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RATE OF CHANGE WITH APPLICATIONS IN PHYSICS PROBLEMS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Master of Natural Sciences

in

The Interdepartmental Program in Natural Sciences

by

Maria Ludu
B.S. Bucharest University, 1983

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ABSTRACT

In this thesis we approach the concept of Rate of Change as an ingredient of mathematical models, and we apply it to a few physics problems. The main goal is to provide materials that help students make logical connections between mathematical concepts and real life applications. We begin with a review of measurements, proportionality and similarity, based on Euclidean geometry. Then, we review linear and affine functions. We introduce a mathematical models involving average and instantaneous rate of change. These are applied to problems from physics. The main point is to show a unifying theme for mathematics and sciences, and the strong connections between the two. This thesis is intended to be a guide for teachers and students in Algebra and Calculus classes. It consists of three main parts:

1) a review of some parts of Euclidian and coordinate geometry and concepts and functions related to proportional reasoning,

2) average rate of change and instantaneous rate of change and

3) applications in physics problems.

INTRODUCTION

In the list of humanity's achievements, calculus is one of the greatest. The origins of calculus go back about 2300 years, to the time when the ancient Greeks found areas using the method of exhaustion, but it was not fully developed until the beginning of the 18^{th} century (or perhaps the late 19^{th} century, if it is a question of logically rigorous foundations). Calculus has long been a prerequisite for careers in science and engineering, but recently it has become part of the curriculum for business and for biological and social sciences. Today, calculators and computers relieve the need for heavy-duty computational skills, but now more than ever, students need a reasonable conceptual understanding of the main ideas of calculus in order to be able to use them in applications. When applying calculus, some students have difficulty making connections between mathematical models and the applications for which they are intended. The question is, what should students learn in high school in order to be well-prepared for the kind of calculus-study that they need for their professions?

Students have an implicit understanding of many mathematical concepts prior to learning any mathematical formulas. They are able to describe in their own words what happens when an object changes its position, or moves further or closer, faster or slower. These understandings need to be integrated with formal, procedural knowledge. Mathematics educator David Tall writes, "…children have an intuitive sense of concepts such as distance, velocity, acceleration", which he suggests can form a foundation for mathematical learning. Yet, as Jihwa Noh of Western Michigan University observes, "understanding of slope and derivative [are] often rule-based: slope as rise/run and derivative as a formula (for example, the derivative of a quadratic function is a linear function)." The learners to whom Noh is referring have not lost the ability to describe what they observe, but they do not make meaningful connections between their natural understandings and the mathematical models. This presents a challenge to teachers. Teachers need to enable students develop deep understandings of new and more abstract mathematical concepts, but they must also show them how to integrate this with prior knowledge.

The building blocks of calculus are functions. One of their fundamental uses is to show how things change. Functions can be represented in graphical, numerical, and analytical ways. The Harvard Calculus Project emphasized the need for integrating these representations. Their

“Rule of Three” required that all functions should be examined through tables, graphs and formulae.

My goal in preparing this thesis has been to discuss and develop some foundational ideas that are closely related to the school curriculum (e.g., measurement, geometry, functions and rate of change) and to present some important ways in which mathematics can be used to model real-world problems, especially those with which high school students may work. My focus is on integrating concepts and modeling procedures. Understanding and being able to make connections is important for all high school students, but it is critical for those intending to go to college. Students need foundational understandings of average rate of change, limits, tangent lines, derivatives, areas under curves, integrals and the Fundamental Theorem of Calculus.

Real applications deal with measured quantities. The connections between mathematics and nature are made through measurement. For this reason I included a short chapter about measurement. Conversions within the same system of units or between the systems are based on proportional reasoning. Euclidian geometry—similarity in particular—is reviewed as a basis for coordinate geometry. Following the geometry review, I present a short introduction to functions, and the four representations that will be used.

The main topics of this paper are average rate of change and instantaneous rate of change, understood as a limiting value of average rate over intervals of decreasing length. After reviewing the mathematics, I present real life applications that are potentially in the high school curriculum. To illustrate average rate of change and instantaneous rate of change, I use intuitively appealing examples related to motion. Once students realize the power of models and how to construct them, we can consider that our instruction was effective, and the students are ready for more advanced calculus topics and their applications.

A report by the Carnegie Corporation states, “Over the coming decades, today’s young people will depend on the skills and knowledge developed from learning math and science to analyze problems, imagine solutions, and bring productive new ideas into being.” .[1]

This thesis seeks to provide tools to help teachers ensure that their students are prepared.

CHAPTER I

MEASUREMENT

1) Introduction

According to Nagel, “if we inquire why we measure in physics, the answer will be that if we do measure, and measure in certain ways, then it will be possible to establish the equations and theories which are the goal of inquiry.”[2]

These days, when we refer to science we refer to measurable quantities. Educated adults think about things almost as though the things came with their measurements attached, automatically associating in their minds a measurement device and a unit used to express the result of the measurement. But it was not always this way, and in this chapter I make a short exposition of the history of measurement, why we need to measure, how we measure in such a way that we can speak in a unified way, the same for everyone. In Nagel’s view, “measurement can be regarded as the delimitation and fixation of our ideas of things...”

2) Practical Measurements

Early recorded examples of measurements are seen in the geometry of ancient Egypt and Babylonia. These peoples used measurements in architectural projects, in astrological observations and in trading. Many ancient units of measurement corresponded to parts of the human body: the hand and foot.

In the Middle Ages, as commerce and trading reached out over increasingly large distances, the need for a standard system of units became more and more apparent. The units of measurement were initially chosen for convenience. Now, people needed to reach agreement on universal units of measurement that were reproducible and easy to use.

In seventeenth and eighteenth centuries, the rise of science and the beginning of the industrial revolution gave further impetus to the development of a universal system of measurement. The definition of the units was modified over the years. The Sistem Internationale (or SI for short) is the most commonly used system in science. It has achieved international acceptance because of the convenience of using the base ten numeration system.

In applications, the process of measurement can be summarized as follows. After we choose the feature of an object or event that is to be measured – length, time, volume, capacity, *etc.*—the measurement process can be viewed as a sequence of steps:

1. Select an appropriate unit of measurement.
2. Compare the unit to the object to be measured, and find how many copies and fractional parts of the unit fit within the object.
3. Express the measurement as the number of units used.

Nagel states, "[In] measurement we attend to certain characters of objects to the exclusion of others." Those that we attend to are "those with which applied mathematics can cope." In order for this to be possible, the characters must fulfill two postulates:

- the possibility of comparison with respect to size, and
- the possibility of addition.

We will state below in more detail what these postulates mean. Nagel calls an object that has a character that meets the required conditions a *magnitude*, and the character itself he calls an *extensive property*. Because measurement involves comparing and adding, all magnitudes belong to families *within which* comparisons and additions can be made. For example, things that have length form a family of magnitudes. The extensive property is length. Similarly, things with weight form a family of magnitudes. In each case, we can compare and we can add.

What then is the measurement procedure? Before we describe it, we introduce one more concept. If we add a magnitude to itself a number of times, we get what is called a *multiple* of the magnitude. Now in measurement, what we do is simply compare multiples of the unit to multiples of the thing we are measuring. If U is the unit and Q is the magnitude we are trying to measure

$$m\mathrm{U} < n\mathrm{Q} \text{ implies Q is greater than } \frac{m}{n}$$

$$m\mathrm{U} = n\mathrm{Q} \text{ implies Q is equal to } \frac{m}{n}$$

$$mU > nQ \text{ implies Q is less than } \frac{m}{n}$$

The axioms for magnitudes were first analyzed with modern rigor by Hölder (ref). His axioms mention an equivalence relation (=), an order relation (<) and a commutative semi -group operation. They are:

1. Either $a > b$, or $a < b$, or $a = b$
2. If $> b$, and $b > c$, then $a > c$

3. For every a there is an a' such that $a = a'$
4. If $a > b$, and $b = b'$, then $a > b'$
5. If $a = b$, then $b = a$
6. For every a there is a b such that $a > b$ (within limits)
7. For every a and b there is a c such that $c = a + b$
8. $a + b > a'$
9. $a + b = a' + b'$
10. $a + b = b + a$
11. $(a + b) + c = a + (b + c)$
12. If $a < b$, there is a number n such that $na > b$ (also within limits)

Since the choice of unit is arbitrary, measurements of the same magnitude by different units are possible. It is often necessary to convert the results of a measurement by one unit to a measurement by another. All measurement conversions are based on proportional reasoning. For example, if we need to convert a length expressed in kilometers to a length expressed in inches, we need to know the measure of a kilometer in inches.

Students need to be aware that some measures are direct and some are derived. For example, when we refer to distance, students need to learn that distance is a length, and it can be measured using the appropriate instruments and units of measurement. The same is true if we speak about time: we use the ticks of a clock to count off time. But, other quantities—rates, in particular—are not measured by direct comparisons to standard units. There is no “unit of speed” that can be put next to a moving car to measure how fast it’s going. The unit for speed is a derived unit, for it depends on the measurement of both distance and time. Nagel calls a property *intensive* if it does not add, but is defined by means of extensive properties. Speed, for example does not add: we cannot find the speed of two cars by adding the speeds of each (unless a very small one is driving on top of another), nor can we find the speed of a trip by summing the speeds of its parts. In this case we need to have relations that reduce the intensive physical quantities to extensive ones. Density is another example; we need to define it from volume (extensive) and mass (extensive).

3) Measurement in Geometry and the Distance Function

Euclid did not pick out a single standard unit of measurement to use in his geometry. Any length might be compared with any other. Euclid never attached numbers to his segments, angles, areas or volumes. In this, his geometry was different from the geometry common in schools. The idea of using a standard unit to measure all distances was introduced by Descartes. He wrote in his *Geometry* (1637) that "taking one line which I shall call unity in order to relate it as closely as possible to numbers and which in general can be chosen arbitrarily," it becomes possible to relate the arithmetic operations of multiplication, division, and extraction of square roots to geometric constructions.

The choice of a single segment to serve as a unit allows us to use in our geometry what mathematicians call a "metric" or a "distance function." Let us consider the Euclidean plane *E*. It is a collection of points and line segments that obey the Postulates of Euclid, which we assume the reader knows. Once the unit has been chosen, then for any pair of points, A, B in the plane, there is a corresponding real number d(A, B), which we may call the distance from A to B (We speak of *the* distance because we are assuming that we have all agreed on the choice of unit, and do not plan to change.) We also speak of "the length of segment AB", which is simply a synonym for d(A, B).

Let us consider a Euclidean Plane *E*, as a collection of points and lines having the properties stipulated by Euclid's Postulates. Some properties that the distance function has are as follows:

a) $d(\mathrm{A, B}) \geq 0$
b) $d(\mathrm{A, B}) = 0 \Leftrightarrow \mathrm{A} = \mathrm{B}$
c) $d(\mathrm{A, B}) = d(\mathrm{B, A})$,
d) $d(\mathrm{A, C}) \leq d(\mathrm{A, B}) + d(\mathrm{B, C})$
e) $AB \cong A'B' \Leftrightarrow d(\mathrm{A, B}) = d(\mathrm{A', B'})$

In modern terminology, a function satisfying (a)—(d) is called a *metric*. Item (e) says that two segments AB and CD are congruent if and only if they have the same length $d(\mathrm{A,B}) = d(\mathrm{C,D})$.

If AB and CD are segments, then we can measure AB by CD. We call the result of this measurement *the ratio of* AB *to* CD. In geometry, this number is also often written in the format AB/CD. It is a fact that for all points A, B, C and D, if C and D are distinct, then

$$\textit{the ratio of } \mathrm{AB} \textit{ to } \mathrm{CD} = d(\mathrm{A,B})/d(\mathrm{C,D})$$

Note that fact 3 implies that if we change units, then all distances change by the same factor. To be more precise, suppose we compare measurements with unit U=PQ and W = RS.

$$d_W(A, B) = AB/RS = d_U(A,B)/d_U(R,S).$$

Thus,

$$d_W(A,B)\ d_U(R,S) = d_U(A,B).$$

The distance function alone is not enough to do geometry. One needs additional structure. We are not going to describe the geometric assumptions that are necessary to create a coordinate system. The first additional piece of information we need is the so-called Ruler Postulate.

