2016

Spectral Properties of Photonic Crystals: Bloch Waves and Band Gaps

Robert Paul Viator Jr
Louisiana State University and Agricultural and Mechanical College

SPECTRAL PROPERTIES OF PHOTONIC CRYSTALS: BLOCH WAVES AND BAND GAPS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Robert P. Viator Jr.
B.S., Louisiana State University, 2009
M.S., Louisiana State University, 2012
August 2016

Acknowledgments

I would first like to thank Professor Robert Lipton for his help and guidance through this project. Without his patience and positive attitude towards mathematics, graduating and starting my career would have been far more difficult and less enjoyable. I would also like to thank Professor Stephen Shipman, whose insight has always revealed something deeper about the mathematics (and its applications) of models and equations I often consider.

My sincerest gratitude goes to my parents, Robert and Patricia Viator, and my siblings, Jesse and Robbie Viator; their support has been unyielding throughout my career.

Finally, I would like to thank the LSU Mathematics department as a whole, for fostering a community of learning, communicating, and sharing mathematics of all flavors.

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Abstract

The author of this dissertation studies the spectral properties of high-contrast photonic crystals, i.e. periodic electromagnetic waveguides made of two materials (a connected phase and included phase) whose electromagnetic material properties are in large contrast. A spectral analysis of 2nd-order divergence-form partial differential operators (with a coupling constant k) is provided. A result of this analysis is a uniformly convergent power series representation of Bloch-wave eigenvalues in terms of the coupling constant k in the high-contrast limit $k \to \infty$. An explicit radius of convergence for this power series is obtained, and can be written explicitly in terms of the Bloch-wave vector α, the Dirichlet eigenvalues of the inclusion geometry, and a lower bound on another spectrum known as the "generalized electrostatic resonances". This lower bound is derived from geometric properties of the inclusion geometry for the photonic crystal.

Chapter 1
Preliminaries

1.1 Framework and Notation

Throughout this thesis, $\mathbb{R}^d$ will denote the d-dimensional real Euclidean space; its standard basis will be denoted $\{\vec{e_j}\}_{j=1}^d$. For a vector $x \in \mathbb{R}^d$, we denote the magnitude of x by $|x| := \sqrt{x_1^2 + \ldots + x_d^2}$. The standard scalar product on $\mathbb{R}^d$ will be given by $x \cdot y := x_1 y_1 + \ldots x_d y_d$.

Integration will be handled in the Lebesgue sense, usually with respect to the standard Lebesgue measure dx. There will often be a need to consider integrals along $d-1$-dimensional submanifolds of $\mathbb{R}^d$, for which the standard surface measure $d\sigma(x)$ (c.f. [11]) will be the measure against which we integrate. In either case, the L^2-norm,

$$\|f\|_2 = \sqrt{\int_X |f(x)|^2 d\mu(x)}$$

for a given measure space X with measure μ, of complex-valued measurable functions $f : X \to \mathbb{C}$ will be extensively used. Of course, the L^2-norm is induced by the L^2-inner-product

$$(f, g) = \int_X f(x)\overline{g(x)} d\mu(x)$$

in the sense that $\|f\|_2 = \sqrt{(f, f)}$. We will denote the set of all (μ-measureable) functions on X with finite L^2-norm by $L^2(X)$: this forms a Hilbert space under the L^2-inner-product.

Much of this thesis will revolve around the analysis of a certain divergence-form partial differential equation. As a result, there will be need for the consideration of derivatives of functions in $L^2(\Omega)$, where Ω is an open domain in $\mathbb{R}$. However,

most functions in $L^2(\Omega)$ aren't even continuous, and, even more problematically, the space $C^\infty(\Omega) \subset L^2(\Omega)$ of smooth functions is not complete with respect to any norm induced by an inner product. Thankfully, $C^\infty(\Omega)$ is *dense* in $L^2(\Omega)$, and by completing (in the sense of Cauchy sequences) $C^\infty(\Omega)$ under a slightly different norm induced by the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \overline{\nabla v} dx + \int_\Omega u\overline{v} dx,$$

$$\|u\|_{1,2} = \sqrt{\langle u, u\rangle},$$

one obtains the well known Sobolev space $H^1(\Omega)$. This space is also a Hilbert Space (with respect to the inner product $\langle, \rangle$), and consists of functions in $L^2(\Omega)$ whose *distributional partial derivatives* are themselves in $L^2(\Omega)$, e.g. those functions $f \in L^2(\Omega)$ such that for each $j = 1, \ldots, d$ there exists a $u_j \in L^2(\Omega)$ such that, for all $\varphi \in C^\infty(\Omega)$ with compact support one has

$$\int_\Omega f \overline{\partial_{x_j} \varphi} dx = -\int_\Omega u_j \overline{\varphi} dx.$$

The function u_j is called the j-th weak partial derivative of f. For more information on the nature of functions in $H^1(\Omega)$, see [10].

This thesis will often consider open, bounded subdomains $\Omega \subset \mathbb{R}^d$; when doing so, it will be necessary to also consider the smoothness of its boundary, which we denote $\partial\Omega := \overline{\Omega} \backslash \Omega$. For a natural number $k \in \mathbb{N}$, we will say that a bounded domain $\Omega \subset \mathbb{R}^d$ has C^k boundary if, for every point $x \in \partial\Omega$ there exists a neighborhood $U \subset \partial\Omega$ of x, an open set $V \subset \mathbb{R}^{d-1}$, and a k-continuously-differentiable, invertible map $\phi : U \to V$ whose inverse is also k-continuously-differentiable. For $\gamma \in (0, 1)$ we say that Ω has $C^{1,\gamma}$ boundary if the maps $\phi : U \to V$ given above have Hölder-continuous gradient with Hölder exponent γ, i.e. for all $x, y \in \partial\Omega$ there exists a

$C > 0$ independent of x and y such that

$$|\nabla\phi(x) - \nabla\phi(y)| \leq C|x-y|^\gamma.$$

When working with partial differential equations on a bounded domain Ω (with suitably smooth boundary), it is natural and often necessary to consider boundary values of solutions to the equation at hand, i.e. the value of functions $u \in H^1(\Omega)$ along $\partial\Omega$. However, functions in $H^1(\Omega)$ are only *equivalence classes* of measurable functions, and therefore may not even be well-defined on a sets of d-dimensional Lebesgue measure zero, such as $\partial\Omega$. To rectify this problem, we have the following well known Trace Theorem, which allows one to make sense of "boundary values" of H^1 functions.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^d$ with C^1 boundary. Then there exists a bounded linear operator $T : H^1(\Omega) \to L^2(\Omega)$ such that*

1. *for $u \in H^1(\Omega) \cap C(\overline{\Omega})$, $Tu = u|_{\partial\Omega}$*
2. *for $u \in H^1(\Omega)$, $\|Tu\|_{L^2(\partial\Omega)} \leq C\|u\|_{1,2}$*

where $C > 0$ depends only on Ω.

We denote the kernel of the Trace operator T by $H^1_0(\Omega)$, which is interpreted as the set of functions in $H^1(\Omega)$ with zero boundary value. Note that T given above is not typically surjective: its range is given by the so-called fractional Sobolev space $H^{1/2}(\partial\Omega)$, which is defined by

$$H^{1/2}(\partial\Omega) := \{u \in L^2(\partial\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{d+1}{2}}} \in L^2(\partial\Omega \times \partial\Omega)\}$$

Indeed, for general $s \in (0,1)$ one may define the fractional Sobolev space

$$H^s(\partial\Omega) := \{u \in L^2(\partial\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{d}{2}+s}} \in L^2(\partial\Omega \times \partial\Omega)\}$$

Its dual is denoted $H^{-s}(\partial\Omega)$. For more information on fractional Sobolev spaces, the interested reader is directed to [9].

1.2 Maxwell's Equations

This thesis will focus primarily on the modeling and analysis of propagation of electromagnetic waves through specific kinds of waveguides. Such wave propagation is governed by the well known Maxwell system of partial differential equations, which, in the absence of point charges and currents, is given by:

$$\begin{aligned} \nabla \cdot D &= 0 \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\tfrac{\partial B}{\partial t} \\ \nabla \times H &= \tfrac{\partial D}{\partial t}. \end{aligned} \tag{1.1}$$

Here $E, B : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ represent the electric and magnetic fields, while D and H represent the electric displacement field and magnetizing field respectively. In order to reduce this system to a solvable one, one needs equations which relate the electric field E to the electric displacement D as well as the magnetic field B to the magnetizing field H. For linear, isotropic, non-dispersive materials (which will be the kind considered in this thesis), the constitutive relations are given by

$$\begin{aligned} D &= \varepsilon E \\ H &= \tfrac{1}{\mu} B, \end{aligned} \tag{1.2}$$

where $\varepsilon, \mu \in \mathbb{C}$ are the *electric permittivity* and *magnetic permeability* of the material respectively. Substituting (1.2) into (1.1) yields the following system of PDE

for E and H:

$$\begin{aligned} \nabla \cdot E &= 0 \\ \nabla \cdot H &= 0 \\ \nabla \times E &= -\mu \tfrac{\partial H}{\partial t} \\ \nabla \times H &= \varepsilon \tfrac{\partial E}{\partial t}. \end{aligned} \tag{1.3}$$

This system of first-order partial differential equations can be transformed into a single second-order partial differential equation for both the H and E fields, through careful manipulation. To obtain the equation for H, we proceed as follows: first, apply the curl operator to both sides of the fourth equation in (1.3) to obtain

$$\nabla \times (\varepsilon^{-1} \nabla \times H) = \nabla \times \frac{\partial H}{\partial t}. \tag{1.4}$$

Using the vector identity

$$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - \nabla \cdot (\nabla F),$$

and using the second equation of (1.3), equation (1.4) becomes

$$\mu \frac{\partial^2 H}{\partial t^2} = \nabla \cdot (\varepsilon^{-1} \nabla H). \tag{1.5}$$

Equation (1.5) is known as a *wave equation*. One can obtain a similar wave equation for the electric field following the same process:

$$\varepsilon \frac{\partial^2 E}{\partial t^2} = \nabla \cdot (\mu^{-1} \nabla E). \tag{1.6}$$

The above two equations for E and H contain both time- and spacial-second-order derivatives. To eliminate the time-dependence, a natural approach is to consider time-harmonic solutions, i.e. solutions of the form

$$\begin{aligned} E(x,t) &= E(x) e^{-i\omega t} \\ H(x,t) &= H(x) e^{-i\omega t}, \end{aligned} \tag{1.7}$$

where $\omega \in \mathbb{R}$ is called the frequency of the wave. Applying (1.7) to (1.3), (1.5), and (1.6), one obtains the following Helmholtz-type equations for H and E,

$$\nabla \cdot (\varepsilon^{-1} \nabla H) = \mu \omega^2 H, \tag{1.8}$$

$$\nabla \cdot (\mu^{-1} \nabla E) = \varepsilon \omega^2 E, \tag{1.9}$$

together with the following differential relation between the two fields:

$$\begin{aligned} \nabla \times E &= i\mu\omega H \\ \nabla \times H &= -i\varepsilon\omega E \end{aligned} \tag{1.10}$$

In the sequel, materials with unitary magnetic permiability μ will be considered. In this case, equation (1.8) is in fact an eigenvalue problem for the operator $\nabla \cdot (\varepsilon^{-1} \nabla)$; it is this perspective which will guide and motivate the analysis that follows.

Chapter 2
Introduction

2.1 Two Physical Examples

The goal of this dissertation is the spectral analysis of Bloch wave modes corresponding to periodic, 2nd-order differential operators of divergence form. The periodic coefficient of these operators will be taken to be very large in one, connected (though not typically bounded) subdomain of $\mathbb{R}^d$, $d = 2, 3$ and unitary in the rest of Euclidean space. To motivate the study of the spectral behavior of these operators, we consider two important models in acoustic and optical materials.

2.1.1 Photonic Crystals

A *photonic crystal* is a periodic array of materials which acts as a waveguide for electromagnetic waves. Though the term "photonic crystal" was not coined until the mid 1980's, periodic dielectric arrays were analyzed as early as the late nineteenth century by Lord Rayleigh (see [30]). In that study, Lord Rayleigh discovered that periodic multi-lyaer dieletric stacks exhibited a (one-dimensional) photonic *band gap*, i.e. a range of frequencies whose propagation through the array is greatly inhibited (also sometimes called a *stop-band*). The study of photonic crystals began to grow rapidly after the publication of two major papers by E. Yablonovich and S. John, which discussed three-dimensional photonic crystals, their band structure, and possible applications of their stop-bands to controlling spontaneous emission by atoms ([32]) and localisation of light ([14])

Since then, photonic crystals have seen development in a variety of designs and applications. In the mid 90's, Figotin and Kuchment produced several papers on

thin-walled photonic crystals, which consist of walled cubic lattices surrounding a low-permittivity material. Indeed, the existence of band-gaps was explicitly proven in [20], [21], [22] for thin-walled lattices as the permittivity of the wall and the reciprocal of the width of the walls tend to infinity. Three-phase periodic media are considered by authors Chen and Lipton in [7]. In that paper, the authors developed a power series expansion to recover dispersion relations for the three-phase material; these dispersion relations rigorously demonstrated backwards wave propagation (i.e. double-negative behavior) across certain frequency intervals. Yet another kind of photonic crystal structure has been analyzed by Hempel and Lienau. In their 2000 paper [13], the authors considered a crystal composed of a connected "host" phase which into which an "inclusion" phase is periodically embedded. There, the authors rigorously prove the existence of band gaps as the ratio of permittivities between the host and the inclusion tends to infinity, which is sometimes called the "high-contrast limit", for a wide class of (fixed) inclusion geometries. This dissertation will focus on crystals of the type considered by [13].

When considering Maxwell's equations in a two-dimensional photonic crystal, it is often helpful to consider specific kinds of electromagnetic modes where the electric and magnetic fields are in some sense perpendicular; these kinds of modes are called *transverse modes*. Since this thesis concerns materials with unitary magnetic permiability, transverse modes with the magnetic field propagating out of the "plane of periodicity" of the crystal will be considered. For clarity, the plane of periodicity will be identified with the standard real plane $\mathbb{R}^2$, while the magnetic field will consist only of a "$\vec{e_3}$" component, i.e.

$$\begin{aligned} H(x) &= h(x_1, x_2)\vec{e_3}, \\ E(x) &= \tfrac{i}{\varepsilon\omega}\nabla \times H(x), \end{aligned} \tag{2.1}$$

for some scalar function h defined on $\mathbb{R}^2$. Applying this representation of H to equation (1.8) with $\mu = 1$ reveals the following eigenvalue problem defined on $\mathbb{R}^2$:

$$\nabla \cdot (\varepsilon^{-1}(x)\nabla h(x)) = \omega^2 h(x). \tag{2.2}$$

2.1.2 Acoustics

Similar to a photonic crystal, an acoustic crystal is a periodic waveguide for acoustic waves (as opposed to electromagnetic waves). The differential equation which governs acoustic wave propogation is given by

$$\Delta p - \frac{1}{c}\frac{\partial^2 p}{\partial t^2} = 0, \tag{2.3}$$

where p is pressure, $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ is the Laplace operator, and c is the speed of sound through the medium; this equation is known as the *acoustic wave equation.* Just as one can consider a *photonic* crystal made of a "host" and "inclusion" phase with widely different perimttivities ε_1 and ε_2 arrayed periodically, one may similarly consider a periodic *acoustic* crystal consisting of two materials with drastically different speeds of sound c_1 and c_2, making the coefficient c^{-1} periodic in $\mathbb{R}^3$. Additionally, the consideration of time-harmonic solutions

$$p(x,t) = p(x)e^{i\omega t} \tag{2.4}$$

reveals another scalar-valued Helmholtz equation

$$c(x)\Delta p(x) + \omega^2 p(x) = 0 \tag{2.5}$$

2.2 Periodicity and the Floquet Transform

In both (2.2) and (2.5), the spectrum of the 2nd-order differential operator $\nabla \cdot (\varepsilon^{-1}(x)\nabla) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ (resp. $c(x)\Delta : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$) is continuous, which creates some difficulty in the analysis of Helmholtz equation. However, for

crystals considered in this paper, the material coefficient will be $\mathbb{Z}^d$-periodic in $\mathbb{R}^d$, $(d = 2, 3)$, which allows for a convenient decomposition of the spectrum of the operator on $L^2(\mathbb{R}^d)$ into a union of spectra over smaller, more well-behaved function spaces of Bloch-waves, i.e. functions $u \in L^2(\mathbb{R}^d)$ such that, for some $\alpha \in (-\pi, \pi]^d$ and some $\mathbb{Z}^d$-periodic function $\psi(x)$ one has

$$u(x) = \psi(x)e^{i\alpha\cdot x} \tag{2.6}$$

In each of these Bloch-wave spaces, the inverse of $\nabla \cdot (\varepsilon^{-1}(x)\nabla)$ (resp. $c(x)\Delta$) becomes a compact operator, so that its spectrum (and hence the spectrum of the original operator) becomes discrete.

