A Finite-Element Analysis of Steady, Two-Dimensional, Incompressible, Laminar Flow.

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Louisiana State University and
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requirements for the degree of
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in
The Department of Engineering Science

by
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## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>ii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>v</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>vi</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I. DEFINITION OF THE PROBLEM</td>
<td>1</td>
</tr>
<tr>
<td>Governing Equations</td>
<td>1</td>
</tr>
<tr>
<td>Non-Dimensional Equations</td>
<td>3</td>
</tr>
<tr>
<td>Statement of Problem</td>
<td>4</td>
</tr>
<tr>
<td>II. HISTORY OF THE PROBLEM</td>
<td>6</td>
</tr>
<tr>
<td>Analytical Investigations</td>
<td>6</td>
</tr>
<tr>
<td>Experimental Investigations</td>
<td>9</td>
</tr>
<tr>
<td>Numerical Investigations</td>
<td>10</td>
</tr>
<tr>
<td>III. THE FINITE ELEMENT METHOD</td>
<td>14</td>
</tr>
<tr>
<td>History</td>
<td>14</td>
</tr>
<tr>
<td>Application</td>
<td>17</td>
</tr>
<tr>
<td>Convergence Criteria</td>
<td>20</td>
</tr>
<tr>
<td>Interpolating Functions</td>
<td>21</td>
</tr>
<tr>
<td>Curved Isoparametric Quadrilaterals</td>
<td>27</td>
</tr>
<tr>
<td>Elimination of Internal Nodes</td>
<td>31</td>
</tr>
<tr>
<td>IV. METHODS OF SOLUTION</td>
<td>35</td>
</tr>
<tr>
<td>Method 1</td>
<td>35</td>
</tr>
<tr>
<td>Method 2</td>
<td>37</td>
</tr>
<tr>
<td>V. FINITE ELEMENT FORMULATION</td>
<td>40</td>
</tr>
<tr>
<td>Method 1</td>
<td>40</td>
</tr>
<tr>
<td>Method 2</td>
<td>46</td>
</tr>
<tr>
<td>VI. DESCRIPTION OF COMPUTER PROGRAMS</td>
<td>53</td>
</tr>
<tr>
<td>Method 1</td>
<td>53</td>
</tr>
<tr>
<td>Method 2</td>
<td>62</td>
</tr>
<tr>
<td>Test Program</td>
<td>69</td>
</tr>
<tr>
<td>Comment</td>
<td>72</td>
</tr>
</tbody>
</table>

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# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Physical Problem, Circular Cylinder</td>
<td>89</td>
</tr>
<tr>
<td>2.</td>
<td>Linear Rectangle</td>
<td>90</td>
</tr>
<tr>
<td>3.</td>
<td>Quadratic Rectangle</td>
<td>91</td>
</tr>
<tr>
<td>4.</td>
<td>Quadratic Isoparametric Quadrilateral</td>
<td>92</td>
</tr>
<tr>
<td>5.</td>
<td>Orientation of Triangular Element</td>
<td>93</td>
</tr>
<tr>
<td>6.</td>
<td>Quadrilateral with Twelve Triangles</td>
<td>94</td>
</tr>
<tr>
<td>7.</td>
<td>Quadrilateral with Four Triangles; Quadrilateral with Sixteen Triangles</td>
<td>95</td>
</tr>
<tr>
<td>8.</td>
<td>Nine Node Quadrilateral</td>
<td>96</td>
</tr>
<tr>
<td>9.</td>
<td>Experimental Elements: Thirteen Node Quadrilateral; Twenty-one Node Quadrilateral</td>
<td>97</td>
</tr>
<tr>
<td>10.</td>
<td>Circular Cylinder, Surface Pressure Distribution</td>
<td>98</td>
</tr>
<tr>
<td>11.</td>
<td>Physical Problem, Lens Shaped Cylinder</td>
<td>99</td>
</tr>
<tr>
<td>12.</td>
<td>Lens Shaped Cylinder, Surface Pressure Distribution</td>
<td>100</td>
</tr>
<tr>
<td>13.</td>
<td>Method 1 Results for a Circular Cylinder</td>
<td>102</td>
</tr>
<tr>
<td>14.</td>
<td>Method 2 Results for a Circular Cylinder</td>
<td>113</td>
</tr>
<tr>
<td>15.</td>
<td>Test Program Results</td>
<td>119</td>
</tr>
<tr>
<td>16.</td>
<td>Method 1 Results for a Lens Shaped Cylinder</td>
<td>121</td>
</tr>
</tbody>
</table>

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ABSTRACT

A solution of the Navier-Stokes and continuity equations for steady, two dimensional, incompressible, laminar flow employing an iteration technique is presented. The technique begins with an initial guess for the velocity and pressure fields, which is used to compute approximations to the non-linear terms in the equations of motion. Thus, treating the non-linear terms as known functions, the finite element method is employed to solve the resulting linear equations for new approximations to the velocities and pressure. Using the solution as a new guess, the non-linear terms are approximated again and the procedure is repeated until two successive solutions are "close."

Presented in the text are two formulations of the governing equations. One consists of the classical form of two equations of motion and the continuity equation, while the other replaces the continuity equation with a combination of the first derivatives of the equations of motion and the continuity equation.

The agreement with published results for flow past a circular cylinder is excellent, although the results of the present work are restricted to small Reynolds numbers. Success is achieved with the first formulation, but not the second.
CHAPTER I

DEFINITION OF THE PROBLEM

A problem of obvious practical importance is the description of the steady motion of a viscous, incompressible fluid. The present paper analyzes the external flow of such a fluid past a cylindrical body normal to the free stream, resulting in a two dimensional problem. Some simplifying assumptions are made about the body, the governing equations and the boundary conditions are developed, and a method of solution is presented.

Governing Equations

An Eulerian or global formulation of Newton's second law yields the vector equation of motion of a continuum. If the continuum is assumed to be homogeneous, isotropic, and if it obeys Stokes' relation that the shear stress is proportional to the velocity gradient, the resulting equation of motion is the Navier-Stokes equation which, for two dimensional, incompressible flow, has the following components:

\[ uu_x + vu_y = \frac{1}{\rho}(X - P_x) + v \nabla^2 u \]

\[ uv_x + vv_y = \frac{1}{\rho}(Y - P_y) + \nabla^2 v \]

1

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where,

\( \rho \) is the fluid density,

\( \nu \) is the kinematic viscosity,

\( \overline{X} \) is the body force per unit volume in the \( x \)-direction,

\( \overline{Y} \) is the body force per unit volume in the \( y \)-direction,

\( u \) is the velocity component in the \( x \)-direction,

\( v \) is the velocity component in the \( y \)-direction,

\( P \) is the pressure,

and subscripts denote partial derivatives. If the pressure is considered as the sum of the hydrostatic pressure, \( P_H \), and another term which is denoted the dynamic pressure, \( P_D \), and it is assumed that the only body force acting is gravity, then with the observation that

\[ \overline{X} = \frac{\partial P_H}{\partial x}, \quad \overline{Y} = \frac{\partial P_H}{\partial y}, \]

it becomes clear that,

\[ \overline{X} - \frac{\partial P}{\partial x} = \overline{X} - \frac{\partial}{\partial x}(P_H + P_D) = -\frac{\partial P_D}{\partial x}, \]

\[ \overline{Y} - \frac{\partial P}{\partial y} = \overline{Y} - \frac{\partial}{\partial y}(P_H + P_D) = -\frac{\partial P_D}{\partial y}. \]

Thus, renaming the pressure \( P \) as the dynamic pressure, i.e. the difference between the total pressure and the hydrostatic pressure, the Navier-Stokes equations of motion become,

\[ uu_x + vv_y = -\frac{1}{\rho} p_x + \nu \nabla^2 u, \]
The principle of conservation of mass applied to a continuum requires that the continuity equation be satisfied,

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \]

For steady, incompressible flow the continuity equation simplifies to,

\[ u_x + v_y = 0 \]

**Non-Dimensional Equations**

To obtain a more useful form of the Navier-Stokes and continuity equations, the variables may be non-dimensionalized using a characteristic length, \( d \), and velocity \( U \), which define the new coordinates,

\[ x' = \frac{x}{d} \quad , \quad y' = \frac{y}{d} \]

velocities,

\[ u' = \frac{u}{U} \quad , \quad v' = \frac{v}{U} \]

and pressure,

\[ p' = \frac{p}{\rho U^2} \]
Substituting these variables into the governing equations, dropping the primes, and rearranging yields,

\[ uu_x + vu_y = - P_x + \frac{v}{Ud} v^2 u \, , \]
\[ uv_x + vv_y = - P_x + \frac{v}{Ud} v^2 v \, , \]
\[ u_x + v_y = 0 \, . \]

Obviously, the only important parameter in this problem is the well known Reynolds Number,

\[ R_E = \frac{Ud}{v} \, . \]

**Statement of Problem**

The problem to be studied is the two-dimensional steady flow of an incompressible fluid, past a cylinder of arbitrary cross section normal to the flow. To simplify the analysis the body is assumed symmetric about an axis parallel to the direction of the free stream. The test problem is the flow past a circular cylinder.

Mathematically, the problem is one of solving a system of three simultaneous, non-linear, elliptic, partial differential equations, subject to a set of forced boundary conditions. The equations are the non-dimensional Navier-Stokes and continuity equations,

\[ uu_x + vu_y = - P_x + \frac{1}{R_E} v^2 u \, , \]
\[ \begin{aligned} \frac{uv}{x} + \frac{uv}{y} &= -\frac{p_y}{1 + \frac{R}{R^2}} \frac{v^2}{v}, \\
\frac{u}{x} + \frac{v}{y} &= 0 , \\
\text{and the boundary conditions are,} \\
y=0 : v=0 , u_y=0 , p_y=0 , \\
r=R(\theta) : u=0 , v=0 , \\
r \to \infty : u \to 1 , v \to 0 , p \to 0 , \\
\text{where } r \text{ is the radial coordinate from the origin,} \\
r^2 = x^2 + y^2 , \\
\text{and } R(\theta) \text{ is the cylinder boundary.} \\
\text{A sketch of the physical problem including the} \\
\text{boundary conditions is shown in Figure 1 [42].} \\
\end{aligned} \]
CHAPTER II

HISTORY OF THE PROBLEM

Flow past a circular cylinder is a problem which has attracted attention for many years. As with most difficult problems, the early stages of development include simplified analytical studies and primitive experimental observation. Subsequently, the early analytical methods are refined and more sophisticated equipment is used in experiments to improve results. In recent years, another step of scientific development has been added, that of numerical investigation with a digital computer, which is responsible for the latest advances in the present problem.

Analytical Investigations

Sir G. G. Stokes obtained the first useful results in 1851 by a simplified analytical investigation [44,45]. Since the inertia forces are quadratic in the velocity, corresponding to the convective terms in the Navier-Stokes equations, he decided the pressure and viscous forces must be the only important forces for small velocities, i.e. at low Reynolds numbers. This simplification results in a biharmonic equation for the stream function, which must satisfy the usual no slip and free stream boundary.
conditions. An analysis of the boundary conditions led Stokes to a solution assumption which resulted in,

$$\psi(r, \theta) \approx C \left[ r \log(r) - \frac{r}{2} + \frac{1}{2r} \right] \sin(\theta),$$

where $\psi$ is the stream function,

$$u = \psi_y, \quad v = -\psi_x,$$

or in polar coordinates,

$$v_{\text{R-dir.}} = \frac{1}{r} \psi_\theta, \quad v_{\theta\text{-dir.}} = -\psi_r$$

$r$ and $\theta$ are the polar coordinates, and $C$ is a constant. The two immediate drawbacks to Stokes' solution are that the velocities become unbounded far from the body, and that there is no value of $C$ which will satisfy all the boundary conditions.

An improvement on Stokes' solution was made by C. W. Oseen in 1910 [37]. Stokes' paradox, the non-existence of a solution far from the body, was explained by Oseen to be a result of the singularity of the flow for small Reynolds numbers. He showed that the ratio of the terms neglected by Stokes, the convective terms, to those retained, the viscous terms, is of the order of the product $Rr$, where $R$ is the body radius, and $r$ is the radial polar coordinate. It is obvious that as $r$ increases the neglected terms are no longer negligible, and Stokes' solution fails. Oseen

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suggested a linearization rather than a neglect of the convective terms. Therefore, the x-component of the equation of motion becomes

\[ U_u x = -P_x + \frac{1}{R_e} \nabla^2 u \]

where \( U \) is the free stream velocity, an obvious improvement on the equation used by Stokes,

\[ 0 = -P_x + \frac{1}{R_e} \nabla^2 u \]

Although Oseen presented a solution for the improved Stokes problem for a sphere only, a year later H. Lamb published the solution for a circular cylinder in terms of rectangular cartesian velocity components [32] which are, in non-dimensional form,

\[
\begin{align*}
    u &= C \left[ \left( \frac{1}{r^2} - 1 \right) \left( \frac{x^2}{r^2} - \frac{1}{2} \right) + \log(r) \right], \\
v &= C \left( \frac{1}{r^2} - 1 \right) \frac{xy}{r^2}
\end{align*}
\]

where,

\[ C = \frac{1}{\log \left( \frac{7.4}{R_e} \right)} \]

x and y are rectangular coordinates, and r is the radial coordinate.

Many analytical methods have been attempted in an
effort to improve the results obtained by Stokes and Oseen, but none have produced a complete solution [5,6,17,18,22,23,26,31,39,42,51,55,58].

Experimental Investigations

The difficulties of experimental research are obvious, any measuring device causes a perturbation in the flow field. The instability of some flows, particularly the pressure distribution, can cause large fluctuations in values due to the perturbation from an experimental instrument.

Most of the early experiments were restricted to high Reynolds number flows, since the accuracy improved with increasing Reynolds number. Some of the first investigators were able to show good agreement with analytical results as early as 1921. The preliminary work was done by Wieselberger [59,60], Relf [40,41], Eisner [14], Thom [48,49], and Homan [21] in the 1920's and 1930's. Increasing the range of Reynolds numbers for which data were available was the primary purpose of most of their research. After a relatively dormant period in the 1940's, experimental research on viscous flow past a circular cylinder was rejuvenated in the 1950's and 1960's by Taneda [47], Acrivos [1], Tritton [52], and others. By this time the experimental equipment had improved considerably in quality, providing more accurate results over a larger range of Reynolds numbers. Additionally, there were more reliable
results to compare with at higher Reynolds numbers than the early analytical work, since the tremendous growth in numerical solutions occurred during this period as a result of the introduction of the digital computer. As one might expect, the improved experimental techniques and equipment produced data which corresponded very closely with the recent numerical solutions, which were restricted to relatively low values of Reynolds numbers.

**Numerical Investigations**

Although numerical methods were used as early as the 1920's to attack the viscous flow problem, they were severely restricted until the high speed digital computer was developed sufficiently in the 1950's, when research with numerical methods realized a tremendous increase. An early numerical technique, which is the basis for many current approaches to the problem and is described below, was introduced in 1928 by A. Thom [48]. He developed a finite difference scheme for solving the governing equations in modified form. Since the ideas of Thom required computing facilities, little research was done in the area for over twenty years until the 1950's, when M. Kawaguti [27] and H. B. Keller [29] made use of the recently developed digital computer to try modified forms of the method introduced by Thom. In the 1960's the power of the finite difference techniques became apparent to many. Thus a large number of articles appear in the literature for the
first time during this period describing numerical solutions to the Navier-Stokes equations [2, 3, 7, 8, 10, 11, 19, 20, 24, 25, 28, 30, 38, 43, 46, 50, 54].

By far, the most popular technique is to convert the Navier-Stokes and continuity equations to an equivalent but simpler mathematical problem of one order higher and solve the system by finite differences, which is basically the method introduced by Thom [48]. Introducing the stream function \( \psi \),

\[
u = \psi_y, \quad v = -\psi_x, \quad \]

and the vorticity \( \omega \),

\[
\omega = \nu_x - u_y ,
\]

the governing equations become,

\[
\nabla^2 \omega = \frac{R}{2} \frac{\partial^2 (\psi, \omega)}{\partial (x, y)},
\]

noting that the definition of the stream function automatically satisfies the continuity equation. A transformation is made to complex polar coordinates where the circular solution domain becomes a rectangle, which is easily and accurately approximated by finite differences. It is rather difficult to choose the location of the external boundary, at which the free stream boundary conditions are imposed. Therefore an extrapolating procedure, consisting

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of many solutions with different external boundary distances, is employed. The solutions obtained are extrapolated to approximate \( r \to \infty \), where \( r \) is the radial distance from the body [30].

Although the previously discussed method is the most popular, there have been attempts to analyze the problem in the primitive variables, \( u, v, P \). The most reliable work utilizing this method was published by A. J. Chorin in 1967 [7,8]. He proposed a finite difference scheme for solving the unsteady Navier-Stokes equations in two or three dimensions in the form,

\[
\begin{align*}
\frac{\partial u}{\partial t} + P_x &= f(u), \\
\frac{\partial v}{\partial t} + P_y &= g(v),
\end{align*}
\]

and the continuity equation in a differentiated form,

\[
\frac{\partial}{\partial t} \left( u_x + v_y \right) = 0.
\]

The time variable is discretized and at each time step an iteration technique is employed to calculate new values. His results seem reliable but somewhat restricted to domains with nice geometry.

The finite difference methods have unquestionably contributed greatly to progress made recently in the investigation of viscous flow, yet a significant restriction is the necessity of having a domain without irregular boundaries.
A considerably more general solution has been attempted recently using the geometrically more powerful finite element method. The simpler problem of inviscid fluid flow has been attacked by several authors with reasonable success [4,12,34,35,56]. M. D. Olson published the first useful results of a finite element analysis of viscous flow in March, 1972 [36]. He develops a variational principle equivalent to the standard stream function equation, with the pressure eliminated,

\[ \nabla^4 \psi + \psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x = 0. \]

The finite element method is employed to obtain an approximation to the stationary value of the functional. Since the plate deflection equation is a biharmonic equation, of the same order as the above governing equation, the author felt an eighteen degree of freedom plate deflection element might yield good results. The element is a three node triangle with the stream function, \( \psi \), and its five first and second derivatives, \( \psi_x, \psi_y, \psi_{xx}, \psi_{xy}, \psi_{yy} \), treated as unknowns at each node. Although the analysis by Olson is a significant contribution, some of the results for flow past a circular cylinder are suspect.
CHAPTER III

THE FINITE ELEMENT METHOD

The finite element method is a discretization process by which a continuous medium, or a system with finite degrees of freedom, is idealized by a set of finite regions, interconnected at their boundaries by nodal points [61]. The values of the desired function at the nodes are the unknowns. Basically a Rayleigh-Ritz technique [9], the method produces a system of simultaneous algebraic equations, whose solution is an approximation to the function which satisfies the governing equation and prescribed boundary conditions.

History

With the introduction of the high speed digital computer, the study of numerical techniques increased greatly in the 1950's. The finite element method is a direct result of two early techniques, structural analysis and finite differences.

Systems with finite degrees of freedom were initially attacked with structural analysis computer programs. Members, or elements, of the system are connected at joints, or nodes, by imaginary pins to allow only forces to be

14

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transmitted between members. Forces applied at the joints of a member, \( e \), are represented as a vector,

\[
\{F\}^e = \begin{bmatrix}
F_{x_1} \\
F_{y_1} \\
F_{x_2} \\
F_{y_2}
\end{bmatrix},
\]

as are the displacements at the nodes,

\[
\{\delta\}^e = \begin{bmatrix}
\delta_{x_1} \\
\delta_{y_1} \\
\delta_{x_2} \\
\delta_{y_2}
\end{bmatrix}.
\]

Imposing a known stiffness relation for each element between the forces and displacements, a system of simultaneous algebraic equations is formed

\[
[K]\{\delta\} = \{F\},
\]

where \([K]\) is the "stiffness" coefficient matrix, \(\{F\}\) is the "force" vector for the \( n \) nodes.
and \( \{F\} \) is the vector of unknown displacements at the nodes,

\[
\{F\} = \begin{bmatrix}
F_{x_1} \\
F_{y_1} \\
\vdots \\
\vdots \\
F_{x_n} \\
F_{y_n}
\end{bmatrix},
\]

and \( \{\delta\} \) is the vector of unknown displacements at the nodes,

\[
\{\delta\} = \begin{bmatrix}
\delta_{x_1} \\
\delta_{y_1} \\
\vdots \\
\vdots \\
\delta_{x_n} \\
\delta_{y_n}
\end{bmatrix}.
\]

The system of algebraic equations is solved for the displacements at the nodes, from which the strains are calculated, and an application of the constitutive relations yields the stresses in a member.

Continuous systems, possessing infinite degrees of freedom, originally were attacked by the finite difference
method [15]. The infinite degrees of freedom are approximated by a finite number, and the differential equation to be satisfied is approximated by a difference equation. A system of algebraic equations is formed from the difference equation at each node in the domain, and these equations are solved for the values of the unknown variable at each node. As the number of nodes is increased the approximation to the solution improves, until the error of computation becomes prohibitive.

A paper by Turner et al. [53] marked the beginning of finite element analysis in 1956. The aerospace industry was the first to employ the method on a large scale. When its power became more universally recognized in the 1960's, the finite element method experienced a great increase in use and analysis by both industrial and academic worlds. Since the introduction of the method, it has served primarily as a means of stress analysis. However, recently its applicability to other fields has become apparent.

Application

Most mathematical problems in engineering may be represented by one of two equivalent formulations. The more traditional approach is to state a governing differential equation to be satisfied and require certain forced boundary conditions to be met. An alternative formulation is to seek an extremum or at least a stationary value of a functional, usually defined as an integral over a domain,
subject to the same set of forced boundary conditions as in the first method. An application of variational calculus to find the function which imposes a stationary value on the functional yields not only the Euler equation, a differential equation in the unknown function, but also "natural" boundary conditions on the function [33,57].

Although the finite difference method directly approximates the differential equation and forced boundary conditions, the finite element method, being a type of Rayleigh-Ritz technique [9], is more suitable to the other formulation, that of requiring a functional to be stationary. The functional $I$ is defined as an integral of the unknown function $\{\phi\}$ and its derivatives over the domain $A$ and its boundary $S$,

$$I = \int_A f(\{\phi\},\{\phi\}_x, \ldots) \, dA + \int_S g(\{\phi\},\{\phi\}_x, \ldots) \, dS .$$

Let $\{\phi\}$ denote the unknown values of the function $\{\phi\}$ at the nodes. Within each element the function $\{\phi\}^e$ is evaluated from the "interpolating" or "shape" functions $[N]$ and the nodal values $\{\phi\}_n$,

$$\{\phi\}^e = [N]\{\phi\}_n .$$

Treating the nodal values $\{\phi\}$ as the "generalized coordinates," the Ritz technique leads to a system of equations necessary for $I$ to be stationary,
If the value of the functional over the whole domain is the sum of the values over each element, i.e.,

$$I = \sum_e I^e,$$

then the set of equations mentioned above becomes,

$$\frac{\partial I}{\partial \{\phi\}} = \sum_e \frac{\partial I^e}{\partial \{\phi\}^e} = 0 .$$

For the special case of $I$ quadratic in $\{\phi\}$ and its derivatives, it can be shown that the equations take on the special form,

$$\sum_e \frac{\partial I^e}{\partial \{\phi\}^e} = \sum_e ([H]^e \{\phi\}^e + \{F\}^e) = 0 ,$$

or simply,

$$[H]\{\phi\} + \{F\} = 0 ,$$

where the matrix $[H]$ and the vector $\{F\}$ are constant such that,
\[
[H_{ij}] = \sum_e [H_{ij}]^e , \\
\{F_i\} = \sum_e \{F_i\}^e .
\]

This set of simultaneous algebraic equations can be solved for the unknown values \{ϕ\}.

**Convergence Criteria**

As the number of elements is increased, the actual domain is more closely approximated by the finite element representation, yet for the solution to converge the functional must satisfy certain conditions, which are imposed on the shape functions \[N\].

The first condition requires the relations \(f\) and \(g\) in the functional integral to be single valued:

"The element shape functions \[N\] must be such that with a suitable choice of \(ϕ\)^e any constant values of \(ϕ\) or its derivatives present in the functional \(I\) should be able to be represented in the limit as element size decreases to zero" [61].

Another condition is needed to insure that

\[
I = \sum_e I^e ,
\]

which is accomplished by requiring the highest derivatives of \(ϕ\) to be finite:
"The element shape functions \([N]\) have to be so chosen that at element interfaces \(\{\phi\}\) and its derivatives, of one order less than that occurring in expressions \(f\) and \(g\) which define the functional, are continuous" [61].

**Interpolating Functions**

Much of the recent research in the field of finite element analysis has involved attempts to develop more sophisticated, and therefore better approximating, interpolating functions \([N]\).

The first popular two dimensional element was the three node linear triangle. Although very simple, the linear triangle has proven satisfactory for many problems, and therefore has received much attention [61]. The linear interpolation between nodes, for both geometry and function variation, is an obvious extension of finite difference methods. In fact, for some cases the same set of algebraic equations is obtained by both methods, providing the nodes are in the same positions. As the linear triangle is examined it becomes obvious that the necessary convergence criteria are satisfied by this element. Not only is the geometry of the element linear, but the variation of a function within the element is assumed to be linear. Let the function sought be a scalar \(\phi\), then,

\[
\phi = d_1 + d_2 x + d_3 y ,
\]

is a linear expression for \(\phi\), where \(d_1, d_2, d_3\) are
constants to be obtained by imposing the conditions at the three nodes,

\[ \phi_i = d_1 + d_2 x_i + d_3 y_i \quad , \quad i = 1,2,3 \]

Solving the three equations for the constants, and substituting in the linear expression for \( \phi \) yields,

\[ \phi = \frac{1}{2A} [(a_1+b_1x+c_1y)\phi_1+(a_2+b_2x+c_2y)\phi_2+(a_3+b_3x+c_3y)\phi_3] \]

where \( A \) is the area of the element, and,

\[ a_1 = x_2y_3 - x_3y_2 \quad , \quad b_1 = y_2 - y_3 \quad , \quad c_1 = x_3 - x_2 \]

and the other constants are obtained by permuting the indices. In matrix notation the function \( \phi \) can be written,

\[ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} e \]

where \( N_i \), \( i=1,2,3 \) are the shape functions,

\[ N_1 = \frac{1}{2A} (a_1+b_1x+c_1y) \quad , \quad \text{etc.} \]

or more simply,

\[ \phi = \langle N \rangle \{\phi\}^e \]
One of the major advantages of the formulation using linear triangles is that numerical integration is unnecessary. A big disadvantage is that representing functions with linear polynomials makes the first derivative in an element constant, and the second derivative zero. Thus the use of linear shape functions may be too crude an approximation of the actual solution surface, especially if the number of elements is small.

