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Twisted Reflection Positivity

Mostafa Ahmad Hayajneh
Louisiana State University and Agricultural and Mechanical College

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Abstract

Reflection positivity has several applications in both mathematics and physics. For example, reflection positivity induces a duality between group representations.

In this thesis, we coin a new definition for a new kind of reflection positivity, namely, twisted reflection positive representation on a vector space. We show that all of the non-compactly causal symmetric spaces give rise to twisted reflection positive representations. We discover examples of twisted reflection positive representations on the sphere and on the Grassmannian manifold which are not unitary, namely, the generalized principle series with the Cos^{λ} transform as an intertwining operator. We give a direct proof for the reflection positivity of the Cos^{λ} transform on $\mathrm{SO}(n)$. On the other hand, we generalize an integrability theorem to the case of non-positive definite distribution. As a result, we give a relation between the non-compactly causal symmetric spaces and the reflection positive distributions. Cocycle conditions are also treated.

We construct a general method to generate twisted reflection positive representations and then we apply it to get twisted reflection positive representations on the Euclidean space.

Finally, we introduce a reflection positive cyclic distribution vector for the circle case. Then we prove that this distribution vector generates a well known reflection positive function.

Chapter 1
Introduction

In functional analysis, many subjects, objects, and theorems are motivated by physics problems. Several problems in the theory of functional analysis have their origin in physics. They start as physics problems and ending up to very important theories in functional analysis. During the process, the physics origin starts to disappear and instead it gets an independent mathematical reasoning. Operator theory, distribution theory and representation theory are examples of this situation. *Reflection positivity*, which is the subject of this thesis, can be considered also as an example that fits in this context.

What is reflection positivity? What do we mean, for example, by reflection positive Hilbert space? What does the adjective "reflection positive" add to the term "Hilbert space"? In short, from mathematical point of view, it is an additional structure that can be added to the original Hilbert space structure, thus creating a new Hilbert space from the first one. This new Hilbert space can be considered as an intrinsic structure or something like a dual for the original one. This of course can be said if we replace the term "Hilbert space" by objects like representations, functions, or distributions. To go deeply in answering these questions, we need to return back to their physics origin.

To approach reflection positivity from the physics point of view, we need to know that relativity and quantum mechanics are two concepts that dominate contemporary physics. Therefore one may ask the question: Are relativity and quantum

theory mathematically compatible? This question was asked by Arthur Jaffe in an article titled *Quantum Theory and Relativity*, see [9]. Unfortunately, in the same article he mentioned that the answer of this question is still unknown about our four-dimensional world.

Constructive quantum field theory is an important topic in physics. It is the framework where such kind of questions can be answered. We recall that in the n-dimensional Euclidean space, the group of affine isometries is $\mathcal{E}_n = \mathrm{O}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and in the n-dimensional Minkowski space (spacetime), the group of affine isometries is the Poincare group $P_n = \mathrm{O}^+_{1,n-1}(\mathbb{R}) \ltimes \mathbb{R}^{1,n-1}$. A quantum field theory consists of two central mathematical objects, namely, a representation $\pi : P_n \to \mathrm{End}(\mathcal{H})$ of the Poincare group and an operator-valued distribution $\varphi : \mathcal{S} \to \mathrm{End}(\mathcal{H})$ called a quantum field operator. Here $\mathcal{H}$ is a Hilbert space of physical states and $\mathcal{S}$ is a Schwartz space. In 1952, Whitman set up his axioms where π and φ must satisfy:

- The representation π is unitary and positive-energy.
- There exists an invariant, vacuum-vector $\Omega = \pi(P_n)\Omega \in \mathcal{H}$.
- Vectors of the form $\varphi(f_1)\cdots\varphi(f_n)\Omega$, for $f_i \in S$ and arbitrary n, span $\mathcal{H}$.
- The field φ transforms covariantly under π.
- The field φ is local.
- The space of invariant vectors Ω is one-dimensional.

Can we pass from quantum field theories to a theory of Euclidean fields, i.e., a theory of generalized functions with Euclidean symmetries? In other words, can

a representation of P_n be analytically continued to a representation of $\mathcal{E}_n$? Recall that the group $\mathrm{O}^+_{1,n-1}(\mathbb{R})$ preserves the Lorentz form $(t,x) \to t^2 - \|x\|^2$ and so maps the forward light cone $\{(t,x) \mid t^2 - \|x\|^2 > 0, t > 0\}$ onto itself. Notice that if we pass to imaginary time $t \to it$, then the Lorentzian signature becomes Euclidean, i.e., the Lorentz form becomes $-t^2 - \|x\|^2 = -\|(t,x)\|^2$. Therefore the answer is yes we can. This was known in the early 1950s to Arthur Wightman and others. So what about the other direction? Can we find sufficient conditions on Euclidean fields so that we can continue them analytically to a mathematical quantum field theory? In their two articles titled *Axioms for Euclidean Green's functions 1, 2*, see [28], [29], Konrad Osterwalder and Robert Schrader made the answer to the previous question possible by setting up their axioms:

- A regularity assumption.
- Euclidean covariance.
- Reflection positivity.
- Clustering.

These axioms were a breakthrough because they made the analytic continuation possible. More precisely, any Euclidean field theory that satisfies Osterwalder-Schrader axioms can be analytically continued to a mathematical quantum field theory. It was proved that Osterwalder-Schrader axioms are equivalent to Wightman axioms. Now, using this approach we can construct mathematical quantum field theories from Euclidean ones.

The reflection positivity axiom is a very important condition because if we provide any Euclidean field theory with a reflection positive bilinear form, then it can be analytically continued to a mathematical quantum field theory. Nowadays, reflection positivity condition has several applications outside the context of quantum theory. It plays a role in establishing fundamental inequalities; it has an application in statistical physics. For further references on the physical side of reflection positivity, see [16], [17], [10], [11], [1]. The article [9] is an excellent introduction to the origin of reflection positivity.

From a purely mathematical point of view, reflection positivity allows us to prove the analytic continuation of group representations when the groups are P_n and $\mathcal{E}_n$. What about the other groups? Can we use the reflection positivity insight to pass between general group representations? More precisely, let G be a Lie group, H a closed subgroup, and τ an involutive automorphism with H as fixed point subgroup. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the corresponding real symmetric Lie algebra. Then $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ is also a real Lie algebra. Let G^c be the simply connected Lie group with the Lie algebra $\mathfrak{g}^c$. If $\pi : G \to \mathrm{End}(\mathcal{E})$ is some nice representation (possibly unitary), then can we construct a continuous unitary representation $\pi^c : G^c \to \mathrm{End}(\widehat{\mathcal{E}})$ such that for the differentiated representations $d\pi$ and $d\pi^c$ we have

(1) $d\pi^c(X) = d\pi(X) \quad \forall X \in \mathfrak{h}$,

(2) $d\pi^c(iY) = id\pi(Y) \quad \forall Y \in \mathfrak{q}$?

The answer to this question led to the Lüscher-Mack theorem which is a noncommutative version of the Hille-Yosida theorem [22], where in functional analysis, the Hille-Yosida theorem characterizes the generators of strongly continuous one-

parameter semigroups of linear operators on Banach spaces. It also led to coining some new notions such as the virtual representation [4] which is called also the local representation, see for example [19], [20], [12], [13], [30], [31]. Reflection positivity conditions are a certain set of conditions imposed on the representation π and on the space $\mathcal{E}$, helping us to construct the space $\widehat{\mathcal{E}}$ and the representation π^c. This application of reflection positivity is the subject of this thesis.

In their remarkable papers [14] and [15], Jorgensen and Ólafsson give us a family of reflection positive representations on the Cayley type symmetric spaces. Thus they were able to use the Lüscher-Mack theorem to analytically continue them to representations on the dual group G^c. Unfortunately, the definition of the reflection positivity there depends on the existence of the Weyl group element w for the Cayley type symmetric spaces. What about the general non-compactly causal symmetric spaces, i.e., the spaces that might not have a Weyl group element w?

In this thesis, we notice some kind of reflection positivity which is still shining there! We coin a new definition for such new kind of reflection positivity, namely, *twisted reflection positive representation on a vector space*. We show that all of the non compactly causal symmetric spaces give rise to θ-twisted reflection positive representations, where θ is the Cartan involution of the corresponding group. In other words, we produce a rich family of twisted reflection positive representations which can be divided into unitary and non-unitary representations. The main tools that will be used to perform the analytic continuation are the Lüscher-Mack theorem and some new integrability theorems introduced in [23].

Amazingly, we connect the results in [27] and [14] together. Then we discover very nice examples of θ-twisted reflection positive representations on the sphere and on the Grassmannian manifold which are not unitary, namely, the generalized principle series with Cos^{λ} transform as an intertwining operator. The Cos^{λ} transform gives us the θ-twist, where θ is the Cartan involution of the corresponding group. This representation can be viewed on both $\mathrm{SO}(n)$ and $\mathrm{SL}(n, \mathbb{R})$. In addition to the indirect proof, we give a direct proof for the reflection positivity of the Cos^{λ} transform on $\mathrm{SO}(3)$ and then we generalize the proof to all $n \geq 3$.

On the other hand, we generalize the integrability theorem given in [23] to the case of non-positive definite distribution. As a result, we give a relation between the non-compactly causal symmetric spaces and the reflection positive distributions. In other words, we build up a θ-twisted reflection positive representation for those spaces using the language of distributions. Cocycle conditions are also treated.

In addition to the non-compactly symmetric spaces examples, we construct a general method to generate σ-twisted reflection positive representations and then we apply it to get σ-twisted reflection positive representations on $\mathbb{R}^n$.

Finally, we introduce a reflection positive cyclic distribution vector for the circle case. Then we prove that this distribution vector generates the well known reflection positive function, see [19] and [25], given by

$$g_{\lambda}(x) = e^{-x\lambda} + e^{-(\beta - x)\lambda}.$$

This thesis is organized as follows: Chapter 2 is devoted to give the background and the basic definitions that will be used frequently throughout this thesis. In Chapter 3, we introduce two important tools which help us to do the analytic continuation, namely, the Lüscher-Mack Theorem, and an integrability theorem introduced in [23]. In Chapter 4, we introduce very effective tools which were introduced in [24], namely, the reflection positive functions and the reflection positive distributions. We conclude this chapter with some examples. Chapter 5 is devoted to present some theorems given in [14]. Some of these theorems are very important in proving the main results of this thesis. In Chapter 6, we introduce the Cos^{λ} transform and we collect some theorems from [27]. Chapter 7 is devoted to the main results of this thesis.

Chapter 2
Definitions and Preliminaries

In this chapter, we introduce the basic definitions that will be used frequently throughout this thesis. The material of this chapter can be found, for example, in [2], and [21].

2.1 Lie Groups and Lie Algebras

Lie groups play an important role in several fields of mathematics and physics such as harmonic analysis, representation theory, geometry and quantum mechanics. They establish relations between many area of mathematics as algebra, geometry, topology and analysis. Actually, they are the core for many of these fields.

This thesis is based on the concept of the Lie group. However, in many theorems and definitions, it is sufficient to use a more general concept. So, let us introduce the definition of a *topological group* and then the definition of a *Lie group*.

Definition 2.1. *A topological group is a group equipped with a topology such that the maps*

$$\begin{aligned} (x,y) &\mapsto xy, \quad G \times G \to G, \\ x &\mapsto x^{-1}, \quad G \to G, \end{aligned}$$

are continuous.

Definition 2.2. *A Lie group is a group and a smooth manifold such that the maps*

$$\begin{aligned} (x,y) &\mapsto xy, \quad G \times G \to G, \\ x &\mapsto x^{-1}, \quad G \to G, \end{aligned}$$

are smooth.

Later on, we will see that every Lie group is associated with a vector space that has an additional structure. So, let us introduce the definition of a *Lie algebra.*

Definition 2.3. *A Lie algebra over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *is a vector space* $\mathfrak{g}$ *equipped with a bilinear map*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X, Y) \mapsto [X, Y],$$

satisfying

$$[X, Y] = -[Y, X],$$
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Definition 2.4. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be Lie algebras. A linear map* $g : \mathfrak{g} \to \mathfrak{h}$ *is called a Lie algebra homomorphism if*

$$g([X, Y]) = [g(X), g(Y)], \ \forall X, Y \in \mathfrak{g}.$$

2.2 Semisimple Lie Groups and Semisimple Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra. Then a Lie *subalgebra* $\mathfrak{h}$ is a subspace with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An *ideal* $\mathfrak{h}$ is a subspace with $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. We say $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

The *killing form* $B(X, Y)$ on $\mathfrak{g}$ is the symmetric bilinear form given by $B(X, Y) = \mathrm{Tr}(\mathrm{ad}\ X\ \mathrm{ad}\ Y)$, where $[\mathrm{ad}\ X](Z) := [X, Z]$.

An *involution* on $\mathfrak{g}$ is a Lie algebra automorphism θ on $\mathfrak{g}$ if $\theta^2 = \mathrm{id}$. If $B_\theta(X, Y) := -B(X, \theta Y)$ is a positive definite bilinear form, then we call θ a Cartan involution on $\mathfrak{g}$.

Definition 2.5. *A finite-dimensional Lie algebra* $\mathfrak{g}$ *is called simple if it is non-abelian and has no proper nonzero ideals.*

Definition 2.6. *A finite-dimensional Lie algebra* $\mathfrak{g}$ *is called semisimple if the killing form for* $\mathfrak{g}$ *is nondegenerate. We call a real Lie group semisimple if its Lie algebra is semisimple.*

Theorem 2.7. *The Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ *with the* $\mathfrak{g}_i$*'s ideals that are each simple Lie algebras.*

Proof. See [21, p. 670]. □

2.3 Examples of Lie Groups and Lie Algebras

Several Lie groups and Lie algebras have been extensively studied in the literature. We introduce below some of them. These groups are important objects that will be used throughout this thesis.

Let $\mathrm{M}(n, \mathbb{R})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{R}$. Let $\mathrm{GL}(n, \mathbb{R})$ be the group of invertible matrices in $\mathrm{M}(n, \mathbb{R})$. We call this group the *linear group*. For $A \in \mathrm{M}(n, \mathbb{R})$, define the norm as

$$\|A\| = \sup_{x \in \mathbb{R}^n, \ \|x\| \leq 1} \|Ax\|,$$

where

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Theorem 2.8. *The group* $\mathrm{GL}(n, \mathbb{R})$*, equipped with the topology inherited from* $\mathrm{M}(n, \mathbb{R})$*, is a topological group.*

Proof. See [2, p. 4]. □

Definition 2.9. *By a linear Lie group, we mean a closed subgroup* $G \subseteq \mathrm{GL}(n, \mathbb{R})$*. For the group* G*, we associate the set*

$$\mathfrak{g} = \mathbf{L}(G) = \{X \in \mathrm{M}(n, \mathbb{R}) | \ \exp(\mathbb{R}X) \subseteq G\},$$

called the Lie algebra of G*.*

Now, we turn to introduce some classical examples for linear Lie groups. Usually, our theorems will be based on these examples.

Example 2.10. *Let* $\mathrm{SL}(n,\mathbb{R})$ *stand for the special linear group defined by*

$$\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) | \det g = 1\}.$$

We can see that it is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$.

Example 2.11. *Let* $\mathrm{O}(n)$ *stand for the orthogonal group defined by*

$$\mathrm{O}(n) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n, \langle gx, gy\rangle = \langle x, y\rangle\},$$

where $\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n$. *We can see that it is a compact subgroup of* $\mathrm{GL}(n,\mathbb{R})$. *Let* $\mathrm{SO}(n)$ *stand for the special orthogonal group defined by*

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}).$$

The more general case is given in the following example.

Example 2.12. *Let* b *be a non-degenerate bilinear form on* $\mathbb{R}^n$ *and let* $\mathrm{O}(b)$ *defined by*

$$\mathrm{O}(b) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n,\ b(gx, gy) = b(x, y)\}.$$

We can find a matrix B *such that* $b(x, y) = y^T Bx$. *Therefore*

$$\mathrm{O}(b) = \left\{g \in \mathrm{GL}(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n,\ g^T Bg = B\right\}.$$

If we take b *to be the symmetric bilinear form*

$$b(x, y) = \sum_{i=1}^{p} x_iy_i - \sum_{i=1}^{q} x_iy_i, \quad p + q = n,$$

then we can write $\mathrm{O}(b) = \mathrm{O}(p, q)$:

$$\mathrm{O}(p, q) = \left\{g \in \mathrm{GL}(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n,\ g^T I_{p,q}g = I_{p,q}\right\},$$

where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

This subgroup is called the pseudo-orthogonal group.

In the following example, we present the Lie algebras for some of the classical Lie groups without computation.

Example 2.13.

$$\begin{aligned}
\mathbf{L}(\mathrm{GL}(n,\mathbb{R})) = \mathfrak{gl}(n,\mathbb{R}) &= \mathrm{M}(n,\mathbb{R}), \\
\mathbf{L}(\mathrm{SL}(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R}) &= \{X \in \mathrm{M}(n,\mathbb{R}) \mid \mathrm{Tr}X = 0\}, \\
\mathbf{L}(\mathrm{SO}(n)) = \mathfrak{so}(n) &= \{X \in \mathrm{M}(n,\mathbb{R}) \mid X^T = -X\}.
\end{aligned}$$

2.4 Haar Measure

In this section, we present the definition of Haar measure for a locally compact topological group. Haar measures are the first step for doing analysis on Lie groups. We use Haar measures to define integrals of the functions defined on Lie groups.

Definition 2.14. *Let G be a locally compact group and let $\mu \geq 0$ be a Radon measure on G. Let $\mathcal{C}_c(G)$ be the space of continuous functions on G with compact support. The measure μ is said to be left invariant if*

$$\int_G f(gx)\ d\mu(x) = \int_G f(x)\ d\mu(x),$$

for every $g \in G$, and for every $f \in \mathcal{C}_c(G)$. This is equivalent to saying that, for every Borel set $E \subseteq G$, and for every $g \in G$,

$$\mu(gE) = \mu(E).$$

Theorem 2.15. *Let G be a locally compact group. Then the following hold.*

(1) There exists a (non-zero) left invariant measure on G which is unique up to positive factor. This measure is called Haar measure.

(2) Let $f \in \mathcal{C}_c(G)$. If $g \in G$, then there is a positive number $\triangle(g)$ such that

$$\int_G f(xg^{-1})\ d\mu(x) = \triangle(g) \int_G f(x)\ d\mu(x).$$

(3) For $f \in \mathcal{C}_c(G)$,

$$\int_G f(x^{-1})\ d\mu(x) = \int_G f(x) \triangle (x^{-1})\ d\mu(x).$$

Proof. See [2, pp. 75-76] . □

Theorem 2.16. *The function $\triangle$ is a continuous group homomorphism,*

$$\triangle : G \to \mathbb{R}_+^*.$$

Proof. See [2, p. 75]. □

We call the function $\triangle$, the *module* of the group G. One can see that, for a Borel set $E \subseteq G$, and for every $g \in G$,

$$\mu(Eg) = \triangle(g)\mu(E).$$

If $\triangle \equiv 1$, then the group G is said to be *unimodular*. Every commutative group is unimodular.

Theorem 2.17. *(1) Every compact group is unimodular.*

(2) Every discrete group is unimodular.

Proof. See [2, p. 75]. □

2.5 Unitary Representations

Representation theory is a very active and dynamic field of research nowadays. It is a very important tool in physics and mathematics. Representations are the most objects that we will focus on in this thesis. In this section, we introduce the basic concepts related to representation theory. Later on, in the next chapters, we will coin and discuss different definitions for the reflection positive representations.

Now, let us present some basic definitions related to representation theory.