CHAPTER II

GEOMETRY

1) Introduction

In this chapter, we will expand on a few Euclidian geometry concepts in order to frame the later discussion, and. then we will introduce the coordinate plane. Coordinates are used for two purposes: 1) to reduce geometric problems to algebraic problems and 2) to visualize algebraic relationships or relationships between the variables in an equation or function. Because these are in some sense reciprocal processes, we can translate back and forth between geometrical representations and algebraic statements.

In school mathematics, graphs are often used to visualize algebraic relations, but the geometry of the plane in which the graph is drawn is taken for granted. In this chapter, our intention is to bring the geometry out explicitly, so that the geometrical assumptions can be examined.

Many of Euclid's results had been stated by earlier mathematicians, but Euclid was the first to bring together all known geometry in a comprehensive deductive system. He organized the geometrical achievements of his time in a treatise composed of thirteen books, called *The Elements*. In his first book, Euclid listed five postulates from which all of the truths of geometry follow. It is a long road from the postulates to the geometric facts that we will isolate and discuss in this chapter. We will not retrace the deductions. The geometry we need all comes out of the theory of similarity, and we state the basic facts of this in the first section.

2) Similar Triangles

Definition. Let ABC and DEF be triangles. We say ΔABC is similar to $\Delta A'B'C'$ if $\measuredangle A = \measuredangle A', \measuredangle B = \measuredangle B', \measuredangle C = \measuredangle C'$ and $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$

Theorem. (AAA- Similarity.) If in two triangles, corresponding angles are equal, then their corresponding sides are proportional, and the triangles are similar. (See Figure 1)

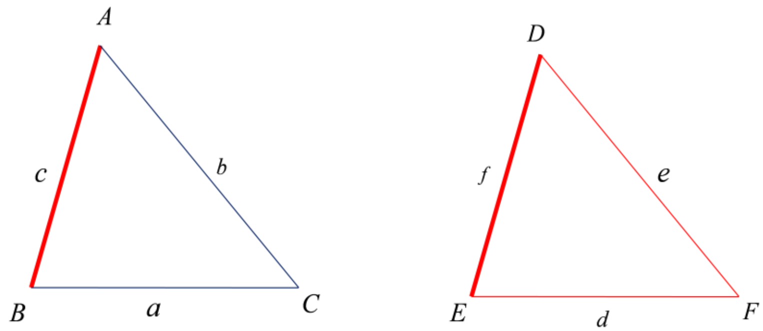

Figure 1. AAA-Similarity

Corollary. (AA-Similarity.) If two angles of a triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

The corollary follows from the theorem because the sum of the angles in any triangle is two right angles.

3) Pythagorean Theorem

The purpose of this section is to prove Pythagorean Theorem using similarity. The Pythagorean Theorem is a fundamental mathematical fact that accounts for how coordinate geometry works. In a sense, it *is* the distance formula of coordinate geometry. The theorem can be so beautifully compressed in an algebraic sentence that many high school students remember the equation $a^2 + b^2 = c^2$ without any awareness of the geometric meaning.
The Pythagorean Theorem can be understood as a consequence of similarity, justifying the claim that similarity underlies coordinate geometry.

Let us consider a right triangle ΔABC, with $\measuredangle C = 90^o$ and $CD \perp AB$, as in Figure 2

Lemma. The altitude to the hypotenuse in a right triangle divides it into two right triangles, each of which is similar to it.

Proof. As shown in the Figure 2, $\measuredangle A \cong \measuredangle A, \measuredangle ADC \cong \measuredangle ACB$.

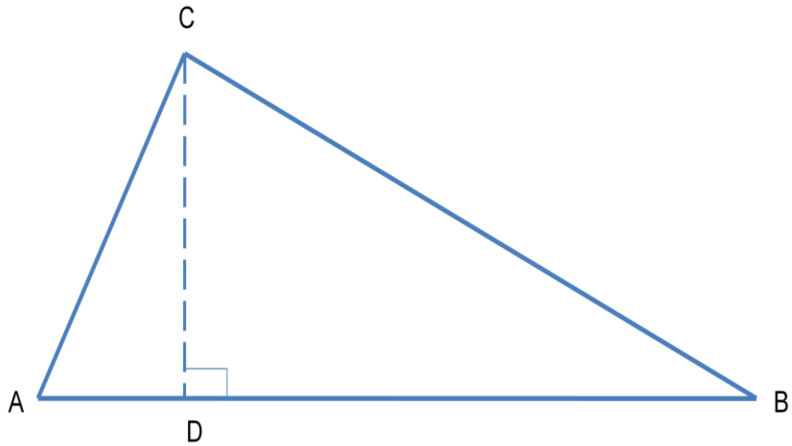

Figure 2. Altitude in a right triangle

By AA-Corollary, $\Delta ACD \sim \Delta ABC$. Therefore, $\measuredangle ADC \cong \measuredangle ACB,\ \measuredangle ACD \cong \measuredangle ABC$.
By AA-Corollary, $\Delta CBD \sim \Delta ABC$, and $\measuredangle BAC \cong \measuredangle BCD,\ \measuredangle BDC \cong \measuredangle BCA$. Q.E.D.

Theorem. (The Pythagorean Theorem.) In any right triangle the square of the length of the hypotenuse is the sum of the squares of the lengths of the other two sides.

Proof. We are going to provide a proof based on similarity. We will name the sides of the triangle a, b, c, as shown in the diagram, Figure 3. By the similarities proved in Theorem 2,

$$\frac{f}{b} = \frac{b}{c}, and\ \frac{a}{g} = \frac{c}{a}$$

$$f = \frac{b^2}{c}, and\ g = \frac{a^2}{c}$$

In $\Delta ACB, c = f + g$, and consequently:

$$c = f + g = \frac{b^2}{c} + \frac{a^2}{c} = \frac{b^2 + a^2}{c}$$

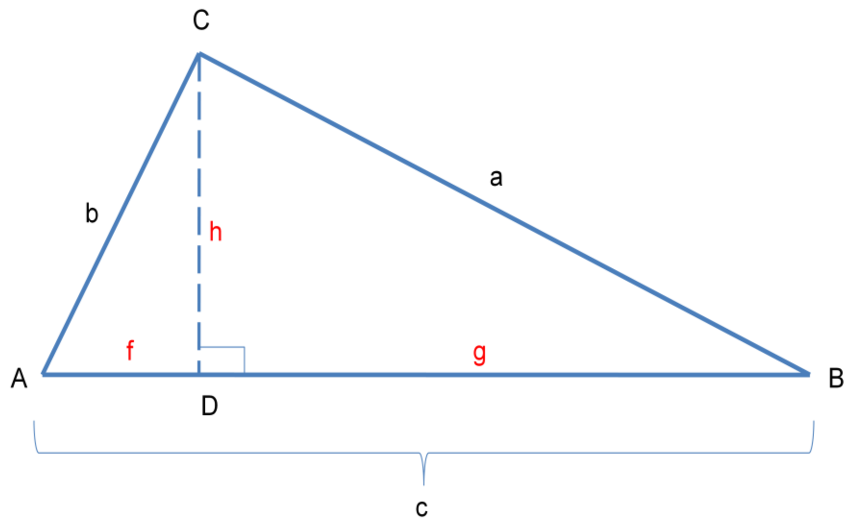

Figure 3. The Pythagorean Theorem

From this, we deduce:

$$c = \frac{b^2+a^2}{c},$$

and after we multiply each side by c, we obtain $c^2 = a^2 + b^2$. Q.E.D.

4) Introduction of the Cartesian System

Each problem that I solved became a rule, which served afterwards to solve other problems

René Descartes

During the seventeenth century, the development of science and industry required new solutions to problems involving curves: the curvature of a lens, the shortest route to a destination, the trajectory of a cannon ball, *etc*. The Greeks had developed ordinary geometric theorems by drawing or visualizing the objects, but classical Greek geometry (as in Euclid) did not have any clear connection to algebra. Two French mathematicians, Rene Descartes (1596-1650) and Pierre de Fermat (1601-1665), independently developed the foundations for analytic geometry. The creation of a coordinate system allowed them to associate points to real numbers and curves to equations. In this section, I describe the foundational ideas that enable analytic geometry. The

first is the Ruler Postulate, the very statement of which requires the distance function that we introduced earlier:

The Ruler Postulate. Given any line l, and any two points A and B on l such that d(A, B) =1, there is a unique bijective function $x: l \longrightarrow$ R such that x(A)=0, x(B)=1 and:

for all points P, Q on l, $|x(P) - x(Q)| = d(P, Q)$.

The Ruler Postulate enables us to equip any given line with a coordinate system. The coordinate system on a line is a function from the line to ℝ. We commonly use the letters x, y (and sometimes z, u, v) to denote the coordinate function.

Our next step is to describe how to create a coordinate system on the plane.

- Step 1: Choose two perpendicular lines, l, and m, intersecting in A.
- Step 2: Choose points B on l, and C on m, each at distance 1 from A.
- Step 3: Let x be the coordinate function on l, and y the coordinate function on m.
- Step 4: Extend the function x to the whole plane as follows: if $P \in E$, draw a line through P that is perpendicular to l. Let D be the foot. Note that x(D)is defined already. Now, let x(P):=x(D). Define y(P) analogously.

Having made such a coordinate system l is called *x-axis*, and m is called *y-axis*. See Figure 4. Any coordinate system made in this way produces a bijection between the points in the plane and the ordered pairs of real numbers. Because of this, once a coordinate system has been made, it is common to use the ordered pairs as names of the points. On the other hand, if a point P is named, its coordinates are x(P) and y(P), as determined by the functions x and y.

5) The Distance Theorem

Theorem. Suppose a coordinate system (x, y) has been chosen. The distance between two points P and Q is given by the formula:

$$d = d(P, Q) = \sqrt{(x(Q) - x(P))^2 + (y(Q) - y(P))^2}.$$

Proof: We are going to use the Pythagorean Theorem to prove distance formula. To be consistent, we will use the same notations as in the previous demonstrations. Let PD, and QE be the perpendiculars from P and Q to the *x-axis* and let PR and QS be the perpendiculars from P and Q to the *y-axis*; see Figures 5 and 6. Now, we consider two cases.

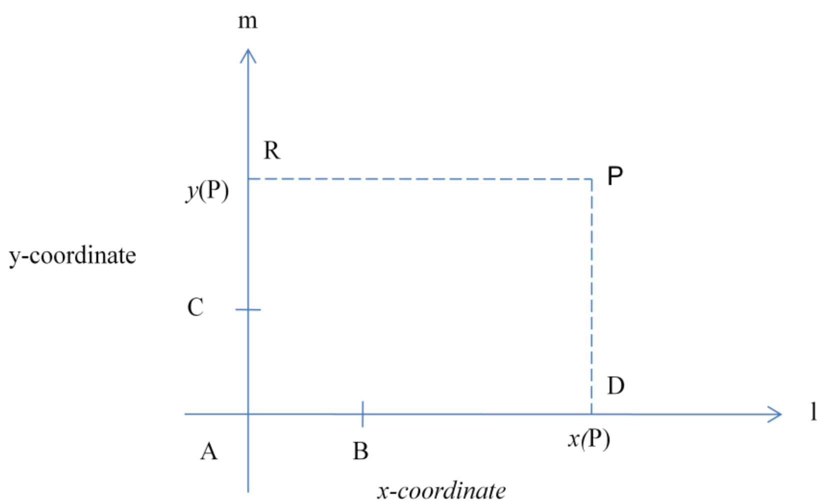

Figure 4. The Cartesian System

Case 1. If $x(\mathrm{P}) = x(\mathrm{Q})$, then x(Q) - x(P) = 0 and $|y(\mathrm{Q}) - y(\mathrm{P})| = d(\mathrm{P,Q})$. Analogously, if $y(\mathrm{P}) = y(\mathrm{Q})$, then $y(\mathrm{Q}) - y(\mathrm{P}) = 0$, and $|x(\mathrm{Q}) - x(\mathrm{P})| = d(\mathrm{P,Q})$, see Figure 5. In either case, the formula is true.