Linear rectangles provide a slight improvement over linear triangles. Consider the simple linear rectangle in Figure 2. Since the element boundaries are parallel to the coordinate axes, the shape function may include products of linear terms,

\[ \phi = d_1 + d_2x + d_3y + d_4xy \]

Note that along an edge the variation in \( \phi \) is linear, since one variable is constant on any edge. In the manner described for linear triangles, the constants \( d_1, d_2, d_3, d_4 \) can be evaluated, and \( \phi \) can be explicitly written as a function of \( x \) and \( y \) in terms of the nodal values of \( \phi \).

A significant improvement on linear elements is obtained by non-linear rectangles. To simplify the analysis a scalar unknown is used as before, and to illustrate the method a quadratic polynomial is employed. But, by no means is the analysis restricted to scalars or quadratic functions! Consider the eight node rectangle shown in
Figure 3. Since there are three nodes on an edge, at which the values of \( \{ \phi \} \) uniquely define a parabola, the shape function may contain terms of increasing degree up to the product of quadratic factors,

\[
\phi = d_1 + d_2x + d_3y + d_4xy + d_5x^2 + d_6y^2 + d_7x^2y + d_8xy^2.
\]

Note, however, that the number of terms is limited to eight by the number of constants which can be evaluated from the eight nodal conditions. Imposing these conditions yields,

\[
\begin{align*}
\phi_1 &= d_1 + d_2x_1 + \ldots + d_8x_1y_1^2, \\
\vdots \\
\phi_8 &= d_1 + d_2x_8 + \ldots + d_8x_8y_8^2,
\end{align*}
\]

or in matrix notation,

\[
\{\phi\}^e = [c]\{d\}.
\]

If there exists an inverse to \([c]\) then,

\[
\{d\} = [c]^{-1}\{\phi\}^e,
\]

and therefore since,

\[
\phi = \langle 1, x, y, \ldots, xy^2 \rangle\{d\},
\]

\(\phi\) can be written,

\[
\phi = \langle 1, x, y, \ldots, xy^2 \rangle [c]^{-1}\{\phi\}^e.
\]
The interpolating functions $N_i(x,y)$ can now be defined,

$$\phi = \langle N_1, N_2, \ldots, N_8 \rangle \phi^e,$$

or in matrix notation,

$$\phi = \langle N \rangle \phi^e.$$

Although this formal procedure for obtaining shape functions is very direct it has disadvantages. An immediate possibility is that the inverse of the matrix $[c]$ may not exist. Additionally there is always considerable difficulty in obtaining the inverse of $[c]$ while maintaining generality with respect to geometry. A close analysis of the shape functions reveals certain characteristics. Since the relation,

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_8 \end{bmatrix} \begin{bmatrix} N_1 \\ \vdots \\ N_8 \end{bmatrix},$$

must hold at any node, it is clear that $N_i$ must vanish at all nodes except node $i$, where $N_i$ must be one. Also the variation on an element edge must be uniquely determined by the nodal values.

In view of these conditions the interpolating functions can be derived by inspection for linear, quadratic, and cubic variation on an edge, but higher degrees are increasingly more difficult. It is convenient to introduce
a set of normalized coordinates,
\[ \xi = \frac{x - x_c}{a}, \quad \eta = \frac{y - y_c}{b}. \]

where \( x_c, y_c \) locate the rectangle's centroid, and \( a \) and \( b \) are half the lengths of the sides, see Figures 2 and 3.

The shape function for the linear rectangle can be expressed as a product of two factors,
\[ N_i = \frac{1}{4} \left( 1 + \xi \xi_i \right) \left( 1 + \eta \eta_i \right), \]

which clearly satisfies the condition of being zero at all nodes except \( i \), where \( N_i = 1 \). Since interpolation between nodes is done with a single valued relation, the first convergence criterion is satisfied. Also, the variation on an element boundary is linear, and there exist two nodes to uniquely determine the values, so the second criterion is satisfied. In a similar manner the interpolating functions for higher degree polynomials can be constructed. The results for quadratic and cubic variation on an edge are listed in Appendix A along with the linear variation.

Although improved shape functions are usually higher degree polynomials, there is nothing to prevent the use of some other functional form, e.g., the trigonometric functions, provided, of course, the function satisfies the convergence criteria.
Curved Isoparametric Quadrilaterals

To closely approximate a physical domain with non-linear boundaries would require many rectangular elements. Although the variation of a function within an element can be represented by sophisticated shape functions to reduce the error, the finite element approximation of the problem is not very effective unless the domain is closely approximated. Thus, it is extremely helpful if the "parent" rectangular elements can be mapped into elements with non-linear edges. The mapping of rectangles into non-linear quadrilaterals gives the finite element method an obvious advantage over the finite difference method in representing the solution domain, particularly if the boundaries are irregular.

By far, the most convenient means of writing the transformation from local $\xi, \eta$ curvilinear coordinates to global $x, y$ coordinates is to employ the already derived shape functions. Considering $x$ and $y$ as functions of $\xi$ and $\eta$, an expression similar to that for the function $\phi$ is obtained,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}^e, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}^e$$

where the functions $N_i'(\xi, \eta)$ are shape functions which

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determine the transformation. If these functions are required to satisfy the convergence conditions, the transformation will be one to one. Although continuity of the function \( \phi \) is evident in the parent element, the values of \( \phi \) in the curved elements must be investigated for continuity. Zienkiewicz summarizes the results of the investigation in two theorems [61]:

**Theorem 1.** "If two adjacent elements are generated from 'parents' in which the shape functions satisfy continuity requirements then the distorted elements will be contiguous."

**Theorem 2.** "If the shape functions \( N_i(\xi,\eta) \) used to define \( \phi \) are such that continuity of \( \phi \) is preserved in the parent coordinates then continuity requirements will be satisfied in distorted elements."

It should be noted that nodal values of \( \phi \) may or may not be associated with the same nodes as those used to specify the geometry. If the number of nodes used to define the geometry is greater than the number used to define the function \( \phi \), then the element is labeled "super-parametric." In the reverse situation the element is called "sub-parametric," which is the more useful type in practice. The third possibility is the more convenient, and is the type used in the present analysis. If the nodes used to define the geometry coincide with those used to define \( \phi \), and if \( N_i = N_i' \), i.e. the interpolating functions are the same for both the geometry definition and function variation, then the elements are denoted "isoparametric."

The basic non-linear element used in the present work is the eight node, quadratic, isoparametric quadrilateral
shown in Figure 4, whose shape functions are listed in Appendix A. The function $\phi$ is given by,

$$\phi = d_1 + d_2 \xi + d_3 \eta + d_4 \xi \eta + d_5 \xi^2 + d_6 \eta^2 + d_7 \xi^2 \eta + d_8 \xi \eta^2$$

which is expressed by the interpolating functions. The geometry transforms according to the same functions,

$$x = (N)\{x\}^e$$
$$y = (N)\{y\}^e$$

For most problems it is necessary to calculate at least the first derivatives of some function with respect to $x$ and $y$. Since the interpolating polynomials, and therefore all functions defined in an element, depend on the local coordinates, it is necessary to make a transformation to calculate $x$ and $y$ derivatives. A simple application of the chain rule results in,

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix},$$

where $[J]$ is the Jacobian matrix,

$$[J] = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}.$$
For admissible transformations, the derivatives with respect to \( x \) and \( y \) can be calculated,

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix}
\]

Some problems require not only first but second derivatives with respect to \( x \) and \( y \). As with first derivatives, the second derivatives with respect to the global coordinates are obtained from the local derivatives,

\[
\begin{bmatrix}
\frac{\partial^2}{\partial x^2} \\
\frac{\partial^2}{\partial x \partial y} \\
\frac{\partial^2}{\partial y^2}
\end{bmatrix} = \begin{bmatrix}
x_\xi^2 & 2x_\xi y_\xi & y_\xi^2 \\
x_\xi x_\eta & x_\xi y_\eta + x_\eta y_\xi & y_\xi y_\eta \\
x_\eta^2 & 2x_\eta y_\eta & y_\eta^2
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial^2}{\partial \xi^2} - x_\xi \frac{\partial}{\partial x} - y_\xi \frac{\partial}{\partial y} \\
\frac{\partial^2}{\partial \xi \partial \eta} - x_\xi \frac{\partial}{\partial x} - y_\eta \frac{\partial}{\partial y} \\
\frac{\partial^2}{\partial \eta^2} - x_\eta \frac{\partial}{\partial x} - y_\eta \frac{\partial}{\partial y}
\end{bmatrix}
\]

where the first derivatives are calculated from the Jacobian matrix as discussed earlier,

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \begin{bmatrix}
y_\eta & -y_\xi \\
-x_\eta & x_\xi
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix}
\]
To calculate each of the matrices above, the derivatives of $x$ and $y$ with respect to $\xi$ and $\eta$ are obtained from,

$$x = \langle N \rangle \{x\}^e, \quad y = \langle N \rangle \{y\}^e,$$

where $\langle N \rangle$ contains the interpolating functions, which depend on the local coordinates.

**Elimination of Internal Nodes**

Nodes within an element need not be included in the overall finite element equations. Equations associated with an internal node may be eliminated from the global equations since they form an independent set of equations. Internal node equations are independent because the finite element neighborhood for an internal node is simply the element surrounding that node. Consider the element formulation for element $e$,

$$\frac{\partial I^e}{\partial \{\phi\}^e} = [H]^e \{\phi\}^e - \{F\}^e,$$

where $\{\phi\}^e$ is the unknown variables at the nodes, and $[H]^e$ and $\{F\}^e$ are the element matrix and vector. Since $\{\phi\}^e$ can be separated into a part which corresponds to nodes common to other elements $\{\overline{\phi}\}^e$, and one which corresponds to nodes within the element $\{\bar{\phi}\}^e$, it becomes clear that one finite element equation is,
\[
\frac{\partial I}{\partial \{\phi\}^e} = \frac{\partial I^e}{\partial \{\phi\}^e} = 0.
\]

Using this relation the element formulation equation can be partitioned to yield,

\[
\frac{\partial I^e}{\partial \{\phi\}^e} = \begin{bmatrix}
\frac{\partial I^e}{\partial \{\phi\}^e} \\
\frac{\partial I^e}{\partial \{\phi\}^e}
\end{bmatrix} = \begin{bmatrix}
[H]^e, [H^*]^e \\
[H^*]^e, [\bar{H}]^e
\end{bmatrix} \begin{bmatrix}
\{\phi\}^e \\
\{\bar{\phi}\}^e
\end{bmatrix} - \begin{bmatrix}
\{F\}^e
\end{bmatrix}
\]

Solving the second matrix equation for \(\{\bar{\phi}\}^e\),

\[
\{\bar{\phi}\}^e = [\bar{H}]^e^{-1} ([H^*]^e \{\phi\}^e - \{F\}^e)
\]

and substituting in the first matrix equation yields,

\[
\frac{\partial I^e}{\partial \{\phi\}^e} = [H^{**}]^e \{\phi\}^e - \{F^*\}^e
\]

where,

\[
[H^{**}]^e = [\bar{H}]^e - [H^*]^e [\bar{H}]^e^{-1} [H^*]^e^T
\]

and

\[
\{F^*\}^e = \{F\}^e - [H^*]^e [\bar{H}]^e^{-1} \{\bar{F}\}^e
\]

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The finite element formulation now contains only the terms corresponding to the boundary nodes for element e. Employing this procedure to eliminate internal nodes requires some additional calculations in equation formation, but reduces significantly the equation solving effort. An obvious simple example is the five node quadrilateral shown in Figure 7. Doherty et al. have investigated this simple element and many others [13]. More sophisticated elements, which are discussed in Chapters VI and VII, appear in Figures 6, 7 and 8. The above elimination scheme is applied to each of the internal nodes. It is worth noting that the element matrix is modified each time a node is eliminated, which is very valuable for some problems.

The generality of the methods described is obvious. An immediate possibility is to extend the analysis to three dimensions. Additionally, the scalar unknown could be replaced by a vector function with little complication. Also, it is apparent that the specification of the interpolating functions is arbitrary, allowing the versatility of changing the functions used without rewriting the program.

Choosing the most efficient element to attack a problem is an art strongly dependent on the engineer's intuition and understanding of the physical problem. However, curved, isoparametric quadrilaterals should be considered in any problem as a means of reducing input, since fewer elements will be needed, and improving accuracy, as a result of
better interpolating functions. The possibilities for element construction and choice of interpolating functions are innumerable, depending only on the engineer's imagination!
CHAPTER IV

METHODS OF SOLUTION

Solutions to the Navier-Stokes and continuity equations are approximated by successively solving a sequence of linear equations by the finite element method. An initial estimate is made of the velocity and pressure fields, from which the non-linear terms are calculated and treated as known functions. New estimates of the velocities and pressures are computed from the linear system. The procedure is repeated until two successive solutions are "close." This iteration technique is applied in two forms, each of which is explained below. Some preliminary investigations using method 1 have been performed by D. L. Garrett [16].

Method 1

The governing equations from Chapter I may be linearized by approximating the non-linear terms,

\begin{align*}
\frac{1}{R_E} \nabla^2 u - p_x &= f(x, y) , \\
\frac{1}{R_E} \nabla^2 v - p_y &= g(x, y) , \\
& u_x + v_y = 0 ,
\end{align*}

35

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where,

\[ f(x,y) = uu_x + vv_y \]
\[ g(x,y) = vv_x + vv_y \]

Assuming a velocity and pressure field, \( u^o, v^o, P^o \), the approximating functions can be evaluated,

\[ f^o(x,y) = u^o u_x^o + v^o u_y^o \]
\[ g^o(x,y) = u^o v_x^o + v^o v_y^o \]

The linear equations,

\[ \frac{1}{Re} v^2 u^i - p_x^i = f^o(x,y) \]
\[ \frac{1}{Re} v^2 v^i - p_y^i = g^o(x,y) \]
\[ u_x^i + v_y^i = 0 \]

and the appropriate forced boundary conditions, are solved by the finite element method for the new velocities and pressures, \( u^i, v^i, P^i \). The procedure is repeated until two successive solutions are close. At the \( n^{th} \) step, the iteration scheme is,
\[
\frac{1}{R_E} \nabla^2 u^n - p^n_x = (u u_x + v v_y)^{n-1},
\]
\[
\frac{1}{R_E} \nabla^2 v^n - p^n_y = (u v_x + v v_y)^{n-1},
\]
\[
u^n_x + v^n_y = 0.
\]

It is hoped that as \( n \) gets large the method will converge to a realistic solution. However, even though two successive solutions may be close, there is no guarantee that the method has converged to a good approximation of the true solution.

**Method 2**

Rewriting the governing equations in a different form yields,
\[
\nabla^2 u = R_E(u u_x + v v_y + P_x),
\]
\[
\nabla^2 v = R_E(u v_x + v v_y + P_y),
\]
\[
u_x + v_y = 0.
\]

If the first equation is differentiated with respect to \( x \), the second with respect to \( y \), and the two are added, the result is,
Due to the linearity of the Laplacian operator the equation may be written,

\[ \frac{1}{R_E} \frac{\partial}{\partial x} \nabla^2 u + \frac{1}{R_E} \frac{\partial}{\partial y} \nabla^2 v = \frac{\partial}{\partial x}(uu_x + vv_y) + \frac{\partial}{\partial y}(uv_x + vv_y) + P_{xx} + P_{yy} . \]

or, since the continuity equation must be satisfied,

\[ \nabla^2 p = - \frac{\partial}{\partial x}(uu_x + vu_y) - \frac{\partial}{\partial y}(uv_x + vv_y) . \]

Combining this equation with those for \( u \) and \( v \) results in a system of three equations in the velocities and pressure. However, since the new system of equations is one order higher than the original system, an extra boundary condition is necessary. Such a condition does become necessary in the finite element formulation of the pressure equation, which is discussed in detail in Chapter V. The new system of governing equations is,

\[ \nabla^2 u = f \ , \quad \nabla^2 v = g \ , \quad \nabla^2 P = h \ , \]

where,
\[ f = R_E(uu_x + vu_y + P_x) , \quad g = R_E(uv_x + vv_y + P_y) , \]
\[ h = -\frac{\partial}{\partial x}(uu_x + vu_y) - \frac{\partial}{\partial y}(uv_x + vv_y) , \]

and the boundary conditions are the original set of physical conditions, plus the additional condition on the pressure needed for the new system to be equivalent to the original. As with method 1, the non-linear functions, \( f \), \( g \), and \( h \), can be calculated from a known velocity and pressure field, which results in a system of three, uncoupled, Poisson equations in \( u \), \( v \), and \( P \).

The iteration technique is essentially the same as that for method 1, except that the equations are different. Assuming an initial velocity and pressure field, \( u^0, v^0, P^0 \), values of \( f^0, g^0 \), and \( h^0 \), are computed, and the three Poisson equations are solved by the finite element method for new values of the velocities and pressures, \( u^1, v^1, P^1 \). As before, the procedure is repeated until two successive solutions are "close." Again, it is hoped that the iteration scheme will converge to a realistic solution.

It should be emphasized that no attempt is made to prove, or disprove, convergence of the iteration schemes. This task may be not only difficult, but possibly intractable!
CHAPTER V

FINITE ELEMENT FORMULATION

The general methods of Chapter III will now be applied to the solutions of particular problems. Explicit expressions will be obtained for the simultaneous algebraic equations formed by the finite element method, and techniques of imposing some boundary conditions will be discussed. The two methods described in Chapter IV will be formulated. Additionally, the simple linear triangle will receive some special attention.

Method 1

As stated in Chapter IV the finite element method is used to solve the system of linear equations,

$$\frac{1}{R_E}(u_{xx} + u_{yy}) - P_x - f = 0$$

$$\frac{1}{R_E}(v_{xx} + v_{yy}) - P_y - g = 0$$

$$u_x + v_y = 0$$

where $f$ and $g$ are functions of $x$ and $y$, and $u$, $v$, and $P$ are subject to forced boundary conditions. An equivalent
The mathematical formulation is to seek functions $u$, $v$, and $P$, which will give a stationary value to the functional,

$$I = \int_{A} \left[ \frac{1}{2\mathcal{R}_E} \left( u_x^2 + u_y^2 + v_x^2 + v_y^2 \right) + uf + vg - P(u_x + v_y) \right] \, dx \, dy.$$  

An application of variational calculus to the functional $I$, yields three Euler equations, which correspond exactly to the above governing equations, and, in addition, the natural boundary conditions,

$$\frac{1}{\mathcal{R}_E} u_x - P = 0 \quad \text{for } x \text{ constant},$$

$$\frac{1}{\mathcal{R}_E} v_y - P = 0 \quad \text{for } y \text{ constant},$$

$$u_y = 0 \quad \text{for } y \text{ constant},$$

$$v_x = 0 \quad \text{for } x \text{ constant}.$$  

Similar expressions are obtained for boundaries which are not parallel to the coordinate axes. By imposing the forced boundary conditions,

$$r = R(\theta) : u = v = 0,$$

$$y = 0 : v = u_y = p_y = 0,$$

$$r \rightarrow \infty : u \rightarrow 1, v \rightarrow 0, P \rightarrow 0,$$

the natural boundary conditions are overridden.

The finite element formulation is obtained by
employing a Ritz technique to find the functions u, v, and P which render the functional I stationary. Treating the values of the functions at the M nodes as the unknown variables, the finite element method requires the equations,

$$\sum_{e=1}^{n} \frac{\partial I^e}{\partial u_k} = 0 , \quad \sum_{e=1}^{n} \frac{\partial I^e}{\partial v_k} = 0 , \quad \sum_{e=1}^{n} \frac{\partial I^e}{\partial P_k} = 0 , \quad k=1, 2, ... M,$$

to be satisfied. Here \(u_k, v_k, P_k\) are values of the functions at node \(k\), \(n\) is the number of elements in the so-called "finite element neighborhood" of node \(k\), and \(I^e\) is the functional \(I\) integrated only over the part of the domain occupied by element \(e\). The "finite element neighborhood" is the group of elements which contain node \(k\). It is now necessary to evaluate the derivatives of \(I^e\) with respect to the values \(u_k, v_k, P_k\),

$$\frac{\partial I^e}{\partial u_k} = \left\{ \frac{1}{R_E} \left[ (u_x \frac{\partial u_x}{\partial u_k} + u_y \frac{\partial u_y}{\partial u_k}) + f \frac{\partial u}{\partial u_k} - p \frac{\partial u_x}{\partial u_k} \right] \right\} dx dy ,$$

$$\frac{\partial I^e}{\partial v_k} = \left\{ \frac{1}{R_E} \left[ (v_x \frac{\partial v_x}{\partial v_k} + v_y \frac{\partial v_y}{\partial v_k}) + g \frac{\partial v}{\partial v_k} - p \frac{\partial v_y}{\partial v_k} \right] \right\} dx dy ,$$

$$\frac{\partial I^e}{\partial P_k} = \left\{ \left[ - \frac{3p}{\partial P_k} (u_x + v_y) \right] \right\} dx dy ,$$

where it should be recognized that \(v\) and \(P\) do not depend on the nodal values of \(u\), and likewise for the other variables. Recalling the fact that the variables are written,
\[ u = \langle N \rangle \{u\}^e, \quad v = \langle N \rangle \{v\}^e, \quad P = \langle N \rangle \{P\}^e, \]

the derivatives above can be evaluated,

\[
\frac{\partial I^e}{\partial u_k} = \left[ \frac{1}{R} \int_{e} \left( \frac{\partial N_k}{\partial x} \langle \frac{\partial N_1}{\partial x}, \ldots \rangle + \frac{\partial N_k}{\partial y} \langle \frac{\partial N_1}{\partial y}, \ldots \rangle \right) \, dx \, dy \right] \{u\}^e

- \left[ \int_{e} \frac{\partial N_k}{\partial x} \langle N \rangle \, dx \, dy \right] \{P\}^e + \int_{e} f N_k \, dx \, dy,
\]

a similar expression for \( \frac{\partial I^e}{\partial v_k} \) and,

\[
\frac{\partial I^e}{\partial P_k} = \left[ - \int_{e} N_k \langle \frac{\partial N_1}{\partial x}, \ldots \rangle \, dx \, dy \right] \{u\}^e + \left[ - \int_{e} N_k \langle \frac{\partial N_1}{\partial y}, \ldots \rangle \, dx \, dy \right] \{v\}^e.
\]

In matrix notation these expressions can be written,

\[
\begin{bmatrix}
\frac{\partial I^e}{\partial u_1} \\
\frac{\partial I^e}{\partial v_1} \\
\frac{\partial I^e}{\partial P_1} \\
\frac{\partial I^e}{\partial u_2} \\
\vdots
\end{bmatrix}
= [H]^e
\begin{bmatrix}
u_1 \\
v_2 \\
p_1 \\
u_2 \\
\vdots
\end{bmatrix} - \{F\}^e,
\]

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where \([H]^e\) is the coefficient matrix for element \(e\), and \({F}^e\) is the constant vector for element \(e\), for which explicit expressions are found in Appendix B. Having expressions for \(\frac{\partial I^e}{\partial u_k}, \frac{\partial I^e}{\partial v_k}, \frac{\partial I^e}{\partial p_k}\), the finite element equations are formed,

\[
\begin{bmatrix}
\sum_{e=1}^{n} \frac{\partial I^e}{\partial u_k} \\
\sum_{e=1}^{n} \frac{\partial I^e}{\partial v_k} \\
\sum_{e=1}^{n} \frac{\partial I^e}{\partial p_k}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
k=1, 2, \ldots, M,
\end{bmatrix}
\]

which can be written in matrix notation as,

\[
[H]{W} - {F} = 0,
\]

where \([H]\) is the overall coefficient matrix, \({W}\) is the vector of unknown velocities and pressures at the nodes, and \({F}\) is the overall constant vector. It is worth noting that the components of the overall vector \({F}\) are simply sums of the components of the element vectors \({F}^e\) corresponding to a particular node, as is true for the overall matrix \([H]\).

The system of linear, simultaneous, algebraic equations is solved for the components of the vector \({W}\). Thus, an approximation of the functions \(u(x,y), v(x,y), P(x,y)\) is

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given by the values \( u_k, v_k, P_k \); \( k = 1, 2, \ldots, M \), where \( M \) is the number of nodes.

Having calculated a new estimate of \( u, v, \) and \( P \), the approximating functions \( f \) and \( g \) can be computed from the new values. Within an element the values of the functions \( u \) and \( v \), and their derivatives, are calculated from the nodal values,

\[
\begin{align*}
u &= \langle N \rangle \{u\}^e, \\
u_x &= \langle N_x \rangle \{u\}^e, \quad u_y = \langle N_y \rangle \{u\}^e, \\
\end{align*}
\]

and similar expressions for \( v \). Recalling the functions \( f \) and \( g \),

\[
\begin{align*}
f &= uu_x + vu_y, \quad g = uv_x + vv_y,
\end{align*}
\]

the new values of the functions can be calculated from the new estimates for the velocities at the nodes,

\[
\begin{align*}
f &= (\langle N \rangle \{u\}^e)(\langle N_x \rangle \{u\}^e) + (\langle N \rangle \{v\}^e)(\langle N_y \rangle \{u\}^e), \\
g &= (\langle N \rangle \{u\}^e)(\langle N_x \rangle \{v\}^e) + (\langle N \rangle \{v\}^e)(\langle N_y \rangle \{v\}^e).
\end{align*}
\]

From the new values of \( f \) and \( g \) the governing equations can again be solved for another approximation of \( u, v, \) and \( P \). As described in Chapter IV the procedure is repeated until convergence is achieved.
Method 2

The second approach employs the finite element method to solve the system of Poisson equations,

\[ \nabla^2 u = f(x,y) , \]
\[ \nabla^2 v = g(x,y) , \]
\[ \nabla^2 p = h(x,y) , \]

subject to the same set of forced boundary conditions stated for method 1.