Definition 2.18. *Let G be a topological group. Let V be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{L}(V)$ be the algebra of bounded operators on V. A map*

$$\begin{aligned} \pi : G &\to \mathcal{L}(V), \\ g &\mapsto \pi(g), \end{aligned}$$

is said to be a representation of G on V if it satisfies the following:

(1) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2),\ \pi(e) = \mathrm{id}$,

(2) for every $v \in V$, the map

$$\begin{aligned} G &\to V, \\ g &\mapsto \pi(g)v, \end{aligned}$$

is continuous.

Definition 2.19. *Let π be a representation of G on V. Then a subspace $W \subseteq V$ is said to be invariant if, for every $g \in G,\ \pi(g)W \subseteq W$.*

Definition 2.20. *Let π be a representation of G on V and let $W \subseteq V$ be an invariant subspace. Then $\pi_0(g) := \pi(g)|_W$ is called a subrepresentation of π.*

Definition 2.21. *Let π be a representation of G on V. Then π is said to be irreducible if the only invariant closed subspaces are $\{0\}$ and V.*

Definition 2.22. *Let (π_1, V_1) and (π_2, V_2) be two representations of G and let $A : V_1 \to V_2$ be a continuous linear map. Then A is said to be an intertwining operator if*

$$A\pi_1(g) = \pi_2(g)A,$$

for every $g \in G$. We say that the representations (π_1, V_1) and (π_2, V_2) are equivalent if there exists an intertwining operator $A : V_1 \to V_2$ which is also an isomorphism.

Definition 2.23. *Let $\mathcal{H}$ be a Hilbert space. An operator A is said to be unitary if $(Av, Aw) = (v, w)$ for every $v, w \in \mathcal{H}$. A representation π of G on $\mathcal{H}$ is said to be unitary if, for every $g \in G$, $\pi(g)$ is a unitary operator, i.e.,*

$$\forall g \in G,\ \forall v, w \in \mathcal{H},\ (\pi(g)v, \pi(g)w) = (v, w).$$

Let V be a vector space and let $\mathrm{End}(V)$ be the Lie algebra of all endomorphisms on V.

Definition 2.24. *A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map*

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

which is a Lie algebra homomorphism. Sometimes we say that V is a module over $\mathfrak{g}$.

Theorem 2.25. *Let π be a representation of a compact group G on a finite dimensional vector space V. Then there exists on V a Euclidean inner product for which π is unitary.*

Proof. See [2, p. 96]. □

Theorem 2.26. *(Schur's Lemma) Let $\mathcal{H}$ be a Hilbert space. Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary representation of a topological group G. Then every intertwining operator $T \in \mathrm{Hom}(\pi, \pi)$ for the representation π can be written $T = \lambda \, \mathrm{id}_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}$.*

Proof. See [3, p. 71]. □

2.6 Complexification and Real Forms

If we start with an $\mathbb{R}$-vector space V then the *complexification* $V_{\mathbb{C}}$ of V is the tensor product $V_{\mathbb{C}} := \mathbb{C} \bigotimes_{\mathbb{R}} V$. The scalar multiplication

$$\lambda \cdot (z \otimes v) := \lambda z \otimes v$$

makes $V_{\mathbb{C}}$ a complex vector space. We identify V with the subspace $1 \otimes V$ of $V_{\mathbb{C}}$. Then each element of $V_{\mathbb{C}}$ can be written in a unique way as $z = x + iy$ with $x, y \in V$. For any real basis $\{v_1, \cdots, v_n\}$ for V, we have the complex basis $\{1 \otimes v_1, \cdots, 1 \otimes v_n\}$ for $V_{\mathbb{C}}$. For more details, see [7, p. 86].

Theorem 2.27. *Let $\mathfrak{g}$ be a real Lie algebra.*

1. *$\mathfrak{g}_{\mathbb{C}}$ is a complex Lie algebra with respect to the complex bilinear Lie bracket, defined by*

$$[x + iy, x' + iy'] := ([x, x'] + [y + y']) + i\,([x, y'] + [y, x'])$$

and satisfying

$$[z \otimes v, z' \otimes v'] = zz' \otimes [v, v'].$$

2. *$[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \cong [\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$ as complex Lie algebras.*

Proof. See [7, p. 86]. □

Definition 2.28. *Let $\mathfrak{g}$ be a complex Lie algebra. A real Lie algebra $\mathfrak{h}$ with $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{g}$ is called a real form of $\mathfrak{g}$.*

It is important to know that there exist nonisomorphic real Lie algebras with isomorphic complexifications.

2.7 Iwasawa Decomposition and the Generalized Principal Series Representations

The material of this section can be found in [8] and [21]. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{b}$ a finite-dimensional abelian Lie algebra over the field $\mathbb{K}$. Given a finite-dimensional $\mathfrak{b}$-module V, we denote the corresponding representation of $\mathfrak{b}$ on V by π. If $\lambda \in \mathbb{K}$ and $T \in \mathrm{End}(V)$, we define

$$V(\lambda, T) := \{v \in V \mid Tv = \lambda v\}.$$

For $X \in \mathfrak{b}$ we set

$$V(\lambda, X) := V(\lambda, \pi(X)).$$

For $\alpha \in \mathfrak{b}^*$, we define $V(\alpha, \mathfrak{b})$ by

$$V(\alpha, \mathfrak{b}) := \{v \in V \mid \forall X \in \mathfrak{b} : \pi(X)v = \alpha(X)v\} = \bigcap_{X \in \mathfrak{b}} V(\alpha(X), X).$$

We set

$$V^{\mathfrak{b}} := V(0, \mathfrak{b}) = \{v \in V \mid \pi(\mathfrak{b})v = 0\}.$$

Sometime, if the role of $\mathfrak{b}$ is obvious, we abbreviate $V(\alpha, \mathfrak{b})$ by V_α. Let

$$\Delta(V, \mathfrak{b}) := \{\alpha \in \mathfrak{b}^* \backslash \{0\} \mid V_\alpha \neq \{0\}\}.$$

Then the elements of $\Delta(V, \mathfrak{b})$ are called *weights*. Set

$$V(\Gamma) := \bigoplus_{\alpha \in \Gamma} V_\alpha, \;\; \phi \neq \Gamma \subset \Delta(V, \mathfrak{b}).$$

Let G be a semisimple connected Lie group with Cartan involution θ. Let $K = G^\theta$ be the corresponding group of θ-fixed points in G. The Lie algebra of K is given

by

$$\mathfrak{k} = \mathfrak{g}(1, \theta) := \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Let $\mathfrak{s} = \mathfrak{g}(-1, \theta) := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then we have the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}.$$

For $X, Y \in \mathfrak{g}$, set

$$(X, Y) := -B(X, \theta(Y)),$$

where $B(\cdot, \cdot)$ is the Killing form of $\mathfrak{g}$. Then $(\cdot, \cdot)$ is an inner product on $\mathfrak{g}$. With respect to this inner product, the transpose $\mathrm{ad}(X)^T$ of $\mathrm{ad}(X)$ is given by $-\mathrm{ad}(\theta(X))$ for all $X \in \mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{s}$ and

$$\mathfrak{m} := \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \{X \in \mathfrak{k} \mid [X, Y] = 0, \ \forall\, Y \in \mathfrak{a}\}.$$

As $\{\mathrm{ad}X \mid X \in \mathfrak{a}\}$ is a commutative family of symmetric endomorphisms of $\mathfrak{g}$, we get

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha.$$

The elements of $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ are called *restricted roots*. For $X \in \mathfrak{a}$ such that $\alpha(X) \neq 0$ for all $\alpha \in \Delta$, one can form a set of *positive restricted roots* to be

$$\Delta^+ := \{\alpha \in \Delta \mid \alpha(X) > 0\}.$$

Set

$$\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha = \mathfrak{g}(\Delta^+).$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra and we have the following *Iwasawa decomposition* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let $N := \exp \mathfrak{n}$ and $A := \exp \mathfrak{a}$. Then N and A are closed subgroups of G and we have the *Iwasawa decomposition* of G:

$$K \times A \times N \ni (k, a, n) \mapsto kan \in G.$$

This map is an analytic diffeomorphism.

Theorem 2.29. *If $\mathfrak{a}$ and $\mathfrak{b}$ are maximal abelian in $\mathfrak{s}$, then there exists $k \in K$ such that*

$$\mathrm{Ad}(k)\mathfrak{b} = \mathfrak{a}.$$

Proof. See [8, p. 248]. $\square$

Let $P_{\min} := MAN$. Then $P_{\min}$ is a group. The groups conjugate to $P_{\min}$ are called *minimal parabolic subgroups* of G. A subgroup of G containing a minimal parabolic subgroup is called a *parabolic subgroup*.

Now, more generally, if we take $\mathfrak{a}$ to be any abelian subalgebra of $\mathfrak{s}$ and change $\mathfrak{m}$ to be

$$\mathfrak{m} := \{X \in \mathfrak{g} \mid [X, Y] = 0, \ \forall\, Y \in \mathfrak{a} \text{ and } X \perp \mathfrak{a}\},$$

then we have again

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha.$$

Let $\mathfrak{p} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then $\mathfrak{p} = N_{\mathfrak{g}}(\mathfrak{n}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{n}] \subset \mathfrak{n}\}$ and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. It is maximal, if $\dim \mathfrak{a} = 1$.

Let $P = N_G(\mathfrak{p}) = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{p} \subset \mathfrak{p}\}$. Then P is a closed subgroup of G with Lie algebra $\mathfrak{p}$. Let M_0 denote the analytic subgroup of G with Lie algebra $\mathfrak{m}$ and $M := Z_K(A)M_0$. Then M is closed subgroup of G with finitely many

connected components, $Z_G(A) = MA$ and the product map $M \times A \times N \to P$, $(m, a, n) \mapsto man$, is an analytic diffeomorphism. Let $L = K \cap M$ and $\mathcal{B} := K/L$. Then

$$G = KP \text{ and } K \cap P = L.$$

Thus $\mathcal{B} = G/P$ and G acts on $\mathcal{B}$. Let

$$g = k(g)a(g)m(g)n(g)$$

with $(k(g), a(g), m(g), n(g)) \in K \times M \times A \times N$. The map $G \ni g \mapsto (a(g), n(g)) \in A \times N$ is analytic and $g \mapsto a(g)$ is right MN-invariant. The elements $k(g)$ and $m(g)$ are not uniquely defined. However, the maps

$$g \mapsto k(g)L \in \mathcal{B} \text{ and } g \mapsto Lm(g) \in L\backslash M$$

are well defined and analytic. Thus G acts on $\mathcal{B} = K/L$ by

$$g \cdot kL = k(gk)L.$$

Let $\overline{N} := \theta(N)$. Then the Lie algebra of $\overline{N}$ is $\overline{\mathfrak{n}} = \theta(\mathfrak{n}) = \bigoplus_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha$. Furthermore, $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and the map $\overline{N} \times M \times A \times N \to P$, $(\overline{n}, m, a, n) \mapsto \overline{n}man$, is an analytic diffeomorphism onto an open and dense subset subset $\overline{N}P$ of G of full measure.

We normalize the invariant measure on $\mathcal{B}$ and compact groups so that the total measure is one. If $f \in \mathcal{C}(\mathcal{B})$ then

$$\int_{\mathcal{B}} f(b)\, db = \int_K f(kL) dk.$$

If V is a real vector space, we set $V_{\mathbb{C}} := \mathbb{C} \bigotimes_{\mathbb{R}} V$. We define the *conjugation of* $V_{\mathbb{C}}$ *relative to* V by

$$\sigma(u + iv) = \overline{u + iv} := u - iv, \quad u, v \in V.$$

If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we define a character $a \mapsto a^\lambda$ on A by

$$a^\lambda := \exp \lambda(X), \ a = \exp X.$$

For $\alpha \in \Delta$, let $m_\alpha := \dim \mathfrak{g}_\alpha$ and let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha \in \mathfrak{a}^*$. Normalize the Haar measure $d\overline{n}$ on $\overline{N}$ such that

$$\int_{\overline{N}} a(\overline{n})^{-2\rho} d\overline{n} = 1.$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ define a continuous representation of G on $L^2(\mathcal{B})$ by

$$[\pi_\lambda(x)f](kL) := a(x)^{-\lambda-\rho} f(x^{-1} \cdot kL) = a(x)^{-\lambda-\rho} f(k(x^{-1}k)L).$$

Note that π_λ is unitary if and only if $\lambda \in i\mathfrak{a}^*$. The representation π_λ is called the generalized principal series.

2.8 Non-Compactly Causal Symmetric Spaces

The material of this section can be found in [14]. Let us start with introducing the definition of a symmetric space.

Definition 2.30. *A symmetric space is a triple (G, H, τ), where*

(1) G is a Lie group,

(2) τ is a nontrivial involution on G, i.e., $\tau : G \to G$ is an automorphism with $\tau^2 = \mathrm{Id}_G$, and

(3) H is a closed subgroup of G such that $G_o^\tau \subset H \subset G^\tau$.

Here the subscript $_o$ means the connected component containing the identity and G^τ denotes the group of τ-fixed points in G. By abuse of notation we also say that (G, H) as well as G/H is a symmetric space.

The infinitesimal version of the above definition is the following definition.

Definition 2.31. *A pair $(\mathfrak{g}, \tau)$ is called a symmetric pair if*

(1) $\mathfrak{g}$ is a Lie algebra,

(2) τ is a nontrivial involution on $\mathfrak{g}$, i.e., $\tau : \mathfrak{g} \to \mathfrak{g}$ is an automorphism with $\tau^2 = \mathrm{Id}_{\mathfrak{g}}$.

Let $(\mathfrak{g}, \tau)$ be a symmetric pair and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If H is a closed subgroup of G, then (G, H) is called *associated* to $(\mathfrak{g}, \tau)$ if τ integrates to an involution on G, again denoted by τ such that (G, H, τ) is a symmetric space. In this case, the Lie algebra of H is denoted by $\mathfrak{h}$ and it is given by

$$\mathfrak{h} = \mathfrak{g}(1, \tau) := \{X \in \mathfrak{g} \mid \tau(X) = X\}.$$

Note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where

$$\mathfrak{q} = \mathfrak{g}(-1, \tau) := \{X \in \mathfrak{g} \mid \tau(X) = -X\}.$$

We have the following relations:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \quad \text{and} \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}. \tag{2.1}$$

From (2.1) it follows that $\mathrm{ad}_{\mathfrak{q}} : \mathfrak{h} \to \mathrm{End}(\mathfrak{q}),\ X \mapsto \mathrm{ad}(X)|_{\mathfrak{q}}$, is a representation of $\mathfrak{h}$. In particular, $(\mathfrak{g}, \mathfrak{h})$ is *reductive* pair in the sense that there exists an $\mathfrak{h}$-stable complement of $\mathfrak{h}$ in $\mathfrak{g}$. On the other hand, if $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair and the commutator relations (2.1) hold, we can define an involution τ of $\mathfrak{g}$ by $\tau|_{\mathfrak{h}} = \mathrm{id}$ and $\tau|_{\mathfrak{q}} = -\mathrm{id}$. Then $(\mathfrak{g}, \tau)$ is a symmetric pair. The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is said to be *irreducible* if the only τ-stable ideals in $\mathfrak{g}$ containing the Lie algebra $\mathfrak{g}^{fix} := \bigcap_{g \in G} \mathrm{Ad}(g)\mathfrak{h}$ are $\mathfrak{g}^{fix}$ and $\mathfrak{g}$.

Let G/H be a semisimple symmetric space and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ. Let $\mathfrak{k} = \mathfrak{g}^\theta$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. Then

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{k} \oplus \mathfrak{p} \\ &= \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_k \oplus \mathfrak{q}_p. \end{aligned}$$

where the subscript denotes the intersection with the corresponding subspace of $\mathfrak{g}$. Let L be a Lie group and $\mathbf{V}$ an L-moudule. We denote by $\mathbf{V}^L$ the subspace of L-fixed points in $\mathbf{V}$.

Definition 2.32. *Let G/H be an irreducible symmetric space. Then*

(1) G/H is called non-compactly causal (NCC) if $\mathfrak{q}_p^{H\cap K} \neq \{0\}$.

(2) G/H is called of Cayley type if $\mathfrak{h}$ has a one-dimensional center contained in $\mathfrak{h}_p$.

If G/H is NCC then $\mathfrak{q}_p^{H\cap K}$ is one-dimensional and there exists an element $X^0 \in \mathfrak{q}_p^{H\cap K}$ such that $\mathfrak{h}_k \oplus \mathfrak{q}_p = \mathfrak{z}_{\mathfrak{g}}(X^0)$.

2.9 Symmetric Lie Groups

Definition 2.33. *Let G be a Banach-Lie group, let τ be an involutive automorphism of G and let H be an open subgroup of $G^\tau = \{g \in G : \tau(g) = g\}$. Then we call the triple (G, H, τ), respectively the pair (G, τ), a symmetric Lie group. The corresponding infinitesimal involution induced on the Lie algebra $\mathfrak{g}$ of G is given by the same notation $\tau : \mathfrak{g} \to \mathfrak{g}$. For $\mathfrak{h} = \mathfrak{g}^\tau = \{X \in \mathfrak{g} : \tau(X) = X\}$, we call the $(\mathfrak{g}, \mathfrak{h}, \tau)$, respectively $(\mathfrak{g}, \tau)$, a symmetric Lie algebra.*

Theorem 2.34. *Let the notation be as above. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{g}^\tau \oplus \mathfrak{g}^{-\tau} = Ker\ (\tau - 1) \oplus Ker\ (\tau + 1)$ be the eigenspace decomposition of $\mathfrak{g}$ with respect to τ. Let*

$\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$ be a subspace of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and let $\tau^c(x+iy) := x - iy$. Then $(\mathfrak{g}^c, \tau^c)$ is a symmetric Lie algebra and is called the c-dual of $(\mathfrak{g}, \tau)$.

Proof. We can see that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$. Thus $\mathfrak{g}^c$ is another real algebra and $\tau^c(x + iy) := x - iy$ leads to the symmetric Lie algebra $(\mathfrak{g}^c, \tau^c)$. $\square$

By Ado's Theorem [5], there is a simply connected Lie group G^c which has $\mathfrak{g}^c$ as its Lie algebra and it is unique up to Lie isomorphism.

2.10 Positive Definite Distributions

We write $\mathcal{D}(M) = C_c^\infty(M, \mathbb{C})$ for the space of compactly supported functions of a manifold M and endow this space with the usual LF topology, i.e., the locally convex direct limit of the Fréchet spaces $\mathcal{D}_X(M)$ of test functions supported in the compact subset $X \subset M$. Its antidual, i.e., the space of continuous antilinear functionals on $\mathcal{D}(M)$, is the space $\mathcal{D}'(M)$ of *distributions* on M, see [32]. This section is taken from [24].

Definition 2.35.