Case 2. If $x(\mathrm{P}) \neq x(\mathrm{Q})$, and $y(\mathrm{P}) \neq y(\mathrm{Q})$, let T be the intersection point of the vertical line through P and the horizontal line through Q, as in Figure 6. Then, ΔPTQ is a right triangle, with $\measuredangle \mathrm{T} = 90^{\circ}$, because PT is parallel to the y-axis and QT is parallel to the x-axis. We can write:

$$d(\mathrm{P,T}) = d(\mathrm{R,S}) = |y(\mathrm{Q}) - y(\mathrm{P})|, \text{ and}$$

$$d(\mathrm{T,Q}) = d(\mathrm{D,E}) = |x(\mathrm{Q}) - x(\mathrm{P})|.$$

By the Pythagorean Theorem in ΔPTQ,

$$d(\mathrm{P,Q})^2 = d(\mathrm{P,T})^2 + d(\mathrm{Q,T})^2$$

If we substitute from the previous equations, we get:

$$d(\mathrm{P,Q})^2 = [(|\mathrm{y(Q)} - \mathrm{y(P)}|)^2] + [(|x(\mathrm{P}) - x(\mathrm{Q})|)^2] = \left(y(\mathrm{Q}) - y(\mathrm{P})\right)^2 + \left(x(\mathrm{Q}) - x(\mathrm{P})\right)^2.$$

By extracting the square root from the each side, we obtain:

$$d(\mathrm{P,Q}) = \sqrt{(y(\mathrm{Q}) - y(\mathrm{P}))^2 + (x(\mathrm{Q}) - x(\mathrm{P}))^2}. \qquad \text{Q.E.D.}$$

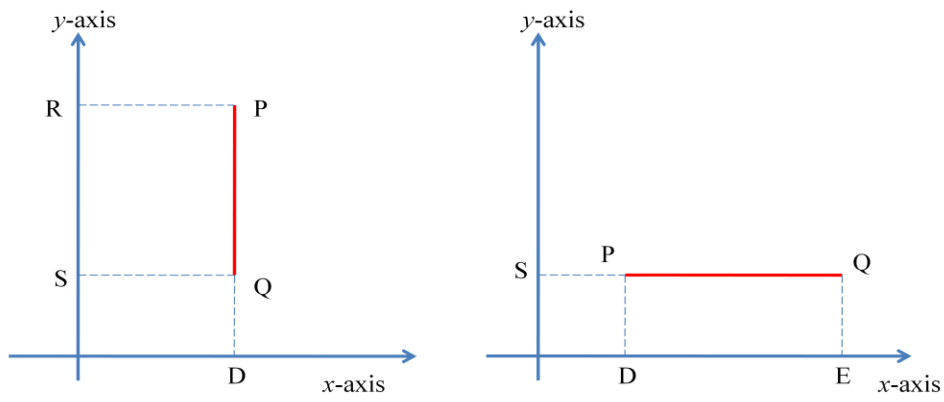

Figure 5 Distance Theorem for Case 1

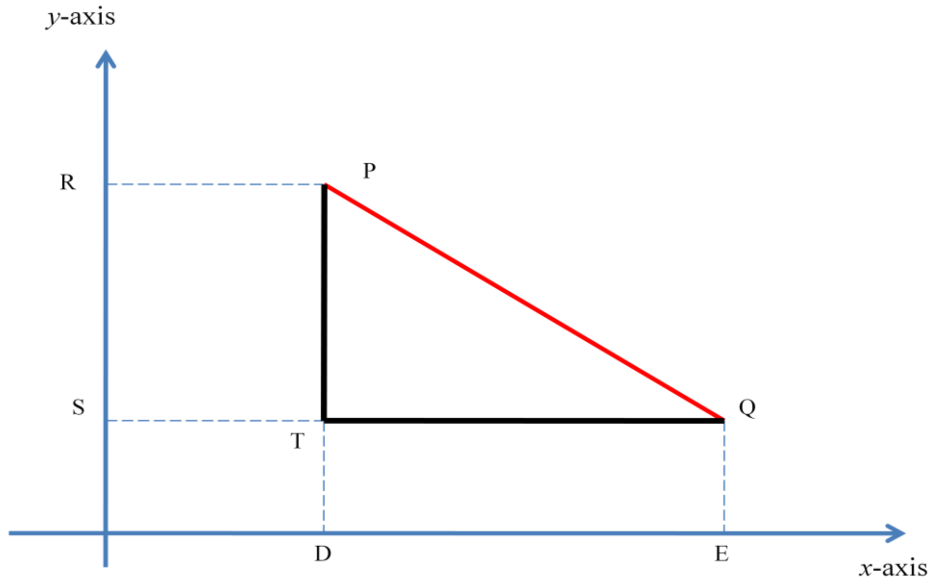

Figure 6. Distance Theorem for Case 2

This relation is very important, because we can express any length of any line as a function of the coordinates, or as the distance between the ending points, in the convenient way, using algebraic procedure. Moreover, we can do any calculations with the algebraic expressions we obtain, instead of manipulating objects in order to see their relationships.

6) Statements with Variables *x, y* and their Graphs

Let $S(x, y)$ be a statement about real numbers x and y. For example:

- "x is an even integer (and y is any real number)."
- "$x^2 + y^2 = 1$"
- "$x^2 + y^2 = 1$ or $x = y$"
- "If x is an even integer, then y is 1."

Definition. If $S(x, y)$ is any statement that is either true or false for any specific real numbers x and y, then the set-theoretic graph of S is the set of all ordered pairs of numbers for which the statement is true:

$$\{ (x, y) \mid S(x, y) \}.$$

Given an x-y- coordinate system in the plane, the geometric graph of S consists of the set of all points P whose coordinates $x(\mathrm{P}), y(\mathrm{P})$ make the statement $S(x, y)$ true:

$$\{ \mathrm{P} \mid S(x(\mathrm{P}), y(\mathrm{P})) \}.$$

In school math, it is common to ignore the distinction between the number pair (x, y) and the point in the plane whose coordinates are x and y. If these two things are not distinguished, then there is no distinction between the set-theoretic graph and the geometric graph. Some textbooks clearly make the distinction between number pairs and points (for example, they use (2,3) to mean the number pair and P(2,3) to mean the point), but they explain so little about the distinction that students are not likely to take it seriously. Very few schoolbooks use the notation $x(\mathrm{P})$ and $y(\mathrm{P})$ that we have described, and therefore the definition of the geometric graph given here might not be appropriate to use in teaching students. Its purpose is to make a distinction that teachers should be aware of, even if it is not incorporated in lessons.

7) Affine Linear Equations and Lines

Definition. By an affine linear equation we understand an equation in two variables x and y of the form $Ax + By + C = 0$, where A, B, C are real numbers, and $A \neq 0$, or$B \neq 0$.

Let E be the Euclidean plane. Assume that we have selected an x-y- coordinate system in E. In this section, we will show that the geometric graph of any affine linear equation is a line

in E and that every line in E is the geometric graph of some affine linear equation. This is a fact that is often taken for granted in school mathematics. However, when the concepts that underlie this basic idea are carefully and explicitly laid out, it turns out that there is really quite a bit of reasoning behind it.[1]

Theorem A. Every line in the plane E is the geometric graph of an affine linear equation

Proof. Let l be a line in E. There is a segment P_1P_2 in E such that l is the perpendicular bisector of this segment; see Figure 7.

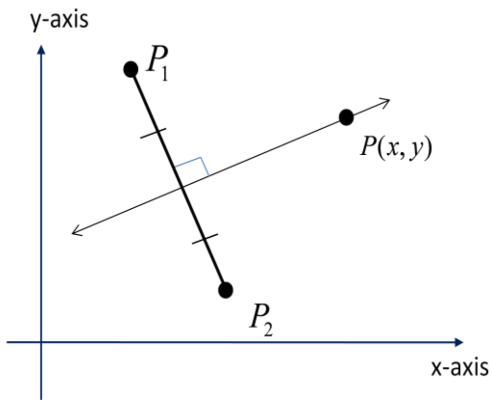

Figure 7. Line *l* bisects P_1P_2.

Thus, l is the set of points P that satisfy the condition $d(P, P_1) = d(P,P_2)$. Using the distance formula, we can rewrite this condition in terms of the coordinates of P, P_1 and P_2:

$$\sqrt{(x(\mathrm{P}) - x(\mathrm{P}_1))^2 + (y(\mathrm{P}) - y(\mathrm{P}_1))^2} = \sqrt{(x(\mathrm{P}) - x(\mathrm{P}_2))^2 + (y(\mathrm{P}) - y(\mathrm{P}_2))^2}$$

If we square both sides of this equation, we obtain:

$$x(\mathrm{P})^2 + x(\mathrm{P}_1)^2 - 2x(\mathrm{P})x(\mathrm{P}_1) + y(\mathrm{P})^2 + y(\mathrm{P}_1)^2 - 2y(\mathrm{P})y(\mathrm{P}_1) =$$
$$x(\mathrm{P})^2 + x(\mathrm{P}_2)^2 - 2x(\mathrm{P})x(\mathrm{P}_2) + y(\mathrm{P})^2 + y(\mathrm{P}_2)^2 - 2y(\mathrm{P})y(\mathrm{P}_2)$$

[1] "Schoolbook ideas that many students master without difficulty and that teachers often regard as routine are often very complex at logical, conceptual or cognitive levels. It is possible that this may help to explain the difficulty some students have in learning some ideas. Of course, there is no guarantee that a logical (or conceptual or cognitive) analysis of a mathematical idea will provide information that will help with the design of curricula, but certainly additional knowledge cannot impair one's ability to teach. And there is the possibility that clarity about the logical foundations might provide something useful." (J. Madden, Personal Communication.)

This implies:

$$x(\mathrm{P}_1)^2 - 2x(\mathrm{P})x(\mathrm{P}_1) + y(\mathrm{P}_1)^2 - 2y(\mathrm{P})y(\mathrm{P}_1) = x(\mathrm{P}_2)^2 - 2x(\mathrm{P})x(\mathrm{P}_2) + y(\mathrm{P}_2)^2 \text{-2y(P)y(}\mathrm{P}_2)$$

Now, use A, B and C to stand for the constants in this equation according to the following scheme,

$$\mathrm{A} = 2[x(\mathrm{P}_2) - x(\mathrm{P}_1)],$$

Grouping terms, we get, Equation (1):

$$-2x(\mathrm{P})x(\mathrm{P}_1) + 2x(\mathrm{P})x(\mathrm{P}_2) - 2y(\mathrm{P})y(\mathrm{P}_1) + 2y(\mathrm{P})y(\mathrm{P}_2) + x(\mathrm{P}_1)^2 - x(\mathrm{P}_2)^2 + y(\mathrm{P}_1)^2 - y(\mathrm{P}_2)^2 = 0$$

$$\mathrm{B} = 2[y(\mathrm{P}_2) - y(\mathrm{P}_1)], \text{ and}$$

$$\mathrm{C} = [x(\mathrm{P}_1)^2 + y(\mathrm{P}_2)^2 - x(\mathrm{P}_2)^2 - y(\mathrm{P}_2)^2].$$

Finally, we show that at least one of A and B is not 0. If both, A=0, and B=0, then: $[x(\mathrm{P}_2) - x\mathrm{P}1=0,$ and $y\mathrm{P}2 - y\mathrm{P}1=0$, implies $x\mathrm{P}2=x\mathrm{P}1, y\mathrm{P}2=y(\mathrm{P}1)$, therefore: $P2=P1$, and P_1 and P_2 are identical, not distinct points as defined. This shows that with *A*, *B* and *C* as defined, $Ax + By + C = 0$ is an affine linear equation. Equation (1) shows that a point P is on *l* if and only if $Ax(\mathrm{P}) + By(\mathrm{P}) + C = 0$. Q.E.D.

Theorem B. The geometric graph of any affine linear equation is a line.

Proof. Suppose that $Ax + By + C = 0$ is an affine linear equation. First, we prove that there are real numbers x_1, x_2, y_1, y_2 that solve the following system of equations:

$$\mathrm{A} = 2(x_2 - x_1)$$

$$\mathrm{B} = 2(y_2 - y_1)$$

$$\mathrm{C} = x_2^2 - x_1^2 + y_2^2 - y_1^2$$

This system is equivalent to:

$$x_2 = \frac{A}{2} + x_1$$

$$y_2 = \frac{B}{2} + y_1$$

$$C = \frac{A^2}{4} + Ax_1 + \frac{B^2}{4} + By_1$$

This system clearly has a solution, since one of A and B is non-zero. Fix any particular solution x_1, x_2, y_1, y_2 , and let P_1 be the point with coordinates (x_1, y_1) and let P_2 be the point with coordinates (x_2, y_2). Then the argument in the previous proof shows that the points P whose coordinates (x, y) solve $Ax + By + C = 0$ all lie on the perpendicular bisector of P_1P_2 and that any point on the perpendicular bisector of P_1P_2 has coordinates that solve this equation. Q.E.D.