Since each equation is the same type, and the equations are independent, it is helpful to consider the solution of a general Poisson equation,

\[ \nabla^2 \phi = \psi(x,y) , \]

where \( \phi \) and \( \psi \) are sufficiently smooth functions of \( x \) and \( y \). The finite element method solves this equation by requiring a stationary value of the functional,

\[ I = \int_A \left[ \frac{1}{2} (\phi_x^2 + \phi_y^2) + \phi \psi \right] \, dx \, dy . \]

A stationary value of \( I \) is obtained, provided the Euler equation,

\[ \nabla^2 \phi = \psi , \]

and the natural boundary condition,

\[ \frac{\partial \phi}{\partial n} = 0 , \]
are satisfied. An analysis of this condition for each variable follows. The finite element equations are,

\[ \sum_{e=1}^{n} \frac{\partial I^e}{\partial \phi_k} = 0 \]

where \( n \) is the number of elements in the finite element neighborhood, and \( k=1, 2, \ldots, M \), where \( M \) is the number of nodes. The derivatives of \( I^e \) are needed,

\[ \frac{\partial I^e}{\partial \phi_k} = \int_{e} \left[ \phi_x \frac{\partial \phi^e_x}{\partial \phi_k} + \phi_y \frac{\partial \phi^e_y}{\partial \phi_k} + \frac{\partial \phi}{\partial \phi_k} \psi \right] \, dx \, dy \]

Again recalling the functions within an element are written in terms of the nodal values,

\[ \phi = \langle N \rangle \{ \phi \}^e \]

\[ \phi_x = \langle N_x \rangle \{ \phi \}^e, \quad \phi_y = \langle N_y \rangle \{ \phi \}^e \]

the derivative of \( I^e \) becomes,

\[ \frac{\partial I^e}{\partial \phi_k} = \int_{e} \left[ \frac{\partial N_k}{\partial x} \left\{ \langle N \rangle \{ \phi \}^e \right\} + \frac{\partial N_k}{\partial y} \left\{ \langle N \rangle \{ \phi \}^e \right\} \right] \, dx \, dy + \int_{e} N_k \psi \, dx \, dy \]

In matrix notation the equations can be written,

\[ \frac{\partial I^e}{\partial \phi_k} = [H]^e \{ \phi \}^e - \{ F \}^e \quad , \quad k=1, 2, \ldots, M, \]
where \([H]^e\) and \({F}^e\) are the coefficient matrix and constant vector for element \(e\), given explicitly in Appendix B. The finite element equations can be formed,

\[
\sum_{e=1}^{N} \frac{\partial I^e}{\partial \phi_k} = \sum_{e=1}^{N} ([H]^e\{\phi\}^e - {F}^e) = 0, \quad k=1,2, \ldots, M,
\]

or in matrix notation,

\[
[H]\{\phi\} - {F} = 0,
\]

where \([H]\) and \({F}\) are the overall coefficient matrix and constant vector, and \({\phi}\) is the vector of unknown values at the nodes,

\[
\{\phi\} = \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\vdots
\end{pmatrix}.
\]

The finite element equations are linear, simultaneous, algebraic equations, which can be solved by a standard method for the values of \(\phi\) at the nodes.

Having a solution method for a general Poisson equation, the governing equations in \(u\), \(v\), and \(P\) can be solved. A simple substitution of the appropriate variable for \(\phi\) in the above analysis yields a system of algebraic equations in the nodal values of that variable, whether it be \(u\), \(v\), or \(P\).
It is important to consider the boundary conditions imposed on a particular variable. As noted above the natural boundary condition on a variable is that the normal derivative is zero. The forced boundary conditions must be imposed on each variable separately. For the \( u \) equation the forced boundary conditions are,

\[
\begin{align*}
    r &= R(\theta) : \quad u = 0 , \\
    r &= R_{\text{max}}(\theta) : \quad u = 1 ,
\end{align*}
\]

where \( R \) is the body radius, and \( R_{\text{max}} \) is the radius to the external boundary. It is clear that allowing the natural boundary condition of zero normal derivative to exist on the remaining part of the boundary, that being the line of symmetry, agrees with the physical requirements. The forced conditions to accompany the \( v \) equation are,

\[
\begin{align*}
    r &= R(\theta) : \quad v = 0 , \\
    r &= R_{\text{max}}(\theta) : \quad v = 0 , \\
    y &= 0 : \quad v = 0 ,
\end{align*}
\]

which includes the entire boundary, so the natural boundary condition is irrelevant. For the pressure equation there is only one forced condition,

\[
    r = R_{\text{max}}(\theta) : \quad P = 0 .
\]

The zero normal derivative condition agrees with physical requirements on the axis of symmetry but not on the body.
Therefore, it is necessary to impose an additional forced boundary condition on the pressure on the body. The extra condition requires the normal derivative of the pressure to correspond to that prescribed by the flow field. Two successive solutions for the velocities and the preceding pressure are used to calculate the pressure gradient in the $x$ and $y$ directions,

\[
    p_x^{n+1} = (p_x + uu_x + vu_y)^n - (uu_x + vu_y)^{n+1},
\]

\[
    p_y^{n+1} = (p_y + uv_x + vv_y)^n - (uv_x + vv_y)^{n+1}.
\]

After first calculating $u^{n+1}$ and $v^{n+1}$ in an iteration, the pressure gradient is approximated. From the two components of the pressure gradient, a normal derivative is calculated. To impose a normal derivative of $q(s)$ on a boundary $S$, it is necessary to add a term to the functional,

\[
    I^{BC} = - \int_S qPds,
\]

where $P$ is the pressure as a function of the location on the boundary. This term requires the specified derivative as a natural boundary condition. The functions $q(s)$ and $P(s)$ can be written,

\[
    q = \langle \bar{N} \rangle \{q\}^S, \quad P = \langle \bar{N} \rangle \{P\}^S,
\]

where $\langle \bar{N} \rangle$ is the row vector containing interpolating functions in one dimensional curvilinear coordinates, and $\{q\}^S$...
and \( \{ P \}^S \) contain the nodal values on the boundary \( S \). The derivative of the extra term needed for the finite element equations is,

\[
\frac{\partial I_{k}^{BC}}{\partial P} = - \int_{S} \langle N \rangle \{ q \}^S N_k \, ds.
\]

By adding this term to column \( K \) of the element constant vector \( \{ F \}^E \), the normal derivative of \( P \) is set equal to \( q \).

The imposing of an extra boundary condition on the pressure is a mathematical necessity, as well as a physical requirement. Recalling that the Poisson equation in \( P \) is obtained from a combination of first derivatives of the motion equations and the continuity equation, it is clear that an extra boundary condition is needed on \( P \), if the two systems of differential equations are to be equivalent.

Having calculated a new estimate of the velocities and pressure at each node, the approximating functions \( f \), \( g \), and \( h \) can be computed. Noting the definitions of the functions,

\[
\begin{align*}
    f &= R_E(\mathbf{u}_x \cdot \mathbf{v}_y + \mathbf{v}_x \cdot \mathbf{u}_y + P_x), \\
    g &= R_E(\mathbf{u}_y \cdot \mathbf{v}_x + \mathbf{v}_y \cdot \mathbf{v}_y + P_y), \\
    h &= -(u_x^2 + v_y^2 + u_{xx} \cdot v_{yy} + v_{yy} \cdot v_{xx} + 2u_y v_x + v_{xy} + u v_{xy} + u v_{xy}),
\end{align*}
\]

within each element \( f \), \( g \), and \( h \) can be written in terms of
the nodal values of $u$, $v$, and $P$,

$$f = \mathcal{R}_E \left[ (\langle N\rangle \{u\}^e \langle N_x \rangle \{u\}^e) + (\langle N\rangle \{v\}^e \langle N_y \rangle \{u\}^e) + \langle N_x \rangle \{P\}^e \right] ,$$

$$g = \mathcal{R}_E \left[ (\langle N\rangle \{u\}^e \langle N_x \rangle \{v\}^e) + (\langle N\rangle \{v\}^e \langle N_y \rangle \{v\}^e) + \langle N_y \rangle \{P\}^e \right] ,$$

$$h = - \left[ (\langle N_x \rangle \{u\}^e)^2 + (\langle N_y \rangle \{v\}^e)^2 + \langle N_x \rangle \{u\}^e \langle N_{xx} \rangle \{u\}^e \right]$$

$$+ \langle N\rangle \{v\}^e \langle N_{yy} \rangle \{v\}^e \right] + 2 \langle N_y \rangle \{u\}^e \langle N_x \rangle \{v\}^e \right]$$

$$+ \langle N\rangle \{v\}^e \langle N_{xy} \rangle \{u\}^e \right] + \langle N\rangle \{u\}^e \langle N_{xy} \rangle \{v\}^e \right] \right] .$$

Using the newly calculated functions $f$, $g$, and $h$, the finite element method is employed again to solve the three Poisson equations for new values of $u_k$, $v_k$, $P_k$, $k=1, 2, \ldots, M$. As described in Chapter IV the procedure is repeated until convergence is achieved.

Since the shape functions for a linear triangle are very simple, the element formulation deserves special attention. Appendix C contains a detailed discussion of the formulation with linear triangles.
CHAPTER VI

DESCRIPTION OF COMPUTER PROGRAMS

Computer programs which implement and test the preceding methods are explained in the following discussion. Program listings appear in Appendix D. Two basic programs, one for each solution method, and a test program are described. Each subroutine is examined separately, and this, coupled with an abundance of comment cards in the listing, should be sufficient to explain the programming logic.

Method 1

Although many programs are used to attack the problem with method one, only one is explained here. Others are discussed in the next chapter. The program examined here employs a special variation of the four node quadrilateral, which is formed by twelve linear triangles connected at nine nodes, five of which are internal nodes, see Figure 6. Using seventy-four elements and ninety-five points requires about 100,000 bytes (hexadecimal) of storage and slightly over one minute of central processing unit time on an IBM 360/65 model computer. Of course, the time requirement depends directly on the number of iterations performed. Most arrays in the program are dimensioned in and passed through labeled or unlabeled common blocks.
Main Program

The main program organizes the calculations, which are done in the subroutines. Input to the program are the problem parameters,

RE---Reynolds Number,
RMAX---Maximum radius on downstream axis,
EXPR---Exponent for radial distribution of nodal points, see subroutine GRID,
EXPT---Exponent for transverse distribution of nodal points, see subroutine GRID,
NELRUP---Number of elements in the radial direction on the upstream axis,
NELRDN---Number of elements in the radial direction on the downstream axis,
NELT---Number of elements in the transverse direction for half the grid, note the total elements in the transverse direction is 2*NELT,
NPREL---Number of points in the radial direction of an element,
NPTEL---Number of points in the transverse direction of an element,
RO---Body radius at each step in the transverse direction, R(θ) in the previous notation.

The finite element grid is generated by calling subroutine GRID, and boundary conditions are set by calling subroutine BOUNDY. An initial guess at the velocity and pressure is calculated in subroutine GUESS, where several guesses are possible. After printing the initial guess and the problem parameters, the iteration scheme is started. After setting the initial value of the relaxation factor GAMMA, the new values of velocity and pressure are
calculated by calling subroutine SOLVE. The values of BIG and SUM are computed,

\[
BIG = \max\{|u_k^n - u_{k-1}^n|, |v_k^n - v_{k-1}^n|, |p_k^n - p_{k-1}^n|, k=1, NP\},
\]

\[
SUM = \sum_{k=1}^{NP} (|u_k^n - u_{k-1}^n| + |v_k^n - v_{k-1}^n| + |p_k^n - p_{k-1}^n|),
\]

where NP is the number of nodal points and superscripts denote iteration numbers. The sum is compared to the previous sum to check convergence. Additionally, as the iteration converges the value of the relaxation factor is increased to a final value of unity. If the method has produced a sufficiently small sum, the iteration stops and the values of u, v, and p at the nodes, which are stored in the vector W, are printed. The Reynolds number is incremented and another problem is worked, until the Reynolds number reaches the desired value. The initial guess for each problem after the first is the previous solution.

Subroutine GRID (NELRUP, NELRDN, NELT, NPR, NPTEL, RMAXDN, EXPR, EXPT)

This subroutine computes the location of the nodal points in terms of the grid parameters, defines the elements by their nodes, and decides which type point each node is. A polynomial of degree express is used to distribute the points on a radial line. And a polynomial of degree expt distributes in the transverse direction. As expr and expt increase the points are grouped more densely near the body.
and near the axis of symmetry respectively. The upstream domain is generated by adding elements in the radial direction, until NELRUP is reached. At the beginning of each radial step the node is marked, in NTYPE, as one on the body, and at the end the node is marked as an external boundary node. Also on the axis of symmetry the nodes are marked. Radial lines of nodal points are added until there are NELT elements in the transverse direction, which will correspond to the domain on the upstream side of the y axis. The downstream domain is generated in a similar manner. However, the number of elements in the radial direction is increased by one until NELRDN is reached. The radial and transverse coordinates are generated, and the rectangular coordinates are computed from them. After the coordinates are calculated, the elements are defined. While stepping out in the radial and transverse directions, the nodes defining each element are stored in the array NODE. Finally the number of nodal points, NP, number of equations, NEQ, the matrix bandwidth, MBAND, and the number of elements, NEL, are computed.

Subroutine BOUNDY (NVAR, NBOUND, NCOND)

The boundary conditions are set in this subroutine according to the flag each node receives in subroutine GRID. NVAR determines the variable to be fixed, either u, v, or P. NBOUND locates the point on the boundary, whether it be on the body, at the external boundary, or on the axis of symmetry. The code of NCOND determines the value that
the variable assumes, which is represented in IB. Setting
the value is discussed in subroutine SOLVE, where IB is
used.

Subroutine GUESS(K)

Here an initial guess for the velocity and pressure
field is calculated. The integer K is used to choose the
guess. For test problems special guesses may be desired as
in the case for the free stream problem to be discussed in
the next chapter. The initial guess for the real problem
is the Oseen solution of the Navier-Stokes equations, which
is presented in Chapter II.

Subroutine SOLVE

In this subroutine the algebraic equations are
formed, the boundary conditions imposed, and a solution to
the equations computed. The overall matrix and vector are
zeroed, and the element matrix and vector are calculated
and added to the overall set by calling subroutine QUAD4.
This system of equations is modified to impose the boundary
conditions. By checking the value of each row in the
vector IB, the value of the variable corresponding to that
row is chosen, and subroutine MODIFY is called to impose
that condition. Finally, subroutine SYMSOL is called to
compute the solution to the set of equations.
Subroutine QUAD4(KK)

This subroutine forms the element matrix, COE, and vector, CON, for a four node quadrilateral, element KK, and introduces them into the overall matrix, COEF, and vector, CONST. First, the local values of the coordinates and variables are defined by picking the correct value from the overall vectors. The coordinates of the internal nodes are defined as averages of the corner nodes, likewise for the variables, see Figure 6. The nine node element matrix and vector are formed by calling subroutine TRI3 twelve times, once for each triangle. In the manner explained in Chapter III, the internal nodes are eliminated. However, before eliminating these nodes a modification of the equations is necessary. Appendix B shows that the three rows and columns corresponding to a node constitute a matrix of the form,

\[
\begin{array}{ccc}
  x & o & x \\
  o & x & x \\
  x & x & o \\
\end{array}
\]

where x's denote numbers not close to zero, and o's denote zeroes or very small numbers. It is clear that the last row, corresponding to \( P \), cannot be eliminated in the usual fashion since a zero exists on the diagonal. Therefore, before an internal node can be eliminated, the pressure at that node must be redefined. At the internal nodes the pressures are defined as weighted averages of the other
nodes. To illustrate this procedure consider the equations,

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \]

\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \]

\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \]

Letting \( x_3 = (x_1 + x_2)/2 \) yields,

\[ (a_{11} + a_{13}/2)x_1 + (a_{12} + a_{13}/2)x_2 + 0 = b_1, \]

\[ (a_{21} + a_{23}/2)x_1 + (a_{22} + a_{23}/2)x_2 + 0 = b_2, \]

\[ (a_{31} + a_{33}/2)x_1 + (a_{32} + a_{33}/2)x_2 + 0 = b_3. \]

It is clear that adding fractions of the coefficients of \( x_3 \) to the other coefficients allows the value of \( x_3 \) to be defined as a combination of the other variables. Also the equation for \( x_3 \), the third equation, may now be ignored. In this manner the pressure at the internal nodes is redefined, and the equations corresponding to these nodes are eliminated by calling ELIM. After the node elimination is completed, the resulting matrix and vector for a four node element is introduced into the overall matrix and vector.

**Subroutine TRI3(N1, N2, N3)**

The matrix and vector for a linear triangle, with node numbers N1, N2, N3, are computed and introduced into the matrix and vector for the nine node quadrilateral.
First the element matrix is initialized. The constants $a_i$, $b_i$, $c_i$, $i=1,2,3$, defined in Chapter III, the element area, the coordinates of the element centroid XBAR, YBAR, and the interpolating functions evaluated at the centroid, AN1, AN2, AN3, all are computed for use in the formulation. As expressed in Appendix B, the components of the element matrix H are formed. The second moments of the area and the values of the velocities at the centroid are calculated. With this information the components of the element vector are formed. Having the element matrix H and vector F, the components are introduced into the appropriate row and column of the matrix COE and vector CON for the nine node quadrilateral.

**Subroutine ELIM (NODE)**

By a standard Gaussian elimination procedure the rows associated with node NODE are eliminated from the nine node quadrilateral matrix COE. The rows to be eliminated are the rows associated with u and v at that node, i.e. rows $3*\text{NODE}-1$, and $3*\text{NODE}-2$.

**Subroutine MODIFY (ROW, VALUE)**

This subroutine modifies the overall matrix, COEF, and vector, CONST, to cause the variable corresponding to row ROW to take on the value VALUE after solution of the equations. To illustrate the method consider the system of linear algebraic equations in $x_1$, $x_2$, $x_3$,
\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \]
\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \]
\[ a_{31}x_1 - a_{32}x_2 + a_{33}x_3 = b_3. \]

To impose \( x_2 = U \) the equations may be written,
\[ a_{11}x_1 + 0 + a_{13}x_3 = b_1 - a_{12}U, \]
\[ 0 + x_2 + 0 = U, \]
\[ a_{31}x_1 + 0 + a_{33}x_3 = b_3 - a_{32}U. \]

The result is two equations in \( x_1 \) and \( x_3 \) identical to the first and third equations with \( x_2 = U \), and the second equation \( x_2 = U \). Subroutine MODIFY applies this method to the banded symmetric matrix COEF and vector CONST.

**Subroutine SYMSOL**

The system of equations,
\[ [\text{COEF}] \{W\} = \{\text{CONST}\} \]

is solved by a standard Gauss elimination method. The method is modified for the special form of storage used to store the banded symmetric matrix in COEF. The diagonal of the matrix is stored in column one of COEF. The line of components parallel to the diagonal are stored in successive columns,
Therefore the matrix COEF is,

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1i} & 0 & 0 \\
a_{21} & a_{22} & & \cdots & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & a_{nn} \\
0 & 0 & \cdots & \cdots & 0
\end{bmatrix}
\]

The number \( i \) is the matrix bandwidth denoted by MBAND, and \( n \) is the number of equations denoted by NEQ.

**Method 2**

The program in Appendix D for method 2 employs the eight node isoparametric element in Figure 4 to solve the Poisson equations. As a result of the simplicity of the equations, little storage and time is needed for the
program. Approximately 50,000 bytes (hexadecimal) of storage and only a few seconds of central processing unit time are needed.

The following subroutines are the same as in method 1:

MODIFY
SYMSOL
GRID
GUESS
BOUNDY

Main Program

As with method 1, the main program controls input and output operations and organizes calculations. Problem parameters, which are explained above, are input. A finite element grid is generated by calling subroutine GRID, and setting the boundary conditions is accomplished by calling subroutine BOUNDY. An initial guess is made for the velocity and pressure, and that guess is printed. The values of the interpolating functions and their derivatives are evaluated in subroutine ONCE. To form the matrix for the Laplacian operator subroutine MATRIX is called. After setting the relaxation factor GAMMA for each variable, the iteration scheme is started. At each iteration step all three Poisson equations are solved. Subroutine SOLVE is used to form and solve the equations. The new solution is compared to the previous solution by calculating SUM and BIG,
BIG = \max \{ |\phi_k^n - \phi_k^{n-1}|, k=1, \text{NP} \} ,

\text{SUM} = \sum_{k=1}^{\text{NP}} |\phi_k^n - \phi_k^{n-1}| ,

where \( \phi \) may be \( u, v, \) or \( P, \) and \( \text{NP} \) is the number of nodal points. A new guess for the velocity and pressure field is made and a check for convergence is performed. Finally the results are printed.

Subroutine ONCE (NGP)

Here the interpolating functions and their derivatives are evaluated at the gauss points. Three sets of gauss points and weighting coefficients are available. By calling subroutine INTERP, the functions and their derivatives are computed, after which they are stored in arrays,

- GN--Interpolating functions,
- DNA--Derivatives of functions with respect to \( A \) (corresponding to \( \xi \) in Chapter III),
- DNB--Derivatives of functions with respect to \( B \) (corresponding to \( \eta \) in Chapter III),
- DNAA--Second derivatives with respect to \( A, \)
- DNAB--Second derivatives with respect to \( A \) and \( B, \)
- DNBB--Second derivatives with respect to \( B. \)

The interpolating functions evaluated at the gauss points along the edge \( B = -1 \) are stored in GNB. Finally the derivatives with respect to \( A \) and \( B \) are evaluated and stored in BDNA and BDNB. The number of gauss points to be used in
each direction is passed through NGP.

**Subroutine INTERP (A, B)**

In this subroutine the interpolating functions and all of the first and second derivatives with respect to A and B are computed explicitly at the coordinates (A,B).

**Subroutine SOLVE**

This subroutine forms the equations, imposes the boundary conditions, and computes a solution. Having the matrix components, the equations are formed by calling the element vector subroutine, VECTOR, once for each element. Imposing the boundary conditions and solving the equations follow the same procedures explained in method 1.

**Subroutine MATRIX (KK)**

The element matrix H, for element KK, is formed and is introduced into the overall matrix COE. First, local coordinates of element KK are defined. Forming the components of the matrix listed in Appendix B requires the derivatives of the interpolating functions with respect to x and y, which are computed in subroutine DERIV. Having evaluated the derivatives, the integrands for the matrix components are stored in HI. Also it is clear that the element area is,

\[
A = \int \int_{e} dx dy = \int_{-1}^{1} \int_{-1}^{1} |J| d\xi d\eta ,
\]

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where $|J|$ is the determinant of the Jacobian transformation matrix from $x,y$ to $\xi,\eta$ coordinates. Therefore, the integrand for the area is $|J|$, which is stored in $A_I$. Having the integrands evaluated at the Gauss points and the weighting coefficients obtained from subroutine ONCE, a simple Gaussian quadrature integration yields the components of $H$. These components are introduced into the matrix COE at the appropriate locations.

**Subroutine VECTOR** (KK)

The element vector $F$, for element KK, is formed and is introduced into the overall vector $\text{CONST}$. Not only are the local coordinates needed, but also the element velocities and pressures at the nodes. Before each iteration on the $u$-equation, the values of the non-linear expressions, $u u_x, u v_x, v u_y, v v_y$

are computed and stored for the elements adjacent to the body. Before each iteration on the $P$-equation, the pressure gradient is approximated as described in Chapter V. From this gradient the normal derivative is computed, see Appendix C. As in subroutine $\text{MATRIX}$, the derivatives of the interpolating functions are obtained from subroutine $\text{DERIV}$. Also needed is the function $\psi$, which is defined in Chapter V, evaluated at the gauss points. This is accomplished by calling subroutine $\text{APPROX}$. Having the interpolating functions, $\text{GN}$, the value of $\psi$, $\text{PSI}$, and the Jacobian

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determinant, DETJ, the integrands may be calculated at the gauss points and stored in FI. With the weighting coefficients and the integrands evaluated at the gauss points, a gaussian quadrature integration may be performed to obtain the components of the element vector F. The required normal derivative conditions are imposed on the pressure equation by adding the appropriate expression to the vector F (see Chapter V). The element vector F is introduced into the proper components of the overall vector CONST.

Subroutine NDERIV (K)

In this subroutine the derivatives of the velocities and pressures, needed for the normal derivative boundary conditions on the pressure, are computed. First the derivatives of x and y are computed with respect to A and B. From these derivatives the determinant of the Jacobian matrix is evaluated at one of the nodes, chosen with K, on the B = -1 element boundary. The derivatives of the interpolating functions with respect to x and y are computed as explained in Chapter III. Finally the needed derivatives,

\[ u_x, u_y, v_x, v_y, p_x, p_y \]

are computed.
Subroutine DERIV (K)

The purpose of this subroutine is to compute the Jacobian determinant and the first and second derivatives of the shape functions with respect to x and y. These quantities are evaluated at the Gauss point defined by K. After determining the derivatives of x and y with respect to A and B, the Jacobian determinant is computed. The first derivatives are computed as described in Chapter III. For the pressure equation the second derivatives are needed. Second derivatives of x and y with respect to A and B are computed. The inverse of the transformation matrix, needed to determine the derivatives with respect to x and y as discussed in Chapter III, is evaluated at gauss point K. Having the transformation matrix, the second derivatives of the shape functions with respect to x and y are computed.

Subroutine APPROX (K)

Here the function ψ, whether it be f, g, or h as discussed in Chapter V, is evaluated at the Gauss point K. First the velocities u and v are evaluated at that point. For each equation the needed factors are computed and the functions defined in Chapter V are evaluated.
Test Program

Since method 2 was unsuccessful, it is necessary to test the theory and programming for errors. This program solves a Poisson equation by the finite element method, using linear triangles in one solution and eight node quadratic elements in another solution. Eight node elements are used to solve,

\[ \nabla^2 \phi = \psi(x,y), \]

where \( \psi \) is assumed to be a constant, \( C \), in one solution, and a function analytically equivalent to \( C \) in another solution,

\[ \psi(x,y) = \frac{\partial^2}{\partial x^2} \left( \frac{C}{\delta} x^2 \right) + \frac{\partial^2}{\partial x \partial y} \left( \frac{C}{\delta} xy \right) + \frac{\partial^2}{\partial y^2} \left( \frac{C}{\delta} y^2 \right). \]

This function is employed as a means of checking the program's ability to compute second derivatives with respect to \( x \) and \( y \). The quadratic element should be capable of computing the derivatives of this quadratic function exactly. Also, linear triangles are used for two solutions. A Gauss elimination scheme is employed to solve the algebraic equations for a solution. Another solution is obtained by Gauss-Seidel iteration. Additionally, the eigenvalues of an amplification matrix are checked.

Some of the subroutines have been explained above. \( \text{ONCE} \) and \( \text{INTERP} \) correspond closely to those in method 2. Subroutine \( \text{DERIV} \) computes the derivatives exactly as done
Main Program

The main program generates the finite element grid, computes the analytical solution, and calls the sub-routines which do the calculations for the finite element analysis. First, the x and y coordinates of the nodes are generated. Next, the analytical solution is computed at each node and stored in AN. The eight node quadrilaterals are defined by reading the array NODE, and subroutine ONCE is called to calculate the interpolating functions and their derivatives at the Gauss points. Forming the algebraic equations is accomplished by calling subroutine MATRIX once for each element. Subroutine MODIFY is used to impose the boundary conditions, after which the equations are solved by a standard Gauss elimination method in subroutine SOLEL. The equations are formed again using the alternative formulation for $\psi$ described above, which is chosen by the flag II. Next, linear triangles are used to attack the problem. The elements are redefined by reading new values for the array NODE. Subroutine MATR is called once for each element to form the equations. Boundary conditions are imposed as before by calling subroutine MODIFY. This set of equations is solved once by Gauss elimination, then recalculated and solved by Gauss-Seidel iteration. Finally, the results are printed. Solution values obtained by each method are printed along with the
eigenvalues of the amplification matrix.

