xi I_\eta \left[ \sin(\Phi) \int_0^{\Phi^*} \frac{\cos(\phi)}{\phi^{1/2}} d\phi - \cos(\Phi) \int_0^{\Phi^*} \frac{\sin(\phi)}{\phi^{1/2}} d\phi \right] \quad (3.18a)
\]

\[
= \pm \sqrt{2\pi} I_\xi I_\eta \left[ \sin(\Phi) C\left(\frac{\sqrt{2\Phi^*}}{\pi}\right) - \cos(\Phi) S\left(\frac{\sqrt{2\Phi^*}}{\pi}\right) \right]
\]

where “+” applies when \( \sin(\Phi - \phi) \geq 0 \) and “−” when \( \sin(\Phi - \phi) < 0 \). Also,

\[
C\left(\frac{\sqrt{2\Phi^*}}{\pi}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(4n+1) \cdot (2n)!} \left(\frac{\sqrt{2\Phi^*}}{\pi}\right)^{4n+1} \quad (3.18b)
\]

\[
S\left(\frac{\sqrt{2\Phi^*}}{\pi}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(4n+3) \cdot (2n+1)!} \left(\frac{\sqrt{2\Phi^*}}{\pi}\right)^{4n+3} \quad (3.18c)
\]

For small values of \( \Phi^* \), Eqs. (3.18b) and (3.18c) yield good approximation for the Fresnel integrals. However, when \( \sin(\Phi - \phi) \) is not positively or negatively defined for \( \phi \in [0, \Phi^*] \), Eq. (3.18a) cannot be used directly. The possible cases are specified in Fig. 3.3(a). When one of the cases occurs, decrease \( \Phi^* \) such that \( \Phi - \Phi^* \) lies on the \( x \) coordinate axis shown in Fig. 3.3(b). Then \( \sin(\Phi - \phi) \) has definite sign (or equals to zero) and Eq. (3.18a) can be used to evaluate \( I_2 \).

Therefore, the solution for the dimensionless surface temperature is obtained by Eq. (3.14a) after \( I_1, I_2 \) are evaluated by Eqns (3.14b) and (3.18a), respectively.

Fig. 3.3 Cases where \( \sin(\Phi - \phi) \) is not positively or negatively defined. (a). Before adjusting \( \Phi^* \). (b) After adjusting \( \Phi^* \). (fig. cont’d)
3.2.3 Elliptical Configuration

Temperature solutions for the elliptical heat source as shown in Fig. 3.2(c) can be evaluated from those of the circular heat source by transforming an elliptical configuration of heat source to a circular one.

Case I. Uniform heat source

The dimensionless temperature solution for a uniform heat source $q(t) = q_0 |\sin(\omega t)|$ and $q_0 = \mu p_c A \omega$ is

$$T(\bar{X}, \bar{Y}, Z, \Phi) = \frac{\epsilon_k \sqrt{\epsilon}}{4\pi^{3/2}} \int_{0}^{\phi} \frac{\sin(\Phi - \phi)}{\phi^{1/2}} d\phi$$

$$\times \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi} \exp \left( \frac{-[\bar{X} + \bar{A}(\Phi - \Phi)] + [\bar{Y} - \epsilon_k \bar{F} + \sin(\theta)]^2 + Z^2}{\phi/\omega} \right) \rho d\theta$$

where $T(\bar{X}, \bar{Y}, Z, \Phi) = Tk / 2aq_0$, and $\epsilon_k = b/a$.

Case II. Parabolic heat source

For the case of a parabolic heat source

$$q(\xi, \eta, t) = q_m \left( 1 - \xi^2/a^2 - \eta^2/b^2 \right)^{1/2} |\sin(\omega t)|$$

where $q_m = \mu p_c A \omega$, and $p_c$ is the maximum contact pressure. The dimensionless temperature is
\[ \bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = \frac{e_E \sqrt{\alpha}}{4\pi^{3/2}} \int_0^\alpha \frac{|\sin(\Phi - \phi)| d\phi}{\phi^{3/2}} \int_{\beta=0}^{2\pi} \int_{\gamma=0}^1 (1 - \bar{r}^2)^{1/2} \times \exp \left\{ -\left[ \bar{X} + \bar{A}\cos(\Phi) - \bar{A}\cos(\Phi - \phi) - \bar{r}\cos(\theta) \right]^2 + \left[ \bar{Y} - e_E \bar{r}\sin(\theta) + \bar{Z} \right]^2 \right\} \bar{r} d\bar{r} d\theta \]  

where \( \bar{T}(\bar{X}, \bar{Y}, \bar{Z}, \Phi) = Tk / 2aq_m \).

Note that the circular heat source is a special case of the elliptical heat source with \( e_E = 1 \).

### 3.3 Numerical Solution Scheme

A computer program is developed to obtain the general solutions of Eq. (3.6) for different types of oscillatory heat sources, i.e. Eqs (3.10) and (3.12) for circular heat source, (3.13) for rectangular heat source, and (3.19) and (3.21) for elliptical heat source. Surface temperature for rectangular heat source can alternatively be evaluated by Eq. (3.14a) with singularity analytically removed. With \( F(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi) \) and \( g(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi, \bar{\xi}, \bar{\eta}) \) denoting the corresponding terms in the equations, the dimensionless temperature can be expressed in the general form of

\[ \bar{T}(\bar{x}, \bar{y}, \bar{z}, \Phi) = \frac{\sqrt{\alpha}}{4\pi^{3/2}} \int_\phi F(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi) d\phi \]  

The integration in Eq. (3.22) is carried out using the so-called Cautious Adaptive Romberg Extrapolation (CADRE) [44] with relative accuracy to four digits as

\[ \bar{T}(\bar{x}, \bar{y}, \bar{z}, \Phi) = \frac{\sqrt{\alpha}}{4\pi^{3/2}} \sum_i^{n_i} \int_{\phi_i}^{\phi_{i+1}} F(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi) d\phi \]  

The dimensionless time interval \( \left[ \phi_0, \phi_{i+1} \right] \) is chosen based on the oscillating frequency. The integration within each of the intervals is carried out by CADRE where the value of the integrand at the singular end point is set to be zero. The algorithm is adaptive and based on extrapolation applied to trapezoidal rule estimates. The integration in Eq. (3.23) is carried out using Gauss quadrature, which is performed over uniform sub-domains as

\[ F(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi) = \sum_{\xi}^{n_\xi} \sum_{\eta}^{n_\eta} \sum_{j=1}^{n_j} \sum_{k=1}^{n_k} w_j w_k g(\bar{x}, \bar{y}, \bar{z}, \Phi, \phi, \bar{\xi}_j, \bar{\eta}_k) \]  

where \( \Delta \bar{\xi}, \Delta \bar{\eta} \) denote the summation taken over the sub-domains of \( \bar{\xi} \) and \( \bar{\eta} \), respectively. The parameters \( \bar{\xi}_j, w_j, n_\xi, \) and \( \bar{\eta}_k, w_k, n_\eta \) represent the abscessas, the weight factors and the number of quadrature points associated with \( \bar{\xi} \) direction and \( \bar{\eta} \) direction, respectively. To ensure that the solution is independent of the number of the quadrature points, different numbers of the quadrature points are used for the same simulation. In the present study, \( n_\xi = n_\eta = 4 \), and the number of sub-domains used in
Gauss quadrature is 6. A steady state condition is assumed once the change in the dimensionless temperature by Eq. (3.22), computed between two consecutive cycles, is less than $6 \times 10^{-7}$.

The procedure described can also be applied to the unidirectional, sliding heat source. When the flux in Eq. (3.1) and displacement $s(t)$ in Eq. (3.2) are evaluated by setting a constant velocity $v(t) = U$, the transient temperature for the unidirectional sliding case under different heat sources can be derived from Eq. (3.6) similarly, and the same steady state results as published literature can be obtained by the same program after a short period of time.

### 3.4 Verification of the Method

In this section the program developed by the present method is verified by applying it into unidirectional sliding contact and oscillating contact. The results are compared with those by Tian & Kennedy [3] for unidirectional sliding case and by Greenwood & Alliston-Greiner [6] for oscillating case.

#### 3.4.1 Unidirectional Sliding Contact

Figure 3.4 shows the comparison of temperature rise under a uniform circular heat source at different Peclet numbers predicted by the present method and the corresponding steady state solutions obtained by Tian & Kennedy [3]. The solution is obtained by setting a constant velocity $U$ and running the program up to the steady state. The temperature distribution is compared along the center line of the heat source in the direction of the sliding motion. The figure shows excellent agreement between the results. The results of other types of heat sources also yield consistent conclusions with those reported in [3]. When Peclet number is small, the temperature distribution is close to the solution for stationary case. The temperature distribution becomes nearly symmetric along the axis $\xi = 0$ and the maximum temperature occurs at the center of the contact area. As Peclet number is increasing, the location of the maximum temperature moves from the center to the edge. The shapes of the heat sources have little effect on the temperature distributions outside the contact area. However, the distribution of the heat flux has a great influence on the temperature distributions within the contact interface.

Figure 3.5 shows the temperature variations up to steady state at the center of contact region under a moving uniform circular heat source at different Peclet numbers. It shows that a lower Peclet number results in a greater steady state temperature, and a longer time-to-steady-state. For example, at $P_c = 0.01$, 99.95% of steady state dimensionless temperature rise, 0.497, is reached within 100s. In contrast, at $P_c = 10$, the dimensionless temperature rise at steady state is 0.125, and the steady state is reached within $2.125 \times 10^{-3}$ s. Similar conclusions can be drawn for other heat sources.
Fig. 3.4 Comparison of surface temperature for a moving uniform circular heat source on a semi-infinite medium. $\xi$ is the axis passing through the center of the heat source along sliding direction. $P_e = UR/2\alpha$, here $R$ is radius of the circular contact area. $q_o = \mu p_o U$.

Fig. 3.5 Temperature variation up to steady state at the center of the contact area for a moving uniform circular heat source on a semi-infinite medium. $P_e = UR/2\alpha$, here $R$ is radius of the circular contact area. $q_o = \mu p_o U$. 
3.4.2 Oscillating Contact

In an analytical treatment of the maximum surface temperatures for fretting contact, Greenwood & Alliston-Greiner [6] considered a distributed heat source whose position was assumed to be stationary. Greenwood & Alliston-Greiner argued that in a typical fretting problem the amplitude of oscillation is small enough that the heat source can be assumed to be stationary. To verify the results, a very small amplitude is used in the present simulations and the solution for uniform circular heat source is considered for an appropriate comparison. The solution for maximum surface temperature in [6] was split into two parts: steady temperature and periodic temperature. The relative parameter controlling the period temperature is: $\gamma = \frac{R\sqrt{\omega}}{\alpha}$.

Figure 3.6 presents a comparison between the steady temperature by the present method and those in reference [6] for different values of $\gamma$. The overall maximum temperatures during the first five cycles are compared in Fig. 3.7. Both solutions are in good agreement. From Fig. 3.6, it can be noted that the smaller the parameter $\gamma$, the shorter is the time-to-steady-state, and a greater the maximum steady temperature becomes. Figure 3.7 shows that the amplitude of temperature oscillation decreases with increasing value of $\gamma$. More details will be discussed in the following section.

![Graph showing comparison of maximum steady temperature variations at different $\gamma$ for oscillating uniform circular heat source on a semi-infinite medium.]

$\gamma = \frac{R\sqrt{\omega}}{\alpha}$. 

Fig. 3.6 Comparison of maximum steady temperature variations at different $\gamma$ for oscillating uniform circular heat source on a semi-infinite medium. $q_0 = \mu p_o A \omega$. 

$\gamma = \frac{R\sqrt{\omega}}{\alpha}$. 

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Fig. 3.7 Comparison of the maximum overall temperature at different $\gamma$ for oscillating uniform circular heat source on a semi-infinite medium. $q_0 = \mu p_0 A \omega$. $\gamma = R\sqrt{\omega/\alpha}$

3.5 Results and Discussion

In the first two parts of this section, a series of results are presented to study the effect of the dimensionless frequency $\bar{\omega} = \omega l^2 / 4\alpha$ and the dimensionless amplitude $\bar{A} = A/l$, hence the Peclet number $P_e$ on the maximum dimensionless temperature. The square heat source is used in these simulations. In the third part analytical expressions for the maximum surface temperatures for different heat sources are provided by a surface-fitting method based on an extensive number of simulations. The surface temperature at the center of the contact is assumed to be maximum, as revealed by Greenwood & Alliston-Greiner [6] and also demonstrated in the following figures of surface temperature distribution. In the first two parts, simulations are carried out by varying only one parameter at a time. A steady state condition is assumed once the change in the dimensionless temperature, computed between two consecutive cycles, is less than $6 \times 10^{-7}$. Table 3.1 shows the input data for the simulations, where the first two columns show the range of values for each parameter, the third column shows the fixed values for the corresponding parameters when the other parameter is considered.
Table 3.1 Dimensionless input data

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
<th>Fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\omega}$</td>
<td>0.001</td>
<td>1140</td>
<td>190</td>
</tr>
<tr>
<td>$\tilde{A}$</td>
<td>0.001</td>
<td>1.6</td>
<td>0.105</td>
</tr>
</tbody>
</table>

3.5.1 Effect of the Dimensionless Frequency $\tilde{\omega}$

Figure 3.8(a) demonstrates the effect of the dimensionless frequency $\tilde{\omega}$ on the maximum dimensionless temperature at a fixed dimensionless amplitude of $\tilde{A} = 0.105$. A uniform square heat source was used in these simulations. The behavior at low value of $\tilde{\omega}$ is best revealed on a log-scale as shown in Fig. 3.8(b). It can be noted that a larger value of dimensionless frequency $\tilde{\omega}$ results in a lower steady state temperature, but requires a larger number of cycles to reach steady state. This behavior is consistent with the effect of Peclet number on a pin-bushing system with relative oscillatory motion investigated by Mansouri & Khonsari [13]. At low values of $\tilde{\omega}$, the maximum temperature is close to that of a stationary case. The dimensionless maximum temperature approaches a constant value, 0.3547. This value is very close to the dimensionless steady temperature presented by Greenwood & Alliston-Greiner [6], who considered a square heat source behaving approximately as a circular heat source and obtained the limit of maximum steady temperature as

$$T_{s,\text{max}} = \frac{1.1222}{\pi} \cdot \frac{2q_0l}{k} \approx 0.3572 \cdot \frac{2q_0l}{k}$$

(3.26)

where $l$ is half side length of the square heat source.

A higher value of $\tilde{\omega}$ corresponds to a higher oscillating velocity, which leads to more heat flowing away along the sliding direction instead of the normal direction. This results in a lower dimensionless steady state temperature.

The effect of the dimensionless frequency $\tilde{\omega}$ on the periodic oscillation of the maximum dimensionless temperature for a uniform square heat source is illustrated in Fig. 3.9, which shows the variations of the maximum overall dimensionless temperature at steady state. At low $\tilde{\omega}$, the periodic oscillation of the maximum temperature plays a dominant role. The dimensionless maximum overall temperature is 0.5527 when $\tilde{\omega} = 0.001$ as shown in Fig. 3.9. This value is close to the limit of the dimensionless overall temperature reported in reference [6] as

$$T_{o,\text{max}} = \frac{1.1222q_0l}{k} = 0.5611 \cdot \frac{2q_0l}{k}$$

(3.27)

The oscillation of periodic temperature decreases with increasing $\tilde{\omega}$ and becomes unimportant when $\gamma > 5$ reported in [6], corresponding to $\tilde{\omega} > 6.25$. 

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Fig. 3.8 Effect of the dimensionless frequency $\bar{\omega}$ on the maximum dimensionless temperature for uniform square heat source. (a) Dimensionless temperature as a function of cycles. $\bar{\omega} = \omega l^2 / 4\alpha$. (b) Steady state versus the dimensionless frequency, $\bar{\omega} = \omega l^2 / 4\alpha$
Fig. 3.9 Maximum dimensionless overall temperature variation at steady state at different dimensionless frequency $\bar{\omega}$ for uniform square heat source. $\bar{\omega} = \omega l^2 / 4\alpha$

Fig. 3.10 Distribution of dimensionless overall temperature for oscillating uniform square heat source along the sliding direction with the center of the contact region as the origin. $\bar{\omega} = \omega l^2 / 4\alpha$
Fig. 3.11 Distribution of dimensionless overall temperature for oscillating uniform circular heat source along the sliding direction with the center of the contact region as the origin. \( \bar{\omega} = \omega l^2 / 4 \alpha \) and \( l = R \).

Fig. 3.12 Distribution of dimensionless overall temperature for oscillating parabolic heat source along the sliding direction with the center of the contact region as the origin. \( q_0 = 2q_m / 3 \) and \( q_m = \mu \rho_m A \omega \). \( \bar{\omega} = \omega l^2 / 4 \alpha \) and \( l = R \).
Figure 3.10 shows the temperature distribution along the direction of the sliding motion when the overall maximum temperature is at the zenith and nadir of the periodic oscillation corresponding to Fig. 3.9 for a uniform square heat source at $\bar{\omega}=0.001$, 1, 6.25 and 47.5. For comparison the temperature distribution for uniform circular heat source and parabolic heat source are presented in Figs. 3.11 and 3.12. It can be noted that the temperature distribution is nearly symmetric along the line $\xi=0$ because the source heats the same contact area periodically during the oscillating process and is almost symmetric with respect to the center of the contact region, which demonstrates that the maximum temperature occurs near the center of the contact region. Strictly speaking, the temperature distributions on the two sides of the contact center are not identically the same. The side that possesses a higher temperature reveals that it has just undergone heating at that instant. In Figs. 3.10-12, the left side is just heated up by the flux there at the time the temperature distribution is picked up, on the other hand the right side experiences a relative longer time of heat dissipation. Therefore, the temperature on the left side is slightly higher than that on the right side, resulting in a slight shifting of the maximum temperature to the left. For the minimum temperature at $\bar{\omega}=0.001$, this effect becomes negligible due to very low sliding velocity. This effect becomes smaller with increasing $\bar{\omega}$ as shown in the figures because the flux with higher oscillating frequency heats both sides more evenly.

The distribution of the steady state temperature in one cycle varies between the two curves corresponding to the minimum and maximum temperatures as shown in Figs. 3.10-12. In addition, comparison of the surface temperature distribution for different heat sources in Figs. 3.10-12 reveals that the temperature distribution within the contact region $|\xi/l|<1$ is significantly affected by the distribution of the heat sources. The parabolic heat source results in a higher temperature concentration at the center of the contact region, hence a higher maximum temperature. The shape of the heat source has little effect on the temperature distribution outside the contact region. These are consistent with those for sliding case with constant velocity.

### 3.5.2 Effect of the Dimensionless Amplitude $\bar{A}$

The effect of dimensionless oscillation amplitude $\bar{A}$ on the maximum dimensionless surface temperature for a uniform square heat source at a fixed dimensionless frequency $\bar{\omega}=190$ is illustrated in Fig. 3.13(a). Increasing the amplitude yields a lower surface temperature. This is consistent with the trends of results in [13]. The cycles-to-steady-state is almost the same, only slightly decreasing with $\bar{A}$ increasing as shown in Fig. 3.13(b). At low amplitudes, the maximum dimensionless temperature approaches a constant value, 0.3416, which is close to 0.3572 in reference [6] based on the solution for a stationary heat source. Physically, with increasing amplitude, the frictional heat is dissipated over a larger surface, resulting in a lower dimensionless surface temperature. Figure 3.14 shows the variation of the maximum dimensionless overall temperature at different dimensionless amplitudes. It is noted that a higher amplitude yields a greater
temperature oscillation. When $\overline{A} > 1.0$, the center of the contact region is heated intermittently, thus contributing to a greater oscillation of the surface temperature.

Fig. 3.13 Effect of the dimensionless amplitude $\overline{A}$ on the maximum dimensionless temperature for uniform square heat source at $\bar{\omega} = 190$. $\overline{A} = \frac{A}{l}$. (a) Dimensionless temperature as a function of cycles. (b) Steady state versus the dimensionless amplitude.
Fig. 3.14 Maximum dimensionless overall temperature variation at steady state at different dimensionless amplitude $\bar{A}$ for uniform square heat source. $\bar{A} = A/l$

### 3.5.3 Analytical Expressions for Maximum Temperature by Surface-Fitting

In this section, analytical expressions for maximum dimensionless steady temperature and maximum dimensionless overall temperature for the different heat sources are presented by the a surface-fitting method using the Hearne Scientific Software, TableCurve 3D, with the fitting function as

$$
\overline{T}_{\text{max}} = \frac{a_1 + a_2 \ln(\bar{\omega}) + a_3 \ln^2(\bar{\omega}) + a_4 \bar{A} + a_5 \bar{A}^2}{1 + a_6 \ln(\bar{\omega}) + a_7 \bar{A}}
$$

(3.28)

The coefficients for different heat sources are summarized in Table 3.2 for $\bar{\omega} \in [5,1140]$, $\bar{A} \in [0.001,1]$. The difference between the overall temperature and the steady temperature, which is close to the oscillation amplitude, represents the oscillation magnitude of the maximum steady state temperatures for a given pair of $\bar{\omega}$ and $\bar{A}$ to a certain extent. The range of validity of the equations is limited to $\bar{\omega} \geq 5$ in order to obtain an accurate solution. $\overline{T}_{\text{max}}$ increases with $\bar{\omega}$ decreasing shown in Fig. 3.8(b) and decreases with $\bar{A}$ increasing shown in Fig. 3.13(b). The final effect of $\bar{\omega}$ and $\bar{A}$ on $\overline{T}_{\text{max}}$ makes the surface $\overline{T}_{\text{max}} = \overline{T}_{\text{max}}(\bar{\omega}, \bar{A})$ curve up near $\bar{A} = 0.001$ and curve down near $\bar{A} = 1$.
when \( \bar{\omega} \) becomes small as shown in Fig. 3.15(a), which presents the maximum dimensionless steady temperature versus \( \bar{\omega} \) and \( \bar{A} \) for parabolic heat source. These opposite changes of \( T_{\text{max}} \) makes it difficult to do an accurate surface fitting. Also, compared with the simulation under high frequency oscillation, it is relatively easy for the reader to get the solution by using Eqs. (3.10-13, 3.19 and 3.21) when \( \bar{\omega} \) is small. In Fig. 3.15(a), the solid circles denote the results by using Eq. 3.12 or 3.21, and the short line between the circles and the surfaces represents the residuals, which are clearly presented in Fig. 3.15(b). It is similar for the uniform circular heat source and the uniform square heat source. The residuals for the different heat sources are within \( \pm 0.0065 \). The greater residual error occurs when \( \bar{\omega} \) is small and \( \bar{A} \) is large.

Figure 3.16 shows the comparison of the maximum dimensionless overall temperature versus dimensionless frequency \( \bar{\omega} \) for the parabolic heat source by the surface-fitting solution along with the results of Greenwood & Alliston-Greiner [6] simulated at \( \bar{A} = 0.001 \). The results are in good agreement. The error is, in part, due to the surface fitting and, in part, due to the assumption of stationary heat source in [6]. The difference becomes slightly greater with increasing \( \bar{\omega} \) as shown in Fig. 3.16.

Figure 3.17 presents the comparison of the maximum dimensionless overall temperature versus dimensionless amplitude \( \bar{A} \) for the parabolic heat source by the surface-fitting solution along with the results of Greenwood & Alliston-Greiner [6] simulated at \( \bar{\omega} = 5 \). It can be seen that the solution in [6] cannot catch the effect of dimensionless amplitude on the dimensionless temperature for the parabolic source. The oscillation amplitude affects not only the magnitude of the heat flux but also the heating area on the surface. With increasing the amplitude, the frictional heat is dissipated over a larger surface, resulting in a lower dimensionless surface temperature as shown in Fig. 3.13(b). The amplitude in reference [6] only affects the calculation of the heat flux. The extension of the contact area due to the oscillation is not considered. When the calculated temperature is normalized, the amplitude is cancelled out and does not appear in the dimensionless temperature equations. Therefore, in Fig. 3.17 the maximum dimensionless overall temperature predicted by reference [6] is constant and its difference from the result by the present method becomes greater with increasing \( \bar{A} \). On the other hand, due to the extension of the contact area, more heat is dissipated over the extended contact area with increasing \( \bar{\omega} \), resulting in a lower dimensionless temperature as shown in Fig. 3.16. In fact, the limit of the dimensionless steady temperature and the dimensionless overall temperature reported in [6] is well established when \( \bar{\omega} = 0.0025 \) and \( \bar{A} = 0.001 \) based on the result by the presented method as shown in Table 3.3. Therefore, the two curves in Fig. 3.16 should have an intersection at a very small value of \( \bar{\omega} \). Similarly, the two curves in Fig. 3.17 should coincide with each other at a low amplitude when \( \bar{\omega} \) decreases to a very small value. The same results can be obtained for uniform circular heat source and the uniform square heat source.
Table 3.2 Analytical expressions for maximum temperature for different heat sources

<table>
<thead>
<tr>
<th>Temperature Solution</th>
<th>$\bar{T}_{\text{max}} = \frac{a_1 + a_2 \ln(\bar{\omega}) + a_3 \ln^2(\bar{\omega}) + a_4 \bar{A} + a_5 \bar{A}^2}{1 + a_6 \ln(\bar{\omega}) + a_7 \bar{A}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Steady Temperature</td>
<td>Maximum Overall Temperature</td>
</tr>
<tr>
<td>Uniform Circular Heat Source</td>
<td>$a_1 = 3.1573178 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$a_2 = 1.5264059 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$a_3 = -5.2393705 \times 10^{-4}$</td>
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<td></td>
<td>$a_4 = -2.3758877 \times 10^{-1}$</td>
</tr>
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<td></td>
<td>$a_5 = -1.0167623 \times 10^{-2}$</td>
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<tr>
<td></td>
<td>$a_6 = 4.5647159 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$a_7 = -7.0922520 \times 10^{-1}$</td>
</tr>
<tr>
<td>Parabolic Heat Source</td>
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</tr>
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<td></td>
<td>$a_2 = 1.7950268 \times 10^{-2}$</td>
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<td></td>
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<tr>
<td></td>
<td>$a_3 = -8.4682800 \times 10^{-4}$</td>
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<td>$a_5 = -2.1187807 \times 10^{-2}$</td>
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<td></td>
<td>$a_6 = 6.0160174 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$a_7 = -7.3841274 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Note: $\bar{\omega} = \omega l^2 / 4\alpha$, $\bar{A} = A / l$, and the solution is valid for $\bar{\omega} \in [5,1140]$, $\bar{A} \in [0.001,1]$; for circular and parabolic heat source, $l$ is the radius of the heat source; for square heat source, $l$ is half of the contact length along sliding direction; $q_0 = \mu p_0 A \omega$, where $p_0$ is the average contact pressure over the interface.
Fig. 3.15 (a) Maximum dimensionless steady temperature versus $\bar{\omega}$ and $\bar{A}$ for parabolic heat source. (b) Residuals of maximum dimensionless steady temperature versus $\bar{\omega}$ and $\bar{A}$ for parabolic heat source.
Fig. 3.16 Comparison of maximum dimensionless overall temperature predicted by fitting solution and by Greenwood & Alliston-Greiner [10] with different $\overline{\omega}$ for the parabolic heat source.

Fig. 3.17 Comparison of maximum dimensionless overall temperature predicted by fitting solution and by Greenwood & Alliston-Greiner [10] with different $\overline{A}$ for the parabolic heat source.
Table 3.3. Comparison of maximum temperature by the presented method and that in [6]

<table>
<thead>
<tr>
<th>Heat Sources</th>
<th>Temperature Type</th>
<th>Results reported in [6]</th>
<th>Results by the Present Method at ( \tilde{\omega} = 0.0025 ), ( \tilde{A} = 0.001 )</th>
<th>Relative Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular Heat Source</td>
<td>Max. Steady Temperature</td>
<td>( \frac{T_{r,\max} k}{2 R q_0} = \frac{1}{\pi} = 0.3183 )</td>
<td>0.3186</td>
<td>0.09</td>
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<tr>
<td></td>
<td>Max. Overall Temperature</td>
<td>( \frac{T_{o,\max} k}{2 R q_0} = 0.5000 )</td>
<td>0.4913</td>
<td>1.74</td>
</tr>
<tr>
<td>Square Heat Source</td>
<td>Max. Steady Temperature</td>
<td>( \frac{T_{r,\max} k}{2 l q_0} = \frac{1.1222}{\pi} = 0.3572 )</td>
<td>0.3602</td>
<td>0.84</td>
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<tr>
<td></td>
<td>Max. Overall Temperature</td>
<td>( \frac{T_{o,\max} k}{2 l q_0} = 0.5611 )</td>
<td>0.5500</td>
<td>1.98</td>
</tr>
<tr>
<td>Parabolic Heat Source</td>
<td>Max. Steady Temperature</td>
<td>( \frac{T_{r,\max} k}{2 R q_0} = \frac{1}{4} \Rightarrow \frac{T_{r,\max} k}{2 R q_0} = 0.3750 )</td>
<td>0.3754</td>
<td>0.11</td>
</tr>
</tbody>
</table>

3.6 Conclusions

In this study, the temperature rise in contact problems with different types of heat sources (circular, rectangular and parabolic) either moving unidirectionally or oscillating on a semi-infinite body is analyzed. Numerical solution is obtained by using the Cautious Adaptive Romberg Extrapolation (CADRE) with automatic integration.

The temperature solution for an oscillatory heat source shares some features with that for unidirectional, moving heat source. For example, the temperature distribution within the heat source is significantly affected by the distribution of the heat source. The shape of the heat source has little effect on the temperature distribution outside the contact region. However, due to the time varying heat source and the periodic sliding motion, the temperature solution in oscillatory sliding differs from that in unidirectional sliding: temperature distribution along the direction of the sliding motion is symmetric with respect to the center of the contact region, the maximum temperature occurs at the center of the contact region, and the steady state temperature oscillates periodically, whose magnitude depends on the dimensionless frequency \( \tilde{\omega} \) and dimensionless amplitude \( \tilde{A} \).