8) Lines and Slopes

In the following, we assume that we are working in a plane with a standard orthogonal coordinate system, with coordinate functions *x* and *y*. As above, if P is a point, *x*(P) denotes the *y*-coordinate of P and *y*(P) denotes the *y*-coordinate of P .

Definition. If P and Q are any points with different *x-coordinates*, then the pair (P, Q) has a slope defined as follows:

$$slope(\mathrm{P}, \mathrm{Q}) = \frac{y(\mathrm{Q}) - y(\mathrm{P})}{x(\mathrm{Q}) - x(\mathrm{P})}$$

Proposition A. Let *l* be a line that has affine linear equations $Ax + By + C = 0$ with *B* not 0. If P and Q are distinct points on *l* then $slope(\mathrm{P}, \mathrm{Q}) = -\frac{\mathrm{A}}{\mathrm{B}}$.

Proof: If P and Q are points on the same line, then the equations of line through P and Q are:

$$Ax(\mathrm{P}) + By(\mathrm{P}) + C = 0$$

$$Ax(\mathrm{Q}) + By(\mathrm{Q}) + C = 0, \text{ for } A, B, C \text{ real numbers, and } B \text{ not } 0.$$

$$A(x(\mathrm{P}) - x(\mathrm{Q})) + \mathrm{B}(y(\mathrm{P}) - y(\mathrm{Q})) = 0$$

$$-\frac{\mathrm{A}}{\mathrm{B}} = \frac{y(\mathrm{Q}) - y(\mathrm{P})}{x(\mathrm{Q}) - x(\mathrm{P})}$$

Comment. Proposition 1 enables us to define the slope of a line that is not parallel to the *y*-axis to be the slope of any pair of points on it. It follows from Proposition 1 that the slope of such a line is $-A/B$, if $Ax + By + C = 0$ is *any* equation for the line.

The content of Proposition 1 is dealt with in a different way in some textbooks (*e.g.*, the CME Project, *Algebra I*, prepared by the Education Development Center in Newton MA). These

books approach the topic of lines and their equations by *starting* with the definition of the slope between points. If this is taken as a first step, then to define the slope of a line it is necessary first to prove Proposition 2, below. It is only after this that these books develop equations for lines. We include this proposition and its proof to illustrate some of the complications that this approach involves.

Lemma. $slope(\mathrm{Q},\mathrm{P}) = \frac{y(\mathrm{P})-y(\mathrm{Q})}{x(\mathrm{P})-x(\mathrm{Q})} = \frac{y(\mathrm{Q})-y(\mathrm{P})}{x(\mathrm{Q})-x(\mathrm{P})} = slope(\mathrm{P},\mathrm{Q})$.

Proposition B. Let *l* be a line that is not parallel to the *y*-axis. If P and Q are distinct points on l and, $\mathrm{P}'$and Q' are also distinct points on l, then $slope(\mathrm{P},\mathrm{Q}) = slope(\mathrm{P}',\mathrm{Q}')$.

Proof. Note that *if l is* horizontal, the conclusion is trivial, since then, $slope(\mathrm{P},\mathrm{Q}) = slope(\mathrm{Q},\mathrm{P}) = 0$. We can assume, therefore, that $x(\mathrm{P}) < x(\mathrm{Q})$, and $x(\mathrm{P}') < x(\mathrm{Q}')$. Now, consider the triangle PQR, where R is the intersection of the line through P perpendicular to the *y-axis*, and of the line through Q parallel to the *y-axis.* We obtain triangle P*'*Q*'*R*'* analogously. We assert that the triangles are similar.(see Figure 8). To see this, note the following and apply AA:

- RQ||R'Q'(both are perpendiculars on *x-axis*)
- PR||P'R' (both parallel to *x-axis*)
- $\measuredangle$QPR $\cong$ $\measuredangle$Q$'$P$'$R$'$ (corresponding angles)
- $\measuredangle PQR \cong \measuredangle P'Q'R'$ (corresponding angles)

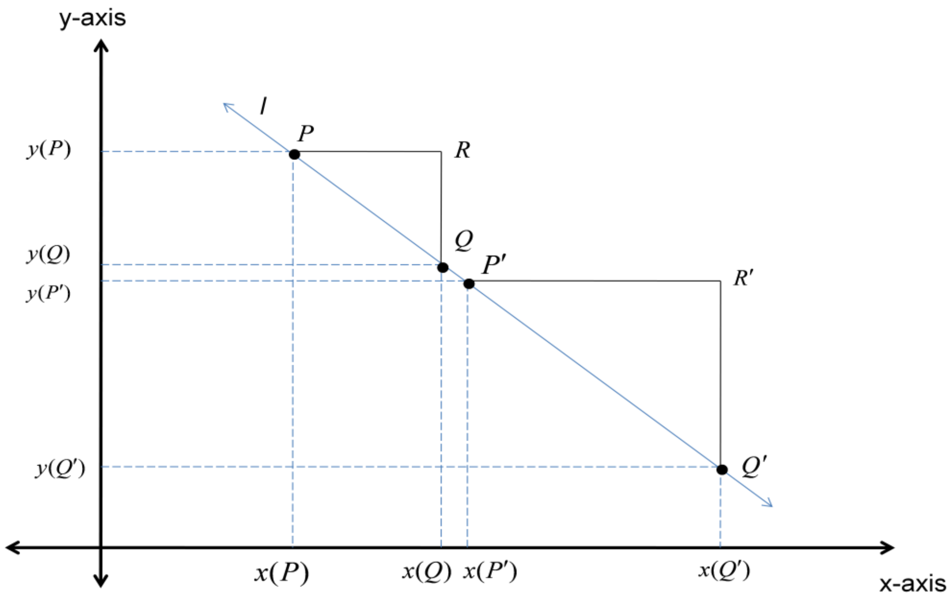

Figure 8. Similar Triangles

Having verified that the triangles are similar, we can say that the sides of the triangles are proportional: $\frac{\mathrm{QR}}{\mathrm{PR}} = \frac{\mathrm{Q'R'}}{\mathrm{P'R'}}$

where

$$\mathrm{QR} = y(\mathrm{R}) - y(\mathrm{Q}),$$
$$\mathrm{PR} = x(\mathrm{R}) - x(\mathrm{P}),$$
$$\mathrm{Q'R'} = y(\mathrm{R'}) - y(\mathrm{Q'}), and$$
$$\mathrm{P'R'} = x(\mathrm{R'}) - x(\mathrm{P'})$$

By applying the 'Distance Theorem', and the slope definition in ΔPQR, and $\Delta P'Q'R'$, we can write:

$slope(\mathrm{P},\mathrm{Q}) = \frac{y(\mathrm{Q})-y(\mathrm{P})}{x(\mathrm{Q})-x(\mathrm{P})} = \frac{\mathrm{RQ}}{\mathrm{RP}} = \frac{\mathrm{Q'R'}}{\mathrm{P'R'}} = \frac{y(\mathrm{R'})-y(\mathrm{Q'})}{x(\mathrm{R'})-x(\mathrm{P'})} = slope(\mathrm{P'},\mathrm{Q'})$. Q.E.D.

Definition. Suppose line l is not parallel to the y-axis. Then, the slope of the line l is defined to be $slope(\mathrm{P},\mathrm{Q})$, where P and Q are any two points on l.

CHAPTER III

FUNCTIONS

Functions have an important role in mathematics, sciences, engineering, and economics. This is because they can represent the relationships between quantities, and they can be used whether these quantities have been directly measured, observed, or even were part of a real event. Functions are easy to work with and offer us a great tool to make predictions.

The notation for functions we use today came from Leibniz, in the late seventeenth century. He named it *functio,* and he used it to describe a variable y whose value depended on a changing variable x. Explicit formulas looked like $y = x^2$ for example. In the eighteenth century, the expression took a more general formulation, $y = f(x)$. In calculus, this function was linked to its graph, which was considered a set of points $(x, f(x))$ in the Cartesian plane.

To serve as a point of reference, we present the following widely-accepted mathematical definition of function.

Definition: Let A and B be sets. A function f from A to B is a rule that assigns, to each element x in A, a unique element in B. The element assigned to x is called $f(x)$. A is called the domain of f and B is called the *codomain*.

Most functions that arise in school mathematics have subsets of the real numbers as domain and codomain. From now on, unless we say otherwise, we use the word function in this sense. A function is a rule that assigns to each real number x in its domain a unique real number $f(x)$.

1) Various Representations of Functions

Functions are the fundamental objects of study in calculus. They arise when we want to show how a quantity depends on another. Functions can be represented in different ways. Regardless the representation, the function is the same; it is only "visualized" differently. Using functions effectively depends on knowing how to choose the most appropriate representation, and to describe the relationships among all the possible representations.

Students are exposed to *verbal* representations of functions, even before the elementary grades; for example: “to the candies you have now in your hand, add one.” Verbal representations stimulate discussion and reasoning. Far more complicated verbal representations occur at all levels; for example: “the surface area of a sphere of radius x”.

Algebraic representations are the most common, concise, and powerful. The algebraic representation for the above will be $f(x) = x + 1$, where x is the number of candies in the child’s hand, and $f(x)$ is the number of candies after adding one. This form can help us predict, calculate, generalize, *etc.* It is not always easy to obtain. The surface area of a sphere is $f(x) = 4\,\pi\, x^2$, which is clearly nor obvious. In many real-life situations, there is no algebraic form.

Numerical representations are presented as a list of ordered pairs x, f(x) or as a table with *input*, *output* values. Students are exposed to tables of numerical values also starting as early as the elementary grades. They are intuitive, and they help students see patterns. Obviously, however, a table cannot fully represent any function with infinite domain. So, tables can only suggest many functions, they cannot *fully represent* them.

Graphical representation provides a way to visualize how the value of one quantity varies with respect to the other. The graph of a function of a function *f* with domain A is the set of ordered pairs:

$$\text{graph}(f) := \{ (x, f(x)) \mid x \in A \}.$$

This is a special case of the idea presented at the end of the previous chapter. In the terminology described there, graph(f) is the set-theoretic graph of the statement, ‘$y = f(x)$’. As mentioned, this set is commonly identified with its geometric representation: the set of points whose x-y-coordinates satisfy $y = f(x)$, (*i.e.*, $\{ \mathrm{P} \mid y(\mathrm{P}) = f(x(\mathrm{P}))\}$).

In the twentieth century the graph of the *general* function $f\colon A \longrightarrow B$, ($A$ and B being any sets) was defined as the set of ordered pairs $\{(x, y) \in A \times B | x \in A, y = f(x)\}$. This made possible the set-theoretic definition: *A function is any set* G of ordered pairs

$G = \{(x, y) \in A \times B | x \in A,\ y \in B\}$,

Where for every $x \in A$ there is a unique $y \in B$, such that $(x, y) \in G$.

This last condition can be rephrased as follows:

for all $x \in A$ there is $y \in B$ such that $(x, y) \in G$, and if both $(x, y_1) \in G$ and $(x, y_2) \in G$, then $y_1 = y_2$

2) Linear Functions and their Graphs

A linear function (or to be more precise, an affine linear function) is a function of the form

$$f(x) = m\,x + b$$

where m and b are constants. In the present section, we use the results from the last several sections of the previous chapter to analyze the graphs of linear functions.

Proposition. The graph of $y = mx + b$ is the line through $(0, b)$ with slope m.

Proof. The set of solutions of $y = mx + b$ is the same as the set of solutions to the affine linear equation $mx - y + b = 0$, and we have shown earlier that the graph of this equation is a line with slope m.

CHAPTER IV

AVERAGE RATE OF CHANGE

A problem well-stated is a problem half-solved.
John Dewey

The processes studied in the physical and social sciences involve understanding how one quantity varies as another quantity changes. Discovering and using functions that describe the dependence of one quantity on another is called modeling. In this section we will describe models of certain kinds of motion. We introduce the important concept of *average rate of change* in this context and illustrate its properties.

Definition: Suppose $f: \mathcal{R} \longrightarrow \mathcal{R}$ is a function defined *at* a and b, $a \neq b$ then the average rate of change o function f from a to b is $\frac{f(b)-f(a)}{b-a}$.