Subroutine MATRIX (KK)

In this subroutine the element matrix $H$ and vector $F$ for the eight node quadratic element $KK$ is formed and introduced into the overall matrix $COEF$ and vector $CONST$. First, the nodal coordinates of the element are defined. Subroutine DERIV is called to compute the derivatives of the shape functions with respect to $x$ and $y$ and the Jacobian determinant, which is the area integrand $AI$. $II$ is checked to determine which form of the function $PSI$ to calculate. Having these quantities, the integrands for the matrix, $HI$, and vector, $FI$, are evaluated. A Gaussian quadrature integration yields the element area $A$, and the components of the matrix $H$ and vector $F$. These components are added to the matrix $COEF$ and vector $CONST$ at the appropriate location.

Subroutine MATR (KK)

Here the element matrix $H$ and vector $F$ for the three node linear triangular element $KK$ is evaluated and added to the global matrix $COEF$ and vector $CONST$. After the nodal coordinates are defined, the needed constants $a_i$, $b_i$, $c_i$, $i=1,2,3$, area, and coordinates of the centroid may be computed. The matrix and vector components are calculated as explained in Chapter V and Appendix C and introduced into the matrix $COEF$ and vector $CONST$. 

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Subroutine MODIFY (ROW, VALUE)

The procedure explained in method 1 is applied directly since the matrix is stored in standard order for this program.

Subroutine SOLEL

This subroutine computes the solution of a set of linear simultaneous equations by a standard Gauss elimination scheme.

Subroutine SOLIT (ITMAX, EPS)

The solution of the algebraic equations is obtained by Gauss-Seidel iteration. ITMAX is the maximum number of iterations allowed, and EPS is the measure of closeness needed for convergence.

Comment

The author recognizes that some inefficiencies exist in the programming. However, since the time and storage requirements are not excessive for any of the programs, these inefficiencies are tolerated for the sake of clarity.
CHAPTER VII

NUMERICAL EXPERIMENTATION AND RESULTS

Immediate success is not achieved with a direct application of the previously introduced methods. Some limited success results after many variations of the original programs are tested. Although not all of the numerical experiments are worth mentioning, most of the major modifications, with comments on the results obtained, are explained below.

Although each subroutine is tested before being incorporated into the major program, a check is needed for the complete program. An obvious test is to assume a known solution and perform the iteration to determine if the solution is a fixed point. A convenient test problem is obtained by imposing the boundary condition,

\[ r = R(\theta) : u = 1 \]

The solution to this hypothetical problem is the "free stream" flow,

\[ u = 1 , \quad v = 0 , \quad P = 0 , \]

throughout the domain. To show that the solution is a fixed point, this convenient field is assumed and the iteration method is applied. If the same results are
obtained in one step, the solution is a fixed point. To investigate if the solution is a point of attraction, i.e. if a "close" guess will converge to the free stream, the free stream is perturbed, and used as an initial guess. An example of a possible perturbation is

\[ u = 1 - \frac{0.1}{r^2}, \quad v = \frac{0.1}{r^2}, \quad p = \frac{0.1}{r^2}, \]

where \( r \) is the radial coordinate, \( r > 1 \). If the initial guess is corrected by the iteration scheme, the free stream solution is a point of attraction for this problem.

After performing these two simple tests the "real" problem is attempted. Imposing the realistic boundary condition,

\[ r = R(\theta) : u = 0, \]

results in a physically meaningful problem. The Oseen approximation discussed in Chapter II is the initial guess for a very low Reynolds number. Applying the iteration scheme until convergence is achieved results in a corrected velocity and pressure field, which is the initial guess for another problem at a slightly higher Reynolds number. It is hoped that problems of successively higher Reynolds numbers can be solved in this fashion.
Method 1

Linear Elements

After considerable experimentation a computer program which employs method 1 has been constructed such that the results agree with known solutions for test problems, but the program's applicability is restricted. D. L. Garrett attempts solutions using simple linear triangles with little success [16]. Although the method successfully solves the two test problems and does converge for the real problem, the solution obviously is incorrect. Subsequent work reveals a more reasonable solution is obtained with a five node quadrilateral, which has an internal node that is eliminated, see Figure 7. A simple analysis of the algebraic equations explains the improvement. An inspection of the components of the coefficient matrix reveals that zeros are positioned near the diagonal in the equations corresponding to u and v, and are located on the diagonal in every third equation, those corresponding to P. The lack of diagonal dominance in the matrix signifies an unstable set of equations. As indicated in Chapter III, eliminating the internal node modifies the ill-conditioned matrix. Fortunately this modification includes placing non-zero numbers on the diagonal of the P equations. Thus, since the zeros no longer exist after node elimination and the solution obtained is an improvement of the linear triangle solution, it seems that the elimination of internal nodes

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not only reduces the subsequent calculations required, but also improves a matrix which lacks diagonal dominance. In fact, this particular property has been investigated at some length. For at least five types of elements with internal nodes, the diagonal term is compared to the sum of the off-diagonal terms in that row before and after internal nodes are eliminated. In each case the diagonal dominance improves after the elimination of nodes. The ratio of the number of internal nodes to the number of boundary nodes seems to have a direct effect on the improvement of the matrix. As the ratio increases, the diagonal dominance becomes stronger.

In an effort to benefit from the advantages of node elimination, solutions of the flow past a circular cylinder are attempted with many different elements possessing internal nodes and composed of linear triangles. Three that are worthy of discussion are the five node quadrilateral mentioned above and shown in Figure 7, the nine node quadrilateral in Figure 6, for which a program appears in Appendix D, and the thirteen node quadrilateral in Figure 7. As expected, the nine node quadrilateral produces the matrix with the strongest diagonal dominance of the three elements, and the five node element matrix has the weakest diagonal dominance. As a result, the program employing the nine node quadrilateral not only has the fastest rate of convergence, but also produces the smoothest and best approximating flow field as compared to experimental and
other numerical work. Also, the program utilizing the five node element is the slowest and yields the worst approximation. A serious restriction on all of the elements attempted is that the convergence rate decreases as the Reynolds number increases, and divergence is encountered before the Reynolds number reaches two. However, for small Reynolds numbers the results compare favorably with previous work. A plot of the pressure on the body versus the angle from the upstream axis is presented in Figure 10. Plotted are the results of the nine node element described above at Reynolds numbers of 0.5 and 1.0, solutions by Keller and Takami at $R_E = 2$ [30], and Dennis and Shimshoni at $R_E = 1$ [10]. These plots show that the finite element method curves are not only of the correct shape, but the variation with the Reynolds number is in the right direction. Additionally the two curves for $R_E = 1.0$ are nearly coincident, indicating excellent agreement with published results. A complete listing of velocities and pressures is presented in Figure 13.

In view of the accurate results obtained for flow past a circular cylinder at low Reynolds numbers, it is reasonable to have confidence in the solution obtained for "creeping" flow past a body of arbitrary cross section. The velocity and pressure fields may be computed for any symmetric body by inputing the proper body radius at each angle to form the surface $R(\theta)$. The ability to analyze an arbitrarily shaped symmetric body is felt to be a
significant contribution. Employing the same program used for the circular cylinder, the flow past a lens shaped cylinder, shown in Figure 11, is analyzed. The results of the computation are the velocities and pressures of Figure 16, and the plot of the pressure distribution on the body surface in Figure 12.

Non-Linear Elements and Other Experiments

Solutions are attempted with non-linear isoparametric elements without success. The eight node quadratic element in Figure 4 is the first element employed. Because of the nature of the matrix produced by this element, division by zero is attempted in the Gauss elimination solution. This difficulty is easily overcome by utilizing the nine node element in Figure 8. Eliminating the internal node modifies the matrix and thus allows the calculations to proceed without attempts to divide by zero. Performing the tests mentioned above shows the free stream solution to be a fixed point but not a point of attraction. For a close guess at the free stream solution, the iteration corrects only the velocities. Although the pressure is changed from the initial guess, the final pressure is incorrect. Similar results are obtained with the thirteen and twenty-one node quadrilaterals in Figure 9, which are developed from eight node quadrilaterals with the internal nodes eliminated.
After experimenting with many elements, it is clear that the quadrilaterals composed of linear triangles and possessing internal nodes produce the best results. In an attempt to expand the range of Reynolds numbers for which convergence is achieved, many modifications are investigated. A relaxation factor for the new guess at each step of the iteration improves convergence slightly. Feeling that the solution method may "overshoot" the result, each new guess is computed using some fraction of the calculated correction. Constant relaxation factors, and step functions depending on the rate of convergence are attempted. The result is a slight improvement in convergence, but the range of Reynolds numbers is not increased significantly. Various boundary conditions are tested to improve the answers or increase the region of convergence, neither of which occurs. As discussed previously, there are certain natural boundary conditions which result from the finite element formulation. If these conditions are allowed to exist on the external boundary of the domain rather than impose the forced conditions the convergence is improved slightly, as is predicted by Forsythe and Wasow [15], but the final result is essentially unchanged. The same result is obtained when the forced conditions are imposed on the velocities while the natural boundary condition on the pressure at the external boundary is permitted.

Additional experiments include another means of eliminating the zero diagonal term. As a substitute for
the continuity equation consider

\[ u_{x}^{n+1} + v_{y}^{n+1} + C p^{n+1} = C p^{n} \]

where \( C \) is an arbitrarily specified constant. It is clear that as \( |p^{n} - p^{n+1}| \rightarrow 0 \), this equation approximates the continuity equation. The finite element formulation using this equation allows a non-zero coefficient of \( P \) in the third equation at each node, which eliminates the zero on the diagonal. However, after several tests it appears that the optimum value of \( C \) is zero, thus the technique is discarded.

To investigate the iteration properties of the overall coefficient matrix, the eigenvalues of an amplification matrix are computed. The results seem to indicate that convergence is not likely. The test is discussed in detail in the next section.

**Method 2**

Since many numerical experiments achieve only limited success, and since node elimination does not sufficiently improve the ill-conditioned coefficient matrix, a new formulation of the equations is sought. The result is method 2 presented in Chapter IV. Although possessing distinct advantages as compared to method 1, this method fails to converge under any conditions tested. The free stream solution is shown to be neither a fixed point nor a point of
attraction. Figure 14 shows the divergence from the free stream solution. Linear triangles, five node quadrilaterals composed of four triangles, and eight node quadratic elements are all employed without success.

A test program is written to check the important subroutines. A Poisson equation is solved with linear triangles and eight node isoparametric elements for the values of the unknown at the nodes, which are compared to the values obtained by an analytical solution. Using the same number of nodes for each solution, and therefore more triangles than quadrilaterals, the results indicate the nonlinear quadrilateral element is the more accurate. To investigate the iteration properties of the matrix an amplification matrix is formed and its eigenvalues are computed. Consider the system of equations,

\[ [A] \{x\} = \{B\} \]

where \([A]\) and \(\{B\}\) contain known constants, and \(\{x\}\) is a vector of unknowns. Writing \([A]\) as the sum of a diagonal matrix \([D]\) and another matrix \([C]\) the system of equations can be written,

\[ [D] \{x\} = \{B\} - [C] \{x\} \]

After solving this equation for \(\{x\}\), the Gauss-Seidel iteration scheme may be introduced,

\[ \{x^{n+1}\} = [D]^{-1}\{B\} - [D]^{-1}[C] \{x^n\} \]
If the eigenvalues of the amplification matrix \([D]^{-1}[C]\) have magnitudes less than one, then convergence is guaranteed. The eigenvalues of this matrix are computed in the test program for a rectangular domain. The magnitudes of the eigenvalues are all less than one with linear triangles, and all but one are less than unity with parabolic quadrilaterals. Shown in Figure 15 are the analytical solution, and two solutions each with triangles and quadrilaterals. Eight node quadrilaterals are used to solve

\[ \nabla^2 \phi = C, \]

where \(C\) is a constant, for which the analytical result is printed. Additionally, to check the program's ability to compute second derivatives, the quadrilaterals are employed to solve an analytically identical equation,

\[ \nabla^2 \phi = \frac{\partial^2}{\partial x^2} \left( \frac{C}{6} x^2 \right) + \frac{\partial^2}{\partial x \partial y} \left( \frac{C}{3} xy \right) + \frac{\partial^2}{\partial y^2} \left( \frac{C}{6} y^2 \right). \]

The results indicate both solutions are identical and that they agree with analytical results to approximately three digits. The matrix formed with linear triangles is solved by Gauss elimination once and Gauss-Seidel iteration a second time. As expected, since the eigenvalues of the amplification matrix have magnitudes less than one, the iteration scheme converges to the same values obtained by elimination, which correspond to the analytical solution to approximately two digits. Although accurate solutions of
test problems are only necessary and not sufficient conditions of correctness, it is felt that the test program described here indicates that the subroutines of the program for method 2 are correct.

A test of the iteration properties of the matrix formed by method 2 is performed in the same manner as that for the test program. Since the equation,

\[ [A]\{x\} = \{B\} \]

formed by the iteration scheme of method 2, has \( \{B\} \) dependent on \( \{x\} \), convergence cannot be guaranteed easily. However, if \( \{x\} \) is close to the final solution, the vector \( \{B\} \) can be approximated as a constant provided its variation with \( \{x\} \) is continuous. Thus, some indication of the convergence properties can be obtained by computing the eigenvalues of the amplification matrix. As might be expected, since the method does not converge, the eigenvalues of the amplification matrix for the problems tested do not all have magnitudes less than one. Additionally, the eigenvalues of the amplification matrix for method 1 are computed with the same result, which likewise is as expected.

Another test program for method 2 uses eight node isoparametric elements to attack the classical Couette and Poiseuille flows. Consider the flow of a viscous fluid between two plates, one stationary and one moving at a velocity \( U \). The fluid velocity may be calculated analytically as a function of the distance from the bottom plate.
in terms of the velocity of the moving plate and the pressure gradient, which is assumed constant. By fixing the pressure at each point in the domain the pressure gradient is imposed, and the values of the velocity at the moving plate are fixed as a boundary condition. As might be expected for this linear problem, since no iteration is needed, the program computes the correct velocity field.

It is worth noting, however, that the application of the method to corner flow, employing eight node parabolic elements, and allowing unprescribed pressure and non-linear velocity terms, is not successful, and that even this slight modification of geometry produces an amplification matrix with several eigenvalues having magnitudes greater than unity.
CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

Although the results of the present work are restricted, several conclusions may be drawn. It is clear that certain viscous flow problems may be analyzed by the finite element method. The simplified flows which result in linear equations are solved easily. However, attacking non-linear problems with an iteration scheme is not easily accomplished, and the results may not be accurate. Two iteration formulations are presented in the present paper and the results are discussed below. For each method additional work is needed to further investigate the convergence properties of the iteration. Clearly, alternative formulations are possible for which the convergence properties may be better than those of the methods presented in this paper. One possibility is the direct formulation of the non-linear governing equations, which requires the solution of a non-linear system of algebraic equations. This was considered but abandoned due to the lack of truly dependable schemes for solving such systems.
Method 1

Since some success has been achieved for small Reynolds numbers, it may be concluded that method 1 is applicable to "creeping flow" past a body of arbitrary cross section. From the numerical experiments performed, it is obvious that the pressure is the most difficult of the variables to calculate accurately. Although this fact causes some problems, it does demonstrate a correlation with other methods. Many experimental and numerical investigators claim their techniques are pressure sensitive. The major difficulty with method 1 is the instability of the algebraic equations. This instability is caused by the inherent zero on the diagonal in every third equation, corresponding to the pressure equation.

Some additional work with method 1 might prove worthwhile. Of most value is the investigation of possibilities to improve the diagonal dominance of the algebraic equations. Some methods such as node elimination and approximating the continuity equation are described in the previous chapter; others might exist. Another important problem is the dependence of the rate of convergence on the Reynolds number, a difficulty encountered by other numerical methods also. Further investigation of this dependence may be of value.
Method 2

Since little success is achieved with method 2 very few conclusions are possible. Solving Poisson's equation by the finite element method is nothing new, but the test program for method 2 does illustrate the applicability of the subroutines. The correct solution of certain test problems further demonstrates the abilities of method 2. However, for the flow past a cylinder divergence is encountered for both the free stream and the realistic boundary conditions. In fact, the iteration scheme does not converge for any test problems attempted.

In view of these results it is clear that additional work is needed to investigate the cause of the divergence. Test problems should be devised which would test the specific parts of the method separately. Further analysis of the eigenvalues of the amplification matrix may reveal useful information. To avoid any problems inherent in the finite element method, finite difference schemes may be used to attack the Poisson equation of method 2, which, of course, restricts the test problems to "nice" domains.
Figure 1. Physical Problem, Circular Cylinder
Figure 2. Linear Rectangle
Figure 3. Quadratic Rectangle
Figure 4. Quadratic Isoparametric Quadrilateral
Figure 5. Orientation of Triangular Element
Figure 6. Quadrilateral with Twelve Triangles
Figure 7

Quadrilateral with Four Triangles

Quadrilateral with Sixteen Triangles
Figure 8. Nine Node Quadrilateral
Figure 9. Experimental Elements

Twelve Node Quadrilateral

Thirteen Node Quadrilateral
Figure 10. Circular Cylinder, Surface Pressure Distribution
Figure 11. Physical Problem, Lens Shaped Cylinder

u=1, v=0, p=0
v=0, u=0, y=0, p=0

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Figure 12. Lens Shaped Cylinder, Surface Pressure Distribution
Figure 13. Method 1 Results for a Circular Cylinder
NO* OF ELEMENTS ON THE UPSTREAM AXIS = 8
NO* OF ELEMENTS ON THE DOWNSTREAM AXIS = 8
NO* OF ELEMENTS IN THE TRANSVERSE DIRECTION = 8
EACH ELEMENT HAS 2 POINTS IN THE RADIAL DIRECTION
AND 2 POINTS IN THE TRANSVERSE DIRECTION
TOTAL NUMBER OF ELEMENTS = 64
TOTAL NUMBER OF POINTS = 81
MATRIX BANDWIDTH = 33
MAXIMUM RADIUS ON THE DOWNSTREAM AXIS = 12.00
REYNOLDS NUMBER = 0.50

ITERATION NUMBER 1 SUM = 1.3E+01 BIG = 1.2E+00
ITERATION NUMBER 2 SUM = 2.8E+00 BIG = 6.6E-01
ITERATION NUMBER 3 SUM = 6.4E-01 BIG = 3.1E-01
ITERATION NUMBER 4 SUM = 1.4E-01 BIG = 3.1E-01
ITERATION NUMBER 5 SUM = 2.7E-02 BIG = 5.9E-02
ITERATION NUMBER 6 SUM = 4.9E-03 BIG = 2.7E-02
ITERATION NUMBER 7 SUM = 3.0E-04 BIG = 6.7E-03
ITERATION NUMBER 8 SUM = 2.3E-05 BIG = 1.6E-03
ITERATION NUMBER 9 SUM = 2.6E-06 BIG = 5.1E-04

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**INITIAL GUESS**

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NO. OF ELEMENTS IN THE TRANSVERSE DIRECTION = 8
EACH ELEMENT HAS 2 POINTS IN THE RADIAL DIRECTION
AND 2 POINTS IN THE TRANSVERSE DIRECTION
TOTAL NUMBER OF ELEMENTS = 64
TOTAL NUMBER OF POINTS = 81
MATRIX BANDWIDTH = 33
MAXIMUM RADIUS ON THE DOWNSTREAM AXIS = 12.00
REYNOLDS NUMBER = 1.00

* * * * * * * * * * * * * * * * * * * * * *

ITERATION NUMBER 1  SUM = 1.4E+01  BIG = 1.4E+00
ITERATION NUMBER 2  SUM = 1.5E+01  BIG = 4.8E-01
ITERATION NUMBER 3  SUM = 2.2E-01  BIG = 2.4E-01
ITERATION NUMBER 4  SUM = 7.4E-02  BIG = 1.1E-01
ITERATION NUMBER 5  SUM = 3.0E-02  BIG = 6.2E-02
ITERATION NUMBER 6  SUM = 8.2E-03  BIG = 2.9E-02
ITERATION NUMBER 7  SUM = 4.6E-03  BIG = 1.7E-02
ITERATION NUMBER 8  SUM = 3.3E-03  BIG = 1.4E-02
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ITERATION NUMBER 10 SUM = 1.5E-03  BIG = 7.7E-03
ITERATION NUMBER 11 SUM = 9.6E-04  BIG = 7.5E-03
ITERATION NUMBER 12 SUM = 4.4E-04  BIG = 5.8E-03
ITERATION NUMBER 13 SUM = 3.3E-04  BIG = 5.5E-03
ITERATION NUMBER 14 SUM = 2.2E-04  BIG = 3.4E-03

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Figure 14. Method 2 Results for a Circular Cylinder
NO. OF ELEMENTS ON THE UPSTREAM AXIS = 4
NO. OF ELEMENTS ON THE DOWNSTREAM AXIS = 4
NO. OF ELEMENTS IN THE TRANSVERSE DIRECTION = 6
EACH ELEMENT HAS 3 POINTS IN THE RADIAL DIRECTION
AND 3 POINTS IN THE TRANSVERSE DIRECTION
TOTAL NUMBER OF ELEMENTS = 24
TOTAL NUMBER OF POINTS = 93
MATRIX BANDWIDTH = 17
MAXIMUM RADIUS ON THE DOWNSTREAM AXIS = 19.00
REYNOLDS NUMBER = 1.00

EXACT DOMAIN AREA = 626.75 COMPUTED DOMAIN AREA = 626.65

ITERATION NUMBER 1
U-VELOCITY NORM = 3.7E-04 LARGEST TERM = 1.1E-05
V-VELOCITY NORM = 0.0 LARGEST TERM = 0.0
PRESSURE NORM = 1.9E-03 LARGEST TERM = 3.6E-05

ITERATION NUMBER 2
U-VELOCITY NORM = 3.2E-03 LARGEST TERM = 9.5E-05
V-VELOCITY NORM = 2.1E-03 LARGEST TERM = 7.8E-05
PRESSURE NORM = 3.3E-03 LARGEST TERM = 9.2E-05

ITERATION NUMBER 3
U-VELOCITY NORM = 3.7E-04 LARGEST TERM = 1.8E-05
V-VELOCITY NORM = 4.2E-04 LARGEST TERM = 1.6E-05
PRESSURE NORM = 4.7E-02 LARGEST TERM = 1.8E-03

ITERATION NUMBER 4
U-VELOCITY NORM = 6.3E-02 LARGEST TERM = 1.9E-03
V-VELOCITY NORM = 5.4E-02 LARGEST TERM = 1.9E-03
PRESSURE NORM = 1.9E-01 LARGEST TERM = 9.5E-03

ITERATION NUMBER 5
U-VELOCITY NORM = 1.2E-01 LARGEST TERM = 6.3E-03
V-VELOCITY NORM = 9.3E-02 LARGEST TERM = 4.3E-03
PRESSURE NORM = 9.9E+00 LARGEST TERM = 3.9E-01

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Figure 15. Test Program Results

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### Computed Domain Areas

**Quadrilateral**

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**Triangle**

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### Solution

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Figure 16. Method 1 Results for a Lens Shaped Cylinder

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* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

No. of elements on the upstream axis = 8
No. of elements on the downstream axis = 8
No. of elements in the transverse direction = 8

Each element has 2 points in the radial direction and 2 points in the transverse direction

Total number of elements = 64
Total number of points = 81
Matrix bandwidth = 33
Maximum radius on the downstream axis = 12.00
Reynolds number = 6.50

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**INITIAL GUESS**

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NO. OF ELEMENTS ON THE DOWNSTREAM AXIS = 8
NO. OF ELEMENTS IN THE TRANSVERSE DIRECTION = 8
EACH ELEMENT HAS 2 POINTS IN THE RADIAL DIRECTION
AND 2 POINTS IN THE TRANSVERSE DIRECTION

TOTAL NUMBER OF ELEMENTS = 64
TOTAL NUMBER OF POINTS = 81
MATRIX BANDWIDTH = 33
MAXIMUM RADIUS ON THE DOWNSTREAM AXIS = 12.00
REYNOLDS NUMBER = 1.00

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BIG = 1.5E+04

ITERATION NUMBER 2  
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ITERATION NUMBER 3  
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ITERATION NUMBER 4  
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ITERATION NUMBER 5  
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ITERATION NUMBER 6  
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APPENDIX A

INTERPOLATING FUNCTIONS

Linear Four Node Element (Figure 3):

\[ N_i = \frac{1}{4}(1 + \xi \xi_i)(1 + \eta \eta_i) \]

Quadratic Eight Node Element (Figure 4):

Mid-side Nodes,

\[ \xi_i = 0, \quad N_i = \frac{1}{2}(1 - \xi^2)(1 + \eta \eta_i) \]
\[ \eta_i = 0, \quad N_i = \frac{1}{2}(1 + \xi \xi_i)(1 - \eta^2) \]

Corner Nodes,

\[ N_i = \frac{1}{4}(1 + \xi \xi_i)(1 + \eta \eta_i)(\xi \xi_i + \eta \eta_i - 1) \]

Cubic Twelve Node Element:

Mid-side Nodes,

\[ \xi_i = \pm 1 \quad \text{and} \quad \eta_i = \pm \frac{1}{3} \]

\[ N_i = \frac{9}{32}(1 + \xi \xi_i)(1 - \eta^2)(1 + 9\eta \eta_i) \]

(swap variables to get others)

Corner Nodes,

\[ N_i = \frac{1}{32}(1 + \xi \xi_i)(1 + \eta \eta_i)[-10 + 9(\xi^2 + \eta^2)] \]

137
APPENDIX B

COMPONENTS OF THE ELEMENT COEFFICIENT MATRIX \([H]^e\) AND CONSTANT VECTOR \({F}^e\)

Method 1

Let \(r = 3i - 2\) and \(s = 3j - 2\) where \(i = 1, n\) and \(j = 1, n\), and \(n\) is the number of nodes in an element.