To that end, Bloch waves on a given unit cell $Y = (0, 1]^d$ will be defined, as well as the Floquet Transform, which will act as a map between the spaces $L^2_{\#}(\alpha, Y)$ of Bloch waves and $L^2(\mathbb{R}^d)$.

2.2.1 The Floquet Transform

Let $f \in L^2(\mathbb{R}^d)$, and define the *Floquet Transform* of f by

$$U[f](x, \alpha) = \sum_{n\in\mathbb{Z}^d} f(x - n)e^{i\alpha\cdot n} \tag{2.7}$$

The Fourier-like variable α is typically referred to as the *quasi-periodicity* or *quasi-momentum*. Shifting x by $k \in \mathbb{Z}^d$ reveals the quasi-periodicity condition

$$U[f](x + k, \alpha) = e^{i\alpha\cdot k}U[f](x, \alpha) \tag{2.8}$$

This shows that $\varphi(x) := U[f](x, \alpha)e^{-i\alpha\cdot x}$ is periodic, so that $U[f](x, \alpha)$ is determined completely by its behavior on the unit cell $Y = (0, 1]^d$ and thus belongs to the space

$$L^2(\alpha, Y) = \{u \in L^2(Y) : u(x)e^{-i\alpha\cdot x} \text{ is } Y\text{-periodic}\} \tag{2.9}$$

of L^2 Bloch-waves with quasi-momentum α. As with $L^2(Y)$, this is a a Hilbert Space under the usual L^2 inner product $(\cdot, \cdot)$. Moreover, $U[f](x, \alpha)$ is periodic with

respect to the quasi-momentum α, i.e. for $k \in \mathbb{Z}^d$ one has

$$U[f](x, \alpha + 2\pi k) = U[f](x, \alpha) \tag{2.10}$$

Thus, in order to obtain complete information on $U[f](x, \alpha)$, one needs only consider α in the domain $B = (-\pi, \pi]^2$, hereafter referred to as the (first) Brillouin zone.

Suppose that B and the measure $d\alpha$ on B are both normalized. Then we have the following theorem, which acts as an analogue of the well known Plancherel theorem for the Fourier Transform.

Theorem 2. *The Floquet Transform*

$$U : L^2(\mathbb{R}^d) \to L^2(B, L^2(Y)) \tag{2.11}$$

is an isometry. Its inverse is given by

$$U^{-1}[g](x) = \int_B g(x, \alpha) d\alpha, \tag{2.12}$$

where $g(x, \alpha) \in L^2(B, L^2(Y))$ is extended from Y to $\mathbb{R}^d$ by condition (2.8)

There are many references which provide greater detail on the nature and structure of the Floquet Transform; the interested reader is directed to [16], [31].

A fascinating feature feature of the Floquet Transform is its interactions with linear differential operators with $\mathbb{Z}^d$-periodic coefficients. As a relevant example, consider the differential operator $L_\varepsilon := \nabla \cdot (\varepsilon^{-1}(x)\nabla)$ found in equation (2.2). Considering $L_\varepsilon : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, it is easy to see that the singular continuous spectrum of L_ε is empty. Though the question of the presence of eigenvalues (elements in the point spectrum) remains an open one, many physicists are in agreement that the point spectrum of L_ε on $L^2(\mathbb{R}^2)$ should be empty as well: though widely believed, this statement has been shown to be very difficult to prove (see [23]).

However, there is a way to decompose the operator L_ε on $L^2(\mathbb{R}^2)$ into a family of operators (each with discrete spectrum) acting on smaller domains of Bloch Waves. First, one notes that, due to the periodicity of the coefficient $\varepsilon^{-1}(x)$, the operator L_ε commutes with the Floquet Transform. Indeed, for any $f \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned} U[L_\varepsilon f](x,\alpha) &= \sum_{n\in\mathbb{Z}^2} \nabla \cdot (\varepsilon^{-1}(x-n)\nabla f(x-n))e^{i\alpha\cdot n} \\ &= \sum_{n\in\mathbb{Z}^2} \nabla \cdot (\varepsilon^{-1}(x)\nabla f(x-n))e^{i\alpha\cdot n} \\ &= \nabla \cdot (\varepsilon^{-1}(x)\nabla \sum_{n\in\mathbb{Z}^2} f(x-n)e^{i\alpha\cdot n}) \\ &= L_\varepsilon U[f](x,\alpha) \end{aligned} \tag{2.13}$$

A similar proof works for any differential operator L on $L^2(\mathbb{R}^d)$ with $\mathbb{Z}^d$-periodic coefficients for $d = 2, 3$. in Thus, for fixed $\alpha \in B$, the a periodic operator L can be thought of as acting only on those functions $f \in L^2(\alpha, Y)$. Denoting this operator restricted to α-Bloch Waves by

$$L^\alpha : L^2(\alpha, Y) \to L^2(\alpha, Y) \tag{2.14}$$

and making use of the Plancherel-type theorem 2, we see that the Floquet Transform expands L into a direct integral of operators

$$\int_B^{\oplus} L^\alpha d\alpha. \tag{2.15}$$

For symmetric L with real coefficients (e.g. those considered in (2.2) and (2.5)), L will be a self-adjoint operator: in that case, one can prove that the decomposition (2.15) yields a similar decomposition for the spectrum of L:

$$\sigma(L) = \bigcup_{\alpha\in B} \sigma(L^\alpha) \tag{2.16}$$

This allows the spectral problems (2.2) and (2.5), originally posed on $L^2(\mathbb{R}^d)$, to instead be posed on the smaller spaces $L^2(\alpha, Y)$. Once the problem has been solved

on $L^2(\alpha, Y)$ for each $\alpha \in B$, one needs only take the union given by (2.16) to recover the solution to the original problem. In light of this result, this dissertation will focus entirely on solutions to (2.2) belonging to $L^2(\alpha, Y)$ for α in the first Brillioun zone.

2.3 Layer Potentials and Boundary Value Problems for the Laplacian

For the kinds of crystals considered in this dissertation, the material coefficient will be $\mathbb{Z}^d$-periodic (as discussed above) as well as piecewise-constant, taking on a value $k^{-1} \ll 1$ in the connected phase of the material while being unitary in the "inclusion" phase D, which will consist of a disjoint union of bounded, separated domains periodically arranged (with $\mathbb{Z}^d$-periodicity) in $\mathbb{R}^d$. As a preliminary step to understanding the spectral properties of the operators L^α coming from the differential operators in (2.2) and (2.5), it will be necessary to obtain information on the boundary value problem

$$\begin{aligned} &\Delta u = 0 && \text{in } D, Y \setminus \overline{D} \\ &u|_+ = u|_- && \text{on } \partial D \\ &u \text{ is } \alpha - \text{quasiperiodic in } Y \end{aligned} \tag{2.17}$$

Here, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator (also known as the Laplacian), and

$$u|_\pm(x) = \lim_{t \to 0^+} u(x \pm t\nu_x) \tag{2.18}$$

where $x \in \partial D$ and ν_x is the *outward* pointing normal vector at x on ∂D. The key to unlocking the nature of solutions to this problem will revolve around the theory of layer potentials, which are integral operators defined for functions on ∂D whose kernels are intrinsically related to Poisson's equation

$$\Delta u = f \tag{2.19}$$

for suitable boundary conditions prescribed to u. For now, the boundary values for u will be ignored, and the theory of layer potentials will be introduced for the free-space Laplacian (i.e. the Laplace operator for functions on $\mathbb{R}^d$).

2.3.1 The Newtonian Potential

For $d \geq 2$, we define the Newtonian Potential on $\mathbb{R}^d$ by

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} log|x| & d = 2 \\ \frac{1}{(2-d)\omega_d}|x|^{d-2} & d \geq 2 \end{cases}, \tag{2.20}$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$. $\Gamma(x)$ is sometimes known as the *fundamental solution* of the Laplacian, as it satisfies the equation

$$\Delta\Gamma(x) = \delta(x), \tag{2.21}$$

where $\delta(x)$ is the Dirac-delta distribution. This distribution is characterized by the following identity for $\varphi \in C_0^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \delta(x)\varphi(x)dx = \varphi(0) \tag{2.22}$$

We define the following integral operator for $f \in L^2(\mathbb{R}^d)$:

$$L[f](x) := \int_{\mathbb{R}^d} \Gamma(x-y)f(y)dy \tag{2.23}$$

From equations (2.21) and (2.22), and since L is a convolution operator, one readily checks that $L[f]$ satisfies

$$\begin{aligned} \Delta L[f](x) &= \int_{\mathbb{R}^d} \Delta\Gamma(x-y)f(y)dy \\ &= \int_{\mathbb{R}^d} \delta(x-y)f(y)dy \\ &= f(x) \end{aligned} \tag{2.24}$$

for $x \in \mathbb{R}^d$. Thus we may view L as the inverse of the Laplace operator, i.e. $L = \Delta^{-1}$, in the sense that, if we would like to solve Poisson's equation (2.19) for

u given $f \in L^2(\mathbb{R}^d)$, we may simply apply L to f to obtain the solution

$$u(x) = \Delta^{-1}[f](x) = \int_{\mathbb{R}^d} \Gamma(x-y)f(y)dy \tag{2.25}$$

There are many valuable sources which expound further on the fundamental solution $\Gamma(x)$; the interested reader is directed to e.g. [11], [10].

2.3.2 Layer Potentials

Consider a (bounded, locally connected and locally path-connected) domain $\Omega \subset \mathbb{R}^d$, and suppose that Ω has finitely many connected components $\Omega_1, \Omega_2, \ldots, \Omega_n$. Let $\rho \in L^2(\partial\Omega)$, and let $\Gamma(x)$ be the Newtonian Potential defined in the previous section. The single-layer potential of ρ is defined by the boundary integral formula

$$S_\Omega[\rho](x) := \int_{\partial\Omega} \Gamma(x-y)\rho(y)d\sigma(y),\ x \in \mathbb{R}^d. \tag{2.26}$$

Similarly, the double-layer potential of ρ is defined

$$D_\Omega[\rho](x) := \int_{\partial\Omega} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\rho(y)d\sigma(y),\ x \in \mathbb{R}^d \setminus \partial\Omega, \tag{2.27}$$

where ν_y is the outward pointing normal vector to y on $\partial\Omega$. As a consequence of the choice of kernels of the integral operators in (2.26) and (2.27), one can readily check that, if $u = S_\Omega[\rho]$ and $v = D_\Omega[\rho]$, then

$$\Delta u = \Delta v = 0 \text{ in } \Omega, \mathbb{R}^d \setminus \overline{\Omega} \tag{2.28}$$

Another equally important property of the layer potential operators is their behavior along $\partial\Omega$. First, define the following operator from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$:

$$K_\Omega[\rho](x) := \text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\rho(y)d\sigma(y),\ x \in \partial\Omega, \tag{2.29}$$

and its adjoint

$$(K_\Omega)^*[\rho](x) := \text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial \nu_x}\Gamma(y-x)\rho(y)d\sigma(y),\ x \in \partial\Omega. \tag{2.30}$$

Here, p.v. denotes the Cauchy principal value. The operators defined in (2.29) and (2.30) are known to be bounded on $L^2(\partial\Omega)$. Indeed, for smooth enough boundaries ($C^{1,\gamma}$ for $\gamma \in (0,1)$), these operators are in fact compact. Proofs of compactness of these operators for $C^{1,\gamma}$-class domains can be found in [25]; a somewhat simpler proof for the case of C^2 boundaries can be found in [11].

The following *jump relations* along $\partial\Omega$ hold for the layer potentials (2.26) and (2.27):

Lemma 3. *If Ω is a domain with smooth enough boundary, then for every $\rho \in L^2(\partial\Omega)$ and almost every $x \in \partial\Omega$,*

$$\begin{aligned} S_\Omega[\rho]|_+(x) &= S_\Omega[\rho]|_-(x) \\ D_\Omega[\rho]|_\pm(x) &= (\mp I + K_\Omega)[\rho](x) \\ \tfrac{\partial}{\partial\nu} S_\Omega[\rho]|_\pm(x) &= (\pm I + (K_\Omega)^*)[\rho](x) \end{aligned} \tag{2.31}$$

Thus one can see that the trace of double-layer potentials onto $\partial\Omega$, as well as the normal derivatives of single-layer potentials on $\partial\Omega$, are not continuous *across* $\partial\Omega$. Proofs for these jump relations can again be found in [11] (for C^2 boundaries) or [25] (for $C^{1,\gamma}$ boundaries).

Chapter 3

Quasi-Periodic Resonances and Bloch Wave Band Structure

Consider a Bloch wave $h(x)$ with Bloch eigenvalue ω^2 propagating through a two or three dimensional crystal lattice characterized by the periodic coefficient $a(x) = a(x+p)$, $p \in \mathbb{Z}^d$, $d = 2, 3$, with unit cell $Y = (0, 1]^d$. The Bloch wave satisfies the differential equation,

$$-\nabla \cdot (a(x)\nabla h(x)) = \omega^2 h(x), \ x \in \mathbb{R}^d, \, d = 2, 3 \tag{3.1}$$

together with the α quasi-periodicity condition $h(x+p) = h(x)e^{i\alpha\cdot p}$. Here α lies in the first Brillouin zone of the reciprocal lattice given by $Y^\star = (-\pi, \pi]^d$. As discussed in previous chapters, Equation (3.1) describes transverse magnetic (TM) wave propagation through a two dimensional photonic crystal.

We examine Bloch wave propagation through high contrast crystals made from periodic configurations of two materials. One material occupies disjoint inclusions and is completely contained within each period cell and surrounded by the second material.The coefficient is taken to be 1 inside the inclusions and $k > 0$ outside.

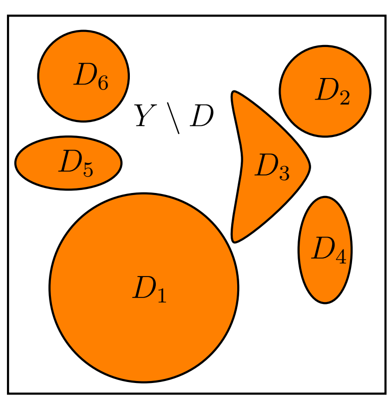

Figure 3.1: **Period Cell.**

The domain occupied by the union of all the inclusions $D_1, D_2, \ldots, D_n$ inside Y is denoted by D, see figure 3.1. The coefficient is specified on the unit period cell by $a(x) = (k\chi_{Y\setminus D}(x) + \chi_D(x))$ where χ_D and $\chi_{Y\setminus D}$ are indicator functions for the sets D and $Y \setminus D$ and are extended by periodicity to $\mathbb{R}^d$. In this paper we consider periodic crystals made from finite collections of separated inclusions each with $C^{1,\gamma}$ boundary.

To proceed we complexify the problem and consider $k \in \mathbb{C}$. Now $a(x)$ takes on complex values inside $Y \setminus D$ and the divergence form operator $-\nabla\cdot(k\chi_{Y\setminus D}+\chi_D)\nabla$ is no longer uniformly elliptic. Our approach develops an explicit representation formula for $-\nabla\cdot(k\chi_{Y\setminus D}+\chi_D)\nabla$ that holds for complex values of k. We identify the subset $z = 1/k \in \Omega_0$ of $\mathbb{C}$ where this operator is invertible. The explicit formula shows that the solution operator $(-\nabla\cdot(k\chi_{Y\setminus D}+\chi_D)\nabla)^{-1}$ may be regarded more generally as a meromorphic operator valued function of z for $z \in \Omega_0 = \mathbb{C} \setminus S$, see section 4.1 and Lemma 10. Here the set S is discrete and consists of poles lying on the negative real axis with only one accumulation point at $z = -1$. For the problem treated here we expand about $z = 0$ and the distance between $z = 0$ and the set S is used to bound the radius of convergence for the power series. The spectral representation for $-\nabla\cdot(k\chi_{Y\setminus D}+\chi_D)\nabla$ follows from the existence of a complete orthonormal set of quasi-periodic functions associated with the *quasi-periodic resonances of the crystal*, i.e., quasi periodic functions v and real eigenvalues λ for which

$$-\nabla\cdot(\chi_D)\nabla v = -\lambda\Delta v. \tag{3.2}$$

These resonances are shown to be connected to the spectra of Neumann-Poincaré operators associated with quasi-periodic double layer potentials. For $\alpha = 0$ these are the well known electrostatic resonances identified in [5], [4], [27], and [26]. Both

Neumann-Poincaré operators and associated electrostatic resonances have been the focus of theoretical investigations [15], [28] and applied in analysis of plasmonic excitations for suspensions of noble metal particles [24] and electrostatic breakdown [3]. The explicit spectral representation for the operator $-\nabla \cdot (k\chi_{Y\setminus D} + \chi_D)\nabla$ is crucial for elucidating the interaction between the contrast k and the quasi-periodic resonances of the crystal, see (3.39), (3.36), and (3.37).