1. *If G is a Lie group, then $\mathcal{D}(G)$ is an involutive algebra with respect to the convolution product and $\varphi^*(g) := \overline{\varphi(g^{-1})} \, \triangle_G \, (g^{-1})$, where $\triangle_G$ is the modular function from Theorem 2.15. Accordingly, we call a distribution $D \in \mathcal{D}'(G)$ positive definite, if it is a positive functional on this algebra, i.e.,*

$$D(\varphi^* * \varphi) \geqslant 0 \quad \textit{for} \quad \varphi \in \mathcal{D}(G).$$

2. *If τ is an involution on G and $S \subset G$ is an open subsemigroup invariant under $s \mapsto s^\sharp := \tau(s)^{-1}$, then $\mathcal{D}(S)$ is a *-algebra with respect to the convolution product and the *-operation $\varphi^\sharp := \varphi^* \circ \tau$. Accordingly, we call a distribution $D \in \mathcal{D}'(S)$ positive definite, if*

$$D(\varphi^\sharp * \varphi) \geqslant 0 \quad \textit{for} \quad \varphi \in \mathcal{D}(S).$$

Definition 2.36. *(Distribution vectors) Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the Lie group G on the Hilbert space $\mathcal{H}$. We write $\mathcal{H}^\infty$ for the linear subspace of smooth vectors, i.e., of all elements $v \in \mathcal{H}$ for which the orbit map $\pi^v : G \to \mathcal{H}, g \mapsto \pi(g)v$ is smooth. Identifying $\mathcal{H}^\infty$ with the closed subspace of equivariant maps in the Fréchet space $C^\infty(G, \mathcal{H})$, we obtain a natural Fréchet space structure on $\mathcal{H}^\infty$ for which the G-action on this space is smooth and the inclusion $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ (corresponding to evaluation in $1 \in G$) is a continuous linear map.*

We write $\mathcal{H}^{-\infty}$ for the space of antilinear functionals on $\mathcal{H}^\infty$ the space of distribution vectors, and note that we have a natural linear embedding $\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}, v \mapsto \langle v, .\rangle$. Accordingly, we also write $\langle \alpha, v\rangle = \overline{\langle v, \alpha\rangle}$ for $\alpha(v)$ for $\alpha \in \mathcal{H}^{-\infty}$ and $v \in \mathcal{H}^\infty$. The group G acts naturally on $\mathcal{H}^{-\infty}$ by

$$(\pi^{-\infty}(g)\alpha)(v) := \alpha(\pi(g)^{-1}v),$$

so that we obtain a G-equivariant chain of continuous inclusions

$$H^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}. \tag{2.2}$$

It is $\mathcal{D}(G)$-equivariant, if we define the representation of $\mathcal{D}(G)$ on $\mathcal{H}^{-\infty}$ by

$$(\pi^{-\infty}(\varphi)\alpha)(v) := \int_G \varphi(g)\alpha(\pi(g)^{-1}v)\ d\mu_G(g) = \alpha(\pi(\varphi^*)v).$$

Definition 2.37. *For any $\varphi \in \mathcal{D}(G)$ and $\alpha \in \mathcal{H}^{-\infty}$, we have $\pi^{-\infty}(\varphi)\alpha \in \mathcal{H}^{-\infty}$ in the sense of (2.2). We can see that*

$$\pi^\alpha(\varphi) := \langle \alpha, \pi^{-\infty}(\varphi)\alpha\rangle$$

defines a distribution on G. This distribution is positive definite because of the fact that

$$\pi^\alpha(\varphi^* * \varphi) = \langle \alpha, \pi^{-\infty}(\varphi^* * \varphi)\alpha\rangle = \langle \pi^{-\infty}(\varphi)\alpha, \pi^{-\infty}(\varphi)\alpha\rangle \geqslant 0.$$

Definition 2.38. *We say that* $\alpha \in \mathcal{H}^{-\infty}$ *is cyclic if* $\pi^{-\infty}(\mathcal{D}(G))\alpha$ *is a dense subspace of* $\mathcal{H}$*.*

The vector-valued GNS construction yields for every positive definite distribution $D \in \mathcal{D}^{'}(G)$ a corresponding Hilbert space $\mathcal{H}_D$ which is contained in the space $\mathcal{D}^*(G)$ of all antilinear functionals on $\mathcal{D}(G)$.

Theorem 2.39. *Let* $D \in \mathcal{D}^{'}(G)$ *be a positive definite distribution on the Lie group* G *and* $\mathcal{H}_D$ *be the corresponding reproducing kernel Hilbert space with kernel* $K(\varphi, \psi) := D(\psi^* * \varphi)$ *obtained by completing* $\mathcal{D}(G) * D$ *with respect to the scalar product* $\langle \psi * D, \varphi * D \rangle = D(\psi^* * \varphi)$*. Then the following assertions hold:*

1. $\mathcal{H}_D \subset \mathcal{D}^{'}(G)$ *and the inclusion* $\gamma_D : \mathcal{H}_D \to \mathcal{D}^{'}(G)$ *is continuous.*

2. *We have a unitary representation* $(\pi_D, \mathcal{H}_D)$ *of* G *by* $\pi_D(g)E = g * E$*, where* $(g * E)(\varphi) := E(\varphi \circ \lambda_g)$ *and the integrated representation of* $\mathcal{D}(G)$ *on* $\mathcal{H}_D$ *is given by* $\pi_D(\varphi)E = \varphi * E$*.*

3. *There exists a unique distribution vector* $\alpha_D \in \mathcal{H}_D^{-\infty}$ *with* $\alpha_D(\varphi * D) = D(\phi)$ *and* $\pi^{-\infty}(\varphi)\alpha_D = \phi * D$ *for* $\varphi \in \mathcal{D}(G)$*. It satisfies* $\pi^{\alpha_D} = D$*.*

4. γ_D *extends to a* $\mathcal{D}(G)$*-equivariant injection* $\mathcal{H}_D^{-\infty} \hookrightarrow \mathcal{D}^{'}(G)$ *mapping* α_D *to* D*.*

Proof. See [24, Proposition 2.8, p. 2188]. $\square$

Chapter 3

Lüscher-Mack and Integrability Theorems

To perform the analytic continuation for some reflection positive representations on some Lie group, thus getting representations on its dual Lie group, we need to use some tools. The first tool is the Lüscher-Mack Theorem, which deals with representations of semigroups. The second tool which was introduced in [23], is an integrability theorem. This tool helps us to pass from the Lie algebra level to the Lie group level. More precisely, this tool allows us to integrate a family of Lie algebra representations to get Lie group representations. Using this tool, one can perform the analytic continuation for some reflection positive representations on some Lie group to get a representation on its dual Lie group. Instead of using semigroups, as in the case of the Lüscher-Mack Theorem, we will use the subgroup H. In this chapter, we introduce and discuss these important tools.

3.1 Lüscher-Mack Theorem

The material of this section can be found in [7, p. 291] and in [14]. Let (G, τ) be a symmetric Lie group. Let $\mathfrak{h} = \mathfrak{g}^\tau$ and let $\mathfrak{q} = \mathfrak{g}^{-\tau}$. Let $C \subseteq \mathfrak{q}$ be an H-invariant pointed generating regular cone, i.e., it satisfies the following conditions:

- $\mathrm{Ad}(H)\mathfrak{q} \subset \mathfrak{q}$.
- There exists $X \in \mathfrak{q}$ such that for all $Y \in \mathfrak{q}\backslash\{0\}$, we have $\langle X, Y\rangle > 0$.
- $C \cap -C = \phi$ and $C - C = \mathfrak{q}$.

Then $S(C) = H \exp C$ is a closed semigroup, by Lawson's Theorem, see [7, pp.198-199].

Theorem 3.1. *Let π be a strongly continuous contractive representation of $S(C)$ on the Hilbert space $\mathbf{H}$ such that $\pi(s)^* = \pi(\tau(s)^{-1})$. Let G^c be the simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. Then there exists a continuous unitary representation $\pi^c : G^c \to U(\mathbf{H})$, extending π, such that for the differentiated representations $d\pi$ and $d\pi^c$ we have*

(1) $d\pi^c(X) = d\pi(X) \quad \forall X \in \mathfrak{h}$,

(2) $d\pi^c(iY) = id\pi(Y) \quad \forall Y \in C$.

Proof. See [6, p. 292]. □

3.2 The Integrability Theorem

The material of this section and the integrability theorem are taken from [23].

3.2.1 Local Flows on Locally Convex Manifolds

Definition 3.2. *A smooth manifold M modeled on a locally convex space is called a locally convex manifold.*

Let $\mathcal{V}(M)$ denote the Lie algebra of smooth vector fields on M.

Let

$$I_m := \{t \in \mathbb{R} : (t, m) \in \mathcal{D}\} \text{ and } M_t := \{m \in M : (t, m) \in \mathcal{D}\},$$

where $\mathcal{D} \subset \mathbb{R} \times M$, $x \in M$ and $t \in \mathbb{R}$.

Definition 3.3. *Let M be a locally convex manifold. A local flow on M is a smooth map $\Phi : \mathcal{D} \to M,\ (t, x) \mapsto \Phi_t(x)$, where $\mathcal{D} \subset \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$, such that for each $x \in M$, the set I_x is an interval containing 0, and*

$$\Phi_0 = id_M \quad \text{and} \quad \Phi_t\Phi_s(x) = \Phi_{s+t}(x)$$

hold for all t, s, x for which both sides are defined. The maps

$$\gamma_x : I_x \to M, \quad t \mapsto \Phi_t(x)$$

are called the flow lines.

Definition 3.4. *Let $\Phi : \mathcal{D} \to M$ be a local flow. Then Φ is said to be global if $\mathcal{D} = \mathbb{R} \times M$.*

If $\Phi : \mathcal{D} \to M$ is a local flow, then

$$X^{\Phi}(x) := \frac{d}{dt}\bigg|_{t=0} \Phi_t(x) = \gamma_x^{'}(0)$$

is smooth vector field on M and is called the *velocity field* of Φ.

Definition 3.5. *Let M be a locally convex manifold.*

1. *A smooth vector field $X \in \mathcal{V}(M)$ is called locally integrable if it is the velocity field of some smooth local flow.*

2. *Let $I \subset \mathbb{R}$ be an open interval containing 0. A differentiable map $\gamma : I \to M$ is called an integral curve of X if*

$$\gamma'(t) = X(\gamma(t)) \quad \textit{for each} \quad t \in I.$$

3. *If $J \supseteq I$ is an interval containing I, then an integral curve $\eta : J \to M$ is called an extension of γ if $\eta|_I = \gamma$. An integral curve is said to be maximal if it has no proper extension.*

Now, we introduce the definition of the Lie derivative of a function with respect to a vector field over a locally convex manifold.

Definition 3.6. *For a locally integrable vector field $X \in \mathcal{V}(M)$, let $\Phi^X : \mathcal{D}_X \to M$ be its maximal local flow. Let $f \in C^\infty(M)$ and $t \in \mathbb{R}$. Then $f \circ \Phi_t^X \in C^\infty(M_t)$. Now, we define*

$$\mathcal{L}_X f := \lim_{t \to 0} \frac{1}{t}\left(f \circ \Phi_t^X - f\right) \in C^\infty(M).$$

3.2.2 Smooth Right Actions and Compatible Distributions

The following definition is needed to state the main theorem of this section.

Definition 3.7. *Suppose that $D \in \mathcal{D}'(M \times M)$ is a distribution on $M \times M$. For $\varphi, \psi \in \mathcal{D}(M)$, define*

$$[\mathcal{L}^1_X D](\varphi \otimes \psi) := -D([\mathcal{L}_X \varphi] \otimes \psi)$$

and

$$[\mathcal{L}^2_X D](\varphi \otimes \psi) := -D(\varphi \otimes [\mathcal{L}_X \psi]).$$

A vector field $X \in \mathcal{V}(M)$ is said to be D-symmetric if

$$\mathcal{L}^1_X D = \mathcal{L}^2_X D,$$

and D-skew-symmetric if

$$\mathcal{L}^1_X D = -\mathcal{L}^2_X D.$$

Here the superscripts indicate whether the Lie derivative acts on the first or the second argument.

Definition 3.8. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra, and let $\beta : \mathfrak{g} \to \mathcal{V}(M)$ be a homomorphism. A positive definite distribution $D \in \mathcal{D}'(M \times M)$ is said to be β-compatible if the vector fields in $\beta(\mathfrak{h})$ are D-skew-symmetric and the vector fields in $\beta(\mathfrak{q})$ are D-symmetric, i.e., if for $x \in \mathfrak{g}$, we have*

$$\mathcal{L}^1_{\beta(x)} D = -\mathcal{L}^2_{\beta(\tau(x))} D.$$

Definition 3.9. *Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. A smooth right action of the pair $(\mathfrak{g}, H)$ on a (locally convex) manifold M is a pair (β, σ), where*

1. *$\sigma : M \times H \to M$ is a smooth right action,*

2. *and* $\beta : \mathfrak{g} \to \mathcal{V}(M)$ *is a homomorphism of Lie algebra for which*

$$\widehat{\beta} : \mathfrak{g} \times M \to TM, \quad (x, m) \mapsto \beta(x)(m)$$

is smooth,

3. $\dot{\sigma}(x) = \beta(x)$ *for* $x \in \mathfrak{h}$,

4. *each vector field* $\beta(x),\ x \in \mathfrak{q}$, *is locally integrable.*

3.2.3 The Integrability Theorem

Theorem 3.10. *Let* $D \in \mathcal{D}'(M \times M)$ *be a positive definite distribution compatible with the smooth right action* (β, σ) *of the pair* $(\mathfrak{g}, H)$ *on* M, *where* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *is a symmetric Banach-Lie algebra and* H *is a connected Lie group with Lie algebra* $\mathfrak{h}$. *Let* G^c *be the simply connected Lie group with Lie algebra* $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. *Then there exists a unique smooth unitary representation* $(\pi^c, \mathcal{H}_D)$ *of* G^c *such that*

1. $\overline{d\pi^c}(x) = \mathcal{L}_x$ *for each* $x \in \mathfrak{h}$.

2. $\overline{d\pi^c}(iy) = i\mathcal{L}_y$ *for each* $y \in \mathfrak{q}$.

Proof. See [23, p. 31]. □

3.2.4 Reflection Positive Distributions and Representations

We can see this section as an application of the integrability theorem. The material of this section is taken from [23].

Definition 3.11. *Let* M *be a smooth finite dimensional manifold and* $D \in \mathcal{D}'(M \times M)$ *be a positive definite distribution. Suppose further that* $\theta : M \to M$ *is an involutive diffeomorphism and that* $M_+ \subset M$ *is an open subset such that the distribution* D_+ *on* $M_+ \times M_+$ *defined by*

$$D_+(\varphi) := D(\varphi \circ (\theta \times \mathrm{id}_M))$$

is positive definite. We say that D *is reflection positive with respect to* (M, M_+, θ).

Theorem 3.12. *Let M be a smooth finite dimensional manifold and $D \in \mathcal{D}'(M \times M)$ be a positive definite distribution which is reflection positive with respect to (M, M_+, θ). Let (G, H, τ) be a symmetric Lie group acting on M such that $\theta(g.m) = \tau(g).\theta(m)$ and $H.M_+ = M_+$. We assume that D is invariant under G and τ. Let G^c be the simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + \mathfrak{q}$ and define $\mathcal{L}_x,\ x \in \mathfrak{g}$, on a maximal domain in the Hilbert subspace $\mathcal{H}_{D_+} \subset \mathcal{D}'(M_+)$. Then there exists a unique smooth unitary representation $(\pi^c, \mathcal{H}_{\mathcal{D}_+})$ of G^c such that*

(i) $\overline{\mathrm{d}\,\pi^c}(x) = \mathcal{L}_x$ *for* $x \in \mathfrak{h}$.

(ii) $\overline{\mathrm{d}\,\pi^c}(iy) = i\mathcal{L}_y$ *for* $y \in \mathfrak{q}$.

Proof. See [23, p. 34]. □

Chapter 4

Reflection Positivity: Hilbert Spaces, Representations, Functions, and Distributions

Many techniques and tools were developed in the last two decades to generate reflection positive representations. In this chapter, we present a very effective tool which was introduced in [24], namely, the reflection positive functions and the reflection positive distributions. The main ingredient for this tool is the reproducing kernel Hilbert spaces. The material of this chapter is taken mostly from [24] and the material of Section 4.5 is taken from [25].

4.1 Reflection Positive Hilbert Space and Reflection Positive Representation on a Hilbert Space

Now, we introduce the definitions of reflection positive Hilbert space and reflection positive representation on a Hilbert space.

Definition 4.1. *(Reflection Positive Hilbert Space, [23, Definition 7.11, p. 51]) Let $\mathcal{E}$ be a complex Hilbert space. Let J be a unitary involution on $\mathcal{E}$, i.e., $J^2 = \mathrm{id}$ and $\langle Jv, w\rangle = \langle v, Jw\rangle$ for $v, w \in \mathcal{E}$. We call a closed subspace $\mathcal{E}_+ \subseteq \mathcal{E}$ J-positive if*

$$\langle Jv, v\rangle \geq 0$$

for $v \in \mathcal{E}_+$. We then say that the triple $(\mathcal{E}, \mathcal{E}_+, J)$ is a reflection positive Hilbert space. We write

$$\mathcal{N} := \{v \in \mathcal{E}_+ : \langle Jv, v\rangle = 0\} = \{v \in \mathcal{E}_+ : (\forall w \in \mathcal{E}_+)\langle Jw, v\rangle = 0\},$$

$q : \mathcal{E}_+ \longrightarrow \mathcal{E}_+/\mathcal{N}, v \longmapsto \widehat{v} = q(v)$ for the quotient map and $\widehat{\mathcal{E}}$ for the Hilbert completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm $\|\widehat{v}\|_{\widehat{\mathcal{E}}} := \sqrt{\langle Jv, v\rangle}$.

Definition 4.2. *(Reflection Positive Representation on a Hilbert Space, [23, Definition 7.12, p. 51]) Let (G, H, τ) be a symmetric Lie group and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be*

the corresponding symmetric Lie algebra. Let $(\mathcal{E}, \mathcal{E}_+, J)$ *be a reflection positive Hilbert space. A continuous representation* π *of* G *is said to be reflection positive on* $(\mathcal{E}, \mathcal{E}_+, J)$ *if the following conditions hold:*

(RP0) $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ *for every* $v, w \in \mathcal{E}$ *and* $g \in G$*, i.e.,* π *is a unitary representation.*

(RP1) $\pi(\tau(g)) = J\pi(g)J$ *for every* $g \in G$*.*

(RP2) $\pi(h)\mathcal{E}_+ = \mathcal{E}_+$ *for every* $h \in H$*.*

(RP3) There exists a subspace $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$*, dense in* $\mathcal{E}_+$*, such that* $d\pi(X)\mathcal{D} \subseteq \mathcal{D}$ *for every* $X \in \mathfrak{q}$*.*

Theorem 4.3. *Let* $(\mathcal{E}, \mathcal{E}_+, J)$ *be a reflection positive Hilbert space. Let* (G, τ) *be a symmetric Lie group and let* $S \subset G$ *be a subsemigroup which is invariant under the involution* $g \mapsto g^\sharp := \tau(g)^{-1}$*, i.e.,* $S^\sharp \subset S$*. Let* $(\pi, \mathcal{E})$ *be unitary representation of* G *such that*

(1) $\pi(S)\mathcal{E}_+ \subset \mathcal{E}_+$*,*

(2) $\pi(\tau(g)) = J\pi(g)J$ *for every* $g \in G$*.*

Then the following assertions hold:

(1) The representation π *of* G *is reflection positive on* $(\mathcal{E}, \mathcal{E}_+, J)$*.*

(2) There exists a continuous unitary representation $\pi^c : G^c \to U(\widehat{\mathcal{E}})$*, extending* π*, such that for the differentiated representations* $d\pi$ *and* $d\pi^c$ *we have:*

  (a) $d\pi^c(X) = d\pi(X) \quad \forall X \in \mathfrak{h}$*,*

  (b) $d\pi^c(iY) = id\pi(Y) \quad \forall Y \in \mathfrak{q}$*.*

Proof. See [24] and [23] and note that Part 2 follows form Theorem 3.1. □

4.2 Reflection Positive Functions

Definition 4.4. *Let $(S, *)$ be an involutive semigroup and $\mathcal{H}$ be a Hilbert space. An operator-valued function $\varphi : X \to B(\mathcal{H})$ is said to be positive definite if, for every finite sequence $(s_1, v_1), \cdots, (s_n, v_n)$ in $S \times \mathcal{H}$,*

$$\sum_{j,k=1}^{n} \langle \varphi(s_j s_k^*) v_k, v_j) \rangle \geq 0.$$

Definition 4.5. *If (G, τ) is a symmetric Lie group and $S \subset G$ be a an open $\sharp$-invariant subsemigroup and $\mathcal{H}$ be a Hilbert space, then we call an operator-valued function $\varphi : G \to B(\mathcal{H})$ reflection positive with respect to (G, τ, S) if the following conditions are satisfied:*

(RP1) φ *is positive definite,*

(RP2) $\varphi \circ \tau = \varphi$, *and*

(RP3) $\varphi|_S$ *is positive definite as a function on the involutive semigroup $(S, \sharp)$.*

Definition 4.6. *A triple $(\pi, \mathcal{E}, \mathcal{F})$, where $(\pi, \mathcal{E})$ is a unitary representation of $G_\tau := G \rtimes \{\mathbf{1}, \tau\}$ and $\mathcal{F} \subseteq \mathcal{E}$ is a G-cyclic subspace fixed pointwise by $\theta := \pi(\tau)$, is said to be a reflection positive $\mathcal{F}$-cyclic representation if the closed subspace $\mathcal{E}_+ := \overline{\mathrm{span}\, \pi(S)\mathcal{F}}$ is θ-positive. If in addition, $\mathcal{F} = \mathbb{C}v_0$ is one-dimensional, then we call the triple $(\pi, \mathcal{E}, v_0)$ a reflection positive cyclic representation.*

Theorem 4.7. *Let (G, τ) be a symmetric Lie group and $S \subset G$ be a $\sharp$-invariant subsemigroup. Then the following assertions hold:*

1. *If $(\pi, \mathcal{E}, \mathcal{F})$ is an $\mathcal{F}$-cyclic reflection positive representation of G_τ and $P : \mathcal{E} \to \mathcal{F}$ is the orthogonal projection, then $\varphi(g) := P\pi(g)P^*$ is a reflection positive function on G with $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$.*

2. *Let* $\varphi : G \to B(\mathcal{F})$ *be a reflection positive function on* G *with* $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{F}}$ *and* $\mathcal{H}_\varphi \subseteq \mathcal{F}^G$ *be a Hilbert subspace with reproducing kernel* $K(x, y) := \varphi(xy^{-1})$ *on which* G *acts by* $(\pi_\varphi(g)f)(x) := f(xg)$ *and* τ *by* $(\tau f)(x) := f(\tau(x))$*. We identify* $\mathcal{F}$ *with the subspace* $ev_{\mathbf{1}}^*\mathcal{F} \subseteq \mathcal{H}_\varphi$*. Then* $(\pi_\varphi, \mathcal{H}_\varphi, \mathcal{F})$ *is an* $\mathcal{F}$*-cyclic reflection positive representation and we have an* S*-equivariant unitary map*

$$\Gamma : \widehat{\mathcal{E}} \to \mathcal{H}_{\varphi|_S}, \quad \Gamma([f]) = f|_S.$$

Proof. See [24, Proposition 1.11, p. 2184]. □

4.3 Reflection Positive Distributions

We adopt the notations in Section 2.10. Let us introduce the definition of Reflection Positive Distribution.