Simulation results show that increasing values of dimensionless frequency \( \tilde{\omega} \) or dimensionless amplitude \( \tilde{A} \), hence increasing Peclet number \( P_e \), results in lower maximum dimensionless steady state temperature. Number-of-cycles to steady state is greatly affected by \( \tilde{\omega} \), but not much influenced by \( \tilde{A} \). At lower \( \tilde{\omega} \) and \( \tilde{A} \), the maximum dimensionless temperature approaches to a constant value. Lower \( \tilde{\omega} \) or high \( \tilde{A} \) yield a
greater oscillation of the overall surface temperature. Within the range of oscillatory amplitude typical for a fretting contact, 10-100 μm [49], the oscillation of the temperature becomes unimportant when \( \bar{\omega} > 6.25 \), which is consistent with the simplified analytical solution derived in reference [6].

Analytical expressions for maximum surface temperature for different heat sources are presented by a surface-fitting method, which can be conveniently used to predict maximum surface temperature or to determine the allowable operating parameters to

3.7 Nomenclature

\( a \) = Axis of elliptical heat source in sliding direction
\( a_i \) = Coefficients for surface-fitting function \( (i = 1, 2, \cdots, 7) \)
\( A \) = Oscillating amplitude (m)
\( \bar{A} \) = Dimensionless oscillating amplitude
\( b \) = Axis of elliptical heat source perpendicular to sliding direction (m)
\( c \) = Specific heat (J/kg K)
\( k \) = Thermal conductivity (w/m K)
\( l \) = Half contact length along sliding direction (m)
\( l_x, l_y \) = Half length of a rectangular heat source in \( x \) and \( y \) direction (m)
\( p, p_0, p_m \) = Instant contact pressure, average contact pressure over interface, maximum contact pressure for parabolic contact (Pa)
\( P_e \) = Peclet number, \( P_e = \frac{UL}{2\alpha} \) for unidirectional sliding case and \( P_e = \bar{\omega}\bar{A} \) for oscillating case
\( q, q_0, q_m \) = Instant heat flux, average heat flux over interface, maximum heat flux for parabolic contact (W/m²)
\( R \) = Radius of a circular heat source (m)
\( s(t) \) = Displacement of heat source (m)
\( T, T_{\text{max}} \) = Temperature rise, maximum temperature rise including maximum steady temperature rise \( T_{s\text{max}} \) and maximum overall temperature rise \( T_{o\text{max}} \) (°C)
\( T_{\text{max}} \) = Maximum dimensionless temperature rise including maximum dimensionless steady temperature rise \( T_{s\text{max}} \) and maximum dimensionless overall temperature rise \( T_{o\text{max}} \) (°C)
\( U \) = Constant sliding velocity (m/s)
\( \nu(t) \) = Oscillatory velocity (m/s)
\( \alpha \) = Thermal diffusivity (m²/s)
\( \varepsilon_E \) = Length ratio for elliptical heat source, \( \varepsilon_E = b/a \)
\( \varepsilon_R \) = Length ratio for rectangular heat source, \( \varepsilon_R = \frac{l_y}{l_x} \)
\( \gamma \) = Dimensionless number, \( \gamma = R\sqrt{\omega/\alpha} \)
\( \mu \) = Friction coefficient
$\phi, \Phi, \Phi^t$ = Dimensionless time, given dimensionless time, setting value of dimensionless time
$
\rho$ = Mass density (kg/m$^3$)
$\omega$ = Angular oscillating frequency (rad/s)
$\bar{\omega}$ = Dimensionless frequency, $\bar{\omega} = \frac{\omega l^2}{4 \alpha}$
$\Omega(t)$ = Contact area at time $t$
$r, \theta$ = Polar coordinates
$\xi, \eta$ = Local coordinates
4 Transient Heat Conduction in Rolling/Sliding Components by a Dual Reciprocity Boundary Element Method

This chapter develops a transfer matrix method associated with the Dual Reciprocity Boundary Element Method (DRBEM) for the study of transient heat conduction problems in presence of moving heat sources. In this method, the time integration is processed by an iteration transfer matrix method, the coefficient matrices are calculated only once, and no domain integration is required. It is shown that the application of DRBEM results in considerable savings in computation time and data preparation. Numerical examples are presented to demonstrate the efficiency and accuracy of the method by comparing the computed results with either published results or solutions using the finite element method. While only two dimensional problems are presented in the chapter, the method can be readily extended to three-dimensional problems to handle more complicated contact problems.

4.1 Introduction

Moving heat source problems occur in a wide variety of industrial applications where two surfaces are in relative sliding motion. Examples include the interaction of the rolling elements and races in roller bearings, shafts in journal bearings, and steel rolls in rolling mills, where the interacting bodies undergo one or more moving heat sources due to rubbing. Relative to the heat source, the stationary body — the outer race of a roller bearing or the sleeve of a journal bearing — is subjected to a fixed heat source. The sliding body — the roller and the shaft that experience moving heat source — is periodically heated over a small contact surface area while cooled over all or part of surface area by convention.

There are volumes of published papers dealing with the thermal behavior of rotating cylinder subjected to surface heating and convective cooling. The classic work was done by Jaeger [50], who developed an analytical transient solution for an adiabatic cylinder subject a rotating heat source on its surface. DesRuisseaux and Zerkle [51] extended Jaeger’s solution by assuming that a convective cooling occurs over the entire cylindrical surface. Ling [52] outlined the quasi-stationary solution for a cylinder subject to cooling and heating. Patula [53] presented a quasi-stationary solution related to cold rolling with the cylinder cooled and heated on parts of its surface but insulated on the rest. Yuen [54] extended the boundary condition formulation to nonuniform heating and cooling, but without presenting nonuniform cooling case. Ulysse and Khonsari [55] developed an analytical solution for steady-state temperature distribution in a cylinder undergoing uniform heating and nonuniform cooling. By neglecting the axial heat conduction due to high rotation speed, Gecim and Winer [56, 57] presented the steady temperature solution in a rotating cylinder subject to surface heating and convective cooling. All these

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3 Paper related to this chapter is being accepted in International Journal of Heat and Mass Transfer with minor changes.
temperature models are analytical in nature. To facilitate such treatment, many simplifying assumptions had to be made.

In addition to the analytical solutions, significant efforts have been devoted to study thermal behavior of rotating cylinder by numerical techniques. Tseng [58] used a first-order upwind differencing scheme to study the steady state heat transfer behavior of a two dimensional rotating roll. Bennon [59] developed a three dimensional finite difference model to predict the temperature distribution of the work roll in both axial and circumferential directions. Hwang [60] performed a two-dimensional finite element analysis of steady state heat transfer that included staggered thermomechanical coupling. Lee [61] presented a three dimensional finite element solution to the heat transfer problem in hot rolling operation. For the finite element method and the finite difference analysis, a very fine mesh is generally required to obtain an accurate temperature solution because the high temperature gradient is localized in a thin layer near the surface.

Compared to the finite element method and the finite difference method, the most important feature of the boundary element method is that it only requires discretization of the boundary rather than that of the whole volume, which reduces the dimensionality of the problem by one. There are many published papers where researchers apply the boundary element method to solve the convective-diffusion equation for quasi-steady moving heat source problems. See, for example, papers dealing with the constant velocity machining or welding process [62-67]. It has been reported that using the BEM scheme, the convection term can be modeled with high precision than that by upwind differencing in finite difference, the sharp temperature gradient over the domain can be easily captured [62]. However, the presence of domain integrals makes the BEM inefficient when the method is applied to diffusion problems with source terms.

One of the most frequently used techniques for converting the domain integral into a boundary one is the so-called Dual Reciprocity Boundary Element Method (DRBEM). This method was initially developed by Nardini and Brebbia [68] in the analysis of elastodynamics. It has been extended to deal with a variety of other problems [69-73]. However, when dealing with transient heat conduction, a temporal derivative is involved and a time marching schemes is generally required, which can be quite time consuming when the solution for large duration of time is desired. Zhu et al. [74] showed an efficient methodology called the Laplace Transform Dual Reciprocity Method (LTDRM) for problems involving time marching. Amado et al [75] studied the application of the LTDRM in the modeling of the laser heat treatment. They showed that while the lengthy computations for a large time marching scheme are avoided, a large number of Laplace solutions is needed to obtain a reliable inversion from the Laplace domain temperature to the time one, particularly when pulse-like thermal cycles are induced.

Vick et al. [76] developed a boundary element model to analyze the surface temperatures generated by friction in sliding contact, and in the subsequent study [77] applied it to examine the effects of surface coatings on the temperatures produced by friction due to sliding contact. A moving full space Green’s function was used as the fundamental solution in both studies. It was a time-dependant fundamental solution
approach, the scheme 2 in Ref. [78], where the time integration was performed within the entire span of time interval between the initial time and the desired time level. Although in this approach the time integration still needs to be carried out, if the span of time integration is quite large, no temperature values at internal nodes need to be computed and stored at each intermediate time step. This method may yield saving of computational time if no unknown temperature values are needed at intermediate time steps. However, if the temperature variations up to steady state are of interest, this method becomes time consuming because the time integration always restarts from the initial time for the temperatures at every desired time.

A survey of the literature reveals that there is very little published research on the application of DRBEM dealing with the heat conduction in the rolling elements, especially when they are subjected to an oscillating motion such as in the so-called pin-joint assembly found in heavy construction and earth-moving machines. In this chapter, a transfer matrix method associated with the Dual Reciprocity Boundary Element Method (DRBEM) is developed for the study of transient heat conduction problems in presence of moving heat sources. In this method, the time integration is processed by an iteration transfer matrix method, the coefficient matrices are calculated only once, and no domain integration is required. It is shown that the application of DRBEM results in considerable savings in computation time and data preparation. Numerical examples are presented to demonstrate the efficiency and accuracy of the method by comparing the computed results with either published results or solutions using the finite element method. While only two dimensional problems are presented in the chapter, the method can be readily extended to three-dimensional problems to handle more complicated contact problems.

4.2 Formulation

Consider a transient temperature field \( T(x,t) \) in a domain \( \Omega \) with a boundary \( \Gamma = \Gamma_T \cup \Gamma_q \cup \Gamma_h \) shown in Fig. 4.1. The equation governing the heat conduction is:

\[
\nabla^2 T(x,t) = \frac{1}{\alpha} \dot{T}(x,t), \quad x \in \Omega, \ t > 0
\]

Fig. 4.1 Problem configuration.

(4.1)

with the initial condition
\( T(x,0) = T^{(0)}(x), \quad x \in \Omega \) \tag{4.1a}

and the boundary conditions are given by

Dirichlet: \( T(x,t) = \overline{T}(x,t), \quad x \in \Gamma_T \) \tag{4.1b}

Neumann: \( q(x,t) = \overline{q}(x,t), \quad x \in \Gamma_q \) \tag{4.1c}

Robin: \( q(x,t) = h(T_{\infty} - T), \quad x \in \Gamma_h \) \tag{4.1d}

where \( x \) is the coordinate vector. The parameter \( t \) denotes the time, and the super-dot stands for the derivative with respect to time. \( \alpha \) is the thermal diffusivity of the material. \( q(x,t) \) is the heat flux defined as \( q(x,t) = k \frac{\partial T}{\partial n} \), where \( k \) is the thermal conductivity of the material, and \( n \) is the unit normal outward to the boundary \( \Gamma \). \( T^{(0)}(x) \) represents the initial temperature distribution. \( \overline{T}(x,t) \) and \( \overline{q}(x,t) \) are the prescribed boundary temperature and flux, respectively. The parameter \( h \) is the heat transfer coefficient, and \( T_{\infty} \) stands for the ambient temperature.

In the dual reciprocity formulation [70], the temperature is approximated in the following manner:

\[
T(x,t) \approx \sum_{j=1}^{N_B+N_I} \alpha_j(t) f_j(x) \tag{4.2}
\]

where \( N_B \) is the number of boundary nodes defining the discretization of the boundary \( \Gamma \), and \( N_I \) is the number of internal nodes. The \( N_B+N_I \) different nodes make up of the DRM collocation points. \( \alpha_j(t) \) are the unknown time dependant functions. \( f_j(x) \) are known interpolation functions linked to a set of particular solutions \( \hat{T}_j(x) \) through the relation

\[
\nabla^2 \hat{T}_j(x) = f_j(x), \quad j = 1,2,\ldots,N_B+N_I \tag{4.3}
\]

where \( \hat{T}_j(x) \) can be interpreted as a pseudo-temperature with an associated heat flux defined on the boundary as \( \hat{q}_j(x) = k \frac{\partial \hat{T}_j(x)}{\partial n} \). \( f_j(x) \) is chosen to be the distance functions, e.g. \( f_j(x) = 1 + r_j \) in the current study with \( r_j = \|x - x_j\| \) being the distance between the field point \( x \) and the collocation points \( x_j \).

Evaluating Eq. (4.3) at every points \( x_i, \quad i = 1,2,\ldots,N_B+N_I \), the resulting set of equations can be expressed in matrix form as

\[
\mathbf{T} = \mathbf{F} \mathbf{\alpha} \tag{4.4}
\]

where \( \mathbf{T} = \{T_i\} \) is the nodal temperature vector of \( T \), and \( \mathbf{\alpha} = \{\alpha_j\} \) is a unknown vector. \( \mathbf{F} \) is a square matrix with its entries defined as \( F_{ij} = f_j(x_i) \).

Taking derivative of Eq. (4.4) with respect to time yields

\[
\dot{\mathbf{T}} = \mathbf{F} \dot{\mathbf{\alpha}} \tag{4.5}
\]

Inversion of Eq. (4.5) results in
\[ \dot{\alpha} = F^{-1} \hat{T} \]  
(4.6)

where super -1 denotes the matrix inversion.

Applying the usual dual reciprocity procedure [70], i.e. substituting Eq. (4.2) into Eq. (4.1), the result is multiplied by the steady state fundamental solution \( \hat{T} \), and then integrating twice by parts or using Green’s second identity, yields the standard DRBEM integral equation:

\[
c(\xi)T(\xi, t) + \int_t \tilde{q}(\xi, x)T(x, t)d\Gamma(x) - \int_t \hat{T}(\xi, x)q(x, t)d\Gamma(x)
\]

\[
= \frac{1}{\alpha} \sum_{j=1}^{N_B + N_i} \dot{\alpha}_j(t) \left\{ c(\xi)\hat{T}_j(\xi) + \int_T \tilde{q}(\xi, x)\hat{T}_j(x)d\Gamma(x) - \int_T \hat{T}(\xi, x)\dot{q}_j(x)d\Gamma(x) \right\}
\]
(4.7)

where \( \xi \) is the current point located on the boundary or within the domain \( \Omega \). \( c(\xi) = 1 \) when \( \xi \) is within the domain. Otherwise, it is equal the fraction of the angle with vertex at \( \xi \) subtended within the domain. \( \tilde{q}(x) \) is defined as \( \tilde{q}(x) = k \frac{\partial \hat{T}(x)}{\partial n} \).

Using the standard boundary element discretization procedure [69], i.e. the local approximation of both the shape of the boundary and the distributions of appropriate functions within boundary elements, the application of Eq. (4.7) at the boundary and internal nodes \( \xi_i, i=1,2,\cdots,N_B + N_i \), results in the following systems of equations in the matrix form as

\[
HT - GQ = \frac{1}{\alpha} \sum_{j=1}^{N_B + N_i} \dot{\alpha}_j(t) \left\{ H\hat{T}_j - G\dot{q}_j \right\}
\]
(4.8)

where \( H \) and \( G \) are the matrices of boundary integrals with kernels \( \tilde{q} \) and \( \hat{T} \), respectively. \( T \) and \( Q \) are nodal vectors of \( T \) and \( q \), respectively; \( \hat{T}_j \) and \( \dot{q}_j \) are nodal vectors of \( \hat{T}_j \) and \( \dot{q}_j \), respectively. Note that the term \( c(\xi) \) has been placed on the leading diagonal of matrix \( H \). The matrix \( T \) contains temperature at both internal and boundary nodes. The heat flux \( q \) is associated with the boundary nodes, but not with the internal nodes. The internal nodes are not always needed. When introduced, they are independent of each other, i.e. there is no internal mesh.

Collecting \( \hat{T}_j \) and \( \dot{q}_j \) vectors in Eq. (4.8) into matrices, and dropping the summation yields:

\[
HT - GQ = \frac{1}{\alpha} \left\{ H\hat{T} - G\dot{q} \right\} \alpha
\]
(4.9)

Substituting Eq. (4.6) into Eq. (4.9) results in

\[
HT - GQ = CT
\]
(4.10a)

with

\[
C = \frac{1}{\alpha} \left\{ H\hat{T} - G\dot{q} \right\} F^{-1}
\]
(4.10b)
By applying the prescribed boundary conditions in Eqs. (4.1b-d), Eq (4.10a) can be rewritten in the following form:

$$\textbf{B} \textbf{X}(t) = \textbf{D} \textbf{R}(t) + \textbf{C} \dot{\textbf{T}}(t)$$ \hspace{1cm} (4.11)

where \textbf{X}(t) contains the unknown nodal values of temperature for the Neumann or the Robin boundary nodes, and heat flux for the Dirichlet boundary nodes. \textbf{R}(t) contains the known nodal values of temperature or heat flux for the Dirichlet or the Neumann boundary nodes. For the Robin boundary nodes, \( T_\infty \) is stored and the corresponding columns of the matrix \( \textbf{G} \) are multiplied by \( h \). Matrices \( \textbf{H}, \textbf{G} \) and \( \textbf{C} \) are only dependent upon the model geometry, and thus they are needed to be calculated only once. The matrices \( \textbf{B} \) and \( \textbf{D} \) are composed of the columns of the matrices \( \textbf{G} \) and \( \textbf{H} \), depending on the prescribed boundary conditions.

Equation (4.11) represents a system of equations of mixed type because \( \textbf{X}(t) \) may include both the unknown temperature and heat flux depending on the boundary conditions. In what follows, we first obtain a general solution to Eq. (4.11) subject to different boundary conditions given in Eqs. (4.1b-d) and subsequently extend the method to the moving boundary problems.

### 4.2.1 Neumann and Robin Boundary Conditions

When the boundary conditions are only of Neumann and Robin type, \( \textbf{X}(t) \) contains only unknown temperature. Then Eq. (4.11) can be rewritten in the form of

$$\textbf{B} \textbf{T}(t) = \textbf{D} \textbf{R}(t) + \textbf{C} \dot{\textbf{T}}(t)$$ \hspace{1cm} (4.12)

The solution of Eq. (4.12) is [79]:

$$\textbf{T}(t) = \exp(\textbf{C}^{-1}\textbf{B}t) \textbf{T}^{(0)} - \int_0^t \exp\left[\textbf{C}^{-1}\textbf{B}(t-s)\right] \textbf{C}^{-1}\textbf{D}\textbf{R}(s) ds$$ \hspace{1cm} (4.13)

Let \( t_k \) and \( t_{k+1} = t_k + \tau \) be the time instants of two continuous steps \( k \) and \( k+1 \), respectively. Then, referring to Eq. (4.13) the temperature at time \( t_k \) can be obtained as:

$$\textbf{T}^{(k)} = \exp(\textbf{C}^{-1}\textbf{B}t_k) \textbf{T}^{(0)} - \int_0^\tau \exp\left[\textbf{C}^{-1}\textbf{B}(t_k-s)\right] \textbf{C}^{-1}\textbf{D}\textbf{R}(s) ds$$ \hspace{1cm} (4.14)

Hence, the temperature at time \( t_{k+1} \) is:

$$\textbf{T}^{(k+1)} = \exp(\textbf{C}^{-1}\textbf{B}t_{k+1}) \textbf{T}^{(0)} - \int_{t_k}^{t_{k+1}} \exp\left[\textbf{C}^{-1}\textbf{B}(t_{k+1}-s)\right] \textbf{C}^{-1}\textbf{D}\textbf{R}(s) ds$$

$$= \exp(\textbf{C}^{-1}\textbf{B}\tau) \textbf{T}^{(k)} - \int_0^\tau \exp\left[\textbf{C}^{-1}\textbf{B}(\tau-s)\right] \textbf{C}^{-1}\textbf{D}\textbf{R}(t_k+s) ds$$ \hspace{1cm} (4.15)

Whether the integrand in Eq. (4.15) is analytically integrable or not depends on the expression of the prescribed boundary conditions \( \textbf{R}(t) \). If not analytically integrable, a small time step \( \tau \) must be chosen such that from time \( t_k \) to \( t_{k+1} \), \( \textbf{R}(t) \) can be treated to be constant, equal to its value at time \( t_k \), i.e. \( \textbf{R}^{(k)} \). Then, performing the related integration, Eq. (4.15) yields:

$$\textbf{T}^{(k+1)} = \textbf{A} \textbf{T}^{(k)} + \textbf{M} \textbf{R}^{(k)}$$ \hspace{1cm} (4.16)

with

$$\textbf{A} = \exp(\textbf{C}^{-1}\textbf{B}\tau)$$ \hspace{1cm} (4.16a)
\[ \mathbf{M} = (\mathbf{I} - \mathbf{A}) \mathbf{B}^{-1} \mathbf{D} \]  

(4.16b)

where \( \mathbf{I} \) is the identity matrix. The parameters \( \mathbf{A} \) and \( \mathbf{M} \) are the coefficient matrices. For a constant step size \( \tau \), they are constants and need to be calculated only once.

Equation (4.16) is the iteration solution for the transfer matrix method. Starting from the initial temperature \( T^{(0)} \), the temperature \( T(x,t) \) at each time instant \( t_k \) can be calculated efficiently until the desired time level is reached. For the system that reaches the steady state after long enough time period, \( \mathbf{A} \) in Eq. (4.16a) approaches to zero when giving a large enough value of \( \tau \), and thus Eq. (4.16) becomes:

\[ T^{(ss)} = \mathbf{B}^{-1} \mathbf{D} \mathbf{R} \]  

(4.17)

where \( T^{(ss)} \) represents the node temperatures at the steady state. Equation (4.17) is consistent with the steady state solution derived from Eq. (4.12) when \( \dot{T}(t) = 0 \).

The important step is to evaluate the matrix exponential \( \mathbf{A} \) accurately, for which the so-called Precise Time Integration (PTI) proposed by Zhong [80] is used. The method, which is unconditionally stable, uses the \( 2^N \) algorithm described as follows. Let \( N = 20 \), \( m = 2^N = 1,048,576 \), then \( \Delta \tau = \tau / m \) is an extremely small time interval. Using the superposition of exponential function,

\[ \mathbf{A} = \exp(\mathbf{C}^{-1} \mathbf{B} \tau) = \left[ \exp(\mathbf{C}^{-1} \mathbf{B} \Delta \tau) \right]^{2^N} \]  

(4.18)

and the Taylor expansion,

\[ \exp(\mathbf{C}^{-1} \mathbf{B} \Delta \tau) \approx \mathbf{I} + \mathbf{C}^{-1} \mathbf{B} \Delta \tau + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^2}{2!} + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^3}{3!} + \cdots + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^p}{p!} = \mathbf{I} + \mathbf{A}_a \]  

(4.19)

with

\[ \mathbf{A}_a = \mathbf{C}^{-1} \mathbf{B} \Delta \tau + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^2}{2!} + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^3}{3!} + \cdots + \frac{(\mathbf{C}^{-1} \mathbf{B} \Delta \tau)^p}{p!} \]  

(4.20)

results in

\[ \mathbf{A} = (\mathbf{I} + \mathbf{A}_a)^{2^N} \]  

(4.21)

Taking into account the relationship

\[ (\mathbf{I} + \mathbf{A}_a)^2 = \mathbf{I} + (2\mathbf{A}_a + \mathbf{A}_a \cdot \mathbf{A}_a) \]  

(4.22)

It can be shown that, by substituting \( \mathbf{A}_a \) by \( (2\mathbf{A}_a + \mathbf{A}_a \cdot \mathbf{A}_a) \), \( \mathbf{I} + \mathbf{A}_a \) becomes

\[ \mathbf{I} + (2\mathbf{A}_a + \mathbf{A}_a \cdot \mathbf{A}_a) = (\mathbf{I} + \mathbf{A}_a)^2, \quad \text{and} \quad [\mathbf{I} + (2\mathbf{A}_a + \mathbf{A}_a \cdot \mathbf{A}_a)]^2 = [\mathbf{I} + \mathbf{A}_a]^2 = (\mathbf{I} + \mathbf{A}_a)^2. \]

Repeating such substitution for \( N \) times yields the solution of Eq. (4.21).

It is noted that the matrix \( \mathbf{A}_a \) is small compared to identity matrix \( \mathbf{I} \) due to a very small time interval \( \Delta \tau \). When calculating matrix \( \mathbf{A} \) by the above algorithm, first \( N \) times of substitution is executed to the small matrix \( \mathbf{A}_a \) and then the result is added to the identity matrix \( \mathbf{I} \), which avoids the direct additions of the identity matrix \( \mathbf{I} \) and small matrix, and thus the loss of significant digits during the computations.
Alternatively, Chen et al. [81] propose an adaptive PTI algorithm, which determines the value of $N$ and $p$ adaptively based on the problem characteristics and the prescribed computational error tolerance. It saves much computer time for calculating matrix $A$. In this chapter, the adaptive PTI algorithm is used to evaluate the coefficient matrices.

### 4.2.2 Dirichlet Boundary Conditions

If the problem is subject to Dirichlet boundary condition, $X(t)$ in Eq. (4.11) includes both the unknown temperature and heat flux, and Eq. (4.11) becomes a system of equations of mixed type, which require a special treatment as follows.

From Eq. (4.2), the heat flux on the boundary where the Dirichlet boundary condition is prescribed can obtained as [82]

$$
q(x,t) \approx k \sum_{j=1}^{N_p+N_i} \alpha_j(t) \frac{\partial f_j(x)}{\partial n}
$$

(4.23)

then the vector of unknowns $X(t)$ in Eq. (4.11) can be expressed as

$$
X(t) = F'(x)A(t)
$$

(4.24a)

where $F'(x)$ is similar to $F$ in Eq. (4.4) except that the entries corresponding to the Dirichlet boundary nodes $x_i$ become $k \frac{\partial f_j(x_i)}{\partial n}$ instead of $f_j(x_i)$, that is:

$$
F'(x) = \begin{cases} 
  f_j(x_i), & \text{Neumann or Robin boundary condition is prescribed at node } x_i \\
  k \frac{\partial f_j(x_i)}{\partial n}, & \text{Dirichlet boundary condition is prescribed at node } x_i
\end{cases}
$$

(4.24b)

Inversion of Eq. (4.4) yields

$$
A = F'^{-1}T
$$

(4.25)

Substitution of Eq. (4.25) into Eq. (4.24a) results in

$$
X(t) = F'F'^{-1}T
$$

(4.26)

Applying Eq. (4.26), Eq. (4.11) can be expressed as

$$
B'T(t) = DR(t) + CT(t)
$$

(4.27a)

with

$$
B' = BF'F'^{-1}
$$

(4.27b)

Note that Eq. (4.27a) can be obtained by replacing $B$ in Eq. (4.12) by $B'$. Therefore, using the same method as in the previous case, Eq. (4.27a) can be solved.

### 4.2.3 Solution for Moving Boundary Problems

In this section the method is extended to solve the heat conduction equation subject to the moving heat source. Consider a body subject to the heat source $\bar{q}(x,t)$ moving at velocity $\nu(t)$ along its boundary $\Gamma$. The moving cycle of the heat source is discretized into $N_{mov}$ continuous steps such that within each step $i$ ($i = 1, 2, \cdots, N_{mov}$), the heat source can be treated to be stationary and acts on the body for time period $\tau_i$, defined as:
\[
\tau_i = \frac{\text{Moving distance in the } i\text{th step}}{\text{Average velocity in the } i\text{th step}}
\] (4.28)

Therefore, the problem within the time period \(\tau_i\) is the same as that in section 4.2.1 with the solution at the previous moving step \(i-1\) within the time period \(\tau_{i-1}\) as a pseudo-initial condition. Thus, the following iteration equation can be obtained

\[
T^{(k+1)} = A_i T^{(k)} + M_i R^{(k)}, \quad i = 1, 2, \ldots, N_{\text{mov}}
\] (4.29)

where \(A_i\) and \(M_i\) are the same as those in Eqs. (4.16a) and (4.16b). However, the matrices \(B_i\) and \(D_i\) are obtained by applying the corresponding boundary conditions within time period \(\tau_i\).

---

Input model data (including material properties, node coordinates, element information, starting boundary conditions and initial temperature), simulation time, the number of the moving steps \(N_{\text{mov}}\) and corresponding time period \(\tau_i\) (\(i = 1, 2, \ldots, N_{\text{mov}}\)).

Integrate boundary to obtain the matrices \(H\) and \(G\).

Shift the starting boundary conditions to get the \(N_{\text{mov}}\) different boundary conditions for each moving step \(i\) and calculate the corresponding pairs of \(A_i\) and \(M_i\).

Compute \(T^{(k+1)} = A_i T^{(k)} + M_i R^{(k)}\) up to the desired time according to different load steps due to the motion of the heat source.

Output node temperature variations

Fig. 4.2 A flowchart of the basic steps of the simulation algorithm for moving heat source problems.

There are at most \(N_{\text{mov}}\) pairs of the coefficient matrices \(A_i\) and \(M_i\) to be evaluated. Each pair corresponds to a specified moving step of the heat source or a certain load condition on the boundary due to the motion of the flux. The calculations of different pairs of \(A_i\) and \(M_i\) are based on the same matrices \(H\) and \(G\) in Eq. (4.8) by using the corresponding boundary condition at the step \(i\) without introducing any additional boundary integration. According to different load steps due to the motion of the heat source, the matrices \(A_i\) and \(M_i\) (\(i = 1, 2, \ldots, N_{\text{mov}}\)) are consecutively substituted into Eq.
(4.29) to simulate the variations of the field temperature. It is noted that in one cycle, some pairs of $A_i$ and $M_i$ may be the same if heat source is subject to periodic sliding or oscillatory motion. In such case, less than $N_{mov}$ pairs of different coefficient matrices are evaluated. Figure 4.2 shows a flowchart of the main steps of the simulation algorithm for moving heat source problems.

Generally, the internal nodes in the DRBEM are not always needed [82]. However, it is noted that for the moving boundary problems, they are necessary no matter whether the boundary condition is of Dirichlet type or not. In any event, even when internal nodes are introduced, they are independent of each other and thus no internal meshing would be required.

4.3 Results and Discussion

Based on the formulation described in section 4.2, a computer program is developed to treat transient heat conduction problems with different types of boundary conditions [83-85]. The utility of the method is illustrated by application of the program to three problems. Examples are presented to validate the results and provide evidence for the efficiency and accuracy of the method.