The chapter is organized as follows: In the next section we introduce the Hilbert space formulation of the problem and the variational formulation of the quasi-static resonance problem. The completeness of the eigenfunctions associated with the quasi-static spectrum is established and a spectral representation for the operator $-\nabla \cdot (k\chi_{Y\setminus D} + \chi_D)\nabla$ is obtained. These results are collected and used to continue the frequency band structure into the complex plane, see Theorem 9 of section 3.2. Spectral perturbation theory [17] is applied to recover the power series expansion for Bloch spectra in section 4.1. The leading order spectral theory is developed for quasi-periodic $\alpha \neq 0$ and periodic $\alpha = 0$ problems in sections 4.2 and 4.3. The main theorems on radius of convergence and separation of spectra given by Theorems 13 and 14 are presented in section 5.1. The class of buffered inclusions is introduced in section 5.2 and the explicit radii of convergence for a random suspension of buffered disks is presented in section 5.3. Explicit formulas for each term of the power series expansion is recovered and expressed in terms of layer potentials in section 6.1. In section 6.2 the explicit formula for the first order correction in the power series is presented in the form of the Dirichlet energy of the solution of a transmission boundary value problem. This formula follows from the layer potential representation for the first term and agrees with the first order correction obtained in the work of [1]. The explicit formulas for the convergence radii are derived in

section 7 as well as hands on proofs of Theorems 13 and 14 and the explicit error estimates for the series truncated after N terms.

3.1 Hilbert space setting, quasi-periodic resonances and representation formulas

We denote the spaces of all α quasi-periodic complex valued functions belonging to $L^2_{loc}(\mathbb{R}^d)$ by $L^2_{\#}(\alpha, Y)$ and the L^2 inner product over Y is written

$$(u, v) = \int_Y u\bar{v}\, dx. \tag{3.3}$$

For $\alpha \neq 0$ the eigenfunctions h for (3.1) belong to the space

$$H^1_{\#}(\alpha, Y) = \{h \in H^1_{loc}(\mathbb{R}^d) : h \text{ is } \alpha \text{ quasiperiodic}\}. \tag{3.4}$$

The space $H^1_{\#}(\alpha, Y)$ is a Hilbert space under the inner product

$$\langle u, v\rangle = \int_Y \nabla u(x) \cdot \nabla \bar{v}(x) dx. \tag{3.5}$$

When $\alpha = 0$, the pair $h(x) = 1$, $\omega^2 = 0$ is a solution to (3.1). For this case the remaining eigenfunctions associated with nonzero eigenvalues are orthogonal to 1 in the $L^2(Y)$ inner product. These eigenfunctions are periodic and belong to $L^2_{loc}(\mathbb{R}^d)$. The set of Y periodic functions with zero average over Y belonging to $L^2_{loc}(\mathbb{R}^d)$ is denoted by $L^2_{\#}(0, Y)$. The periodic eigenfunctions of (3.1) associated with nonzero eigenvalues belong to the space

$$H^1_{\#}(0, Y) = \{h \in H^1_{loc}(\mathbb{R}^d) : h \text{ is periodic}, \textstyle\int_Y h\, dx = 0\}. \tag{3.6}$$

The space $H^1_{\#}(0, Y)$ is also Hilbert space with the inner product $\langle u, v\rangle$ defined by (3.5).

For any $k \in \mathbb{C}$, the the variational formulation of the eigenvalue problem (3.1) for h and ω^2 is given by

$$B_k(h, v) = \omega^2(h, v) \tag{3.7}$$

for all v in $H^1_{\#}(\alpha, Y)$ where $B_k : H^1_{\#}(\alpha, Y) \times H^1_{\#}(\alpha, Y) \longrightarrow \mathbb{C}$ is the sesquilinear form

$$B_k(u, v) = k \int_{Y \setminus D} \nabla u(x) \cdot \nabla \bar{v}(x) dx + \int_D \nabla u(x) \cdot \nabla \bar{v}(x) dx. \tag{3.8}$$

The linear operator $T^\alpha_k : H^1_{\#}(\alpha, Y) \longrightarrow H^1_{\#}(\alpha, Y)$ associated with B_k is defined by

$$\langle T^\alpha_k u, v \rangle := B_k(u, v). \tag{3.9}$$

In what follows we decompose $H^1_{\#}(\alpha, Y)$ into invariant subspaces of source free modes and identify the associated quasi-periodic resonance spectra. This decomposition will provide an explicit spectral representation for the operator T^α_k, see Theorem 8. We first address the case $\alpha \in Y^\star \setminus \{0\}$. Let $W_1 \subset H^1_{\#}(\alpha, Y)$ be the completion in $H^1_{\#}(\alpha, Y)$ of the subspace of functions with support away from D, and let $W_2 \subset H^1_{\#}(\alpha, Y)$ be the subspace of functions in $H^1_0(D)$ extended by zero into Y. Clearly W_1 and W_2 are orthogonal subspaces of $H^1_{\#}(\alpha, Y)$, so define $W_3 := (W_1 \oplus W_2)^\perp$. We therefore have

$$H^1_{\#}(\alpha, Y) = W_1 \oplus W_2 \oplus W_3. \tag{3.10}$$

The orthogonal decomposition and integration by parts shows that elements $u \in W_3$ are harmonic separately in D and $Y \setminus D$.

Now consider $\alpha = 0$ and decompose $H^1_{\#}(0, Y)$. Let $W_1 \subset H^1_{\#}(0, Y)$ be the completion in $H^1_{\#}(0, Y)$ of the subspace of functions with support away from D. Here let $\tilde{H}^1_0(D)$ denote the subspace of functions $H^1_0(D)$ extended by zero into

$Y \setminus D$ and let 1_Y be the indicator function of Y. We define $W_2 \subset H^1_{\#}(0, Y)$ be the subspace of functions given by

$$W_2 = \{u = \tilde{u} - \left(\int_D \tilde{u} dx \right) 1_Y \ \mid \ \tilde{u} \in \tilde{H}^1_0(D)\}. \tag{3.11}$$

Clearly W_1 and W_2 are orthogonal subspaces of $H^1_{\#}(0, Y)$, and $W_3 := (W_1 \oplus W_2)^{\perp}$. As before we have

$$H^1_{\#}(0, Y) = W_1 \oplus W_2 \oplus W_3 \tag{3.12}$$

and W_3 is identified with the subspace of $H^1_{\#}(0, Y)$ functions that are harmonic inside D and $Y \setminus D$ respectively. The orthogonality between W_2 and W_3 follows from the identity $\int_{\partial D} \partial_n w \, ds = 0$ for $w \in W_3$. We summarize with the following observation.

Lemma 4. *For every $\alpha \in Y^{\star}$, if $u \in W_3$ then u is harmonic in $Y \setminus D$ and D separately.*

To set up the spectral analysis observe that Lemma 4, together with uniqueness of traces onto ∂D of functions in $H^1_{\#}(\alpha, Y)$ for $\alpha \in Y^*$, implies that elements of W_3 can be represented in terms of single layer potentials supported on ∂D. We introduce the d-dimensional α-quasi-periodic Green's function, $d = 2, 3$ given by, see, e.g., [2],

$$G^{\alpha}(x, y) = -\sum_{n \in \mathbb{Z}^d} \frac{e^{i(2\pi n + \alpha)\cdot(x-y)}}{|2\pi n + \alpha|^2} \text{ for } \alpha \neq 0 \tag{3.13}$$

and the periodic Green's function given by

$$G^0(x, y) = -\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{i2\pi n\cdot(x-y)}}{|2\pi n|^2} \text{ for } \alpha = 0. \tag{3.14}$$

In both cases $d = 2, 3$, one can show that $G^{\alpha}(x, y) - \Gamma(x - y)$ is harmonic is both $x-$ and $y-$harmonic in Y, where $\Gamma(x - y)$ is the free-space Newtonian Potential defined in (2.20) for any $\alpha \in Y^*$. For $\alpha \in Y^* \setminus \{0\}$, the α-quasi-periodic Green's

function $G^\alpha(x,y)$ acts as a fundamental solution of the Laplace operator, in the sense that

$$-\Delta_x G^\alpha(x,y) = \delta(x-y) \ ,x,y \in Y. \tag{3.15}$$

Note also that, for any quasi-periodic $f \in L^2(Y)$ with quasimomentum α, we have

$$u(x) := \int_Y G^\alpha(x,y) f(y) dy \ ,x \in Y \tag{3.16}$$

is in $H^1_\#(\alpha, Y)$. Thus, denoting the negative Laplacian acting on $H^1_\#(\alpha, Y)$ by $(-\Delta_\alpha)$, we may identify its inverse with the integral operator

$$(-\Delta_\alpha)^{-1} f(x) := u(x) := \int_Y G^\alpha(x,y) f(y) dy \ ,x \in Y \tag{3.17}$$

The periodic Green's function is not exactly a fundamental solution of the Laplace operator, but it is in some sense very close to one. Indeed, the $G^0(x,y)$ satisfies the following differential equation:

$$-\Delta_x G^0(x,y) = \delta(x-y) - 1_Y, \tag{3.18}$$

where 1_Y is the constant function on Y with value 1. For periodic, mean-zero functions $f \in L^2(Y)$, one still obtains the identity

$$-\Delta_x \int_Y G^0(x,y) f(y) dy = \int_Y \delta(x-y) f(y) dy = f(x) \tag{3.19}$$

since f is mean-zero on Y, and

$$u(x) := \int_Y G^0(x,y) f(y) dy \ ,x \in Y \tag{3.20}$$

is in $H^1_\#(0, Y)$ as well. Thus, as in the quas-periodic case, we identify the inverse of $-\Delta$ acting on $H^1_\#(0, Y)$, denoted $-\Delta_0$, with

$$(-\Delta_0)^{-1} f(x) := \int_Y G^0(x,y) f(y) dy \ ,x \in Y \tag{3.21}$$

Let $H^{1/2}(\partial D)$ be the fractional Sobelev space on ∂D defined in the usual way, and denote its dual by $(H^{1/2}(\partial D))^* = H^{-1/2}(\partial D)$. For $\rho \in H^{-1/2}(\partial D)$, and $\alpha \in Y^\star$ define the single layer potential

$$S_D[\rho](x) = \int_{\partial D} G^\alpha(x,y)\rho(y)d\sigma(y),\ x \in Y. \tag{3.22}$$

It follows from [8], that for any $\rho \in H^{-1/2}(\partial D)$

$$\begin{aligned} \Delta S_D\rho &= 0 \text{ in } D \text{ and } Y \setminus D, \\ S_D\rho \mid^+_{\partial D} &= S_D\rho \mid^-_{\partial D}, \\ \frac{\partial}{\partial \nu} S_D\rho \mid^\pm_{\partial D} &= \pm\frac{1}{2}\rho + (\tilde{K}_D^{-\alpha})^*\rho, \end{aligned} \tag{3.23}$$

where ν is the outward directed normal vector on ∂D and $(\tilde{K}_D^{-\alpha})^*$ is the Neumann Poincaré operator defined by

$$(\tilde{K}_D^{-\alpha})^*\rho(x) = \text{ p. v. } \int_{\partial D} \frac{\partial G^\alpha(x,y)}{\partial \nu(x)}\rho(y)d\sigma(y),\ x \in \partial D, \tag{3.24}$$

and $\tilde{K}_D^\alpha$ is the Neumann Poincaré operator

$$\tilde{K}_D^\alpha\rho(x) = \text{ p. v. } \int_{\partial D} \frac{\partial G^\alpha(y,x)}{\partial \nu(y)}\rho(y)d\sigma(y),\ x \in \partial D. \tag{3.25}$$

Before continuing, note that while the Layer Potentials defined in (2.26) and (2.27) were defined as integral operators for L^2 densities ρ on the boundary of the domain, the layer potentials defined above for the present problem are defined for densities in $H^{-1/2}(\partial D)$. The primary reason for this adjustment is to represent the trace of every function in W_3 as the image of a single layer potential. In order to obtain this one-to-one correspondence, it is necessary to widen the domain of S_D to the larger domain $H^{-1/2}(\partial D)$; indeed, if we consider $S_{\partial D} := S_D|_{\partial D}$ as a map on $L^2(\partial D)$, then it can be shown that the domain of its inverse is $H^1(\partial D)$, which is smaller than the target space of $H^{1/2}(\partial D)$ ([2]). That the domain of $S_{\partial D}$ can be

widened from $L^2(\partial D)$ to $H^{-1/2}(\partial D)$ follows from the identity

$$(S_{\partial D}\rho, \rho)_{L^2(\partial D)} = \|S_D\rho\|^2_{H^1_\#(\alpha,Y)} \tag{3.26}$$

A complete proof of this result can be found in [28].

In what follows we assume the boundary ∂D is $C^{1,\gamma}$, for some $\gamma > 0$. Here the layer potentials $\tilde{K}^\alpha_D$, and $(\tilde{K}^{-\alpha}_D)^*$ are continuous linear mappings from $L^2(\partial D)$ to $L^2(\partial D)$ and compact, since $\frac{\partial G^\alpha(x,y)}{\partial\nu(x)}$ is a continuous kernel of order $d-2$ in dimensions $d = 2, 3$ (since $G^\alpha(x, y) - \Gamma(x - y)$ is smooth). The operator S_D is a continuous linear map from $H^{-1/2}(\partial D)$ into $H^1_\#(\alpha, Y)$ and we define $S_{\partial D}\rho = S_D\rho\,|_{\partial D}$ for all $\rho \in H^{-1/2}(\partial D)$. Here $S_{\partial D} : H^{-1/2}(\partial D) \longrightarrow H^{1/2}(\partial D)$ is continuous and invertible, see [8].

One readily verifies the symmetry

$$G^\alpha(x, y) = G^{-\alpha}(y, x), \tag{3.27}$$

and application delivers the Plemelj symmetry for $\tilde{K}^{-\alpha}$, $(\tilde{K}^{-\alpha})^*$ and $S_{\partial D}$ as operators on $L^2(\partial D)$ given by

$$\tilde{K}^{-\alpha}S_{\partial D} = S_{\partial D}(\tilde{K}^{-\alpha})^*. \tag{3.28}$$

Moreover as seen in [28] the operator $-S_{\partial D}$ is positive and selfadjoint in $L^2(\partial D)$ and in view of (3.28) $(\tilde{K}^{-\alpha}_D)^*$ is a compact operator on $H^{-1/2}(\partial D)$.

Let $G : W_3 \longrightarrow H^{1/2}(\partial D)$ be the trace operator, which is bounded and onto.

Lemma 5. *$S_D : H^{-1/2}(\partial D) \longrightarrow W_3$ is a one-to-one, bounded linear map with bounded inverse $S_D^{-1} = S_{\partial D}^{-1}G$.*

Proof. Let $\rho \in H^{-1/2}(\partial D)$, and set $f = S_D\rho$. Then by the first equation of (3.23), f is harmonic in D and $Y \setminus D$ separately, and so for any $v_1 \in W_1$ and $v_2 \in W_2$ we have

$$\langle f, v_1\rangle = 0 = \langle f, v_2\rangle. \tag{3.29}$$

But $W_3 = (W_1 \oplus W_2)^\perp$, so $f = S_D\rho \in W_3$ for every $\rho \in H^{-1/2}(\partial D)$.

Now suppose $u \in W_3$, and consider $Gu = u\mid_{\partial D} \in H^{1/2}(\partial D)$. For all $x \in Y$ define $w(x) = S_D(S_{\partial D}^{-1}Gu)$. Since $u, w \in W_3$, it follows that $w - u \in W_3$ as well. Since $Gu = Gw$, we have that $G(w-u) = 0$, and so $w - u \in (W_1 \oplus W_2)$. But $W_3 = (W_1 \oplus W_2)^\perp$, so $w = u$ as desired. □

We introduce an auxiliary operator $T : W_3 \longrightarrow W_3$, given by the sesquilinear form

$$\langle Tu, v\rangle = \frac{1}{2}\int_{Y\setminus D} \nabla u(x) \cdot \nabla \bar{v}(x)dx - \frac{1}{2}\int_D \nabla u(x) \cdot \nabla \bar{v}(x)dx. \tag{3.30}$$

The next theorem will be useful for the spectral decomposition of T_k^α and in the proof of Theorem 15.

Theorem 6. *The linear map T defined in equation* (3.30) *is given by*

$$T = S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}$$

and is compact and self-adjoint.