Definition 4.8. *If* (G, τ) *is a symmetric Lie group and* S *is open and* $\sharp$*-invariant subsemigroup, then we call a distribution* $D \in \mathcal{D}^{'}(G)$ *reflection positive for* (G, τ, S) *if the following conditions are satisfied:*

(RP1) D *is positive definite, i.e.,* $D(\varphi^* * \varphi) \geq 0$ *for* $\varphi \in \mathcal{D}(G)$*.*

(RP2) $\tau D = D$*, i.e.,* $D(\phi \circ \tau) = D(\varphi)$ *for* $\varphi \in \mathcal{D}(G)$*, and*

(RP3) $D|_S$ *is positive definite as a distribution on the involution semigroup* $(S, \sharp)$*, i.e.,* $D(\varphi^\sharp * \varphi) \geq 0$ *for* $\varphi \in \mathcal{D}(S)$*.*

Definition 4.9. *A triple* $(\pi, \mathcal{H}, \alpha)$*, where* $(\pi, \mathcal{H})$ *is a unitary representation of* G_τ *and* $\alpha_0 \in \mathcal{H}^{-\infty}$ *a cyclic distribution vector fixed under* $\theta := \pi(\tau)$*, is said to be a reflection positive distribution cyclic representation if the closed subspace* $\mathcal{E}_+ := \overline{\operatorname{span} \pi^{-\infty}(\mathcal{D}(S))\alpha_0}$ *is* θ*-positive.*

Theorem 4.10. *For (G, τ, S) as above, the following assertions hold:*

1. *If $(\pi, \mathcal{H}, \alpha)$ is a distribution cyclic reflection positive representation of G_τ, then $\pi^\alpha(\varphi) := \alpha(\pi^{-\infty}(\varphi)\alpha)$ is a reflection positive distribution on G.*

2. *If D is a reflection positive distribution on G, then $(\pi_D, \mathcal{H}_D, \alpha_D)$ is a reflection positive distribution cyclic representation, where τ acts on $\mathcal{D}'(G)$ by $(\tau E)(\varphi) := E(\varphi \circ \tau)$.*

Proof. See [24, Proposition 2.12, p. 2191]. □

4.3.1 Reflection Positivity on the Real Line

In this subsection, we give some examples of the reflection positive functions and the reflection positive distributions.

Theorem 4.11. *Let $(G, \tau, S) = (\mathbb{R}, -\mathrm{id}_\mathbb{R}, \mathbb{R}_+)$. Let $\mathcal{F}$ be a Hilbert space and $\varphi : \mathbb{R} \to B(\mathcal{F})$ be positive definite and strongly continuous. Then φ is reflection positive if and only if there exists a finite* $\mathrm{Herm}(\mathcal{F})_+$*-valued Borel measure Q on $[0, \infty)$ such that*

$$\varphi(x) = \int_0^\infty e^{-\lambda|x|}\ dQ(\lambda).$$

Proof. See [24, Proposition 3.1, p. 2192]. □

Corollary 4.12. *Let $(G, \tau, S) = (\mathbb{R}, -\mathrm{id}_\mathbb{R}, \mathbb{R}_+)$. A continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ is reflection positive if and only if it has an integral representation of the form*

$$\varphi(x) = \int_0^\infty e^{-\lambda|x|}\ d\mu(\lambda),$$

where μ is a finite positive Borel measure on $[0, \infty)$.

Proof. The proof follows directly from Theorem 4.11. Also, see [24, Corollary 3.3 p. 2194]. □

Example 4.13. *Let* $(G, \tau, S) = (\mathbb{R}_+, 1/\mathrm{id}_{\mathbb{R}}, (0,1))$*. Let* $\varphi_\lambda : \mathbb{R}_+ \to \mathbb{C}$ *be given by*

$$\varphi_\lambda(x) = e^{-\lambda|\log x|} = \begin{cases} x^{-\lambda}, & x \geq 1 \\ x^{\lambda}, & x \leq 1. \end{cases}$$

Then we can prove that φ_λ *is a reflection positive function with respect to the triple* (G, τ, S)*.*

Example 4.14. *Let* $(G, \tau, S) = (\mathbb{R}, -\mathrm{id}_{\mathbb{R}}, \mathbb{R}_+)$*. Then we can prove that for* $0 < s < 1$*, we have*

$$D(\varphi) := \int_{\mathbb{R}} \varphi(x)\ |x|^{-s}\ dx$$

is a reflection positive distribution with respect to the triple (G, τ, S)*.*

Example 4.15. *Let* $(G, \tau, S) = (\mathbb{R}^{\times}, 1/\mathrm{id}_{\mathbb{R}}, (-1,1))$*. Then we can prove that for* $0 < s < 1$*, we have*

$$D(\varphi) := \int_{\mathbb{R}} \varphi(x)\ |x|^{\frac{s}{2}} |x-1|^{-s}\ dx$$

is a reflection positive distribution with respect to the triple (G, τ, S)*.*

4.4 Reflection Positivity for Complementary Series Representations of the Conformal Group

This section can be considered as an application of the previous sections.

Definition 4.16. *Let* $\mu_{\mathbb{S}}$ *be the* $O_{n+1}(\mathbb{R})$*-invariant measure on* $\mathbb{S}^n$*, which, in stereographic coordinates, is given by*

$$d\mu_{\mathbb{S}}(x) = \frac{2^n}{(1 + \|x\|^2)^n} dx.$$

Lemma 4.17. *For* $Q(x,y) := (1 - \langle x, y \rangle)^{-s/2}$ *and* $0 \leq s < n$*, the measure*

$$d\mu(x,y) := Q(x,y) d\mu_{\mathbb{S}}(x) d\mu_{\mathbb{S}}(y)$$

on $\mathbb{S}^n \times \mathbb{S}^n$ *is a finite positive Radon measure which is positive definite.*

Proof. See [24, Lemma 5.5, p. 2211]. □

Lemma 4.18. *For $f \in C(\mathbb{S}^n)$, the function*

$$\Gamma(f) : \mathbb{S}^n \to \mathbb{C}, \quad \Gamma(f)(y) := \int_{\mathbb{S}^n} f(x) Q(x,y) d\mu_{\mathbb{S}}(x)$$

is continuous.

Proof. See [24, Lemma 5.6, p. 2211]. □

Definition 4.19. *(The Hilbert spaces $\mathcal{H}_s$, $0 < s < n$) Let $\mathcal{H}_s := \mathcal{H}_\mu \subseteq \mathcal{M}(\mathbb{S}^n)$ be the Hilbert space of measures which corresponds to the measure μ. It is the completion of the range of the map*

$$T_\mu : C(\mathbb{S}^n) \to \mathcal{M}(\mathbb{S}^n), \quad f \mapsto \Gamma(f)\mu_{\mathbb{S}},$$

with respect to the inner product

$$\langle \Gamma(f_1)\mu_{\mathbb{S}}, \Gamma(f_2)\mu_{\mathbb{S}} \rangle = \langle f_1, f_2 \rangle_\mu.$$

Lemma 4.20. *The prescription*

$$\pi_s(g)\nu := J_{g^{-1}}^{s/2-n} g_* \nu$$

defines a unitary representation of $G = O^+_{1,n+1}(\mathbb{R})$ on the Hilbert subspace $\mathcal{H}_s \subseteq \mathcal{M}(\mathbb{S}^n)$.

Proof. See [24, Lemma 5.8, p. 2212]. □

Theorem 4.21. *The function $\|x\|^{-s}$ is positive definite on the involutive semigroup $(\mathbb{R}^n_+, \sharp)$ if and only if $s = 0$ or $s \geqslant \max(0, n-2)$.*

Proof. See [24, Proposition 6.1, p. 2215]. □

Theorem 4.22. *The kernel*

$$R(x,y) := (1 - 2\langle x, y\rangle + \|x\|^2\|y\|^2)^{-s/2}$$

on the open unit ball $\mathcal{D} \subseteq \mathbb{R}^n$ is positive definite if and only if

$$s = 0 \quad or \quad s \geqslant \max(0, n-2).$$

Proof. See [24, Proposition 6.2, p. 2215]. □

Theorem 4.23. *Let $\sigma \in G$ be the conformal reflection in $\partial\mathcal{D}$. Then*

$$S_{\mathcal{D}} = H \exp(C), \quad where \quad H = O^{+}_{1,n}(\mathbb{R}) \subseteq G^{\sigma},$$

and $C \subseteq \mathfrak{q}$ is a closed convex $\mathrm{Ad}(H)$-invariant cone.

Proof. See [24, Proposition 6.5, p. 2217]. □

Lemma 4.24. *if $s = 0$ or $s \geq n-2$, then the closed subspace*

$$\widetilde{\mathcal{E}_+} := \overline{\{\Gamma(f)\mu_{\mathbb{S}} : \mathrm{Supp}(f) \subseteq \mathcal{D}\}} \subseteq \mathcal{H}_s$$

is θ-positive.

Proof. See [24, Lemma 6.6, p. 2218]. □

Theorem 4.25. *For $x \in \mathbb{S}^{n-1} = (\mathbb{S}^n)^{\sigma}$ let $\delta_x \in \mathcal{H}_s^{-\infty}$ be the delta measure in x. The triple $(\pi_s, \mathcal{H}_s, \delta_x)$ is a reflection positive distribution cyclic representation for $(G, \tau, S^0_{\mathcal{D}})$ if $s = 0$ or $n-2 \leq s < n$.*

Proof. See [24, Theorem 6.7, p. 2219]. □

4.5 Reflection Positivity on the Circle Group

The material of this section is taken from [25]. Let $G = \mathbb{T}_\beta := \mathbb{R}/\beta\mathbb{Z}$ and let $[t] := t + \beta\mathbb{Z}$. Then $\mathbb{T}^+_\beta := \{[t] \in \mathbb{T}_\beta : 0 < t < \beta/2\}$. We fix the involution $\tau_\beta(z) = z^{-1}$.

Definition 4.26. *Let $\mathcal{H}$ be a Hilbert space. A weak operator continuous function $\varphi : \mathbb{T}_\beta \to B(\mathcal{H})$ is called reflection positive with respect to $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$ if and only if it is positive definite and the kernel $(\varphi(t+s))_{0<t,s<\beta/2}$ is positive definite, which is equivalent to the positive definiteness of the kernel*

$$K_\varphi(t,s) := \varphi\left(\frac{t+s}{2}\right),\ 0 < t, s < \beta.$$

Theorem 4.27. *A β-periodic weak operator continuous function $\varphi : \mathbb{T}_\beta \to B(\mathcal{H})$ is reflection positive with respect to $(\mathbb{T}_\beta, \mathbb{T}_\beta^+, \tau_\beta)$ if and only if there exists a Herm$(\mathcal{H})_+$-valued measure μ_+ on $[0,\infty)$ such that*

$$\varphi(t) = \int_0^\infty e^{-x\lambda} + e^{-(\beta-x)\lambda}\ d\mu_+(\lambda) \quad \text{for} \quad 0 \le t \le \beta.$$

Then the measure μ_+ is uniquely determined by φ.

Proof. See [25, p. 3]. □

Chapter 5
Reflection Positivity and Causal Spaces

In this chapter, many theorems in [14] were rewritten so that we can use them easily. The main theorem of this chapter gives us a family of reflection positive representations on the Cayley type symmetric spaces. The main tool that will be used in this chapter to perform the analytic continuation is the Lüscher-Mack theorem.

5.1 Reflection Positivity for Complementary Series

We adopt the notations in Section 2.8. If G/H is NCC then $\mathfrak{q}_p^{H\cap K}$ is one-dimensional and there exists an element $X^0 \in \mathfrak{q}_p^{H\cap K}$ such that $\mathfrak{h}_k \oplus \mathfrak{q}_p = \mathfrak{z}_{\mathfrak{g}}(X^0)$. We can normalize X^0 such that $\mathrm{ad}X^0$ has eigenvalues $0, 1$ and -1. Let $\mathfrak{a} := \mathbb{R}X^0$,

$$\mathfrak{n} := \{X \in \mathfrak{g} \mid [X^0, X] = X\},$$

and

$$\overline{\mathfrak{n}} := \{X \in \mathfrak{g} \mid [X^0, X] = -X\}.$$

Let

$$\mathfrak{m} := \{X \in \mathfrak{z}_{\mathfrak{g}}(X^0) \mid B(X, X^0) = 0\}$$

where B is the Killing form of $\mathfrak{g}$. Then

$$\mathfrak{p}_{\max} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is the maximal parabolic subalgebra of $\mathfrak{g}$.

Assume from now on that $G \subseteq G_{\mathbb{C}}$ where $G_{\mathbb{C}}$ is the simply connected, connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We will assume that $H = G^{\tau}$. Then $H \cap K =$

$Z_K(X^0)$. Let $A := \exp \mathfrak{a}$, $N := \exp \mathfrak{n}$ and $\overline{N} := \exp \overline{\mathfrak{n}}$. Let M_o be the analytic subgroup of G corresponding to $\mathfrak{m}$ and let $M := (H \cap K)M_o$. Then M is a closed and τ-stable subgroup of G, $M \cap A = \{1\}$ and $MA = Z_G(A)$. Let $P_{\max} := N_G(\mathfrak{p}_{\max})$. Then $P_{\max} = MAN$.

Theorem 5.1. *$HP_{\max}$ is open in G and contained in $\overline{N}P_{\max}$.*

Proof. See [14, Lemma 5.3, p. 39]. □

Let $\mathfrak{a}_q$ be a maximal abelian subalgebra of $\mathfrak{p}$ containing X^0. Then $\mathfrak{a}_q \subset \mathfrak{q}_p$ and $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{q}$. Let Δ be the set of roots of $\mathfrak{a}_q$ in $\mathfrak{g}$. Then $\Delta = \Delta_0 \cup \Delta_+ \cup \Delta_-$, where $\Delta_0 = \{\alpha \in \Delta \mid \alpha(X^0) = 0\}$ and $\Delta_\pm = \{\alpha \in \Delta \mid \alpha(X^0) = \pm 1\}$. Choose a positive system Δ_0^+ in Δ_0, and let $\Delta^+ = \Delta_0^+ \cup \Delta_+$. Two roots $\alpha, \beta \neq \pm\beta$ are called *strongly orthogonal* if $\alpha \pm \beta$ is not a root. Choose a maximal set of strongly orthogonal roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ in Δ_+ such that γ_r is the maximal root in Δ_+, γ_{r-1} is the maximal root in Δ_+ strongly orthogonal in to γ_r, γ_{r-2} is the maximal root in Δ_+ strongly orthogonal in to γ_r and γ_{r-1}, etc. Choose $H_j \in \mathfrak{a}_q$ such that $\langle \gamma_i, H_j \rangle = 2\delta_{ij}$ and $H_j \in [\mathfrak{g}_{\gamma_j}, \mathfrak{g}_{-\gamma_j}]$. Choose $X_j \in \mathfrak{g}_{\gamma_j}$, such that with $X_{-j} := \tau(X_j) = -\theta(X_j)$, we have $H_j = [X_j, X_{-j}]$. Let $\log := (\exp|_{\overline{\mathfrak{n}}})^{-1} : \overline{N} \to \overline{\mathfrak{n}}$. Define $\zeta : \overline{N}P_{\max}/P_{\max} \to \overline{\mathfrak{n}}$ by

$$\zeta(\overline{n}P_{\max}) = \log(\overline{n}).$$

Theorem 5.2. *Let $\Omega = \mathrm{Ad}(L)\{\sum_{j=1}^r t_j X_{-j} \mid \forall j : -1 < t_j < 1\}$. Then Ω is convex,*

$$HP_{\max} = (\exp \Omega)P_{\max},$$

and ζ induces an H-isomorphism $H/L \cong \Omega$.

Proof. See [14, Theorem 5.6, p. 41]. □

Definition 5.3. *Let*

$$S(H, P_{\max}) := \{g \in G \mid gH \subset HP_{\max}\}.$$

Then $S(H, P_{\max})$ is a closed semigroup invariant under $s \mapsto s^\sharp = \tau(s^{-1})$. For $g \in G$ and $X \in \overline{\mathfrak{n}}$ such that $g \exp X \in \overline{N} P_{\max}$ define $g \cdot X \in \overline{\mathfrak{n}}$ and $a(g, X) \in A$ by $g \exp X \in \exp(g \cdot X) M a(g, X) N$. Then $g \cdot X = \zeta(g \exp X)$ and $a(g, X)$ is defined for $g \in S(H, P_{\max})$ and $X \in \Omega$. The map $(g, X) \mapsto g \cdot X$ transfers the canonical action on $G/P_{\max}$, restricted to the open set $HP_{\max}/P_{\max}$, to Ω.

Theorem 5.4. *(1) Let $s, r \in S(H, P_{\max})$ and $X \in \Omega$. Then $(sr) \cdot X = (s \cdot (r \cdot X))$ and $a(sr, X) = a(s, r \cdot X) a(r, X)$.*

(2) Let $g = ma \in MA$ and $X \in \overline{\mathfrak{n}}$. Then $g \exp X \in \overline{N} P_{\max}$, $g \cdot X = \mathrm{Ad}(g) X$, and $a(g, X) = a$.