The components of the coefficient matrix \([H]^e\) are:

\[
H^e(r, s) = \frac{1}{RE} \int_\Omega \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \, dx \, dy
\]

\[
H^e(r, s + 1) = 0
\]

\[
H^e(r, s + 2) = - \int_\Omega \frac{\partial N_i}{\partial x} N_j \, dx \, dy
\]

\[
H^e(r + 1, s) = 0
\]

\[
H^e(r + 1, s + 1) = H^e(r, s)
\]

\[
H^e(r + 1, s + 2) = - \int_\Omega \frac{\partial N_i}{\partial y} N_j \, dx \, dy
\]

\[
H^e(r + 2, s) = H^e(s, r + 2)
\]

\[
H^e(r + 2, s + 1) = H^e(s + 1, r + 2)
\]

\[
H^e(r + 2, s + 2) = 0
\]
The components of the constant vector \( \{F\}^e \) are:

\[
F^e(r) = \int_{e} fN_i \, dx\,dy \\
F^e(r+1) = \int_{e} gN_i \, dx\,dy \\
F^e(r+2) = 0
\]

**Method 2**

Let \( i = 1,n \), \( j = 1,n \) where \( n \) is the number of nodes.

The components of the coefficient matrix \([H]^e\) are:

\[
H^e(i,j) = \int_{e} \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \, dx\,dy
\]

The components of the constant vector \( \{F\}^e \) are:

\[
F^e(i) = -\int_{e} N_i \psi \, dx\,dy
\]
APPENDIX C

FINITE ELEMENT FORMULATION
WITH LINEAR TRIANGLES

As a result of linear shape functions, much of the formulation with linear triangles may be extended to final form with little extra effort. Not only may the matrix and vector components be integrated analytically, but imposing the normal derivative boundary conditions may be done algebraically.

The basis for the simplifications is the linear shape functions,

$$N_i(x,y) = \frac{1}{2A} (a_i + b_i x + c_i y), \quad i=1,2,3,$$

where $A$ is the element area, and $a_i$, $b_i$, $c_i$ are constants calculated from the nodal coordinates, see Chapter III. To demonstrate the simplifications, the formulation for the Poisson equation is developed. From Appendix B the components of the element matrix $[H]^e$ are obtained,

$$H^e(i,j) = \int_e \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \, dx \, dy .$$

A substitution of the interpolating functions yields,

$$H^e(i,j) = \int_e \left( \frac{1}{2A} \right)^2 (b_i b_j + c_i c_j) \, dx \, dy .$$

140

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But noting that $A$, $b_i$, $b_j$, $c_i$, $c_j$ are all constants the integral simplifies,

$$\Pi^e(i,j) = \frac{1}{4A} (b_i b_j + c_i c_j) .$$

In a similar manner the element constant vector $\{F\}^e$ components,

$$F^e(i) = - \int_e N_1 \psi dx dy$$

may be integrated. Assuming a linear formulation for $\psi$,

$$\psi(x,y) = R_1 + R_2 x + R_3 y ,$$

and multiplying inside the integral yields,

$$F^e(i) = - \int_e \frac{1}{2A} (a_i R_1 + a_i R_2 x + a_i R_3 y + b_i R_1 x + b_i R_2 x^2 + b_i R_3 xy + c_i R_1 y + c_i R_2 xy + c_i R_3 y^2) dx dy .$$

Recognizing that $A$, $a_i$, $b_i$, $c_i$, $R_1$, $R_2$, $R_3$ are constants, and recalling the definitions of centroids and second moments of areas, the expression becomes,

$$F^e(i) = - \frac{a_i R_1}{2} - \frac{\bar{x}^2}{2} (a_i R_2 + b_i R_1) - \frac{\bar{y}^2}{2} (a_i R_3 + c_i R_1)
- c_i R_3 \frac{I_{XX}}{2A} - \frac{I_{XY}}{2A} (c_i R_2 + b_i R_3) - b_i R_2 \frac{I_{YY}}{2A} ,$$

where $\bar{x}, \bar{y}$ are the coordinates of the centroid, and $I_{XX}$, $I_{XY}$, $I_{YY}$ are the second moments of the element's area.
Having assumed \( \psi(x,y) \) to be linear in the previous discussion, it is now necessary to compute the coefficients \( R_1, R_2, R_3 \). For the Poisson equation in \( u \) the function \( \psi(x,y) \) is defined as,

\[
\psi(x,y) = R_E(uu_x + vu_y + P_x) .
\]

Substituting the representations of \( u, u_x, \) and \( u_y \),

\[
u = N\{u\}^e ,
\]

\[
u_x = \langle b_1, b_2, b_3 \rangle\{u\}^e , \quad u_y = \langle c_1, c_2, c_3 \rangle\{u\}^e ,
\]

and similar expressions for the other variables, the function \( \psi(x,y) \) becomes,

\[
\psi(x,y) = \frac{R_E}{4A^2} \left[ (N\{u\}^e)(b_1\{u\}^e) + (N\{v\}^e)(c_1\{u\}^e) \\
+ 2A\langle b_1 \rangle\{P\}^e \right] .
\]

Recognizing the coefficients of \( x \) and \( y \) reveals the constants \( R_1, R_2, R_3 \),

\[
R_1 = \frac{R_E}{4A^2} \left[ (a_1\{u\}^e)(b_1\{u\}^e) + (a_1\{v\}^e)(c_1\{u\}^e) \\
+ 2A \langle b_1 \rangle\{P\}^e \right] ,
\]

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In exactly the same manner expressions for $R_1$, $R_2$, $R_3$ are calculated for the function $\psi(x,y)$ in the $v$ and $P$ equations.

The normal derivative boundary condition on $P$ is easy to attack with linear triangles. Recall the components of the pressure gradient are computed from the previous pressure and two successive values for the velocities,

$$p_{x}^{n+1} = (p_{x} + uu_{x} + vu_{y})^n - (uu_{x} + vu_{y})^{n+1},$$

and a similar equation for $p_{y}^{n+1}$. By substituting the product of the interpolating functions, or their derivatives, with the nodal values, an algebraic expression is obtained for the components $p_{x}$ and $p_{y}$. Locating the triangle at an arbitrary angle $\theta$ to the $x$ axis, see Figure 5, the normal derivative of $P$ is computed,

$$p_{n} = p_{y} \cos \theta - p_{x} \sin \theta.$$

Writing $\cos \theta$ and $\sin \theta$ in terms of the nodal coordinates, the normal derivative becomes,

$$p_{n} = p_{y} \frac{x_2 - x_1}{L} - p_{x} \frac{y_2 - y_1}{L},$$

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where,

\[ L = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \]

Recognizing the constants,

\[ c_3 = x_2-x_1, \quad b_3 = y_2-y_1, \]

the final expression is,

\[ p_n = p_y \frac{c_3}{L} + p_x \frac{b_3}{L}. \]

To impose this normal derivative, the extra term introduced in Chapter V is considered,

\[ I^{BC} = - \int_s qPds. \]

If \( q \) and \( P \) are allowed to vary linearly along the edge,

\[ q(s) = q_1 + \frac{s}{L}(q_2-q_1), \quad P(s) = P_1 + \frac{s}{L}(P_2-P_1), \]

where \( q_1 \) and \( P_1 \) are the nodal values and \( s \) is measured along the edge, the integral becomes,

\[ I^{BC} = - \int_s \left[ q_1 + \frac{s}{L}(q_2-q_1) \right] \left[ P_1 + \frac{s}{L}(P_2-P_1) \right] ds. \]
Carrying out the integration yields,

\[ I^{BC} = - p_1 \frac{L}{6}(2q_1 + q_2) - p_2 \frac{L}{6}(2q_2 + q_1) \]

To form the finite element equations the derivatives of \( I^{BC} \) are needed,

\[ \frac{\partial I^{BC}}{\partial p_1} = - \frac{L}{6}(2q_1 + q_2) \quad \text{and} \quad \frac{\partial I^{BC}}{\partial p_2} = - \frac{L}{6}(2q_2 + q_1) \]

By adding these expressions to the element constant vector \( \{F\}^e \), in row one and two respectively, the normal derivative of \( P \) will be set equal to \( q \).
C ***************************************************************************
C
C COMMON COEF(250,40),CONST(250),MBAND,NEQ
C /BLK10/NTYPE(100),IB(300),NP,NEL
C /BLK1/RE
C /BLK3/X(100),Y(100),W(300)
C /BLK6/R0(15),R(100),T(100)
C /BLKII/MM(300),NIT
C DIMENSION P(30),TT(30)

C INPUT PARAMETERS
C
C READ (5,20)RE,RMAX,EXPR,EXPT,NELRUP,NELRDN,NELT,NPREL,NPTEL
C READ (5,22)R0

C GENERATE THE FINITE ELEMENT GRID
C
C CALL GRID(NELRUP,NELRDN,NELT,NPREL,NPTEL,RMAX,EXPR,EXPT)

C SET THE BOUNDARY CONDITIONS
C
C DO 2 I=1,NEQ
C 2 IB(I) = -1
C CALL BOUNDY(2,0,0)
C CALL BOUNDY(1,1,0)
C CALL BOUNDY(2,1,0)
C CALL BOUNDY(1,2,1)
C CALL BOUNDY(2,2,0)
C CALL BOUNDY(3,2,0)

C SET INITIAL GUESS FOR THE VELOCITY AND PRESSURE FIELD
C
CALL GUESS()

DO 4 I=1,NEQ
  4 WW(I) = W(I)
  6 WRITE (6,24)
  6 WRITE (6,34)
  WRITE (6,30) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=1,43)
  WRITE (6,50)
  WRITE (6,30) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=44,NP)

C
C ITERATION SCHEME
C

NN = 2 * NELT
WRITE (6,26) NELRUP,NELRDN,NN,NPREL,NPTNL,NEL,NP,MBAND,RMAX,RE

OLSUM = 1.E+4

GAMMA = .6
DO 12 J=1,15

NIT = J
CALL SOLVE

BIG = 0.

SUM = 0.

DO 8 I=1,NEQ

DIF = CONST(I) - W(I)
IF (BIG .LT. ABS(DIF)) BIG = ABS(DIF)

8 SUM = SUM + DIF * DIF

WRITE (6,32) J,SUM,BIG
IF (OLSUM .LT. SUM) GO TO 14

OLSUM = SUM

DO 10 I=1,NEQ

10 WW(I) = W(I) + GAMMA * (CONST(I) - W(I))
IF (SUM .LT. 1.E-5) GO TO 14
IF (SUM .LT. 1.E-1) GAMMA = .8
IF (SUM .LT. 1.E-4) GAMMA = 1.

CONTINUE

CONTINUE
C RESULTS

WRITE (6,28)
WRITE (6,34)
WRITE (6,30) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=1,43)
WRITE (6,50)
WRITE (6,30) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=44,NP)
RE = RE + .5
IF (RE .GT. 1.1) STOP
GO TO 6

C

C FORMAT STATEMENTS

20 FORMAT (4F10.0,5I5)  GRW 84
22 FORMAT (16F5.0)  GRW 85
24 FORMAT (1H1,4(/),36X,29(* ')/53X,'INITIAL GUESSES'/  GRW 86
& 36X,29(* ')/)  GRW 87
26 FORMAT (1H1,4(/),36X,29(* ')/  GRW 88
& 36X,*NO. OF ELEMENTS ON THE UPSTREAM AXIS =',I3//  GRW 89
& 36X,*NO. OF ELEMENTS ON THE DOWNSTREAM AXIS =',I3//  GRW 90
& 36X,*NO. OF ELEMENTS IN THE TRANSVERSE DIRECTION =',I3//  GRW 91
& 36X,*EACH ELEMENT HAS',I3,* POINTS IN THE RADIAL DIRECTION//  GRW 92
& 45X,*AND',I3,* POINTS IN THE TRANSVERSE DIRECTION//  GRW 93
& 36X,*TOTAL NUMBER OF ELEMENTS =',I3//  GRW 94
& 36X,*TOTAL NUMBER OF POINTS =',I3//  GRW 95
& 36X,*MATRIX BANDWIDTH =',I3//  GRW 96
& 36X,*MAXIMUM RADIUS ON THE DOWNSTREAM AXIS =',F6.2//  GRW 97
& 36X,*REYNOLDS NUMBER =',F5.2//36X,29(* ')/)  GRW 98
28 FORMAT (1H1,4(/),36X,29(* ')/50X,'CORRECTED',  GRW 99
& 'VALUES'/36X,29(* ')/)  GRW 100
30 FORMAT (36X,I3,2F9.2,3F12.6)  GRW 101
32 FORMAT (36X,*ITERATION NUMBER',I3,5X,*SUM =',1PE8.1,  GRW 102
& 5X,*BIG =',1PE8.1/)  GRW 103
34 FORMAT (36X,*NODE',5X,*R',8X,*T',5X,*U-VELOCITY',2X,  GRW 104
& *V-VELOCITY',3X,*PRESSURE*/)  GRW 105
SUBROUTINE GRID (NELRUP, NELRDN, NELT, NPREL, NPTEL, RMAXDN, EXPR, EXPT) GRW 1
C C THIS SUBROUTINE GENERATES THE FINITE ELEMENT GRID C
C CCMDGN COEF(250,40), CONST(250), MBAND, NEQ
C &/NO/NODE(80,4)
C &/BLK10/NTYPE(100), IB(300), NP, NEL
C &/BLK3/X(100), Y(100), W(300)
C &/BLK6/RO(15), R(100), T(100)
C DIMENSION RR(20), TT(20)
C C CODE FOR NODAL POINT TYPES C
C IF THE POINT IS ON THE X-AXIS, NTYPE=0
C IF THE POINT IS ON THE BODY, NTYPE=1
C IF THE POINT IS AT RMAX, NTYPE=2
C C CALCULATE CONSTANTS AND INCREMENT THE RADIUS AND ANGLE C
C PI = 3.141592
C NPRUP = NELRUP*(NPREL - 1) + 1
C NPRDN = NELRDN*(NPREL - 1) + 1
C NPT = NELT*(NPTEL - 1) + 1
C NP = NPRUP*(NELT*2+1) + (NPTEL-2)*(NELT*2)*(NELRUP+1)
C IF (NELRUP .EQ. NELRDN) GO TO 4
C NN = NELRDN - NELRUP
C DO 2 I=1,NN
C 2 NP = NP + NPREL - 1 + (NELT-I+1)*(NPTEL+NPREL-3)
C CONTINUE
C DO 6 I=1,NP
C 6 NTYPE(I) = -1
C DO 30 J=1,NPRUP
C 30 NTYPE(J) = 0
C DO 32 J=1,NPRDN
C 32 NTYPE(NP-J+1) = 0
C A = NPRDN - 1

151
DO 8 I=1,NPRDN
B = I - 1
8 RR(I) = RMAXDN*(B/A)**EXPR
A = NPT - 1
DO 10 I=1,NPT
B = I - 1
10 TT(I) = PI/2.*((B/A)**EXPT
C
C THE UPSTREAM DOMAIN
C
NP = 0
NTH = 0
II = 0
NPT1 = NPT - 1
DO 18 I=1,NPT1
NTYPE(NP+1) = 1
NTH = NTH + 1
THETA = TT(I)
S = SIN(THETA)
C = COS(THETA)
II = II + 1
IF (II .EQ. 1 .OR. II .EQ. NPTEL) GO TO 12
NM = NPREL - 1
GO TO 14
12 NM = 1
II = 1
14 DO 16 J=1,NPRUP,NM
NP = NP + 1
RADIUS = RR(J) + R0(NTH)
R(NP) = RADIUS
T(NP) = THETA * 180. / 3.141592
X(NP) = - RADIUS * C
16 Y(NP) = RADIUS * S
18 NTYPE(NP) = 2
C THE DOWNSTREAM DOMAIN
C
NN = NPRUP
DO 28 I=1,NPT
NTYPE(NP+I) = 1
NTH = NTH + 1
THETA = PI - TT(NPT-I+1)
S = SIN(THETA)
C = COS(THETA)
KK = 0
II = II + 1
IF (II .EQ. NPTEL) GO TO 20
NM = NPREL - 1
GO TO 22
20 NM = 1
II = 1
IF (NN .EQ. NPRDN) GO TO 22
NN = NN + NPREL - 1
KK = 1
22 CONTINUE
DO 24 J=1,NN,NM
NP = NP + 1
RADIUS = R(R(J)) + RO(NTH)
R(NP) = RADIUS
T(NP) = THETA * 180. / 3.141592
X(NP) = - RADIUS * C
24 Y(NP) = RADIUS * S
IF (KK .EQ. 0) GO TO 28
NM = NPREL - 1
DO 26 J=1,MM
26 NTYPE(NP-J) = 2
28 NTYPE(NP) = 2
C
C DEFINE THE ELEMENTS BY THEIR NODES
C
C
NEL = 0
N1 = NELRUP
NELT2 = 2*NELT
DO 50 J=1, NELT2
IF (N1 *LT. NELRDN *AND. J *GT. NELT) N1 = N1 + 1
N2 = N1*(NPREL - 1) + 1
N3 = N1 + 1
DO 50 I=1, N1
NEL = NEL + 1
IF (I *EQ. 1) GO TO 34
NODE(NEL+1) = NODE(NEL-1,1) + NPREL - 1
GO TO 38
34 IF (J *GT. 1) GO TO 36
NODE(1,1) = 1
GO TO 38
36 NODE(NEL,1) = NODE(NEL-1,NPTEL+NPREL) + 1
38 CONTINUE
NODE(NEL,2) = NODE(NEL,1) + N2 - (I-1)*(NPREL-1) + I - 1
IF (NPTEL *EQ. 2) GO TO 42
DO 40 II=3,NPTEL
40 NODE(NEL,II) = NODE(NEL,II-1) + N3
NODE(NEL,NPTEL) = NODE(NEL,NPTEL) + (NPREL-1)*(I-1) - I + 1
42 CONTINUE
MM = NPREL - 1
DO 44 II=1,MM
44 NODE(NEL,NPTEL+II) = NODE(NEL,NPTEL) + II
NN = 2*(NPTEL + NPREL - 2)
DO 46 II=1,MM
46 NODE(NEL,NN-MM+II) = NODE(NEL,1) + NPREL - II
NN = NPTEL - 2
IF (NN *EQ. 0) GO TO 50
DO 48 II=1,NN
48 NODE(NEL,NPTEL+MM+II) = NODE(NEL,NPTEL-II) + 1
50 CONTINUE
MBAND = 3*(NELRDN*(2*NPTEL-3) + 2*NPREL - 1)
SUBROUTINE BOUNDY(NVAR,NBOUND,NCOND)

C THIS SUBROUTINE SETS THE BOUNDARY CONDITIONS
C
COMMON/BLK10/NTYPE(100),IB(300),NP,NEL
C
C IF NVAR = 1 THE U-VELOCITY IS FIXED
C IF NVAR = 2 THE V-VELOCITY IS FIXED
C IF NVAR = 3 THE PRESSURE IS FIXED
C
C IF NBOUND=0 THE VARIABLE IS FIXED ALONG Y=0
C IF NBOUND=1 THE VARIABLE IS FIXED ON THE BODY
C IF NBOUND=2 THE VARIABLE IS FIXED AT RMAX
C IF NBOUND=3 THE VARIABLE IS FIXED AT ALL POINTS
C
C IF NCOND=0 THE VARIABLE IS SET TO 0
C IF NCOND=1 THE VARIABLE IS SET TO 1
C IF NCOND=2 THE VARIABLE RETAINS IT'S PRESENT VALUE
C IF NCOND=3 THE VARIABLE RETAINS THE INITIAL GUESS
C IF NCOND=4 THE VARIABLE IS FREE
C
NV = 3 - NVAR
IF (NBOUND .EQ. 3) GO TO 4
DO 2 I=1,NP
2 IF (NTYPE(I) .EQ. NBOUND) IB(3*I-NV) = NCOND
RETURN
4 DO 6 I=1,NP
6 IB(3*I-NV) = NCOND
RETURN
END
SUBROUTINE GUESS(K)

C THIS SUBROUTINE SETS THE INITIAL GUESS FOR THE VELOCITY FIELD.

CCM0N/BLK1/RE,IVAR
& /BLK3/X(100),Y(100),W(300)
& /BLK10/NTYPE(100),IB(300),NP,NEL
GO TO (4,8),K

C OSEEN'S APPROXIMATION

CC = 3.7/RE
CC = 2.0/ALOG(CC)
DO 2 I=1, NP
  J = 3*I
  RR = X(I)*X(I) + Y(I)*Y(I)
  RS = SQRT(RR)
  W(J-2) = -.5*CC*((1./RR - 1.)*(1.5 - X(I)*X(I)/RR) - ALOG(RS))
  W(J-1) = -.5*CC*(X(I)*Y(I)/RR) *(1. - 1./RR)
2 W(J) = -. CC*X(I)/(RE*RR)
RETURN

C FREE STREAM SOLUTION

4 DO 6 I=1, NP
  W(3*I-2) = 1.
  W(3*I-1) = 0.
6 W(3*I) = 0.
RETURN

C PERTURBED FREE STREAM

8 DO 10 I=1, NP
  RR = .1/(X(I)*X(I) + Y(I)*Y(I))
  W(3*I-2) = 1. - RR
10 RETURN
10 \text{RETN}
\text{END}
SUBROUTINE SOLVE
C
C THIS SUBROUTINE FORMS THE EQUATIONS AND SOLVES THEM.
C
COMMON COEF(250,40), CONST(250), MBAND, NEQ
&/BLK10/, NT (100), IB(300), NP, NEL
&/BLK3/, X(100), Y(100), W(300)
&/BLK11/, WW(300), NIT
C
C FORM THE EQUATIONS
C
DO 2 I=1, NEQ
   CONST(I)=0.
   DO 2 J=1, MBAND
   2 COEF(I,J)=0.
   DO 4 I=1, NEL
   4 CALL QUAD4(I)
C
C IMPOSE BOUNDARY CONDITIONS
C
DO 6 I=1, NEQ
   IF (IB(I) .EQ. 0) CALL MODIFY(I,0.)
   IF (IB(I) .EQ. 1) CALL MODIFY(I,1.)
   IF (IB(I) .EQ. 2) CALL MODIFY(I,W(I))
   IF (IB(I) .EQ. 3) CALL MODIFY(I,WW(I))
   6 CONTINUE
C
C SOLVE THE EQUATIONS
C
   CALL SYMSOL
RETURN
END
SUBROUTINE QUAD4(N)

C THIS SUBROUTINE FORMS THE COEFFICIENT MATRIX, COE(I,J), AND CONSTANT
C CON(I), FOR A FOUR NODE QUADRILATERAL, AND INTRODUCES THEM INTO
C OVERALL COEFFICIENT MATRIX AND CONSTANT VECTOR.

COMMON COEF(250,40),CONST(250),MBAND,NEQ
&/BLK3/XX(100),YY(100),W(300)
&/NO/NOD(80,4)
&/BLK5/X(9),Y(9),U(9),V(9),CON(27),COE(27,27)
DIMENSION ID(12)
INTEGER RCW,CCLM
DO 2 I=1,4
K = NOD(N,I)
ID(3*I-2) = 3*K-2
ID(3*I-1) = 3*K-1
ID(3*I) = 3*K
X(I) = XX(K)
Y(I) = YY(K)
U(I) = W(3*K-2)
2 V(I) = W(3*K-1)

C INTRODUCE INTERNAL NODES

C X(9) = (X(1) + X(2) + X(3) + X(4))/4.
Y(9) = (Y(1) + Y(2) + Y(3) + Y(4))/4.
U(9) = (U(1) + U(2) + U(3) + U(4))/4.
V(9) = (V(1) + V(2) + V(3) + V(4))/4.
X(5) = (X(1) + X(2) + X(3))/3.
Y(5) = (Y(1) + Y(2) + Y(3))/3.
U(5) = (U(1) + U(2) + U(3))/3.
V(5) = (V(1) + V(2) + V(3))/3.
X(6) = (X(2) + X(3) + X(9))/3.
Y(6) = (Y(2) + Y(3) + Y(9))/3.
U(6) = (U(2) + U(3) + U(9))/3.
V(6) = (V(2) + V(3) + V(9))/3.

C

160
\[ V(6) = \frac{(V(2) + V(3) + V(9))}{3}. \]
\[ X(7) = \frac{(X(3) + X(4) + X(9))}{3}. \]
\[ Y(7) = \frac{(Y(3) + Y(4) + Y(9))}{3}. \]
\[ U(7) = \frac{(U(3) + U(4) + U(9))}{3}. \]
\[ V(7) = \frac{(V(3) + V(4) + V(9))}{3}. \]
\[ X(8) = \frac{(X(4) + X(1) + X(9))}{3}. \]
\[ Y(8) = \frac{(Y(4) + Y(1) + Y(9))}{3}. \]
\[ U(8) = \frac{(U(4) + U(1) + U(9))}{3}. \]
\[ V(8) = \frac{(V(4) + V(1) + V(9))}{3}. \]

C FORM COEFFICIENT MATRIX AND CONSTANT VECTOR FOR FIVE NODE QUADRILATER

C

DO 1 I=1,27
CON(I)=0.
DO 1 J=1,27
1 COE(I,J)=0.
CALL TRI3(1,2,5)
CALL TRI3(2,3,6)
CALL TRI3(3,4,7)
CALL TRI3(4,1,8)
CALL TRI3(1,5,8)
CALL TRI3(2,6,5)
CALL TRI3(3,7,6)
CALL TRI3(4,8,7)
CALL TRI3(8,5,9)
CALL TRI3(5,6,9)
CALL TRI3(6,7,9)
CALL TRI3(7,8,9)

C REDEFINE PRESSURE AND ELIMINATE INTERNAL NODES

C

NODE 9

C

DO 4 I=1,27
DO 4 J=1,4
4 COE(I,3*J) = COE(I,3*J) + COE(I,27)/4.
   DO 6 I=1,27
   DO 6 J=1,4
6 COE(3*J,I) = COE(I,3*J)
   CALL ELIM(9)
C
C NODE 8
C
   DO 22 I=1,24
   COE(I,3) = COE(I,3) + COE(I,24)/3.
   COE(I,6) = COE(I,6) + COE(I,24)/6.
   COE(I,9) = COE(I,9) + COE(I,24)/6.
22 COE(I,12) = COE(I,12) + COE(I,24)/3.
   DO 24 I=1,24
   COE(3,1) = COE(I,3)
   COE(6,1) = COE(I,6)
   COE(9,1) = COE(I,9)
24 COE(12,1) = COE(I,12)
   CALL ELIM(8)
C
C NODE 7
C
   DO 26 I=1,21
   COE(I,3) = COE(I,3) + COE(I,21)/6.
   COE(I,6) = COE(I,6) + COE(I,21)/6.
   COE(I,9) = COE(I,9) + COE(I,21)/3.
26 COE(I,12) = COE(I,12) + COE(I,21)/3.
   DO 28 I=1,21
   COE(3,1) = COE(I,3)
   COE(6,1) = COE(I,6)
   COE(9,1) = COE(I,9)
28 COE(12,1) = COE(I,12)
   CALL ELIM(7)
C
C NODE 6
C
C
DO 30 I=1,18
COE(I,3) = COE(I,3) + COE(I,18)/6.
COE(I,6) = COE(I,6) + COE(I,18)/3.
COE(I,9) = COE(I,9) + COE(I,18)/3.
30 COE(I,12) = COE(I,12) + COE(I,18)/6.
DO 32 I=1,18
COE(3,I) = COE(I,3)
COE(6,I) = COE(I,6)
COE(9,I) = COE(I,9)
32 COE(12,I) = COE(I,12)
CALL ELIM(6)

C
C NODE 5
C
DO 34 I=1,15
COE(I,3) = COE(I,3) + COE(I,15)/3.
COE(I,6) = COE(I,6) + COE(I,15)/3.
COE(I,9) = COE(I,9) + COE(I,15)/3.
34 COE(I,12) = COE(I,12) + COE(I,15)/6.
DO 36 I=1,15
COE(3,I) = COE(I,3)
COE(6,I) = COE(I,6)
COE(9,I) = COE(I,9)
36 COE(12,I) = COE(I,12)
CALL ELIM(5)

C
C INTRODUCE INTO OVERALL COEFFICIENT MATRIX AND CONSTANT VECTOR
C
DO 12 I=1,12
ROW = IO(I)
CONST(ROW) = CONST(ROW) + CON(I)
DO 12 J=1,12
COLM = ID(J) - ROW + 1
IF (COLM .LE. 0) GO TO 12
\[
\text{COEF(ROW, COLM) = COEF(ROW, COLM) + COE(I, J)}
\]

12 CONTINUE
RETURN
END
SUBROUTINE TRI3(N1,N2,N3)

C THIS SUBROUTINE FORMS THE MATRIX AND VECTOR FOR A THREE NODE TRIANGLE

C WITH NODES N1, N2, N3.