Proof. For $u, v \in W_3$, consider

$$\langle S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u, v\rangle = \int_Y \nabla[S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u] \cdot \nabla\bar{v}. \tag{3.31}$$

Since $\Delta S_D\rho = 0$ in D and $Y \setminus D$ for any $\rho \in H^{-1/2}(\partial D)$, an integration by parts yields

$$\langle S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u, v\rangle = \int_{\partial D} \bar{v}\Big(\frac{\partial[S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u]}{\partial\nu}\Big|_{\partial D}^{-} - \frac{\partial[S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u]}{\partial\nu}\Big|_{\partial D}^{+}\Big)d\sigma.$$

Applying the jump conditions from (3.23) yields

$$\langle S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}u, v\rangle = -\int_{\partial D} (\tilde{K}_D^{-\alpha})^* S_D^{-1}u\bar{v}d\sigma. \tag{3.32}$$

Note that by the same jump conditions

$$(\tilde{K}_D^{-\alpha})^* S_D^{-1} u = \frac{1}{2}\left(\frac{\partial u}{\partial \nu}\Big|_{\partial D}^{-} + \frac{\partial u}{\partial \nu}\Big|_{\partial D}^{+}\right). \tag{3.33}$$

Application of (3.33) to equation (3.32) and an integration by parts yields the desired result. Compactness follows directly from the properties of S_D and $(\tilde{K}^{-\alpha})^*$.

□

Rearranging terms in the weak formulation of (3.2) and writing $\mu = 1/2 - \lambda$ delivers the equivalent eigenvalue problem for quasi-periodic electrostatic resonances.

$$\langle Tu, v\rangle = \mu\langle u, v\rangle,\ u, v \in W_3.$$

Since T is compact and self adjoint on W_3, there exists a countable subset $\{\mu_i\}_{i\in\mathbb{N}}$ of the real line with a single accumulation point at 0 and an associated family of orthogonal finite-dimensional projections $\{P_{\mu_i}\}_{i\in\mathbb{N}}$ such that

$$\langle \sum_{i=1}^{\infty} P_{\mu_i} u, v\rangle = \langle u, v\rangle,\ u, v \in W_3$$

and

$$\langle \sum_{i=1}^{\infty} \mu_i P_{\mu_i} u, v\rangle = \langle Tu, v\rangle,\ u, v \in W_3.$$

Moreover, it is clear by (3.30) that

$$-\frac{1}{2} \leq \mu_i \leq \frac{1}{2}.$$

The upper bound $1/2$ is the eigenvalue associated with the eigenfunction $\Pi \in H^1_\#(\alpha, Y)$ such that $\Pi = 1$ in D and is harmonic on $Y \setminus D$. In section 5.2 an explicit lower bound μ^- is identified such that the inequality $-1/2 < \mu^- \leq \mu_i$, holds for a generic class of geometries uniformly with respect to $\alpha \in Y^\star$.

Lemma 7. *The eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ of T are precisely the eigenvalues of the Neumann-Poincaré operator $(\tilde{K}_D^{-\alpha})^*$ associated with quasi-periodic double layer potential restricted to ∂D.*

Proof. If a pair (μ, u) belonging to $(-1/2, 1/2] \times W_3$ satisfies $Tu = \mu u$ then $S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1} u = \mu u$. Multiplication of both sides by S_D^{-1} shows that $S_D^{-1}u$ is an eigenfunction for function for $(\tilde{K}_D^{-\alpha})^*$ associated with μ. Suppose the pair (μ, w) belongs to $(-1/2, 1/2] \times H^{-1/2}(\partial D)$ and satisfies $(\tilde{K}_D^{-\alpha})^* w = \mu w$. Since the trace map from W_3 to $H^{1/2}(\partial D)$ is onto then there is a u in W_3 for which $w = S_D^{-1}u$ and $(\tilde{K}_D^{-\alpha})^* S_D^{-1} u = \mu S_D^{-1} u$. Multiplication of this identity by S_D shows that u is an eigenfunction for T associated with μ. $\square$

Finally, we see that if $u_1 \in W_1$ and $u_2 \in W_2$, then

$$\langle Tu_1, v\rangle = \frac{1}{2}\langle u_1, v\rangle,$$
$$\langle Tu_2, v\rangle = -\frac{1}{2}\langle u_2, v\rangle$$

for all $v \in H^1_\#(\alpha, Y)$.

Let Q_1, Q_2 be the orthogonal projections of $H^1_\#(\alpha, Y)$ onto W_1 and W_2 respectively, and define $P_1 := Q_1 + P_{1/2}$, $P_2 := Q_2$. Here $P_{1/2}$ is the projection onto the one dimensional subspace spanned by the function $\Pi \in H^1_\#(\alpha, Y)$. Then $\{P_1, P_2\} \cup \{P_{\mu_i}\}_{-\frac{1}{2}<\mu_i<\frac{1}{2}}$ is an orthogonal family of projections, and

$$\langle P_1 u + P_2 u + \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} P_{\mu_i} u, v\rangle = \langle u, v\rangle$$

for all $u, v \in H^1_\#(\alpha, Y)$.

We now recover the spectral decomposition for T_k^α associated with the sesqualinear form (3.9).

Theorem 8. *The linear operator* $T_k^\alpha : H^1_\#(\alpha, Y) \longrightarrow H^1_\#(\alpha, Y)$ *associated with the sesqualinear form* B_k *is is given by*

$$\langle T_k^\alpha u, v\rangle = \langle kP_1 u + P_2 u + \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} [k(1/2+\mu_i) + (1/2-\mu_i)]P_{\mu_i} u, v\rangle$$

for all $u, v \in H^1_\#(\alpha, Y)$.

Proof. For $u, v \in H^1_\#(\alpha, Y)$ we have

$$B_k(P_{\mu_i} u, v) = k\int_{Y\setminus D} \nabla P_{\mu_i} u \cdot \nabla \bar{v} + \int_D \nabla P_{\mu_i} u \cdot \nabla \bar{v}.$$

Since $P_{\mu_i} u$ is an eigenvector corresponding to $\mu_i \neq \pm\frac{1}{2}$, we have

$$\int_{Y\setminus D} \nabla P_{\mu_i} u \cdot \nabla \bar{v} = \frac{(1/2+\mu_i)}{(1/2-\mu_i)} \int_D \nabla P_{\mu_i} u \cdot \nabla \bar{v}$$

and so we calculate

$$B_k(P_{\mu_i} u, v) = [k\frac{(1/2+\mu_i)}{(1/2-\mu_i)} + 1] \int_D \nabla P_{\mu_i} u \cdot \nabla \bar{v}.$$

But we also know that

$$\int_D \nabla P_{\mu_i} u \cdot \nabla \bar{v} = (1/2-\mu_i) \int_Y \nabla P_{\mu_i} u \cdot \nabla \bar{v}$$

and so

$$B_k(P_{\mu_i} u, v) = [k(1/2+\mu_i) + (1/2-\mu_i)] \int_Y \nabla P_{\mu_i} u \cdot \nabla \bar{v}.$$

Since we clearly have

$$B_k(P_1 u, v) = k\int_{Y\setminus D} \nabla P_1 u \cdot \nabla \bar{v},$$

$$B_k(P_2 u, v) = \int_D \nabla P_2 u \cdot \nabla \bar{v},$$

and the projections P_1, P_2, P_{μ_i} are mutually orthogonal for all $-\frac{1}{2} < \mu_i < \frac{1}{2}$, the proof is complete. □

It is evident that $T_k^\alpha : H^1_\#(\alpha, Y) \longrightarrow H^1_\#(\alpha, Y)$ is invertible whenever

$$k \in \mathbb{C} \setminus Z \text{ where } Z = \{\frac{\mu_i - 1/2}{\mu_i + 1/2}\}_{\{-\frac{1}{2} \leq \mu_i \leq \frac{1}{2}\}} \tag{3.34}$$

and for $z = k^{-1}$,

$$(T_k^\alpha)^{-1} = zP_1 u + P_2 u + \sum_{-\frac{1}{2} < \mu_i < \frac{1}{2}} z[(1/2 + \mu_i) + z(1/2 - \mu_i)]^{-1} P_{\mu_i}. \tag{3.35}$$

For future reference we also introduce the set S of $z \in \mathbb{C}$ for which T_k^α is not invertible given by

$$S = \{\frac{\mu_i + 1/2}{\mu_i - 1/2}\}_{\{-\frac{1}{2} < \mu_i < \frac{1}{2}\}} \tag{3.36}$$

which also lies on the negative real axis. In section 5.2 we will provide explicit upper bounds on S that depend upon the geometry of the inclusions.

Collecting results, the spectral representation of the operator $-\nabla \cdot (k\chi_{Y \setminus D} + \chi_D)\nabla$ on $H^1_\#(\alpha, Y)$ is given by

$$-\nabla \cdot (k\chi_{Y \setminus D} + \chi_D)\nabla = -\Delta_\alpha T_k^\alpha, \tag{3.37}$$

in the sense of linear functionals over the space $H^1_\#(\alpha, Y)$. Here $-\Delta_\alpha$ is the Laplace operator associated with the bilinear form $\langle \cdot, \cdot \rangle$ defined on $H^1_\#(\alpha, Y)$. This formulation is useful since it separates the effect of the contrast k from the underlying geometry of the crystal.

3.2 Band Structure for Complex Coupling Constant

We set $\omega^2 = \lambda$ in (3.1) and extend the Bloch eigenvalue problem to complex coefficients k outside the set Z given by (3.34). The operator representation is applied to write the Bloch eigenvalue problem as

$$-\nabla \cdot (k\chi_{Y \setminus D} + \chi_D)\nabla u = -\Delta_\alpha T_k^\alpha u = \lambda u. \tag{3.38}$$

We characterize the Bloch spectra by analyzing the operator

$$B^{\alpha}(k) = (T_k^{\alpha})^{-1}(-\Delta_{\alpha})^{-1}, \tag{3.39}$$

where the operator $(-\Delta_{\alpha})^{-1}$ defined for all $\alpha \in Y^*$ is given by

$$(-\Delta_{\alpha})^{-1}u(x) = -\int_Y G^{\alpha}(x,y)u(y)\,dy. \tag{3.40}$$

The operator $B^{\alpha}(k) : L^2_{\#}(\alpha, Y) \longrightarrow H^1_{\#}(\alpha, Y)$ is easily seen to be bounded for $k \notin Z$, see Theorem 28. Since $H^1_{\#}(\alpha, Y)$ embeds compactly into $L^2_{\#}(\alpha, Y)$ we find by virtue of Poincare's inequality that $B^{\alpha}(k)$ is a bounded compact linear operator on $L^2_{\#}(\alpha, Y)$ and therefore has a discrete spectrum $\{\gamma_i(k,\alpha)\}_{i\in\mathbb{N}}$ with a possible accumulation point at 0, see Remark 29. The corresponding eigenspaces are finite dimensional and the eigenfunctions $p_i \in L^2_{\#}(\alpha, Y)$ satisfy

$$B^{\alpha}(k)p_i(x) = \gamma_i(k,\alpha)p_i(x) \text{ for } x \text{ in } Y \tag{3.41}$$

and also belong to $H^1_{\#}(\alpha, Y)$. Note further for $\gamma_i \neq 0$ that (3.41) holds if and only if (3.38) holds with $\lambda_i(k,\alpha) = \gamma_i^{-1}(k,\alpha)$, and $-\Delta_{\alpha}T_k^{\alpha}p_i = \lambda_i(k,\alpha)p_i$. Collecting results we have the following theorem

Theorem 9. *Let Z denote the set of points on the negative real axis defined by* (3.34). *Then the Bloch eigenvalue problem* (3.1) *for the operator $-\nabla(k\chi_{Y\setminus D}+\chi_D)\nabla$ associated with the sesquilinear form* (3.8) *can be extended for values of the coupling constant k off the positive real axis into $\mathbb{C} \setminus Z$, i.e., for each $\alpha \in Y^{\star}$ the Bloch eigenvalues are of finite multiplicity and denoted by $\lambda_j(k,\alpha) = \gamma_j^{-1}(k,\alpha)$, $j \in \mathbb{N}$ and the band structure*

$$\lambda_j(k,\alpha) = \omega^2,\ j \in \mathbb{N} \tag{3.42}$$

extends to complex coupling constants $k \in \mathbb{C} \setminus Z$.

Chapter 4

Power Series Expansion of Bloch Eigenvalues

Now that the band structure of the Bloch eigenvalues of the photonic crystal has been identified, the next goal is to obtain a power series representation of the eigenvalues $\gamma_j(k,\alpha), j \in \mathbb{N}$ in equation (3.41) in reciprocal powers of the high contrast parameter k. In order to accomplish this goal, we will use the expansion (3.39) of operator $(T_k^\alpha)^{-1}$ in terms of $z = k^{-1}$ and develop a power series expansion in terms of z using standard perturbation theory, see e.g. [18], [19], [17].

4.1 Power Series Representation of Bloch Eigenvalues for High Contrast Periodic Media

In what follows we set $\gamma = \lambda^{-1}(k,\alpha)$ and analyze the spectral problem

$$B^\alpha(k)u = \gamma(k,\alpha)u \tag{4.1}$$

Henceforth we will analyze the high contrast limit by by developing a power series in $z = \frac{1}{k}$ about $z = 0$ for the spectrum of the family of operators associated with (4.1).

$$\begin{aligned}
B^\alpha(k) &:= (T_k^\alpha)^{-1}(-\Delta_\alpha)^{-1} \\
&= (zP_1 + P_2 + z\textstyle\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}}[(1/2+\mu_i) + z(1/2-\mu_i)]^{-1}P_{\mu_i})(-\Delta_\alpha)^{-1} \\
&= A^\alpha(z).
\end{aligned}$$

Here we define the operator $A^\alpha(z)$ such that $A^\alpha(1/k) = B^\alpha(k)$ and the associated eigenvalues $\beta(1/k,\alpha) = \gamma(k,\alpha)$ and the spectral problem is $A^\alpha(z)u = \beta(z,\alpha)u$ for $u \in L^2_\#(\alpha, Y)$.

It is easily seen from the above representation that $A^\alpha(z)$ is self-adjoint for $k \in \mathbb{R}$ and is a family of bounded operators taking $L^2_\#(\alpha, Y)$ into itself and we have the following:

Lemma 10. *$A^\alpha(z)$ is holomorphic on $\Omega_0 := \mathbb{C} \setminus S$. Where $S = \cup_{i\in\mathbb{N}} z_i$ is the collection of points $z_i = (\mu_i + 1/2)/(\mu_i - 1/2)$ on the negative real axis associated with the eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$. The set S consists of poles of $A^\alpha(z)$ with only one accumulation point at $z = -1$.*

In the sections 5.2 and 5.3 we develop explicit lower bounds $-1/2 < \mu^- \leq \mu^-(\alpha) = \min_i\{\mu_i\}$, that hold for generic classes of inclusion domains D and for every $\alpha \in Y^\star$. The corresponding upper bound z^+ on S is written

$$\max_i\{z_i\} = \frac{\mu^-(\alpha) + 1/2}{\mu^-(\alpha) - 1/2} = z^* \leq z^+ < 0. \tag{4.2}$$

Let $\beta_0^\alpha \in \sigma(A^\alpha(0))$ with spectral projection $P(0)$, and let Γ be a closed contour in $\mathbb{C}$ enclosing β_0^α but no other $\beta \in \sigma(A^\alpha(0))$. The spectral projection associated with $\beta^\alpha(z) \in \sigma(A^\alpha(z))$ for $\beta^\alpha(z) \in \mathrm{int}(\Gamma)$ is denoted by $P(z)$. We write $M(z) = P(z)L^2_\#(\alpha, Y)$ and suppose for the moment that Γ lies in the resolvent of $A^\alpha(z)$ and $\dim(M(0)) = \dim(M(z)) = m$, noting that Theorems 13 and 14 provide explicit conditions for when this holds true. Now define $\hat{\beta}^\alpha(z) = \frac{1}{m}\mathrm{tr}(A^\alpha(z)P(z))$, the weighted mean of the eigenvalue group $\{\beta_1^\alpha(z), \ldots \beta_m^\alpha(z)\}$ corresponding to $\beta_1^\alpha(0) = \ldots = \beta_m^\alpha(0) = \beta_0^\alpha$. We write the weighted mean as

$$\hat{\beta}^\alpha(z) = \beta_0^\alpha + \frac{1}{m}\mathrm{tr}[(A^\alpha(z) - \beta_0^\alpha)P(z)]. \tag{4.3}$$

Since $A^\alpha(z)$ is analytic in a neighborhood of the origin we write

$$A^\alpha(z) = A^\alpha(0) + \sum_{n=1}^\infty z^n A_n^\alpha. \tag{4.4}$$

The explicit form of the sequence $\{A^\alpha_n\}_{n\in\mathbb{N}}$ is given in Section 5.1. Define the resolvent of $A^\alpha(z)$ by

$$R(\zeta, z) = (A^\alpha(z) - \zeta)^{-1},$$

and expanding successively in Neumann series and power series we have the identity

$$\begin{aligned} R(\zeta, z) &= R(\zeta, 0)[I + (A^\alpha(z) - A^\alpha(0))R(\zeta, 0)]^{-1} \\ &= R(\zeta, 0) + \textstyle\sum_{p=1}^{\infty}[-(A^\alpha(z) - A^\alpha(0))R(\zeta, 0)]^p \\ &= R(\zeta, 0) + \textstyle\sum_{n=1}^{\infty} z^n R_n(\zeta), \end{aligned} \tag{4.5}$$

where

$$R_n(\zeta) = \sum_{k_1+\ldots k_p=n, k_j\geq 1} (-1)^p R(\zeta, 0)A^\alpha_{k_1} R(\zeta, 0)A^\alpha_{k_2} \ldots R(\zeta, 0)A^\alpha_{k_p}$$

for $n \geq 1$.