4.3.1 Fixed Boundary Conditions

The first case involves the transient heat transfer in a square-shaped geometry shown in Fig. 4.3. The problem is treated as a two dimensional problem by the present method although because on the nature of the boundary conditions, it could be treated as a one dimensional problem. A Dirichlet boundary condition, $T = 0$ is applied at side $x = 0$ and Robin boundary condition at side $x = 1m$ with convective heat transfer coefficient $h = 1000W/m^2\cdot K$ and ambient temperature $T_\infty = 20^\circ C$. The other two sides are subject to Neumann boundary conditions, $\overline{q} = 0$ (insulation). Zero initial temperature is assumed. The thermal conductivity of the material $k = 52W/m \cdot K$, and the thermal diffusivity $\alpha = 10^{-5} m^2/s$. The analytical solution for temperature distribution of such a problem can be derived [35] as

\[
T(x,t) = \frac{HT_x}{1+H} \left\{ x + 4 \sum_{n=1}^{\infty} \beta_m \cos(\beta_m x) - \sin(\beta_m x) \frac{\exp(-\alpha \beta_m^2 t) \sin(\beta_m x)}{\beta_m \left[ 2 \beta_m - \sin(2 \beta_m) \right]} \right\} 
\]

where $H = h/k$, and $\beta_m$ are the positive roots of equation $\beta_m \cot(\beta_m) = -H$

As shown in Fig. 4.3, the boundary of the unit square is discretized into 40 equally sized linear elements with 40 boundary nodes, i.e. $N_g = 40$. To implement the Dirichlet boundary condition, 25 internal nodes are involved, i.e. $N_i = 25$. The comparison of temperature variation at location $x = 1, y = 0.5$ by the present method along with the analytical solution is shown in Fig. 4.4(a). The comparison of temperature distribution between the present method and the analytical solution along the $x$ direction at $t = 10000s$ is plotted in Fig. 4.4(b). It can be seen from Figs. 4.4 that the results from
both methods agree very well. The time taken by the present method to do the simulation is comparable to the analytical solution on the same computer. Also the present method is unconditional stable. Changing the time step with different values, the exactly same results are obtained. When taking a large time step $\tau = 10000s$, the results shown in Fig. 4.4(b) are directly obtained by one step iteration. This feature is important for the moving boundary problem with variable velocity.

\begin{align*}
\frac{\partial T}{\partial y} &= 0 \\
T &= 0 \\
\frac{\partial T}{\partial y} &= 0
\end{align*}

Fig. 4.3 Numerical model of the unit square.

Fig. 4.4 (a) Comparison of temperature variation at location $x = 1, y = 0.5$ by the present method along with the analytical solution. (b) Comparison of temperature distribution between the present method and the analytical solution along the $x$ direction at $t=10000s$. (fig. cont’d)
4.3.2 Unidirectional Moving Heat Source

In this case, a long cylinder with radius \( R \) is subjected to a unidirectional moving heat source \( \vec{q} \) along its surface with angular velocity \( \dot{\phi} \). The heating is assumed to occur over the width \( 2\phi_0 = 0.2805 \) rad. The rest of the surface is subjected to the convective boundary condition with heat transfer coefficient \( h \) and ambient temperature \( T_\infty \). The Peclet number \( \text{Pe} = \omega R^2 / \alpha = 200 \), and the Biot number \( \text{B} = hR / k = 0.5 \), where \( k \) and \( \alpha \) are the thermal conductivity and the thermal diffusivity of the material, respectively.

The numerical model by the present method is shown in Fig. 4.5(a). There are 112 boundary nodes and 196 internal nodes, i.e. \( N_B = 112 \), \( N_I = 196 \). The boundary is discretized into 112 equally sized elements. In one cycle, the motion of the heat flux is divided into 112 steps, \( N_{\text{mov}} = 112 \), with the same time length \( \tau = 2\pi / 112\dot{\phi} \). Using the corresponding boundary condition within each step due to the flux motion, 112 pairs of coefficient matrices can be obtained. Substituting them into Eq. (4.29) and starting from the initial condition, temperature variations are simulated. For comparison, the finite element results are also presented. Figure 4.5(b) shows the FEM model. It can be easily seen that the model scale is greatly reduced by the DRBEM compared with the FEM model.
Fig. 4.5 Numerical model of the roller component. (a) BEM model. (b) FEM model.

Fig. 4.6 Comparison of steady state surface temperature distribution by the present method and ABAQUS.
Fig. 4.7 (a) Comparison of surface temperature variation up to 100s by the present method and ABAQUS. (b) Comparison of surface temperature variation in two cycles at steady state by the present method and ABAQUS.
The comparisons of the calculated results by the present method along with the results by ABAQUS are shown in Figs. 4.6-7. Figure 4.6 presents the comparison of temperature distribution along the outer surface at steady state, Fig. 4.7(a) the comparison of the surface temperature variation up to 100s, and Fig. 4.7(b) the comparison of the surface temperature variation in two cycles at steady state. All the results are in good agreement as shown in the Figs. 4.6-7. In addition, the time taken by the present method is greatly less than that by ABAQUS for the same simulation on the same computer, i.e. 25 minutes for 1000s simulation by the present method, while 2 hours 18 minutes for 420s simulation by ABAQUS. Such calculation efficiency will become more prominent when much longer time of temperature history is simulated. Most of time required by the present method is mainly in the calculation of the coefficient matrices $A_i (i = 1, 2, \cdots, N_{\text{mov}})$, i.e. the matrix exponential. Once the pairs of the coefficient matrices are determined, temperature variation can be obtained efficiently by the iteration Eq. (4.29).

### 4.3.3 Oscillating Heat Source

Fig. 4.8 Numerical model of a rectangular domain subject to an oscillating heat source.

Considering the same example in Ref. [27] for the case of oscillating heat source, a rectangular domain $0.3 \times 0.1\text{m}$ is subject to an oscillating heat source $\overline{q}$ on its top surface with heating width $0.1\text{m}$ and oscillation velocity $\nu = 1.2\text{m/s}$. The rest of the surface is kept at constant temperature $T_0$. The initial temperature is $T_i$. The numerical model is shown in Fig. 4.8. 80 boundary nodes and 33 internal nodes are included. The top surface is discretized into 30 elements. The motion of the heat flux is divided into 40 steps in one cycle from the left to the right side. In every step the heat flux slides across one element size within time $\tau = 0.3/30/\nu = 0.00833$ seconds. One oscillation cycle includes 40 heating positions of the flux, i.e. $N_{\text{mov}} = 40$. However, there are only 21 different heating steps in one cycle, corresponding 21 different load conditions from the left side to the right side. That is, some of the coefficient matrices are used twice in one cycle. Only the 21 pairs of coefficient matrices need to be evaluated by using Eqns. (4.16a) and (4.16b). Substituting them into Eq. (4.29) according the oscillating motion of the heat source and starting from the initial condition, temperature variations are obtained.
The same parameters as in [27] are used in the simulation. Figure 4.9 shows the comparison of the temperature rise by the present method and by the analytical solution at the locations (0.15, 0.1), (0.05, 0.1), (0.25, 0.1) and (0.15, 0.05). The both results are in good agreement. The results are efficiently obtained by the present method within 3 minutes, comparable to the analytical solution, whereas it required 11 hours to accomplish the same task using the ABAQUS on the same computer.

Fig. 4.9 Comparison of temperature rise obtained analytically and by the present method at locations (0.15, 0.1), (0.05, 0.1), (0.25, 0.1) and (0.15, 0.05).

4.4 Concluding Remarks

In this study, a new method for transient heat conduction by a combination of the dual reciprocity boundary element method and the transfer matrix method is developed and applied to the study of problems in presence of moving heat sources. The proposed method avoids the use of the time marching schemes for transient problems and limits the discretization of the model to its boundary. Therefore, models based on this method considerably save the computational time and the date preparation.

Once the coefficient matrices are evaluated, the temperature variations can be efficiently simulated, and longer time of simulation does not increases much the time spending. This is important for the analysis of moving heat source problems, which generally takes a long time for the simulation to get to a steady state. Also the temporal derivative involved with transient problems is processed by a so-called Precise Time Integration (PTI) method, which make the developed method unconditional stable, and thus makes it possible to use different time steps in the analysis and to efficiently simulate the moving heat source problems with variable velocity.
Numerical examples demonstrate the efficiency and accuracy of the method. The method presented provides an efficient solution for the moving heat source problems. In this work, only two dimensional problems are presented. However, the procedure can be readily extended to three-dimensional problems as well as more complicated thermal contact problems. Also it is possible to use the method to treat problems with material nonlinearities by using the similar transformation as in Refs. [69, 71, 73, 83]. It is noted that, due to the temperature-dependant material properties, the coefficient matrices $A$ and $M$ need to be calculated at each time step.

4.5 Nomenclature

$B = Biot$ number, $B = hR / k$
$A = Coefficient matrix for TMM$
$B = Assembled boundary integral matrix related to field unknowns$
$B' = Modified $B$ matrix, $B' = BF'F^{-1}$
$c(\xi) = Constant$ in the boundary integration equation

$C = Matrix$ defined by $C_0 = \frac{1}{\alpha} \left( H\hat{T} - G\hat{Q} \right) F^{-1}$

$D = Assembled$ boundary integral matrix related to known boundary temperature or flux

$f_j (x) = Interpolation$ functions (distance functions)
$F = Square$ matrix, $F_{ij} = f_j (x_i)$.
$F'(x) = Modified$ $F$ matrix defined in Eq. (24b)
$G = Matrix$ of boundary integrals with kernel $\tilde{T}$
$h = Heat$ transfer coefficient (w/m$^2$ K)
$H = Matrix$ of boundary integrals with kernel $\tilde{q}$
$I = Identity$ matrix.
$k = Thermal$ conductivity (w/m K)

$M = Coefficient$ matrix for TMM related to boundary conditions
$n = Unit$ normal outward to the boundary
$N_B, N_I = Number$ of boundary and internal nodes
$N_{mov} = Number$ of moving steps in one cycle
$Pe = Peclet$ number, $Pe = \omega R^2 / \alpha$
$\tilde{q} = Prescribed$ boundary flux (W/m$^2$)

$\tilde{q}(x), \hat{q}_j (x) = Heat$ flux from fundamental solution, and heat flux related to DRBEM (W/m$^2$)

$Q, \hat{q}_j = Nodal$ vector of $q$, and nodal vector of $\hat{q}_j$

$\hat{Q} = Matrix$ of vectors $\hat{q}_j$

$r_j = distance$ between the field point $x$ and the collocation points $x_j$, $r_j = \|x - x_j\|$ (m)
$R = Radius$ of cylinder (m)
$R(t) = Vector$ of known boundary conditions
\(T, \bar{T}, T^{(0)}, T_{w}\) = Field temperature, prescribed boundary temperature, initial temperature, and ambient temperature (°C)

\(\tilde{T}(x), \hat{T}_j(x)\) = Fundamental solution, and temperature related to DRBEM (°C)

\(T, \hat{T}_j\) = Nodal temperature vector of \(T\), and nodal temperature vector of \(\hat{T}_j\)

\(\hat{T}\) = Matrix of vectors \(\hat{T}_j\)

\(\nu\) = Moving velocity (m/s)

\(x\) = Coordinate vector of field point (m)

\(X(t)\) = Vector of field unknowns

\(\alpha\) = Thermal diffusivity (m²/s)

\(\alpha_j(t)\) = Unknown time dependant functions

\(\alpha\) = Vector of \(\alpha_j(t)\)

\(\phi(t), \phi_0\) = Angular displacement, and angular semi-heating width (rad)

\(\tau\) = Time step size (s)

\(\xi\) = Coordinate vector of source point (m)

Superscripts:

-1 = Matrix inversion

. = Time derivative

\(k\) = Time index

Subscripts:

\(i\) = Index of node or moving step

\(j\) = Collocation index
5 On the Temperature Rise of Bodies Subjected to Unidirectional or Oscillating Frictional Heating and Surface Convective Cooling

This chapter presents a computationally efficient algorithm for simulating the transient temperature in a body subjected to unidirectional sliding or oscillatory heat source along its boundary. The algorithm utilizes the combination of finite element technique and transfer matrix method. The method does not require direct simulation at each cycle and thus provides significant computation savings without the loss of accuracy. For a given system, the coefficient matrices are calculated only once and then applied to different sets of heat loads experienced by the same system. This method has application in manufacturing and tribological processes where the temperature rise due to surface heating must be taken into consideration. Illustrative examples are presented where a half space is subjected to a unidirectional sliding or oscillating heat source with consideration of convective cooling at the surface.

5.1 Introduction

Surface heating by application of a moving heat source is a common problem in most manufacturing processes that involve cutting, grinding, welding, and surface heat treatment using laser irradiation along the workpiece as well as in many tribological applications where the contact surfaces are in relative sliding motion. Examples include the interaction of the rolling elements and races in roller bearings, shafts in journal bearings under dry lubrication, and steel rolls in rolling mills. A common feature in each of these applications is that the temperature distribution in the heated zone at or near the surface can play an important role in the metallurgical microstructure, thermal shrinkage, thermal cracking, residual stress, and many other performance parameters.

Since the early pioneering works of Block [1] and Jaeger [2] on the treatment of moving heat source acting on a semi-infinite solid, numerous research papers have appeared that study various aspects of problem. The majority of the published works have concentrated on treating the problem of semi-infinite bodies subjected to unidirectional moving heat source [3, 51, 86]. More recently problems dealing with oscillating heat source [5, 6, 28], and rotating cylinders subjected to surface heating and convective cooling [51, 55-57] have been treated. The early contributions resorted to analytical solutions based on a number of restrictive assumptions, among which are the small dimension of heating region in relation to other dimensions of the solid, and the use of the adiabatic boundary condition outside the heated region. Another common approach is to assume that the effect of surface cooling due to convection occupied the entire surface. However, many practical applications, it is difficult to meet these assumptions.

Significant efforts have been devoted by numerical techniques to study the thermal behavior in the solid with complex geometry and subjected to mixed boundary conditions. Using the finite element method, Neder et al. [91] evaluated the contact temperature

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4 Paper related to this chapter is being accepted in Tribology Transactions with minor changes.
distribution between real surfaces in sliding contact, and Li [92] studied the temperature
distribution in workpiece during scratching and grinding. Tseng [58] used a first-order
upwind differencing scheme to study the steady state heat transfer behavior of a two-
dimensional rotating roll. Kennedy et. al. [9-11] performed a finite element analysis of
sliding surface temperatures. Mansouri and Khonsari [13] developed a model to predict
the surface temperature in a pin-bushing system with relative oscillatory motion. Among
these published literature, only studies in [9, 10, 13] focused their attention to oscillatory
heat source by numerical methods.

Finite element method can be applied to handle complex moving heat source problems,
but it often suffers from numerical oscillation at high sliding velocities, so that accurate
prediction of temperature generally requires extremely fine mesh, which is
computationally expensive [9]. This is particularly crucial when dealing with three-
dimensional oscillatory heat source problems where a large number of cycles are needed
for the system to reach a steady state [13, 28].

An alternative approach is the transfer matrix method. This method can reduce the
amount of computer storage and computing time. It is widely used in structure mechanics
[15, 16]. An application of transfer matrix method in heat transfer analysis of periodic
heating problems and its efficiency is reported by Fan & Barber [17] where they
presented a very efficient method for the analysis of linear periodic heating problems, and
studied the temperature profile in a two dimensional media undergoing periodic variable
boundary temperatures.

In this chapter, a computationally efficient method is presented for simulating the
transient temperature in a body subjected to unidirectional sliding or oscillatory heat
source along its boundary. The method extends the transfer matrix method for the
analysis of moving heat source problems, where the body undergoes periodic surface
heating and convective cooling due to the unidirectional sliding or oscillatory motion of
the heat source. The algorithm utilizes the combination of finite element technique and
transfer matrix method. By avoiding the need for direct simulation of every cycle, the
model provides significant computation savings without the loss of accuracy. For a given
system, the coefficient matrices need to be calculated only once and can be applied in
different sets of heat loads thereafter. Therefore, the presented model provides an
efficient solution for predicting the temperature field in many manufacturing and
tribological applications where the temperature rise due to surface heating must be taken
into consideration. A half-space subjected to a unidirectional sliding or oscillating heat
source with consideration of convective cooling is studied to illustrate the capability of
the solution methodology.

5.2 Model Development

5.2.1 Governing Equation and Boundary conditions

Referring to Fig. 5.1, consider a transient temperature field \( T(x,t) \) in a domain \( \Omega \) with
a boundary \( \Gamma \), heated by a moving heat source \( \bar{q}(x,y,z,t) \) with velocity \( \nu(t) \) along its
surface. The rest of the surface undergoes the convective cooling. The equation governing the heat conduction is:

\[ \nabla^2 T(x, t) = \frac{1}{\alpha} \dot{T}(x, t), \quad x \in \Omega, \ t > 0 \quad (5.1) \]

with the initial condition

\[ T(x, 0) = T^{(0)}(x), \quad x \in \Omega \quad (5.1a) \]

and the boundary conditions are given by

\[ T(x, t) = \bar{T}(x, t), \quad x \in \Gamma_T \quad (5.1b) \]
\[ q(x, t) = \bar{q}(x, t), \quad x \in \Gamma_q \quad (5.1c) \]
\[ q(x, t) = h(T_{\infty} - T), \quad x \in \Gamma_h \quad (5.1d) \]

where \( x \) is coordinate vector \((x, y, z)\). The parameter \( t \) denotes the time, and the super-dot stands for the derivative with respect to time. \( \alpha \) and \( k \) are the thermal diffusivity and the thermal conductivity of the material, respectively. The heat flux \( q(x, t) \) is defined as

\[ q(x, t) = k \frac{\partial T}{\partial n}, \] here \( n \) is the unit normal outward to the boundary \( \Gamma \). \( T^{(0)}(x) \) represents the initial temperature distribution. \( \bar{T}(x, t) \) and \( \bar{q}(x, t) \) are the prescribed boundary temperature on surface \( \Gamma_T \) and heat flux on surface \( \Gamma_q \), respectively. The boundary surface \( \Gamma_h \) is subjected to convective cooling with the heat transfer coefficient \( h \), and the ambient temperature \( T_{\infty} \).

![Diagram of a solid subjected to an oscillating heat source](image)

**Fig. 5.1 Solid subjected to an oscillating heat source**

### 5.2.2 Formulation of the Solution

Using the standard finite element discretization procedure [34], Eq. (5.1) results in the following system of equations in the matrix form as

\[ C \dot{T}(t) + K T(t) = F(t) \quad (5.2) \]

where

\[ C = \sum_e \int_{\Omega_e} N^T \rho c N d\Omega \quad (5.2a) \]
\[
K = \sum_e \int_{\Gamma_e} \left( \frac{\partial \mathbf{N}_k^T}{\partial x} k \frac{\partial \mathbf{N}_k^T}{\partial y} + \frac{\partial \mathbf{N}_k^T}{\partial y} k \frac{\partial \mathbf{N}_k^T}{\partial z} + \frac{\partial \mathbf{N}_k^T}{\partial z} k \frac{\partial \mathbf{N}_k^T}{\partial x} \right) d\Omega + \sum_e \int_{\Gamma_e} h \mathbf{N}_k^T \mathbf{N}_k d\Gamma \tag{5.2b}
\]
\[
\mathbf{F} = \sum_e \int_{\Gamma_e} \mathbf{q}_n^T d\Gamma + \sum_e \int_{\Gamma_e} h \mathbf{T}_e \mathbf{N}_k^T d\Gamma \tag{5.2c}
\]
where \( e \) and \( \Omega^e \) denote the element \( e \) and the corresponding element domain, respectively. \( \mathbf{N} \) is the shape function in the element of \( e \). Parameters \( \rho \) and \( c \) represent the density and capacitance of the material, respectively. The superscript \( T \) denotes the matrix transpose.

The general solution of Eq. (5.2) is
\[
\mathbf{T}(t) = e^{\mathbf{H} t} \mathbf{T}^{(0)} + \int_0^t e^{-\mathbf{H}(t-s)} \mathbf{C}^{-1} \mathbf{F}(s) ds \tag{5.3}
\]
where the superscript \( 0 \) denotes the initial time \( t_0 \), and
\[
\mathbf{H} = -\mathbf{C}^{-1} \mathbf{K} \tag{5.3a}
\]
Then the solution at time \( t_k \) is
\[
\mathbf{T}^{(k)} = e^{\mathbf{H} t_k} \mathbf{T}^{(0)} + \int_0^{t_k} e^{\mathbf{H}(t_k-s)} \mathbf{C}^{-1} \mathbf{F}(s) ds \tag{5.4}
\]
The solution at time \( t_{k+1} \) (\( t_{k+1} = t_k + \tau \)) where \( \tau \) is a small time step is:
\[
\mathbf{T}^{(k+1)} = e^{\mathbf{H} t_{k+1}} \mathbf{T}^{(0)} + \int_0^{t_{k+1}} e^{\mathbf{H}(t_{k+1}-s)} \mathbf{C}^{-1} \mathbf{F}(s) ds
\]
\[
= e^{\mathbf{H} \tau} \mathbf{T}^{(k)} + \int_0^\tau e^{\mathbf{H}(\tau-s)} \mathbf{C}^{-1} \mathbf{F}(t_k + s) ds \tag{5.5}
\]
For a small enough time step \( \tau \), the load vector \( \mathbf{F}(t) \) can be treated to be constant from time \( t_k \) to \( t_{k+1} \) and substituted by the load vector at time \( t_k \), denoted as \( \mathbf{F}^{(k)} \). In case of a moving heat source problem, time step \( \tau \) depends on the sliding velocity of the heat source. Performing the integration in Eq. (5.5) yields the following iteration equation for the heat transfer analysis as
\[
\mathbf{T}^{(k+1)} = \mathbf{A} \mathbf{T}^{(k)} + \mathbf{M} \mathbf{F}^{(k)} \tag{5.6}
\]
where
\[
\mathbf{A} = e^{\mathbf{H} \tau} \tag{5.6a}
\]
\[
\mathbf{M} = (\mathbf{I} - \mathbf{A}) \mathbf{K}^{-1} \tag{5.6b}
\]

Physically, the coefficient matrix \( \mathbf{A} \) demonstrates how the heat conducts inside the solid, and the coefficient matrix \( \mathbf{M} \) shows how the surface heat input contributes to the field temperature rise in the solid. The load vector \( \mathbf{F} \) includes two sub-vectors: the vector of surface flux on the contact surface and the vector of the ambient temperature on the convective boundary surfaces. The terms involving the heat fluxes corresponds to each element on the discretized contact surface as shown in Fig. 5.2, where the contact surface is divide into \( N_a \times N_b \) rectangular elements with sizes covering \( a \times b \) area. The heat source distribution is divided into heat source segments acting over the small rectangles locating around the corner nodes. The heat flux on each small segment is applied according to the following Eq. (5.7).
\[
q = \begin{cases} 
\bar{q} & (x, y) \in S \\
\frac{h(T_\infty - T)}{x, y} & (x, y) \notin S 
\end{cases}
\quad (5.7)
\]

where \( S \) denotes the contact area at a certain time, and \( \bar{q} \) is the heat flux. For frictional contact, it is calculated by the friction coefficient \( \mu \), contact pressure \( p \) and relative sliding velocity \( \nu \) between the contact interface in the form of
\[
\bar{q} = \mu p |\nu| \quad (5.7a)
\]

In Fig. 5.2, the contact area \( S \) envelopes the five corner nodes and is approximated by the eleven small shaded segments. The different flux values, evaluated by Eq. 5.7a, are assigned to these element surfaces of the small segments at every time step, according to the motion of the heated region. The rest of the element surfaces outside of contact area \( S \) are subjected to the convective cooling, evaluated by Eq. 5.7 when \( (x, y) \notin S \). The parameter \( T \) is calculated by averaging the temperatures at the four corner nodes of the segment.

![Fig. 5.2 Discretized contact surface.](image)

For a unidirectional sliding heat source moving at a constant velocity \( U \), Eq. (5.7a) yields
\[
\bar{q} = \mu p U \quad (5.8)
\]

For the case of the oscillatory heat source, the heat source is assumed to oscillate sinusoidally according to
\[
\nu(t) = A \omega \sin(\omega t) \quad (5.9)
\]

with
\[
\omega = 2\pi f \quad (5.9a)
\]

where \( f \) is the oscillating frequency, and \( A \) represents the oscillating amplitude. Then, the heat flux for oscillatory heat source can be obtained from Eq. (5.7a) as
\[
\bar{q} = \mu p A \omega |\sin(\omega t)| \quad (5.10)
\]

The other sub-vector of \( F \) contains the values of ambient temperature \( T_\infty \) corresponding to the convective boundary surfaces outside the contact area. It is noted that, except \( \bar{q} \) and \( T_\infty \), the other terms in Eq. (5.2c) are multiplied into the corresponding columns of matrix \( \mathbf{M} \) in Eq. (5.6). Therefore, the vector \( F \) in Eq. (5.6) only contains the
values of the heat fluxes, corresponding to each small segment on the discretized contact surface shown in Fig. 5.2, and the ambient temperatures, corresponding to the boundary surfaces subjected to convective cooling.

Referring to Fig. 5.1, for a model having \( N \) nodes, \( N_h \) convective surfaces, and with the contact surface divided into \( N_q = N_e \cdot N_h \) small segmented elements illustrated in Fig. 5.2, the dimension of vector \( \mathbf{T} \) is \( N \), the dimension of vector \( \mathbf{F} \) is \( N_h + N_q \), and the dimensions of matrices \( \mathbf{A} \) and \( \mathbf{M} \) are \( N \times N \) and \( N \times (N_h + N_q) \), respectively. After the coefficient matrices \( \mathbf{A} \) and \( \mathbf{M} \) are evaluated, the temperature in the solid can be easily simulated by Eq. (5.6). The final temperature is normalized as

\[
\hat{T} = \frac{T_k}{2lq_0}
\]

(5.11)

where \( l \) is the half length of the heat source along the sliding direction. \( q_0 = \mu p_0 U \) for the unidirectional case and \( q_0 = \mu p_0 A \omega \) for the oscillatory case. \( p_0 \) is the average contact pressure for both cases.

### 5.2.3. Calculation of Coefficient Matrices

The coefficient matrices \( \mathbf{A} \) and \( \mathbf{M} \) can be evaluated directly by using Eqs. (5.6a) and (5.6b). Alternatively, a commercial FEM software, ABAQUS, is utilized for this purpose. Examination of Eq. (5.6) reveals that matrices \( \mathbf{A} \) and \( \mathbf{M} \) can be treated as influence matrices. Each of the column vectors of matrix \( \mathbf{A} \) reflects how a unit nodal temperature affects the field temperature inside the solid, and each of the column vectors of matrix \( \mathbf{M} \) reflects how a unit discretized boundary flux or unit ambient temperature contributes to the field temperature rise. Thus, in this study, matrices \( \mathbf{A} \) and \( \mathbf{M} \) are indirectly calculated by using Eq. (5.6). The method is described as follows.

The matrix \( \mathbf{A} \) can be obtained by \( N \) cycles of simulations with the same time period \( \tau \). For the \( i \)th \( (i = 1, 2, \cdots, N) \) simulations, the initial condition and boundary load are set to

\[
(T^{(0)})_i = \{\delta_y\} \text{ and } \mathbf{F} = \{0\}
\]

(5.12)

where \( \delta_y \) is Kronecker delta, and \( j = 1, 2, \cdots, N \) are the index to the entries of the vector \( \mathbf{T}^{(0)} \). The result \( \mathbf{T}^{(i)} \) is the \( i \)th column of matrix \( \mathbf{A} \).

Similarly, the matrix \( \mathbf{M} \) can be evaluated by performing \( N_h + N_q \) cycles of simulations with the same time period \( \tau \) as that used in calculation of the matrix \( \mathbf{A} \). For the \( i \)th \( (i = 1, 2, \cdots, N_h + N_q) \) simulations, the following initial condition and boundary load are applied.

\[
T^{(0)} = \{0\} \text{ and } (\mathbf{F})_i = \{\delta_y\}
\]

(5.13)

where \( j = 1, 2, \cdots, N_h + N_q \) are the index to the entries of the vector \( \mathbf{F} \). The result \( \mathbf{T}^{(i)} \) is the \( i \)th column of matrix \( \mathbf{M} \).
After \(N + N_h + N_q\) cycles of simulations, the coefficient matrices \(A\) and \(M\) are obtained. It is noted that \(A\) and \(M\) keep constant for different heat sources and need to be calculated only once. Once they are determined, the temperature distribution in the solid can be efficiently simulated using Eq. (5.6) by starting from the initial condition \(T^{(0)}\) and, at each time step \(t_k\), assigning the different values to the components of vector \(F^{(k)}\) according to the variable boundary conditions due to the moving contact areas or the time-dependant ambient temperatures. A computer program is developed using the procedure described in Section 5.2 to treat moving heat source problems. Applications of the method are presented in the following section.

5.3. Model Verification

Consider a uniformly distributed square heat source with contact size of \(1 \times 1\) mm on the surface of a semi-infinite media. We will consider both two types of moving heat source: unidirectional sliding and oscillatory motion along the \(x\) direction. The FEM model shown in Fig. 5.3 is used to take into consideration the half symmetry of the moving heat source problem. The dimension of the FEM model is \(11.7 \times 3.15 \times 4\) mm. It contains 29176 nodes and 25895 eight-node hexahedral elements. The contact area is a rectangular surface with very fine meshes on the top surface of the model, bordered by the dashed line, as illustrated in Fig. 5.3. The square heat source either slides inside the contact surface or oscillates back and forth along the \(x\) direction with the neutral position at the center of the analyzed contact surface as shown in Fig. 5.3. At each time step, it acts on a rectangular region of \(1 \times 0.5\) mm inside the contact surface of the half symmetric model. The heated area involves 200 surface elements. The magnitude of the heat source depends on the sliding velocity and is evaluated by Eq. (5.8) for the unidirectional sliding or Eq. (5.10) for the oscillatory sliding. It is assumed that all the heat generated by friction flows into the media.

Zero temperature is prescribed on the bottom surface and the side surfaces. The top surface of the model except the contact region of the heat source is subjected to the convective cooling, specified by the dimensionless Biot number, defined as

\[
Bi = \frac{hl}{k}
\]  \hspace{1cm} (5.14)

The transient temperature is simulated by using Eq. (5.6) up to a steady state. A steady state is assumed once the change in the maximum surface temperature, computed between two consecutive cycles, is less than \(10^{-5}\).