Let $f(x) = mx + b$ be a linear function. Note that the average rate of change of f between any two numbers is the same. It is equal to m, the slope of the graph of f.

1) Describing Motion

Let us consider a particle moving on a straight line, from a point P (at time $t_{initial}$) to a point Q (at time t_{final}). If we choose a coordinate system on the line (using the Ruler Postulate), then every point on the line has a number attached. At any time, t, the particle has a position, described by a number. Let $f(t)$ be that number. We call f the *position function* of the particle. The coordinate of P is $f(t_{initial})$ and the coordinate of Q is $f(t_{final})$.

We will examine this situation by graphic f in a coordinate plane that represents time on the horizontal coordinate and position on the vertical coordinate.
As we apply the general definition average rate of change, we find that the average rate of change of position between $t_{initial}$ and t_{final} is:

$$\frac{f(\mathrm{t}_{final})-f(\mathrm{t}_{initial})}{\mathrm{t}_{final}-\mathrm{t}_{initial}}.$$

Since the denominator represents the elapsed time, and it is always positive, the numerator will be the one to give the sign of the variation.

If f is a linear function between $t_{initial}$ and t_{final}, we call the motion *uniform linear*. If this is the case, then the slope between any two points on the graph with time coordinates between $t_{initial}$ and t_{final} will be the same. In terms of average rate of change, this means that between any two times, the average rate of change is the same. This constant is called the velocity of the particle. (But note that we have defined velocity only for functions that are linear.) The graph of the position function of a particle with uniform linear motion of positive velocity is shown in Figure 9.

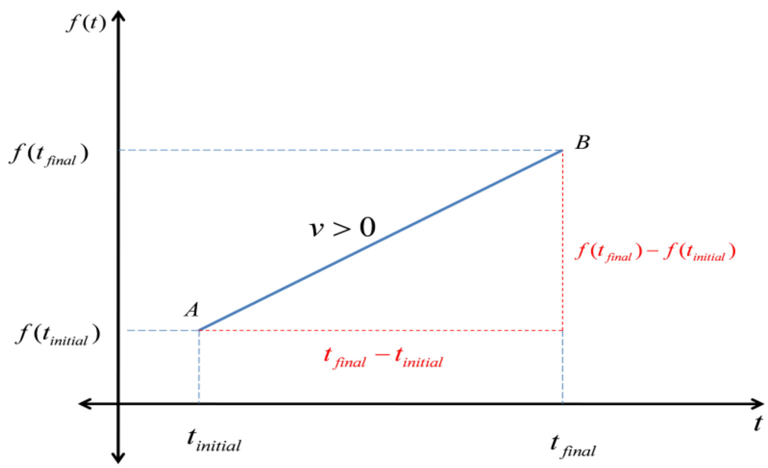

Figure 9. Graph of motion with constant positive velocity

If the motion is uniform linear but $f(t_{initial}) > f(t_{final})$, then the slope is negative. See Figure 10. This situation needs some discussion. What is the meaning of negative average velocity, and how is the particle moving? Accordingly to the diagram, the position of the particle at $t_{initial}$ has a larger coordinate than the position at t_{final}. and the particle simply came back toward the origin of the system of coordinates at a constant rate.

My students tend to be confused by situations where the velocity is zero. They can understand positive velocity, work uncomfortably with negative velocity, but are totally scared of the “at rest” situation. We will combine this case with the other two situations, velocities positive and negative, see Figure 11. There is a lot of information about the the motion of the particle in this diagram. We use A, M, N, Q and B to label points on the graph. Each of these points corresponds to a time (*e.g.*, t(A)) and a position (e.g., $f(t(\text{A}))$).

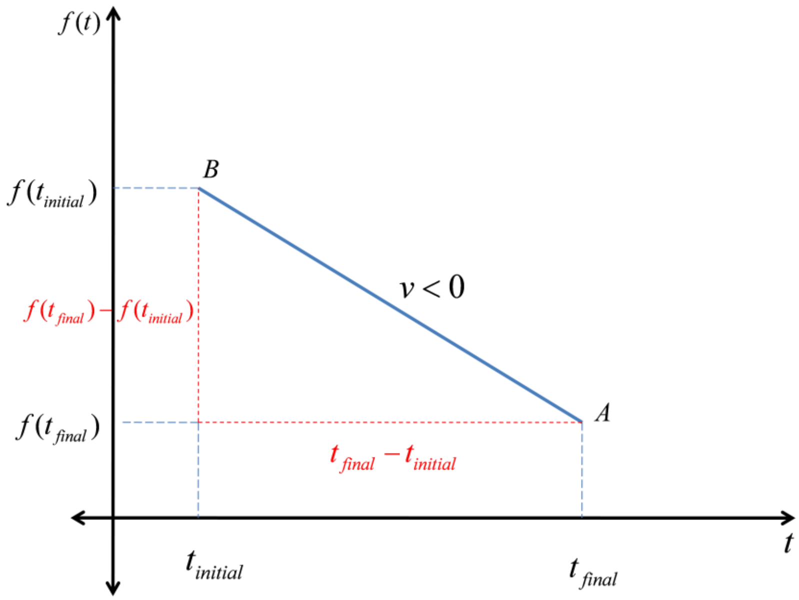

Figure 10 Graph of motion with negative velocity

a) Between time t(A) and time t(M), $v = \frac{f(t(\mathrm{M})) - f(t(\mathrm{N}))}{t(\mathrm{M}) - t(\mathrm{N})} > 0$, and the motion is linear uniform, with positive velocity, just as in the situation described in Figure 9.

b) From time t(M) to time t(N), $v = \frac{f(t(\mathrm{N})) - f(t(\mathrm{M}))}{t(\mathrm{N}) - t(\mathrm{M})} = \frac{0}{t(\mathrm{N}) - t(\mathrm{M})} = 0$. The position does not change; the object is at rest. Its velocity is zero. The time passes (indicated by the event N appearing to the right of the event M), but the particle does not gain any distance, it stays in the same place. This motion (or *non*-motion) is represented by a horizontal line, whose length is determined by the elapsed time.

c) From t(M) to t(Q) the velocity is negative. The object moved toward the origin of the motion, from the last point it reached the plane. It is represented by a line with negative slope, and if it touches the horizontal line at height $f(t_{\text{initial}})$ it means that it came back to the point it started from.[2]

---

[2] It is interesting to think about what it means for the graph to cross the horizontal line: the particle reaches the starting point, and then continues to move back past that point.
Students may have trouble understanding negative velocity, because they confuse the velocity as slope of the line coming toward the origin, and the decreasing velocity to a stop. The first situation addresses the direction, the second addresses the magnitude.

At time t(Q), the velocity changes instantly from negative to positive. In real life, velocity never changes instantly from one value to another. We will expand on this next chapter when dealing with instantaneous rate of change.

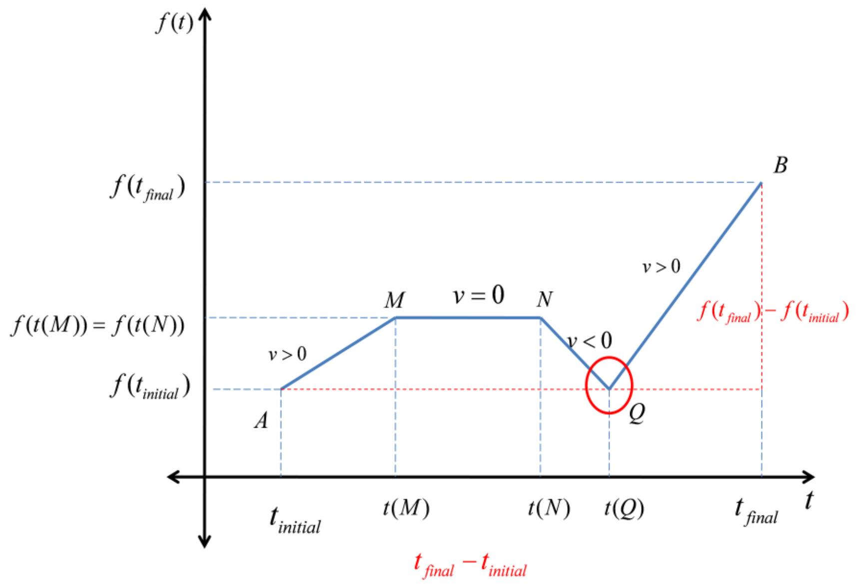

Figure 11 Graph of motion with constant velocities

Let's now consider an even more complex example. Look at Figure 12.

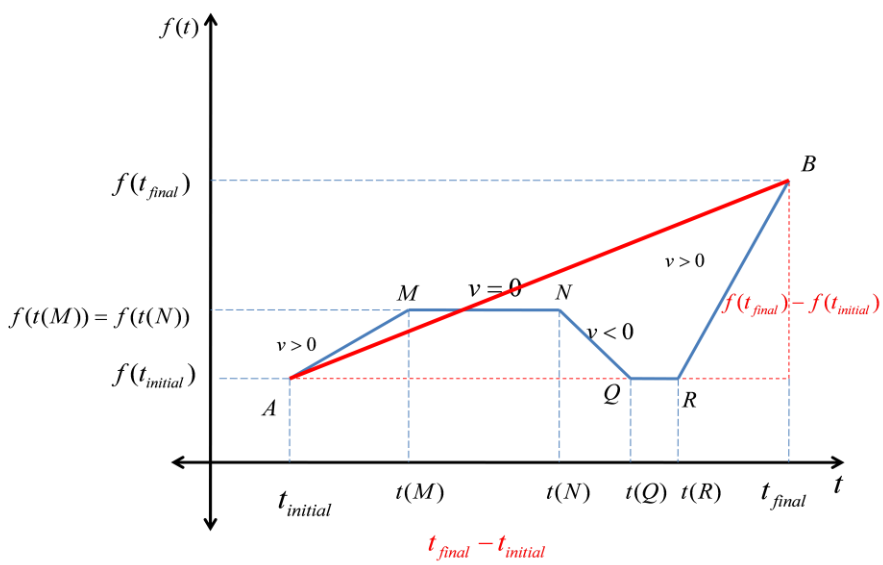

Figure 12. Graph of linear uniform motion

As we can see, there are five different stages to the motion represented by the blue graph (the

crooked line), each of which is a uniform linear motion. The straight red line which has been added to the graph represents a new uniform linear motion that begins and ends at the same times and at the same places as the complex motion. Because they begin at the same time and place and end at the same time and place, both motions have the same average rate of change between $t_{initial}$ and t_{final}.

Let us say more about this situation. If a particle moves smoothly from position $f(t_{initial}) = f(t(\mathrm{A}))$ to position $f(t_{final}) = f(t(\mathrm{B}))$ without any stops, and without increasing or decreasing its velocity, then the slope of the line AB will represent the velocity of this "ideal motion." In other words, the particle starts with a velocity equal to the average velocity v_{AB} and keeps this velocity all the way to the end.

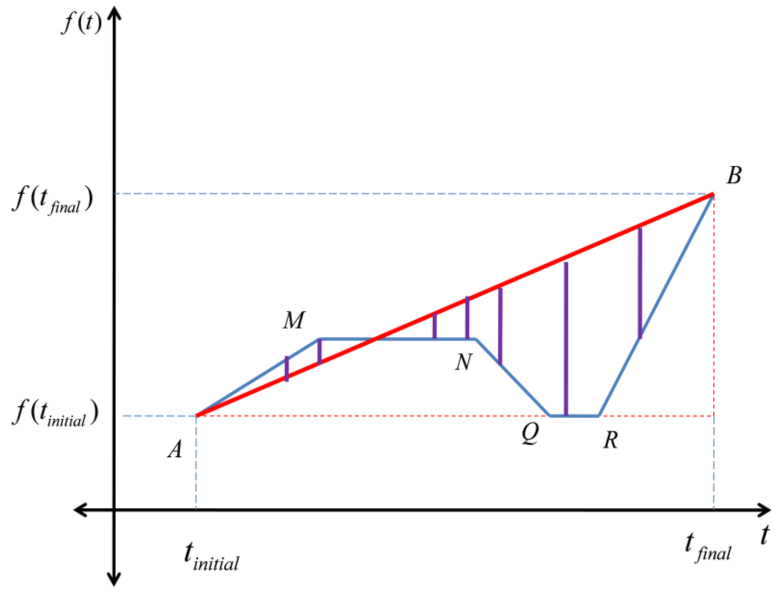

Figure 13 Comparison of complex and simple motions with same average velocity

We can compare the graphs we analyzed so far, to see how well a particle moving with such uniform linear motion approximates the more complex motion. Referring to Figure 13, we can see that the two motions represented are close to each other when the particle moves from t(A) to t(N), but they are quite different during the time interval from t(N) toward t(B). We cannot offer the function whose graph is the segment AB as a good description of the complex motion from N to B: $t(\mathrm{N}) < t < t(\mathrm{B})$.