Application of the contour integral formula for spectral projections [29], [18], [19] delivers the expansion for the spectral projection

$$\begin{aligned} P(z) &= -\tfrac{1}{2\pi i} \textstyle\oint_\Gamma R(\zeta, z)d\zeta \\ &= P(0) + \textstyle\sum_{n=1}^{\infty} z^n P_n \end{aligned} \tag{4.6}$$

where $P_n = -\frac{1}{2\pi i} \oint_\Gamma R_n(\zeta)d\zeta$. Now we develop the series for the weighted mean of the eigenvalue group. Start with

$$(A^\alpha(z) - \beta^\alpha_0)R(\zeta, z) = I + (\zeta - \beta^\alpha_0)R(\zeta, z) \tag{4.7}$$

and we have

$$(A^\alpha(z) - \beta^\alpha_0)P(z) = -\frac{1}{2\pi i} \oint_\Gamma (\zeta - \beta^\alpha_0)R(\zeta, z)d\zeta, \tag{4.8}$$

so

$$\hat{\beta}(z) - \beta_0^\alpha = -\frac{1}{2m\pi i}\mathrm{tr}\oint_\Gamma (\zeta - \beta_0^\alpha) R(\zeta, z) d\zeta. \tag{4.9}$$

Equation (4.9) delivers an analytic representation formula for a Bloch eigenvalue or more generally the eigenvalue group when β_0^α is not a simple eigenvalue. Substituting the third line of (4.5) into (4.9) and manipulation yields

$$\hat{\beta}^\alpha(z) = \beta_0^\alpha + \sum_{n=1}^{\infty} z^n \beta_n^\alpha, \tag{4.10}$$

where

$$\beta_n^\alpha = -\frac{1}{2m\pi i}\mathrm{tr}\sum_{k_1+\cdots+k_p=n} \frac{(-1)^p}{p}\oint_\Gamma A_{k_1}^\alpha R(\zeta,0) A_{k_2}^\alpha \ldots R(\zeta,0) A_{k_p}^\alpha R(\zeta,0) d\zeta;\ n \geq 1. \tag{4.11}$$

4.2 Spectrum in the High Contrast Limit: Quasi-periodic Case

We now identify the spectrum of the limiting operator $A^\alpha(0)$ when $\alpha \neq 0$. Using the representation

$$A^\alpha(z) = (zP_1 + P_2 + z\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} [(1/2+\mu_i) + z(1/2-\mu_i)]P_{\mu_i})(-\Delta_\alpha)^{-1}, \tag{4.12}$$

we see that

$$A^\alpha(0) = P_2(-\Delta_\alpha)^{-1}. \tag{4.13}$$

Denote the spectrum of $A^\alpha(0)$ by $\sigma(A^\alpha(0))$. The following theorem provides the explicit characterization of $\sigma(A^\alpha(0))$.

Theorem 11. *Let $-\Delta_D$ be the negative Laplacian with zero Dirichlet boundary conditions on ∂D with inverse $-\Delta_D^{-1} : L^2(D) \to L^2(D)$. Denote the spectrum of $-\Delta_D^{-1}$ by $\sigma(-\Delta_D^{-1})$. Then $\sigma(A^\alpha(0)) = \sigma(-\Delta_D^{-1})$.*

To establish the theorem we first show that the eigenvalue problem

$$P_2(-\Delta_\alpha)^{-1}u = \lambda u$$

with $\lambda \in \sigma(A^\alpha(0))$ and eigenfunction $u \in L^2_\#(\alpha, Y)$ is equivalent to finding λ and $u \in W_2$ for which

$$(u, v)_{L^2(Y)} = \lambda \langle u, v\rangle, \text{ for all } v \in W_2. \tag{4.14}$$

To conclude we will then show that the set of eigenvalues for (4.14) is given by $\sigma(-\Delta_D^{-1})$. To see the equivalence note that we have $u = P_2 u$ and for $v \in H^1_\#(\alpha, Y)$,

$$\langle P_2(-\Delta_\alpha)^{-1}u, v\rangle = \lambda\langle u, v\rangle = \lambda\langle P_2 u, v\rangle \tag{4.15}$$

hence

$$\langle(-\Delta_\alpha)^{-1}u, P_2 v\rangle = \lambda\langle u, P_2 v\rangle. \tag{4.16}$$

Since $\langle(-\Delta_\alpha)^{-1}u, v\rangle = \int_Y u\overline{v}\,dx = (u, v)_{L^2(Y)}$ for any $u \in L^2_\#(\alpha, Y)$ and $v \in H^1_\#(\alpha, Y)$, equation (4.16) becomes

$$(u, P_2 v)_{L^2(Y)} = \lambda\langle u, P_2 v\rangle, \tag{4.17}$$

and the equivalence follows noting that P_2 is the projection of $H^1_\#(\alpha, Y)$ onto W_2.

To conclude we show that the set of eigenvalues for (4.14) is given by $\sigma(-\Delta_D^{-1})$. Note that $P_2 v$ is supported in D, so

$$\lambda^{-1}\int_D u\overline{P_2 v} = \int_D \nabla u \cdot \nabla\overline{P_2 v}. \tag{4.18}$$

Now since $P_2 : H^1_\#(\alpha, Y) \to W_2 = \tilde{H}^1_0(D)$ is onto, it follows that λ^{-1} is a Dirichlet eigenvalue of the negative Laplacian acting on D and the proof of Theorem 11 is complete.

4.3 Spectrum in the High Contrast Limit: Periodic Case

Recall for the periodic case P_2 is the projection onto W_2 given by (3.11) and the limiting operator $A^0(0)$ is written

$$A^0(0) = P_2(-\Delta_0)^{-1}. \tag{4.19}$$

Here the operator $(-\Delta_0)^{-1}$ is compact and self-adjoint on $L^2_{\#}(0,Y)$ and given by

$$(-\Delta_0)^{-1}u(x) = -\int_Y G^0(x,y)u(y)dy. \tag{4.20}$$

Denote the spectrum of $A^0(0)$ by $\sigma(A^0(0))$. To characterize this spectrum we introduce the sequence of numbers $\{\nu_j\}_{j\in\mathbb{N}}$ given by the positive roots ν of the spectral function $S(\nu)$ defined by

$$S(\nu) = \nu \sum_{i\in\mathbb{N}} \frac{a_i^2}{\nu - \delta_i^*} - 1, \tag{4.21}$$

where $\{\delta_j^*\}_{j\in\mathbb{N}}$ are the Dirichlet eigenvalues for $-\Delta_D$ associated with eigenfunctions ψ_j for which $\int_D \psi_j\, dx \neq 0$ and $a_j = |\int_D \psi_j\, dx|$. The following theorem provides the explicit characterization of $\sigma(A^\alpha(0))$.

Theorem 12. *Let $\{\delta_j'\}_{j\in\mathbb{N}}$ denote the collection of Dirichlet eigenvalues for $-\Delta_D$ associated with eigenfunctions ψ_j for which $\int_D \psi_j\, dx = 0$. Then $\sigma(A^0(0)) = \{\delta_j'^{-1}\}_{j\in\mathbb{N}} \cup \{\nu_j^{-1}\}_{j\in\mathbb{N}}$.*

To establish the theorem we proceed as before to see that the eigenvalue problem

$$P_2(-\Delta_0)^{-1}u = \lambda u$$

with $\lambda \in \sigma(A^0(0))$ and eigenfunction $u \in L^2_{\#}(0,Y)$ is equivalent to finding λ and $u \in W_2$ for which

$$(u,v)_{L^2(Y)} = \lambda\langle u,v\rangle, \text{ for all } v \in W_2. \tag{4.22}$$

To conclude we show that the set of eigenvalues for (4.22) is given by $\{\delta_j'^{-1}\}_{j\in\mathbb{N}} \cup \{\nu_j^{-1}\}_{j\in\mathbb{N}}$. We see that $u \in W_2$ and from (3.11) we have the dichotomy: $\int_D \tilde{u}dx = 0$ and $u = \tilde{u} \in \tilde{H}_0^1(D)$ or $\int_D \tilde{u}dx \neq 0$ and $u = \tilde{u} - \gamma 1_Y$ with $\gamma = \int_D \tilde{u}dx$. It is evident for the first case that the eigenfunction $u \in \tilde{H}_0^1(D)$ and for $v \in W_2$ given by

$$v = \tilde{v} - \left(\int_D \tilde{v}dx\right) 1_Y \text{ for } \tilde{v} \in \tilde{H}_0^1(D) \tag{4.23}$$

the problem (4.22) becomes

$$\int_D u\overline{\tilde{v}} = \lambda \int_D \nabla u \cdot \nabla \overline{\tilde{v}}, \text{ for all } \tilde{v} \in \tilde{H}_0^1(D), \tag{4.24}$$

and we conclude that $\tilde{u}$ is a Dirichlet eigenfunction with zero average over D so $\lambda \in \{\delta_j'^{-1}\}_{j\in\mathbb{N}}$. While for the second, we have $u \in W_2$ and again

$$\int_D u\overline{\tilde{v}} = \lambda \int_D \nabla u \cdot \nabla \overline{\tilde{v}}, \text{ for all } \tilde{v} \in \tilde{H}_0^1(D). \tag{4.25}$$

Writing $u = \tilde{u} - \gamma 1_Y$ and integration by parts in (4.25) shows that $\tilde{u} \in \tilde{H}_0^1(D)$ is the solution of

$$\Delta\tilde{u} + \nu\tilde{u} = -\nu\gamma \text{ for } x \in D. \tag{4.26}$$

We normalize $\tilde{u}$ so that $\gamma = \int_D \tilde{u}dx = 1$ and write

$$\tilde{u} = \sum_{j=1}^{\infty} c_j \psi_j \tag{4.27}$$

where, ψ_j are the Dirichlet eigenfunctions of $-\Delta_D$ associated with eigenvalue δ_j extended by zero to Y. Substitution of (4.27) into (4.26) gives

$$\sum_{j=1}^{\infty} (-\delta_j + \nu) c_j \psi_j = -\nu. \tag{4.28}$$

Multiplying both sides of (4.28) by $\overline{\psi_k}$ over Y and orthonormality of $\{\psi_j\}_{j\in\mathbb{N}}$, shows that $\tilde{u}$ is given by

$$\tilde{u} = \nu \sum_{k\in\mathbb{N}} \frac{\int_D \overline{\psi_k}}{\nu - \delta_k^*} \psi_k, \tag{4.29}$$

where δ_k^* correspond to Dirichlet eigenvalues associated with eigenfunctions for which $\int_D \psi_k \, dx \neq 0$. To calculate ν, we integrate both sides of (4.29) over D to recover the identity

$$\nu \sum_{k \in \mathbb{N}} \frac{a_k^2}{\nu - \delta_k^*} - 1 = 0. \tag{4.30}$$

It follows from (4.30) that $\lambda \in \{\nu_i^{-1}\}_{i \in \mathbb{N}}$ and the proof of Theorem 12 is complete.

Chapter 5

Radius of Convergence and Separation of Bloch Spectra

With the power series expansion being identified in chapter 4, it remains to determine when, exactly, this power series converges. In this chapter, the radius of convergence for the power series (4.10) is given in terms of the quasi-momentum α and a lower bound on the quasi-periodic resonances described in Lemma 7. The lower bound on the resonances plays a critical role in the existence of the radius of convergence, and it is proven to exist for a large class of "buffered geometries." A specific geometry of randomly placed, buffered discs is considered in section 5.3, and a precise lower bound on the radius of convergence for (4.10) is calculated for that geometry.

5.1 Radius of Convergence and Separation of Spectra

Fix an inclusion geometry specified by the domain D. Suppose first $\alpha \in Y^\star$ and $\alpha \neq 0$. Recall from Theorem 11 that the spectrum of $A^\alpha(0)$ is $\sigma(-\Delta_D^{-1})$. Take Γ to be a closed contour in $\mathbb{C}$ containing an eigenvalue $\beta_j^\alpha(0)$ in $\sigma(-\Delta_D^{-1})$ but no other element of $\sigma(-\Delta_D^{-1})$, see Figure 5.1. Define d to be the distance between Γ and $\sigma(-\Delta_D^{-1})$, i.e.,

$$d = \operatorname{dist}(\Gamma, \sigma(-\Delta_D^{-1}) = \inf_{\zeta \in \Gamma}\{\operatorname{dist}(\zeta, \sigma(-\Delta_D^{-1})\}. \tag{5.1}$$

The component of the spectrum of $A^\alpha(0)$ inside Γ is precisely $\beta_j^\alpha(0)$ and we denote this by $\Sigma'(0)$. The part of the spectrum of $A^\alpha(0)$ in the domain exterior to Γ is denoted by $\Sigma''(0)$ and $\Sigma''(0) = \sigma(-\Delta_D^{-1}) \setminus \beta_j^\alpha(0)$. The invariant subspace of $A^\alpha(0)$ associated with $\Sigma'(0)$ is denoted by $M'(0)$ with $M'(0) = P(0)L^2_\#(\alpha, Y)$.

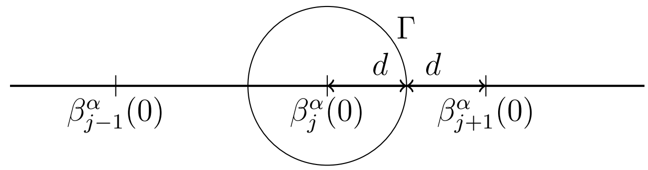

Figure 5.1: Γ

Suppose the lowest quasi-periodic resonance eigenvalue for the domain D lies inside $-1/2 < \mu^-(\alpha) < 0$. It is noted that in the sequel a large and generic class of domains are identified for which $-1/2 < \mu^-(\alpha)$. The corresponding upper bound on the set $z \in S$ for which $A^\alpha(z)$ is not invertible is given by

$$z^* = \frac{\mu^-(\alpha) + 1/2}{\mu^-(\alpha) - 1/2} < 0, \tag{5.2}$$

see (4.2). Now set

$$r^* = \frac{|\alpha|^2 d |z^*|}{\frac{1}{1/2 - \mu^-} + |\alpha|^2 d}. \tag{5.3}$$

Theorem 13. Separation of spectra and radius of convergence for $\alpha \in Y^\star$, $\alpha \neq 0$.
The following properties hold for inclusions with domains D that satisfy (5.2)*:*

1. *If $|z| < r^*$ then Γ lies in the resolvent of both $A^\alpha(0)$ and $A^\alpha(z)$ and thus separates the spectrum of $A^\alpha(z)$ into two parts given by the component of spectrum of $A^\alpha(z)$ inside Γ denoted by $\Sigma'(z)$ and the component exterior to Γ denoted by $\Sigma''(z)$. The invariant subspace of $A^\alpha(z)$ associated with $\Sigma'(z)$ is denoted by $M'(z)$ with $M'(z) = P(z)L^2_\#(\alpha, Y)$.*

2. *The projection $P(z)$ is holomorphic for $|z| < r^*$ and $P(z)$ is given by*

$$P(z) = \frac{-1}{2\pi i} \oint_\Gamma R(\zeta, z)\, d\zeta. \tag{5.4}$$

3. *The spaces $M'(z)$ and $M'(0)$ are isomorphic for $|z| < r^*$.*

4. *The power series* (4.10) *converges uniformly for $z \in \mathbb{C}$ inside $|z| < r^*$.*

Suppose now $\alpha = 0$. Recall from Theorem 12 that the limit spectrum for $A^0(0)$ is $\sigma(A^0(0)) = \{{\delta'_j}^{-1}\}_{j\in\mathbb{N}} \cup \{{\nu_j}^{-1}\}_{j\in\mathbb{N}}$. For this case take Γ to be the closed contour in $\mathbb{C}$ containing an eigenvalue $\beta_j^0(0)$ in $\sigma(A^0(0))$ but no other element of $\sigma(A^0(0))$ and define

$$d = \inf_{\zeta\in\Gamma}\{\mathrm{dist}(\zeta, \sigma(\mathrm{A}^0(0)))\}. \tag{5.5}$$

Suppose the lowest quasi-periodic resonance eigenvalue for the domain D lies inside $-1/2 < \mu^-(0) < 0$ and the corresponding upper bound on S is given by

$$z^* = \frac{\mu^-(0) + 1/2}{\mu^-(0) - 1/2} < 0. \tag{5.6}$$

Set

$$r^* = \frac{4\pi^2 d|z^*|}{\frac{1}{1/2-\mu^-} + 4\pi^2 d}. \tag{5.7}$$

Theorem 14. Separation of spectra and radius of convergence for $\alpha = 0$.