To demonstrate the accuracy of the present method, the results without consideration of surface convection, or \(Bi=0\), are compared with the analytical solutions by Tian and Kennedy [3] for a half space in unidirectional sliding contact and by Wen and Khonsari [28] for a half space in oscillatory contact. The material properties used in the simulation are listed in Table 5.1. All the simulations are based on the same coefficient matrices calculated according to the discussion in Section 5.2.3.
Table 5.1 Material properties used in the simulations

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density $\rho$ (kg/m$^3$)</td>
<td>7865</td>
</tr>
<tr>
<td>Specific heat $c$ (J/kg K)</td>
<td>460</td>
</tr>
<tr>
<td>Thermal conductivity $k$ (W/m K)</td>
<td>58</td>
</tr>
</tbody>
</table>

Fig. 5.3 FEM model of the moving heat source problem

**Unidirectional sliding.** With the origin of the coordinate system set at the center of the heat source, the steady-state temperature of a point $(x,y,z)$ exposed to a uniform square heat source sliding unidirectionally along the $x$ direction on the surface of a half space is [3]:

$$T_{\text{steady}}(x, y, z) = \int_{y=-l}^{l} \int_{x=-l}^{l} \frac{q_0}{2\pi ks} \exp\left(-\frac{U}{2\alpha} \left[s-(x-x')\right]\right) dx' dy'$$

where $s = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$. $q_0 = \mu p_0 U$ and $p_0$ is the average contact pressure.
The results for the unidirectional sliding heat source predicted by the present method and those calculated by Eq. (5.15) are compared under different Peclét numbers, defined as

\[ P_e = \frac{U l}{2 \alpha} \]  

(5.16)

Figure 5.4 shows a comparison of the surface temperature rise for the unidirectional sliding heat source at different Peclét numbers by the present method along with the steady state solution to Eq. (5.15) available in Ref. [3]. The temperature distributions are compared along the center line of the heat source in the direction of the sliding motion. The figure shows excellent agreement between the results.

Fig. 5.4 Comparison of surface temperature distribution for unidirectional sliding case along the sliding direction through the center of the heat source. \( q_0 = \mu p_0 U \), \( p_0 \) is the average contact pressure.

**Sinusoidal Oscillation.** The transient temperature at a point \( (x,y,z,t) \) for the case of a uniform square heat source oscillating along the \( x \) direction according to Eq. (5.9) on the surface of a half space, with the origin of the coordinate system set at the neutral position of the oscillatory motion, is [28]:

\[
T(x,y,z,t) = \frac{q_0}{4\rho c(\pi\alpha)^{3/2}} \left( \int_{\tau=0}^{t} \frac{\sin(\omega(t-\tau))}{\tau^{3/2}} d\tau \right) \\
\times \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \exp \left( -\frac{\left[x-A\cos(\omega(t-\tau)) - x'\right]^2 + (y-y')^2 + z^2}{4\alpha\tau} \right) dx' dy'
\]

(5.17)

where \( q_0 = \mu p_0 A \omega \) and \( p_0 \) is the average contact pressure.
The heat source is considered to oscillate with different oscillating frequencies, \( f \), and amplitudes, \( A \). The amplitude is normalize as

\[
\bar{A} = \frac{A}{l}
\]  

(5.18)

Figures 5.5 and 5.6 present the comparisons between the surface temperature distributions along the oscillating direction as obtained by the present method and those by Eq. (5.17) from ref. [28] when the temperature at the oscillatory center reaches its maximum value. The origin of the coordinate system is located at the neutral position of the oscillatory motion shown in Fig. 5.3. Figure 5.5 compares the temperatures at different oscillatory amplitudes when the frequency \( f = 10 \), and Fig. 5.6 compares the temperatures at different oscillatory frequencies when the amplitude \( \bar{A} = 2.0 \). The prediction results of the present chapter are in good agreement with published literature. It can be easily seen from Figs. 5.5 and 5.6 that, for smaller values of oscillatory amplitude or oscillatory frequency, the temperature distribution along the sliding direction is nearly symmetric along axis \( x = 0 \), and the maximum temperature occurs at the oscillation center. As the oscillating amplitude increases, the location of the maximum temperature moves from the neutral position of the oscillation to the edge. This is similar to that encountered in unidirectional sliding shown in Fig. 5.4 as Peclét number increases. This behavior will be described in depth in Section 5.4.

Similarly, good agreement between the temperature distribution along the \( y \) and \( z \) directions are obtained between the results by the present method and those obtain by the analytical solutions, Eqs. (5.15) or (5.17). Having verified the solution methodology, we will next apply the method to problems involving convective cooling on the contact surface. Note that coexistence of both heating and convective cooling is much more complicated. For this reason, most of the published literature generally assumes that the effect of surface cooling due to convection occupies the entire surface to avoid solving the Fredholm integral equation. The formulation and solution methodology presented in this study easily treats this problem without making such a simplifying assumption.

### 5.4 Results and Discussion

In this section, the same FEM model shown in Fig. 5.3 and the corresponding coefficient matrices as those in Section 5.3 are used to investigate the effect of the surface convective heat transfer on the temperature distribution under unidirectional sliding or oscillating. The same uniformly distributed square heat source as in Section 5.3 is considered.

Table 5.2 shows the input data for the simulations, where the first two columns show the range of values for each parameter, the third column for the oscillatory case shows the fixed values for the corresponding parameters, \( \bar{A} \) or \( f \), when the other parameter is considered. A parameter characterizing the penetration depth, \( d \), defined as the depth where the maximum surface temperature decreases to its half value in the \( z \) direction, is introduced to investigate the thermal layer thickness under the different moving and cooling conditions [92].
Fig. 5.5 Comparison of surface temperature distribution for the oscillatory case along the sliding direction at different oscillatory amplitudes when frequency $f=10$ and $Bi=0$. The origin is located at the neutral position of the oscillatory motion. $q_0 = \mu p_0 A \omega$, $p_0$ is the average contact pressure.

Fig. 5.6 Comparison of surface temperature distribution for oscillatory case along the sliding direction at different oscillatory frequencies when amplitude $\bar{A}=2.0$ and $Bi=0$. The origin is located at the neutral position of the oscillatory motion. $q_0 = \mu p_0 A \omega$, $p_0$ is the average contact pressure.
All the cases are simulated on an Intel 2 Quad CPU 2.4GHz computer. The time taken by the present method is less than two minutes for the unidirectional sliding. For oscillatory sliding, the execution time is between one to two hours. The present method is more efficient than ABAQUS using the same FE model shown in Fig. 5.3 on the same computer. For each simulation step with the same time period, the present method takes less than 1.2 seconds but ABAQUS needs about 4.6 seconds. Therefore, the present method is more than three times faster than ABAQUS. This is very important when dealing with oscillatory heat source problems where a large number of cycles are needed for the system to reach a steady state. Specifically, for the oscillating case when $\bar{A} = 2.5$ and $f = 10$, the present method takes 74 minutes to complete a 4 seconds simulation of 6660 steps with the step time $\tau = 6.006 \times 10^{-4}$ seconds. In contrast, ABAQUS takes over 8 hours to complete the same simulation on the same computer. In current study, the step time $\tau$ is determined in such way that the heat source can only oscillate across at most one element along the sliding direction within the time period $\tau$, in order to catch the temperature variation when the heat flux oscillates near or over the oscillatory center where the maximum surface temperature may occur. The maximum oscillating velocity at the oscillating center determines the step time should be less than $6.367 \times 10^{-4}$ seconds for the case of $\bar{A} = 2.5$ and $f = 10$.

Table 5.2 Input data for the simulations

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
<th>Fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convection</td>
<td>$Bi$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Unidirectional sliding case</td>
<td>$Pe$</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>Oscillating case</td>
<td>$\bar{A}$</td>
<td>0.2</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>2</td>
<td>20</td>
</tr>
</tbody>
</table>

5.4.1 Unidirectional Sliding

For unidirectional sliding case, the heat source slides from the left to right along the x direction as illustrated in Fig. 5.3 at a constant velocity $U$. Different sliding speeds with Peclet ranging from 0.2 to 10 as in Table 2 are studied. Figure 5.7 shows a typical temperature contour due to a uniform square heat source with Peclet number, $Pe = 1.0$ and Biot number, $Bi = 0.5$, where the model is cut by two parallel surfaces perpendicular to the sliding direction in order to zoom into the heated zone. Figures 5.8, 5.9 and 5.10 show the dimensionless temperature distributions at different Peclet numbers along the sliding, transverse and vertical direction, respectively, passing through the point with the maximum surface temperature. It can be found from Figs. 5.8-10 that the heat effect zone becomes more concentrated with increasing values of Peclet number, becoming narrower in width as shown in Fig. 5.9c and yielding a lower penetration depth in Fig 5.10c.
The effect of the surface convection on the temperature distribution varies with different Peclét numbers. For small Peclet numbers, the entire computation region is significantly affected by the convective heat transfer as shown in Figs. 5.8-10a and 5.8-10b. For large Peclét numbers, the convection hardly affects the temperature rise at the points located at $x/l<1$, but significantly affects the other points located at $x/l>1$ shown in Figs. 5.8-10c. This is consistent with the results in ref. [86].

Figure 5.11 shows the maximum surface temperature versus Peclét number simulated with different Biot numbers. The maximum surface temperature decreases with the increasing values of Peclét number. The surface convection has significant influence on the maximum surface temperature for small Peclét numbers, but such effect becomes negligible when $Pe>3$. Similar effect occurs on the thermal penetration depth shown in Fig. 5.12, which presents the penetration depth versus Peclét number under different Biot numbers. As the Peclét number is increasing, the penetration depth decreases and the effect of the surface convection becomes less pronounced and becomes negligible when $Pe>2$.

The present method takes less than two minutes for all the simulations for the unidirectional sliding case on an Intel 2 Quad CPU 2.4GHz computer. It is more efficient than ABAQUS on the same computer, which requires about seven minutes.

Fig. 5.7 Dimensionless temperature contour for unidirectional sliding heat source with $Pe = 1.0$ and $Bi = 0.5$. $l$ is the half length of the heat source along the sliding direction.
Fig. 5.8 Surface temperature distribution for unidirectional sliding case along the sliding direction through the center of the heat source. $q_o = \mu p_o U$, $p_o$ is the average contact pressure. (a) $Pe=0.2$; (b) $Pe=1.0$; (c) $Pe=10$. (fig. cont’d)
Fig. 5.9 Surface temperature distribution for unidirectional sliding case in the transverse direction through the maximum temperature point. $q_0 = \mu p_0 U$, $p_0$ is the average contact pressure. (a) $Pe=0.2$; (b) $Pe=1.0$; (c) $Pe=10$. (fig. cont’d)
Fig. 5.10 Temperature distribution for unidirectional sliding case under the maximum surface temperature point. (a) $Pe=0.2$; (b) $Pe=1.0$; (c) $Pe=10$. $q_0 = \mu p_0 U$, $p_0$ is the average contact pressure. (fig. cont’d)
Fig. 5.11 Maximum surface temperature verse Peclet number for the unidirectional sliding. $q_0 = \mu p_0 U$, $p_0$ is the average contact pressure
5.4.2 Oscillatory motion

In this case, the heat source oscillates sinusoidally as prescribed by Eq. 5.9 along \( x \) direction with the neutral position placed at the center of the analyzed contact surface as illustrated in Fig 5.3. Figures 5.13-16 present the results for the heat source oscillating with the same frequency \( f =10 \), but with different amplitudes. Figures 5.13a and 13b show the dimensionless temperature contour for two different oscillating amplitudes when \( Bi = 0.5 \). The model is cut by two parallel surfaces perpendicular to the sliding direction in order to zoom in the heated zone.

Figures 5.14, 5.15 and 5.16 present the dimensionless temperature distribution along the sliding, transverse and vertical direction, respectively, passing through the point where the surface temperature reach to its maximum value. Generally, the maximum surface temperature for the oscillating heat source occurs at the oscillation center as illustrated in Fig. 5.3, especially for the smaller dimensionless amplitudes. When \( \bar{A} \) is small, the temperature distribution is close to the solution for the stationary case. As shown in Fig. 5.13a, the temperature distribution is nearly symmetric about the surface perpendicular to the sliding direction and through the oscillating center. The temperature distribution along the sliding direction is nearly symmetric along axis \( x=0 \) as shown in Fig. 5.14a. As the oscillating amplitude increases, the thermal trail can be observed as in Fig. 5.13b. This behavior is similar to that encountered in unidirectional sliding shown in Fig. 5.7. The temperature distribution along the oscillating direction becomes close to the profile for a unidirectional sliding case shown in Fig. 5.14c. Figures 5.5 and 5.6 demonstrate the similar trend as the oscillating amplitude or the oscillating frequency increases, respectively. Therefore, similar to the unidirectional sliding, the increasing sliding velocity due to the increase of the amplitude or the frequency will result in the

![Graph showing dimensionless penetration depth versus Peclet number](image)

Fig. 5.12 Penetration depth verse Peclet number for the unidirectional sliding. \( \bar{d} = d/l \).
moving of the maximum surface temperature from the contact center to the edge. The simulations predict that the location of the maximum surface temperature is slight away from the oscillating center when the oscillating amplitude or frequency becomes larger. As revealed by Tian and Kennedy [7], the surface temperature rise for the oscillatory sliding has two components: a steady state mean temperature or normal surface temperature rise, and a local surface temperature rise. The normal surface temperature rise is a cumulative residual temperature rise caused by the reciprocating sliding movement of the heat source. The local surface temperature rise is very similar to that obtained for a unidirectional sliding heat source, especially when the dimensionless amplitude is relatively large. This is why the temperature distribution for the oscillatory heat source experiences similar behavior to that for a unidirectional sliding when the oscillation velocity increases.

Compared to the unidirectional sliding where the heat effect zone is limited to the contact region, especially at the large Peclét numbers, the heat effect zone for the oscillating heat source is slightly greater as shown in Figs. 5.14-16. It is due to the larger heating span along the moving direction, $2(A+l)$, and repetitive passing and heating of the oscillatory heat source over the same surface area. Figures 5.17 and 5.18 show respectively the maximum surface temperature and the half temperature penetration depth verse dimensionless amplitude. Both the dimensionless maximum surface temperature and the penetration depth decrease with the increasing values of the oscillating amplitude when keeping the oscillating frequency at the constant value, $f=10$.

It can be seen from Figs. 5.14-16 that the surface convective heat transfer has nearly the same effect on the temperature distribution in the entire computation region for different oscillating amplitudes. Similarly, the maximum surface temperature and the penetration depth are affected by the convection in the entire studied range of amplitudes shown in Figs. 5.17 and 5.18, respectively. This is very different from its effect on the unidirectional sliding case. As reported in Ref. [7], the surface convection has a significant effect on the normal surface temperature rise. This makes the curves of the temperature distributions in Figs. 5.14-16 go in the same way for the different Biot number $Bi=0, 0.2, 0.5, 1.0, 2.0$.

The results of an extensive set of simulations reveal that similar results can be obtained by changing the oscillatory frequency while keeping the amplitude constant. The details are not provided here due to space limitation. For all the simulations, the execution time is within one to two hours on an Intel 2 Quad CPU 2.4GHz computer. However, ABAQUS takes over 8 hours for the same simulation on the same computer.

Table 5.3 presents the effect of the convection heat transfer on the maximum surface temperature in different oscillating conditions, where the values represent the ratio of the maximum surface temperatures under different surface cooling to that without consideration of convection. It can be seen that, for an oscillatory heat source, a heat conduction analysis without consideration of the surface convection will overestimate the surface temperature due to the surface cooling, especially when the heat transfer coefficient is large.
Fig. 5.13 Dimensionless temperature contour for oscillatory heat source with oscillating frequency $f=10$ and $Bi = 0.5$. (a) $\overline{A} = 0.5$; (b) $\overline{A} = 4.0$.

Fig. 5.14 Surface temperature distribution for oscillatory case along the sliding direction when oscillating frequency $f=10$. The origin is located at the neutral position of the oscillatory motion. $q_0 = \mu p_0 A \omega$, $p_0$ is the average contact pressure. (a) $\overline{A}=0.2$; (b) $\overline{A}=1.0$; (c) $\overline{A}=5$. (fig. cont’d)
Fig. 5.15 Surface temperature distribution for oscillatory case in the transverse direction passing through the point with maximum surface temperature when oscillating frequency $f=10$. $q_0 = \mu p_0 A\omega$, $p_0$ is the average contact pressure. (a) $\bar{A}=0.2$; (b) $\bar{A}=1.0$; (c) $\bar{A}=5$. (fig. cont’d)
Fig. 5.16 Temperature distribution for oscillatory case under the point with maximum surface temperature when oscillating frequency $f=10$. $q_o = \mu \bar{p}_o A \omega$, $p_o$ is the average contact pressure. (a) $\bar{A}=0.2$; (b) $\bar{A}=1.0$; (c) $\bar{A}=5$. (fig. cont’d)
Dimensionless Location, $z/l$

Dimensionless Temperature, $Tk/2q_0$

(b)

Dimensionless Location, $z/l$

Dimensionless Temperature, $Tk/2q_0$

(c)
Fig. 5.17 Maximum surface temperature verse dimensionless amplitude for oscillatory case when oscillating frequency \( f = 10 \). \( q_0 = \mu p_o A \omega \), \( p_o \) is the average contact pressure.

Fig. 5.18 Penetration depth verse dimensionless amplitude for oscillatory case when oscillating frequency \( f = 10 \).
Table 5.3 Effect of the convection on the maximum surface temperature

<table>
<thead>
<tr>
<th>Oscillatory Conditions</th>
<th>Percentage of Max. Surface Temperature in That without Convection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Bi = 0$</td>
</tr>
<tr>
<td>$f = 10$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{A} = 0.2$</td>
<td>100.00</td>
</tr>
<tr>
<td>$\tilde{A} = 1.0$</td>
<td>100.00</td>
</tr>
<tr>
<td>$\tilde{A} = 5.0$</td>
<td>100.00</td>
</tr>
<tr>
<td>$f = 2$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{A} = 2.0$</td>
<td>100.00</td>
</tr>
<tr>
<td>$f = 10$</td>
<td>100.00</td>
</tr>
<tr>
<td>$f = 20$</td>
<td>100.00</td>
</tr>
</tbody>
</table>

5.5 Conclusions

In this study, a computationally efficient method is presented that combines the finite element method and the transfer matrix method to treat transient heat transfer problems involving either unidirectional or oscillatory sliding heat source along its boundary. Avoiding the need for direct simulation of every cycle, the computational scheme results in much savings in computer time. Specifically, the present method is more than three times faster than ABAQUS for each time step. This is very important when dealing with oscillatory heat source problems where a large number of cycles are needed for the system to reach a steady state. Also, the same coefficient matrices allow one to efficiently predict temperature field subject to moving heat source with different magnitude or different motion such as unidirectional or oscillatory sliding at different sliding speeds. All the simulations in current study are based on the same pairs of the coefficient matrices $A$ and $M$, which are calculated only once.

The present method makes it possible to efficiently handle the problems involving thermomechanical interaction, where computations using the direct approach become computationally prohibitive, by treating the thermal and elastic parts of the contact solution independently and then combining them together through a numerical iteration scheme to obtain the thermomechanical solution.

In the case of unidirectional sliding, heat effect zone becomes concentrated with increasing values of Peclét number. The effect of the surface convective heat transfer coefficient on the temperature distribution varies with different Peclét numbers. For small Peclét numbers, the entire computation region is significantly affected by the convective heat transfer. For large Peclét number, convective heat transfer coefficient hardly affects the temperature rise at the points located at $x/l < 1$, but significantly affects the other points located at $x/l > 1$. The surface convection affects significantly the maximum surface temperature and the penetration depth when the Peclét number is small, but such effect becomes negligible when $Pe > 3$ on the maximum surface temperature and when $Pe > 2$ on the penetration depth.
For oscillatory sliding case, the heat effect zone is slightly greater than that for the case of unidirectional sliding due to a larger heating span along the oscillating direction and repetitive passing and heating of the oscillatory heat source over the same surface area. The surface convective heat transfer has significant effect on the temperature distribution in the entire computation region by strongly affecting the normal temperature rise. When the oscillating amplitude or frequency is small, the temperature distribution is close to the solution for the stationary case. The temperature distribution along oscillating direction becomes nearly symmetric along axis $x=0$. As the oscillating amplitude or frequency increases, the thermal trail similar to that encountered in unidirectional sliding occurs and the temperature distribution along the oscillating direction becomes close to the profile for a unidirectional sliding case. The maximum surface temperature and the penetration depth are significantly affected by the convection as well. A heat conduction analysis without consideration of the surface convection will overestimate the surface temperature for an oscillatory heat source due to the surface cooling, especially when the heat transfer coefficient is large.

### 5.6 Nomenclature

$a =$ Size of contact element in $x$ direction (m)

$A$, $\bar{A}$ = Oscillating amplitude (m) and dimensionless amplitude, respectively. $\bar{A} = A/l$

$A =$ Coefficient matrix associated with initial condition

$b =$ Size of contact element in $y$ direction (m)

$Bi =$ Biot number, $Bi = hl/k$

$c =$ Specific heat (J/kg K)

$C =$ Capacitance matrix

$d$, $\bar{d} =$ Half temperature penetration depth (m) and dimensionless penetration depth, respectively. $\bar{d} = d/l$

$f =$ Oscillating frequency (Hz)

$F =$ Heat load vector including heat flux and convection

$h =$ Convective heat transfer coefficient (W/m$^2$K)

$H =$ Exponential matrix

$I =$ Identity matrix

$k =$ Thermal conductivity (w/m K)

$K =$ Conductance matrix

$l =$ Half length of the heat source in the sliding direction (m)

$M =$ Coefficient matrix associated with boundary conditions

$n =$ Unit normal outward to the boundary $\Gamma$

$N =$ Vector of shape functions for element

$N_a$, $N_b =$ Number of discretized segments on the contact surface in $x$ and $y$ direction, respectively

$N$, $N_q$, $N_h =$ Node number of the model, contact element number, and number of convective surfaces, respectively. $N_q = N_a \cdot N_b$

$p$, $p_0 =$ Contact pressure and average contact pressure, respectively (N/m$^2$)

$Pe =$ Peclet number
$q, q_\Gamma, q_0$ = General flux, prescribed boundary condition of heat flux and average flux, respectively, (W/m$^2$)

$S$ = Contact surface

$t, t_k$ = Time and the time at $k$th step (s)

$T, \hat{T}, T$ = Field temperature, Dimensionless temperature and Vector of field temperature $T$, respectively ($^\circ$C)

$T^{(0)}, \hat{T}, T_\infty$ = Initial temperature, prescribed surface temperature, and ambient temperature, respectively ($^\circ$C)

$U$ = Constant sliding velocity (m/s)

$\nu$ = General sliding velocity (m/s)

$x$ = coordinate vector $(x, y, z)$ (m)

$\alpha$ = Thermal diffusivity, $\alpha = k / \rho \alpha_c$ (m$^2$/s)

$\tau$ = Step time (s)

$\rho$ = Density (kg/m$^3$)

$\delta_{ij}$ = Kronecker delta

$\Gamma_T, \Gamma_q, \Gamma_h$ = Surface $\Gamma$ with prescribed temperature, heat flux and convection, respectively

$\mu$ = Friction coefficient

$\omega$ = Angular frequency (rad/s)

$\Omega$ = Domain considered

Superscripts:

$e$ = Element

$(k)$ = Iteration number

$T$ = Transpose of matrix

-1 = Inversion of matrix
6 Thermomechanical Effects on Transient Temperature in Non-Conformal Contacts Experiencing Reciprocating Sliding Motion

In this chapter, a thermomechanical analysis of a semi-infinite elastic solid in contact with a rigid adiabatic sphere subjected to oscillatory sliding motion is conducted to investigate the effects of transient mechanical and thermomechanical loads on the temperature variation in the solid. The interdependent mechanical and thermal analyses are coupled by an iteration scheme using the transfer matrix method combined with the finite element technique. The significance of interaction among heat generation, temperature rise and contact conditions are interpreted in light of numerical results. It is shown that the constraint of the thermal expansion in the indentation direction between the contacting bodies plays a significant role in the increase of contact pressure and heat generation, which in turn affect the temperature rise and thermal expansion.

6.1 Introduction

Frictional heating and resulting temperature rise at sliding interfaces is of great importance in a wide variety of engineering applications, such as journal bearings, pin joints and automotive brake system. The frictional heat is a function of friction coefficient, sliding velocity and contact pressure. As the frictional heat flows into the contacting bodies, the contact area changes due to thermal expansion, resulting in the change of the contact pressure, which in turn further intensifies the heat generation and temperature. The cycle of frictional heating leading to thermal expansion and higher contact pressures and temperature often becomes unstable and causes component failure. Thermoelastic instability, hot spotting, seizure, cracking, and thermal blistering are all in effect a manifestation of unstable thermal runaway brought about as a result of thermomechanical interaction of contacting bodies.

A review of published literature reveals significant advances have been made in the analytical treatment of thermomechanical responses in sliding contact. Pioneering work in this area began by the early contribution of Blok [1] and Jaegar [2] on the treatment of the temperature rise of a semi-infinite body and has continued by many other researchers thereafter. For example, Tian and Kennedy [3], Liu et al. [86] presented the solutions for temperature rise in semi-infinite body with unidirectional sliding motion, and Hirano and Yoshida [5], Wen and Khonsari [28] developed the solutions for the oscillatory sliding case. The thermomechanical analyses of a half space under given heat source with unidirectional sliding motion are presented by Ju and Farris [29], Liu and Wang [87], Martini et al. [88] and Gong and Komvopoulos [89]. In all of these studies, the distributions of the heat sources were decoupled from the mechanical response of the deformed solid. The heat flux was assumed to be unaffected by the mechanical distortion due to the thermal expansion.

To consider the concomitant effects of mechanical and thermal response, Liu and Wang [18, 19] developed two- and three-dimensional thermooelastic contact models of two infinitely large surfaces to account for the thermal effects on the mechanical response for
steady state heat transfer. They later extended the analysis to a three-dimensional thermomechanical model of nonconforming contact [20]. By performing a fully coupled finite element analysis, Ye and Komvopoulos [21] examined the simultaneous effects of mechanical and thermal surface load on the deformation of elastic-plastic layered media. Gong and Komvopoulos [22] analyzed an elastic-plastic media with patterned surface in contact with a rigid sphere to study the effects of friction coefficient, sphere radius and repetitive sliding on the contact stress and deformation fields.

The great majority of published literature contains very useful information and insight into the temperature and thermomechanical fields in solids due to the unidirectional moving heat sources or sliding rough surfaces. Nevertheless, a comprehensive study of transient temperature variation for an oscillatory sliding contact with the thermomechanical effects is still lacking.

In the current study, a thermomechanical analysis of a semi-infinite elastic solid in contact with a rigid adiabatic sphere subjected to oscillatory sliding motion is conducted to investigate the effects of transient heat transfer and thermomechanical loads on the temperature variation in the solid. The interdependent mechanical and thermal analyses are coupled by an iteration method using the developed transfer matrix method combined with the finite element technique. The analysis provides a practical method for solving problems that involve thermomechanical coupling (TMC) in the oscillatory contact. The significance of interaction among heat generation, temperature rise and contact conditions are interpreted in light of numerical results. Simulation results reveal that the constraint of the thermal expansion in the indentation direction between the contact bodies plays a significant role in the increase of contact pressure and heat generation, which in turn affect temperature rise and thermal expansion.

6.2 Model Development

6.2.1 Problem Definition and Finite Element Model

Consider an elastic semi-infinite solid in contact with an adiabatic rigid sphere shown in Fig. 6.1. The load $W$ is applied on the sphere. The sphere oscillates back and forth along the $x$ direction on the surface of the half space. Heat is generated in the contact region due to friction. As the frictional heat flows into the semi-infinite solid, the contact area and the distribution of the contact pressure changes due to thermal expansion, which in turn influences the heat generation and temperature. Therefore, the semi-infinite solid is subjected to interdependent thermal and mechanical loads along its surface, requiring simultaneous consideration of these elements.

In the present study, the thermal and elastic parts of the contact solution are treated independently and then combined together through a numerical iteration scheme to obtain a thermomechanical solution according to the following procedure:

i. Thermoelastic analysis is performed to determine the contact pressure and contact area. For the first step, no thermal load is applied.
ii. Transient heat transfer analysis is conducted by using the contact results from Step i to simulate the temperature distribution in the medium.

iii. The sphere is moved according to a specified frequency and oscillatory amplitude and Steps i and ii are repeated until either a steady state for temperature is reached or the maximum temperature exceeds 700°C, indicating the possibility of thermal induced failure by scuffing.

Figure 6.2 shows the FEM model for the problem. Due to symmetry, only one half of the sphere and the half-space solid are considered. The same exact mesh is used for both the thermoelastic analysis and the transient heat transfer analysis. The radius of the rigid sphere is 5 mm, and the dimensions of the solid are 11.7×3.15×4 mm. The model contains 29176 nodes and 25895 eight-node hexahedral elements. Very fine mesh is used at the contact surface to catch the high temperature gradient near the heating region and increase the accuracy in the calculation of contact area and contact pressure.

![Fig. 6.1 A semi-infinite body in contact with a rigid ball with oscillatory motion.](image)

6.2.2 Nonlinear Thermoelastic Analysis

The oscillating motion of the sphere is simulated by moving the sphere along the sliding direction in an incremental fashion. At each step, corresponding to the location of the oscillating sphere, a static thermoelastic analysis is performed using ABAQUS. The nodes on the bottom surface are fixed against displacement in the x and z directions. The nodes on the symmetric plane are fixed against displacement in the y direction. The sphere is treated as an adiabatic body; therefore, it is only necessary to consider the thermomechanical effects on the semi-infinite solid. Thus, the loads on the contacting bodies include the load on sphere and the thermal load on the semi-infinite body in the form of the nodal temperature obtained by the transient heat transfer analysis. The thermal load results in thermal expansion depending on the material property of the medium. At the first step, there is no temperature variation in the solid and thus no thermal load exists.