(3) Let C be an H-invariant pointed and generating cone in $\mathfrak{q}$ containing X^0. Then $S = H \exp C$ is a closed semigroup acting on Ω by contractions. Furthermore $H \times C^o \ni (h, X) \mapsto h \exp X \in S^o$ is a diffeomorphism.

(4) $S(H, P_{\max}) \subset HP_{\max}$.

Proof. See [14, Lemma 5.9, p. 43]. □

Theorem 5.5. *Let $C = C_{\max}$ be the maximal pointed generating cone in $\mathfrak{q}$ containing X^0. Then the following hold:*

(1) $C^o \cap \mathfrak{a} = \{X \in \mathfrak{a}_q \mid \forall \alpha \in \Delta_+ : \alpha(X) > 0\}$.

(2) $S(H, P_{\max}) = H \exp C_{\max}$.

Proof. See [14, Lemma 5.11, p. 44]. □

The Haar measure on compact groups will always be normalized to have total measure one. Let $L := H \cap K$. We normalized our measures such that the following hold:

(1) Let the measure da on A be given by

$$\int_A f(a)\ da = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(a_t)\ dt, \qquad a_t = \exp 2tX_0.$$

(2) Let dX be the Lebesgue measure on $\overline{\mathfrak{n}}$ such that, for $d\overline{n} = \exp(dX)$, then we have

$$\int_{\overline{\mathfrak{n}}} a(\overline{n})^{-2\rho}\ d\overline{n} = 1,$$

where $\rho(X) = \frac{1}{2}\mathrm{Tr}(\mathrm{ad}(X))|\mathfrak{n}$.

(3) The measure on N is $\theta(d\overline{n})$.

(4) Let dh be a Haar measure on H.

(5) We can normalize the invariant measure on G and M such that for $f \in \mathcal{C}_c(G)$, $\mathrm{Supp}(f) \subset HP_{\max}$, we have

$$\int_G f(g)\ dg = \int_H \int_M \int_A \int_N f(hman)a^{2\rho}\ dn\ da\ dm\ dh.$$

Then the invariant measure $d\dot{x}$ on G/H is given by

$$\int_G f(x)\ dx = \int_{G/H} \int_H f(xh)\ dh\ d\dot{x}, \quad f \in \mathcal{C}_c(G),$$

and similarly for K/L.

(6) We fix the Haar measure on M such that $dg = a^{2\rho}\ dk\ dm\ da\ dn$.

Theorem 5.6. *Let the measures be normalized as above. Let $L := H \cap K$. Then the following hold:*

(1) Let $f \in \mathcal{C}_c(\overline{N}MAN)$. Then

$$\int_G f(g)\ dg = \int_{\overline{N}} \int_M \int_A \int_N f(\overline{n}man)a^{2\rho}\ dn\ da\ dm\ d\overline{n}.$$

(2) Let $f \in \mathcal{C}_c(\overline{N})$. For $y \in \overline{N}MAN$ write $y = \overline{n}(y)m_{\overline{N}}(y)\alpha(y)n_{\overline{N}}(y)$. Let $x \in G$. Then

$$\int_{\overline{N}} f(\overline{n}(x\overline{n}))\alpha(x\overline{n})^{-2\rho}\ dg = \int_{\overline{N}} f(\overline{n})\ d\overline{n}.$$

(3) Let $f \in \mathcal{C}(K/L)$. For $g \in G$ write $g = k(g)m(g)a(g)n(g)$ according to $G = KMAN$. Then

$$\int_{K/L} f(\dot{k})\ d\dot{k} = \int_{\overline{N}} f(k(\overline{n})b)a(\overline{n})^{-2\rho}\ d\overline{n}.$$

(4) Let $f \in \mathcal{C}(K/L)$ and let $x \in G$. Then

$$\int_{K/L} f(\dot{k})\ d\dot{k} = \int_{K/L} f(k(xk)b)a(xk)^{-2\rho}\ d\dot{k}.$$

(5) Assume that $\mathrm{Supp}(f) \subset H/L \subset K/L$. *Then*

$$\int_{K/L} f(\dot{k})\ d\dot{k} = \int_{H/L} f(k(h)b)a(h)^{-2\rho}\ d\dot{h}.$$

(6) Let $f \in \mathcal{C}_c(\overline{N})$, $\mathrm{Supp}(f) \subset \exp(\Omega) \subset \overline{N}$. *Then*

$$\int_{\overline{N}} f(\overline{n})\ d\overline{n} = \int_{H/L} f(\overline{n}(h))\alpha(h)^{-2\rho}\ d\dot{h}.$$

(7) For $x \in HP_{\max}$ write $x = h(x)m_H(x)a_H(x)n_H(x)$ with $h(x) \in H$, $m_H(x) \in M$, $a_H(x) \in A$, and $n_H(x) \in N$. Let $f \in \mathcal{C}_c^\infty(H/L)$ and let $x \in G$ be such that $xHP_{\max} \subset HP_{\max}$. Then

$$\int_{H/L} f(h(xh)b)a_H(xh)^{-2\rho}\ d\dot{h} = \int_{H/L} f(\dot{h})\ d\dot{h}.$$

Proof. See [14, Lemma 5.12, p. 45]. □

Definition 5.7. *We identify* $\mathfrak{a}_{\mathbb{C}}^*$ *with* $\mathbb{C}$ *by*

$$\mathfrak{a}_{\mathbb{C}}^* \ni \lambda \mapsto 2\lambda(X^0) \in \mathbb{C}.$$

Then ρ *corresponds to* $\dim \mathfrak{n}$. *For* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, *let* $\mathcal{C}^\infty(\lambda)$ *be the space of* $\mathcal{C}^\infty$*-functions* $f : G \to \mathbb{C}$ *such that, for* $a_t = \exp t(2X^0)$,

$$f(gma_t n) = e^{-(\lambda+\rho)t} f(g) = a_t^{-(\lambda+\rho)} f(g).$$

Thus we have

$$\begin{aligned} \mathcal{C}^\infty(G) &:= \{f : G \to \mathbb{C} \mid f \text{ is } \mathcal{C}^\infty\text{-function }\}, \\ \mathcal{C}^\infty(\lambda) &:= \{f \in \mathcal{C}^\infty(G) \mid f(gman) = a^{-(\lambda+\rho)} f(g)\}. \end{aligned}$$

Define an inner product on $\mathcal{C}^\infty(\lambda)$ *by*

$$\langle f, g \rangle_{L^2} := \int_K \overline{f(k)} g(k)\ dk = \int_{K/L} \overline{f(k)} g(k)\ d\dot{k}.$$

Then $\mathcal{C}^\infty(\lambda)$ *becomes a pre-Hilbert space. We denote by* $\mathcal{H}_\lambda$ *the completion of* $\mathcal{C}^\infty(\lambda)$. *Thus we have*

$$\mathcal{H}_\lambda \ := \ \mathcal{C}^\infty(\lambda)^\sim \quad \text{with respect to} \ \ \langle \cdot, \cdot \rangle_{L^2}.$$

Here, $^\sim$ *stands for the completion of the pre-Hilbert space. Define* π_λ *by*

$$[\pi_\lambda(x) f](g) := f(x^{-1} g), \quad x, g \in G, \quad f \in \mathcal{C}^\infty(\lambda).$$

Then $\pi_\lambda(x)$ *is bounded, so it extends to a bounded operator on* $\mathcal{H}_\lambda$, *which we denote by the same symbol. Furthermore* π_λ *is a continuous representation of* G *which is unitary if and only if* $\lambda \in i\mathbb{R}$. *We can realize* $\mathcal{H}_\lambda$ *as* $\mathbf{L}^2(K/L)$ *and as* $\mathbf{L}^2(\overline{N}, a(\overline{n})^{2Re(\lambda)} d\overline{n})$ *by restriction. In the first realization the representation* π_λ *becomes*

$$[\pi_\lambda(x) f](k) := a(x^{-1}k)^{-\lambda-\rho} f(k(x^{-1}k))$$

and in the second

$$[\pi_\lambda(x)f](\overline{n}) := \alpha(x^{-1}\overline{n})^{-\lambda-\rho} f(\overline{n}(x^{-1}\overline{n})).$$

Theorem 5.8. *The pairing*

$$\mathcal{H}_\lambda \times \mathcal{H}_{-\overline{\lambda}} \ni (f,g) \mapsto \langle f,g\rangle_{L^2} := \int_K \overline{f(k)}g(k)\ dk = \int_{K/L} \overline{f(k)}g(k)\ d\dot{k}$$

is G-invariant, i.e.,

$$\langle \pi_\lambda(x)f, g\rangle_{L^2} = \langle f, \pi_{-\overline{\lambda}}(x^{-1})g\rangle_{L^2}.$$

Proof. See [14, Lemma 5.13, p. 47]. □

Remark 5.9. *We can now see that $(\pi_\lambda, \mathcal{H}_\lambda)$ is unitary if $-\overline{\lambda} = \lambda$.*

Theorem 5.10. *Let $\mathcal{H}(H,\lambda) := \overline{\{f \in \mathcal{C}^\infty(\lambda) \mid \operatorname{Supp}(f) \subset HP_{\max}\}}$. Then the following assertions hold.*

(1) The restriction map induces an isometry of $\mathcal{H}_\lambda$ onto $\mathbf{L}^2(\overline{N}, a(\overline{n})^{2Re(\lambda)}d\overline{n})$.

(2) On $\overline{N}$ the invariant pairing $\langle\cdot,\cdot\rangle_{L^2}$ is given by

$$\langle f,g\rangle_{L^2} = \int_{\overline{N}} \overline{f(\overline{n})}g(\overline{n})\ d\overline{n}, \quad f \in \mathcal{H}_\lambda, \quad g \in \mathcal{H}_{-\overline{\lambda}}.$$

(3) The map

$$\mathcal{H}(H,\lambda) \ni f \mapsto f|_H \in \mathbf{L}^2(H/L, a(h)^{2\rho}d\dot{h})$$

is an isometry.

(4) Let $f \in \mathcal{H}_\lambda,\ g \in \mathcal{H}_{-\overline{\lambda}}$ and assume that $\operatorname{Supp}(fg) \subset HP_{\max}$. Then

$$\langle f,g\rangle_{L^2} = \int_{H/L} \overline{f(h)}g(h)\ d\dot{h}.$$

Proof. See [14, Lemma 5.15, p. 48]. □

Theorem 5.11. *G/H is of Cayley type if and only if G/K is a tube-domain $G/K \cong \mathbb{R}^n + i\Omega$, where Ω is an open self-dual cone isomorphic to H/L. Thus G/H is locally isomorphic to one of the following spaces (where we denote by the subscript $+$ the group of elements having positive determinant):*

$$\begin{aligned}
&\mathrm{SU}(n,n)/\mathrm{GL}(n,\mathbb{C})_+,\\
&\mathrm{SO}^*(4n)/SU^*(2n)\mathbb{R}_+,\\
&\mathrm{Sp}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})_+,\\
&\mathrm{SO}(2,n)/\mathrm{SO}(1,n-1)\mathbb{R}_+,\\
&E_{7(-25)}/E_{6(-25)}\mathbb{R}_+.
\end{aligned}$$

Proof. See [14, Lemma 5.21, p. 51]. □

Theorem 5.12. *Assume that G/H is of Cayley type. There exists commuting homomorphisms $\varphi_j^{\mathbb{C}} : \mathrm{SL}(2,\mathbb{C}) \to G$ such that $\varphi_1^{\mathbb{C}}(\mathrm{SU}(1,1)) \subset H$ and $X^0 = \frac{1}{2}\sum_j \varphi_j^{\mathbb{C}}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\right)$. Let*

$$w = \varphi_1^{\mathbb{C}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdots \varphi_r^{\mathbb{C}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Then $\mathrm{Ad}(w)(X^0) = -X^0$.

Proof. See [14, Lemma 5.20, p. 50]. □

Theorem 5.13. *Let $A_w^\lambda : \mathcal{H}_\lambda \to \mathcal{H}_{-\lambda}$ be given by*

$$[A_w^\lambda f](x) := \int_{\overline{N}} f(xw\overline{n})\ d\overline{n}.$$

Then for Re(λ) "big", A_w^λ is an intertwining operator.

Proof. See [14, p. 50]. □

Definition 5.14. *Define a new invariant bilinear form on* $\mathcal{C}^\infty(\lambda)$ *by*

$$\langle f, g\rangle := \langle f, A_w^\lambda g\rangle_{L^2}.$$

If there exists a (maximal) constant $R > 0$ *such that the invariant bilinear form* $\langle\cdot,\cdot\rangle$ *is positive definite for* $|\lambda| < R$*, then we call the resulting unitary representation the complementary series. Otherwise we set* $R = 0$*. In the cases where* $\langle\cdot, A_w^\lambda\cdot\rangle_{L^2}$ *is positive definite we complete* $\mathcal{C}^\infty(\lambda)$ *with respect to this new product and we denote the resulting space by* $\mathcal{E}_{\lambda,w}$*. Thus we have*

$$\mathcal{E}_{\lambda,w} \quad := \quad \mathcal{C}^\infty(\lambda)^\sim \quad \textit{with respect to} \quad \langle\cdot, A_w^\lambda\cdot\rangle_{L^2}.$$

Theorem 5.15. *For the Cayley-type symmetric spaces the constant* R *is given by*

$$\begin{aligned}
\mathrm{SU}(n,n) \quad &: \quad R = \begin{cases} n & n \textit{ odd} \\ 0 & n \textit{ even}, \end{cases} \\
\mathrm{SO}^*(4n) \quad &: \quad R = n, \\
\mathrm{Sp}(n,\mathbb{R}) \quad &: \quad R = \begin{cases} n/2 & n \textit{ odd} \\ 0 & n \textit{ even}, \end{cases} \\
\mathrm{SO}_o(n,2) \quad &: \quad R = \begin{cases} 0 & n \equiv 0 \textit{ mod } 4 \\ 1 & n \equiv 1,3 \textit{ mod } 4 \\ 2 & n \equiv 2 \textit{ mod } 4, \end{cases} \\
E_{7(-25)} \quad &: \quad R = 3.
\end{aligned}$$

Proof. See [14, Lemma 5.21, p. 51]. □

Theorem 5.16. *For* $f \in \mathcal{E}_{\lambda,w}$ *let* $J_w(f)(x) := f(\tau(xw))$*. Then the following hold:*

(1) $J_w(f)(x) := f(\tau(x)w^{-1})$*.*

(2) $J_w(f) \in \mathcal{E}_{\lambda,w}$ *and* $A_w^\lambda J_w = J_w A_w^\lambda$*.*

(3) $J_w(f) : \mathcal{E}_{\lambda,w} \to \mathcal{E}_{\lambda,w}$ *is a unitary isomorphism.*

(4) $(J_w)^2 = \mathrm{id}$.

(5) For $x \in G$*, we have* $J_w \circ \pi_\lambda(x) = \pi_\lambda(\tau(x)) \circ J_w$.

Proof. See [14, Lemma 5.23, p. 52]. □

Definition 5.17. *Assume that* G/H *is non-compactly causal. Then we can define* $A_w^\lambda J_w$ *directly (even if individual operators* A_w^λ *and* J_w *do not exists) by*

$$[A_w^\lambda J_w](f)(x) := \int_{\overline{N}} f(\tau(x)\overline{n})\ d\overline{n}.$$

Theorem 5.18. *Assume that* G/H *is non-compactly causal. Then* $A_w^\lambda J_w$ *intertwines* π_λ *and* $\pi_{-\lambda} \circ J_w$ *if* $A_w^\lambda J_w$ *has no poles at* λ.

Proof. See [14, Lemma 5.24, p. 53]. □

Theorem 5.19. *Let* $f \in \mathcal{C}^\infty(\lambda)$*. Then*

(1)

$$[A_w^\lambda J_w](f)(\overline{n}) = \int_{\overline{N}} f(x)\alpha(\tau(\overline{n})^{-1}x)^{\lambda-\rho}\ dx.$$

(2) If $\mathrm{Supp}(f) \subset HP_{\max}$*, then for* $h \in H$

$$[A_w^\lambda J_w](f)(h) = \int_{H/L} f(x)\alpha(h^{-1}x)^{\lambda-\rho}\ d\dot{x}.$$

Proof. See [14, Theorem 5.25, p. 53]. □

Theorem 5.20. *Let* $f, g \in \mathcal{C}^\infty(\lambda)$*. Then*

(1)

$$\langle f, g\rangle_{J_w} = \int_{\overline{N}} \int_{\overline{N}} \overline{f(x)} g(y)\alpha(\tau(x)^{-1}y)^{\lambda-\rho}\ dxdy.$$

(2) If f and g have support in $HP_{\max}$, then

$$\langle f, g\rangle_{J_w} = \int_{H/L}\int_{H/L} \overline{f(h)}g(k)\alpha(h^{-1}k)^{\lambda-\rho}\, dhdk.$$

Proof. See [14, Corollary 5.23, p. 52]. □

Now define the following spaces that we need them in the upcoming theorems.

$$\begin{aligned}
\mathcal{C}_c^\infty(\Omega,\lambda) &:= \text{closure}\left\{ f \in \mathcal{C}^\infty(\lambda) \;\middle|\; \begin{matrix} \mathrm{Supp}(f) \subset \exp(\Omega)P_{\max} = HP_{\max}, \\ \mathrm{Supp}(f|_{\exp(\Omega)}) \text{ is compact} \end{matrix} \right\}, \\
\mathcal{E}^+_{\lambda,w} &:= \text{closure } \mathcal{C}_c^\infty(\Omega,\lambda) \text{ in } \mathcal{E}_{\lambda,w}, \\
\widehat{\mathcal{E}_{\lambda,w}} &:= (\mathcal{E}^+_{\lambda,w}/\mathcal{N})^\sim \quad \text{with respect to } \langle\cdot, A_w^\lambda J_w\ \cdot\rangle_{L^2}, \\
\mathcal{E}^+_\lambda &:= \mathcal{C}_c^\infty(\Omega,\lambda)^\sim \quad \text{with respect to } \langle\cdot, A_w^\lambda J_w\cdot\rangle_{L^2}, \\
\widehat{\mathcal{E}_\lambda} &:= (\mathcal{E}^+_\lambda/\mathcal{N})^\sim \quad \text{with respect to } \langle\cdot, A_w^\lambda J_w\ \cdot\rangle_{L^2}.
\end{aligned}$$

Theorem 5.21. *Assume that G/H is non-compactly causal. Let $s \in S$ and $f \in \mathcal{C}_c^\infty(\Omega,\lambda)$. Then $\pi_\lambda(s)f \in \mathcal{C}_c^\infty(\Omega,\lambda)$, i.e., $\mathcal{C}_c^\infty(\Omega,\lambda)$ is S-invariant.*

Proof. See [14, Lemma 5.27, p. 54]. □

Let $(\pi, \mathbf{H})$ be an admissible representation of G^c and let $\mathbf{H}_{K^c}$ be the space of K^c-finite elements in $\mathbf{H}$. For $\delta \in \widehat{K^c}$, let $\mathbf{H}(\delta)$ be the subspace of K^c-finite vectors of type δ, i.e.,

$$\mathbf{H}(\delta) = \bigcup_{T\in \mathrm{Hom}_{K^c}(\mathbf{H}_\delta,\mathbf{H})} T(\mathbf{H}_\delta),$$

where $\mathbf{H}_\delta$ is the representation space of δ.