The following properties hold for inclusions with domains D that satisfy (5.6):

1. *If $|z| < r^*$ then Γ lies in the resolvent of both $A^0(0)$ and $A^0(z)$ and thus separates the spectrum of $A^0(z)$ into two parts given by the component of spectrum of $A^0(z)$ inside Γ denoted by $\Sigma'(z)$ and the component exterior to Γ denoted by $\Sigma''(z)$. The invariant subspace of $A^0(z)$ associated with $\Sigma'(z)$ is denoted by $M'(z)$ with $M'(z) = P(z)L^2_{\#}(\alpha, Y)$.*

2. *The projection $P(z)$ is holomorphic for $|z| < r^*$ and $P(z)$ is given by*

$$P(z) = \frac{-1}{2\pi i}\oint_\Gamma R(\zeta, z)\, d\zeta. \tag{5.8}$$

3. *The spaces $M'(z)$ and $M'(0)$ are isomorphic for $|z| < r^*$.*

4. *The power series (4.10) converges uniformly for $z \in \mathbb{C}$ inside $|z| < r^*$.*

Next we provide an explicit representation of the integral operators appearing in the series expansion for the eigenvalue group.

Theorem 15. Representation of integral operators in the series expansion for eigenvalues

Let P_3 be the projection onto the orthogonal complement of $W_1 \oplus W_2 \oplus \mathrm{span}\{\Pi\}$ and let I denote the identity on $L^2(\partial D)$, then the explicit representation for for the operators A_n^α in the expansion (4.10), (4.11) *is given by*

$$A_1^\alpha = [S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1}S_D^{-1}P_3 + P_1](-\Delta_\alpha)^{-1} \text{ and}$$
$$A_n^\alpha = S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1}S_D^{-1}[S_D((\tilde{K}_D^{-\alpha})^* - \frac{1}{2}I)((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1}S_D^{-1}]^{n-1}P_3(-\Delta_\alpha)^{-1}. \tag{5.9}$$

We have a corollary to Theorems 13 and 14 regarding the error incurred when only finitely many terms of the series 4.10 are calculated.

Theorem 16. Error estimates for the eigenvalue expansion.

1. *Let $\alpha \neq 0$, and suppose D, z^*, and r^* are as in Theorem 13. Then the following error estimate for the series* (4.10) *holds for $|z| < r^*$:*
$$\left|\hat{\beta}^\alpha(z) - \sum_{n=0}^{p} z^n \beta_n^\alpha\right| \leq \frac{d|z|^{p+1}}{(r^*)^p(r^* - |z|)}. \tag{5.10}$$
2. *Let $\alpha = 0$, and suppose D, z^*, and r^* are as in Theorem 14. Then the following error estimate for the series* (4.10) *holds for $|z| < r^*$:*
$$\left|\hat{\beta}^0(z) - \sum_{n=0}^{p} z^n \beta_n^0\right| \leq \frac{d|z|^{p+1}}{(r^*)^p(r^* - |z|)}. \tag{5.11}$$

We summarize results in the following theorem.

Theorem 17. *The Bloch eigenvalue problem* (3.1) *is defined for the coupling constant k extended into the complex plane and the operator $-\nabla \cdot (k\chi_{Y\setminus D} + \chi_D)\nabla$ with domain $H^1_\#(\alpha, Y)$ is holomorphic for $k \in \mathbb{C} \setminus Z$. The associated Bloch spectra is given by the eigenvalues $\lambda_j(k, \alpha) = (\beta_j^\alpha(1/k))^{-1}$, for $j \in \mathbb{N}$. For $\alpha \in Y^\star$ fixed, the eigenvalues are of finite multiplicity. Moreover for each j and $\alpha \in Y^\star$, the eigenvalue group is analytic within a neighborhood of infinity containing the disk $|k| > r^{*-1}$ where r^* is given by* (5.3) *for $\alpha \neq 0$ and by* (5.7) *for $\alpha = 0$.*

The proofs of Theorems 13, 14 and 16 are given in section 7. The proof of Theorem 15 is given in section 6.1.

5.2 Radius of Convergence and Separation of Spectra for Periodic Scatterers of General Shape

We start by identifying an explicit condition on the inclusion geometry that guarantees a lower bound μ^- on the quasi-periodic spectra that holds uniformly for $\alpha \in Y^*$, i.e., $-\frac{1}{2} < \mu^- \leq \mu^-(\alpha) = \min_i\{\mu_i\} \leq \frac{1}{2}$.

Let $D \Subset Y$ be a union of simply connected sets (inclusions) D_i, $i = 1, \dots, N$ with C^2 boundary. Recall that, for any eigenpair (μ, w) of $T|_{W_3}$ and all $v \in H^1_\#(\alpha, Y)$,

$$\frac{1}{2}\int_{Y\setminus D} \nabla w \cdot \nabla \bar{v} - \frac{1}{2}\int_{D} \nabla w \cdot \nabla \bar{v} = \mu \int_{Y} \nabla w \cdot \nabla \bar{v}. \tag{5.12}$$

Adding $\frac{1}{2}\int_Y \nabla w \cdot \nabla \bar{v}$ to both sides yields

$$\int_{Y\setminus D} \nabla w \cdot \nabla \bar{v} = (\mu + \frac{1}{2}) \int_{Y} \nabla w \cdot \nabla \bar{v}. \tag{5.13}$$

We will show that there exists a $\rho > 0$ such that $\mu_i + \frac{1}{2} > \rho$ independent of $i \in \mathbb{N}$ and $\alpha \in Y^*$. If such a ρ exists, then clearly $\mu_i > \rho - \frac{1}{2}$ for all i and α, providing an explicit lower bound $\mu^- = \rho - \frac{1}{2}$ satisfying the desired inequality.

Theorem 18. *Let $\mu^-(\alpha)$ be the lowest eigenvalue of T in $W_3 \subset H^1_{\#}(\alpha, Y)$. Suppose there is a $\theta > 0$ such that for all $u \in W_3$ we have*

$$\|\nabla u\|^2_{L^2(Y\setminus D)} \geq \theta \|\nabla u\|^2_{L^2(D)}. \tag{5.14}$$

Let $\rho = \min\{\frac{1}{2}, \frac{\theta}{2}\}$. Then $\mu^-(\alpha) + \frac{1}{2} > \rho$ for all $\alpha \in Y^*$.

Proof. We proceed by contradiction: suppose that $\mu^-(\alpha)+\frac{1}{2} < \frac{1}{2}$ and $\mu^-(\alpha)+\frac{1}{2} < \frac{\theta}{2}$. Let u^- be the normalized eigenvector of T associated with $\mu^-(\alpha)$. Then we have

$$\int\limits_{Y\setminus D} |\nabla u^-|^2 < \frac{1}{2} \tag{5.15}$$

and

$$\frac{\theta}{2} > \int\limits_{Y\setminus D} |\nabla u^-|^2 \geq \theta \int\limits_{D} |\nabla u^-|^2. \tag{5.16}$$

Thus we have

$$\int\limits_{D} |\nabla u^-|^2 < \frac{1}{2}. \tag{5.17}$$

Inequalities (5.15) and (5.17) yield

$$\|\nabla u^-\|^2_{L^2(Y)} < 1.$$

But u^- was normalized so that

$$\|\nabla u^-\|^2_{L^2(Y)} = 1,$$

completing the proof. $\square$

Clearly the parameter θ is a geometric descriptor for D. The class of periodic distributions of inclusions for which Theorem (18) holds for a fixed positive value of θ is denoted by P_θ and we have the corollary given by:

Corollary 19. *For every inclusion domain D belonging to P_θ Theorems 14 through 17 hold with z^* replaced with z_θ^+ given by*

$$z_\theta^+ = \frac{\mu^- + 1/2}{\mu^- - 1/2} < 0, \tag{5.18}$$

where $\mu^- = \min\{\frac{1}{2}, \frac{\theta}{2}\} - \frac{1}{2}$.

Now we introduce a wide class of inclusion shapes with $\theta > 0$ that satisfy (5.14). Consider an inclusion domain $D = \cup_{i=1}^N D_i$. Suppose we can surround each D_i by a buffer layer R_i so that each inclusion D_i together with its buffer does not intersect with the any of the other buffered inclusions, i.e., $D_i \cup R_i \cap D_j \cup R_j = \emptyset$, $i \neq j$. The set of such inclusion domains will be called *buffered geometries*, see Figure 5.2. We now denote the operator norm for the Dirichlet to Neumann map for each inclusion by $\|DN_i\|$ and the Poincare constant for each buffer layer by C_{R_i} and we have the following theorem.

Theorem 20. *The buffered geometry lies in P_θ provided*

$$\theta^{-1} = \max_i \{(1 + C_{R_i})\|DN_i\|\} < \infty. \tag{5.19}$$

Proof. To prove this theorem it suffices to consider one of the components D_i denoted by D and its buffer R_i denoted by R. The union of inclusion and buffer is denoted by $D' = D \cup R$. We now show for any function $w' \in H^1(R)$ there is a $w \in H^1(D')$ such that

$$w(x) = w'(x),\ x \in R$$

and

$$\int_D |\nabla w|^2 dx \leq \theta^{-1} \int_R |\nabla w'|^2, \tag{5.20}$$

where $\theta^{-1} = \{1 + C_R \|DN\|\}$ and DN is the Dirichlet to Neumann map for D.

Let $w \in H^1(D')$ such that $w = w'$ in R and $\Delta w = 0$ in D with boundary condition $w|_{\partial D} = w'$. Note that since w is harmonic in D, we have

$$\int_{\partial D} \partial_\nu w d\sigma = 0,$$

where ν is the outward pointing normal vector on ∂D. Thus

$$\begin{aligned} \int_D |\nabla w|^2 &= \int_{\partial D} \partial_\nu w \bar{w} = \int_{\partial D} \partial_\nu w \overline{(w - (w')^*)} \\ &= \int_{\partial D} \partial_\nu w \overline{(w' - (w')^*)}, \end{aligned} \tag{5.21}$$

where $(w')^*$ is the average of w' over R, given by

$$(w')^* = \frac{1}{|R|} \int_R w' dx. \tag{5.22}$$

Taking DN as the Dirichlet-to-Neumann map on $H^{1/2}(\partial D)$, we have

$$\begin{aligned} \int_{\partial D} \partial_\nu w \overline{(w' - (w')^*)} &\leq {}_{H^{-1/2}(\partial D)}\langle DN|_{\partial D} w, w' - (w')^* \rangle_{H^{1/2}(\partial D)} \\ &= {}_{H^{-1/2}(\partial D)}\langle DN|_{\partial D} [w' - (w')^*], w' - (w')^* \rangle_{H^{1/2}(\partial D)} \\ &\leq \|DN|_{\partial D}\| \|w' - (w')^*\|^2_{H^{1/2}(\partial D)}. \end{aligned} \tag{5.23}$$

The second line of (5.23) holds since $w = w'$ on ∂D and $Ker(DN)$ is simply the constant functions on ∂D. Let C_R be the Poincaré constant of R, i.e.

$$\|q - (q)^*\|^2_{L^2(R)} \leq C_R \|\nabla q\|^2_{L^2(R)} \tag{5.24}$$

for all $q \in H^1(R)$. Then we calculate

$$\begin{aligned} \|w' - (w')^*\|^2_{H^{1/2}(\partial D)} &\leq \|w' - (w')^*\|^2_{H^{1/2}(\partial R)} \\ &= \inf_{v|_{\partial R} = w' - (w')^*} \|v\|^2_{H^1(R)} \\ &\leq \|w' - (w')^*\|^2_{H^1(R)} \leq (1 + C_R)\|\nabla w'\|^2_{L^2(R)}. \end{aligned} \tag{5.25}$$

Substituting the last line of (5.25) into the last line of (5.23) and setting $\theta^{-1} = \|DN\|(1 + C_R)$, we obtain inequality (5.20) as desired.

Let $u \in W_3$, and set $w' = u$ in R. Then the w arising from the above theorem is a harmonic function in D satisfying $w|_{\partial D} = u$. Since u is also harmonic in D, we have that $u = w$ in D by uniqueness of solutions to Laplace's equation with Dirichlet boundary conditions, and inequality (5.20) becomes

$$\theta \int_D |\nabla u|^2 \leq \int_R |\nabla u|^2 \leq \int_{Y \setminus D} |\nabla u|^2. \tag{5.26}$$

$\square$

5.3 Radius of Convergence and Separation of Spectra for Disks

We now consider Bloch spectra for crystals in $\mathbb{R}^2$ with each period cell containing an identical random distribution of N disks D_i, $i = 1, \ldots, N$ of radius a. We suppose that the smallest distance separating the disks is $t_d > 0$. The buffer layers

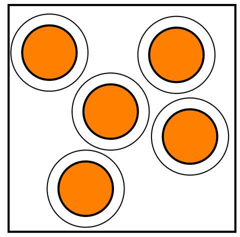

Figure 5.2: **Random buffered suspension.**

R_i are annuli with inner radii a and outer radii $b = a + t$ where $t \leq t_d/2$ and is chosen so that the collection of buffered disks lie within the period cell. For this case the constant θ is computed in [6] and is given by

$$\theta = \frac{b^2 - a^2}{b^2 + a^2}. \tag{5.27}$$

Since $a < b$, we have that

$$0 < \theta < 1. \tag{5.28}$$

We also note that when D_i is a disc of radius $a > 0$, we can recover an explicit formula for d from equation 5.1. In particular, any eigenvalue $\beta_j^\alpha(0)$ of $-\Delta_D^{-1}$, for $\alpha \neq 0$. may be written

$$\beta_j^\alpha(0) = \left(\frac{\eta_{n,k}}{a}\right)^{-2}, \tag{5.29}$$

where $\eta_{n,k}$ is the kth zero of the nth Bessel function $J_n(r)$. Let $\tilde{\eta}$ be the minimizer of

$$\min_{m,j \in \mathbb{N}} |(\eta_{n,k})^{-2} - (\eta_{m,j})^{-2}|. \tag{5.30}$$

Then we may choose Γ from section 5.1 so that

$$d = \frac{1}{2}|(\frac{a}{\eta_{n,k}})^2 - (\frac{a}{\tilde{\eta}})^2|. \tag{5.31}$$

We apply explicit form for θ to obtain a formula for r^* in terms of a,b, d given above, and α. Recall that ρ from Theorem 18 is given by $\rho = \min\{\frac{1}{2}, \frac{\theta}{2}\}$. In light

of inequality (5.28), we have that

$$\rho = \frac{1}{2}\left(\frac{b^2 - a^2}{b^2 + a^2}\right), \tag{5.32}$$

and we calculate the lower bound μ^-:

$$\mu^- = \rho - \frac{1}{2} = -\frac{a^2}{b^2 + a^2}. \tag{5.33}$$

Recalling that

$$|z^*| \le |z^+| = \frac{\mu^- + 1/2}{1/2 - \mu^-} \ ,$$

we obtain an explicit radius of convergence r^* in terms of a, b, $\eta_{n,k}$, $\tilde{\eta}$, and α for $\alpha \neq 0$,

$$r^* = \frac{|\alpha|^2|(\frac{a}{\eta_{n,k}})^2 - (\frac{a}{\tilde{\eta}})^2|(b^2 - a^2)}{4(b^2 + a^2) + |\alpha|^2|(\frac{a}{\eta_{n,k}})^2 - (\frac{a}{\tilde{\eta}})^2|(b^2 + 3a^2)}. \tag{5.34}$$

When $\alpha = 0$ Theorem 12 shows that the limit spectrum consists of a component given by the roots ν_{0k} of

$$1 = N\nu \sum_{k \in \mathbb{N}} \frac{a_{0k}^2}{\nu - (\eta_{0k}/a)^2}, \tag{5.35}$$

where N is the number of discs and $a_{0k} = \int_{B_a(0)} u_{0k}\,dx$ are averages over discs of radius a of the rotationally symmetric normalized eigenfunctions u_{0k} given by

$$u_{0k} = J_0(r\eta_{0k}/a)/(a\sqrt{\pi}J_1(\eta_{0k})). \tag{5.36}$$

The other component is comprised of the eigenvalues exclusively associated with mean zero eigenfunctions. The collection of these eigenvalues is given by $\{\cup_{n\neq 0,k}(\eta_{nk}/a)^2\}$ The elements λ_{nk} of the spectrum $\sigma(A^0(0))$ are given by the set $\{\cup_{n\neq 0,k}(\eta_{nk}/a)^2\} \cup \{\cup_k \nu_{0k}\}$. Now fix an element λ_{nk} and let $\tilde{\eta}$ be the minimizer of

$$\min_{m,j \in \mathbb{N}} |(\lambda_{n,k})^{-1} - (\lambda_{m,j})^{-1}|. \tag{5.37}$$

Then as before we may choose Γ from section 5.1 so that

$$d = \frac{1}{2}|(\lambda_{n,k}{}^{-1} - \tilde{\eta}^{-1}| \tag{5.38}$$

and in terms of a, b, $\lambda_{n,k}$, and $\tilde{\eta}$ for $\alpha = 0$:

$$r^* = \frac{4\pi^2|(\lambda_{n,k})^{-1} - \tilde{\eta}^{-1}|(b^2 - a^2)}{4(b^2 + a^2) + 4\pi^2|(\lambda_{n,k})^{-1} - \tilde{\eta}^{-1}|(b^2 + 3a^2)}. \tag{5.39}$$

The collection of suspensions of N buffered disks is an example of a class of buffered inclusion geometries and collecting results we have the following:

Corollary 21. *For every suspension of buffered disks with θ given by* (5.27)*: Theorem 13 holds with r^* given by* (5.34) *for $\alpha \in Y^\star$, $\alpha \neq 0$, and Theorem 14 holds with r^* given by* (5.39) *for $\alpha = 0$.*

Chapter 6

High Order Terms and Correctors for Power Series Representations

Now that the radius of convergence has been found for expansion (4.10), a natural course of action is to determine higher order terms in the series. To that effect, the following section calculates explicit representations of the operators found in the contour-integral formulation (4.11) of higher-order correctors in terms of layer potential operators. The resulting representations rely again on the $H^1_{\#}(\alpha, Y)$-orthogonal decomposition (3.39) of $(T^\alpha_k)^{-1}$. In section 6.2, the representations obtained in section 6.1 are used to recover an explicit representation of the first two terms in the k^{-1}-power series expansion for α-Bloch eigenvalues of the high-contrast photonic crystal.