C

CCMCN/BLK1/RE
&/BLK5/X(9),Y(9),U(9),V(9),CON(27),COE(27,27)
DIMENSION ID(9),H(9,9),F(9)
INTEGER RCW,COLM

C INITIALIZE

C

DO 2 I=1,9
F(I) = 0.

DO 2 J=1,9
2 H(I,J) = 0.

C CALCULATE FACTORS

C

A1 = X(N2)*Y(N3) - X(N3)*Y(N2)
A2 = X(N3)*Y(N1) - X(N1)*Y(N3)
A3 = X(N1)*Y(N2) - X(N2)*Y(N1)
B1 = Y(N2) - Y(N3)
B2 = Y(N3) - Y(N1)
B3 = Y(N1) - Y(N2)
C1 = X(N3) - X(N2)
C2 = X(N1) - X(N3)
C3 = X(N2) - X(N1)
AREA = (A1 + A2 + A3)/2.
XBAR = (X(N1) + X(N2) + X(N3))/3.
YBAR = (Y(N1) + Y(N2) + Y(N3))/3.
T = 4.*AREA
AN1 = (A1 + B1*XBAR + C1*YBAR)/T
AN2 = (A2 + B2*XBAR + C2*YBAR)/T
AN3 = (A3 + B3*XBAR + C3*YBAR)/T
\[ T = T \times \text{RE} \]

C

C FORM MATRIX ABOVE DIAGONAL

C

\[
\begin{align*}
H(1,1) &= (B1*B1 + C1*C1)/T \\
H(1,4) &= (B1*B2 + C1*C2)/T \\
H(1,7) &= (B1*B3 + C1*C3)/T \\
H(1,3) &= -B1*AN1 \\
H(1,6) &= -B1*AN2 \\
H(1,9) &= -B1*AN3 \\
H(2,3) &= -C1*AN1 \\
H(2,6) &= -C1*AN2 \\
H(2,9) &= -C1*AN3 \\
H(3,4) &= -B2*AN1 \\
H(3,7) &= -B3*AN1 \\
H(3,5) &= -C2*AN1 \\
H(3,8) &= -C3*AN1 \\
H(4,4) &= (B2*B2 + C2*C2)/T \\
H(4,7) &= (B2*B3 + C2*C3)/T \\
H(4,6) &= -B2*AN2 \\
H(4,9) &= -B2*AN3 \\
H(5,6) &= -C2*AN2 \\
H(5,9) &= -C2*AN3 \\
H(6,7) &= -B3*AN2 \\
H(6,8) &= -C3*AN2 \\
H(7,7) &= (B3*B3 + C3*C3)/T \\
H(7,9) &= -B3*AN3 \\
H(8,9) &= -C3*AN3 \\
H(2,2) &= H(1,1) \\
H(2,5) &= H(1,4) \\
H(2,8) &= H(1,7) \\
H(5,5) &= H(4,4) \\
H(5,8) &= H(4,7) \\
H(8,8) &= H(7,7)
\end{align*}
\]
C FILL IN MATRIX BELOW DIAGONAL

DO 4 I=1,8
    K = I + 1
    DO 4 J=K,9
    4 H(J,I) = H(I,J)

C FORM VECTOR

XX = AREA*(((Y(N1)-YBAR)*(Y(N1)-YBAR) + (Y(N2)-YBAR)*(Y(N2)-YBAR)) / 12. + YBAR*YBAR)

YY = AREA*(((X(N1)-XBAR)*(X(N1)-XBAR) + (X(N2)-XBAR)*(X(N2)-XBAR)) / 12. + XBAR*XBAR)

XY = AREA*(((X(N1)-XBAR)*(Y(N1)-YBAR) + (X(N2)-XBAR)*(Y(N2)-YBAR)) / 12. + XBAR*YBAR)

U1 = A1*U(N1) + A2*U(N2) + A3*U(N3)
U2 = B1*U(N1) + B2*U(N2) + B3*U(N3)
U3 = C1*U(N1) + C2*U(N2) + C3*U(N3)
V1 = A1*V(N1) + A2*V(N2) + A3*V(N3)
V2 = B1*V(N1) + B2*V(N2) + B3*V(N3)
V3 = C1*V(N1) + C2*V(N2) + C3*V(N3)
AA = U1*U2 + V1*U3
BB = U2*U2 + V2*U3
CC = U3*U2 + V3*U3

T = 2.*AREA

TT = T*T


& /((2.*TT) + (XY*(B1*CC + C1*BB) + B1*BB*YY + C1*CC*XX)/(TT*T))


F(7) = (A3*AA + XBAR*(A3*BB + B3*AA) + YBAR*(A3*CC + C3*AA))

& /((2.*TT) + (XY*(B3*CC + C3*BB) + B3*BB*YY + C3*CC*XX)/(TT*T))

AA = U1*V2 + V1*V3
BB = U2*V2 + V2*V3
CC = U3*V2 + V3*V3
\[ + \frac{(XY*(B1*CC + C1*BB) + B1*BB*YY + C1*CC*XX)/(TT*TT)}{\beta(2*TT) + (XY*(B1*CC + C1*BB) + B1*BB*YY + C1*CC*XX)/(TT*TT)} \]


\[ F(8) = (A3*AA + XBAR*(A3*BB + B3*AA) + YBAR*(A3*CC + C3*AA)) \]
\[ + \frac{(XY*(B3*CC + C3*BB) + B3*BB*YY + C3*CC*XX)/(TT*TT)}{\beta(2*TT) + (XY*(B3*CC + C3*BB) + B3*BB*YY + C3*CC*XX)/(TT*TT)} \]

C
C INTRODUCE INTO OVERALL MATRIX AND VECTOR
C
I1 = 3*N1
I2 = 3*N2
I3 = 3*N3
DO 6 I=1,3
   ID(I) = I1 + I - 3
6   ID(I+6) = I3 + I - 3
DO 8 I=1,9
   ROW = ID(I)
   CON(ROW) = CON(ROW) - F(I)
7 DO 8 J=1,9
   COLM = ID(J)
   COE(ROW, COLM) = COE(ROW, COLM) + H(I,J)
RETURN
END
SUBROUTINE ELIM(NODE)

C THIS SUBROUTINE ELIMINATES INTERNAL NODE EQUATIONS FOR U AND V

C

COMMON/BLKS/X(9),Y(9),U(9),V(9),CON(27),COE(27,27)

LL = 3*NODE - 1
DO 10 L=1,2
  K = LL - L
  DO 10 I=1,K
    C=COE(I,K+1)/COE(K+1,K+1)
    CON(I)=CON(I)-C*CON(K+1)
  10 CONTINUE
DO 10 J=1,K
  10 COE(I,J)=COE(I,J)-C*COE(K+1,J)
RETURN
END
SUBROUTINE MODIFY(ROW, VALUE)

C THIS SUBROUTINE MAKES CONST(ROW) = VALUE
C
COMMON COEF(250, 40), CONST(250), MBAND, NEQ
INTEGER ROW
DO 4 J = 2*MBAND
    MOD I F C O L U M N
    I = ROW - J + 1
    IF (I .LE. 0) GO TO 2
    CONST(I) = CONST(I) - COEF(I, J)*VALUE
    COEF(I, J) = 0.

C MOD I F R O W
    2 I = ROW + J - 1
    IF (I .GT. NEQ) GO TO 4
    CONST(I) = CONST(I) - COEF(ROW, J)*VALUE
    COEF(ROW, J) = 0.
    4 CONTINUE

C FIX UNKNOWN AND NORMALIZE DIAGONAL
C
    CONST(ROW) = VALUE
    COEF(ROW, 1) = 1.
    RETURN
END
SUBROUTINE SYMSOL

C THIS SUBROUTINE SOLVES A SYSTEM OF EQUATIONS
C WITH A BANDED SYMMETRIC MATRIX
C
C CGMMCN COEF(250,40),CONST(250),MBAND,NEQ
C
C REDUCE MATRIX
C
DO 8 N=1,NEQ
DO 6 L=2,MBAND
C = COEF(N,L)/COEF(N,1)
I = N + L - 1
IF (I .GT. NEQ) GO TO 6
J = 0
DO 4 K=L,MBAND
J = J+1
4 COEF(I,J) = COEF(I,J) - C*COEF(N,K)
6 COEF(N,L) = C
8 CONTINUE
C
C REDUCE VECTOR
C
DO 12 N=1,NEQ
DO 10 L=2,MBAND
I = N + L - 1
IF (I .GT. NEQ) GO TO 12
10 CONST(I) = CONST(I) - COEF(N,L)*CONST(N)
12 CONST(N) = CONST(N)/COEF(N,1)
C
C BACK SUBSTITUTION
C
DO 16 I=2,NEQ
N = NEQ - I + 1
DO 16 K = 2, MBAND
   L = N + K - 1
   IF (L .LT. NEQ) GO TO 16
   CONTINUE = CONS(N) = COEF(N,K) * CONST(L)
16  CONTINUE = CONS(N) = COEF(N,K) * CONST(L)
       RETURN
       END
C
C
C ** * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C COMMON COEF(100,20),CONST(100),MBAND,NEQ
C & /BLK1/RE,IVAR
C & /BLK2/AREA,AL(16),AO
C & /BLK3/X(100),Y(100),W(300)
C & /BLK6/R0(15),R(100),T(100)
C & /BLK10/NTYPE(100),IB(300),NP,NREL
C & /BLK16/COE(100,20)
C DIMENSION GAMMA(3),WW(300)
C
C INPUT PARAMETERS
C
C READ (5,34)RE,RMAX,EXPR,EXPT,NELRUP,NELRDN,NELT,NPREL,NPTEL
C READ (5,30)R0
C
C GENERATE THE FINITE ELEMENT GRID
C
C CALL GRID(NELRUP,NELRDN,NELT,NPREL,NPTEL,RMAX,EXPR,EXPT)
C
C SET THE BOUNDARY CONDITIONS
C
C NN = 3*NP
C DO 2 I=1,NN
C 2 IB(I) = -1
C CALL BOUNDY(1,0,0)
C CALL BOUNDY(2,1,1)
C CALL BOUNDY(1,1,0)
C CALL BOUNDY(2,2,1)
C CALL BOUNDY(1,2,0)
C CALL BOUNDY(0,2,0)
C SET INITIAL GUESS FOR THE VELOCITY AND PRESSURE FIELD
C
CALL GUESS(1)
DO 4 I=1,NN
4 WW(I) = W(I)
WRITE (6,36)
WRITE (6,38)
WRITE (6,40) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=1,43)
WRITE (6,50)
WRITE (6,40) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=44,NP)
NN = 2*NELT
WRITE (6,32) NE,RUP,NERDN,NN,NPREL,NPTEL,NEL,NP,MBAND,RMAX,RE
C FORM THE MATRIX
C
DO 6 I=1,NP
DO 6 J=1,MBAND
6 COE(I,J) = 0.
CALL ONSCE(3)
AREA = 0.
IVAR = 3
DO 8 I=1,NEL
8 CALL MATRIX(I)
WRITE (6,42) AO,AREA
C ITERATION SCHEME
C
DO 10 I=1,3
10 GAMMA(I) = 1.
DO 22 K=1,5
WRITE (6,26)K
ICOUNT = 0
IVAR = 3
12 IVAR = IVAR - 1
FORM EQUATIONS AND SOLVE THEM

DO 14 I=1,NP
   DO 14 J=1,MBAND
   14 CDEF(I,J) = CDE(I,J)
   CALL SOLVE

COMPUTE CHANGE FROM LAST ITERATION

BIG = 0.
SUM = 0.
DO 16 I=1,NP
   DIF = ABS(WW(3*I-IVAR) - CONST(I))
   SUM = SUM + DIF
16 IF (BIG .T. DIF) BIG = DIF
   IF (IVAR .EQ. 2) WRITE (6,44) SUM,BIG
   IF (IVAR .EQ. 1) WRITE (6,46) SUM,BIG
   IF (IVAR .EQ. 0) WRITE (6,48) SUM,BIG
   IF (BIG .GT. 1.E+3) STOP
   IF (BIG .LT. 1.E-2) GAMMA(IVAR+1) = 1.
   IF (BIG .LT. 1.E-3) GAMMA(IVAR+1) = 1.

MAKE NEW GUESS FOR NEXT ITERATION

DO 20 I=1,NP
   J = 3*I - IVAR
   WW(J) = CONST(I)
   IF (IB(J) .T. 0) GO TO 18
   WW(J) = WW(J) + GAMMA(IVAR+1)*(CONST(I) - WW(J))
   GO TO 20
18 WW(J) = CONST(I)
20 CONTINUE
   IF (BIG .LT. 1.E-5) ICOUNT = ICOUNT + 1
   IF (ICOUNT .EQ. 3) GO TO 24
IF (IVAR GT 0) GO TO 12
22 CONTINUE
24 CONTINUE

C RESULTS
C
WRITE (6,28)
WRITE (6,38)
WRITE (6,40) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=1,43)
WRITE (6,50)
WRITE (6,40) (J,R(J),T(J),W(3*J-2),W(3*J-1),W(3*J),J=44,NP)

C FORMAT STATEMENTS
C
26 FORMAT (/36X,'_ITERATION NUMBER',I3)
28 FORMAT (1H1,4(/),36X,29(* ')
& 'CORRECTED',/50X,'NAMES',/36X,29(* ')/)
30 FORMAT (16F5.0)
32 FORMAT (1H1,4(/),36X,29(* ')
& 'NUMBER OF ELEMENTS ON THE UPSTREAM AXIS =',I3/
& 'NUMBER OF ELEMENTS ON THE DOWNSTREAM AXIS =',I3/
& 'NUMBER OF ELEMENTS IN THE TRANSVERSE DIRECTION =',I3/
& 'EACH ELEMENT HAS',I3,' POINTS IN THE RADIAL DIRECTION/'
& 'AND',I3,' POINTS IN THE TRANSVERSE DIRECTION//'
& 'TOTAL NUMBER OF ELEMENTS =',I3/
& 'TOTAL NUMBER OF POINTS =',I3/
& 'MATRIX BANDWIDTH =',I3/
& 'MAXIMUM RADIUS ON THE DOWNSTREAM AXIS =',F5.2/
& 'REYNOLDS NUMBER =',F5.2/36X,29(* ')/)
34 FORMAT (4F10.5,515)
36 FORMAT (1H1,4(/),36X,29(* ')/53X,'INITIAL GUESS'/
& 'PRESSURE'//)
38 FORMAT (36X,'NODE',5X,'R',8X,'T',5X,'U VELOCITY',2X,
& 'V VELOCITY',3X,'PRESSURE'//)
40 FORMAT (36X,I3,2F9.2,3F12.6)
42 FORMAT(*36X,*EXACT DOMAIN AREA =* ,F7.2,2X,* 
& *COMPUTED DOMAIN AREA =* ,F7.2/)
44 FORMAT(*41X,*U-VELOCITY NORM =* ,1PE8,1,5X,*LARGEST TERM =* ,1PE8,1)
46 FORMAT(*41X,*V-VELOCITY NORM =* ,1PE8,1,5X,*LARGEST TERM =* ,1PE8,1)
48 FORMAT(*41X,* PRESSURE NORM =* ,1PE8,1,5X,*LARGEST TERM =* ,1PE8,1)
50 FORMAT(*41H1//////)
STOP
END
SUBROUTINE GRID(NELRUP, NELRDN, NELT, NPTEL, RMAXDN, EXPR, EXPT)  
C  
C THIS SUBROUTINE GENERATES THE FINITE ELEMENT GRID,  
C  
COMM N COEF(100, 20) , CON ST(100) , MB AND, NEQ  
& /BLK2/AREA, AI(16), AO  
& /BLK3/X(100), Y(100), W(300)  
& /BLK4/NODE(50, 9)  
& /BLK6/R0(15), R(100), T(100)  
& /BLK1C/NTYPE(100), IB(300), NP, NEL  
DIMENSION RR(20), TT(20)  
C  
C CALCULATE CONSTANTS  
C  
PI = 3.141592  
AD = ((RMAXDN + 1e)*(RMAXDN + 1e) - 1e)*PI/2,  
NPTEL = 2*(NPTEL - NPTEL) - 3  
NP = NELRUP*NPTEL + 1  
NPRD = NELRDN*NPTEL + 1  
NPT = NELT*(NPTEL - 1) + 1  
NP = NELRUP*(NELT*2 + 1) + (NPTEL - 2)*(NELT*2)*(NELTUP + 1)  
IF (NELRUP = EQ NELRDN) GO TO 4  
NN = NELRDN - NELRUP  
DO 2 I=1, NN  
2 NP = NP + NPTEL - 1 + (NELT - I + 1)*(NPTEL + NPTEL - 3)  
4 CONTINUE  
C  
C CODE FOR NODAL POINT TYPES  
C  
IF THE POINT IS ON THE X-AXIS, NTYPE=0  
C  
IF THE POINT IS ON THE BODY, NTYPE=1  
C  
IF THE POINT IS AT RMAX, NTYPE=2  
C  
DO 6 I=1, NP  
6 NTYPE(I) = -1  
DO 30 J=1, NPROUP  
30
30 NTYPE(J) = 0
32 NTYPE(NP-J+1) = 0
C
C INCREMENT RADIUS AND ANGLE
C
8 RR(I) = RMAXDV*(B/A)**EXPR
A = NPT - 1
B = I - 1
10 TT(I) = PI/2*(B/A)**EXPT
C
C THE UPSTREAM DOMAIN
C
NP = 0
NTH = 0
II = 0
NPT1 = NPT - 1
DO 18 I=1,NPT1
NTYPE(NP+1) = 1
NTH = NTH + 1
THETA = TT(I)
S = SIN(THETA)
C = COS(THETA)
II = II + 1
IF (II .EQ. 1 .OR. II .EQ. NPTEL) GO TO 12
NM = NPREL - 1
GO TO 14
12 NM = 1
II = 1
14 DO 16 J=1,NPRUP,NM
NP = NP + 1
RADIUS = RR(J) + RO(NTH)
R(NP) = RADIUS
T(NP) = THETA * 180 / 3.141592
X(NP) = - RADIUS * C
16 Y(NP) = RADIUS*S
18 NTYPE(NP) = 2

C THE DOWNSTREAM DOMAIN
C
NN = NPRUP
DO 28 I=1,NPT
NTYPE(NP+1) = 1
NTH = NTH + 1
THETA = PI - TT(NPT-I+1)
S = SIN(THETA)
C = COS(THETA)
KK = 0
II = II + 1
IF (II .EQ. NPTEL) GO TO 20
NM = NPREL - 1
GO TO 22
20 NM = 1
II = 1
IF (NN .EQ. NPRDN) GO TO 22
NN = NN + NPREL - 1
KK = I
22 CONTINUE
DO 24 J=1,NN,NM
NP = NP + 1
RADIUS = RR(J) + RO(NTH)
R(NP) = RADIUS
T(NP) = THETA * 180 / 3.141592
X(NP) = - RADIUS * C
24 Y(NP) = RADIUS*S
IF (KK .EQ. 0) GO TO 28

MM = NPREL - 1
DO 26 J = 1, MM
26 NTYPE(NP = J) = 2
28 NTYPE(NP) = 2

C
C DEFINE THE ELEMENTS BY THEIR NODES
C
DO 52 I = 1, 50
52 NODE(I, NP1) = 0
NEL = 0
N1 = NELRUP
NELT2 = 2 * NELT
DO 50 J = 1, NELT2
IF (N1 .LT. NELRDN .AND. J .GT. NELT) N1 = N1 + 1
N2 = N1 * (NPREL - 1) + 1
N3 = N1 + 1
NEL = NEL + 1
DO 50 I = 1, N1
NEL = NEL + 1
IF (I .EQ. 1) GO TO 34
NODE(NEL + 1) = NODE(NEL - 1, 1) + NPREL - 1
GO TO 38
34 IF (J .GT. 1) GO TO 36
NODE(1, I) = 1
NODE(1, NP1) = 1
GO TO 38
36 NODE(NEL + 1) = NODE(NEL - 1, NPTEL + NPREL) + 1
NODE(NEL, NP1) = 1
38 CONTINUE
NODE(NEL + 2) = NODE(NEL, 1) + N2 - (I - 1) * (NPREL - 1) + I - 1
IF (NPTEL .EQ. 2) GO TO 42
DO 40 I = 3, NPTEL
40 NODE(NEL + I) = NODE(NEL + I - 1) + N3
NODE(NEL, NPTEL) = NODE(NEL, NPTEL) + (NPREL - 1) * (I - 1) - I + 1
42 CONTINUE
MM = NPREL - 1
DO 44 I=1/MM
44 NODE(NEL*NPTEL+II) = NODE(NEL*NPTEL) + II
NN = 2*(NPTEL + NPRL - 2)
DO 46 II=1/MM
46 NODE(NEL*NPTEL+MM+II) = NODE(NEL*II) + NPRL - II
NN = NPTEL + 2
IF (NN.EQ.0) GO TO 50
DO 48 II=1/MM
48 NODE(NEL*NPTEL+MM+II) = NODE(NEL*NPTEL-II) + 1
50 CONTINUE = VELRDY*(2*NPTEL-3) + 2*NPRL - 1
NEQ = NP
RETURN
END
SUBROUTINE BOUNDY(IVAR,NBOUND,NCOND)
C
C THIS SUBROUINIE SETS THE BOUNDARY CONDITIONS.
C
COMMON/BLK10/NTYPE(100),IB(300),NP,NEL
C
C IF IVAR = 2 THE U-VELOCITY IS FIXED
C IF IVAR = 1 THE V-VELOCITY IS FIXED
C IF IVAR = 0 THE PRESSURE IS FIXED
C
C IF NBOUND=0 THE VARIABLE IS FIXED ALONG Y=0
C IF NBOUND=1 THE VARIABLE IS FIXED ON THE BODY
C IF NBOUND=2 THE VARIABLE IS FIXED AT RMAX
C IF NBOUND=3 THE VARIABLE IS FIXED AT ALL POINTS
C
C IF NCOND=0 THE VARIABLE IS SET TO 0
C IF NCOND=1 THE VARIABLE IS SET TO 1
C IF NCOND=2 THE VARIABLE RETAINS IT'S PRESENT VALUE
C IF NCOND=3 THE VARIABLE RETAINS THE INITIAL GUESS
C IF NCOND=4 THE VARIABLE IS FREE
C
IF (NBOUND .EQ. 3) GO TO 4
DO 2 I=1,NP
 2 IF (NTYPE(I) .EQ. NBOUND) IB(3*I-IVAR) = NCOND
RETURN
DO 6 I=1,NP
 6 IB(3*I-IVAR) = NCOND
RETURN
END
SUBROUTINE GUESS(K)

COMMON/BLK1/IVAR
& /BLK3/X(100),Y(100),W(300)
& /BLK10/TYPE(100),IB(300),NP,NEL
GO TO (4,8),K

OSEEN'S APPROXIMATION

CC = 3*R/R
CC = 2*R/ALOG(CC)
DO 2 I=1,NP
J = 3*I
RR = X(I)*X(I) + Y(I)*Y(I)
RS = SQRT(RR)
W(J-2) = -S*CC*(1.0/RR-1.0)*(S - X(I)*X(I)/RR) - ALG(RS))
W(J-1) = -S*CC*(X(I)*Y(I)/RR)*(1.0 - 1.0/RR)
2 W(J) = -CC*X(I)/(RE*RR)
RETURN

FREE STREAM SOLUTION

4 DO 6 I=1,NP
W(3*I-2) = 1.0
W(3*I-1) = 0.0
6 W(3*I) = 0.0
RETURN

PERTURBED FREE STREAM

8 DO 10 I=1,NP
RR = 1.0/(X(I)*X(I) + Y(I)*Y(I))
W(3*I-2) = 1.0 - RR
w(3*I-1) = RR
10 w(3*I) = RR
RETURN
END

GR# 36
GR# 37
GR# 38
GR# 39
SUBROUTINE ONCE(NGP)
C
C THIS SUBROUTINE EVALUATES THE INTERPOLATING FUNCTIONS
C AND THEIR DERIVATIVES AT THE GAUSS POINTS;
C
C COMMON/N lbs/N(B),DNDA(B),DNDB(B),DNDAA(B),DNDB(B),DNDBB(B)
C /BLK7/WT(4),NG,NNG
C /BLKB/GN(8,16),DNA(8,16),DNB(8,16),
C /DNAA(8,16),DNAB(8,16),DNBB(8,16)
C /BLK11/GNB(3,4),BDNA(8,3),BDNB(8,3)
REAL NP,N,PT(4)
NG = NGP
NNG = NG*NG
C
C GAUSS POINTS AND WEIGHTING COEFFICIENTS
C
IF (NG .EQ. 3) GO TO 2
IF (NG .EQ. 4) GO TO 4
C
C TWO POINTS IN EACH DIRECTION
C
PT(1) = - 5773502691
PT(2) = - PT(1)
WT(1) = 1.
WT(2) = 1.
GO TO 6
C
C THREE POINTS IN EACH DIRECTION
C
2 PT(1) = - 7745966692
PT(2) = 0.
PT(3) = - PT(1)
WT(1) = 5555555556
WT(2) = 8888888889
WT(3) = WT(1)
GO TO 6