2) A Numerical Example Involving Average Rate of Change

A person starts driving at 7:30 in the morning to go to his job. He drives for 6 miles, and at 7:42 he stops for about 15 minutes to have some breakfast and refill the tank with gas, spending a total of $42.75. He then realizes that he has forgotten some documents at home. It takes him about 8 minutes to return home, and other 5 minutes to find the papers. After he finds them, he drives towards his office, but he is behind the regular schedule now. He takes the interstate, arriving in his office at 8:30 am. Assume his office is 23 miles away from his home. DO the following:

1) Draw a graph to show the relationship between distance traveled and time. Show on the graph the nature of the rate of change (velocity) for each part of the trip.
2) Plot on the graph significant events, *e.g.*, leave home, arrive gas station, leave gas station, arrive office, *etc*.
3) Find the average rate of change from home to the office, and explain its significance.
4) Discuss each part of the trip, and compare the average velocity from t(home) to t(office)

We can see the sequence of time and distance traveled in Table 1. In Figures 14 and 15, we can see the graphs of average velocities for the simple and complex motion. This is consistent with the mathematical model described before.

Table 1 Distance and Time for Problem 1

location	*t= time*	*distance between two locations*
home	7:30	0
Gas station	7:42	6
Gas station	7:57	6
Home	8:05	0
home	8:10	0
office	8:30	23

At the start of the trip the velocity gradually increases from zero to the travelling velocity, but we will assume that this change takes place so quickly that it appears as a corner in the graph.

We'll also assume that the velocity is nearly constat during each part of the trip, so position is modeled by a straight line. we can see that the velocity itself is a function constant with regard to time; But, if we multiply the value of fuction constant in the median points of each interval, and the width of the time interval, we have the formula for the distance as a function of constant velocity, and time.

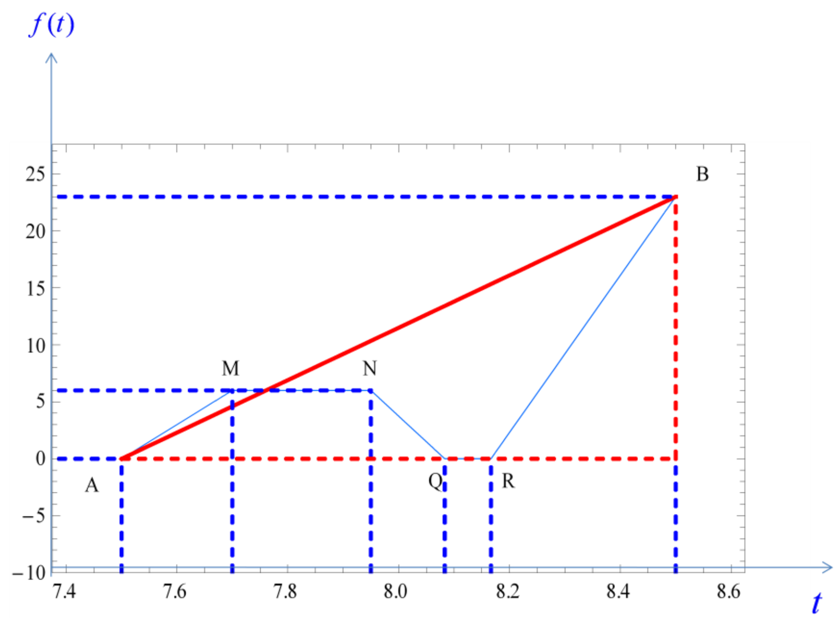

Figure 14. Graph of the motion for problem 1

We can calculate the average velocity as a slope of two points, see Table 2.

Again, we can see the signification of the average velocity, as the velocity of an ideal motion.

Another version to this problem could be: "after the person left the house the second time, he was driving slowly because of the traffic. On the interstate, he could speed up to make up for the time he lost." This will result in two different positive velocities, see the Figure 16.

Table 2 Velocity Calculated as Slope

Position(t, f(t))	*Velocity in miles per hour*
$A(7.5, 0)$	
$M(7.7, 6)$	$\frac{6-0}{7.7-7.5}=30$
$N(7.95, 6)$	$\frac{6-6}{7.95-7.7}=0$
$Q(8.0833, 0)$	$\frac{0-6}{8.0833-7.95}=-45$
$R(8.167, 0)$	$\frac{0-0}{8.167-8.0833}=0$
$B(8.5, 23)$	$\frac{23-0}{8.5-7.5}=23$

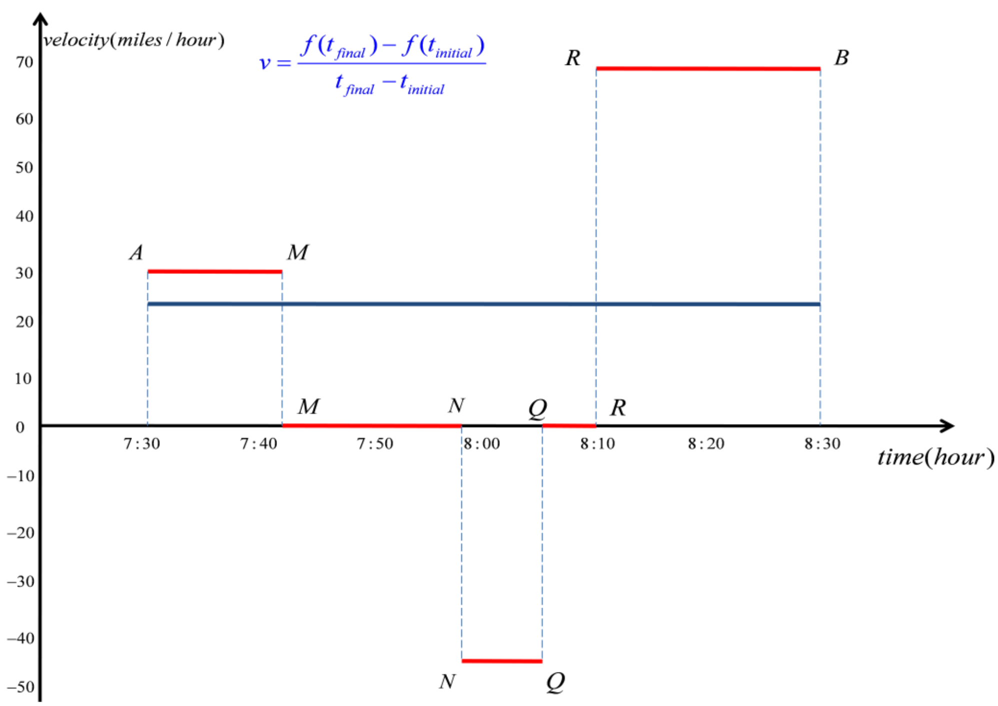

Figure 15. Graph of velocities

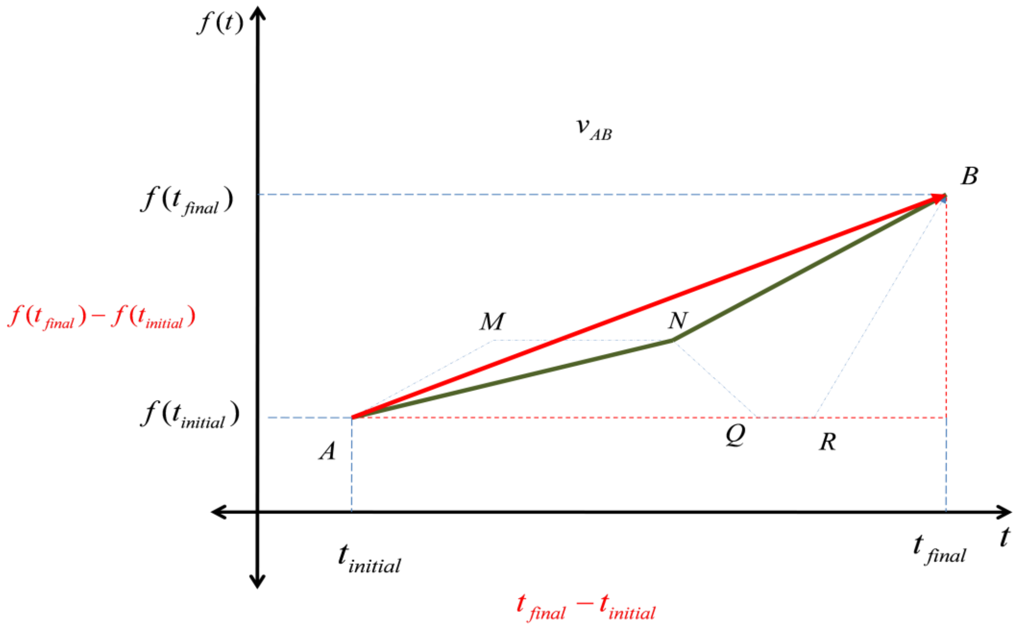

Figure 16. Uniform motion with two positive velocities

Note: We need to make the students aware that when we refer to the *distance* between two points, we refer to the points on the graph determined by t(A) and t(B), and the actual *lengths* are along the y axis. Some students make confusion between the distance traveled by a particle moving from t(A) to t(B), called displacement, and the geometrical segment AB on the graph.

CHAPTER V

INSTANTANEOUS RATE OF CHANGE

It's much easier to point out the problem than it is to say just how it should be solved.

John Kenneth Galbraith

One of the most important concepts in mathematical analysis is the concept of derivative. The foundations of the derivative are attributed to Leibniz and Newton. The derivative has numerous applications in the sciences, but it is challenging for students to understand it.

In this chapter we show how the derivative is related to the concept of average rate of change: it is, in fact a limit of average rates over intervals of decreasing length. After explaining the origin of the concept, we give some illustrations of its use.

1) Theoretical Approach

In the previous chapter we analyzed the average rate of change for a linear function, and we observed that the concept does not model other functions well. We observed that the average rate of change is the slope of the graph, when the graph is a line, but when the graph is more complex, the average rate of change may not be a slope visible in the graph.

The geometric description of a tangent in Euclid's geometry, Book III, Definition 2: "A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle." In Figure 17, below, we have the visual representation of the tangent as it was described by Euclid; the tangent t intersects C only once, in P.

Now, let us take a more complicated curve, not a circle, and see if the Euclid's definition is still valid; see Figure 18: As we can see, there are two lines through the point P. Line l intersects curve C only once, in point P, but it is a it is a secant, not a tangent. Line t, on the other hand, is tangent to the curve at point P, but it intersects the curve C twice.

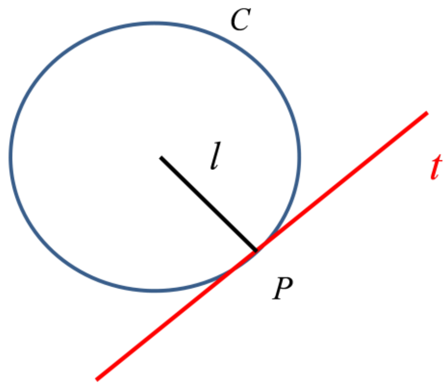

Figure 17. Euclid's description of a tangent

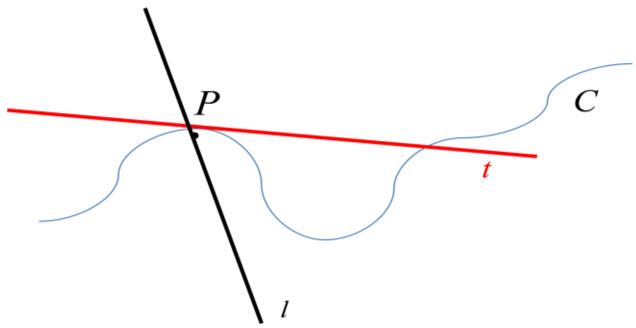

Figure 18. Tangent to a curve

Now the tangent to a circle is easy to find. By Euclid Book III, Proposition 18, a line is tangent to a circle at P if and only if it is perpendicular to the radius to P. But in the case of the curve, there are no obvious radii. The question is, by what properties doe we recognize the tangent to a curve, and how can we find it?