6.1 Layer Potential Representation of Operators in Power Series

In this section we identify explicit formulas for the operators A^α_n appearing in the power series (4.11). It is shown that A^α_n, $n \neq 0$ can be expressed in terms of integral operators associated with layer potentials and we establish Theorem 15.

Recall that $A^\alpha(z) - A^\alpha(0)$ is given by

$$(zP_1 + \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} z[(1/2+\mu_i) + z(1/2-\mu_i))^{-1}P_{\mu_i})(-\Delta_\alpha^{-1}). \tag{6.1}$$

Factoring $(1/2+\mu_i)^{-1}$ from the second summand, we expand in power series

$$[(1/2+\mu_i) + z(1/2-\mu_i)]^{-1} = (1/2+\mu_i)^{-1} \sum_{n=0}^{\infty} z^n \left(\frac{\mu_i - 1/2}{\mu_i + 1/2}\right)^n, \tag{6.2}$$

and

$$A^\alpha(z) - A^\alpha(0) = (zP_1 + \sum_{n=1}^{\infty} z^n \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i + 1/2)^{-1} \left(\frac{\mu_i - 1/2}{\mu_i + 1/2} \right)^{n-1} P_{\mu_i} P_3)(-\Delta_\alpha^{-1}). \tag{6.3}$$

It follows that

$$A_1^\alpha = [P_1 + \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (1/2 + \mu_i)^{-1} P_{\mu_i} P_3](-\Delta_\alpha^{-1}) \tag{6.4}$$

and

$$A_n^\alpha = [\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i + 1/2)^{-1} \left(\frac{\mu_i - 1/2}{\mu_i + 1/2} \right)^{n-1} P_{\mu_i} P_3](-\Delta_\alpha^{-1}). \tag{6.5}$$

Recall also that we have the resolution of the identity

$$I_{H^1_\#(\alpha,Y)} = P_1 + P_2 + P_3 \text{ with } P_3 = \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} P_{\mu_i}, \tag{6.6}$$

and the spectral representation

$$\begin{aligned} \langle Tu, v \rangle &= \langle (S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1}) P_3 u + \tfrac{1}{2} P_1 u - \tfrac{1}{2} P_2 u, v \rangle \\ &= \langle \sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} \mu_i P_{\mu_i} u + \tfrac{1}{2} P_1 u - \tfrac{1}{2} P_2 u, v \rangle. \end{aligned} \tag{6.7}$$

Adding $\frac{1}{2}I$ to both sides of the above equation, we obtain

$$\begin{aligned} \langle (T + \tfrac{1}{2} I) u, v \rangle &= \langle (\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i + \tfrac{1}{2}) P_{\mu_i} + P_1) u, v \rangle \\ &= \langle ((S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1} + \tfrac{1}{2} P_3) P_3 + P_1) u, v \rangle \\ &= \langle ((S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2} \tilde{I}) S_D^{-1}) P_3 + P_1) u, v \rangle, \end{aligned} \tag{6.8}$$

where $\tilde{I}$ is the identity on $H^{-1/2}(\partial D)$. Now from (6.8) we see that

$$
\begin{aligned}
\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\tfrac{1}{2}+\mu_i)^{-1} P_{\mu_i} P_3 &= (S_D(\tilde{K}_D^{-\alpha})^* S_D^{-1} + \tfrac{1}{2}P_3)^{-1} P_3 \\
&= (S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}\tilde{I})S_D^{-1})^{-1} P_3 \qquad (6.9) \\
&= (S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}\tilde{I})^{-1} S_D^{-1}) P_3.
\end{aligned}
$$

Combining the first line of (6.4) and (6.9), we obtain

$$
A_1^\alpha = [S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}\tilde{I})^{-1} S_D^{-1} P_3 + P_1](-\Delta_\alpha)^{-1}. \qquad (6.10)
$$

We now turn to the higher-order terms. By the mutual orthogonality of the projections P_{μ_i}, we have that

$$
\begin{aligned}
&\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i + 1/2)^{-1} \left(\tfrac{\mu_i - 1/2}{\mu_i + 1/2}\right)^{n-1} P_{\mu_i} \\
&= \left(\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (1/2+\mu_i)^{-1} P_{\mu_i}\right) \left(\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} \left(\tfrac{\mu_i - 1/2}{\mu_i + 1/2}\right) P_{\mu_i}\right)^{n-1} \\
&= \left(\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (1/2+\mu_i)^{-1} P_{\mu_i}\right) \left(\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i - 1/2) P_{\mu_i}\right)^{n-1} \left(\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i + 1/2) P_{\mu_i}\right)^{1-n}.
\end{aligned}
\qquad (6.11)
$$

As above, we have that

$$\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (1/2+\mu_i)^{-1} P_{\mu_i} P_3 \quad = \quad S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}\tilde{I})^{-1} S_D^{-1} P_3,$$

$$\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (1/2+\mu_i) P_{\mu_i} P_3 \quad = \quad S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}\tilde{I}) S_D^{-1} P_3, \tag{6.12}$$

$$\sum_{-\frac{1}{2}<\mu_i<\frac{1}{2}} (\mu_i - 1/2) P_{\mu_i} P_3 \quad = \quad S_D((\tilde{K}_D^{-\alpha})^* - \tfrac{1}{2}\tilde{I}) S_D^{-1} P_3.$$

Combining (6.12), (6.11), and (6.4), we obtain the layer-potential representation for A_n^α, proving Theorem 15:

$$A_n^\alpha = S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1} S_D^{-1} [S_D((\tilde{K}_D^{-\alpha})^* - \frac{1}{2}I)((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1} S_D^{-1}]^{n-1} P_3(-\Delta_\alpha)^{-1}. \tag{6.13}$$

6.2 Explicit First Order Correction to the Bloch Band Structure in the High Contrast Limit

In this section we develop explicit formulas for the second term in the power series

$$\beta_j^\alpha(z) = \beta_j^\alpha(0) + z\beta_{j,1}^\alpha + z^2\beta_{j.2}^\alpha + ... \tag{6.14}$$

for simple eigenvalues. We use the analytic representation of $A^\alpha(z)$ and the Cauchy Integral Formula to represent $\beta_{j,1}^\alpha$

$$\begin{aligned} \beta_{j,1}^\alpha &= \tfrac{1}{2\pi i m} \mathrm{tr} \oint_\Gamma A_1^\alpha R(0,\zeta) d\zeta \\ &= \tfrac{1}{2\pi i m} \mathrm{tr} \left(A_1^\alpha \oint_\Gamma R(0,\zeta) d\zeta \right) \\ &= \mathrm{tr}\,(A_1^\alpha P(0)) = \tfrac{1}{m} \sum_{k=1}^m \langle \varphi_k, A_1^\alpha P(0) \varphi_k \rangle_{L^2_\#(\alpha,Y)}. \end{aligned} \tag{6.15}$$

Here $P(0)$ is the $L^2_{\#}(\alpha, Y)$ projection onto the eigenspace corresponding to the Dirichlet eigenvalue $(\beta_j^\alpha(0))^{-1}$ of $-\Delta$ on D. For simple eigenvalues consider the normalized eigenvector $P(0)\varphi = \varphi$ and

$$\beta_{j,1}^\alpha \quad = \langle \varphi, A_1^\alpha P(0)\varphi \rangle_{L^2_{\#}(\alpha,Y)}. \tag{6.16}$$

We apply the integral operator representation of A_1^α to deliver an explicit formula for the first order term $\beta_{j,1}^\alpha$ in the series for $\beta_j^\alpha(z)$. The explicit formula is given by the following theorem.

Theorem 22. *Let $\beta_j^\alpha(z)$ be an eigenvalue of $A^\alpha(z)$. Then for $|z| < r^*$ there is a $\beta_j(0) \in \sigma(-\Delta^{-1}|_D)$ with corresponding eigenfunction φ_j such that*

$$\beta_j^\alpha(z) = \beta_j(0) + z(\beta_j(0))^2 \int_{Y \setminus D} |\nabla v|^2 + z^2 \beta_{j.2}^\alpha + ... \tag{6.17}$$

Where v takes α-quasi periodic boundary conditions on ∂Y, is harmonic in $Y \setminus D$, and takes the Neumann boundary condition on ∂D given by

$$\partial_n v|_{\partial D^+} = \partial_n \varphi|_{\partial D^-},$$

where ∂_n is the normal derivitave on ∂D with normal vector n pointing into $Y \setminus D$.

Remark 23. *Recall from Theorem 17 that the eigenvalues $\lambda_j^\alpha(k) = (\beta_j^\alpha(1/k))^{-1}$, for $j \in \mathbb{N}$. The high coupling limit expansion for $\lambda_j^\alpha(k)$ is written in terms of the expansion $\beta_j^\alpha(z) = \beta_j(0) + z\beta_{j,1}^\alpha + \cdots$ as*

$$\begin{aligned} \lambda_j^\alpha(k) &= (\beta_j(0))^{-1} - \frac{1}{k}(\beta_j(0))^{-2}\beta_{j,1}^\alpha + \cdots \\ &= \lambda_j(0) - \frac{1}{k}\int_{Y \setminus D} |\nabla v|^2 + \cdots , \end{aligned} \tag{6.18}$$

where $\lambda_j(0) = (\beta_j(0))^{-1}$ is the j^{th} Dirichlet eigenvalue for the Laplacian on D. This naturally agrees with the formula for the leading order terms presented in [2].

Proof. Recall from the previous section that

$$\begin{aligned} A_1^\alpha &= [S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}I)^{-1} S_D^{-1} P_3 + P_1](-\Delta_\alpha)^{-1} \\ &= K_1^\alpha(-\Delta_\alpha)^{-1}, \end{aligned} \tag{6.19}$$

where $K_1^\alpha := S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)S_D^{-1}P_3 + P_1$. Moreover,

$$(-\Delta_\alpha)^{-1} f = -\int_Y G^\alpha(x,y) f(y) dy. \tag{6.20}$$

Since φ is a Dirichlet eigenvector of D with eigenvalue $(\beta_j(0))^{-1}$ and $\varphi = 0$ in $Y \setminus D$, we have

$$\varphi = -\beta_j(0)\chi_D(-\Delta\varphi). \tag{6.21}$$

Now from (6.21)

$$\begin{aligned} -\Delta_\alpha^{-1}\varphi &= \beta_j(0)\int_Y G^\alpha(x,y)\chi_D(\Delta_y\varphi)dy \\ &= \beta_j(0)\int_D G^\alpha(x,y)(\Delta_y\varphi)dy \\ &= \beta_j(0)(\int_D \nabla_y\cdot(G^\alpha(x,y)\nabla_y\varphi)\,dy - \int_D \nabla_y G^\alpha(x,y)\cdot\nabla_y\varphi\,dy) \\ &= \beta_j(0)(S_D[\partial_n\varphi_{|_{\partial D^-}}](x) - R(x)), \end{aligned} \tag{6.22}$$

where the last equality follows from the divergence theorem and definition of the single layer potential S_D and

$$R(x) = \int_D \nabla_y G^\alpha(x,y)\cdot\nabla_y\varphi\,dy. \tag{6.23}$$

Hence

$$A_1^\alpha\varphi = K_1^\alpha\beta_j(0)(S_D[\partial_n\varphi_{\partial D^-}](x) - R(x)). \tag{6.24}$$

Now we aply the definition of K_1^α and compute $P_1R(x)$ and $P_3R(x)$. Integrating by parts, we find

$$
\begin{aligned}
R(x) &= \textstyle\int_D \nabla_y G^\alpha(x,y)\cdot\nabla_y\varphi\,dy \\
&= \textstyle\int_D \nabla_y\cdot(\nabla_y G^\alpha(x,y)\varphi)dy - \int_D -\Delta_y G^\alpha(x,y)\varphi\,dy \qquad (6.25)\\
&= \varphi(x).
\end{aligned}
$$

Thus $P_1 R(x) = P_3 R(x) = 0$ since $\varphi \in W_1$. Combining this result, (6.15), (6.24), and the definition of K_1^α we obtain

$$
\begin{aligned}
\beta_{j,1}^\alpha = \operatorname{tr}(A_1^\alpha P(0)) &= \langle\varphi, A_1^\alpha P(0)\varphi\rangle_{L^2_\#(\alpha,Y)} \\
&= \langle\varphi, \beta_j(0) S_D((\tilde{K}_D^{-\alpha})^* + \tfrac{1}{2}I)^{-1}[\partial_n\varphi_{|_{\partial D^-}}]\rangle_{L^2_\#(\alpha,Y)}.
\end{aligned}
\tag{6.26}
$$

Let $v \in H^1_\#(\alpha,Y)$ be defined

$$
v := S_D((\tilde{K}_D^{-\alpha})^* + \frac{1}{2}I)^{-1}[\partial_n\varphi_{|_{\partial D^-}}]. \tag{6.27}
$$

Then v is harmonic in D and $Y\setminus D$, and

$$
\partial_n v|_{\partial D^+} = \partial_n\varphi|_{\partial D^-}. \tag{6.28}
$$

On applying (6.21), (6.27), and (6.28) equation (6.26) becomes

$$
\begin{aligned}
\beta_{j,1}^\alpha &= \beta_j(0)\langle\varphi, v\rangle = -(\beta_j(0))^2\int_D v\Delta\overline{\varphi}\,dy \qquad (6.29)\\
&= -(\beta_j(0))^2\Big(\int_{\partial D}\partial_n\varphi_{|_{\partial D^-}}\bar{v}d\sigma - \int_D\nabla\varphi\cdot\nabla\bar{v}\Big)\\
&= -(\beta_j(0))^2\Big(\int_{\partial D}\partial_n v_{|_{\partial D^+}}\bar{v}d\sigma - \int_D\nabla\varphi\cdot\nabla\bar{v}\Big).
\end{aligned}
$$

Last an integration by parts yields

$$
\begin{aligned}
\int_D \nabla\varphi\cdot\nabla\bar{v} &= \int_D \nabla\cdot(\nabla\bar{v}\varphi) - \Delta\bar{v}\varphi \\
&= \int_{\partial D}\partial_n\bar{v}_{|_{\partial D^-}}\varphi d\sigma = 0.
\end{aligned}
\tag{6.30}
$$

Combining this result with the last line of (6.29) and integrating by parts a final time reveals a representation of the second term in (6.14)

$$\beta_{j,1}^{\alpha} = (\beta_j(0))^2 \int_{Y \setminus D} |\nabla v|^2 dx, \tag{6.31}$$

and the theorem follows. $\square$

Chapter 7

Derivation of the Convergence Radius and Separation of Spectra

Here we prove Theorems 13 and 14. To begin, we suppose $\alpha \neq 0$ and recall that the Neumann series (4.5) and consequently (4.6) and (4.10) converge provided that

$$\|(A^\alpha(z) - A^\alpha(0))R(\zeta, 0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} < 1. \tag{7.1}$$

With this in mind we will compute an explicit upper bound $B(\alpha, z)$ and identify a neighborhood of the origin on the complex plane for which

$$\|(A^\alpha(z) - A^\alpha(0))R(\zeta, 0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} < B(\alpha, z) < 1, \tag{7.2}$$

holds for $\zeta \in \Gamma$. The inequality $B(\alpha, z) < 1$ will be used first to derive a lower bound on the radius of convergence of the power series expansion of the eigenvalue group about $z = 0$. It will then be used to provide a lower bound on the neighborhood of $z = 0$ where properties 1 through 3 of Theorem 13 hold.