C FOURSE POINTS IN EACH DIRECTION
C
4 PT(1) = - .8611363115
PT(2) = - .3399810435
PT(3) = - PT(2)
PT(4) = - PT(1)
WT(1) = .3478548451
WT(2) = .6521451548
WT(3) = WT(2)
WT(4) = WT(1)

C STORE INTERPOLATING FUNCTIONS AND DERIVATIVES AT EACH GAUSS POINT
C
6 DO 8 I=1,NG
   A = PT(I)
   DO 8 J=1,NG
   B = PT(J)
   CALL INTERP(A,B)
   KK = J + NG*(I-1)
   DO 8 K=1,8
   GN(K,KK) = v(<)
   DNA(K,KK) = DNDA(K)
   DNB(K,KK) = DNDB(K)
   DNAA(K,KK) = DNDAA(K)
   DNAB(K,KK) = DNDAB(K)
   DNBB(K,KK) = DNDBB(K)
C
C STORE INTERPOLATING FUNCTIONS AT GAUSS POINTS ALONG AN EDGE
C
   DO 10 J=1,NG
   A = PT(J)
   CALL INTERP(A,-1.)
   DO 10 I=1,3
10 GNB(I,J) = N(J)

C STORE DERIVATIVES AT NODES
C

DO 12 K=1,3
   A = K - 2
   CALL INTERP(A,-1.)
DO 12 I=1,8
   BDNA(I,K) = DNDA(I)
12 BDNB(I,K) = DNDB(I)
RETURN
END
SUBROUTINE INTERP(A,B)

C THIS SUBROUTINE EVALUATES THE INTERPOLATING FUNCTIONS
C AND THEIR DERIVATIVES AT (A,B).
C
COMMON//h5/n(8),DNDA(8),DNDB(8),DNDAAB(8),DNDBB(8)
REAL N
C
C THE INTERPOLATING FUNCTIONS
C
N(1)=.25*(1.-A)*(1.-B)*(-A-B-1.)
N(2)=.50*(1.-A*A)*(1.-B)
N(3)=.25*(1.+A)*(1.-B)*(A-B-1.)
N(4)=.50*(1.+A)*(1.-B*B)
N(5)=.25*(1.+A)*(1.+B)*(A+B-1.)
N(6)=.50*(1.-A*A)*(1.+B)
N(7)=.25*(1.-A)*(1.+B)*(-A+B-1.)
N(8)=.50*(1.-A)*(1.-B*B)
C
C THE DERIVATIVES WITH RESPECT TO A
C
DNDA(1)=.25*(1.-B)*(2.*A+B)
DNDA(2)=-A*(1.-B)
DNDA(3)=.25*(1.-B)*(2.*A-B)
DNDA(4)=.50*(1.-B*B)
DNDA(5)=.25*(1.+B)*(2.*A+B)
DNDA(6)=-A*(1.+B)
DNDA(7)=.25*(1.+B)*(2.*A-B)
DNDA(8)=.50*(1.-B*B)
C
C THE DERIVATIVES WITH RESPECT TO B
C
DNDB(1)=.25*(1.-A)*(2.*B+A)
DNDB(2)=-.50*(1.-A*A)
DNDB(3)=.25*(1.+A)*(2.*B-A)
DNDB(4) = -B*(1e+A)
DNDB(5) = 25*(1e+A)*(2e*B+A)
DNDB(6) = 50*(1e-A*A)
DNDB(7) = 25*(1e-A)*(2e*B-A)
DNDB(8) = -B*(1e-A)
IF (B = 0 OR -1e) RETURN

C
C SECOND DERIVATIVES WITH RESPECT TO A
C
DNDA(1) = 5*(1e - B)
DNDA(2) = B - 1e
DNDA(3) = 5*(1e + B)
DNDA(4) = 0
DNDA(5) = 5*(1e + B)
DNDA(6) = -(1e + B)
DNDA(7) = 5*(1e + B)
DNDA(8) = 0

C
C SECOND DERIVATIVES WITH RESPECT TO B
C
DNDB(1) = 5*(1e - A)
DNDB(2) = 0
DNDB(3) = 5*(1e + A)
DNDB(4) = -(1e + A)
DNDB(5) = 5*(1e + A)
DNDB(6) = 0
DNDB(7) = 5*(1e - A)
DNDB(8) = A - 1e

C
C SECOND DERIVATIVES WITH RESPECT TO A AND B
C
DNDA(1) = 25 - 5*(A + B)
DNDA(2) = A
DNDA(3) = 5*(B - A) - 25
DNDA(4) = -B
DNDAB(5) = .25 + .5*(A + B)
DNDAB(6) = -A
DNDAB(7) = .5*(A - B) - .25
DNDAB(8) = B
RETURN
END
SUBROUTINE SOLVE

C THIS SUBROUTINE FORMS THE VECTOR, IMPOSES BOUNDARY CONDITIONS, AND SOLVES THE EQUATIONS.

C COMMON COEF(100,20),CONST(100),MBAND,NEQ
C /BLK1/RE,IVAR
C /BLK3/X(100),Y(100),W(300)
C /BLK10/NTYPE(100),IB(300),NP,NEL

C FORM THE VECTOR
C DO 2 I=1,NP
    2   CONST(I) = 0,
    DO 4 I=1,NEQ
    4   CALL VECTOR(I)

C SET BOUNDARY CONDITIONS
C DO 6 I=1,NP
    IF (IB(3*I-IVAR) .EQ. 0) CALL MODIFY(I,0)
    IF (IB(3*I-IVAR) .EQ. 1) CALL MODIFY(I,1)
    6 CONTINUE

C SOLVE THE EQUATIONS
C CALL SYMSOL
RETURN
END
SUBROUTINE MATRIX(KK)

C THIS SUBROUTINE FORMS THE LAPLACIAN MATRIX FOR AN ELEMENT
C AND INTRODUCES IT INTO THE OVERALL MATRIX,
C
COMMON/BLK2/AREA, AI(16), AO
& /BLK3/XX(100), YY(100), WI(300)
& /BLK4/NODE(50, 9)
& /BLK7/WT(4), NG, NNG
& /BLK9/X(8), Y(8), U(8), V(8), P(8)
& /BLK14/DETJ, DNX(8), DNY(8), DNXX(8), DNXY(8), DNYY(8)
& /BLK16/CDE(100, 20)
DIMENSION HI(8, 8, 16), H(8, 8)

C DEFINE LOCAL VALUES
C
DO 2 I=1, 8
K = NODE(KK, I)
X(I) = XX(K)
2 Y(I) = YY(K)

C FORM THE INTEGRANDS
C
DO 4 K=1, NNG
CALL DERIV(K)
AI(K) = DETJ
DO 4 I=1, 8
DO 4 J=1, 8
4 HI(I, J, K) = (DNX(I)*DNX(J) + DNY(I)*DNY(J))*DETJ

C INTEGRATE
C
A = 0
DO 6 I=1, NG
DO 6 J=1, NG

GR# 1
GR# 2
GR# 3
GR# 4
GR# 5
GR# 6
GR# 7
GR# 8
GR# 9
GR# 10
GR# 11
GR# 12
GR# 13
GR# 14
GR# 15
GR# 16
GR# 17
GR# 18
GR# 19
GR# 20
GR# 21
GR# 22
GR# 23
GR# 24
GR# 25
GR# 26
GR# 27
GR# 28
GR# 29
GR# 30
GR# 31
GR# 32
GR# 33
GR# 34
GR# 35
6 A = A + WT(I)*WT(J)*AI(J+NG*(I-1))
    AREA = AREA + A
    DO 10 I=1,8
    DO 10 J=1,8
    SUM = 0:
    DO 8 K=1,NG
    DO 8 L=1,NG
    8 SUM = SUM + WT(_) * WT(K) * HI(I,J,L+NG*(K-1))
    10 H(I,J) = SUM

C INTRODUCE INTO OVERALL MATRIX
C
    DO 12 I=1,8
    K = NODE(KK,I)
    DO 12 J=1,8
    L = NODE(KK,J) - K + 1
    IF (L LE 0) GO TO 12
    COE(K*L) = COE(K*L) + H(I,J)
    12 CONTINUE
    RETURN
    END
SUBROUTINE VECTOR(KK)

C THIS SUBROUTINE FORMS THE VECTOR FOR AND ELEMENT
C AND INTRODUCES IT INTO THE OVERALL VECTOR,
C
COMMON COEF(100,20),CONST(100),MBAND,NEQ
 & /BLK1/RE,IVAR
 & /BLK3/XX(100),YY(100),W(300)
 & /BLK4/NODE(50,9)
 & /BLK7/WT(4),NG,NNG
 & /BLK8/GN(8,16),DNA(8,16),DNB(8,16),
 & DNAA(8,16),DNAB(8,16),DNB(8,16)
 & /BLK9/X(8),Y(8),U(8),V(8),P(8)
 & /BLK11/GNB(3,4),BDNA(8,3),BDNB(8,3)
 & /BLK12/PSI
 & /BLK13/U(8),UX(50,3),VY(50,3),UX(50,3),VY(50,3)
 & /BLK14/DETJ,DXX(8),DNY(8),DNXX(8),DNXY(8),DNYY(8)
 DIMENSION F(8,16),F(8),PX(3),PY(3),PN(3),
 & UUX(50,3),UVX(50,3),VUY(50,3),VYV(50,3)

C DEFINE LOCAL VALUES
C
DO 2 I=1,8
K = NODE(KK,I)
X(I) = XX(K)
Y(I) = YY(K)
U(I) = W(3*K-2)
V(I) = W(3*K-1)
2 P(I) = W(3*K)
C
C COMPUTE THE NON-LINEAR TERMS FROM THE LAST SOLUTION
C
IF (IVAR .NE. 2 .OR. NODE(KK,9) .EQ. 0) GO TO 6
DO 4 K=1,3
CALL NDERIV(K)
UU(XK,K) = U(K)*UX
UV(XK,K) = U(K)*VX
VU(YK,K) = V(K)*UY
4 VV(YK,K) = V(K)*VV
6 CONTINUE

C COMPUTE THE NEEDED PRESSURE NORMAL DERIVATIVES
C
IF (IVAR NE 0 OR NDDE(KK,9) EQ 0) GO TO 10
DO 8 K=1,3
 CALL NDERIV(K)
 PX(K) = DPX + UUX(KK,K) + VUX(KK,K) - U(K)*UX - V(K)*UY
 PY(K) = DPY + U(VX(KK,K) + VV(KK,K) - U(K)*VX - V(K)*VY
 PN(1) = (PX(1)*(X(2) - X(1)) + PY(1)*(Y(2) - Y(1)))/
 & SQRT((X(2) - X(1))*(X(2) - X(1)) + (Y(2) - Y(1))*(Y(2) - Y(1)))
 PN(2) = (PX(2)*(X(3) - X(1)) + PY(2)*(Y(3) - Y(1)))/
 & SQRT((X(3) - X(1))*(X(3) - X(1)) + (Y(3) - Y(1))*(Y(3) - Y(1)))
 PN(3) = (PX(3)*(X(3) - X(2)) + PY(3)*(Y(3) - Y(2)))/
 & SQRT((X(3) - X(2))*(X(3) - X(2)) + (Y(3) - Y(2))*(Y(3) - Y(2)))
8 CONTINUE
C
C FORM INTEGRANDS
C
DO 12 K=1,NNG
 CALL DERIV(K)
 CALL APPRX(K)
 DO 12 I=1,8
12 FI(I,K) = GN(I,K)*PSI*DETJ
C
C INTEGRATE
C
DO 16 I=1,8
 SUM = 0
 DO 14 K=1,NG
 DO 14 L=1,NG
14 SUM = SUM + WT(C)*WT(L)*FI(I,L+NG*(K-1))
16 F(I) = SUM

C
C IMPOSE NORMAL DERIVATIVE BOUNDARY CONDITION ON PRESSURE
C
IF ( IVAR .NE. 0 .OR. NODE(KK,9) .EQ. 0 ) GO TO 24
   DO 20 I=1,3
      DO 20 K=1,NG
         SUM = 0.
         DO 18 J=1,3
            18 SUM = SUM + P4(J)*GNB(J,K)
      20 FI(I,K) = SUM*GNB(I,K)
         DO 22 K=1,NG
            22 F(I) = F(I) + WT(K)*FI(I,K)
   24 DO 26 I=1,8
      K = NODE(KK,I)
  26 CONST(K) = CONST(K) + F(I)
      RETURN
   END
SUBROUTINE NDERIV(K)

! THIS SUBROUTINE CALCULATES THE DERIVATIVES OF THE VELOCITIES AND PRESSURE
! NEEDED FOR THE NORMAL DERIVATIVE BOUNDARY CONDITION ON THE BODY

COMMON/BLK9/X(8),Y(8),U(8),V(8),P(8)
& /BLK11/GBA(3,4),BDNA(8,3),BDNB(8,3)
& /BLK13/UX,UY,UX,UY,DPX,DPY
& /BLK14/DETJ,DNX(8),DNY(8),DNXX(8),DNXY(8),DNYY(8)

DERIVATIVES WITH RESPECT TO A AND B

XA = 0
XB = 0
YA = 0
YB = 0

DD 2 I=1,8
XA = XA + BDNA(I,K)*X(I)
XB = XB + BDNB(I,K)*X(I)
YA = YA + BDNA(I,K)*Y(I)
YB = YB + BDNB(I,K)*Y(I)

JACOBIAN DETERMINANT

DETJ = XA*YB - XB*YA
IF (DETJ *LT* 1E-7) GO TO 8

DERIVATIVES WITH RESPECT TO X AND Y

DD 4 I=1,8
DNX(I) = (BDNA(I,K)*YB - BDNB(I,K)*YA)/DETJ
DNY(I) = (BDNB(I,K)*X - BDNA(I,K)*XB)/DETJ

DERIVATIVES OF VELOCITIES AND PRESSURE
UY = 0
UX = 0
VY = 0
DPX = 0
DPY = 0

DO 6 I = 1, 8
UX = UX + DNX(I) * U(I)
UY = UY + DNY(I) * U(I)
VX = VX + DNX(I) * V(I)
VY = VY + DNY(I) * V(I)
DPX = DPX + DNX(I) * P(I)
DPY = DPY + DNY(I) * P(I)
6 RETURN

10 WRITE (5, 10) DETJ
8 FORMAT (5D9.4, 'JACOBIAN DETERMINANT =', E10.2)

STOP
END

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SUBROUTINE DERIV(K)

C THIS SUBROUTINE CALCULATES THE JACOBIAN DETERMINANT
AND THE DERIVATIVES WITH RESPECT TO X AND Y,

C COMMON/BLK1/RE,IVAR
& /BLK8/GN(8,16),DNA(8,16),DNB(8,16),
& DNAA(8,16),DNAB(8,16),DNBB(8,16)
& /BLK9/X(8),Y(8),U(8),V(8),P(8)
& /BLK14/DETJ,DNX(8),DNY(8),DNXX(8),DNXY(8),DNYY(8)
DIMENSION A(3,3),AI(3,3),C(3,8)

C CALCULATE FACTORS

XA = 0.
XB = 0.
YA = 0.
YB = 0.
DO 2 I=1,8
XA = XA + DNA(I,K)*X(I)
XB = XB + DNB(I,K)*X(I)
YA = YA + DNA(I,K)*Y(I)
2 YB = YB + DNB(I,K)*Y(I)

C DETERMINANT OF THE JACOBIAN

DETJ = XA*YB - XB*YA
IF (DETJ .LT. 1.E-7) GO TO 12

C FIRST DERIVATIVES

DO 4 I=1,8
DNX(I) = (DNA(I,K)*YB - DNB(I,K)*YA)/DETJ
4 DNY(I) = (DNB(I,K)*XA - DNA(I,K)*XB)/DETJ
IF (IVAR .NE. 0) RETURN

200
C  CALCULATE FACTORS
   C
   XAA = 0.0
   XAB = 0.0
   XBB = 0.0
   YAA = 0.0
   YAB = 0.0
   YBB = 0.0
   DO 6 I=1,S
   XAA = XAA + DNAA(I,K)*X(I)
   XAB = XAB + DNAB(I,K)*X(I)
   XBB = XBB + DNBB(I,K)*X(I)
   YAA = YAA + DNAA(I,K)*Y(I)
   YAB = YAB + DNAB(I,K)*Y(I)
   6 YBB = YBB + DNBB(I,K)*Y(I)
C  TRANSFORMATION MATRIX AND VECTOR
   C
   A(1,1) = XA*XA
   A(1,2) = 2.0*XAYA
   A(1,3) = YAYA
   A(2,1) = XAXB
   A(2,2) = XAYB + XBYA
   A(2,3) = YAYB
   A(3,1) = XBYB
   A(3,2) = 2.0*XBYB
   A(3,3) = YBYB
   DET = A(1,1)*(A(2,2)*A(3,3) - A(2,3)*A(3,2)) - A(1,2)*(A(2,1)*A(3,3) - A(2,3)*A(3,1)) + A(1,3)*(A(2,1)*A(3,2) - A(2,2)*A(3,1))
   E - A(1,2)*(A(2,1)*A(3,3) - A(2,3)*A(3,1))/DET
   &  + A(1,3)*(A(2,1)*A(3,2) - A(2,2)*A(3,1))/DET
   AI(1,1) = (A(2,2)*A(3,3) - A(2,3)*A(3,2))/DET
   AI(1,2) = (A(1,3)*A(3,2) - A(1,2)*A(3,3))/DET
   AI(1,3) = (A(1,2)*A(3,2) - A(1,3)*A(2,2))/DET
   AI(2,1) = (A(2,3)*A(3,1) - A(2,1)*A(3,3))/DET
   GR#  36
   GR#  37
   GR#  38
   GR#  39
   GR#  40
   GR#  41
   GR#  42
   GR#  43
   GR#  44
   GR#  45
   GR#  46
   GR#  47
   GR#  48
   GR#  49
   GR#  50
   GR#  51
   GR#  52
   GR#  53
   GR#  54
   GR#  55
   GR#  56
   GR#  57
   GR#  58
   GR#  59
   GR#  60
   GR#  61
   GR#  62
   GR#  63
   GR#  64
   GR#  65
   GR#  66
   GR#  67
   GR#  68
   GR#  69
   GR#  70
\[
\begin{align*}
AI(2,2) &= (A(1,1)A(3,3) - A(1,3)A(3,1))/\text{DET} \\
AI(2,3) &= (A(1,3)A(2,1) - A(1,1)A(2,3))/\text{DET} \\
AI(3,1) &= (A(2,1)A(3,2) - A(2,2)A(3,1))/\text{DET} \\
AI(3,2) &= (A(1,2)A(3,1) - A(1,1)A(3,2))/\text{DET} \\
AI(3,3) &= (A(1,1)A(2,2) - A(1,2)A(2,1))/\text{DET} \\
\text{DO 8 I=1,8} \\
C(1,I) &= \text{DNAA(I,K)} - XAA*\text{DNX(I)} - YAA*\text{DNY(I)} \\
C(2,I) &= \text{DNAB(I,K)} - XAB*\text{DNX(I)} - YAB*\text{DNY(I)} \\
8 \quad C(3,I) &= \text{DNBB(I,K)} - XBB*\text{DNX(I)} - YBB*\text{DNY(I)}
\end{align*}
\]
SUBROUTINE APPROX(K)

C THIS SUBROUTINE CALCULATES THE APPROXIMATING FUNCTION, PSI,
C
COMMON/BLK1/RE,IVAR
C /BLK8/GN(8,16),DNA(8,16),DNB(8,16),
C /BLK9/DA(8,16),DNAB(8,16),DNBB(8,16)
C /BLK10/X(8),Y(8),U(8),V(8),P(8)
C /BLK12/PSI
C /BLK14/DETJ,DXX(8),DNY(8),DNXX(8),DNYY(8)

C VELOCITIES

UN = 0.
VN = 0.
DO 2 I=1,8
UN = UN + GN(I,K)*U(I)
2 VN = VN + GN(I,K)*V(I)
GO TO (6,10),IVAR

C EQUATION

UX = 0.
UY = 0.
VX = 0.
VY = 0.
UXX = 0.
UXY = 0.
VXY = 0.
VYY = 0.
DO 4 I=1,8
UX = UX + DNX(I)*U(I)
UY = UY + DNY(I)*U(I)
VX = VX + DNX(I)*V(I)
VY = VY + DNY(I)*V(I)
UXX = UXX + D\text{\(X\)}X(I)\text{*U}(I)
UXY = UXY + D\text{\(X\)}Y(I)\text{*U}(I)
VXY = VXY + D\text{\(X\)}Y(I)\text{*V}(I)
4 VYY = VYY + D\text{\(X\)}Y(I)\text{*V}(I)
PSI = UX*UX + VY*VY + UN*UXX + VN*VYY + 2*Uy*VX + VN*Uxy + UN*Vxy
RETURN
C
C V EQUATION
C
6 PY = 0
VX = 0
VY = 0
DO 8 I=1,8
PY = PY + D\text{\(X\)}Y(I)\text{*P}(I)
VX = VX + D\text{\(X\)}Y(I)\text{*V}(I)
8 VY = VY + D\text{\(X\)}Y(I)\text{*V}(I)
PSI = - RE*(PY + UN*VX + VN*VY)
RETURN
C
C U EQUATION
C
10 PX = 0
UX = 0
UY = 0
DO 12 I=1,8
PX = PX + D\text{\(X\)}X(I)\text{*P}(I)
UX = UX + D\text{\(X\)}X(I)\text{*U}(I)
12 UY = UY + D\text{\(X\)}Y(I)\text{*U}(I)
PSI = - RE*(PX + UN*UX + VN*UY)
RETURN
END
SUBROUTINE MODIFY(ROW, VALUE)

C THIS SUBROUTINE MAKES CONST(ROW) = VALUE.

C COMMON COEF(100, 20), CONST(100), MBAND, NEQ
INTEGER ROW

C MODIFY COLUMN
C
DO 4 J = 2, MBAND
   I = ROW - J + 1
   IF (I LE 0) GO TO 2
   CONST(I) = CONST(I) - COEF(I, J) * VALUE
   COEF(I, J) = 0

C MODIFY ROW
C
   2 I = ROW + J - 1
   IF (I GT NEQ) GO TO 4
   CONST(I) = CONST(I) - COEF(ROW, J) * VALUE
   COEF(ROW, J) = 0

4 CONTINUE

C FIX UNKNOWN AND NORMALIZE DIAGONAL
C
CONST(ROW) = VALUE
COEF(ROW, 1) = 1.
RETURN
END
SUBROUTINE SYMSDL
C
C THIS SUBROUTINE SOLVES A SYSTEM OF EQUATIONS
C STORED IN A Banded, SYMMETRIC MATRIX.
C
COMMON COEF(100,20),CONST(100),MBAND,NEQ
C
C REDUCE MATRIX
C
DO 6 N=1,NEQ
DO 4 L=2,MBAND
C = COEF(N,L)/COEF(N,1)
I = N + L - 1
IF (I .GE. NEQ) GO TO 4
J = 0
DO 2 K=L,MBAND
J = J+1
2 COEF(I,J) = COEF(I,J) - C*COEF(N,K)
4 COEF(N,L) = C
6 CONTINUE
C
C REDUCE VECTOR
C
DO 10 N=1,NEQ
DO 8 L=2,MBAND
I = N + L - 1
IF (I .GE. NEQ) GO TO 10
8 CONST(I) = CONST(I) - COEF(N,L)*CONST(N)
10 CONST(N) = CONST(N)/COEF(N,1)
C
C BACK SUBSTITUTION
C
DO 12 I=2,NEQ
N = NEQ - I + 1
DO 12 K=2,MBAND
L = N + K - 1
IF (L .GE. NEQ) GO TO 12
CONST(N) = CONST(N) - COEF(N,K)*CONST(L)
12 CONTINUE
RETURN
END
This program solves Poisson's equation by the finite element method, using eight node quadrilaterals and three node triangles.