Definition 5.22. *Let $(\pi, \mathbf{H})$ is called the highest-weight representation of G^c (with respect to Δ^+) if there exists a $\delta \in \widehat{K^c}$ (we call δ for the minimal K^c-type of π) such that*

(1) $d\pi(\mathfrak{n}_{\mathbb{C}})\mathbf{H}(\delta) = 0$*,*

(2) $d\pi(U(\overline{\mathfrak{n}}))\mathbf{H}(\delta) = \mathbf{H}_{K^c}$*.*

Theorem 5.23. *(Vergne-Rossi, Wallach). Assume that G/H is non-compactly causal and that G^c is simple. Let $\lambda_0 \in \mathfrak{a}^*$ be such that $\langle \lambda_0, H_r \rangle = 1$. Let $\gamma = \langle \lambda_0, 2X^0 \rangle$ and let*

$$L_{pos} := -\frac{\gamma(r-1)d}{2}.$$

Then the following hold:

(1) For $\lambda - \rho \leqslant L_{pos}$ there exists a unitary irreducible highest weight representation $(\rho_\lambda, \mathbf{K}_\lambda)$ of G^c with one-dimensional minimal K^c-type $\lambda - \rho$.

(2) If G/H is of Cayley-type, then $\gamma = r$. Furthermore $\lambda \leqslant L_{pos}$ if and only if $\lambda \leqslant r$.

Proof. See [14, Theorem 5.29, p. 55]. □

Theorem 5.24. *For the Cayley-type symmetric spaces the highest weight representation $(\rho_\lambda, \mathbf{K}_\lambda)$ exists for λ in the following half-line:*

$$\begin{aligned} \mathrm{SU}(n,n) &\ :\ \lambda \leqslant n \\ \mathrm{SO}^*(4n) &\ :\ \lambda \leqslant 2n \\ \mathrm{Sp}(n,\mathbb{R}) &\ :\ \lambda \leqslant n \\ \mathrm{SO}_o(n,2) &\ :\ \lambda \leqslant 2 \\ E_{7(-25)} &\ :\ \lambda \leqslant 3. \end{aligned}$$

In particular we have that $(\rho_\lambda, \mathbf{K}_\lambda)$ is defined for $\lambda \in [-R, R]$.

Proof. See [14, Lemma 5.30, p. 56]. □

Theorem 5.25. *Assume that G/H is non-compactly causal. For $\lambda - \rho \leqslant L_{pos}$ there exists a unitary irreducible highest weight representation $(\rho_\lambda, \mathbf{K}_\lambda)$ of G^c and a lowest K^c-type vector u of norm one such that for every $h \in H$*

$$\alpha(h)^{\lambda-\rho} = \langle u, \rho_\lambda(h)u \rangle.$$

Hence the kernel

$$(H \times H) \ni (h,k) \mapsto \alpha(k^{-1}h)^{\lambda-\rho} \in \mathbb{R}$$

is positive semidefinite. In particular $\langle .|. \rangle_{J_w}$ is positive semidefinite on $\mathcal{C}_c^\infty(\Omega, \lambda)$ for $\lambda - \rho \leqslant L_{pos}$.

Proof. See [14, Lemma 5.32, p. 57]. □

5.2 Main Theorems

In this section, we introduced the main theorems given in [14].

Theorem 5.26. *Assume that G/H is non-compactly causal and such that there exists $w \in K$ such that $\mathrm{Ad}(w)|_{\mathfrak{a}} = -1$. Let π_λ be a complementary series such that $\lambda - \rho \leqslant L_{pos}$. Let C be the minimal H-invariant cone in $\mathfrak{q}$ such that $S(C)$ is contained in the contraction semigroup of $HP_{\max}$ in $G/P_{\max}$. Let Ω be the bounded realization of H/L in $\overline{\mathfrak{n}}$. Let*

$$J_w(f)(x) := f(\tau(x)w^{-1}).$$

Let $\mathcal{E}_{\lambda,w}^+$ be the closure of $\mathcal{C}_c^\infty(\Omega,\lambda)$ in $\mathcal{E}_{\lambda,w}$. Then the following hold:

(1) $(G, \tau, \pi_\lambda, C, J_w, \mathcal{E}_{\lambda,w}^+)$ satisfies the positivity conditions (RP0)-(RP2) in Definition 4.2 and the assumptions of Theorem 4.3.

(2) π_λ defines a contractive representation $\widehat{\pi}_\lambda$ of $S(C)$ on $\widehat{\mathcal{E}}_\lambda$ such that

$$\widehat{\pi}_\lambda(s)^* = \widehat{\pi}_\lambda(\tau(s)^{-1}).$$

(3) *There exists a unitary representation* $\widehat{\pi}_\lambda^c$ *of* G^c *such that*

(i) $d\widehat{\pi}_\lambda^c(X) = d\widehat{\pi}_\lambda(X)\ \forall X \in \mathfrak{h}$.

(ii) $d\widehat{\pi}_\lambda^c(iY) = id\widehat{\pi}_\lambda(Y)\ \forall Y \in C$.

Proof. See [14, Theorem 5.33, p. 58]. $\square$

Theorem 5.27. *Assume that* $\lambda - \rho \leqslant L_{pos}$. *Let* ρ_λ, $\mathbf{K}_\lambda$ *and* u *be as specified in Theorem 5.25, and let* $f, g \in \mathcal{C}_c^\infty(\Omega, \lambda)$ *and* $s \in S(C)$. *Define* $\rho_\lambda(f)u := \int_{H_o} f(h)\rho_\lambda(h)u\, dh$. *Then the following hold:*

(1) $\langle f, [A_w^\lambda J_w](g)\rangle_{L^2} = \langle \rho_\lambda(f)u, \rho_\lambda(g)u\rangle$,

(2) $\rho_\lambda(\pi_\lambda(s)f)u = \rho_\lambda(s)\rho_\lambda(f)u$,

(3) $\pi_\lambda(s)$ *passes to a contractive operator* $\widehat{\pi}_\lambda(s)$ *on* $\widehat{\mathcal{E}_\lambda}$ *such that*

$$\widehat{\pi}_\lambda(s)^* = \widehat{\pi}_\lambda(\tau(s)^{-1}).$$

Proof. See [14, Lemma 5.34, p. 58]. $\square$

Theorem 5.28. *(Identification Theorem) Assume that* G/H *is non-compactly causal and that* $\lambda - \rho \leqslant L_{pos}$. *Let* ρ_λ, $\mathbf{K}_\lambda$ *and* $u \in \mathbf{K}_\lambda$ *be as in Theorem 5.25. Then the following hold:*

(1) *There exists a continuous contractive representation* $\widehat{\pi}_\lambda$ *of* $S_o(C)$ *on* $\widehat{\mathcal{E}_\lambda}$ *such that* $\widehat{\pi}_\lambda(s)^* = \widehat{\pi}_\lambda(\tau(s)^{-1}), \quad \forall s \in S_o(C)$.

(2) *There exists a unitary representation* $\widehat{\pi}_\lambda^c$ *of* G^c *such that*

(i) $d\widehat{\pi}_\lambda^c(X) = d\widehat{\pi}_\lambda(X)\ \forall X \in \mathfrak{h}$.

(ii) $d\widehat{\pi}_\lambda^c(iY) = id\widehat{\pi}_\lambda(Y)\ \forall Y \in C$.

(3) The map

$$\mathcal{C}_c^\infty(\Omega,\lambda) \ni f \mapsto \rho_\lambda(f)u \in \mathbf{K}_\lambda$$

extends to an isometry $\widehat{\mathcal{E}_\lambda} \cong \mathbf{K}_\lambda$ *intertwining* $\widehat{\pi}_\lambda^c$ *and* ρ_λ*. In particular* $\widehat{\pi}_\lambda^c$ *is irreducible and isomorphic to* ρ_λ*.*

Proof. See [14, Theorem 5.35, p. 60]. $\square$

Chapter 6

The Cos^λ Transform

In this chapter, we collect some theorems and properties for an important transform introduced in [27], namely, the Cos^λ transform. This transform will show up in the next chapter and will play an important role in the main theorem of this thesis. We shall see that Cos^λ transform appears as an intertwining operator for the complementary series. Also, it appears as a reflection positive kernel. The material of this chapter is taken from [27].

6.1 The Standard Intertwining Operator

We adopt the notations in Section 2.7. Write, for $x \in G$, $x = k(x)m(x)a(x)n(x)$ according to $G = KMAN$. Also, write, for $x \in \overline{N}MAN$, $x = \overline{n}(x)m_{\overline{N}}(x)\alpha(x)n_{\overline{N}}(x)$. Let $L = K \cap M$ and let $\mathcal{B} = K/L$.

Recall that for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the generalized principal series representation of G on $L^2(\mathcal{B})$ is given by

$$[\pi_\lambda(x)f](kL) := a(x)^{-\lambda-\rho} f(x^{-1} \cdot kL) = a(x)^{-\lambda-\rho} f(k(x^{-1}k)L).$$

For $f \in \mathcal{C}^\infty(\mathcal{B})$, define the standard intertwining operator A^λ by

$$[A^\lambda f](kL) := \int_{\overline{N}} f(k\overline{n} \cdot L)\, d\overline{n} = \int_{\overline{N}} f(k(k\overline{n})L)\, d\overline{n} \tag{6.1}$$

whenever the integral exists, for more details see [27].

The following theorem gives some facts about the intertwining operator A^λ.

Theorem 6.1. *(Vogan-Wallach). For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, write $\lambda = \lambda_R + i\lambda_I$ with $\lambda_R, \lambda_I \in \mathfrak{a}^*$. Then the following holds:*

(1) There exists a constant a_P such that if

$$\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P) := \{\mu \in \mathfrak{a}_{\mathbb{C}}^* \mid (\forall \alpha \in \Delta^+)\langle \mu_R, \alpha \rangle \geqslant a_P\}$$

then the integral 6.1 converges absolutely. Furthermore, there exists a constant $C > 0$ such that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$ and $f \in C^\infty(\mathcal{B})$

$$\|A^\lambda f\|_\infty \leqslant C\|f\|_\infty.$$

(2) If $f \in C^\infty(\mathcal{B})$, then

$$\mathfrak{a}_{\mathbb{C}}^*(a_P) \ni \lambda \mapsto A^\lambda f \in C^\infty(\mathcal{B})$$

is continuous and holomorphic on the interior $\{\mu \in \mathfrak{a}_{\mathbb{C}}^* \mid (\forall \alpha \in \Delta^+)\langle \mu_R, \alpha \rangle > a_P\}$.

(3) The operator A^λ intertwines π_λ and $\pi_{-\lambda}^\theta$. Thus, if $x \in G$ and if $f \in C^\infty(\mathcal{B})$, then

$$A^\lambda(\pi_\lambda(x)f) = \pi_{-\lambda}^\theta(x)(A^\lambda f).$$

Proof. See [27, p. 275] □

Now, as in [27], we can take a_P to be

$$a_P := \max_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle > 0.$$

The following theorem writes the intertwining operator A^λ in a useful form that we need it in the upcoming chapter.

Theorem 6.2. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$ and let $f \in L^p(\mathcal{B})$. Then for $k \in K$*

$$A^\lambda f(k) = \int_K f(h)\ \alpha(k^{-1}h)^{\lambda-\rho}\ dh = \int_{\mathcal{B}} f(b)\ \alpha(k^{-1}b)^{\lambda-\rho}\ db.$$

In particular, A^λ is a convolution operator on $L^2(\mathcal{B})$ if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$.

Proof. See [27, p. 276] □

6.2 The Cos^λ Transform as Intertwining Operators Between Generalized Principal Series Representations of $\mathrm{SL}(n+1,\mathbb{K})$

The material of this section is taken from [27]. Let $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$, or the skew field $\mathbb{H}$ of quaternions. Let $G = \mathrm{SL}(n+1,\mathbb{K})$. Let $\mathrm{Gr}_p(\mathbb{K})$ be the Grassmann manifold of k-dimensional subspaces of $\mathbb{K}^{n+1}$. Set $q := n+1-p$. Define an invariant $\mathbb{R}$-bilinear form on $\mathfrak{g}$ by

$$\langle X, Y\rangle := \frac{n+1}{pq}\mathrm{Re}(\mathrm{Tr}(XY)).$$

Let $z \mapsto \overline{z}$ be the conjugation in $\mathbb{K}$ and let $x = (x_{\nu\mu}) \in \mathrm{M}(n+1,\mathbb{K})$. Let $\overline{x} := (\overline{x}_{\nu\mu})$ and let $x^* := (\overline{x}_{\mu\nu}) = \overline{x}^t$. The homomorphism $\theta : G \to G,\ x \mapsto (x^{-1})^*$ is a Cartan involution on G . The corresponding Cartan involution on $\mathfrak{g}$ is $\theta(X) = -X^*$. We have

$$\begin{aligned} K &= \mathrm{SU}(n+1,\mathbb{K}), \\ \mathfrak{k} &= \{X \in \mathrm{M}(n+1,\mathbb{K}) \mid X^* = -X \text{ and } \mathrm{Tr}(X) = 0\}, \\ \mathfrak{s} &= \{X \in \mathrm{M}(n+1,\mathbb{K}) \mid X^* = X \text{ and } \mathrm{Tr}(X) = 0\}. \end{aligned}$$

We adopt the notation in [5], where we have $G = \mathrm{SL}(n{+}1,\mathbb{R})$ and $K = \mathrm{SO}(n{+}1)$ for $\mathbb{K} = \mathbb{R}$, and $G = \mathrm{SL}(n{+}1,\mathbb{C})$ and $K = \mathrm{SU}(n{+}1)$ for $\mathbb{K} = \mathbb{C}$, and $G = \mathrm{SU}^*(2(n{+}1))$ and $K = \mathrm{Sp}(n+1)$ for $\mathbb{K} = \mathbb{H}$. Let

$$H_0 = \begin{pmatrix} \frac{q}{n+1}I_p & 0 \\ 0 & \frac{-p}{n+1}I_q \end{pmatrix} \in \mathfrak{s}$$

and let $\mathfrak{a} := \mathbb{R}H_0$. Define $\mathfrak{m} = \{X \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) | \langle X, H_0\rangle = 0\}$. Then $\mathfrak{z}(\mathfrak{m}) \cap \mathfrak{s} = \mathfrak{a}$. We have $\Delta = \{\alpha, -\alpha\}$ where $\alpha(H_0) = 1$. Let $\Delta^+ = \{\alpha\}$. Then

$$\begin{aligned} P &= \left\{ p(a,b;X) := \begin{pmatrix} a & X \\ 0 & b \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathrm{GL}(p,\mathbb{K}) \\ b \in \mathrm{GL}(q,\mathbb{K}) \end{matrix} \text{ and } \begin{matrix} \det a \det b = 1, \\ X \in \mathrm{M}(p \times q,\mathbb{K}) \end{matrix} \right\}, \\ L &= \mathrm{S}(\mathrm{U}(p,\mathbb{K}) \times \mathrm{U}(q,\mathbb{K})). \end{aligned}$$

Let $e_1, \cdots, e_{n+1}$ be the standard basis for $\mathbb{K}^{n+1}$ and let $b_0 = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_p \in \mathrm{Gr}_p(\mathbb{K})$. Then $\mathrm{Gr}_p(\mathbb{K}) = K \cdot b_0 \cong K/L = G/P$.

Now, let us give the definition of the cosine function $\mathrm{Cos} : \mathrm{Gr}_p(\mathbb{K}) \times \mathrm{Gr}_p(\mathbb{K}) \to \mathbb{R}$.

Definition 6.3. *Let $b \in \mathrm{Gr}_p(\mathbb{K})$ and view b as a p-dimensional real vector space. Let $E \subset b$ be a convex subset containing the zero vector, such that the volume of E is one. For $c \in \mathrm{Gr}_p(\mathbb{K})$, let $P_c : \mathbb{K}^n \to c$ denote the orthogonal projection onto c. Then define*

$$|\operatorname{Cos}(b,c)| := \mathrm{Vol}_{\mathbb{R}} \left(P_c\left(E\right)\right)^{1/d}.$$

Remark 6.4. *(a) We can see that the definition is independent of E. To see this, let $k \in K$ and note that k act as an orthogonal transformation. Thus $|\operatorname{Cos}(k \cdot b, c)| = |\operatorname{Cos}(b, k^{-1} \cdot c)|$. Now, let $b = k \cdot b_0$ and $c = h \cdot b_0$ with $k, h \in K$. Then*

$$|\operatorname{Cos}(b,c)| = |\operatorname{Cos}(h^{-1} \cdot b, b_0)| = \mathrm{Vol}_{\mathbb{R}} \left(P_{b_0}\left(h^{-1}E\right)\right)^{1/d}.$$

(b) The reason behind calling this function by the name cosine is that if we take $p = 1$, $\mathbb{K} = \mathbb{R}$ and $b = \mathbb{R}x, c = \mathbb{R}y \in \mathrm{Gr}_1(\mathbb{R})$, then

$$|\operatorname{Cos}(b,c)| = \frac{|(x,y)|}{\|x\|\|y\|} = |\cos(\angle(x,y))|,$$

where $\angle(x,y)$ denotes the angle between x and y.

We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by

$$\lambda \mapsto \frac{n+1}{pq}\lambda(H_0) \quad \text{with inverse} \quad z \mapsto z\frac{pq}{n+1}\alpha.$$

Define the Cos^λ-transform $C^\lambda : \mathcal{C}^\infty(\mathcal{B}) \to \mathbb{C}$ by

$$C^\lambda f(k) = \int_{\mathcal{B}} f(b)\, |\operatorname{Cos}(k \cdot b_0, b)|^{\lambda - \rho}\, db,$$

and recall that for $f \in \mathcal{C}^\infty(\mathcal{B})$ and $k \in K$

$$A^\lambda f(k) = \int_{\mathcal{B}} f(b)\ \alpha(k^{-1}b)^{\lambda-\rho}\ db.$$

The following theorem is a very important step towards connecting the cosine transform with the standard intertwining operator. It implies that the Cos^λ transform is nothing but an intertwining operator for the generalized principal series.

Theorem 6.5. *For $x \in G$ and $\lambda \in \mathbb{C}$, we have*

$$\alpha(x)^\lambda = \big|\,\mathrm{Cos}(x \cdot b_0, b_0)\big|^\lambda.$$

In particular, if $b = k \cdot b_0$ and $c = h \cdot b_0 \in \mathcal{B}$, then

$$\alpha(h^{-1}k)^\lambda = |\,\mathrm{Cos}(b, c)|^\lambda,$$

and so, if $\mathrm{Re}\lambda > \rho$, then $C^\lambda = A^\lambda$.

Proof. See [27, p. 285]. □

Chapter 7
Main Results

This chapter is devoted to the main results of this thesis. The main theorem of [14] gives us a family of reflection positive representations. Unfortunately, they were restricted on the Cayley type symmetric spaces, the symmetric spaces depend on the existence of the Weyl group element w. In this chapter, in spite of the nonexistence of the Weyl group element w in the general non compactly causal symmetric spaces, we notice some kind of reflection positivity still shines there! We coin a new definition for such new kind of reflection positivity, namely, *twisted reflection positive representation on a vector space.*

We use some results in [14] and change everything into the compact picture. Then we prove that all of the non compactly causal symmetric spaces give rise to θ-twisted reflection positive representations, where θ is the Cartan involution of the corresponding group. In other words, we produce a rich family of twisted reflection positive representations which can be divided into unitary and non-unitary representations. To be more precise, once we have a positive definite form, we get a unitary representation as in the case of the Cayley type spaces. On the other hand, in the case of the general non-compactly symmetric spaces, the form need not be positive definite and so the representation need not be unitary.

The main tools that will be used in this chapter to perform the analytic continuation are the Lüscher-Mack theorem and some integrability theorems introduced in [23]. We present a new reflection positive representation on the sphere which

is not unitary. We give an elementary proof for the reflection positiveness. This representation can be viewed on both $\mathrm{SO}(n)$ and $\mathrm{SL}(n, \mathbb{R})$.

In Subsection 7.1.1, we introduce the definitions of reflection positive vector space and σ-twisted reflection positive representation on a vector space. Note that the form here is not positive definite. If the form is positive definite, then we get the definition of reflection positive Hilbert space as usual.