In the present study, two types of loads are considered. In the first case, the sphere is oscillating under a constant force $W$ as illustrated in Fig. 6.1, where the sphere is allowed
to move up and down freely depending on the equilibrium between the applied load \( W \) and the reaction force in the contact interface, such as in the pin-on-disk device used in a tribometer. In the second case, the sphere is pressed into the semi-infinite solid up to a very small specified depth, \( d \), and oscillates back and forth while keeping the sphere fixed against displacement in the vertical direction. In the contact region, the thermal expansion of the half-space along the indentation direction is strictly constrained by the rigid sphere. An example of the second situation is a journal bearing in which the bushing OD is affixed to an external housing. In this case, the thermal expansion of the bushing is constrained by the housing.

6.2.3 Transient Heat Transfer by Transfer Matrix Method

Since the sphere is treated as an adiabatic body, heat conduction occurs only in the half-space solid. In the transient heat transfer analysis, zero temperature is prescribed on the bottom surface and the side surfaces of the solid as shown in Fig. 6.2. The contact surface is subject to the frictional heating and the rest of the top surface undergoes the convective cooling. The transient temperature distribution is simulated by the transfer matrix method.
6.2.3.1 Formulation of the Transfer Matrix Method

The general equation governing a transient temperature field \(T(x,t)\) in a domain \(\Omega\) with a boundary \(\Gamma\) as illustrated in Fig. 6.1 or Fig. 6.2 is:

\[
\nabla^2 T(x,t) = \frac{1}{\kappa} \dot{T}(x,t), \quad x \in \Omega, \: t > 0
\]

with the initial condition

\[
T(x,0) = T^{(0)}(x), \quad x \in \Omega
\]

and the boundary conditions are given by

\[
T(x,t) = \bar{T}(x,t), \quad x \in \Gamma_T
\]

\[
q(x,t) = \bar{q}(x,t), \quad x \in \Gamma_q
\]

\[
q(x,t) = h(T_\infty - T), \quad x \in \Gamma_h
\]

where \(x\) is coordinate vector \((x,y,z)\). The parameter \(t\) denotes the time, and the super-dot stands for the derivative with respect to time. Parameters \(\kappa\) and \(k\) are the thermal diffusivity and the thermal conductivity of the material, respectively. The heat flux \(q(x,t)\) is defined as \(q(x,t) = k \frac{\partial T}{\partial n}\), where \(n\) is the unit normal outward to the boundary \(\Gamma\). The parameter \(T^{(0)}(x)\) represents the initial temperature distribution. \(\bar{T}(x,t)\) and \(\bar{q}(x,t)\) are the prescribed boundary temperature on surface \(\Gamma_T\) and heat flux on surface \(\Gamma_q\), respectively. The boundary surface \(\Gamma_h\) is subjected to convective cooling with the heat transfer coefficient \(h\), and the ambient temperature \(T_\infty\).

Using the standard finite element discretization procedure [34], Eq. (6.1) yields the following system of equations in the matrix form as

\[
C \dot{T}(t) + KT(t) = F(t)
\]

where

\[
C = \sum_e \int_{\Omega^e} N^T \rho c N d\Omega
\]

\[
K = \sum_e \int_{\Omega^e} \left( \frac{\partial N^T}{\partial x} k \frac{\partial N}{\partial x} + \frac{\partial N^T}{\partial y} k \frac{\partial N}{\partial y} + \frac{\partial N^T}{\partial z} k \frac{\partial N}{\partial z} \right) d\Omega + \sum_e \int_{\Gamma^e} h N^T N d\Gamma
\]

\[
F = \sum_e \int_{\Gamma^e} \bar{q} N^T d\Gamma + \sum_e \int_{\Gamma^e} \bar{h} T_\infty N^T d\Gamma
\]

where \(N\) is the shape function in the element of \(e\). \(\Omega^e\) and \(\Gamma^e\) denote the element domain and the element surface if it is on the boundary, respectively. Parameters \(\rho\) and \(c\) represent the density and specific heat of the material, respectively. The superscript T denotes the matrix transpose.

Solving Eq. (6.2) yields the following iteration equation for the heat transfer analysis as

\[
T^{(k+1)} = AT^{(k)} + MF^{(k)}
\]

with

94
\[ A = e^{H \tau} \]  
\[ M = (I - A)K^{-1} \]  
(6.3a)  
(6.3b)

and

\[ H = -C^{-1}K \]  
(6.3c)

where the superscript \( k \) represents the time step and \( \tau \) is the time period from time \( t_k \) to \( t_{k+1} \). \( A \) and \( M \) are the coefficient matrices and \( I \) is the identity matrix. The superscript \( -1 \) denotes the matrix inversion. \( F \) is the vector of the discretized heat load. For a model having \( N \) nodes, \( N_h \) convective surfaces, and with the contact surface divided into \( N_q \) small segmented elements, the dimension of vector \( T \) is \( N \), the dimension of vector \( F \) is \( N_h + N_q \), and the dimensions of matrices \( A \) and \( M \) are \( N \times N \) and \( N \times (N_h + N_q) \), respectively.

Physically, the coefficient matrix \( A \) demonstrates how heat conducts inside the solid, and the coefficient matrix \( M \) shows how the surface heat input contributes to the field temperature rise in the solid. The load vector \( F \) includes two parts. One of them contains the terms involved with the heat flux on the contact surface due to the frictional heating and convective cooling. The other part consists of the terms of the ambient temperature on the convective boundary surfaces. In the present method, except \( q \) and \( T_\infty \), the other terms in Eq. (6.2c) are multiplied into the corresponding columns of matrix \( M \) in Eq. (6.3). Therefore, vector \( F \) in Eq. (6.3) only contains the values of the heat fluxes, corresponding to each small surface element on the discretized contact surface, and the ambient temperatures, corresponding to the boundary surfaces subjected to convective cooling. The heat flux on each small surface element is evaluated by the following Eq. (6.4).

\[ q = \begin{cases} \bar{q} & \text{Elements \& Contact region} \\ h(T_\infty - T) & \text{Elements \&\not\& Contact region} \end{cases} \]  
(6.4)

where the parameter \( T \) represents the average temperatures of small segmented elements on the contact surface. Whether a surface element lies inside the contact region or not is determined by the contact pressure obtained in the thermoelastic analysis. If the contact pressure is not equal to zero, the corresponding surface element is in contact with the sphere and the frictional heat \( \bar{q} \) is applied on it, otherwise, the surface is not in contact with the sphere and undergoes the convective cooling.

For a sinusoidal oscillation in the form of

\[ \nu(t) = A \omega \sin(\omega t) \]  
(6.5)

with

\[ \omega = 2\pi f \]  
(6.5a)

where \( f \) is the oscillating frequency and \( A \) is the oscillating amplitude.

If all the heat dissipated in the frictional contact is converted into heat, the frictional heat flux and the total heat generated in the contact interface can be given as Eq. (6.6a) and (6.6b), respectively.

\[ \bar{q} = \mu p A \omega \left| \sin(\omega t) \right| \]  
(6.6a)
\[ Q = \mu W_c A \omega |\sin(\omega t)| \]

(6.6b)

where \( \mu \) denotes the friction coefficient. \( p \) is the contact pressure and \( W_c \) represents the contact force.

### 6.2.3.2 Calculation of the Coefficient Matrices

The calculation of the coefficient matrices \( A \) and \( M \) is based on using the commercial software ABAQUS. Examination of Eq. (6.3) reveals that matrices \( A \) and \( M \) can be treated as influence matrices. Each of the column vectors of matrix \( A \) reflects how a unit nodal temperature affects the field temperature inside the solid, and each of the column vectors of matrix \( M \) reflects how a unit discretized boundary flux or unit ambient temperature contributes to the field temperature rise. Thus, in this chapter, matrices \( A \) and \( M \) are indirectly calculated by using Eq. (6.3). The method is described as follows.

The matrix \( A \) can be obtained by \( N \) cycles of simulations with the same time period \( \tau \). For the \( i \)th (\( i = 1, 2, \cdots, N \)) simulations, the initial condition and boundary load are set to

\[ (T^{(0)})_i = \{\delta_{ij}\} \text{ and } F = \{0\} \]

(6.7)

where \( \delta_{ij} \) is the Kronecker delta, and \( j = 1, 2, \cdots, N \) are the index to the entries of the vector \( T^{(0)} \). The result \( T^{(i)} \) is the \( i \)th column of matrix \( A \).

Similarly, the matrix \( M \) can be evaluated by performing \( N_h + N_q \) cycles of simulations with the same time period \( \tau \) as that used in calculation of matrix \( A \). For the \( i \)th (\( i = 1, 2, \cdots, N_h + N_q \)) simulations, the following initial condition and boundary load are applied

\[ T^{(0)} = \{0\} \text{ and } (F)_i = \{\delta_{ij}\} \]

(6.8)

where \( j = 1, 2, \cdots, N_h + N_q \) are the index to the entries of the vector \( F \). The result \( T^{(i)} \) is the \( i \)th column of matrix \( M \).

After \( N + N_h + N_q \) cycles of simulations, the coefficient matrices \( A \) and \( M \) are obtained. It is noted that \( A \) and \( M \) keep constant for different heat sources and need to be calculated only once. Once they are determined, the transient temperature distribution in the half space at each time step \( t_k \) is easily calculated from Eq. (6.3). The procedure is to start from the initial condition \( T^{(0)} \) and to assign different values of heat fluxes to the components of vector \( F^{(k)} \) at each time step \( t_k \). This can be evaluated by Eq. (6.4), according to the oscillatory motion of the sphere, which has been evaluated in the thermoelastic analysis. The final temperature is normalized using the following relationship.

\[ \hat{T} = \frac{T_k}{2Rq_0} \]

(6.9)
where $R$ is the contact radius. $q_0 = \mu p_0 A \omega$. The parameter $p_0$ is the average contact pressure. For thermoelastic analysis, $p_0$ and $R$ will be the initial contact pressure and corresponding contact radius due to the indentation, respectively.

6.3. Solution Algorithm

A computer program is developed using the procedure described in Section 6.2.1-6.2.3. Figure 6.3 presents the flow chart of the basic steps of the simulation algorithm. First, the coefficient matrices $A$ and $M$ are evaluated. Then, beginning with a static thermoelastic analysis with consideration of thermal expansion (no temperature changes and thus no thermal expansion at the first step), the results of contact pressure are used to calculate the heat flux by Eq. (6.6a), which is assigned to the corresponding components of the load vector $F$ according to Eq. (6.4). At the same time, using the temperature result at the previous step as the pseudo-initial temperature or using the initial condition for the first step, the transient temperature after one cycle with the time period $\tau$ is obtained by solving Eq. (6.3). The thermomechanical response in the semi-infinite solid is simulated by moving the sphere according the specified oscillating motion, updating the nodal

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**Fig. 6.3 Flow chart of the basic steps of the simulation algorithm.**

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temperature load in the semi-infinite solid, and repeating the thermoelastic analysis followed by a thermal analysis, until either a steady state for temperature is reached or the maximum temperature gets to over 700°C.

6.4. Model Validation

To evaluate the accuracy of the present model, a normal contact and an oscillatory sliding heat source are simulated. Table 6.1 lists the parameters used in the simulations. Figure 6.4 shows the comparison between the finite element results and Hertz analytical solution [90] for an elastic half space indented by a rigid sphere under different loads. The contact pressure is normalized by the maximum Hertz contact pressure \( P_{\text Hertz} \) and the coordinate by Hertz contact radius \( R_{\text Hertz} \). Both results are in good agreement.

If the sphere oscillates back and forth along the surface of the half space according to Eq. (6.5), the contact pressure shown in Fig. 6.4 yields a parabolic heat source described by the following expression:

\[
q(r, t) = \frac{3}{2} q_0 \left(1 - \frac{r^2}{R^2}\right)^{1/2} \left|\sin(\omega t)\right| \tag{6.10}
\]

where \( R \) is the contact radius. \( q_0 \) is the average heat flux as

\[
q_0 = \frac{\mu W A \omega}{\pi R^2} \tag{6.10a}
\]

Without thermomechanical consideration, and with the origin of the coordinate system set at the neutral position of the oscillatory motion, the transient temperature at a point \((x, y, z)\) for a parabolic heat source, Eq. (6.10), oscillating along the \(x\) direction according to Eq. (6.5) on the surface of a half spaces is given in [28] as:

\[
T(x, y, z, t) = \frac{3q_0}{8\rho c(\pi \alpha)^{3/2}} \int_{\tau=0}^{2\pi} \left|\sin\left(\omega(t-\tau)\right)\right| d\tau \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \left(1 - \frac{r}{R}\right)^{1/2} \left[ -\frac{x - A\cos(\omega(t-\tau)) - r\cos(\theta)}{4\alpha \tau} + \frac{(y - r\sin(\theta))^2 + z^2}{4\alpha \tau} \right] r dr d\theta \tag{6.11}
\]

The temperature distribution of a half space under a oscillating parabolic heat source in Eq. (6.10) is simulated using the present transfer matrix method. The result is compared with the analytical solution by Eq. (6.11) from Ref. [28]. Figure 6.5 presents the comparison of the surface temperature distributions along the oscillating direction as obtained by the present method and those by Eq. (6.11) when the temperature at the oscillatory center reaches its maximum value. In this figure, as well as in the subsequent figures, the temperature is normalized by \( 2Rq_0/k \), and the coordinate is normalized by the contact radius \( R \). It can be seen from Fig. 6.5 that both results are in good agreement.

Having verified the solution methodology, we will next combine the thermoelastic analysis and the transient heat transfer analysis to obtain the thermomechanical solution for the problem proposed in Section 6.2.
Table 6.1 Parameters used in the analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, ( \rho ) (kg/m³)</td>
<td>7865</td>
</tr>
<tr>
<td>Specific heat, ( c ) (J/kg K)</td>
<td>460</td>
</tr>
<tr>
<td>Thermal conductivity, ( k ) (W/m·K)</td>
<td>58</td>
</tr>
<tr>
<td>Elastic modulus, ( E ) (GPa)</td>
<td>200</td>
</tr>
<tr>
<td>Poisson ratio, ( \nu )</td>
<td>0.3</td>
</tr>
<tr>
<td>Thermal expansion, ( a ) (1/K)</td>
<td>1.15×10⁻⁵</td>
</tr>
<tr>
<td>Heat transfer coefficient, ( h ) (W/m² K)</td>
<td>40</td>
</tr>
<tr>
<td>Oscillating frequency, ( f ) (Hz)</td>
<td>10</td>
</tr>
<tr>
<td>Oscillating amplitude, ( A ) (mm)</td>
<td>2.0</td>
</tr>
<tr>
<td>Friction coefficient, ( \mu )</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Fig. 6.4 Comparison of normalized contact pressure for different loads between finite element and Hertz solutions.

Fig. 6.5 Comparison of temperature distribution along \( x \) direction through the center of the heat source by the present transfer matrix method and the analytical solution. \( R \) is the contact radius. \( q_o \) is the average heat flux.
6.5. Results and Discussion

In this section, the thermomechanical response of the semi-infinite solid in contact with a sphere oscillating back and forth is studied. Two types of loads on the sphere are considered: constant applied force, \( W = 1000 \text{N} \), and fixed the displacement of the sphere in the vertical direction after pressing it into the semi-infinite solid up to a very small specified depth. Both loads on the sphere yield the same contact condition at the initial time; that is, the same distribution of contact pressure in the same contact area. During the oscillation, the sphere in the first case is allowed to move freely in the vertical direction depending on the equilibrium between the applied force and the resulted contact force. But in the second case, the sphere is fixed against displacement in the vertical direction. An example of first case is commonly met in a typical pin-on-disk device used in the tribology research laboratory. There are also many applications that experience conditions that can be modeled as the second case such as a journal bearing with its bushing fixed inside an external housing in the manner that its thermal expansion is constrained.

The results for the heat transfer analysis without the thermomechanical interaction are presented to establish a reference for comparison with the results of the thermomechanical analysis. The semi-infinite solid in the heat transfer analysis without the thermomechanical interaction is subjected to the same oscillatory heat source on its top surface as the initial frictional heat flux in the thermoelastic cases. All the cases can be simulated until a steady state for temperature is reached and the results at the steady state are presented. For the periodic oscillatory contact, the temperature result when the sphere or the frictional heat source oscillates from the extreme left to the extreme right is symmetric with corresponding result when the sphere oscillates from the extreme right to the extreme left along the axis through the oscillatory center and perpendicular to the sliding direction. Therefore, only the results corresponding to sphere oscillating from the left to the right are presented in the following Case I and II. The same parameters listed in Table 6.1 are used.

Case I. Oscillatory Contact under Constant Load \( W = 1000 \text{N} \)

In this case, the sphere oscillates along the \( x \) direction in contact with a semi-infinite solid under the constant load \( W = 1000 \text{N} \) as illustrated in Fig. 6.2. The sphere can move freely in the vertical direction depending on the contact force. The results are compared with those for the heat transfer without the thermomechanical consideration.

Figure 6.6(a) shows the dimensionless temperature contour for the heat transfer without thermomechanical interaction. Figure 6.6(b) includes thermomechanical coupling. Figure 6.6 illustrates that the contact temperature is slight higher when the thermomechanical interaction is taken into account. The same characteristic is illustrated in Fig. 6.7, which presents the comparison of the surface temperature distributions along the sliding direction with and without TMC when the sphere oscillates to the five different locations, \( x_n = -A, -0.5A, 0, 0.5A, A \), from the left to the right. Figure 6.8 shows more clearly the distribution of the contact temperature when the sphere is at the
oscillatory center. It can be seen from Figs. 6.6-8 that the thermal expansion in Case I only has a little effect on the temperature rise in the contact region, a slight rise and small shift of the maximum contact temperature to the left as shown in Fig. 6.8. The slight increase of the maximum contact temperature due to the thermal expansion becomes negligible when the sphere oscillates to the locations at or near both oscillatory ends as shown in Fig. 6.7.

Fig. 6.6 Dimensionless temperature contours when the heat source/sphere oscillates to the oscillatory center from the left to the right at steady state. The temperature is normalized by \( 2Rq_0/k \). \( R \) and \( q_0 \) are the initial contact radius and the initial average heat flux, respectively. (a) Oscillatory contact without TMC. (b) Oscillatory contact under constant applied force with TMC.

Figure 6.9 shows that the variation of the indentation depth or the downward displacement of the sphere and the maximum contact pressure with the dimensionless maximum contact temperature, respectively. The initial indentation depth \( d_i \) and the initial maximum contact pressure \( p_{mi} \) are used to normalize the indentation depth and the pressure, respectively. Figure 6.9(a) reveals a linear decrease in the indentation depth with the maximum contact temperature to values less than its initial value. Figure 6.9(b) demonstrates a linear increase in the peak contact pressure with the maximum contact temperature to values greater than its initial value.
Fig. 6.7 Comparison of the surface temperature distributions of Case I along with the heat transfer results without TMC along the sliding direction when the sphere/heat source at the different locations. The origin is located at the neutral position of the oscillatory motion. $R$ and $q_0$ are the initial contact radius and the initial average heat flux, respectively. $A$ is the oscillatory amplitude.

Fig. 6.8 Comparison of the surface temperature distributions between Case I and the heat transfer without TMC along the sliding direction when the sphere/heat source is at the oscillatory center. The origin is located at the neutral position of the oscillatory motion. $R$ and $q_0$ are the initial contact radius and the initial average heat flux, respectively.
Fig. 6.9 Variation of (a) Indentation depth and (b) Maximum contact pressure with the dimensionless maximum contact temperature for the oscillatory contact under constant applied force. $p_{mi}$ and $d_i$ is the initial maximum contact pressure the initial indentation depth, respectively. $R$ and $q_0$ are the initial contact radius and the initial average heat flux, respectively.
In the case of the oscillatory contact under the constant load \( W \), the thermal expansion due to the temperature rise in the half space results in an outward expansion of the contact surface. This outward expansion of the contact surface lifts the sphere up, reduces the indentation depth as shown in Fig. 6.9(a) and leads to a slight decrease of the contact area. The simulation results reveal that the thermal expansion of the half space results in a decrease of the contact area by 3.8%, except at the locations close to both oscillatory ends where the contact area keeps nearly the same as its initial value due to the low temperature rise as shown in Fig 6.7 at \( x_R = \pm A \). The slight decrease of the contact area results in the increase of the maximum value of the contact pressure shown in Fig. 6.9(b), which in turn affects the contact temperature illustrated in Figs. 6.6-8. Since the total heat generated in the contact interface, evaluated by Eq. 6.6(b), keeps unchanged in comparison with the heat transfer analysis without thermomechanical interaction, the thermal expansion has only little effect on the temperature in the half-space solid. Such effects mainly focus in the contact region shown in Fig. 8 when the sphere oscillates to the locations near or at the oscillatory center shown in Fig. 6.7.

**Case II. Oscillatory Contact with the Sphere Fixed Against Displacement in Vertical Direction**

Similar simulation as in Case I is performed, where the sphere oscillates along the \( x \) direction in contact with a semi-infinite solid, but the sphere is fixed against in vertical direction after pressed into the semi-infinite solid up to a very small specified depth. The indentation yields the same initial contact pressure as that in Case I and the same contact force \( W_c = 1000N \). The results are compared with those in Case I.

Figure 6.10 shows the dimensionless temperature contour for Case II when the sphere oscillates from the left to the right relative to the neutral position. The same behaviors as those in Fig. 6.6 are observed in Fig. 6.11 but with a higher surface temperature. Figure 6.11 presents the comparison of the surface temperature distributions between Case I and Case II when the sphere oscillates to the five different locations, \( x_R = -A, -0.5A, 0, 0.5A, A \), from the left to the right. Figure 6.12 shows the effect of the thermal expansion in Case II on the temperature distribution compared with the temperature distribution in Case I and the result without TMC along the oscillatory direction when the sphere is at the oscillatory center. Figures 6.10-12 illustrate that the thermal expansion in Case II affects the surface temperature in the entire studied region and such effect becomes more significant when the sphere oscillates close to the oscillatory center as shown in Figs. 6.10-12.

Figure 6.13 shows the variation of the maximum contact pressure and the contact force with the dimensionless maximum contact temperature, respectively. The pressure is normalized by the initial maximum contact pressure \( p_{mi} \), and the force by the initial contact force \( W_c \). Figure 6.15(a) demonstrates a faster rate of increase in the peak contact pressure with the maximum contact temperature to greater values in Case II than in Case I. The thermal expansion in Case II additionally results in a significant increase of the
contact force as revealed in Fig. 6.15(b). The increase of the contact force will result in an increase in the friction force, and thus an increase of the heat generation and the power loss.

For the case with sphere fixed against displacement in the vertical direction, the outward expansion of the contact surface due to the temperature rise tends to lift the sphere up and reduce the indentation depth in the half space. However, such a trend is constrained by the sphere, which is fixed against displacement in the vertical direction, resulting in the greater increase of the contact pressure shown in Fig. 6.13(a) and concomitant rise in the contact force shown in Figs. 6.13(b). Also, the outward expansion of the contact surface due to the thermal expansion brings additional points on the half space surface in contact with the sphere and thus increases the contact area. The simulation results reveal that there is a maximum 4.4% increase of the contact area. Due to more significant increase of the contact force shown Fig. 6.13(b), the contact pressure and the heat generation increase greatly, resulting in a higher rise of the surface temperature.

Table 6.2 presents the comparison of the thermomechanical effects on the contact area, maximum contact pressure, contact force and temperature between Case I and Case II, where the values represent the increase percentage of the parameters by comparing with those obtained without consideration of the thermomechanical effects. It can be seen that the thermal expansion in Case II plays a significant role on the temperature rise, much more than that in Case I. Therefore, more care should be taken when the thermal expansion in an oscillatory contact is constrained due to the structural configuration of the system. For example, when dealing with journal bearings that experiences oscillatory motion, with the busing constrained by an external housing, the thermal expansion of the shaft and bushing could result in the loss of the clearance. A complete loss of the designed clearance between the shaft and bushing may occur with the catastrophic seizure [24, 26].

Fig. 6.10 Dimensionless temperature contour when the sphere oscillates to the oscillatory center from the left to the right for the oscillatory contact with the sphere fixed against displacement in vertical direction. The temperature is normalized by $2Rq_0/k$. $R$ and $q_0$ are the initial contact radius and the initial average heat flux, respectively.
Fig. 6.11 Comparison of the surface temperature distributions between Case I and Case II along the sliding direction with the sphere at the different locations. The origin is located at the neutral position of the oscillatory motion. \( R \) and \( q_0 \) are the initial contact radius and the initial average heat flux, respectively. \( A \) is the oscillatory amplitude.

Fig. 6.12 Comparison of the surface temperature distributions along the sliding direction among Case I, Case II and the heat transfer without TMC when the sphere/heat source is at the oscillatory center. The origin is located at the neutral position of the oscillatory motion. \( R \) and \( q_0 \) are the initial contact radius and the initial average heat flux, respectively.
Fig. 6.13 Variation of (a) Maximum contact pressure and (b) Contact force with the dimensionless maximum contact temperature for the oscillatory contact with the sphere fixed against displacement in vertical direction. $p_{mi}$ and $W_{ci}$ is the initial indentation depth and the initial contact force, respectively. $R$ and $q_0$ are the initial contact radius and the initial average heat flux, respectively.
In the above two cases, Case I and II, the steady state for temperature can be reached. There also exist some other situations where a very large contact temperature is developed and that steady state temperature is not attained. The large temperature can be indicative of the possibility of scuffing failure [93, 94], which may occur in the contacting bodies relatively oscillating at a high speed under heavy load. In Case II, when increasing the oscillatory frequency up to 13Hz and keeping the rest operation parameters the same, the maximum contact temperature can rise to over 700°C after four cycles. The contour of the absolute temperature is presented in Fig. 6.14, where the maximum temperature is 726°C.

![Fig. 6.14 Contour of the absolute temperature rise for the case of possible scuffing failure.](image)

<table>
<thead>
<tr>
<th></th>
<th>Contact Area (%)</th>
<th>Max. Contact Pressure (%)</th>
<th>Max. Contact Force (%)</th>
<th>Max. Temperature (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>-3.8</td>
<td>4.8</td>
<td>0</td>
<td>2.1</td>
</tr>
<tr>
<td>Case II</td>
<td>+4.4</td>
<td>12.0</td>
<td>21.3</td>
<td>14.8</td>
</tr>
</tbody>
</table>

**6.6. Conclusions**

A thermomechanical model is developed to study the interaction of a semi-infinite elastic solid in contact with a rigid adiabatic sphere subjected to oscillatory sliding motion. The thermal and mechanical parts of the contact solution are treated independently and then combined together through a numerical iteration scheme to obtain a thermo-mechanical solution. In the presented simulations, the distribution of the transient temperature is calculated efficiently by the transfer matrix method. The model makes it practical to handle the problems involving thermomechanical interaction in
oscillatory contacts, where simulations using the direct approach could become computationally prohibitive.

In the case of the oscillatory contact under a constant applied force, the thermal expansion due to the temperature rise in the half space results in an outward expansion of the contact surface. This expansion tends to lift the sphere up, reduces the indentation depth and thus leads to a slight decrease of the contact area. The slight decrease of the contact area results in the increase of the maximum value of the contact pressure and the change of the pressure distribution, which in turn affects the distribution of the heat flux and the temperature rise in the contact region. Since the total heat generated in the contact interface remains unchanged in comparison with the heat transfer without thermomechanical effects, the thermal expansion has only little effect on the temperature rise in the half space. Such effects mainly focus in the contact region when the sphere oscillates to the locations near or at the oscillatory center.

For the case of the oscillatory contact with the sphere fixed against direction in the indentation direction, the outward expansion of the contact surface due to the thermal expansion tends to lift the sphere up and reduce the indentation depth of the sphere into the half space. However, this trend is constrained by the sphere, which is fixed against in the indentation direction, resulting in the significant increase of the contact pressure and contact force. Also, the outward expansion of the contact surface due to the thermal expansion brings additional points on the half space surface in contact with the sphere and thus increases the contact area. Due to the significant increase of the contact force, the contact pressure and the heat generation increase greatly, resulting in the higher rise of the surface temperature.

Therefore, whether the thermal expansion in the contact components is constrained or not in the indentation direction plays a significant role in the thermomechanical interaction in the oscillatory contact. Care should be taken when the thermal expansion in an oscillatory contact is constrained due to the structural configuration of the system to ensure that loss of clearance does not occur. This is a particularly important problem in journal bearings undergoing oscillatory motion with their bushings constrained by an external housing. Failure of proper consideration of thermomechanical interaction can result in a complete loss of the operating clearance and a catastrophic seizure.

6.7 Nomenclature

\( A \) = Oscillating amplitude (m) and dimensionless amplitude
\( A \) = Coefficient matrix associated with initial condition
\( c \) = Specific heat (J/kg K)
\( C \) = Capacitance matrix
\( d, d_i \) = Indentation depth (m) and initial indentation depth, respectively
\( f \) = Oscillating frequency (Hz)
\( F \) = Heat load vector including heat flux and convection
\( h \) = Convective heat transfer coefficient (W/m\(^2\) K)
\( H \) = Exponential matrix
\( I = \) Identity matrix
\( k = \) Thermal conductivity \((\text{W/m K})\)
\( K = \) Conductance matrix
\( M = \) Coefficient matrix associated with boundary conditions
\( n = \) Unit normal outward to the boundary \( \Gamma \)
\( N = \) Vector of shape functions for element
\( N, N_q, N_h = \) Node number of the model, contact element number, and number of convective surfaces, respectively.
\( p, p_0 = \) Contact pressure and average contact pressure, respectively \((\text{N/m}^2)\)
\( q, \overline{q}, q_0 = \) General flux, prescribed boundary condition of heat flux and average flux, respectively, \((\text{W/m}^2)\)
\( t, t_k = \) Time and the time at \( k \)th step \((\text{s})\)
\( T, \text{T}, T^{(0)}, \overline{T}, T_\infty = \) Field temperature, Vector of field temperature \( T \), Initial temperature, prescribed surface temperature, and ambient temperature, respectively \((\circ\text{C})\)
\( \nu = \) Oscillating velocity \((\text{m/s})\)
\( x = \) Coordinate vector \((x, y, z)\) \((\text{m})\)
\( \alpha = \) Thermal expansion \((1/\text{K})\)
\( \kappa = \) Thermal diffusivity, \( \kappa = k / \rho c \) \((\text{m}^2/\text{s})\)
\( \tau = \) Step time \((\text{s})\)
\( \rho = \) Density \((\text{kg/m}^3)\)
\( \delta_{ij} = \) Kronecker delta
\( \Gamma_T, \Gamma_q, \Gamma_h = \) Surface \( \Gamma \) with prescribed temperature, heat flux and convection, respectively
\( \mu = \) Friction coefficient
\( \omega = \) Angular frequency \((\text{rad/s})\)
\( \Omega = \) Domain considered

Superscripts:
\( e = \) Element
\( (k) = \) Iteration number
\( T = \) Transpose of matrix
\( -1 = \) Inversion of matrix
\( i = \) Initial value of parameters
Chapter 7 develops a method for treating the thermomechanical interaction of bodies that undergo relative oscillatory motion. The approach utilizes a combination of the transfer matrix and the finite element methods. The thermomechanical coupling (TMC) process between the contacting bodies involves a transient solution scheme where the frictional heat is automatically partitioned between the contacting surfaces. The coupling between thermal and mechanical interaction is treated iteratively. An application of the proposed model in the study of thermomechanical behavior of journal bearings with oscillatory motion undergoing thermally induced seizure (TIS) is presented. The results of a wide range of operating parameters are presented. The significance of applied load, contact clearance, friction coefficient, oscillation parameters, convective heat transfer and variable load directions condition in the thermally induced seizure are discussed in light of the numerical results.