Following Leibniz's approach to this problem, let us consider a curve in the coordinate plane given as the geometric graph of an equation in x and y . Consider a fixed point P on the curve, and the tangent to the curve in that fixed point, as in the Figure 19. We assume that the tangent is not vertical. Let its slope be m_P . Now, on the curve C, take another point Q, such that Q and P are distinct, and draw a line through both points. The line is a secant to the curve. Finally, imagine that we move point Q on the curve in such a way that Q gets closer to the fixed point P. The secant rotates about P accordingly, getting closer and closer to the tangent, see figure 20.

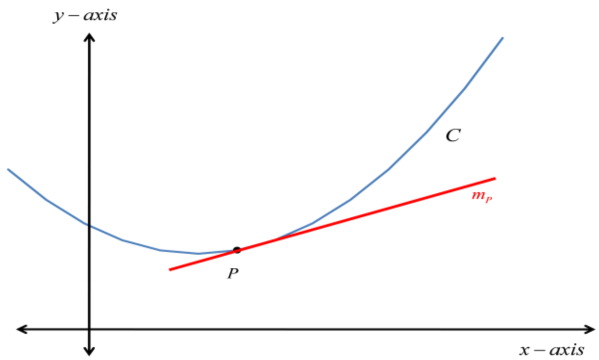

Figure 19. Tangent to a curve; Leibniz approach

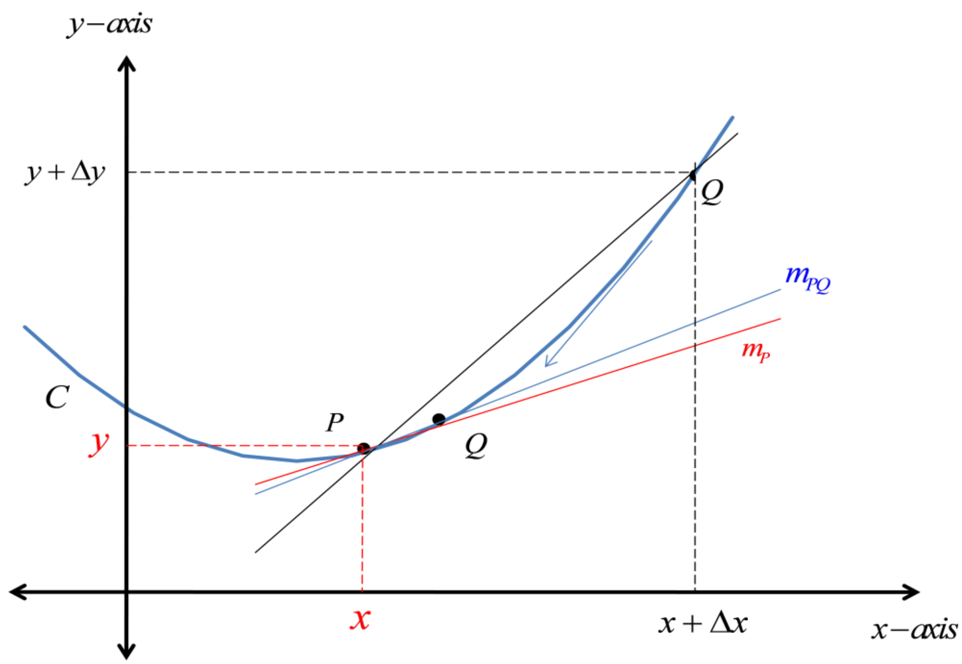

Figure 20. Graph of secants approaching tangent

We can say that as Q gets closer to P, the slope of the line PQ takes values that are closer to the value of the slope of the tangent at P.

We now describe an algebraic approach to finding m_P. We will considered two points P, with coordinates (x, y), and Q with coordinates $(x+\Delta x, y+\Delta y)$. T he differences between the x-coordinates, and y-coordinates are:

$$\Delta x = |x - (x + \Delta x)|, \text{ and } \Delta y = |y - (y + \Delta y)|$$

We can find the slope of the line PQ as follows:

$$m_{\mathrm{PQ}} = \frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \frac{\Delta y}{\Delta x}$$

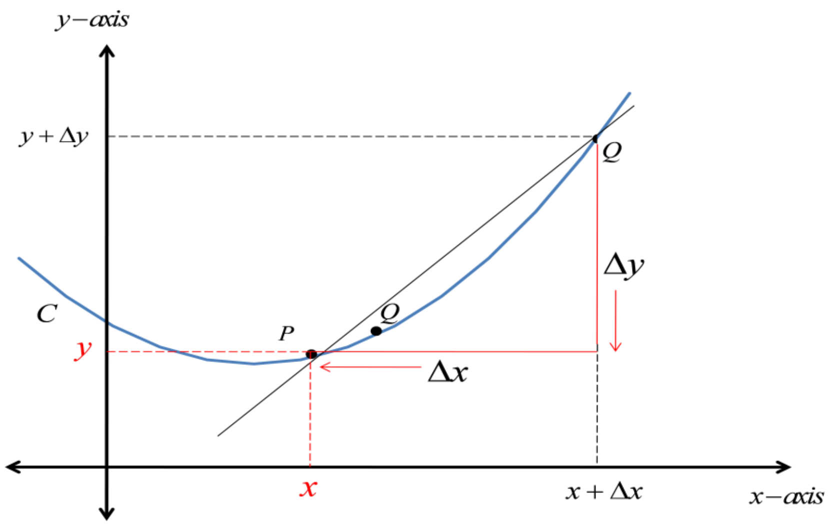

Figure 21 Graph of Δ*x*, *and* Δ*y*

Now, as point Q gets closer to the point P, the difference between the coordinates decreases $\Delta x \rightarrow 0$, and the changing number m_{PQ} gets closer to the fixed number m_{P}.

Leibnitz created a notation to show the process, Figure 21:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = m_P$$

Note that Δx decreases toward 0 , but is never equal to 0. If Δx becomes zero, then Δy is 0 as well; P =Q, and there is only one point, not two.

The payoff is that this enables us to find the slope of the tangent line at any point P(*x*, *y*) on the curve. Since its slope is the limiting value of a family of average rates of change, all measured from the same starting point but extending over shorter and shorter intervals, the slope can be viewed as the instantaneous rate of change for the curve around the chosen point.

Let us see how this works for a parabola. We will consider a parabola with a general equation $y = Ax^2 + Bx + C$, where *A*, *B*, *C* are real numbers. Then, $\Delta y = 2Ax\Delta x + A\Delta x^2 + B\Delta x$, and the slope of the tangent will be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2Ax\Delta x + A\Delta x^2 + B\Delta x}{\Delta x} = 2Ax + B$$

To find the slope, we have an algebraic procedure, and we know that it is consistent with the geometrical representation.

Up to this time, we have been assuming that the tangent was obviously "there to be found' and that the only problem was to compute its slope. But now that we have a procedure for computing the slope, we can dispense with this assumption, and simply use the procedure to define the tangent.

Definition. *The tangent line to the curve (a, f(a)) at a point* P*(a, f(a)) is a line through the point* P *with a slope* $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Definition. T*he derivative of a function f at a number a, is* $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Now, if we have a fixed number *a* and a function *f*, we can find the equation of the tangent to the graph of *f* at the point (*a*, *f*(*a*)):

$$y - f(a) = f'(a)(x - a)$$

In summary, the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through that point whose slope is equal to $f'(a)$, the derivative of function f at a . The quantity, $f'(a)$, is the instantaneous rate of change of the function $y = f(x)$ with respect to x when $x = a$.

2) Applications to Instantaneous Rate of Change

a. Galilei Project

As we mentioned, the technological revolution brought with it a lot of problems that needed a new mathematical model. Some of the problems were old problems, but they needed answers that fit reality, and allowed for prediction of future behavior. One of these problems was the path of a canon ball. Aristotle created a model for the canon ball path based on linear paths, but this model was just plain wrong. Galileo Galilei set up a device to study accelerated motion on an inclined plane, and he used the same device to study the projectile motion. He was able to determine that the path of the projectile is parabolic.

In the effort to understand the curve, several questions arose. What would be the maximum height? For what launch angle would the ball travel furthest? What will the velocity at the point of inpact be?

In the Figure 22 we can see height of the projectile represnted as a function of time. Interesting, this graph also shows the shape of the trajectory, since the horizontal position is proportional to the time, since no force acts horizontally.

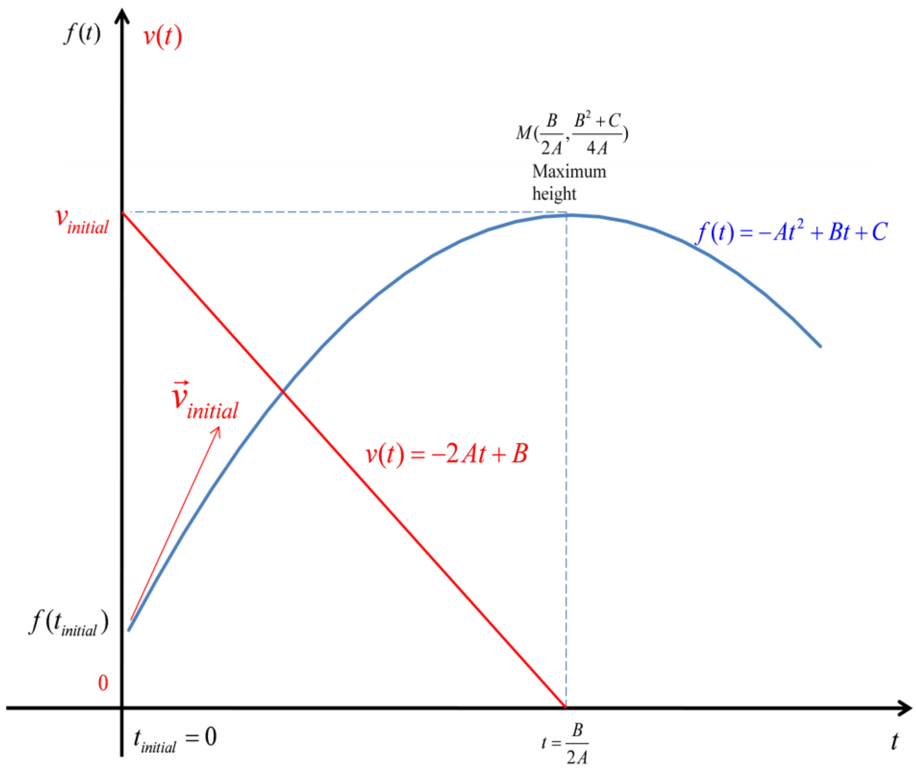

Figure 22. Trajectory of a projectile

Let us consider a projectile launched at the time t_{initial}, $= 0$ with an initial vertical velocity, v_{initial} > 0. The function that gives the height is $f(t) = -Ax^2 + Bx + C$ for some constants A, B and C. We want to find out at what specific time, t, the projectile reaches the maximum height. For this, we will calculate the derivative of f(t) to find the vertical velocity:

$$v(t) = f'(t) = \lim_{\Delta t \to 0} \frac{\Delta t(-2At - A\Delta t + B)}{\Delta t} = -2At + B$$

At the maximum point of the trajectory, the projectile will change the sense of motion, but this is possible only after the velocity goes through zero. Therefore, at the top most point of the curve, the velocity will be zero. If we input this condition into the equation for velocity, we get

$$0 = -2At + B$$

and from there we can find $t = \frac{B}{2A}$

We can see that the velocity itself is a function of time. We can measure the velocity's rate of change. This is called the acceleration. In this problem, the acceleration is

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = -2A.$$

The acceleration is negative for the case considered, because actually this would be the gravitational acceleration, which is oriented toward the center of the Earth. Otherwise, the projectile would fly away, instead of coming back toward the ground.