In Subsection 7.1.2, we start with giving some lemmas and then transforming some theorems into the compact picture. This picture helps us tremendously, proving that all of the non compactly causal symmetric spaces give rise to θ-twisted reflection positive representations, where θ is the Cartan involution of the corresponding group.

Amazingly, in Subsection 7.1.3, we connect results in [27] and [14] together. We discover very nice examples of θ-twisted reflection positive representations, namely, the generalized principle series with Cos^{λ} transform as an intertwining operator. The Cos^{λ} transform gives us the θ-twist, where θ is the Cartan involution of the corresponding group.

Unlike Subsection 7.1.3, where we used some well known facts and indirect proofs, in Subsection 7.1.4, we give a direct proof for the reflection positivity of the Cos^{λ} transform on $\mathrm{SO}(3)$. In Subsection 7.1.5, we generalize the proof given in Subsection 7.1.3 for all $n \geq 3$. More precisely, we prove directly that the Cos^{λ} transform is reflection positive on $\mathrm{SO}(n)$, for all $n \geq 3$.

To integrate a representation from a group to its dual, we use the Lüscher-Mack theorem or some integrability theorems. In Section 7.2, we generalize the integrability theorem given in Section 3.2.3 to the case of non positive definite distribution.

Section 7.4 gives the relation between the non-compactly causal symmetric spaces and the reflection positive distributions. In other words, it builds up a reflection positive representation for those spaces using the language of the distributions.

Section 7.5 introduces the definition of the σ-twisted reflection positive distribution and it gives us a way to construct σ-twisted reflection positive representation. At the end of this section, we give some examples of σ-twisted reflection positive representations on $\mathbb{R}^n$.

In Section 7.6, we introduce a reflection positive cyclic distribution vector for the circle case. We prove that this distribution vector generates the well known reflection positive function, see [19] and [25], given by

$$g_\lambda(x) = e^{-x\lambda} + e^{-(\beta-x)\lambda}.$$

7.1 Twisted Reflection Positivity

7.1.1 σ-Twisted Reflection Positive Representation on a Vector Space

Now, we introduce the definitions of reflection positive vector space and σ-twisted reflection positive representation on a vector space. Note that the form here need not be positive definite. If the form is positive definite, then we get the usual definition of the reflection positive Hilbert space.

Definition 7.1. *(Reflection Positive Vector Space) Let $\mathcal{E}$ be a complex vector space with a continuous Hermitian form $\beta(\cdot,\cdot) = \langle\cdot,\cdot\rangle$. Let J be an involution on $\mathcal{E}$ such*

that $\langle Jv, w\rangle = \langle v, Jw\rangle$ for $v, w \in \mathcal{E}$. We call a closed subspace $\mathcal{E}_+ \subseteq \mathcal{E}$ J-positive if $\langle Jv, v\rangle \geq 0$ for $v \in \mathcal{E}_+$. We then say that the quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ is a reflection positive vector space. If the Hermitian form β is positive definite, then β becomes an inner product inducing a topology on $\mathcal{E}$. In fact $\mathcal{E}$ becomes a pre-Hilbert space. If we denote its completion by the same letter $\mathcal{E}$, then we say that $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ is a reflection positive Hilbert space. We write

$$\mathcal{N} := \{v \in \mathcal{E}_+ : \langle Jv, v\rangle = 0\} = \{v \in \mathcal{E}_+ : (\forall w \in \mathcal{E}_+)\langle Jw, v\rangle = 0\},$$

$q : \mathcal{E}_+ \longrightarrow \mathcal{E}_+/\mathcal{N}, v \longmapsto \widehat{v} = q(v)$ for the quotient map and $\widehat{\mathcal{E}}$ for the Hilbert completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm $\|\widehat{v}\|_{\widehat{\mathcal{E}}} := \sqrt{\langle Jv, v\rangle}$.

Definition 7.2. *(σ-Twisted Reflection Positive Representation on a Vector Space) Let (G, H, τ) be a symmetric Lie group and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the corresponding symmetric Lie algebra. Let $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ be a reflection positive vector space and let $\langle \cdot, \cdot\rangle = \beta(\cdot, \cdot)$. Let σ be an involution on G such that $\sigma\tau = \tau\sigma$. A continuous representation is said to be σ-twisted reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ if the following conditions hold:*

(RP0) $\langle \pi(\sigma(g))v, \pi(g)w\rangle = \langle v, w\rangle$ *for every* $v, w \in \mathcal{E}$.

(RP1) $\pi(\tau(\sigma(g))) = J\pi(g)J$ *for every* $g \in G$.

(RP2) $\pi(h)\mathcal{E}_+ = \mathcal{E}_+$ *for every* $h \in H$.

(RP3) There exists a subspace $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$, *dense in* $\mathcal{E}_+$, *such that* $d\pi(X)\mathcal{D} \subseteq \mathcal{D}$ *for every* $X \in \mathfrak{q}$.

Taking σ to be the identity involution in the above definition, we obtain the usual reflection positive representation.

7.1.2 Non-Compactly Causal Symmetric Spaces and Twisted Reflection Positivity

We adopt the notations in Section 5.1. Write, for $x \in G$, $x = k(x)m(x)a(x)n(x)$ according to $G = KMAN$. Also, write, for $x \in \overline{N}MAN$, $x = \overline{n}(x)m_{\overline{N}}(x)\alpha(x)n_{\overline{N}}(x)$. Identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by

$$\mathfrak{a}_{\mathbb{C}}^* \ni \lambda \mapsto 2\lambda(X^0) \in \mathbb{C},$$

where X^0 is given in Section 5.1. Then ρ corresponds to $\dim \mathfrak{n}$. Let $L := H \cap K$. Define an inner product on $\mathbf{L}^2(K/L)$ by

$$\langle f, g\rangle := \int_K \overline{f(k)}g(k)\ dk = \int_{K/L} \overline{f(k)}g(k)\ d\dot{k}.$$

Define π_λ by

$$[\pi_\lambda(x)f](k) := a(x^{-1}k)^{-\lambda-\rho}f(k(x^{-1}k)).$$

Note that π_λ is unitary if and only if $\lambda \in i\mathbb{R}$. Let $\mathfrak{a}_{\mathbb{C}}^*(a_P)$ be defined as in Theorem 6.1 and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$. Then recall that the intertwining operator in Theorem 6.2, $A^\lambda : L^2(K/L) \to L^2(K/L)$ can be written as

$$A^\lambda f(kb) = \int_{K/L} f(xb)\alpha(k^{-1}x)^{\lambda-\rho}\ d(xb). \tag{7.1}$$

Now, define the bilinear form $\langle\cdot,\cdot\rangle_\lambda : L^2(K/L) \times L^2(K/L) \to \mathbb{C}$ by

$$\begin{aligned} \beta_\lambda(f,h) = \langle f, h\rangle_\lambda &:= \left\langle A^\lambda f, h\right\rangle_{L^2} \\ &= \int_{K/L} \overline{A^\lambda f(yb)}h(yb)\ \ d(yb) \\ &= \int_{K/L}\int_{K/L} \overline{f(xb)}h(yb)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb). \end{aligned}$$

Now, let us introduce our main theorem of this section which says that for any non-compactly causal symmetric space, we get a reflection positive vector space. It proves that the generalized principal series is θ-twisted reflection positive representation, where θ is the Cartan involution. Note that this theorem applies for the Cayley type as well as the others.

Theorem 7.3. *Assume that G/H is non-compactly causal and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ and assume that $\lambda - \rho \leqslant L_{pos}$. Let $S := S(H, P_{\max}) = H \exp C_{\max}$. Let $\mathcal{E} := L^2(K/L)$ and $\mathcal{E}_+ := L^2(H/L) = L^2(HP/L)$. Let $Jf(kb) := f(J(kb)) = f(\tau(k)b)$ and $\beta_\lambda(\cdot,\cdot) = \langle \cdot,\cdot \rangle_\lambda$ be as above. Then the following hold:*

(1) $\langle f, Jf \rangle_\lambda \geq 0, \quad \forall\ f \in \mathcal{E}_+$.

(2) $J\pi_\lambda(g) = \pi_\lambda(\tau(\theta(g)))J, \ \ \forall \ \ g \in G$.

(3) $\langle \pi_\lambda(g) f, h \rangle_\lambda = \langle f, \pi_\lambda(\theta(g^{-1}))h \rangle_\lambda, \ \ \forall \ \ g \in G, \ \forall\ f, h \in \mathcal{E}$.

(4) $\pi_\lambda(S)\mathcal{E}_+ \subset \mathcal{E}_+$.

(5) $\|\pi_\lambda(s) f\|_J \leq \|f\|_J, \ \ \forall\ s \in S, \ \forall\ f \in \mathcal{E}_+$.

In particular, the quadruple $(\mathcal{E}, \beta_\lambda, \mathcal{E}_+, J)$ is a reflection positive vector space and the generalized principal series π_λ is θ-twisted reflection positive representation on $(\mathcal{E}, \beta_\lambda, \mathcal{E}_+, J)$.

Proof. Parts (1), (2), (3), (4) and (5) are nothing but Lemmas 7.13, 7.5, 7.7, 7.8 and 7.14, respectively. □

The following lemmas will be needed to prove our main theorem of this section.

Lemma 7.4. *Let $x \in G$ and let $c : G \to G$ be given by $c(x) := \theta\tau(x)$. Then*

1. $k(c(x)) = c(k(x))$,
2. $a(c(x)) = a(x)$.

Proof. We know that $\theta(N) = \tau(N) = \overline{N}$, $\theta(M) = \tau(M) = M$, and $\tau(K) \subset K$. Also, we know that if $a \in A$, then $\tau(a) = \theta(a) = a^{-1}$, see Section 5.1. Thus we

have $c(N) = N$ and $c(M) = M$. Let $x \in G$. Then

$$\begin{aligned} c(x) &= c(k(x)ma(x)n) \\ &= c(k(x))m'c(a(x))n', \end{aligned}$$

and

$$c(x) = k(c(x))m''a(c(x))n''.$$

Thus, we have $k(c(x)) = c(k(x))$ and $a(c(x)) = c(a(x)) = a(x)$. $\square$

The following lemma states that the operator J intertwines between the generalized prinsipal series representations with some twist.

Lemma 7.5. *Let $x \in G$ and let $c(x) = \theta\tau(x)$ as above. Then $J\pi_\lambda(g) = \pi_\lambda(c(g))J$.*

Proof. Let $x \in G$ and let $k \in K$. Then

$$\begin{aligned} [J\pi_\lambda(g)f](kb) &= [\pi_\lambda(g)f](\tau(k)b) \\ &= a(g^{-1}\tau(k))^{-\lambda-\rho} f\left(k(g^{-1}\tau(k))b\right) \\ &= a(c(c(g^{-1})k))^{-\lambda-\rho} f\left(k(c(c(g^{-1})k))b\right) \\ &= a(c(g^{-1})k)^{-\lambda-\rho} f\left(c(k(c(g^{-1})k))b\right) \text{ by Lemma 7.4} \\ &= a(c(g^{-1})k)^{-\lambda-\rho}[Jf]\left(k(c(g^{-1})k)b\right) \\ &= [\pi_\lambda(c(g))Jf](kb). \end{aligned}$$

This completes the proof. $\square$

The following Lemma computes the adjoint of the principal series representation under the L^2-inner product and for the sake of completeness we provide the proof.

Lemma 7.6. *Let $g \in G$ and let $f, h \in L^2(K/L)$. Then*

$$\langle \pi_{-\lambda}(g)f, \pi_\lambda(g)h \rangle_{L^2} = \langle f, h \rangle_{L^2}.$$

Proof. Let $g \in G$ and let $f, h \in L^2(K/L)$. Then

$$\begin{aligned}
\langle \pi_{-\lambda}(g)f, \pi_\lambda(g)h\rangle_{L^2} &= \int_{K/L} \overline{\pi_{-\lambda}(g)f(xb)}\pi_\lambda(g)h(xb) \;\; d(xb) \\
&= \int_{K/L} \overline{a(g^{-1}x)^{\lambda-\rho} f(k(g^{-1}x)b)} a(g^{-1}x)^{-\lambda-\rho} h(k(g^{-1}x)b) d(xb) \\
&= \int_{K/L} \overline{f(g^{-1}\cdot xb)} h(g^{-1}\cdot xb) \; a(g^{-1}x)^{-2\rho} \;\; d(xb) \\
&= \int_{K/L} \overline{f(xb)} h(xb) \; d(xb) \quad \text{ by Part 4 of Lemma 5.6} \\
&= \langle f, h\rangle_{L^2}.
\end{aligned}$$

This completes the proof. □

The following lemma computes the adjoint for the generalized principal series representation under the hermitian form $\langle \cdot, \cdot \rangle_\lambda$.

Lemma 7.7. *Let $g \in G$ and let $f, h \in L^2(K/L)$. Then*

$$\langle \pi_\lambda(g)f, h\rangle_\lambda = \left\langle f, \pi_\lambda(\theta(g^{-1}))h\right\rangle_\lambda.$$

Proof. Let $g \in G$ and let $f, h \in L^2(K/L)$. Then

$$\begin{aligned}
\langle \pi_\lambda(g)f, h\rangle_\lambda &= \left\langle A^\lambda \pi_\lambda(g)f, h\right\rangle_{L^2} \\
&= \left\langle \pi_{-\lambda}(\theta(g))A^\lambda f, h\right\rangle_{L^2} \quad \text{ by Part 3 of Theorem 6.1} \\
&= \left\langle A^\lambda f, \pi_\lambda(\theta(g^{-1}))h\right\rangle_{L^2} \quad \text{by Lemma 7.6} \\
&= \left\langle f, \pi_\lambda(\theta(g^{-1}))h\right\rangle_\lambda.
\end{aligned}$$

This completes the proof. □

The following lemma proves that $\mathcal{E}_+$ is invariant under the semigroup S.

Lemma 7.8. *The subspace $\mathcal{E}_+$ is S invariant, i.e., $\pi_\lambda(S)\mathcal{E}_+ \subset \mathcal{E}_+$.*

Proof. Let $f \in \mathcal{E}_+ = L^2(H/L)$.

Then

$$\begin{aligned}
\pi_\lambda(s)f(kb) \neq 0 &\implies f(k(s^{-1}k)b) \neq 0 \\
&\implies k(s^{-1}k)b \in \mathrm{Supp}(f) \\
&\implies s^{-1} \cdot kb \in \mathrm{Supp}(f) \\
&\implies kb \in s \cdot \mathrm{Supp}(f).
\end{aligned}$$

Thus

$$\mathrm{Supp}(\pi_\lambda(s)f) = \overline{\{kb \in K/L \mid \pi_\lambda(s)f(kb) \neq 0\}} \quad \subset \quad \overline{s \cdot \mathrm{Supp}(f)} = s \cdot \mathrm{Supp}(f).$$

Hence

$$\begin{aligned}
\mathrm{Supp}(\pi_\lambda(s)f) &\subset s \cdot \mathrm{Supp}(f) \\
&\subset s \cdot (HP/L) \\
&\subset HP/L.
\end{aligned}$$

The last inclusion follows from the fact that $SH \subset HP$, see Part 4 of Theorem 5.4. Therefore $\pi_\lambda(s)f \in L^2(H/L)$. This completes the proof. $\square$

The following technical lemma is needed in proving several theorems that comes after.

Lemma 7.9. *Let $x, y \in G$. Then $\alpha(\tau(k(x))^{-1}k(y)) = a(x)^{-1}a(y)^{-1}\alpha(\tau(x)^{-1}y)$.*

Proof. We know that if $a \in A$ and $m \in M$, then $aN = Na$, $mN = Nm$, $a\overline{N} = \overline{N}a$, $m\overline{N} = \overline{N}m$, $aM = Ma$, $\tau(a) = a^{-1}$. Also recall that $\tau(N) = \overline{N}$ and $\tau(M) \subset M$. Let $x = k(x)ma(x)n$ and $y = k(y)m'a(y)n'$. Let $X = \tau(k(x))^{-1}k(y)$ and write it

as $X = \overline{n}(X)m''\alpha(X)n''$. Then

$$\begin{aligned}
\tau(x)^{-1}y &= \tau(k(x)ma(x)n)^{-1}k(y)m'a(y)n' \\
&= \tau(n)^{-1}\tau(a(x))^{-1}\tau(m)^{-1}\tau(k(x))^{-1}k(y)m'a(y)n' \\
&= \tau(n)^{-1}\tau(a(x))^{-1}\tau(m)^{-1}Xm'a(y)n' \\
&= \tau(n)^{-1}\tau(a(x))^{-1}\tau(m)^{-1}\overline{n}(X)m''\alpha(X)n''m'a(y)n' \\
&= \overline{n}\widetilde{m}a(x)a(y)\alpha(X)\widetilde{n} \\
&\qquad \text{for some } \overline{n} \in \overline{N}, \widetilde{m} \in M \text{ and } \widetilde{n} \in N.
\end{aligned}$$

Thus $\alpha(\tau(x)^{-1}y) = a(x)a(y)\alpha(X)$. This completes the proof. □

Corollary 7.10. *Let $x, y \in G$. Then $a(\theta(x))\ \alpha(k(\theta(x))^{-1}k(y))\ a(y) = \alpha(x^{-1}y)$.*

Proof. Replacing x by $\tau(x)$ in the satament of Lemma 7.9, we get

$$a(\tau(x))\ \alpha(\tau(k(\tau(x)))^{-1}k(y))\ a(y) = \alpha(x^{-1}y).$$

Now using Parts 1 and 2 of Lemma 7.4, we get $\tau(k(\tau(x))) = k(\theta(x))$ and $a(\tau(x)) = a(\theta(x))$. This completes the proof. □

Lemma 7.11. *Let $x \in \overline{N}MAN$. Then*

1. $k(x) = k(\overline{n}(x))$,

2. $a(x) = a(\overline{n}(x))\alpha(x)$.

Proof. Write x as $x = \overline{n}(x)m\alpha(x)n$. Let $\overline{n}(x) = k(\overline{n}(x))m'a(\overline{n}(x))n'$. Then

$$\begin{aligned}
x &= \overline{n}(x)m\alpha(x)n \\
&= k(\overline{n}(x))m'a(\overline{n}(x))n'm\alpha(x)n \\
&= k(\overline{n}(x))\widetilde{m}a(\overline{n}(x))\alpha(x)\widetilde{n} \quad \text{for some } \widetilde{m} \in M \text{ and } \widetilde{n} \in N.
\end{aligned}$$

Thus $k(x) = k(\overline{n}(x))$ and $a(x) = a(\overline{n}(x))\alpha(x)$. This completes the proof. □

Lemma 7.12. *Let $x, y \in \overline{N}MAN$. Then*

$$\alpha(\tau(\overline{n}(y))^{-1}\overline{n}(x)) = \alpha(y)^{-1}\alpha(x)^{-1}\alpha(\tau(y)^{-1}x).$$

Proof. We know that if $a \in A$ and $m \in M$, then $aN = Na$, $mN = Nm$, $a\overline{N} = \overline{N}a$, $m\overline{N} = \overline{N}m$. Let $x = \overline{n}(x)m\alpha(x)n$ and $y = \overline{n}(y)m'\alpha(x)n'$. Let $X = \tau(\overline{n}(x))^{-1}\overline{n}(y)$ and write it as $X = \overline{n}(X)m''\alpha(X)n''$. Then

$$\begin{aligned}
\tau(x)^{-1}y &= \tau(\overline{n}(x)m\alpha(x)n)^{-1}\overline{n}(y)m'\alpha(x)n' \\
&= \tau(n)^{-1}\tau(\alpha(x))^{-1}\tau(m)^{-1}\tau(\overline{n}(x))^{-1}\overline{n}(y)m'\alpha(x)n' \\
&= \tau(n)^{-1}\tau(\alpha(x))^{-1}\tau(m)^{-1}Xm'\alpha(x)n' \\
&= \tau(n)^{-1}\tau(\alpha(x))^{-1}\tau(m)^{-1}\overline{n}(X)m''\alpha(X)n''m'\alpha(x)n' \\
&= \widetilde{\overline{n}}\widetilde{m}\alpha(x)\alpha(y)\alpha(X)\widetilde{n} \\
&\qquad \text{for some } \widetilde{\overline{n}} \in \overline{N}, \widetilde{m} \in M \text{ and } \widetilde{n} \in N.
\end{aligned}$$

Thus $\alpha(\tau(x)^{-1}y) = \alpha(x)\alpha(y)\alpha(X)$. This completes the proof. □

The following theorem is the main ingredient in proving that non-compactly causal symmetric space has a reflection positive vector space. It is important to know that the theorem deals with the compact picture.