COMMON COEF(21,22),CONST(21),NEQ

& /BLK1/PSI,11
& /BLK2/NODE(24,8)
& /BLK3/X(21),Y(21)
& /BLK4/AREA,A1(16)
& /BLK11/VAL2(21)

DIMENSION AN(21),SDL1(21),SDL2(21),SDL3(21),VAL1(21)

NEQ = 21
PSI = - .01
ITMAX = 100
EPS = 1e-E-6

X AND Y COORDINATES

DO 2 I=1,5
X(I) = 5*(I-1)
Y(I) = 0e
X(I+8) = 5*(I-1)
Y(I+8) = 10e
X(I+16) = 5*(I-1)
2 Y(I+16) = 20e

DO 4 I=6,8
X(I) = 10*(I-6)
Y(I) = 5e
X(I+8) = 10*(I-6)
4 Y(I+8) = 15e

ANALYTICAL SOLUTION

DO 8 I=1,NEQ
SUMO = 0e
SUM = 0
DO 6 J=1,20
M = J - 1
A = (2*M + 1)*3.141592
A1 = 8*M/A**3
A2 = A*Y(I)/40*
A3 = A/2*
A4 = A*X(I)/40*
SUM = SUM + ((-1)**M) * A1 * (COSH(A2)/COSH(A3)) * COS(A4)
IF (ABS(SUM-SUM0) .LT. 1.E-9) GO TO 8
6 SUM0 = SUM
8 AN(I) = 2*(1.0 - (X(I)/20.0)**2) - 8.0*SUM

C SOLUTION BY EIGHT NODE QUADRILATERALS
C
NEL = 4
READ (5,10) (I,(NODE(I,J),J=1,8),K=1,NEL)
10 FORMAT (915)
CALL ONCE(2)
II = 0
12 CONTINUE
DO 14 I=1,NEQ
CONST(I) = 0.
DO 14 J=1,NEQ
14 COEF(I,J) = 0.
AREA = C *
DO 16 I=1,NEL
16 CALL MATRIX(I)
A1 = AREA
DO 18 I=17,21
18 CALL MODIFY(I,0.)
CALL MODIFY(5.0.)
CALL MODIFY(8.0.)
CALL MODIFY(13.0.)
CALL MODIFY(16.0.)
IF (II .EQ. 0) CALL CHECK
DO 19 I=1,NEQ
19 VAL1(I) = VAL2(I)
   CALL SOLEL
   IF (II .NE. 0) GO TO 22
   DO 20 I=1,NEQ
20 SOL1(I) = CONST(I)
   III = 1
   GO TO 12
22 CONTINUE
   DO 24 I=1,NEQ
24 SOL2(I) = CONST(I)
C
C SOLUTION BY LINEAR TRIANGLES
C
NEL = 24
II = 0
READ (5,26) (I, (NODE(I,J),J=1,3),K=1,NEL)
26 FORMAT (4I5)
DO 30 I=1,NEQ
   CONST(I) = 0
30 COEF(I,J) = 0
   AREA = 0
   DO 32 I=1,NEL
32 CALL MATR(I)
   A2 = AREA
   DO 34 I=17,21
34 CALL MODIFY(I,A2)
   CALL MODIFY(5,A2)
   CALL MODIFY(8,A2)
   CALL MODIFY(13,A2)
   CALL MODIFY(16,A2)
   IF (II .EQ. 1) GO TO 38
   II = 1
CALL SOLEX
DO 36 I=1,NEQ
36 SOL3(I) = CONST(I)
    GO TO 28
38 CONTINUE
    CALL CHECK
    DO 40 I=1,NEQ
    COEF(I,NEQ+1) = CONST(I)
40 CONST(I) = 1.
    CALL SOLIT(ITMAX,EPS)

C
C
C RESULTS
C
WRITE (6,42) A1,A2
42 FORMAT (1H1,'/',5S,'COMPUTED DOMAIN AREAS','//50X,'QUADRILATERAL',GRW 121
   & 10X,'TRIANGLE'/'41X,'2F20.5,4(/,5X,'S O L U T I O N'/'
   & 5X,'QUADRILATERALS'/'21X,'TRIANGLES'/'45X,28(''-''),5X,28(''-'')/
   & 45X,'EIGEN'/'27X,'EIGEN'/'25X,'NODE'/'2X,'ANALYTICAL'/'4X,
   & 'VALUES'/'4X,'PSI=C,'3X,'PSI=F(X,Y)'/'5X,'VALUES'/'1X,
   & 'ELIMINATION'/'1X,'ITERATION'/)
WRITE (6,44) (I,AN(I),VAL1(I),SOL1(I),SOL2(I),GRW 127
   & VAL2(I),SOL3(I),CONST(I),I=1,NEQ)
44 FORMAT (26X,I2,F11.5,F11.2,2F11.5,F11.2,2F11.5)
STOP
END
SUBROUTINE ONCE(NGP)
C THIS SUBROUTINE EVALUATES THE INTERPOLATING FUNCTIONS
C AND THEIR DERIVATIVES AT THE GAUSS POINTS.
C
COMMON/BLK7/GN(8,16),WT(4),NG,NNG
& /BLK8/DNA(8,16),DNB(8,16),DNAA(8,16),DNAB(8,16),DNBB(8,16)
& /BLK9/N(8),DNDA(8),DNDB(8),NDAA(8),NDAB(8),NDABB(8)
REAL NP,N,PT(4)
NG = NGP
NNG = NG*NG
C GAUSS POINTS AND WEIGHTING COEFFICIENTS
C
IF (NG .EQ. 3) GO TO 2
IF (NG .EQ. 4) GO TO 4
C
C TWO POINTS IN EACH DIRECTION
C
PT(1) = - .5773502691
PT(2) = - PT(1)
WT(1) = 1.
WT(2) = 1.
GO TO 6
C
C THREE POINTS IN EACH DIRECTION
C
2 PT(1) = -.7745966692
PT(2) = .0.
PT(3) = - PT(1)
WT(1) = .5555555556
WT(2) = .8888888889
WT(3) = WT(1)
GO TO 6
C
C FOUR POINTS IN EACH DIRECTION

4 PT(1) = -1.8611363115
 PT(2) = -0.3399810435
 PT(3) = -PT(2)
 PT(4) = -PT(1)
 WT(1) = 0.3478548451
 WT(2) = 0.6521451548
 WT(3) = WT(2)
 WT(4) = WT(1)

C STORE INTERPOLATING FUNCTIONS AND DERIVATIVES AT EACH GAUSS POINT

6 DO 8 I=1,NG
   A = PT(I)
   DO 8 J=I,NG
   B = PT(J)
   CALL INTERP(A,B)
   KK = J + NG*(I-1)
   DO 8 K=1,8
   GN(K,KK) = N(K)
   DNA(K,KK) = DNDA(K)
   DNB(K,KK) = DNDB(K)
   DNAA(K,KK) = DNDA(K)
   DNAB(K,KK) = DNAB(K)
   8 DNBB(K,KK) = DNDBB(K)
RETURN
END
SUBROUTINE INTERP(A, B)

C THIS SUBROUTINE EVALUATES THE INTERPOLATING FUNCTIONS
   AND THEIR DERIVATIVES AT (A, B).

C COMMON/BLK9/N(8), DNDAB(8), DNDAA(8), DNDDB(8)
   REAL N

C THE INTERPOLATING FUNCTIONS

C
N(1) = \cdot 25 \cdot (1 \cdot A \cdot (1 \cdot B \cdot (-A-B-1))
N(2) = \cdot 50 \cdot (1 \cdot A \cdot A \cdot (1 \cdot -B))
N(3) = \cdot 25 \cdot (1 \cdot A \cdot (1 \cdot B \cdot (A-B-1))
N(4) = \cdot 50 \cdot (1 \cdot A \cdot A \cdot (-B*B))
N(5) = \cdot 25 \cdot (1 \cdot A \cdot (1 \cdot B \cdot (A+B-1))
N(6) = \cdot 50 \cdot (1 \cdot A \cdot A \cdot (1 \cdot +B))
N(7) = \cdot 25 \cdot (1 \cdot A \cdot (1 \cdot B \cdot (-A+B-1))
N(8) = \cdot 50 \cdot (1 \cdot A \cdot (1 \cdot B*B))

C THE DERIVATIVES WITH RESPECT TO A

C
DNDAB(1) = \cdot 25 \cdot (1 \cdot B \cdot (2 \cdot A+B))
DNDAA(2) = \cdot A \cdot (1 \cdot -B)
DNDAB(3) = \cdot 25 \cdot (1 \cdot B \cdot (2 \cdot A+B))
DNDAA(4) = \cdot 50 \cdot (1 \cdot B*B)
DNDAB(5) = \cdot 25 \cdot (1 \cdot B \cdot (2 \cdot A+B))
DNDAA(6) = \cdot A \cdot (1 \cdot +B)
DNDAB(7) = \cdot 25 \cdot (1 \cdot B \cdot (2 \cdot A-B))
DNDAA(8) = \cdot 50 \cdot (1 \cdot B*B)

C THE DERIVATIVES WITH RESPECT TO B

C
DNDDB(1) = \cdot 25 \cdot (1 \cdot B \cdot (2 \cdot B+A))
DNDDB(2) = \cdot 50 \cdot (1 \cdot B \cdot A)
DNDDB(3) = \cdot 25 \cdot (1 \cdot A \cdot (2 \cdot B-A))
SECOND DERIVATIVES WITH RESPECT TO A

| DNDB(4) | = -B*(1e + A) |
| DNDB(5) | = .25*(1e + A)*(2e*B + A) |
| DNDB(6) | = 50*(1e - A*A) |
| DNDB(7) | = .25*(1e - A)*(2e*B - A) |
| DNDB(8) | = -B*(1e - A) |

SECOND DERIVATIVES WITH RESPECT TO B

| DNDB(1) | = .5*(1e - B) |
| DNDB(2) | = B - 1e |
| DNDB(3) | = .5*(1e - B) |
| DNDB(4) | = 0e |
| DNDB(5) | = .5*(1e + B) |
| DNDB(6) | = -(1e + B) |
| DNDB(7) | = .5*(1e + B) |
| DNDB(8) | = 0e |

SECOND DERIVATIVES WITH RESPECT TO A AND B

| DNDB(1) | = .25 - .5*(A + B) |
| DNDB(2) | = A |
| DNDB(3) | = .5*(B - A) - .25 |
| DNDB(4) | = -B |
| DNDB(5) | = .25 + .5*(A + B) |
DNDAB(6) = A
DNDAB(7) = 5*(A - B) - .25
DNDAB(8) = B
RETURN
END
SUBROUTINE MATRIX(KK)

C THIS SUBROUTINE FORMS THE MATRIX AND VECTOR FOR AN EIGHT NODE QUADRIL
C AND INTRODUCES THEM INTO THE OVERALL MATRIX AND VECTOR.

COMMON COEFF(21,22),CONST(21),NEQ
& /BLK1/CC. II
& /BLK2/NODE(24,8)
& /BLK3/XX(21),YY(21)
& /BLK4/AREA,II(16)
& /BLK5/X(8),Y(8)
& /BLK6/DEJT,DXN(8),DNY(8),DNXX(8),DNXY(8),DNYY(8)
& /BLK7/GN(8,16),WT(4),NG,NNG

DIMENSION HI(8,8,16),H(8,8),F(8,16),F(8)

C DEFINE LOCAL COORDINATES
C
DO 2 I=1,8
  K = NODE(KK,I)
  X(I) = XX(K)
  Y(I) = YY(K)
C
C COMPUTE INTEGRANDS
C
DO 10 K=1,NNG
  CALL DERIV(K)
  AI(K) = DEJT
  IF (II .EQ. 0) GO TO 6
  PSI = 0
  DO 4 I=1,8
    4 PSI = PSI + DNXX(I)*CC*X(I)*X(I)/6. + DNYY(I)*CC*Y(I)*Y(I)/6.
       & + DNXY(I)*CC*X(I)*Y(I)/3.
  GO TO 8
  6 PSI = CC
  8 CONTINUE
DO 10 I=1,8
FI(I,K) = -GN(I,K)*PSI*DETJ
DO 10 J=1,8
10 HI(I,J,K) = (DNX(I)*DNX(J) + DNY(J)*DNY(I))*DETJ

C INTEGRATE
C
A = 0  
DO 12 I=1,NG
DO 12 J=1,NG
12 A = A + WT(I)*WT(J)*AI(J+NG*(I-1))
AREA = AREA + A
DO 18 I=1,8
SUM = 0  
DO 14 K=1,NG
DO 14 L=1,NG
14 SUM = SUM + WT(K)*WT(L)*FI(I,L+NG*(K-1))
F(I) = SUM
DO 18 J=1,8
SUM = 0  
DO 16 K=1,NG
DO 16 L=1,NG
16 SUM = SUM + WT(L)*WT(K)*HI(I,J,L+NG*(K-1))
18 HI(I,J) = SUM

C INTRODUCE INTO OVERALL MATRIX AND VECTOR
C
DO 20 I=1,8
K = NODE(KK,I)
CONST(K) = CONST(K) + F(I)
DO 20 J=1,8
L = NODE(KK,J)
20 COEF(K,L) = COEF(K,L) + HI(I,J)
RETURN
END
SUBROUTINE DERIV(K)
C THIS SUBROUTINE CALCULATES THE JACOBIAN DETERMINANT AND
C THE DERIVATIVES OF THE SHAPE FUNCTIONS WITH RESPECT TO X AND Y.
C COMMON/BLK5/X(8),Y(8)
& /BLK6/DETJ,DNX(8),DNY(8),DNXX(8),DNXY(8),DNYY(8)
& /BLK8/DNA(8,16),DNB(8,16),DNAA(8,16),DNAB(8,16),DNBB(8,16)
DIMENSION A(3,3),AI(3,3),C(3,8)
C CALCULATE FACTORS
C
XA = 0.
XB = 0.
YA = 0.
YB = 0.
DO 2 I=1,8
XA = XA + DNA(I,K)*X(I)
XB = XB + DNB(I,K)*X(I)
YA = YA + DNA(I,K)*Y(I)
2 YB = YB + DNB(I,K)*Y(I)
2
C DETERMINANT OF THE JACOBIAN
C
DETJ = XA*YB - XB*YA
IF (DETJ .LT. 1.E-7) GO TO 12
C FIRST DERIVATIVES
C
DO 4 I=1,8
DNX(I) = (DNA(I,K)*YB - DNB(I,K)*YA)/DETJ
4 DNY(I) = (DNB(I,K)*XA - DNA(I,K)*XB)/DETJ
C CALCULATE FACTORS
C
XAA = 0.0
XAB = 0.0
XBB = 0.0
YAA = 0.0
YAB = 0.0
YBB = 0.0
DO 6 I=1,8
XAA = XAA + DNA(X)X(I)
XAB = XAB + DNA(B)X(I)
XBB = XBB + DNA(A)X(I)
YAA = YAA + DNA(A)Y(I)
YAB = YAB + DNA(B)Y(I)
6 YBB = YBB + DNA(B)Y(I)

C TRANSFORMATION MATRIX AND VECTOR

C

A(1,1) = XA*XA
A(1,2) = 2*XA*YA
A(1,3) = YA*YA
A(2,1) = XA*XB
A(2,2) = XA*YB + XB*YA
A(2,3) = YA*YB
A(3,1) = XB*XB
A(3,2) = 2*XB*YB
A(3,3) = YB*YB

DET = A(1,1)*(A(2,2)*A(3,3) - A(2,3)*A(3,2)) - A(1,2)*A(2,1)*A(3,3) + A(1,3)*A(2,2)*A(3,1)

C
\[ A(3,2) = (A(1,2)A(3,1) - A(1,1)A(3,2))/\text{DET} \]
\[ A(3,3) = (A(1,1)A(2,2) - A(1,2)A(2,1))/\text{DET} \]

DO 8 I=1,8
C(1,1) = DNAA(I,K) - XAA*DNX(I) - YAA*DNY(I)
C(2,1) = DNAB(I,K) - XAB*DNX(I) - YAB*DNY(I)
8 C(3,1) = DNB(I,K) - XBB*DNX(I) - YBB*DNY(I)

C
SECOND DERIVATIVES

DO 10 I=1,8
DNXX(I) = AI(1,1)*C(1,1) + AI(1,2)*C(2,1) + AI(1,3)*C(3,1)
DNXY(I) = AI(1,2)*C(1,1) + AI(2,2)*C(2,1) + AI(2,3)*C(3,1)
10 DNYY(I) = AI(3,1)*C(1,1) + AI(3,2)*C(2,1) + AI(3,3)*C(3,1)
RETURN
12 WRITE (6,14) DETJ
14 FORMAT (///5D14,*JACOBIAN DETERMINANT =*E10*2)
STOP
END
SUBROUTINE MATR(KK)

C THIS SUBROUTINE FORMS THE MATRIX AND VECTOR FOR A THREE NODE TRIANGLE

C AND INTRODUCES THEM INTO THE OVERALL MATRIX AND VECTOR.

C COMMON COEF(21,22),CONST(21),NEQ

C /BLK1/PSI,II

C /BLK2/NODE(24,8)

C /BLK3/XX(21),YY(21)

C /BLK4/AR,AL(16)

DIMENSION F(3),H(3),A(3),B(3),C(3),X(3),Y(3)

C DEFINE LOCAL COORDINATES

C DO 2 I=1,3

K = NODE(KK,I)

X(I) = XX(K)

Y(I) = YY(K)

C CALCULATE FACTORS

A(1) = X(2)*Y(3) - X(3)*Y(2)

A(2) = X(3)*Y(1) - X(1)*Y(3)

A(3) = X(1)*Y(2) - X(2)*Y(1)

B(1) = Y(2) - Y(3)

B(2) = Y(3) - Y(1)

B(3) = Y(1) - Y(2)

C(1) = X(3) - X(2)

C(2) = X(1) - X(3)

C(3) = X(2) - X(1)

AREA = (A(1) + A(2) + A(3))/2.

AR = AR + AREA

T = 4.*AREA

C LOCATE THE CENTROID
C
XBAR = (X(1) + X(2) + X(3))/3
YBAR = (Y(1) + Y(2) + Y(3))/3
C
FORM THE MATRIX AND VECTOR
C
DO 4 I=1,3
   F(I) = -(A(I) + B(I)*XBAR + C(I)*YBAR)*PSI/2
DO 4 J=1,3
   H(I,J) = (B(I)*B(J) + C(I)*C(J))/T
C
INTRODUCE INTO OVERALL MATRIX AND VECTOR
C
DO 6 I=1,3
   K = NODE(KK,I)
   CONST(K) = CONST(K) + F(I)
DO 6 J=1,3
   L = NODE(KK,J)
   6 COEF(K,L) = COEF(K,L) + H(I,J)
RETURN
END
SUBROUTINE MODIFY(ROW, VALUE)

C THIS SUBROUTINE SETS CONST(ROW) = VALUE.

C COMMON COEF(21,22),CONST(21),NEQ
INTEGER ROW

C ZERO ROW AND COLUMN AND MODIFY VECTOR

C DO 2 I=1,NEQ
C CONST(I) = CONST(I) - COEF(I,ROW)*VALUE
C COEF(I,ROW) = 0.
C 2 COEF(ROW,I) = 0.

C SET REQUIRED VALUE AND-normalize diagonal

C CONST(ROW) = VALUE
C COEF(ROW,ROW) = 1.
RETURN
END
SUBROUTINE SOLVE
C
C THIS SUBROUTINE SOLVES A SYSTEM OF EQUATIONS
C BY GAUSS ELIMINATION.
C
COMMON A(21,22),B(21),NEQ
REAL*8 SUM,DOUBLE
L = NEQ-1
C
C FIND THE LARGEST PIVOT ELEMENT
C
DO 14 K=1,L
JJ = K
BIG = ABS(A(K,K))
K1 = K + 1
DO 2 I=K1,NEQ
AB = ABS(A(I,K))
IF (BIG .GE. AB) GO TO 2
BIG = AB
JJ = I
2 CONTINUE
C
C ROW INTERCHANGE
C
IF (JJ .EQ. K) GO TO 6
DO 4 J=K,NEQ
TEMP = A(JJ,J)
A(JJ,J) = A(K,J)
4 A(K,J) = TEMP
TEMP = B(JJ)
B(JJ) = B(K)
B(K) = TEMP
6 CONTINUE
IF (ABS(A(K,K)) .GT. 1.E-07) GO TO 10
WRITE (6,8) (I,K,A(I,K),I=1,NEQ)

1 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
2 WRITE (6,8) (I,K,A(I,K),I=1,NEQ)
3 WRITE (6,9) (I,100,5,2,3)
4 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
5 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
6 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
7 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
8 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
9 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
10 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
11 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
12 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
13 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
14 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
15 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
16 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
17 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
18 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
19 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
20 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
21 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
22 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
23 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
24 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
25 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
26 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
27 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
28 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
29 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
30 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
31 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
32 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
33 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
34 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
35 WRITE (6,9) (I,K,A(I,K),I=1,NEQ)
8 FORMAT (40X, 'LARGEST PIVOT NUMBER IS A(*,I2,,*,I2,*) = 'E10.2)  
10 CONTINUE

C REDUCE MATRIX
C
DO 12 I=K1,NEQ
   RATIO = A(I,K)/A(K,K)
   B(I) = B(I) - RATIO*B(K)
12  DO 14 J=K1,NEQ
14  A(I,J) = A(I,J) - RATIO*A(K,J)
   DO 14 I=K1,NEQ
   A(I,K) = 0.

C BACK SUBSTITUTION
C
B(NEQ) = B(NEQ)/A(NEQ,NEQ)
DO 18 M=1,L
   SUM = 0.
   I = NEQ-M
   I1 = I + 1
   DO 16 J=I,NEQ
16   SUM = SUM + DBLE(A(I,J))*DBLE(B(J))
18   B(I) = (B(I)-SUM)/A(I,I)
RETURN
END
SUBROUTINE SOLIT(ITMAX,EPS)
C
C THIS SUBROUTINE SOLVES A SYSTEM OF EQUATIONS
C BY GAUSS-SEIDEL ITERATION.
C
C COMMON A(21,22),X(21),NEQ
C
C NORMALIZE WITH RESPECT TO THE DIAGONAL
C
N1 = NEQ + 1
DO 2 I=1,NEQ
AA = A(I,I)
DO 2 J=1,N1
2 A(I,J) = A(I,J)/AA
C
C ITERATE ON THE VECTOR
C
DO 8 IT = 1,ITMAX
II = 1
DO 6 I=1,NEQ
XX = X(I)
X(I) = A(I,N1)
DO 4 J=1,NEQ
4 IF (I NEQ J) GO TO 4
X(I) = X(I) - A(I,J)*X(J)
8 CONTINUE
C
C CHECK FOR CONVERGENCE
C
IF (ABS(XX-X(I)) LE EPS) GO TO 6
II = 0
6 CONTINUE
IF (II NEQ 1) GO TO 10
8 CONTINUE
10 CONTINUE
SUBROUTINE CHECK

This subroutine forms the amplification matrix and computes its eigenvalues.

COMMON COEF(21,22),CONST(21),NEQ
& /BLK10/DIC(21,21)
DIMENSION C(21,21),D(21)

SEPARATE MATRIX INTO A DIAGONAL MATRIX PLUS ANOTHER MATRIX

DO 2 I=1,NEQ
   D(I) = 1./COEF(I,I)
   DO 2 J=1,NEQ
      2 C(I,J) = COEF(I,J)
   DO 4 I=1,NEQ
      4 C(I,I) = 0.

FORM AMPLIFICATION MATRIX

DO 6 I=1,NEQ
   DO 6 J=1,NEQ
      6 DiC(I,J) = D(I)*C(I,J)

COMPUTE THE EIGENVALUES OF DIC

CALL WHOLD(NEQ)
CALL EVAL(NEQ)
RETURN
END
SUBROUTINE HHOLD(N)
C
C THIS SUBROUTINE REDUCES A SYMMETRIC MATRIX TO
C TRIDIAGONAL FORM WITH A HOUSEHOLDER TRANSFORMATION,
C
COMMON/BLK10/A(21,21)
& /BLK12/AA(21),D(21)
DIMENSION Q(21)
REAL*8 DABS,DSIGN
REAL*8 QQ,S,DBLE
N2 = N-2
DO 10 K=1,N2
C
C FORM SUBDIAGONAL VECTOR D, CALCULATE CONSTANT 1/(2*B*B),
C AND STORE U VECTOR IN A(I,K)
C
SQ = 0
K1 = K+1
DO 2 I=K1,N
T = A(I,K)
2 SQ = SQ + T*T
D(K+1) = -SIGN(SQRT(SQ),A(K1,K))
CON = SQ + ABS(A(K1,K))*SQRT(SQ)
CONI = 1.
IF (ABS(CON) .GT. 1.E-9) CONI = 1./CON
A(K1,K) = A(K1,K) - D(K+1)
C
C FORM Q VECTOR, AND SCALAR S
C
S = 0
DO 6 I=K1,N
QQ = 0
DO 4 J=K1,N
DUM = A(I,J)
4 IF (J .GT. I) DUM = A(J,I)
6 S = S + DUM*

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4 QQ = QQ + DBLE(DUM)*DBLE(A(J,K))
Q(I) = QQ*CONI
6 S = S + DBLE(Q(I))*DBLE(A(I,K))
S = S*CONI
  IF (DABS(S) .LT. 1.E-20 .AND. S .NE. 0.) S = DSIGN(1.D-20, S)
C REDUCE MATRIX
C
DO 8 I=KL,N
  DO 8 J=KL,1
  8 A(I,J) = A(I,J) - A(I,K)*Q(J) - A(J,K)*Q(I) + S*A(I,K)*A(J,K)
C 10 CONTINUE
C FORM DIAGONAL VECTOR AA
C
  D(1) = 0
  D(N) = A(N*N-1)
  DO 12 I=1,N
  12 AA(I) = A(I,I)
RETURN
END
SUBROUTINE EVAL(N)

THIS SUBROUTINE COMPUTES THE EIGENVALUES OF A TRIDIAGONAL MATRIX BY THE BISECTION TECHNIQUE.

COMMON/BLK12/A(21),B(21)
& /BLK11/VALUE(21)
DIMENSION BS(21),H(21),L(21)
REAL MAX,MIN,LOW

DEFINE INTERVAL CONTAINING EIGENVALUES

H(I) = A(I) + ABS(B(2))
H(N) = A(N) + ABS(B(N))
L(I) = A(I) - ABS(B(2))
L(N) = A(N) - ABS(B(N))
M = N-1
DO 2 I=2,M
H(I) = A(I) + ABS(B(I)) + ABS(B(I+1))
2 L(I) = A(I) - ABS(B(I)) - ABS(B(I+1))
HI = H(I)
LOW = L(I)
DO 4 I=2,N
IF (H(I) .GE. HI) HI = H(I)
4 IF (L(I) .LT. LOW) LOW = L(I)
DO 6 I=1,N
6 BS(I) = B(I)*B(I)
EPS = ABS(A(1))
DO 8 I=2,N
8 EPS = EPS + ABS(A(I)) + ABS(B(I))
EPS = EPS*16**(-6)

BISECT INTERVAL TO FIND EIGENVALUES

DO 18 I=1,N
MAX = HI
MIN = LOW
NIT = 0
DIF = ABS(MAX-MIN)

10 X = .5*(MAX+MIN)
NIT = NIT + 1
IF (NIT .GE. 30) GO TO 18
K = 0
Q = 1.
DO 12 J=1,N
   IF (ABS(Q) .LT. EPS) Q = SIGN(EPS*Q)
   Q = A(J) - X - BS(J)/Q
12 IF (Q .LT. 0.) K = K+1
   IF (K .GE. 1) GO TO 14
MIN = X
GO TO 16
14 MAX = X
   DIF = ABS(MAX-MIN)
16 IF (DIF .GT. EPS) GO TO 10
18 VALUE(I) = MAX
RETURN
END
VITA

Born October 21, 1946, Gary Richard Wooley lived in the New Orleans area until entering college. After being graduated from East Jefferson High School in May, 1964, he enrolled at McNeese State College in Lake Charles, Louisiana, in September, 1964, and transferred to Louisiana State University in Baton Rouge in January, 1965. In January, 1969, he received the Bachelor of Science degree in Mechanical Engineering from Louisiana State University, after which he entered graduate school in the Department of Engineering Science as a research assistant. He received the Master of Science degree in Engineering Science from Louisiana State University in May, 1970. While pursuing an education he worked four summers for various companies in the oil industry, was a National Science Foundation Trainee for three years, and was a graduate teaching assistant for three and a half years. He is presently a Louisiana State University Dissertation Year Fellow and a candidate for the degree of Doctor of Philosophy in the Department of Engineering Science.
EXAMINATION AND THESIS REPORT

Candidate: Gary Richard Wooley

Major Field: Engineering Mechanics

Title of Thesis: A Finite Element Analysis of Steady, Two Dimensional, Incompressible Laminar Flow

Approved:

[Signature]
Major Professor and Chairman

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Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]

[Signature]

[Signature]

Date of Examination:

November 22, 1972