We can find the maximum height reached. We found that this occurs when the independent variable has value $\frac{B}{2A}$. By the equation of the motion

$$f(t) = -At^2 + Bt + C,$$

$$f\left(\frac{B}{2A}\right) = -A\left(\frac{B}{2A}\right)^2 + B\left(\frac{B}{2A}\right) + C$$

$$f_{maximum\ height} = \frac{B^2 + C}{4A}.$$

Eventually, the projectile will reach the ground, and we want to find the vertical velocity at that instant. The projectile will reach the ground when the value of f is 0. Because $f(t) = -At^2 + Bt + C$ we can find the two possible roots. The larger represents the time of impact. We simply find *f'* at this time.

b. Numerical Model for Galilei Problem

Let consider the motion of the projectile described above, $f(t) = -At^2 + Bt + C$, with initial velocity of $10\,\frac{m}{s}$ and acceleration for free fall $9.8\frac{m}{s^2}$. For simplicity, we consider the beginning of the motions when the timer is zero, and without prior elevation. We do not know the time of landing, but we assume the equation above describes the motion prior to that moment.
The velocity of the projectile at a moment of time $0 < t_{landing}$ will be the instantaneous rate of change of $f(t)$ with respect to time, and it is the derivative of the function $f(t)$ in t

$$f'(t) = -2At + B =$$
$$-2\left(\frac{9.8}{2}\frac{m}{s^2}\right)t + 10\frac{m}{s} =$$
$$-9.8\frac{m}{s^2}t + 10\frac{m}{s}$$

We can make a graphical representation of this function, see Figure 23. The solid red line is the graph of the height function $f(t)$ We can see a maxim height at the midpoint of the graph. The function has two zeros: one at the initial moment and the other one when the bullet reaches the ground.

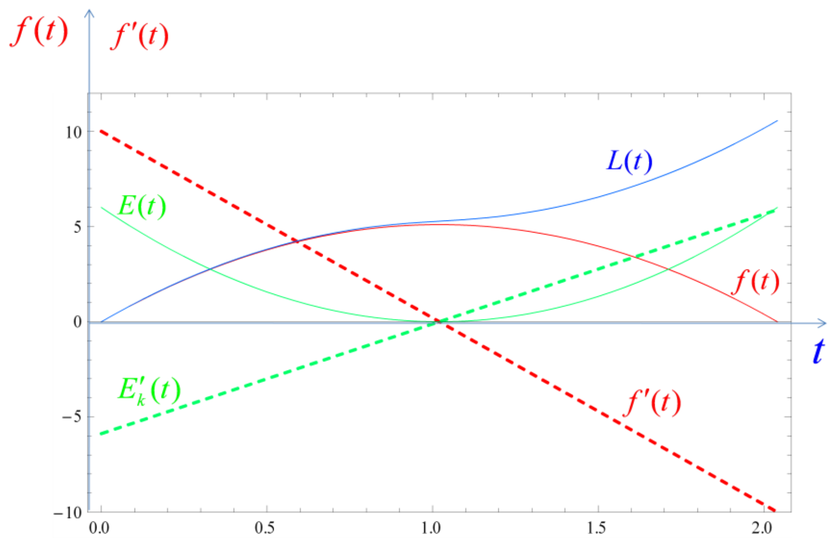

Figure 23. Trajectory and velocity in Galilei problem

The solid blue line is the graph of the function that gives the total vertical distance travelled. We add the length of the vertical trajectory going up, and then coming back down. We can see that it slows down on accumulating length around the maximum point of the trajectory. The dashed red line represents the velocity function $f'(t)$ It has a maximum magnitude in the initial moment, then its magnitude decreases to zero at the moment the highest point of the trajectory is reached.

c. Fencing a Garden

Problem. A gardener has $P=140$ ft. of material to fence a rectangular vegetable garden.

1. Find a function to model the area of the garden, given the width.
2. For what width is the area greater than 825 ft^2 ($A>825 \text{ ft}^2$)
3. Can the gardener fence a garden with area $A= 1250\text{ft}^2$?
4. Find the dimensions of the largest area the gardener can fence.

Solution. Let us consider a and b to be the width and length of the garden in feet. They are unknown, but we know a relationship that connect them. $140 = 2a + 2b$, so $b = 70 - a$. In the Figure 24 we have a visual representation of the garden. As requested in part 1 of the problem, we have to find a function to represent the area for different possibilities of fencing, keeping constant the length of the fence. It means that the gardener will not buy extra material, and he will use the whole fence. The area will be

$$A = a\, b = a\,(70 - a).$$

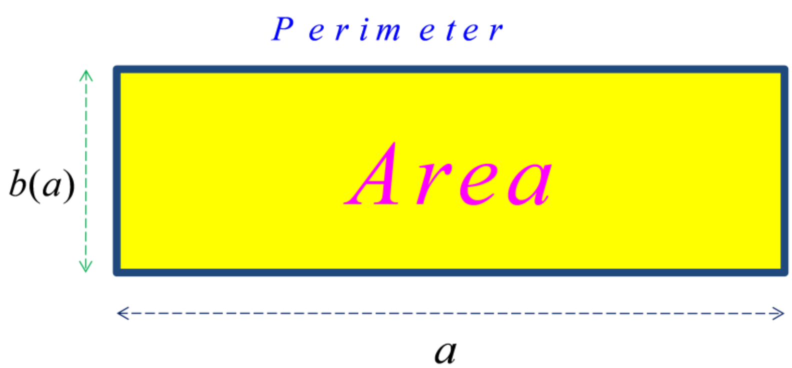

Figure 24. Visual representation of the garden

Now, we have the area as a function of one of the side, and we can graph it. See Figure 25. Now let us consider part 2. Here, we have another restriction, $A \geq 1250\text{ft}^2$ That means we keep the initial conditions, and look on the graph for areas that are larger than 825 ft^2 .

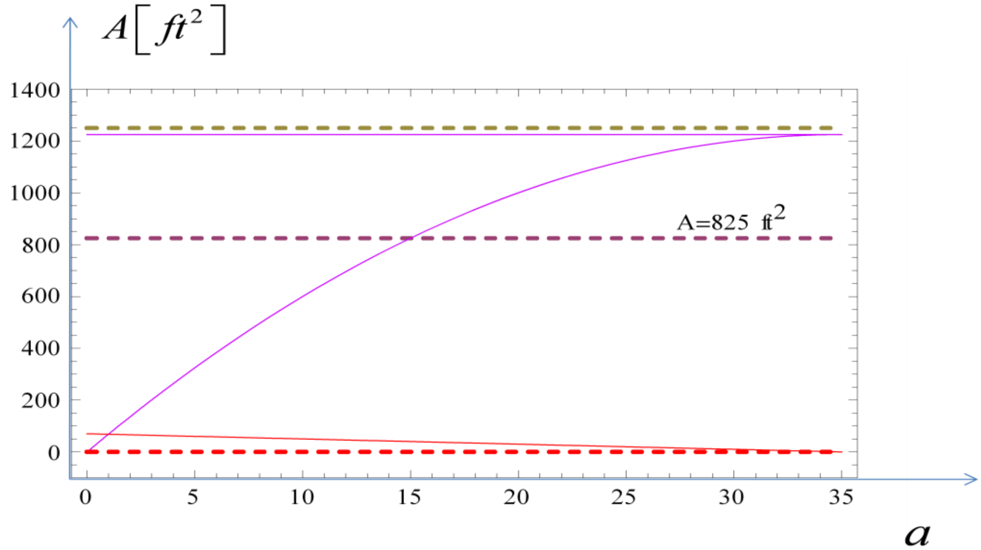

Figure 25. The area greater than 825 ft^2

Analytically, we can calculate interval values for a by setting up an inequality:

$$a\,(70 - a) \geq 825.$$

From it, we find $15 < a < 55$, and if we are looking on the graph, this is consistent with the shape of the curve. To answer part 3 is easy now, because we can see that the maximum of the curve is below 1200. Finally, we address part 4. We can find the maximum area analytically, using the derivative. From the condition $\frac{dA}{da} = 0$ or for maximum value there is a minimum variation of area with regard to a, we obtain $\frac{dA}{da} = 0 \rightarrow \frac{d(-a^2+70a)}{da} = -2a + 70$ and $a = 35$. If one of the sides is a quarter of perimeter it means that the garden is a square. So, the maximum area is obtained when the garden is a square.

CONCLUSIONS

Measurement, geometry, proportional reasoning, coordinates, graphing, functions rate of change and derivatives are major themes in the mathematics curriculum. In this thesis, we have illustrated some deep connections between these themes. Geometry is connected to measurement via the distance function, a basic fact of contemporary geometry. To justify the distance function to students, it is useful to refer to ideas that first become clear when we look at measurement abstractly and recognize the need for a unit. The coordinate plane is the bridge to algebraic formalism, based on the powerful geometric concept of similarity and its consequence, the Pythagorean Theorem.

There is great emphasis on linear functions in the high school curriculum, because they are a gateway to calculus. They are easy to understand, but have a restricted applicability, and limited capacity for modeling real life situations. When we teach our students, we want to inspire them to learn mathematics concepts with deep understanding, all the while making meaningful connections with the real world. For that, every piece of mathematical information that we offer to our students should be presented together with its conceptual foundations, and with an eye to real life application.

This material presented in this thesis invites one to teach toward conceptual understanding, procedural fluency, and strategic competence. Only quality instruction can shape our students for society's demands. Today's students will be engaged in many different jobs, for many of which they will need to master current and emerging technologies. In order to achieve their career goals, our students will need to prove themselves proficient in problem-solving, and creative, independent, and imaginative.

Teaching mathematics in connection to sciences creates a base for our students to understand physical concepts through rigorous mathematical models, and to be able to apply them to the real life situations offered by the sciences. There are a lot of powerful examples outside of our classroom. We only need to bring them inside in the proper way, dissect ideas to the point of understanding, and show strong bridges between concept and application. Then we will be able to offer our students the knowledge and independence they need to participate productively in future scientific work.

The reality in our schools shows that one of the greatest concerns for students and teachers is passing the standardized tests in mathematics and science. This brings up the

question: where are the gaps in current practice? We are all aware that there are many gaps we need to close when speaking about education. I consider one of them is between teaching to master procedures. Are the assessments we are using really designed to promote proficiency and concept understanding?

Having stated my conclusions, allow me to reflect on the state of education. Students need mind openers to see the reality that is behind the stressed formulas. The final, published formula is itself the product of reasoning, based on theories and their proof. We cannot shorten the understanding process by giving our students the final product. We have to bring them to an understanding of the building blocks before they are able to use the formal tools for other applications.

Even students who are relatively proficient in classroom work are often not able to apply mathematics to problems in the science class. They are stuck using numerical examples, and are unable to make verbal, functional and graphical representations. We have to help our students to see the applicability of mathematical models and become proficient in using them. Many times, teaching through modeling takes short cuts in the classroom, under the assumption that the students have understood the concept if they have mastered the process. Reality shows us that this is assumption is wrong. The process is a succession of steps. We as teachers have to model the process thoroughly, building clear, simple, yet rigorous understanding blocks.

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VITA

Maria Ludu was born in December 1957, in Rimnicu Vilcea, Romania.
She graduated high school in 1977 at High School No. 2, in Rimnicu Vilcea.
Mrs. Ludu has a Bachelor in Science from Bucharest University, College of Physics.
In 1980, she married Andrei Ludu, and in 1988 they had their daughter, Delia Ludu.
Mrs. Ludu started to teach in Romania, in 1983;
In 1998, she and her family moved permanently to the U.S.
From 2001-2007, she taught mathematics in East Baton Rouge parish, when she worked as Math. Specialist at Capitol High., where she participated in the “Vertical Alignment Project”.
In 2008 she was promoted to Math Coordinator/Trainer position.
From 2007-2009, she is a graduate student at Louisiana State University, for a Master of Natural Sciences.
Maria’s philosophy, as a teacher, is that the challenges in teaching mathematics and sciences include training student’s attention, developing skills and the pursuit of problem-solving. Therefore, it is necessary that students be challenged and trained to maintain high self-expectations and competition in their own environment. As a Mathematics Coordinator, Mrs. Ludu has the opportunity to work on structuring and developing the activity of the department of mathematics. She helps the teachers to model their lessons and find appropriate ways to differentiate instruction such that each student could be assisted to their needs, while maintaining the quality of instruction, and high level expectations
Maria considers that rising teachers’ ability to understand students’ deficiencies and misconceptions is the first step on building more effective ways of teaching.
Maria Ludu currently resides in Baton Rouge, Louisiana.