7.1 Introduction

Most widely used mechanical components such as journal bearings, seals, brakes and clutches are susceptible to the frictional heating due to the rubbing of the contact surfaces with relative sliding motion. The frictional heat results in an increase of temperature in the contacting bodies and adversely affects the performance of the machine, or even results in failure of the system. A prominent example of this type of failure is the so-called thermally induced seizure (TIS) in journal bearings. This phenomenon occurs as a result of the structural distortion caused by frictional heating where the bearing loses its designed clearance relative to the shaft.

There is a large volume of research dealing with the thermomechanical behavior of journal bearings undergoing thermally induced seizure. Bishop and Ettles [98] analyzed the thermoelastic interaction of a journal in a plastic bearing that was interference-fit with the shaft. Dufrane and Kannel [23] derived an analytical solution based on a one dimensional analysis of the catastrophic seizure due to dry friction for predicting seizure times of journal bearings. Khonsari and Kim [24] performed a comprehensive two-dimensional numerical study of thermally induced seizure in journal bearings during startup. Hazlett and Khonsari [25, 26] developed a finite element model to study the nature of thermally induced seizure in journal bearings. More recently, Krithivasan and Khonsari [12] performed a comprehensive finite element study of seizure in journal bearings and extended the work in their parallel study [99] to include provisions for study of thermomechanical interactions of heavily loaded pin-bushing assembly in oscillating motion.

The majority of published works [12, 23-26, 98] have focused their attention to unidirectional rotating journal bearings in which the load is constant and the velocity is unidirectional. An oscillatory journal bearing is analyzed in Ref. [99], but in this work both the load and the oscillation speed are constant. Yet, there are many applications in
which the load varies dynamically and the velocity changes magnitudes and directions in an oscillatory fashion. In addition, in many practical applications the shaft is stationary while the bushing oscillates. Examples of such a sling system include the pin-bushing assemblies in heavy construction machinery, and bearings in reciprocating components of internal combustion engines as well as in reciprocating compressors.

Significant advances in analysis of thermal responses in oscillatory contact have been recently reported [5, 7-10, 13, 28]. However, these works assume that there is no change in the contact status. A complete analysis with accurate provision for thermomechanical coupling is required to account for the journal bearings with oscillatory motion undergoing thermally induced seizure.

In this chapter, an efficient method for treating the thermomechanical interaction of bodies that undergo relative oscillatory motion is reported. The approach utilizes a combination of the transfer matrix and the finite element methods. The thermomechanical coupling (TMC) process between the shaft and bushing involves a transient solution scheme where the frictional heat is automatically partitioned between the contacting surfaces. The coupling between thermal and mechanical interaction is treated iteratively. An application of the proposed model in the study of thermomechanical behavior of journal bearings with oscillatory motion undergoing thermally induced seizure (TIS) is presented. The results of a wide range of operating parameters are presented for oscillating shaft, oscillating bushing, and for variable load directions. The influence of applied load, contact clearance, friction coefficient, oscillation parameters, convective heat transfer condition and variable load directions on the thermally induced seizure are explored and discussed in light of the numerical results.

### 7.2 Model Description

Consider a cylindrical shaft of radius $R_{so}$ oscillating internally against a cylindrical bushing of inner radius $R_{bi}$ and outer radius $R_{bo}$ as illustrated in Fig. 7.1(a). The applied load $W$ is directed vertically downwards through the center of the shaft. The contact position changes as the shaft oscillates relative to the bushing at a specified amplitude $A$ and frequency $f$. The oscillating velocity is described by a sinusoidal function of time $\dot{\phi}(t)$. The coordinate system $or\theta$ is fixed on the stationary part. The contact pressure is assumed to be uniform along the axial direction and a two dimensional analysis is performed to treat the problem.

As the shaft oscillates on the inner surface of the bushing under the applied load, heat is generated in the contact region due to friction. The total interfacial heat is partitioned between the shaft and bushing, resulting in a rise in the temperature of the contacting solids. The thermal expansion that takes place in both the shaft and the bushing as a function of time is examined to determine the change in the operating clearance and its effect on the torque. Research reveals that when operating with unidirectional sliding motion, at some point in time, the clearance between the shaft and the bushing may drop to a dangerously low level as the shaft encroaches the bushing [26]. This results in an increase of the contact force and the formation of extra contact area. An increase of the
contact force raises the heat generation and sets up a positive feedback that accelerates to
the stage where the operating clearance vanishes. To analysis this behavior, a transient
thermomechanical interaction between the shaft and the bushing has to be employed
synchronously. This type of a problem is often classified as thermomechanical coupling
(TMC).

Note that in a typical journal bearing, the bushing is stationary while the shaft
oscillates. This configuration is shown in Fig. 7.1(a). However, there are many
applications, such as the pin-joint assemblies found in undercarriage systems used in
mining or heavy construction, where the bushing oscillates and the shaft remains
stationary as illustrated in Fig. 7.1(b). Also, the direction of applied load may vary
periodically according to a certain load cycle. All of these cases are analyzed in this
chapter.

In this study, the thermomechanical analysis of the oscillatory bearing is conducted in
a stepwise linear fashion through the following procedure:

i. A static contact analysis with consideration of thermal expansion is conducted to
determine the contact variables: the contact pressure distribution and the clearance
between the shaft and bushing. Appropriate mechanical load is applied according
the load cycle if specified. For the first step, no thermal load is applied.

ii. A transient heat transfer analysis is performed by using the contact results from
Step i to simulate the temperature distribution in the shaft and bushing. The heat
division between the contacting bodies is fully accounted for by satisfying the heat
equilibrium and the temperature continuity at the interactive surfaces of the shaft
and the bushing.

iii. The oscillating body (either the shaft or the bushing) is rotated according to a
specified frequency and oscillatory amplitude and Step i and Step ii are repeated
until the convergence criteria is satisfied.

Figure 7.2 shows the finite element model used in this chapter for the case of either
oscillating shaft or oscillating bushing. The exact same finite element mesh with four-
node quadrilateral element and three-node triangular element for central part of the solid
shaft is used for the thermomechanical and transient heat transfer analyses. This ensures
that the results of the thermal analysis can be directly exported to the thermomechanical
analysis. In the following sections, the procedures of the thermomechanical analysis and
the transient heat transfer analysis are discussed in depth.

7.2.1 Nonlinear Thermomechanical Analysis

The oscillating motion of the moving component, either the shaft or the bushing, is
simulated by oscillating the moving component in an incremental fashion. The
thermomechanical analysis at each oscillatory step is performed by using the commercial
software ABAQUS. The loads on the journal bearing consist of the thermal load applied
as nodal temperature and the mechanical load as an applied force $W$. The time dependant
thermal load is obtained from the calculated results by the transient thermal contact
analysis. The coefficient of thermal expansion for the shaft and bushing are the same in
current study. The thermal expansion due to the thermal load results in changes in configurations of the shaft and the bushing, and thus affects the contact status and the contact pressure. At the first step, there is no temperature variation and thus no thermal load is applied. The mechanical load \( W \) is applied according to the load cycle if specified; otherwise, a constant vertical load is applied. In either case for oscillating shaft or oscillating bushing, the load \( W \) is always applied on the oscillatory part, and the stationary part is fixed against displacement.

Since the outer surface of the bushing is affixed inside a rigid housing, then the bushing is constrained from outward expansion. To model this situation, all nodes in the outer boundary of the bushing are connected in all displacement components, using the *KINEMATIC COUPLING option of ABAQUS, to a reference node. The center of the bushing, \( o_B \), is chosen as the reference node. This node defines the outer surface of the bushing as a rigid body. The boundary constraint or the load on the bushing is applied on the reference node.

Therefore, for the case of oscillating shaft and stationary bushing, the load \( W \) is applied on the center of the shaft \( o_S \), and the center of the bushing is fixed in all degrees of freedom as illustrated in Fig. 7.1(a). In the case of oscillating bushing and stationary shaft, the load \( W \) is applied on node \( o_B \), and node \( o_S \) is fixed as illustrated in Fig. 7.1(b).

\[
\begin{align*}
\phi(t) & \quad \Gamma g \quad \Delta T \quad h_T \\
R_{oS} & \quad R_{bi} \\
\theta = \frac{3\pi}{2} & \quad \phi = 0
\end{align*}
\]

Fig. 7.1 A configuration of the journal bearing: (a) Oscillating shaft and stationary bushing. (b) Oscillating bushing and stationary shaft. (fig. cont’d)
7.2.2 Transient Heat Transfer by Transfer Matrix Method

The transient heat transfer analysis is conducted using the transfer matrix method. The temperature continuity and the heat equilibrium are satisfied in the contact area to determine the heat division between the shaft and bushing. The outer surface of the

Fig. 7.2 A finite element model of the journal bearing.
bushing and the inner surface of the shaft, if it is hollow, are subjected to convective cooling.

### 7.2.2.1 Formulation of the Transfer Matrix Method

The general equation governing the transient heat conduction in a domain $\Omega$ with a boundary $\Gamma$ as illustrated in Fig. 7.1 is:

$$\nabla^2 T(x,t) = \frac{1}{\kappa} \dot{T}(x,t), \quad x \in \Omega, \ t > 0 \quad (7.1)$$

with the initial condition

$$T(x,0) = T^0(x), \quad x \in \Omega \quad (7.1a)$$

and subjected to the boundary conditions

$$T(x,t) = \bar{T}(x,t), \quad x \in \Gamma_T \quad (7.1b)$$

$$q(x,t) = \bar{q}(x,t), \quad x \in \Gamma_q \quad (7.1c)$$

$$q(x,t) = h(T_\infty - T), \quad x \in \Gamma_h \quad (7.1d)$$

where $x$ is coordinate vector $(x,y,z)$. The parameter $t$ denotes the time, and the super-dot stands for the derivative with respect to time. Parameters $\kappa$ and $k$ are the thermal diffusivity and the thermal conductivity of the material, respectively. $T(x,t)$ is the temperature field. The heat flux $q(x,t)$ is defined as $q(x,t) = k \frac{\partial T}{\partial n}$, where $n$ is the unit normal outward to the boundary $\Gamma$. $T^0(x)$ represents the initial temperature distribution. $\bar{T}(x,t)$ and $\bar{q}(x,t)$ are the prescribed boundary temperature on surface $\Gamma_T$ and heat flux on surface $\Gamma_q$, respectively. The boundary surface $\Gamma_h$ is subjected to convective cooling with the heat transfer coefficient $h$, and the ambient temperature $T_\infty$.

After space discretization by the finite element procedure, the transient heat transfer equation, Eq. (7.1), can be expressed in a matrix form [34] as

$$C \ddot{T}(t) + K T(t) = F(t) \quad (7.2)$$

where $T$, $C$, $K$ and $F$ are the temperature vector, capacitance matrix, conductance matrix and heat load vector, respectively, defined as

$$C = \sum_e \int_{\Omega^e} \mathbf{N}^T \rho_c \mathbf{N} d\Omega \quad (7.2a)$$

$$K = \sum_e \int_{\Omega^e} \left( \frac{\partial \mathbf{N}^T}{\partial x} k \frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial y} k \frac{\partial \mathbf{N}}{\partial y} + \frac{\partial \mathbf{N}^T}{\partial z} k \frac{\partial \mathbf{N}}{\partial z} \right) d\Omega + \sum_e \int_{\Gamma^e} h \mathbf{N}^T \mathbf{N} d\Gamma \quad (7.2b)$$

$$F = \sum_e \int_{\Gamma^e} \bar{q} \mathbf{N}^T d\Gamma + \sum_e \int_{\Gamma^e} hT_\infty \mathbf{N}^T d\Gamma \quad (7.2c)$$

where $\mathbf{N}$ is the shape function in the element of $e$. $\Omega^e$ and $\Gamma^e$ denote the element domain and the element surface if it is on the boundary, respectively. Parameters $\rho$ and $c$ represent the density and capacitance of the material, respectively. The superscript $T$ denotes the matrix transpose.
Solving Eq. (7.2) and performing the related integration results in the following iteration equation for heat transfer analysis as
\[ T^{(k+1)} = AT^{(k)} + MF^{(k)} \]  
(7.3)
where
\[ A = e^{HR} \]  
(7.3a)
\[ M = (I - A)K^{-1} \]  
(7.3b)
and
\[ H = -C^{-1}K \]  
(7.3c)
where the superscript \( k \) represents the time step and \( \tau \) is the time period from time \( t_k \) to \( t_{k+1} \). \( A \) and \( M \) are the coefficient matrices and \( I \) is the identity matrix. The superscript -1 denotes the matrix inversion. \( F \) is the vector of the discretized heat load. For a model having \( N \) nodes, \( N_h \) convective surfaces, and with the contact surface divided into \( N_q \) small segmented elements, the dimension of vector \( T \) is \( N \), the dimension of vector \( F \) is \( N_h + N_q \), and the dimensions of matrices \( A \) and \( M \) are \( N \times N \) and \( N \times (N_h + N_q) \), respectively.

Physically, the coefficient matrix \( A \) determines how heat conducts inside the solid, and the coefficient matrix \( M \) shows how the surface heat input contributes to the field temperature rise in the solid. The load vector \( F \) includes two parts: one part contains the terms involved with the heat flux on the contact surface due to the frictional heating and convective cooling. The other part consists of the terms involving the ambient temperature on the convective boundary surfaces. In the present method, except \( q \) and \( T_\infty \), the other terms in Eq. (7.2c) are multiplied into the corresponding columns of matrix \( M \) in Eq. (7.3). Therefore, vector \( F \) in Eq. (7.3) only contains the values of the heat fluxes, corresponding to each small surface element on the discretized contact surface, and the ambient temperatures, corresponding to the boundary surfaces subjected to convective cooling.

7.2.2.2 Calculation of the Coefficient Matrices

The calculation of the coefficient matrices \( A \) and \( M \) is based on using the commercial software ABAQUS. Examination of Eq. (7.3) reveals that matrices \( A \) and \( M \) can be treated as influence matrices. Each of the column vectors of matrix \( A \) reflects how a unit nodal temperature affects the field temperature inside the solid, and each of the column vectors of matrix \( M \) reflects how a unit discretized boundary flux or unit ambient temperature contributes to the field temperature rise. Thus, in this chapter, matrices \( A \) and \( M \) are indirectly calculated by using Eq. (7.3). The method is described as follows.

The matrix \( A \) can be obtained by \( N \) cycles of simulations with the same time period \( \tau \). For the \( i \)th \( (i = 1, 2, \cdots, N) \) simulations, the initial condition and boundary load are set to
\[ (T^{(0)})_i = \{\delta_y\} \]  
and
\[ F = \{0\} \]  
(7.4)
where $\delta_j$ is Kronecker delta, and $j = 1, 2, \cdots, N$ are the index to the entries of the vector $T^{(0)}$. The result $T^{(1)}$ is the $i$th column of matrix $A$.

Similarly, the matrix $M$ can be evaluated by performing $N_h + N_q$ cycles of simulations with the same time period $\tau$ as that used in calculation of the matrix $A$. For the $i$th ($i = 1, 2, \cdots, N_h + N_q$) simulations, the following initial condition and boundary load are applied

$$T^{(0)} = \{0\} \text{ and } (F)_i = \{\delta_j\}$$

(7.5)

where $j = 1, 2, \cdots, N_h + N_q$ are the index to the entries of the vector $F$. The result $T^{(1)}$ is the $i$th column of matrix $M$.

After $N + N_h + N_q$ cycles of simulations, the coefficient matrices $A$ and $M$ are obtained. It is noted that $A$ and $M$ keep constant for different heat sources and need to be calculated only once. Once they are determined, the transient temperature distribution in the domain at each time step $t_k$ is easily calculated from Eq. (7.3). The procedure is to start from the initial condition $T^{(0)}$ and to assign different values of heat fluxes to the components of vector $F^{(k)}$ at each time step $t_k$. For the journal bearing shown in Fig. 7.1, evaluation of the vector $F^{(k)}$ requires to couple the heat and the temperature between the contact surfaces of the shaft and the bushing.

### 7.2.2.3 Formulation of the Thermal Contact

For a system of journal bearing shown in Fig. 7.1 subjected to relative oscillation, Eq. (7.3) is applied to both the shaft and the bushing as

$$T^{(k+1)}_l = A_l T^{(k)}_l + M_l F^{(k)}_l$$

(7.6)

where subscript $l = \{S, B\}$ denotes the shaft or the bushing. The heat load on the shaft $F_S$ consists of the frictional heat generated in the contact surface with the bushing, and the convective cooling by the air within the clearance. If the shaft is hollow, the heat load also includes the internal convective cooling. For the bushing, besides the frictional heating in the contact surface and the convective cooling within the clearance, the heat load $F_B$ includes the convective cooling on the outer surface.

In the contact region, the frictional heat generated in the contact interface is partitioned between the shaft and the bushing. The following temperature continuity and the heat equilibrium, Eqns. (7.7) and (7.8), should be satisfied:

$$T_S = T_B$$

(7.7)

$$\bar{q}_S + \bar{q}_B = q$$

(7.8)

where the total heat generation $q$ is given as

$$q(t) = \mu p R_s |\dot{\phi}(t)|$$

(7.8a)

with the oscillation velocity given by a sinusoidal function as
\[
\dot{\phi}(t) = A\omega \sin(2\pi ft)
\]

(7.8b)

where \(\mu\) is the friction coefficient. The parameter \(p\) is the contact pressure obtained from the thermomechanical analysis. Parameters \(A\) and \(f\) are the oscillating amplitude and frequency, respectively.

Within the clearance, the heat dissipations of the shaft and the bushing by the convective cooling are evaluated using the following expression:

\[
q_c = h_c (T_g - T_i)
\]

(7.9)

where \(h_c\) is the convective heat transfer coefficient. \(T_g\) is the surrounding temperature within the clearance calculated by averaging the temperatures of the open contact pairs between the shaft and the bushing as

\[
T_g = \frac{1}{2}(T_s + T_B)
\]

(7.9a)

Introducing Eq. (7.6) into Eq. (7.7) and using the relationship in Eq. (7.8), and solving the resulting system of equations enables one to evaluate the divisions of the frictional heat to the shaft and the bushing, \(q_s\) and \(q_b\), respectively. Whether an element on the outer surface of the shaft or the inner surface of the bushing lies inside the contact region or not is determined by the contact pressure obtained in the thermomechanical analysis. If the contact pressure is not equal to zero, the corresponding surface element is in contact and the frictional heat \(q_f\) is applied on it; otherwise, the surface element is not in the contact area and undergoes the convective cooling evaluated by Eq. (7.9).

For the transient heat transfer analysis, each time step corresponds to a certain oscillatory location of the journal bearing. The results from the previous step are treated as a pseudo-initial condition for current step. The total frictional heat generated in the contact surface is calculated from Eq. (7.8a) by using the contact results obtained from the thermomechanical analysis. Applying the matching conditions, Eqs. (7.7) and (7.8), the heat partition in the contact surface between the shaft and the bushing, and thus the heat flowing into both contacting bodies can be determined. Also the convective heat within the clearance can be evaluated by Eq. (7.9). The convective heat transfer condition on the outer surface of the bushing is specified. Similarly, if the shaft is hollow, the heat conduction coefficient on the inter surface of the shaft must be specified. Thus, the thermal loads on the shaft and bushing, \(F_S\) and \(F_B\), can be evaluated. The temperature distributions in the shaft and the bushing are simulated using Eq. (7.3).

### 7.2.3 Convergence Criteria

Similar seizure criteria as that for the journal bearing with continuous rotation [12] is used for the present study. Frictional torque is the torque resisting the driving torque. An increase in the frictional torque will result in power loss and decrease in efficiency. When the frictional torque reaches at least 50 times of its original value, the simulation is stopped since failure is imminent. The frictional torque is evaluated by summing the element frictional forces in the contact region at each simulation step in the transient
\[ Torque = \mu R_s \sum_i (P_i L_i) \]  

(7.10)

where \( P_i \) and \( L_i \) are contact pressure and the length of the \( i \)th element in the contact area.

### 7.3 Solution Algorithm

Using the procedure described in Section 7.2.1-7.2.3, a computer program is developed that efficiently treats TMC procedure involving oscillatory system. Figure 7.3 presents the flow chart of the basic steps of the simulation algorithm. First, the coefficient matrices \( A \) and \( M \) for the shaft and the bushing are evaluated, respectively. Then, starting with a thermomechanical analysis (no temperature variation and thus no temperature load at the first step), the results of contact pressure is used to calculate the frictional heat flux by Eq. (7.8), which is partitioned according to the matching conditions, Eqs. (7.7) and (7.8). Thus, the load vectors \( F \) for the shaft and the bushing are determined. At the same time, the transient heat transfer analysis is performed to determine the temperature distribution in the contact zone.
time, using the temperature results at the previous step as the pseudo-initial temperature or using the initial condition for the first step, the temperature after one cycle with the time period $\tau$ is obtained by Eq. (7.3). Updating the temperature loads and repeating the thermal mechanical analysis followed by a thermal analysis, the thermal mechanical response in the journal bearing can be simulated until the convergence criteria is satisfied.

7.4. Results and Discussion

Simulations are performed for a wide range of operating parameters listed in Table 7.1. A total of 21 different cases were analyzed, 17 of which correspond to oscillating shaft and stationary bushing, and 4 cases involving oscillating bushing with stationary shaft. These include three cases involving hollow shafts oscillating inside a stationary bushing and one case of variable load directions (Case 15S). The results are summarized in Table 7.2, where the letter in the case number denotes whether the shaft is solid (S) or hollow (H). A discussion of the results is presented in the following Sections 7.4.1-7.4.3.

Table 7.1 Parameters used in the analysis

<table>
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<th>Parameter</th>
<th>Value</th>
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<td>Radius of shaft, $R_{os}$ (mm)</td>
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<tr>
<td>Radius of hollow shaft, $R_{is}$ (mm)</td>
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<td>Length of bushing, $L$ (mm)</td>
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<td>Elastic modulus of shaft, $E_s$ (GPa)</td>
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<td>Oscillating frequency, $f$ (Hz)</td>
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7.4.1 General Behavior of the Thermally Induced Seizure

In order to gain insight into the seizure phenomenon in the oscillatory contact, Case 2S (stationary bushing and solid oscillating shaft) in Table 7.2 is taken as an example to discuss in details to employ the thermomechanical interaction between the shaft and bushing. In Case 2S, a solid shaft oscillates inside a bushing as shown in Fig. 7.1(a) with the following specifications: \( W=4400 \text{N}, \ C=0.025 \text{mm}, \ \mu =0.15, \ A=30^\circ, \ f=10\text{Hz} \) and \( h_e=80 \text{W/m}^2 \text{K} \).

Temperature Rise and Ovalization. Figure 7.4 shows the temperature rise in the journal bearing after 300, 379 and 391 cycles. Figure 7.4(a) presents the temperature rise after 300 cycles. Note that the maximum temperature rise at this stage is 82.2°C, which occurs at the contact interface between the shaft and the bushing. At this stage, the clearance decreases due to thermal expansion, but the contact is still limited to the oscillation area at \( \theta = 3\pi / 2 \) as shown in Fig. 7.4(a). Figure 7.4(b) shows the temperature rise after 379 cycles, where the initiation of the extra contact at the top of the bushing \( \theta = \pi / 2 \) can be observed. The formation of the extra contact area intensifies the contact pressure and the heat generation, resulting in a larger temperature rise. The temperature rise at the top of the bushing goes from about 5°C in Fig. 7.4(a) to 28.9°C in Fig. 7.4(b), and to over 200°C in just 12 more cycles as shown in Fig. 7.4(c). At the same time, the bottom of the bushing continues to experience more heating and the maximum temperature rise reaches over 300°C. It is noted that the temperature distribution in the shaft shown in Fig. 7.4 is different from that for the unidirectional rotation as reported in Refs. [12, 26]. In the case of the unidirectional rotation, the temperature in the shaft is nearly uniform along the circumferential direction when the rotation speed is very high, thus the shaft experiences a uniform outward expansion and the “ovalization” only occurs in the bushing. However, for the oscillatory journal bearing, the ovalization occurs in both the shaft and the bushing. A further discussion of this phenomenon is given next.

Figure 7.5 illustrates the variations of the contact clearance and the profiles of the outer surface of the shaft and inner surface of the bushing after 300, 379 and 391 cycles. Figure 7.5 reveals that the thermally induced seizure (TIS) in the oscillatory journal bearing is initiated by both the inward expansion of the bottom inner surface of the bushing as shown in Fig. 7.5(b) and the ovalization of the shaft as shown in Fig. 7.5(c), resulting in an extra contact between the top of the shaft and the inner bushing surface, a zero clearance as shown in Fig. 7.5(a). This leads to the increase in the contact pressure and the formation of extra contact area. The increase of the contact pressure raises the heat generation and set up a positive feedback that accelerates the loss of the clearance. Also the increase of the contact pressure leads to an increase of the frictional torque. During the final stage of TIS, the shaft encroaches and squeezes the bushing, as there is further loss of the clearance as illustrated in Fig. 7.5 at Cycle 391.

Figure 7.6 shows the variations of the temperatures over time for the shaft and bushing at contact points A (\( \theta = \pi / 2 \)) and B (\( \theta = 3\pi / 2 \)) (see in Fig. 7.2). Figure 7.6 demonstrates that after 379 cycles the temperatures at A (\( \theta = \pi / 2 \)) for the shaft and the
bushing reach the same value due to the extra contact at the top of the bushing and increases abruptly in a very short time. This occurs in only 12 oscillating cycles or 1.2 seconds before TIS occurs.

**Contact Pressure and Torque.** Figure 7.7 shows the variations of the contact pressure after 300, 379 and 391 cycles, compared with the initial contact pressure. Before the establishment of the extra contact at the top of the bushing, $\theta = \pi / 2$, the contact area at the bottom of the bushing, $\theta = 3\pi / 2$, becomes smaller than the initial area due to the inward expansion of the contact surface at the bottom of the bushing. Also the peak value of the contact pressure increases slightly as shown in Fig. 7.7. However, once the top contact pairs between the shaft and bushing are in intimate contact, the contact area at both the top and the bottom of the bushing propagates quickly, resulting in an abrupt increase of the contact pressure and the frictional force. Figure 7.8 shows the variation of the frictional torque over time. All the simulations reveal that the frictional torque reaches very high values in less than 5 seconds after the establishment of the extra area of contact at the top of the bushing. This is comparable to the short time of 3 seconds for the unidirectional rotation in Refs. [12, 26].

![Diagram](image)

(a)

Fig. 7.4 Temperature rise for Case 2S at (a) Cycle 300 (b) Cycle 379 and (c) Cycle 391 when seizure occurs. $W=4400N$, $C=0.025\text{mm}$, $\mu =0.15$, $A=30^\circ$, $f=10\text{Hz}$, $h_c=80\text{ W/m}^2\text{K}$ and with the solid oscillating shaft. (fig. cont’d)
Fig. 7.5 Variation of (a) Clearance along contact surface, (b) Profile of inner surface of the bushing, and (c) Profile of outer surface of the shaft at Cycles 300, 379 and 391 when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\text{ W/m}^2\text{K}$ with the solid oscillating shaft in Case 2S. (fig. cont’d)
Fig. 7.6 Temperature variations at contact points A(θ = π/2) and B(θ = 3π/2) when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_v=80\text{ W/m}^2\text{K}$ with the solid oscillating shaft in Case 2S.
Fig. 7.7 Distributions of contact pressure along contact surface at beginning, Cycles 300, 379 and 391 when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\text{ W/m}^2\text{ K}$ with the solid oscillating shaft in Case 2S.

Fig. 7.8 Variation of frictional torque up to seizure when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\text{ W/m}^2\text{ K}$ with the solid oscillating shaft in Case 2S.
7.4.2 Parametric Study

In this part, the influence of the key parameters such as applied load, contact clearance, friction coefficient, oscillation parameters, convective heat transfer condition and variable load directions on TIS is investigated and discussed.

**Applied load** $W$, **friction coefficient** $\mu$, **oscillatory amplitude** $A$ and **frequency** $f$. Referring to Table 7.2, an increase in the load (Cases 1S, 2S and 3S), an increase in friction coefficient (Cases 4S, 5S and 2S), an increase in oscillation amplitude (Cases 6S, 2S and 7S) or an increase in the oscillation frequency (Cases 8S, 2S and 9S) all result in a significant decrease in the available time to seizure, that is the time that the journal bearing runs satisfactorily before it encounters seizure. All these parameters are directly related to the heat generation; see Eq. (7.8a). Increasing any one of them will raise the frictional heat generation and the thermal expansion of the shaft and the bushing, yielding to an earlier occurrence of seizure. The same oscillation velocity between Cases 6S and 8S, or Cases 7S and 9S leads to an approximately the same time to seizure.

**Clearance** $C$. A larger clearance provides more space to compensate the clearance loss due to the thermal expansion of the shaft and bushing, postponing the seizure as revealed by comparison of Cases 5S and 16S.