We have the basic estimate given by

$$\begin{aligned} &\|(A^\alpha(z) - A^\alpha(0))R(\zeta, 0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} \leq \\ &\|(A^\alpha(z) - A^\alpha(0))\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]}\|R(\zeta, 0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]}. \end{aligned} \tag{7.3}$$

Here $\zeta \in \Gamma$ as defined in Theorem 13 and elementary arguments deliver the estimate

$$\|R(\zeta, 0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} \leq d^{-1}, \tag{7.4}$$

where d is given by (5.1).

Next we estimate $\|(A^\alpha(z) - A^\alpha(0))\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]}$. Denote the energy seminorm of u by

$$\|u\| = \|\nabla u\|_{L^2(Y)}. \tag{7.5}$$

To proceed we introduce the Poincare estimate for functions belonging to $H^1_\#(\alpha, Y)$ for $\alpha \neq 0$:

Lemma 24.

$$\|u\|_{L^2(Y)} \leq |\alpha|^{-1}\|u\|. \tag{7.6}$$

Proof. A straight forward calculation using (3.13) gives the upper bound

$$(-\Delta_\alpha^{-1} v, v)_{L^2(Y)} \leq |\alpha|^{-2}\|v\|^2_{L^2(Y)} \tag{7.7}$$

and we have the Cauchy inequality

$$\|v\|^2_{L^2(Y)} = \langle -\Delta_\alpha^{-1} v, v\rangle \leq \| - \Delta_\alpha^{-1} v\|\|v\|. \tag{7.8}$$

Applying (7.7) we get

$$\| - \Delta_\alpha^{-1} v\| = (\langle -\Delta_\alpha^{-1} v, -\Delta_\alpha^{-1} v\rangle)^{1/2} = ((-\Delta_\alpha^{-1} v, v))^{1/2} \leq |\alpha|^{-1}\|v\|_{L^2(Y)} \tag{7.9}$$

and the Poincare inequality follows from (7.8) and (7.9). □

For any $v \in L^2_\#(\alpha, Y)$, we apply (7.6) to find

$$\begin{aligned} &\|(A^\alpha(z) - A^\alpha(0))v\|_{L^2(Y)} \\ &\leq |\alpha|^{-1}\|(A^\alpha(z) - A^\alpha(0))v\| \\ &= |\alpha|^{-1}\|((T_k^\alpha)^{-1} - (T_0^\alpha)^{-1})(-\Delta_\alpha)^{-1} v\| \\ &\leq |\alpha|^{-1}\|((T_k^\alpha)^{-1} - P_2)\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}\| - \Delta_\alpha^{-1} v\|. \end{aligned} \tag{7.10}$$

Applying (7.9) and (7.10) delivers the upper bound:

$$\|(A^\alpha(z) - A^\alpha(0))\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} \leq |\alpha|^{-2}\|((T_k^\alpha)^{-1} - P_2)\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}. \tag{7.11}$$

The next step is to obtain an upper bound on $\|((T_k^\alpha)^{-1} - P_2)\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}$. For all $v \in H^1_\#(\alpha, Y)$, we have

$$\frac{\|((T_k^\alpha)^{-1} - P_2)v\|}{\|v\|} \leq |z|\{w_0 + \sum_{i=1}^{\infty} w_i|(1/2 + \mu_i) + z(1/2 - \mu_i)|^{-2}\}^{1/2}, \tag{7.12}$$

where $w_0 = \|P_1 v\|^2/\|v\|^2$, $w_i = \|P_i v\|^2/\|v\|^2$, and $w_0 + \sum_{i=1}^{\infty} w_i = 1$. So maximizing the right hand side is equivalent to calculating

$$\begin{aligned} &\max_{w_0+\sum w_i=1}\{w_0 + \sum_{i=1}^{\infty} w_i|(1/2 + \mu_i) + z(1/2 - \mu_i)|^{-2}\}^{1/2} \\ &= \sup\{1, |(1/2 + \mu_i) + z(1/2 - \mu_i)|^{-2}\}^{1/2}. \end{aligned} \tag{7.13}$$

Thus we maximize the function

$$f(x) = |\frac{1}{2} + x + z(\frac{1}{2} - x)|^{-2} \tag{7.14}$$

over $x \in [\mu^-(\alpha), \mu^+(\alpha)]$ for z in a neighborhood about the origin. Let $Re(z) = u$, $Im(z) = v$ and we write

$$\begin{aligned} f(x) &= |\tfrac{1}{2} + x + (u + iv)(\tfrac{1}{2} - x)|^{-2} \\ &= ((\tfrac{1}{2} + x + u(\tfrac{1}{2} - x))^2 + v^2(\tfrac{1}{2} - x)^2)^{-1} \\ &\leq (\tfrac{1}{2} + x + u(\tfrac{1}{2} - x))^{-2} = g(Re(z), x), \end{aligned} \tag{7.15}$$

to get the bound

$$\|((T_k^\alpha)^{-1} - P_2)\|_{L(H^1_\#(\alpha,Y))} \leq |z| \sup\{1, \sup_{x\in[\mu^-(\alpha),\mu^+(\alpha)]} g(u, x)\}^{1/2}. \tag{7.16}$$

We now examine the poles of $g(u, x)$ and the sign of its partial derivative $\partial_x g(u, x)$ when $|u| < 1$. If $Re(z) = u$ is fixed, then $g(u, x) = ((\frac{1}{2} + x) + u(\frac{1}{2} - x))^{-2}$

has a pole when $(\frac{1}{2}+x)+u(\frac{1}{2}-x)=0$. For u fixed this occurs when

$$\hat{x}=\hat{x}(u)=\frac{1}{2}\left(\frac{1+u}{u-1}\right). \tag{7.17}$$

On the other hand, if x is fixed, g has a pole at

$$u=\frac{\frac{1}{2}+x}{x-\frac{1}{2}}. \tag{7.18}$$

The sign of $\partial_x g$ is determined by the formula

$$\partial_x g(u,x) \quad = \quad N/D, \tag{7.19}$$

where $N=-2(1-u)^2x-(1-u^2)$ and $D:=((\frac{1}{2}+x)+u(\frac{1}{2}-x))^4\geq 0$. Calculation shows that $\partial_x g<0$ for $x>\hat{x}$, i.e. g is decreasing on $(\hat{x},\infty)$. Similarly, $\partial_x g>0$ for $x<\hat{x}$ and g is increasing on $(-\infty,\hat{x})$.

Now we identify all $u=Re(z)$ for which $\hat{x}=\hat{x}(u)$ satisfies

$$\hat{x}<\mu^-(\alpha)<0. \tag{7.20}$$

Indeed for such u, the function $g(u,x)$ will be decreasing on $[\mu^-(\alpha),\mu^+(\alpha)]$, so that $g(u,\mu^-(\alpha))\geq g(u,x)$ for all $x\in[\mu^-(\alpha),\bar{\mu}]$, yielding an upper bound for (7.16).

Lemma 25. *The set U of $u\in\mathbb{R}$ for which $-\frac{1}{2}<\hat{x}(u)<\mu^-(\alpha)<0$ is given by*

$$U:=[z^*,1]$$

where

$$-1\leq z^*:=\frac{\mu^-(\alpha)+\frac{1}{2}}{\mu^-(\alpha)-\frac{1}{2}}<0.$$

Proof. Note first that $\mu^-(\alpha)=\inf_{i\in\mathbb{N}}\{\mu_i\}\leq 0$ follows from the fact that zero is an accumulation point for the sequence $\{\mu_i\}_{i\in\mathbb{N}}$ so it follows that $-1\leq z^*$. Noting $\hat{x}=\hat{x}(u)=\frac{1}{2}\frac{u+1}{u-1}$, we invert and write

$$u=\frac{\frac{1}{2}+\hat{x}}{\hat{x}-\frac{1}{2}}. \tag{7.21}$$

We now show that

$$z^* \leq u \leq 1 \tag{7.22}$$

for $\hat{x} \leq \mu^-(\alpha)$. Set $h(\hat{x}) = \frac{\frac{1}{2}+\hat{x}}{\hat{x}-\frac{1}{2}}$. Then

$$h'(\hat{x}) = \frac{-1}{(\hat{x}-\frac{1}{2})^2}, \tag{7.23}$$

and so h is decreasing on $(-\infty, \frac{1}{2})$. Since $\mu^-(\alpha) < \frac{1}{2}$, h attains a minimum over $(-\infty, \mu^-(\alpha)]$ at $x = \mu^-(\alpha)$. Thus $\hat{x}(u) \leq \mu^-(\alpha)$ implies

$$z^* = \frac{\mu^-(\alpha)+\frac{1}{2}}{\mu^-(\alpha)-\frac{1}{2}} \leq u \leq 1 \tag{7.24}$$

as desired. □

Combining Lemma 25 with inequality (7.16), noting that $-|z| \leq Re(z) \leq |z|$ and on rearranging terms we obtain the following corollary.

Corollary 26. *For* $|z| < |z^*|$*:*

$$\|(A^\alpha(z) - A^\alpha(0))\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} \leq |\alpha|^{-2}|z|(-|z|-z^*)^{-1}(\frac{1}{2}-\mu^-(\alpha))^{-1}. \tag{7.25}$$

From Corollary 26, (7.3), and (7.4) we easily see that

$$\begin{aligned} &\|(A^\alpha(z) - A^\alpha(0))R(\zeta,0)\|_{\mathcal{L}[L^2_\#(\alpha,Y);L^2_\#(\alpha,Y)]} \leq \\ &B(\alpha,z) = |\alpha|^{-2}|z|(-|z|-z^*)^{-1}(\frac{1}{2}-\mu^-(\alpha))^{-1}d^{-1}. \end{aligned} \tag{7.26}$$

a straight forward calculation shows that $B(\alpha, z) < 1$ for

$$|z| < r^* := \frac{|\alpha|^2 d|z^*|}{\frac{1}{\frac{1}{2}-\mu^-(\alpha)} + |\alpha|^2 d} \tag{7.27}$$

and property 4 of Theorem 13 is established since $r^* < |z^*|$.

Now we establish properties 1 through 3 of Theorem 13. First note that inspection of (4.5) shows that if (7.1) holds and if $\zeta \in \mathbb{C}$ belongs to the resolvent of

$A^\alpha(0)$ then it also belongs to the resolvent of $A^\alpha(z)$. Since (7.1) holds for $\zeta \in \Gamma$ and $|z| < r^*$, property 1 of Theorem 13 follows. Formula (4.6) shows that $P(z)$ is analytic in a neighborhood of $z = 0$ determined by the condition that (7.1) holds for $\zeta \in \Gamma$. The set $|z| < r^*$ lies inside this neighborhood and property 2 of Theorem 13 is proved. The isomorphism expressed in property 3 of Theorem 13 follows directly from Lemma 4.10 ([17], Chapter I §4) which is also valid in a Banach space.

The proof of 14 proceeds along identical lines. To prove Theorem 14, we need the following Poincaré inequality for $H^1_\#(0, Y)$.

Lemma 27.

$$\|v\|_{L^2_\#(0,Y)} \leq \frac{1}{2\pi}\|v\|. \tag{7.28}$$

This inequality is established using (3.14) and proceeding using the same steps as in the proof of Lemma 24. Using (7.28) in place of (7.6) we argue as in the proof of Theorem 13 to show that

$$\|(A^0(z) - A^0(0))R(\zeta, 0)\|_{\mathcal{L}[(L^2_\#(0,Y);L^2_\#(0,Y)]} < 1 \tag{7.29}$$

holds provided $|z| < r^*$, where r^* is given by (5.7). This establishes Theorem 14.

The error estimates presented in Theorem 16 are easily recovered from the arguments in ([17] Chapter II, §3); for completeness, we restate them here. We begin with the following application of Cauchy inequalities to the coefficients β^α_n of (4.10) from ([17] Chapter II, §3, pg 88):

$$|\beta^\alpha_n| \leq d(r^*)^{-n}. \tag{7.30}$$

It follows immediately that, for $|z| < r^*$,

$$\left|\hat{\beta}^\alpha(z) - \sum_{n=0}^{p} z^n \beta^\alpha_n\right| \leq \sum_{n=p+1}^{\infty} |z|^n |\beta^\alpha_n| \leq \frac{d|z|^{p+1}}{(r^*)^p(r^* - |z|)}, \tag{7.31}$$

completing the proof.

For completeness we establish the boundedness and compactness of the operator $B^\alpha(k)$.

Theorem 28. *The operator $B^\alpha(k) : L^2_\#(\alpha, Y) \longrightarrow H^1_\#(\alpha, Y)$ is bounded for $k \notin Z$.*

To prove the theorem for $\alpha \neq 0$ we observe for $v \in L^2_\#(\alpha, Y)$ that

$$\begin{aligned} &\|B^\alpha(k)v\| = |(T^\alpha_k)^{-1}(-\Delta_\alpha)^{-1}v\| \leq \\ &\leq \|((T^\alpha_k)^{-1}\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}\| - \Delta^{-1}_\alpha v\| \\ &\leq |\alpha|^{-1}\|((T^\alpha_k)^{-1}\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}\|v\|_{L^2(Y)}, \end{aligned} \tag{7.32}$$

where the last inequality follows from (7.9). The upper estimate on $\|((T^\alpha_k)^{-1}\|_{\mathcal{L}[H^1_\#(\alpha,Y);H^1_\#(\alpha,Y)]}$ is obtained from

$$\frac{\|T^\alpha_k)^{-1}v\|}{\|v\|} \leq \{|z|\hat{w} + \tilde{w} + |\sum_{i=1}^{\infty} w_i|(1/2 + \mu_i) + z(1/2 - \mu_i)|^{-2}\}^{1/2}, \tag{7.33}$$

where $\hat{w} = \|P_1 v\|^2/\|v\|^2 =$, $\tilde{w} = \|P_2 v\|^2/\|v\|^2$, $w_i = \|P_i v\|^2/\|v\|^2$. Since $\hat{w} + \overline{w} + \sum_{i=1}^{\infty} w_i = 1$ one recovers the upper bound

$$\frac{\|T^\alpha_k)^{-1}v\|}{\|v\|} \leq M, \tag{7.34}$$

where

$$M = \max\{1, |z|, \sup_i\{|(1/2 + \mu_i) + z(1/2 - \mu_i)|^{-1}\}\}, \tag{7.35}$$

and the proof of Theorem 28 is complete. An identical proof can be carried out when $\alpha = 0$.

Remark 29. *The Poincare inequalities* (7.6) *and* (7.28) *together with Theorem 28 show that $B^\alpha(k) : L^2_\#(\alpha, Y) \longrightarrow L^2_\#(\alpha, Y)$ is a bounded linear operator mapping $L^2_\#(\alpha, Y)$ into itself. The compact embedding of $H^1_\#(\alpha, Y)$ into $L^2_\#(\alpha, Y)$ shows the operator is compact on $L^2_\#(\alpha, Y)$.*

Conclusions

To conclude, a thorough perturbative spectral analysis of certain divergence form 2nd-order operators acting on quasi-periodic and periodic Hilbert Spaces arising from modelling of electromagnetic wave propagation through photonic crystals is achieved. For a fixed quasi-momentum α, Theorem 13 describes the band structure of Bloch-eigenvalues in the high-contrast limit $k \to \infty$: for large enough contrast k, the α-Bloch-eigenvalues may be represented by a uniformly convergent power series with radius of convergence (5.3). Moreover, for k^{-1} smaller than this radius, the α-Bloch spectrum remains separated in the sense that no α-Bloch eigenvalues "run into eachother" as $k \to \infty$. Theorem 14 reveals a similar result for the periodic case with radius of convergence given by equation (5.7).

The first-order correction to the Bloch-eigenvalue power series is recovered in section 6.2, which agrees with similar results obtained in the literature (c.f. [2]). This recovery is obtained via an integral operator representation of the terms in the z-power series expansion of the family $A^\alpha(z) = (T_k^\alpha)^{-1}(-\Delta_\alpha)^{-1}$: this integral operator represenation is found in section 6.1.

These spectral results were driven primarily by a different kind of spectral representation (see (3.39)) of the divergence form operator in terms of the gradient inner product on Y, which in turn is driven by the problem (3.2). These "generalized electrostatic resonances" are found to be exactly the eigenvalues of the Neumann-Poincaré operator $(\tilde{K}_D^{-\alpha})^*$. For certain types of inclusion geometries, these resonances are found to have a lower bound (see Theorems 18 and 20), which is used to obtain the radius via Neumann series, a Poincaré inequality, and an optimization problem on the real line (see section 7).

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Vita

Robert P. Viator Jr. was born in August 1987 in Baton Rouge, Louisiana. He finished his undergraduate studies at Louisiana State University in May 2009. He returned to Louisiana State University for graduate school in mathematics in August 2009, and earned a master of science degree in mathematics in December 2012. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2016.