**Internal cooling in a hollow shaft** $h_i$. A hollow shaft exposed to the same convective cooling rate on the inside and around its perimeter (Case 12H) seizes quicker than a solid shaft (Case 2S). If the shaft is hollow and subjected to internal cooling, an increase of the internal cooling rate (Cases 13H and 14H) can delay the seizure. Only substantially higher internal heat transfer rate in Case 14H can postpone the seizure to a great extent. Figure 7.9 presents the comparison of the temperature rise at the 300th cycle between Cases 12H and 14H. Figure 7.10 shows the comparison of the contact clearance and the profile of the contact surface between the solid (Case 2S) and hollow shaft (Cases 12H and 14H) after 300 cycles. Compared with the solid shaft shown in Fig. 7.4(a), the hollow shaft with the lower internal cooling rate results in a higher temperature rise as shown in Fig. 7.9(a), and more thermal expansion on the bushing and shaft, respectively as shown in Figs. 7.10(b) and (c). Physically, the loss of the thermal capacity in the hollow shaft makes it difficult for the frictional heat to conduct away from the contact interface as compared to the solid shaft. At the same time the lower internal convection cannot provide equivalent function to take away the frictional heat. However, improving the internal cooling can lead to more heat dissipated through the shaft and decrease the temperature rise as shown in Fig. 7.9(b) and lower the thermal expansion of the journal bearing as shown in Figs. 7.10(b) and (c), thus limit the clearance loss as shown in Fig. 7.10(a) (or even eliminate seizure).

**External cooling** $h_e$. External cooling has different effects depending on whether the shaft or the bushing is oscillating. These are described as follows.
(a) Oscillating shaft. Cases 2S, 10S and 11S illustrate that the external heat convection on the outer surface of the bushing has a negligible effect on the seizure time.

(b) Oscillating bushing. The external heat convection has a significant effect on the seizure time in the case of oscillating bushing as revealed by Cases 18S, 19S and 20S. High values of the heat transfer coefficient delays the occurrence of seizure. Also under the same operation condition, oscillating bushing promotes longer seizure time than does the oscillating shaft. Figure 7.11 shows the temperature rise in the journal bearing after 300 oscillatory cycles in Case 18S. Compared with Fig. 4(a) with oscillating shaft (Case 2S), Fig. 7.11 illustrates a smaller heating span on the shaft and greater heating span on the bushing. This provides for more heat flow into the bushing, resulting in lower temperature rise in the shaft and higher temperature rise in the bushing. Figure 7.11 reveals that a larger area of the bushing with higher temperature rise is beneficial in the heat dissipation by the external cooling.

Figure 7.12 shows the comparison of the contact clearance and the profile of the contact surface between the cases of oscillating shaft, Case 2S, and oscillating bushing, Case 18S. Figure 7.12(b) demonstrates a more inward expansion on the bottom surface of the bushing due to the higher temperature rise, but significantly less ovalization on the shaft as shown in Fig. 7.12(c) for oscillating bushing due to less temperature rise. Eventually, there is bigger clearance left for the oscillating bushing than that for the oscillating shaft after 300 cycles as shown in Fig. 7.12(a), which postpones the seizure.

Fig. 7.9 Temperature rise under oscillating hollow shaft with (a) \( h_e = 80 \text{ W/m}^2 \text{ K} \) in Case 12H and (b) \( h_e = 1000 \text{ W/m}^2 \text{ K} \) in Case 14H at Cycle 300 when \( W = 4400 \text{N}, C = 0.025 \text{mm}, \mu = 0.15, A = 30^\circ, f = 10 \text{Hz}, \) and \( h_e = 80 \text{ W/m}^2 \text{ K} \). (fig. cont’d)
Fig. 7.10 Comparison between cases of solid (Cases 2S) and hollow (Cases 12H and 14H) oscillating shaft in (a) Clearance along contact surface, (b) Profile of inner surface of the bushing, and (c) Profile of outer surface of the shaft at Cycle 300 when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\text{W/m}^2\text{K}$. (fig. cont’d)
Fig. 7.11 Temperature rise for Case 18S at Cycle 300 under oscillating bushing when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_v=80 \text{W/m}^2\text{K}$ with solid shaft.

Fig. 7.12 Comparison between cases of oscillating shaft (Case 2S) and oscillating bushing (Case 18S) in (a) Clearance along contact surface, (b) Profile of inner surface of the bushing, and (c) Profile of outer surface of the shaft at Cycle 300 when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_v=80 \text{W/m}^2\text{K}$ with the solid shaft. (fig. cont’d)
Variable load directions. The variation cycle of the load direction is presented in Fig. 7.13, where the load direction is measured by the angle $\phi$ as illustrated in Fig. 7.1.
Referring to Table 7.2, the journal bearing under the variable load directions (Case 15S) sei zes slower than under a constant load (Case 2S). Figure 7.14 shows the comparison of the temperature rise at the 300th cycle between the constant load (Cases 2S) and variable load cases (Case 15S). Compared with Fig. 7.4(a) under the constant load (Case 2), Fig. 7.14 demonstrates a larger heating span on both the shaft and the bushing. This improves the conduction of the frictional heat away from the contact interface inside the shaft and bushing and contributes to the heat dissipation by the external cooling, resulting in lower temperature rise in the journal bearing as shown in Fig. 7.14.

Figure 7.15 shows the comparison of the contact clearance and the profile of the contact surface between Case 2S and Cases 15S after 300 cycles. Figure 7.15 demonstrates a smaller inward expansion on the bottom surface of the bushing as shown in Fig. 7.15(b) and less ovalization on the shaft as shown in Fig. 7.15(c) and thus bigger clearance as shown in Fig. 7.15(a) for the variable load directions compared to the constant load after 300 cycles due to the lower temperature rise. Also, Fig. 7.15(b) illustrates that the inward expansion on the bottom surface of the bushing for the variable load direction is more uniform along the perimeter but smaller than that for the constant load due to the larger heating span and lower temperature rise on the bushing.

Fig. 7.13 Variation of load directions in one load cycle for Case 15S.
Fig. 7.14 Temperature rise for Case 15S at Cycle 300 under oscillating shaft when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\ \text{W/m}^2\ \text{K}$ with solid shaft.

Fig. 7.15 Comparison between Case 2S under constant load and 15S with variable load directions in (a) Clearance along contact surface, (b) Profile of inner surface of the bushing, and (c) Profile of outer surface of the shaft at Cycle 300 when $W=4400\text{N}$, $C=0.025\text{mm}$, $\mu=0.15$, $A=30^\circ$, $f=10\text{Hz}$, and $h_e=80\ \text{W/m}^2\ \text{K}$ with the solid shaft. (fig. cont’d)
7.4.3 Cases without Seizure

If the journal bearing is operating in low oscillating frequency under good lubrication (low oscillatory velocity, low friction coefficient, and thus low heat generation) and the contact clearance is large enough, a steady state for the temperature may be reached and seizure may not occur. In this section, two cases for oscillating shaft of Case 17S and oscillating bushing of Case 21S are studied. The operating conditions are $W=4400$N, $C=0.05$mm, $\mu =0.05$, $A=30^\circ$, $f=5$Hz, and $h_e=80$ W/m$^2$K with solid shaft. Note that the friction coefficient is much smaller than the cases in which TIS occurs.

Figure 7.16 shows the temperature variations at two contact points A and B as illustrated in Fig. 2 over time for the cases of the oscillating shaft, Case 17S, and the oscillating bushing, Case 21S. Figure 7.17 presents the temperature contours at the steady state for these two cases. Figure 7.18 shows the comparison of the clearance at the steady state. Note that in Fig. 7.18, the clearance profile for the oscillating bushing is rotated by $180^\circ$ for clearer comparison. Figures 7.16 and 7.17 illustrate that lower steady state temperature rise is obtained in the case of the oscillating bushing than oscillating shaft. Lower temperature rise for the case of oscillating bushing results in smaller thermal expansion and larger contact clearance at the steady state as shown in Fig. 7.18. The temperature gradient in the shaft is very small for both cases. Almost uniform temperature distribution occurs in the shaft because the frictional heat cannot be easily dissipated out of the solid shaft. Therefore, improvement of the heat dissipation in the shaft by using an internally liquid cooled bearing can reduce the steady state temperature.

Fig. 7.16 Temperature variation over oscillatory cycle for the cases of (a) Oscillating shaft in Case 17S and (b) Oscillating bushing in Case 21S when $W=4400$N, $C=0.05$mm, $\mu =0.05$, $A=30^\circ$, $f=5$Hz, and $h_e=80$ W/m$^2$K with solid shaft. (fig. cont’d)
Fig. 7.17 Temperature contours at steady state for (a) Oscillating shaft in Case 17S and (b) Oscillating bushing in Case 21S when $W=4400\text{N}$, $C=0.05\text{mm}$, $\mu=0.05$, $A=30^\circ$, $f=5\text{Hz}$, and $h_c=80\text{W/m}^2\text{K}$ with solid shaft. (fig. cont’d)
Fig. 7.18 Comparison of the contact clearance at steady state between the cases of oscillating shaft (Case 17S) and oscillating bushing (Case 21S) when $W=4400\,N$, $C=0.05\,mm$, $\mu=0.05$, $A=30^\circ$, $f=5\,Hz$, and $h_e=80\,W/m^2\,K$ with solid shaft.
Fig. 7.19 Percentage of the frictional heat into the shaft for (a) Oscillating shaft in Case 17S and (b) Oscillating bushing in Case 21S when $W=4400\text{N}$, $C=0.05\text{mm}$, $\mu =0.05$, $A=30^\circ$, $f=5\text{Hz}$, and $h_c=80\text{ W/m}^2\text{K}$ with solid shaft.

Figure 7.19 presents the percentage of the frictional heat into the shaft. For the cases of oscillating shaft, the bushing is heated on the same area, resulting in the contact
temperature on the bushing is higher than that on the shaft. Thus, at very beginning more frictional heat, about 65.5%, transfers into the shaft as shown in Fig. 19(a). As the shaft accumulates more frictional heat and this heat cannot be easily dissipated out, the heat into the shaft decreases. At the steady state, only about 10% frictional heat flows into the shaft as shown in Fig. 19(a). Figure 19(b) shows that more heat conducts into the bushing in the case of oscillating bushing. At the starting stage, because of the temperature rise in the bushing, the heat flowing into the bushing decreases slightly, resulting in a slight increase of the frictional heat flowing into the shaft as shown in Fig. 19(b). However, when the temperature in the bushing rises to such extent that the heat dissipation by the external cooling on the outer bushing surface becomes dominant, the heat flowing into the shaft begins to decrease. At the steady state, there is about 5% heat flowing into the shaft, which is lower than that for oscillating shaft.

7.5 Conclusions

In this chapter, the thermomechanical interaction between the shaft and bushing is studied. The thermomechanical coupling analysis is performed using an iterative scheme. The heat division between the contacting bodies is fully accounted for. Simulations under a wide range of operating parameters are presented with solid and hollow oscillating shaft, a stationary shaft with oscillating bushing as well as variable load directions.

It is shown that thermally induced seizure occurs as a result of the inward expansion of the bottom inner surface of the bushing and the ovalization of the shaft due to the thermal expansion of the bushing and the shaft, yielding extra contact area between the top of the shaft and inner surface of the bushing. Once the extra area of the top contact is established, the contact pressure and the frictional torque increase sharply, which in turn results in an abrupt increase of the heat generation and the formation of extra contact area. In a very short time afterwards, the frictional torque increases to a high value and seizure occurs.

The increase of the applied load, friction coefficient, oscillatory amplitude or frequency results in a significantly earlier occurrence of the seizure due to the increase of the heat generation. External cooling on the outer surface of the bushing has a negligible effect on the seizure time for the case of oscillating shaft. On the other hand, the external cooling has a significant effect on the seizure time for the case of oscillating bushing. An oscillating bushing can more efficiently dissipate the heat by the external cooling than a stationary bushing. Improving internal cooling contributes to limit the clearance loss. Variable load directions enhance heat conduction inside the journal bearing and promote more heat dissipation by the external cooling, thus result in a longer time to seizure than a constant load.

Lowering the oscillation frequency, improving the lubrication to reduce the frictional heat, increasing designed clearance to ensure a reasonable working clearance or providing good internal or external cooling to increase heat dissipation can prolong seizure time or eliminate TIS completely.
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<th>Oscillatory Frequency $f$ (Hz)</th>
<th>Outer Bushing Convection $h_e$ (W/m$^2$ K)</th>
<th>Inner Shaft Convection $h_i$ (W/m$^2$ K)</th>
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*VLD means variable load directions.
7.6 Nomenclature

\( A = \) Angular oscillation amplitude (Rad)  
\( A = \) Coefficient matrix associated with initial condition  
\( a = \) Specific heat (J/kg K)  
\( a = \) Capacitance matrix  
\( f = \) Oscillation frequency (Hz)  
\( F = \) Heat load vector including heat flux and convection  
\( h = \) Convective heat transfer coefficient (W/m\(^2\) K)  
\( H = \) Exponential matrix  
\( I = \) Identity matrix  
\( k = \) Thermal conductivity (W/m K)  
\( K = \) Conductance matrix  
\( M = \) Coefficient matrix associated with boundary conditions  
\( n = \) Unit normal outward to the boundary \( \Gamma \)  
\( N, N_q, N_h = \) Node number of the model, contact node number, and number of convective surfaces, respectively  
\( p = \) Contact pressure (N/m\(^2\))  
\( q, q = \) Heating/cooling flux at interface, prescribed boundary flux, respectively (W/m\(^2\))  
\( R = \) Radius (m)  
\( t, t_k = \) Time and the time at \( k \)th step (s)  
\( T, T, T^{(0)}, T, T_\infty = \) Field temperature, Vector of field temperature \( T \), Initial temperature, prescribed surface temperature, and ambient temperature, respectively (°C)  
\( x = \) coordinate vector \((x, y, z) (m)\)  
\( \alpha = \) Thermal expansion (1/K)  
\( \kappa = \) Thermal diffusivity, \( \kappa = k / \rho c \) (m\(^2\)/s)  
\( \tau = \) Step time (s)  
\( \rho = \) Density (kg/m\(^3\))  
\( \phi(t) = \) Oscillation angle (Rad)  
\( \dot{\phi}(t) = \) Angular oscillating velocity (Rad/s)  
\( \Gamma_T, \Gamma_q, \Gamma_h = \) Surface \( \Gamma \) with prescribed temperature, heat flux and convection, respectively  
\( \mu = \) Friction coefficient  
\( \Omega = \) Domain considered

Superscripts:
\( (k) = \) Iteration number  
\( T = \) Transpose of matrix  
\( -1 = \) Inversion of matrix

Subscripts:
\( e = \) External  
\( B = \) Bushing
\( i \) = internal
\( o \) = outer
\( S \) = Shaft
8 Three-Dimensional Heat Transfer Analysis of Pin-bushing System with Oscillatory Motion: Theory and Experiment

Chapter 8 presents a three dimensional computational model for heat transfer analysis of a journal bearing with experimental verification. The computational model extends the transient heat transfer model developed in Chapter 7 to a 3D case with consideration of the axial distribution of the frictional heat. Two methods to couple the heat and temperature at the contact interface are presented. The first one can automatically account for the heat division between contacting bodies by satisfying the heat equilibrium and temperature continuity at interactive surfaces. In the second method, the frictional heat is partitioned by introducing a fictitious layer between contacting bodies with a specified so-called gap conductance. Both methods get to equivalent results. The model is capable of predicting the transient temperature field in journal bearings. It is can also be used to determine the maximum contact temperature which is difficult to be measured experimentally. Comparison with experimental tests conducted in a laboratory is presented.

8.1 Introduction

Journal bearings are widely used in various mechanical devices, particularly in rotating machinery, where the load is constant and the velocity is unidirectional. Yet, there are also many industrial applications in which the journal bearing experiences oscillatory motion and the load changes dynamically. These include pin-bushing assemblies found in undercarriage systems that are used in mining and heavy construction equipment. Other examples include journal bearings in engine piston-pins as well as those utilized in forging and coining presses. In these applications the journal bearing is expected to operate under much heavier load condition than unidirectional rotating ones. Thus, more frictional heat is generated at the contact interface between the shaft and bushing. The frictional heat and the resulting high temperature adversely affect the performance, or even cause failure of the journal bearing. Therefore, it is necessary to develop analytical tools to determine the contact temperature for the oscillatory journal bearing under heating loaded operation conditions.

A brief literature survey of pertinent research is as follows. Floquet et al. [95] used a two-dimensional Fourier transform method developed by Ling [52] to calculate the contact temperature in a dry bearing operating with a plastic liner. Later, Floquet and Play [96] extended this technique to three-dimensional problems. Bishop and Ettles [98] analyzed the thermoelastic interaction of a journal in a plastic bearing that was interference-fit with the shaft. Ghosh and Brewe [97] performed a 3D thermal analysis of a journal bearing by using the finite element method. Hazlett and Khonsari [25, 26] developed a finite element model to study the nature of thermally induced seizure in journal bearings. More recently, Krithivasan and Khonsari [12, 99] performed a comprehensive finite element study of seizure in journal bearings.
Most of the literatures studying the thermal or thermomechanical behavior of journal bearings either assumed two dimensional journal bearings [1, 12, 25, 26, 52, 95, 99] or assumed the journal bearing rotates unidirectionally [96, 97]. A three dimensional model to treat journal bearings experiencing oscillatory motion is still lacking.

In this chapter, a three dimensional computational thermal model is developed. The model utilizes a combination of the transfer matrix and finite element methods. The frictional heat generated at the contact interface is instantaneously partitioned between the bushing and the shaft. Two methods to couple the heat and temperature at the contact interface are presented. The first one can automatically account for the heat division between contacting bodies by satisfying the heat equilibrium and temperature continuity at interactive surfaces. In the second method, the frictional heat is partitioned by introducing a fictitious layer between contacting bodies with a specified gap conductance. Application of the model to the heat transfer analysis of journal bearings is presented. Non-uniformly distributed frictional heat along the axial direction is considered. The model is capable of predicting the transient temperature field for journal bearings. It can also be used to determine the maximum contact temperature which is difficult to be measured experimentally. Comparison of the simulated results along with experimental tests conducted in a laboratory is presented.

### 8.2 Model Development

In this section, a three dimensional transient heat transfer model is developed. The approach utilizes a combination of the transfer matrix and finite element method. The heat division between the contacting bodies is taken into account.

#### 8.2.1 Transfer Matrix Heat Transfer Method

Referring to Fig. 8.1, a shaft oscillates on the inner surface of a stationary bushing under the applied load $W$. As the shaft oscillates, heat is generated in the contact region due to friction. The general equation governing the transient heat conduction in the shaft or the bushing as illustrated in Fig. 8.1 is:

$$\nabla^2 T(x,t) = \frac{1}{n} \dot{T}(x,t), \quad x \in \Omega, \ t > 0 \tag{8.1}\$$

with the initial condition

$$T(x,0) = T^0(x), \quad x \in \Omega \tag{8.1a}\$$

and subjected to the boundary conditions

$$T(x,t) = \bar{T}(x,t), \quad x \in \Gamma_T \tag{8.1b}\$$

$$q(x,t) = \bar{q}(x,t), \quad x \in \Gamma_q \tag{8.1c}\$$

$$q(x,t) = h(T(x,t) - T), \quad x \in \Gamma_h \tag{8.1d}\$$

where $\Omega$ and $\Gamma$ are the considered domain and the domain boundary, respectively. $x$ is coordinate vector $(x, y, z)$. The parameter $t$ denotes the time, and the super-dot stands for the derivative with respect to time. Parameters $\kappa$ and $k$ are the thermal diffusivity and the thermal conductivity of the material, respectively. $T(x,t)$ is the temperature field.
The heat flux \( q(x, t) \) is defined as \( q(x, t) = k \frac{\partial T}{\partial n} \), where \( n \) is the unit normal outward to the boundary \( \Gamma \). \( T^0(x) \) represents the initial temperature distribution. \( T(x, t) \) and \( \overline{q}(x, t) \) are the prescribed boundary temperature on surface \( \Gamma_T \) and heat flux on surface \( \Gamma_q \), respectively. The boundary surface \( \Gamma_h \) is subjected to convective cooling with the heat transfer coefficient \( h \), and the ambient temperature \( T_\infty \).

After space discretization by the finite element procedure, the transient heat transfer equation, Eq. (8.1), can be expressed in a matrix form [34] as

\[
CT(t) + KT(t) = F(t)
\]  

(8.2)

where \( T \), \( C \), \( K \) and \( F \) are the temperature vector, capacitance matrix, conductance matrix and heat load vector, respectively, defined as

\[
C = \sum_e \int_{\Omega^e} N^T \rho c N d\Omega
\]  

(8.2a)

\[
K = \sum_e \int_{\Omega^e} \left( \frac{\partial N^T}{\partial x} k \frac{\partial N}{\partial x} + \frac{\partial N^T}{\partial y} k \frac{\partial N}{\partial y} + \frac{\partial N^T}{\partial z} k \frac{\partial N}{\partial z} \right) d\Omega + \sum_e \int_{\Gamma_h} h N^T N d\Gamma
\]  

(8.2b)

\[
F = \sum_e \int_{\Gamma_q} \overline{q} N^T d\Gamma + \sum_e \int_{\Gamma_h} h T_\infty N^T d\Gamma
\]  

(8.2c)

where \( N \) is the shape function in the element of \( e \). \( \Omega^e \) and \( \Gamma^e \) denote the element domain and the element surface if it is on the boundary, respectively. Parameters \( \rho \) and \( c \) represent the density and capacitance of the material, respectively. The superscript \( T \) denotes the matrix transpose.

Solving Eq. (8.2) and performing the related integration results in the following iteration equation for heat transfer analysis as

\[
T^{(k+1)} = AT^{(k)} + MF^{(k)}
\]  

(8.3)
where
\[
\begin{align*}
A &= e^{H \tau} \\
M &= (I - A) K^{-1}
\end{align*}
\]  
(8.3a)
and
\[
H = -C^{-1} K
\]  
(8.3c)

where the superscript \(k\) represents the time step and \(\tau\) is the time period from time \(t_k\) to \(t_{k+1}\). \(A\) and \(M\) are the coefficient matrices and \(I\) is the identity matrix. The superscript \(-1\) denotes the matrix inversion. \(F\) is the vector of the discretized heat load. For a model having \(N\) nodes, \(N_h\) convective surfaces, and with the contact surface divided into \(N_q\) small segmented elements, the dimension of vector \(T\) is \(N\), the dimension of vector \(F\) is \(N_h + N_q\), and the dimensions of matrices \(A\) and \(M\) are \(N \times N\) and \(N \times (N_h + N_q)\), respectively.

Physically, the coefficient matrix \(A\) determines how heat conducts inside the solid, and the coefficient matrix \(M\) shows how the surface heat input contributes to the field temperature rise in the solid. The load vector \(F\) includes two parts: one part contains the terms involved with the heat flux on the contact surface due to the frictional heating and convective cooling. The other part consists of the terms involving the ambient temperature on the convective boundary surfaces. In the present method, except \(\overline{q}\) and \(T_c\), the other terms in Eq. (8.2c) are multiplied into the corresponding columns of matrix \(M\) in Eq. (8.3). Therefore, vector \(F\) in Eq. (8.3) only contains the values of the heat fluxes, corresponding to each small surface element on the discretized contact surface, and the ambient temperatures, corresponding to the boundary surfaces subjected to convective cooling.

It is noted that \(A\) and \(M\) keep constant for different heat sources and need to be calculated only once. Once they are determined, the transient temperature distribution in the domain at each time step \(t_k\) is easily calculated from Eq. (8.3). The procedure is to start from the initial condition \(T^{(0)}\) and to assign different values of heat fluxes to the components of vector \(F^{(k)}\) at each time step \(t_k\) according to the motion of the heat source.

For two bodies in oscillatory contact as shown in Fig. 8.1, evaluation of the vector \(F^{(k)}\) requires to couple the heat and the temperature at the interactive surfaces of the contacting bodies.

**8.2.2 Formulation of the Thermal Contact**

For a system of journal bearing subjected to relative oscillation as illustrated in Fig. 8.1, Eq. (8.3) is applied to both contacting bodies as
\[
T^{(k+1)}_l = A_l T^{(k)}_l + M_l F^{(k)}_l
\]  
(8.4)
where subscript \(l = \{S, B\}\) denotes the shaft or the bushing. For both the shaft and the bushing, the heat load \(F_l\) consists of the frictional heat generated at the contact interface,
the convective cooling by the air within the clearance, and the convective cooling on the outer surfaces.

In current study, the contact surfaces, the inner surface of the bushing and the outer surface of the shaft within the contact length, are meshed in the same way such that, at the contact surfaces, every element/node on the shaft corresponds to a certain element/node on the bushing as shown in Fig. 8.2, which presents a typical contact pair of elements between the shaft and bushing.

![Fig. 8.2 Pair of contact elements between the shaft and bushing](image)

Whether a pair of elements on the outer surface of the shaft or the inner surface of the bushing shown in Fig. 8.2 lies inside the contact region or not is determined by the contact pressure. The contact pressure on the contact pair of elements \( P \) is calculated by averaging the node pressures as:

\[
p = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)
\]

where \( p_i \) \((i = 1, 2, 3, 4)\) denotes the node pressure in the element obtained though a static contact analysis. The contact pressure on node \( p_j \) or on element \( p \) is the same for the contact pairs between the shaft and bushing.

If the contact pressure evaluated by Eq. (8.5) is equal to zero, the corresponding surface element is in the clearance between the shaft and bushing, and undergoes convective cooling. The heat dissipations by the convective cooling within the clearance are evaluated using the following expression:

\[
q_i = h_g (T_g - T_{el})
\]

where

\[
T_g = \frac{1}{2}(T_{es} + T_{eb})
\]
where \( h_c \) is the convective heat transfer coefficient. \( T_s \) is the surrounding temperature within the clearance. \( T_{el} \), representing \( T_{es} \) or \( T_{eb} \), is the temperature of the open element pair for the shaft and the bushing, respectively. \( T_{li} \) (\( i = 1, 2, 3, 4 \)) denotes the element node temperatures as illustrated in Fig. 8.2.

If the contact pressure is not equal to zero, the corresponding surface element is in contact where the frictional heat \( q \) is generated. The heat generation is given as

\[
q(t) = \mu p R_s \left| \dot{\phi}(t) \right|
\]

with the oscillation velocity given by a sinusoidal function as

\[
\dot{\phi}(t) = A \omega \sin(\omega t)
\]

where \( \mu \) is the friction coefficient. \( R_s \) denotes the radius of the shaft. Parameters \( A \) and \( f \) are the oscillating amplitude and frequency, respectively. \( \omega = 2\pi f \) is the angular frequency.

In the contact region, part of the heat in Eq. (8.7) transfers to the shaft and the rest conducts into the bushing. Two methods to treat the thermal contact between the shaft and the bushing are presented. In the first method, the heat division between the shaft and the bushing are automatically account for by satisfying the heat equilibrium and temperature continuity at the contact surfaces. In the second method, a so-called gap conductance in a fictitious layer between contacting bodies are introduced to partition the frictional heat. Both methods are described in this section.

**Automatic heat partition.** In the contact region, the frictional heat generated in the contact interface is partitioned between the shaft and the bushing. The following temperature continuity and the heat equilibrium, Eqns. (8.8) and (8.9), should be satisfied:

\[
T_s = T_B
\]

\[
\bar{q}_s + \bar{q}_B = q
\]

where \( \bar{q}_s \) and \( \bar{q}_B \) are the heat to the shaft and the bushing, respectively.

Introducing Eq. (8.4) into Eq. (8.8) and using the relationship in Eq. (8.9), and solving the obtained system of equations enables one to evaluate the divisions of the frictional heat to the shaft and the bushing, \( \bar{q}_s \) and \( \bar{q}_B \), respectively. This method automatically partitions the frictional heat between the shaft and bushing.

**Heat partition by a gap conductance.** In the contact area, the heat instantaneously flows into the contacting bodies depending on the heat partitioning coefficients for the shaft \( \lambda_s \) and the bushing \( \lambda_B \) (\( \lambda_s + \lambda_B = 1 \)) as

\[
\bar{q}_s = q_c + \lambda_s q
\]

\[
\bar{q}_B = -q_c + \lambda_B q
\]

with
\[ q_c = k_g (T_{eb} - T_{es}) \]  \hspace{1cm} (8.12)

where \( q_c \) is the heat flux across the contact interface due to conduction. \( k_g \) is the gap conductance. The gap conductance represents the thermal conductivity of a fictitious “third body” layer of thickness \( \Delta l \) between the contact surfaces, selected to yield a continuous temperature across the contact interface. It is defined by Gong and Komvopoulos [22] as

\[ k_g = \frac{k}{\Delta l} \]  \hspace{1cm} (8.13)

where \( k \) is the thermal conductivity of the layer. For current problem, \( k \) is equal to the thermal conductivity of the shaft. \( \Delta l \) represents the smallest size of the contact element.

Then the simulation can be performed by setting \( \lambda_s = \lambda_b = 0.5 \), and thus the heat equilibrium is satisfied. The temperature across the contact area is instantaneously adjusted by the conductance flux \( q_c \) through the relationship in Eqns. (8.10) and (8.11) in order to ensure the temperature continuity. When both bodies are at the same temperature in the contact region, the heat flux into each body is equal to \( 0.5q \) and \( q_c = 0 \). If the temperature at the bushing surface is higher than that at the shaft surface, the heat flux \( q_c \) in Eq. (8.12) is positive. More heat than \( 0.5q \) as in Eq. (8.10) is applied on the shaft to raise the lower temperature on it and less heat than \( 0.5q \) as in Eq. (8.11) is applied on the bushing to lower the higher temperature rise, and thus to reduce the temperature difference and make the temperature continuous across the contact interface. On the contrary, if the temperature at the shaft surface is higher than that at the bushing surface, the heat flux \( q_c \) in Eq. (8.12) is negative. A reverse adjustment is conducted.

The first method can automatically ensure both the heat equilibrium and the temperature continuity between contacting bodies. For the second method the gap conductance must be carefully determined to make the temperature continuous across the contact interface. In current study, two coupling methods result in equivalent results.

8.2.3 Solution Algorithm

For the transient heat transfer analysis of a journal bearing system as shown in Fig. 8.1, each time step corresponds to a certain oscillatory location between the shaft and the bushing. The results from the previous step are treated as a pseudo-initial condition for current step. The total frictional heat generated in the contact surface is calculated from Eq. (8.7) by using the contact results obtained from a static contact analysis. Using either of the coupling methods described in Section 8.2.2, Eqns. (8.8) and (8.9) or Eqns. (8.10) and (8.11), the heat flowing into both contacting bodies can be determined. Also the convective heat within the clearance can be evaluated by Eq. (8.6). The convective heat transfer conditions on the outer surfaces of the shaft and bushing are specified. Thus, the thermal loads on the shaft and bushing, \( F_S \) and \( F_B \), can be evaluated. The temperature distributions in the shaft and the bushing are simulated using Eq. (8.4) once the coefficient matrices \( A \) and \( M \) for the shaft and bushing are evaluated.
The calculation of the coefficient matrices $A$ and $M$ is based on using the commercial software ABAQUS. Examination of Eq. (8.3) reveals that matrices $A$ and $M$ can be treated as influence matrices. Each of the column vectors of matrix $A$ reflects how a unit nodal temperature affects the field temperature inside the solid, and each of the column vectors of matrix $M$ reflects how a unit discretized boundary flux or unit ambient temperature contributes to the field temperature rise. Thus, in this chapter, matrices $A$ and $M$ are indirectly calculated by using Eq. (8.3). The method, applicable to both the shaft and the bushing, is described as follows.

![Flow chart of the basic steps of the simulation algorithm.](image)

The matrix $A$ can be obtained by $N$ cycles of simulations with the same time period $\tau$. For the $i$th $(i = 1, 2, \cdots, N)$ simulations, the initial condition and boundary load are set to

$$(T^{(0)})_i = \{\delta_i\} \text{ and } F = \{0\}$$

(8.14)
where $\delta_j^i$ is Kronecker delta, and $j = 1, 2, \cdots, N$ are the index to the entries of the vector $T^{(0)}$. The result $T^{(i)}$ is the $i$th column of matrix $A$.

Similarly, the matrix $M$ can be evaluated by performing $N_h + N_q$ cycles of simulations with the same time period $\tau$ as that used in calculation of the matrix $A$. For the $i$th ($i = 1, 2, \cdots, N_h + N_q$) simulations, the following initial condition and boundary load are applied

$$T^{(0)} = \{0\} \text{ and } (F)_i = \{\delta_j^i\}$$

where $j = 1, 2, \cdots, N_h + N_q$ are the index to the entries of the vector $F$. The result $T^{(i)}$ is the $i$th column of matrix $M$.

After $N + N_h + N_q$ cycles of simulations respectively for the shaft and the bushing, the coefficient matrices $A_i$ and $M_i$ are obtained.

Using the procedure described in Section 8.2.1-8.2.3, a computer program is developed to efficiently treat heat transfer problems involving oscillatory system. Figure 8.3 presents the flow chart of the basic steps of the simulation algorithm.

### 8.3 Model Verification

<table>
<thead>
<tr>
<th>Table 8.1 Parameters used in the simulations</th>
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<tr>
<td>Radius of cylinder (mm)</td>
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<td>Heating width $\phi_0$ (Rad)</td>
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<td>Density $\rho$ (kg/m$^3$)</td>
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<td>Thermal conductivity $k$ (W/m $^\circ$K)</td>
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<td>Specific heat $c$ (J/kg K)</td>
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<td>Surface convection $h$ (W/m$^2$ K)</td>
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<td>Heat flux (W/m$^2$)</td>
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<td>Rotation/Oscillation speed $\dot{\phi}$ (Rad/s)</td>
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<td>Oscillation amplitude $A$ (Deg)</td>
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</table>

In order to demonstrate the capability and accuracy of the present method, a cylinder subjected to a unidirectional or oscillatory heat source is studied. The results are compared with those by ABAQUS. Figure 8.4 shows the three dimensional (3D) model used by the present method, Fig. 8.4(a), and the two dimensional (2D) model by ABAQUS, Fig. 8.4(b). Both ends of the cylinder in Fig. 8.4(a) are insulated. The circumferential surface is heated over the width $\phi_0$ as shown in Fig. 8.4(b). The rest of the circumferential surface is subjected to the convective cooling. The heat source is assumed to be uniformly distributed along the axial direction. Therefore, the problem can
be treated as a two dimensional problem, and the results is verified by ABAQUS using a 2D model. The parameters used in the simulation are listed in Table 8.1. Both types of moving heat source, unidirectional sliding and oscillatory heat sources, are considered. The same coefficient matrices are used for both cases.

![3D FEM model](image1)

![2D FEM model](image2)

Fig. 8.4 (a) 3D FEM model used by the present method. (b) 2D FEM model used by ABAQUS.

**Unidirectional sliding heat source.** For the case of the unidirectional sliding heat source, the comparisons of the simulation results by the present method along with the results by ABAQUS are shown in Figs. 8.5-6. Figure 8.5(a) presents the comparison of the surface temperature variation up to 400s, and Fig. 8.5(b) the comparison of the surface temperature variation in two cycles at steady state. In Fig. 8.6, the steady state temperature distributions obtained by the present method as shown in Fig. 8.5(a) are compared with that obtained by ABAQUS as shown in Fig. 8.6(b). All the results are in good agreement as shown in Figs. 8.5-6.

**Oscillatory heat source.** In the case of the oscillatory heat source, the comparisons of the simulation results by the present method along with the results by ABAQUS are shown in Figs. 8.7-8. Figure 8.7(a) presents the comparison of the surface temperature variation up to 200s, and Fig. 8.7(b) the comparison of the surface temperature variation in two cycles and steady state. The steady state temperature distributions obtained by the present method as shown Fig. 8.8(a) are compared with that obtained by ABAQUS as shown in Fig. 8.8(b). The results by both methods are in good agreement.

The computer time taken by the present method is greatly less than that by ABAQUS on the same computer. For both cases of unidirectional and oscillatory heat sources, the present method takes 48 minutes for 1200s simulations. In contrast, the computation by ABAQUS takes 1 hour 42 minutes for a 1200s simulation in the case of the unidirectional sliding and 1 hour 55 minutes for a 809s simulation in the case of the oscillatory sliding.
Noted that the 3D model (see Fig. 8.4(a)) used by the present method is 6 times in node scale and 5 times in element scale larger than the 2D model (see Fig. 8.4(b)) used by ABAQUS. Such efficiency of the present method is very important when dealing with 3D oscillatory heat source problems, where a large number of cycles are required for the system to reach a steady state. In addition, the same coefficient matrices enable one to simulate heat sources with different motions.

Fig. 8.5 (a) Comparison of surface temperature variation up to 400s and (b) Comparison of surface temperature variation in two cycles at steady state between the present method and ABAQUS for unidirectional heat source.
Fig. 8.6 Temperature rise at steady state (a) by the present method and (b) by ABAQUS for unidirectional sliding heat source.
Fig. 8.7 (a) Comparison of surface temperature variation up to 200s and (b) Comparison of surface temperature variation in two cycles at steady state between the present method and ABAQUS for oscillatory heat source.
Fig. 8.8 Temperature rise at steady state (a) by the present method and (b) by ABAQUS for oscillatory heat source.
8.4 Experiment

8.4.1 Experimental Test Apparatus

Figure 8.9 shows the schematic of experimental apparatus (Lewis LRI-8H). The machine is capable of measuring the coefficient of friction of journal bearings under varying operating conditions such as load and speed, and the temperatures on the outer surface of the bushing.

The shaft is tapered at one end and is centered in the rear supporting base. It is driven by an electrical motor with a maximum speed 3300rpm through a four-bar linkage mechanism. The bushing is fixed inside a housing. The desired vertical load is applied by using a dead weight located on the right hanger device via a lever. The lever scale is 1:10, so that 1N on the hanger is equivalent to 10N on the bearing. The left hanger works to balance the self weight of the level bar and the right one. The journal bearing is lubricated by grease. As the shaft oscillated relative to the bushing, the friction force is translated to compression or tension of a linkage bar and sensed by a load cell. The temperatures at four points on the outer surface of the bushing are measured by four thermal couples. The measure points are located on the central cross-section of the bushing and distributed as shown in Fig. 8.10. The signals from the load cell and the thermal couples are transferred to the computer system for recording and processing. The coefficient of friction, the oscillation frequency and the load are processed by software and displayed on the computer screen. The time interval of data reading is adjustable and is independent of the duration of the test. In the current study, a minimum 20s is taken.

Fig. 8.9 Schematic of experimental apparatus of bushing tester. 1–Load cell; 2–Linkage bar; 3–Thermal couples; 4–Shaft; 5–Lip seal; 6–Housing; 7–Guiding poles; 8–Computer system; 9–Load applying device.
8.4.2 Experiment Procedure

Before any measurement is taken, the system is balanced so that the coefficient of friction is nil when the shaft is in its static position. The system is run-in for one hour under a lower oscillation frequency and smaller load. Upon completion of each test, the system is given enough time to cool down, typically 2 hours. For each test, the history of the coefficient of friction and the temperature is monitored and stored in the computer.

8.5 Computational Models and Boundary Conditions

The computational model consists of housing, bushing and shaft as illustrated in Fig. 8.11. Figure 8.12 shows the finite element model. The exact same finite element mesh with eight-node brick element and six-node triangular prism element for central part of the shaft are used for the static contact and transient heat transfer analyses. This ensures that the contact pressure obtained by the static contact analysis can be directly exported to calculate the frictional heat for the transient heat transfer analysis. For simplicity, the bushing and the housing are treated as one entity but with different material properties. Only the contact between the shaft and the bushing is considered.

The analysis begins with a contact analysis by using ABAQUS to calculate the contact pressure distribution $p$, and thus the total frictional heat generated in the contact interface between the shaft and bushing. After the variation of the frictional heat is determined, a transient heat transfer analysis is performed by using the model developed in Section 8.2. The temperature field in the system is simulated until a steady state is reached.

In the static contact analysis, the conic surface of the shaft is fixed against displacements in all three directions. The outer surface of the housing is coupled to a reference node in all displacement components and the load is applied on the reference node. For the transient heat transfer analysis, the outer surfaces of the shaft and bushing

![Fig. 8.10 Locations of thermal couples on the central cross-section of the bushing](image-url)
are subjected to convective cooling. In contact region, the temperature and the frictional heat are coupled by using the method described in Section 8.2.2.

Fig. 8.11 Modeled components

Fig. 8.12 Finite element model
8.6 Results and Discussion

In this section, the shaft oscillating inside the bushing under 4448N load is simulated. Table 8.2 lists the main dimension and the material properties of the journal bearing and operation conditions in the experiment and in the simulation.

<table>
<thead>
<tr>
<th>Main Dimension</th>
<th>Radius of shaft $R_s$ (mm)</th>
<th>12.25</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inner radius of bushing $R_{bi}$ (mm)</td>
<td>12.7</td>
</tr>
<tr>
<td></td>
<td>Outer radius of housing $R_{ho}$ (mm)</td>
<td>44.4</td>
</tr>
<tr>
<td></td>
<td>Length of bushing $L_{b}$ (mm)</td>
<td>25.5</td>
</tr>
<tr>
<td></td>
<td>Length of Housing $L_{h}$ (mm)</td>
<td>54.0</td>
</tr>
<tr>
<td>Load</td>
<td>Load $W$ (N)</td>
<td>4448</td>
</tr>
<tr>
<td>Oscillation</td>
<td>Oscillation frequency $f$ (Hz)</td>
<td>1.34</td>
</tr>
<tr>
<td></td>
<td>Oscillation amplitude $A$ (°)</td>
<td>45</td>
</tr>
<tr>
<td>Convection</td>
<td>Clearance convection $h_g$ (W/m$^2$ K)</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>Surface convection $h_e$ (W/m$^2$ K)</td>
<td>39</td>
</tr>
<tr>
<td>Shaft &amp; Bushing</td>
<td>Density $\rho_{sb}$ (kg/m$^3$)</td>
<td>7870</td>
</tr>
<tr>
<td>(Carbon Steel)</td>
<td>Thermal conductivity $k_{sb}$ (w/m K)</td>
<td>51.9</td>
</tr>
<tr>
<td></td>
<td>Specific heat $c_{sb}$ (J/kg K)</td>
<td>486</td>
</tr>
<tr>
<td>Housing</td>
<td>Density $\rho_{h}$ (kg/m$^3$)</td>
<td>8000</td>
</tr>
<tr>
<td>(303 S/S)</td>
<td>Thermal conductivity $k_{h}$ (w/m K)</td>
<td>16.2</td>
</tr>
<tr>
<td></td>
<td>Specific heat $c_{h}$ (J/kg K)</td>
<td>500</td>
</tr>
</tbody>
</table>

8.6.1 Frictional Heat

The frictional heat generated at the contact interface between the shaft and bushing is evaluated by Eq. (8.7). The heat flux is dependent on the contact pressure, the coefficient of friction and the oscillatory speed. The contact pressure is determined by a static contact analysis. The friction coefficient and oscillatory speed are obtained by the experiment.

8.6.1.1 Static Contact Analysis

In order to determine the frictional heat according to Eq. (8.7), the contact pressure must be first calculated. A static contact analysis is conducted by using ABAQUS. The results are shown in Figs. 8.13-15. One end of the shaft with the conic surface is fixed. The load is applied closer to the other end. In this fashion the shaft behaves as a cantilever, resulting in non-uniform stress and contact pressure along the axial direction.
Figure 8.13 and 8.14 show the Von Mises stress for the shaft and the bushing, respectively. Figure 8.15 presents the contact pressure on the bushing. It can be seen from Figs. 8.13-15 that the contact between the shaft and the bushing only occurs in a small area along the axial direction. The contact concentrates on the edge of the bushing close to the fix end of the shaft. The other end the bushing is not in contact with the shaft as shown in Fig. 8.15. The non-uniform distribution of contact pressure results in a corresponding non-uniform frictional flux along the axial direction, which makes it necessary to conduct a 3D heat transfer analysis of the oscillatory system.

<table>
<thead>
<tr>
<th>Level</th>
<th>Stress (MPa)</th>
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<tr>
<td>1</td>
<td>391</td>
</tr>
<tr>
<td>2</td>
<td>244</td>
</tr>
<tr>
<td>3</td>
<td>147</td>
</tr>
<tr>
<td>4</td>
<td>489</td>
</tr>
</tbody>
</table>

Fig. 8.13 Von Mises stress for shaft

<table>
<thead>
<tr>
<th>Level</th>
<th>Stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>733</td>
</tr>
<tr>
<td>2</td>
<td>458</td>
</tr>
<tr>
<td>3</td>
<td>275</td>
</tr>
<tr>
<td>4</td>
<td>917</td>
</tr>
</tbody>
</table>

Fig. 8.14 Von Mises stress for bushing
Fig. 8.15 Contact pressure

8.6.1.2 Oscillation Velocity

The oscillatory motion is achieved by using a four-bar-linkage mechanism shown in Fig. 8.16. From the geometric relationship as illustrated in Fig. 8.16, one arrives at

\[
\begin{align*}
\omega_1 l_1 \sin(\theta_1 - \theta_3) &= \omega_2 l_2 \sin(\theta_2 - \theta_1) \\
l_1 \sin(\theta_1) &= l_2 \sin(\theta_2) - l_1 \sin(\theta_1) \\
l_3 \cos(\theta_3) + l_1 \cos(\theta_1) &= l + l_2 \cos(\theta_2)
\end{align*}
\]

where \( l_1, l_2, l_3 \) and \( l_4 \) are the length of bars in the four-bar-linkage mechanism. \( \omega_1 \) is the angular rotating speed of the motor, and \( \omega_2 \) is the output oscillation velocity of the shaft.

Solving Eq. (8.16) yields

\[
\omega_2 = \frac{\omega_1 l_1 (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_3)}{l_2 (\sin \theta_2 \cos \theta_3 - \cos \theta_2 \sin \theta_3)}
\]

where

<table>
<thead>
<tr>
<th>Level</th>
<th>Pressure (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>337</td>
</tr>
<tr>
<td>2</td>
<td>211</td>
</tr>
<tr>
<td>3</td>
<td>127</td>
</tr>
<tr>
<td>4</td>
<td>421</td>
</tr>
</tbody>
</table>
\[ \theta_1 = \omega t \]
\[ \theta_2 = \sin^{-1}\left(\frac{d}{\sqrt{a^2 + b^2}}\right) - \tan^{-1}\left(\frac{b}{a}\right) \]  
\[ \theta_3 = \sin^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) - \tan^{-1}\left(\frac{b}{a}\right) \]

and
\[ a = l_1 \sin(\theta_1) \]
\[ b = l_1 \cos(\theta_1) - l \]
\[ c = \frac{-a^2 - b^2 + l_2^2 - l_3^2}{2l_3} \]
\[ d = \frac{a^2 + b^2 + l_2^2 - l_3^2}{2l_2} \]

Figure 8.17 shows the angular velocity in one oscillatory cycle evaluated by Eq. (8.17) under the experiment condition with \( \omega = 6.28 \pi \) Rad/s, \( l = 644 \text{mm}, \ l_1 = 55.22 \text{mm}, \ l_2 = 77.8 \text{mm}, \) and \( l_3 = 640 \text{mm}. \)

8.6.1.3 Friction coefficient

Figure 8.18 shows the variation of the friction coefficient under the experiment condition at motor speed \( \omega = 6.28 \pi \) Rad/s, and applied load \( W = 4448 \text{N}. \) The bold solid line represents the fitting curve of the friction, which will be used in the following analysis to calculate the frictional heat.
In current study, it is assumed that the contact pressure keep unchanged during oscillation of the shaft. Thermal expansion is not considered. Using Eq. (8.7) and introducing the contact distribution as shown in Fig. 8.15, the oscillation velocity by Eq. (8.17) and the variation of the friction coefficient by the experiment as shown in Fig. 8.18, the time dependent frictional heat can be determined.

8.6.2 Temperature Results

The transient temperature in the journal bearing system is simulated up to a steady state by using the present method. Both coupling methods in Section 8.2.2, the automatic heat partition and the partition by gap conductance, are used. Similar results are obtained. The results with the heat partitioned by gap conductance are presented in the following.

Figure 8.19 and 8.20 shows the temperature rise in the bushing and the shaft at the steady state, respectively. The maximum temperature for the shaft and bushing occurs at the contact region as shown in Figs. 8.19 and 8.20(a) corresponding to the maximum contact pressure shown in Fig. 8.15. Along the axial direction, the inner surface of the bushing experiences a greater temperature gradient due to the edge loading. For the bushing, the maximum contact temperature occurring at its inner surface is about 18°C higher than the maximum temperate at the outer surface as shown in Fig. 8.20.

Figure 8.21 presents the variation of the temperature rise at four measurement points as specified in Fig. 8.10, compared with the experiment results. The maximum difference is less than 5 Celsius degree. The error is, in part, due to simplification of the FEM model and in part due to the temperature measurement in experiment. Figure 8.21 shows that the simulated temperature rises faster than the measured temperature from the 1000th to 5000th cycle. Also the experiment temperature still rises slowly while the simulation has reached a steady state. This is probably because that the components attaching to the
housing and the shaft contribute to the heat dissipation from the bushing and shaft, and decrease the temperature rise. In addition, the temperature rise measured by thermal couples lags behind the actual temperature rise on the bushing surface. Improving the measurement of the surface temperature will reduce this difference.

<table>
<thead>
<tr>
<th>Level</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54.6</td>
</tr>
<tr>
<td>2</td>
<td>49.5</td>
</tr>
<tr>
<td>3</td>
<td>46.2</td>
</tr>
<tr>
<td>4</td>
<td>42.8</td>
</tr>
<tr>
<td>5</td>
<td>39.5</td>
</tr>
<tr>
<td>6</td>
<td>36.2</td>
</tr>
<tr>
<td>7</td>
<td>32.8</td>
</tr>
</tbody>
</table>

Fig. 8.19 Temperature rise for shaft at steady state

<table>
<thead>
<tr>
<th>Level</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54.3</td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
</tr>
<tr>
<td>3</td>
<td>41.4</td>
</tr>
<tr>
<td>4</td>
<td>36.3</td>
</tr>
<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>26.0</td>
</tr>
<tr>
<td>7</td>
<td>20.8</td>
</tr>
</tbody>
</table>

(a)  
(b)

Fig. 8.20 Temperature rise for bushing at steady state: (a) Inner surface temperature; (b) Outer surface temperature
8.7 Conclusions

A computational efficient 3D heat transfer model is developed. The approach utilizes a combination of the finite element and transfer matrix method. The frictional heat generated at the contact interface is instantaneously partitioned between the contacting bodies. The model can be used to handle complicated moving heat source problems involving oscillatory contacts.

A transient heat transfer analysis of a journal bearing system is performed with consideration of the actual distribution of the frictional heat. The temperature comparison between the experimental and computational investigations at the four locations of the outer bushing surface is used to validate the results of the model.

The nature of the load conditions in the experimental setup is such that the shaft behaves as a cantilever, and thus susceptible to edge loading at high imposed loads. Under this condition, the frictional heat distributed non-uniformly along the axial direction and a three dimensional analysis becomes necessary.

The results of the simulations show that under the specified operating conditions, the maximum contact temperature occurs at the edge of the contact corresponding to the maximum contact pressure. The temperature distribution along the axial direction is non-uniform. It is shown that the model and the associated simulation algorithm are capable of predicting the maximum contact temperature, which is difficult to be measured experimentally. The maximum temperature occurring at the inner surface of the bushing
is 18°C higher than the maximum temperature on its outer surface where the thermal
couples are located in the experiment.

8.8 Nomenclature

\( A \) = Angular oscillation amplitude (Rad)
\( \mathbf{A} \) = Coefficient matrix associated with initial condition
\( c \) = Specific heat (J/kg K)
\( \mathbf{C} \) = Capacitance matrix
\( f \) = Oscillation frequency (Hz)
\( \mathbf{F} \) = Heat load vector including heat flux and convection
\( h \) = Convective heat transfer coefficient (W/m\(^2\) K)
\( h_g \) = Heat transfer coefficient in clearance (W/m\(^2\) K)
\( \mathbf{H} \) = Exponential matrix
\( \mathbf{I} \) = Identity matrix
\( k \) = Thermal conductivity (W/m K)
\( k_g \) = Gap conductance (W/m\(^2\) K)
\( \mathbf{K} \) = Conductance matrix
\( l_1, l_2, l_3, l_4 \) = Length of bars in the four-bar-linkage structure (m)
\( \Delta l \) = Smallest size of the contact element
\( \mathbf{M} \) = Coefficient matrix associated with boundary conditions
\( n \) = Unit normal outward to the boundary \( \Gamma \)
\( \mathbf{N} \) = Vector of shape functions for element
\( N, N_q, N_h \) = Node number of the model, contact node number, and number of convective
surfaces, respectively
\( p \) = Contact pressure (N/m\(^2\))
\( q, \bar{q} \) = Heating/cooling flux at interface, prescribed boundary flux, respectively (W/m\(^2\))
\( q_c \) = Heat flux across the contact interface due to conduction (W/m\(^2\))
\( R \) = Radius (m)
\( t, t_k \) = Time and the time at \( k \)th step (s)
\( T, \mathbf{T}, T^{(0)}, \bar{T}, T_\infty \) = Field temperature, Vector of field temperature \( T \), Initial temperature,
prescribed surface temperature, and ambient temperature, respectively (°C)
\( \mathbf{x} \) = coordinate vector (\( x, y, z \)) (m)
\( \kappa \) = Thermal diffusivity, \( \kappa = k / \rho c \) (m\(^2\)/s)
\( \tau \) = Step time (s)
\( \rho \) = Density (kg/m\(^3\))
\( \theta_1, \theta_2, \theta_3 \) = Angular location of bars in the four-bar-linkage structure (Rad)
\( \phi(t) \) = Oscillation angle (Rad)
\( \dot{\phi}(t) \) = Angular oscillating velocity (Rad/s)
\( \Gamma_T, \Gamma_q, \Gamma_h \) = Surface \( \Gamma \) with prescribed temperature, heat flux and convection,
respectively
\( \mu \) = Friction coefficient
\( \omega \) = Angular frequency (1/s)
\( \omega_1, \omega_2 \) = Angular rotating speed of the motor (Rad/s)
\( \lambda \) = Heat partitioning coefficient
\( \Omega \) = Domain considered

Superscripts:
(\( k \)) = Iteration number
T = Transpose of matrix
-1 = Inversion of matrix

Subscripts:
\( e \) = External
B = Bushing
H = Housing
\( i \) = Internal
\( o \) = Outer
S = Shaft
9 Conclusions

9.1 Conclusions

An extensive thermomechanical analysis of contacts experiencing oscillatory motion is presented. The research involved development of efficient algorithms and solution methodology to efficiently treat thermomechanical coupling with interfacial frictional heating. These analyses have wide range of applications in tribology.

The major contributions of the research are summarized as follows.

1) Analytical solutions for transient temperature involving oscillatory heat source.

Two analytical models are developed for determining the transient temperature in oscillatory contacts. In the first model, a mathematical model of heat transfer in the body subjected to oscillatory heat source is developed. The governing equations are analytically solved by using Duhamel’s theorem. The method provides an analytical technique to treat oscillatory heat source problems. To verify the analytical solution, the results are compared with numerical solutions and excellent agreement is reported. In the second model, the solution for temperature distribution due to a point heat source on the surface of a semi-infinite solid was used to develop solutions for three common types of heat sources involving circular, rectangular and parabolic heat sources undergoing oscillatory motion. The appropriate governing equations for different heat sources are derived and solved by an efficient algorithm. The analytical results are validated by comparing to published results for special cases. Based on the model, thermal behavior under different oscillatory heat sources is studied. Finally analytical expressions are provided for predicting the maximum surface temperature based on a surface-fitting method.

2) Numerical modes for moving heat source problems based on the combination of transfer matrix method with dual reciprocity boundary element method and finite element method.

Two computationally-efficient algorithms are developed to handle complex transient moving heat source problems. The first model utilizes a combination of the transfer matrix and dual reciprocity boundary element methods. The domain integrals due to transient heat transfer are converted to a boundary one by the dual reciprocity boundary element method. The time integration is processed by an iteration transfer matrix method. The influence matrices are evaluated by an adaptive precise time integration method, making the method unconditionally stable. The efficiency and accuracy of the method are verified by comparing to analytical solutions or the finite element method. The second numerical model is developed by using the transfer matrix method associated with the finite element method. The approach avoids direct simulation at each cycle and thus provides significant computation savings without the loss of accuracy. A half-space subjected to a unidirectional sliding or oscillatory heat source with consideration of
convective cooling is comprehensively studied, and the efficiency of the solution technique is documented. In both models, the same coefficient matrices enable one to simulate heat sources with different distribution and different motions. Also the numerically stable nature of the developed models makes them applicable to effectively treat a wide range of sliding speeds and operating conditions in oscillatory contacts.

3) Thermomechanical effects on transient temperature in non-conformal contacts experiencing reciprocating sliding motion.

The high efficiency of the developed numerical methods makes it possible to efficiently handle the problems involving thermomechanical interaction. In this dissertation, a numerical model for treating thermomechanical interaction in oscillatory contacts is developed. A semi-infinite elastic solid in contact with a rigid adiabatic sphere subjected to oscillatory sliding motion is studied to investigate the effects of transient mechanical and thermomechanical loads on the temperature variation in the solid. The results show that the constraint of the thermal expansion in the indentation direction between the contacting bodies plays a significant role in the increase of contact pressure and heat generation, which in turn influences the temperature rise and thermal expansion.

4) Thermomechanical coupling in oscillatory system with application to journal bearing seizure

A numerical model for the thermomechanical coupling in oscillatory system is developed. The thermomechanical coupling process between the contacting bodies involves a transient solution scheme where the frictional heat is automatically partitioned between the contact surfaces. The coupling between thermal and mechanical interaction is treated by an iteration method. An application of the proposed model in the study of thermomechanical behavior of journal bearings with oscillatory motion undergoing thermally induced seizure—a catastrophic bearing failure—is provided. The significance of a wide range of operating parameters in the thermally induced seizure is studied with both solid and hollow oscillating shaft. The situations involving stationary bushing and oscillating shaft as well as oscillating bushing and stationary shaft are studied. Also, the effect of variable load direction on seizure is investigated. The simulation results show that thermally induced seizure occurs as a result of the inward expansion of the inner surface of the bushing and the ovalization of the shaft due to the thermal expansion of the bushing and shaft. Lowering the oscillation speed, improving the lubrication, increasing designed clearance or providing good internal or external cooling can prolong seizure time or eliminate thermally induced seizure completely.

5) Three-dimensional heat transfer analysis of pin-bushing and comparison with laboratory experiments

A three-dimensional computational model for heat transfer analysis of a journal bearing is presented with experimental verification. The frictional heat generated at the contact interface is instantaneously partitioned between the oscillating shaft (pin) and the bushing. The model takes into consideration the effect of bushing affixed inside the
housing and the edge loading to mimic the experimental laboratory tests. The friction coefficient is measured and used as an input to the simulation. The predictions of the temperature at four locations around the circumference are compared to the experimental measurements. It is shown that the model can effectively predict the transient temperature variation in journal bearings. The effect of edge loading on the temperature distribution is also investigated.

9.2 Suggestions for Possible Future Research

The literature on the thermomechanical behavior of oscillatory contacts is very limited in scope, particularly when considering surface coating or roughness of the contact surfaces. It would be interesting to understand the effects of oscillatory heat on the surface coating or the effects of roughness of the contact surfaces on the temperature rise in contacting bodies with relative oscillatory motion.

Another area where research is needed is the accurate prediction of the variation of the friction coefficient between the contact surfaces. It would be very challenging but promising to develop mathematical models to investigate the friction between the contact surfaces in relative oscillatory motion so as to determine the frictional heat more accurately.
References


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F: 212-591-7841
E: darchib@asme.org

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Complete List of Authors: Wen J. and Khonsari M. M.
Paper Title (Conference/Journal): Analytical Formulation for the Temperature Profile by Duhamel's Theorem in Bodies Subjected to an Oscillatory Heat Source
Paper Number (Conference): 
Volume Number (Journal): 129
Page(s) in the publication of
the permission request: 236-240
Year of Publication: 2007
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List Table Numbers: 0
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Title of outside publication: Thermomechanical Interaction Analysis of Bodies 
Subjected to Oscillatory Heat Source
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Vita

Jun Wen was born in Wulanchabu, China. He received Bachelor and Master of Science degrees from China University of Petroleum in 1995 and in 1998, respectively. After that, he worked in the same university as a faculty teacher. In August 2003, he started to pursue his doctoral study in the Department of Mechanical Engineering of Louisiana State University. Since then, he has been a doctoral student under the guidance of Dr. Michael M. Khonsari, Dow Chemical Endowed Chair and Professor in the Department of Mechanical Engineering of Louisiana State University. Jun Wen will receive his Doctor of Philosophy degree at the 2008 Fall Commencement.