Theorem 7.13. *Let $f \in \mathcal{E}_+$. Then $\langle f, Jf\rangle_\lambda \geq 0$.*

Proof. Let $f \in \mathcal{E}_+$. Then

$$\begin{aligned}
\langle f, Jf\rangle_\lambda &= \int_{K/L}\int_{K/L} \overline{f(xb)}Jf(yb)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)}f(\tau(y)b)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)}f(yb)\ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb).
\end{aligned}$$

Thus by Part 5 of Theorem 5.6, $\langle f, Jf\rangle_\lambda$ equals to

$$\int_{H/L}\int_{H/L} \overline{f(k(x)b)}f(k(y)b)\ a(x)^{-2\rho}a(y)^{-2\rho}\ \alpha(\tau(k(y))^{-1}k(x))^{\lambda-\rho}\ \ d(xb)\ d(yb).$$

Therefore

$$\begin{aligned}
&\langle f, Jf\rangle_\lambda \\
= \;& \int_{H/L}\int_{H/L} \overline{f(k(x)b)\ a(x)^{-\rho-\lambda}} f(k(y)b)a(y)^{-\rho-\lambda}\ \alpha\left(\tau(y)^{-1}\,x\right)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
& \quad \text{by Lemma 7.9} \\
= \;& \int_{H/L}\int_{H/L} \overline{Sf(x)} Sf(y)\ \alpha\left(\tau(y)^{-1}\,x\right)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
\geq \;& 0 \qquad \text{by Theorem 5.25.}
\end{aligned}$$

Here, $Sf(x) := f(k(x)b)\ a(x)^{-\rho-\lambda}$ and hence $Sf \in \mathcal{C}^\infty_\lambda(HP)$. This completes the proof. $\square$

The following theorem proves that the principal series representation is contraction on $\mathcal{E}_+$.

Theorem 7.14. *Let $f \in L^2(H/L)$. Then $\|\pi_\lambda(s)f\|_J \leq \|f\|_J\ \forall\ s \in S$.*

Proof. Let $f \in L^2(H/L)$. Then

$$\begin{aligned}
\|\pi_\lambda(s)f\|_J &= \int_{K/L}\int_{K/L} \overline{\pi_\lambda(s)f(xb)} J\pi_\lambda(s)f(yb)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{\pi_\lambda(s)f(xb)} \pi_\lambda(s)f(\tau(y)b)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{\pi_\lambda(s)f(xb)} \pi_\lambda(s)f(yb)\ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{Sf(s^{-1}x)} Sf(s^{-1}y)\ \ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{[\pi_\lambda(s)Sf](x)} [\pi_\lambda(s)Sf](y)\ \ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&\qquad \text{here } \pi_\lambda \text{ is considered as in the induced picture.}
\end{aligned}$$

Hence

$$\begin{aligned}
\|\pi_\lambda(s)f\|_J &\le \int_{K/L}\int_{K/L} \overline{Sf(x)}Sf(y)\ \ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&\qquad \text{by Part 3 of Theorem 5.27,} \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)}f(yb)\ \alpha(\tau(y)^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)}f(\tau(y)b)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)}Jf(yb)\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb) \\
&= \|f\|_J.
\end{aligned}$$

Here, $Sf(x) := f(k(x)b)\ a(x)^{-\rho-\lambda}$ and hence $Sf \in \mathcal{C}^\infty_\lambda(HP)$. This completes the proof. $\square$

7.1.3 Cos^λ Transform and Twisted Reflection Positivity

Let $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$, or the skew field $\mathbb{H}$ of quaternions. Let $G = \mathrm{SL}(n+1,\mathbb{K})$. Let $\mathrm{Gr}_p(\mathbb{K})$ be the Grassmann manifold of k-dimensional subspaces of $\mathbb{K}^{n+1}$. Let $z \mapsto \overline{z}$ be the conjugation in $\mathbb{K}$ and let $x = (x_{\lambda\mu}) \in \mathrm{M}(n+1,\mathbb{K})$. Let $\overline{x} := (\overline{x}_{\lambda\mu})$ and let $x^* := (\overline{x}_{\lambda\mu}) = \overline{x}^t$. The homomorphism $\theta : G \to G,\ x \mapsto (x^{-1})^*$ is a Cartan involution on G. We have

$$\begin{aligned}
K &= \mathrm{SU}(n+1,\mathbb{K}), \\
P &= \left\{ \begin{pmatrix} a & X \\ 0 & b \end{pmatrix} \ \middle|\ \begin{matrix} a \in \mathrm{GL}(p,\mathbb{K}) \\ b \in \mathrm{GL}(q,\mathbb{K}) \end{matrix} \text{ and } \det a \det b = 1, X \in \mathbf{M}(p\times q,\mathbb{K}) \right\}.
\end{aligned}$$

Let $e_1, \cdots, e_{n+1}$ be the standard basis for $\mathbb{K}^{n+1}$ and let $b_0 = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_p \in \mathrm{Gr}_p(\mathbb{K})$. Then $\mathrm{Gr}_p(\mathbb{K}) = K \cdot b_0 \cong K/L = G/P$. Let

$$\begin{aligned}
L &= \mathrm{S}(\mathrm{U}(p,\mathbb{K}) \times \mathrm{U}(q,\mathbb{K})), \\
H &= \mathrm{SU}(p,q).
\end{aligned}$$

The following theorem states that the above space is a non-compactly causal symmetric space.

Theorem 7.15. *The space $\mathcal{B} = G/H$ is a non-compactly causal symmetric space (NCC) and the corresponding involution is*

$$\tau(x) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \theta(x) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad \textit{for} \quad x \in G,$$

and $K^\tau = L$ and $G^\tau = H$.

Proof. The proof follows from the table in [8, p. 89]. □

Recall that for $f \in L^p(\mathcal{B})$ and $k \in K$

$$A^\lambda f(k) = \int_{\mathcal{B}} f(b)\ \alpha(k^{-1}b)^{\lambda-\rho}\ db,$$

and

$$C^\lambda f(k) = \int_{\mathcal{B}} f(b)\ |\operatorname{Cos}(b,c)|^{\lambda-\rho}\ db.$$

The following theorem states that the $\operatorname{Cos}^\lambda$ transform is nothing but an intertwining operator for the generalized principal series under some non-compactly causal symmetric spaces.

Theorem 7.16. *For $x \in G$ and $\lambda \in \mathbb{C}$, we have*

$$\alpha(x)^\lambda = \left|\operatorname{Cos}(x \cdot b_0, b_0)\right|^\lambda.$$

In particular, if $b = k \cdot b_0$ and $c = h \cdot b_0 \in \mathcal{B}$, then

$$\alpha(h^{-1}k)^\lambda = |\operatorname{Cos}(b,c)|^\lambda,$$

and so, if Re $\lambda > \rho$, then $C^\lambda = A^\lambda$.

Proof. See [27, p. 285]. □

The following theorem is the main theorem of this section which gives the relation between the $\operatorname{Cos}^\lambda$ transform and the reflection positivity. More precisely, it states that the generalized principal series representation is θ-twisted reflection positive representation on a space built by the $\operatorname{Cos}^\lambda$ transform.

Theorem 7.17. *Let $G = \mathrm{SL}(n+1,\mathbb{K})$. For $x \in G$, let $\theta(x) = (x^{-1})^*$ and*

$$\tau(x) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \theta(x) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Then $L = \mathrm{S}(\mathrm{U}(p,\mathbb{K}) \times \mathrm{U}(q,\mathbb{K}))$ and $H = \mathrm{SU}(p,q)$. Assume that $\lambda - \rho \leqslant L_{pos}$. Let $S := S(H,P)$. Let $\mathcal{E} := L^2(K/L)$ and $\mathcal{E}_+ := L^2(H/L)$. Let $Jf(kb) := f(\tau(k)b)$. Let

$$\begin{aligned} \beta_\lambda(f,h) &:= \left\langle C^\lambda f, h \right\rangle_{L^2} \\ &= \int_{K/L} \int_{K/L} \overline{f(xb)} h(yb) \ |\operatorname{Cos}(xb,yb)|^{\lambda-\rho} \ \ d(xb) \ d(yb). \end{aligned}$$

Then the quadruple $(\mathcal{E}, \beta_\lambda, \mathcal{E}_+, J)$ is a reflection positive vector space and the generalized principal series π_λ is θ-twisted reflection positive representation on the reflection positive vector space $(\mathcal{E}, \beta_\lambda, \mathcal{E}_+, J)$.

Proof. The space G/H is a non-compactly causal symmetric space, by Theorem 7.15. Now, the proof follows directly from Theorem 7.3 and Theorem 7.16. $\square$

7.1.4 A Direct Proof of the Reflection Positivity for $\operatorname{Cos}^\lambda$ Transform (The Case of SO(3))

In the previous section we use some well known facts and indirect proofs to prove that non-compactly causal spaces are twisted reflection positive. This section is devoted to give a direct proof for the reflection positivity of the $\operatorname{Cos}^\lambda$ transform on SO(3).

Let $K = \mathrm{SO}(3)$ and $L = \mathrm{SO}(2)$. Let $\nu = \rho - \lambda$ and let $Q(x) = |\operatorname{Cos}(xb, b)|^{-\nu} = |\cos\theta_x|^{-\nu}$. For $xb \in K/L$, we can write $xb = vg_xb$, where

$$v = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ g_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \end{pmatrix},$$

such that $0 \leq t \leq 2\pi$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. By abuse of notations, let x be the angle between xb and the x-axis. Here $H/L = \{xb = vg_x b \in K/L \mid -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$.

Recall that for $f, h \in L^2(K/L) = L^2(K)^L$, we have

$$\begin{aligned}
\beta_\lambda(f,h) &= \left\langle C^\lambda f, h \right\rangle_{L^2} \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)} h(yb) \; |\operatorname{Cos}(xb,yb)|^{\lambda-\rho} \;\; d(xb)\; d(yb) \\
&= \int_{K/L}\int_{K/L} \overline{f(xb)} h(yb) \; |\operatorname{Cos}(y^{-1}xb,b)|^{\lambda-\rho} \;\; d(xb)\; d(yb) \\
&= \int_K\int_K \overline{f(x)} h(y) \;\; |\cos\theta_{y^{-1}x}|^{\lambda-\rho} \;\; dx\; dy.
\end{aligned}$$

Theorem 7.18. *Let $\varphi \in L^2(H/L)$. Then $\beta_\lambda(\varphi, J\varphi) \geq 0$.*

Proof. Let $\varphi \in L^2(H/L)$. Then

$$\begin{aligned}
\beta_\lambda(\varphi, J\varphi) &= \int_K\int_K \overline{\varphi(x)}\; J\varphi(y)\; Q(y^{-1}x)\; dy\; dx \\
&= \int_K\int_K \overline{\varphi(y)}\; \varphi(x)\; Q(y^\sharp x)\; dy\; dx \\
&= \int_0^\pi\int_0^\pi\int_L\int_L \overline{\varphi(ug_y)}\; \varphi(vg_x)\; Q(g_y u^{-1} v g_x)\; du\; dv\; dy\; dx.
\end{aligned}$$

Now, we have

$$\begin{aligned}
vg_x e_0 &= \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= (\sin t \sin x, \cos t \sin x, \cos x)^T.
\end{aligned}$$

Similarly,

$$ug_{-y}e_0 = (-\sin s \sin y, -\cos s \sin y, \cos y)^T.$$

Hence

$$\begin{aligned}
\cos\theta_{g_y u^{-1} v g_x} &= \langle g_y u^{-1} v g_x e_0, e_0\rangle \\
&= \langle v g_x e_0, u g_{-y} e_0\rangle
\end{aligned}$$

Thus

$$\begin{aligned}\cos\theta_{g_y u^{-1} v g_x} &= \cos x\cos y - (\sin t\sin x\sin s\sin y) - (\cos t\sin x\cos s\sin y)\\ &= \cos x\cos y - \sin x\sin y\,(\sin t\sin s + \cos t\cos s)\\ &= \cos x\cos y - \sin x\sin y\cos(t-s).\end{aligned}$$

Thus we have

$$\begin{aligned}&\beta_\lambda(\varphi, J\varphi)\\ =\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\varphi(ug_y)}\ \varphi(vg_x)\ Q(g_y u^{-1} v g_x)\ du\ dv\ dy\ dx\\ =\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\varphi(ug_y)}\ \varphi(vg_x)\ \left|\cos\theta_{g_y u^{-1} v g_x}\right|^{-\nu}\ du\ dv\ dy\ dx\\ =\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\varphi(ug_y)}\ \varphi(vg_x)\ \frac{1}{\left|\cos x\cos y - \sin x\sin y\cos(t-s)\right|^{\nu}}\ du\ dv\ dy\ dx.\end{aligned}$$

Therefore

$$\begin{aligned}&\beta_\lambda(\varphi, J\varphi)\\ =\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\frac{\varphi(ug_y)}{|\cos y|^\nu}}\ \frac{\varphi(vg_x)}{|\cos x|^\nu}\ \frac{1}{(1-\tan x\tan y\cos(t-s))^\nu}\ du\ dv\ dy\ dx\\ =\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\frac{\varphi(ug_y)}{|\cos y|^\nu}}\ \frac{\varphi(vg_x)}{|\cos x|^\nu}\ \sum_{n=0}^\infty \binom{n-1+\nu}{n}(\tan x\tan y\cos(t-s))^n\ du\ dv\ dy\ dx.\end{aligned}$$

Here we used the fact that $-\frac{\pi}{4} \le x, y \le \frac{\pi}{4}$ and hence $-1 \le \tan x, \tan y \le 1$.

$$\begin{aligned}&\beta_\nu(\varphi, J\varphi)\\ =\ & \sum_{n=0}^\infty \binom{n-1+\nu}{n}\\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\frac{\varphi(ug_y)\tan^n y}{|\cos y|^\nu}}\ \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu}\,(\cos t\cos s + \sin t\sin s)^n\ du\ dv\ dy\ dx\\ =\ & \sum_{n=0}^\infty \binom{n-1+\nu}{n}\sum_{k=0}^n\binom{n}{k}\\ & \int_0^\pi\int_0^\pi\int_L\int_L \overline{\frac{\varphi(ug_y)\tan^n y}{|\cos y|^\nu}}\ \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu}\cos^k t\ \cos^k s\ \sin^{n-k} t\ \sin^{n-k} s\ du\ dv\ dy\ dx\end{aligned}$$

Therefore

$$
\begin{aligned}
& \beta_\nu(\varphi, J\varphi) \\
= & \sum_{n=0}^{\infty} \binom{n-1+\nu}{n} \sum_{k=0}^{n} \binom{n}{k} \left| \int_0^\pi \int_L \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu} \cos^k t \ \sin^{n-k} t \ dv \ dx \right|^2 \\
\geq & \ 0.
\end{aligned}
$$

This completes the proof. □

7.1.5 A Direct Proof of the Reflection Positivity for Cos^λ Transform (The General Case of $\mathrm{SO}(n)$)

In this section, we generalize the above proof of the reflection positivity of the Cos^λ transform for a more general case. More precisely, we prove that the Cos^λ transform is reflection positive on $\mathrm{SO}(n)$, for all $n \geq 3$.

Let $K = \mathrm{SO}(n+1)$ and $L = \mathrm{SO}(n)$. Let $\nu = \rho - \lambda$ and let $Q(x) = |\mathrm{Cos}(xb, b)|^{-\nu} = |\cos\theta_x|^{-\nu}$. For $xb \in K/L$, we can write $xb = vg_x b$, where

$$
v = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \ g_x = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \end{pmatrix}.
$$

such that $v \in \mathrm{SO}(n)$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. By abuse of notations, let x be the angle between xb and the x-axis. Here $H/L = \{xb = vg_x b \in K/L \mid -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$.

Recall that for $f, h \in L^2(K/L) = L^2(K)^L$, we have

$$
\beta_\lambda(f, h) \ = \ \int_K \int_K \overline{f(x)} h(y) \ |\cos\theta_{y^{-1}x}|^{\lambda-\rho} \ \ dx \ dy.
$$

Theorem 7.19. *Let $\varphi \in L^2(K/L)$. Then $\beta_\lambda(\varphi, J\varphi) \geq 0$.*

Proof. Let $\varphi \in L^2(K/L)$. Then we have

$$
\beta_\lambda(\varphi, J\varphi) \ = \ \int_0^\pi \int_0^\pi \int_L \int_L \overline{\varphi(ug_y)} \ \varphi(vg_x) \ Q(g_y u^{-1} v g_x) \ du \ dv \ dy \ dx.
$$

Now, we have

$$
\begin{aligned}
vg_x e_0 &= \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \end{pmatrix} e_0 \\
&= \sin x \ v \ e_{n-1} + \cos x \ e_n.
\end{aligned}
$$

Similarly,

$$
ug_{-y}e_0 = -\sin y \ u \ e_{n-1} + \cos y \ e_n.
$$

Hence

$$
\begin{aligned}
\cos\theta_{g_y u^{-1} v g_x} &= \langle g_y u^{-1} v g_x e_n, e_n \rangle \\
&= \langle v g_x e_n, u g_{-y} e_n \rangle \\
&= \cos x \cos y - \sin x \sin y \langle u e_{n-1}, v e_{n-1} \rangle.
\end{aligned}
$$

Notice that for $w = (w_1, \cdots, w_n)$ and $z = (z_1, \cdots, z_n)$, we have

$$
\langle w, z \rangle^n = \sum_{1 \le i_1, \cdots, i_n \le n} f_{i_1, \cdots, i_n}(w) f_{i_1, \cdots, i_n}(z),
$$

where $f_{i_1,\cdots,i_n}(w) = w_{i_1} \cdots w_{i_n}$. Thus we have

$$
\begin{aligned}
& \beta_\lambda(\varphi, J\varphi) \\
= & \int_0^\pi \int_0^\pi \int_L \int_L \overline{\varphi(ug_y)} \ \varphi(vg_x) \ Q(g_y u^{-1} v g_x) \ du \ dv \ dy \ dx \\
= & \int_0^\pi \int_0^\pi \int_L \int_L \overline{\varphi(ug_y)} \ \varphi(vg_x) \ \left|\cos\theta_{g_y u^{-1} v g_x}\right|^{-\nu} \ du \ dv \ dy \ dx \\
= & \int_0^\pi \int_0^\pi \int_L \int_L \overline{\varphi(ug_y)} \ \varphi(vg_x) \ \frac{1}{\left|\cos x \cos y - \sin x \sin y \langle ue_{n-1}, ve_{n-1}\rangle\right|^\nu} \ du \ dv \ dy \ dx \\
= & \int_0^\pi \int_0^\pi \int_L \int_L \frac{\overline{\varphi(ug_y)}}{|\cos y|^\nu} \ \frac{\varphi(vg_x)}{|\cos x|^\nu} \ \frac{1}{(1 - \tan x \tan y \langle ue_{n-1}, ve_{n-1}\rangle)^\nu} \ du \ dv \ dy \ dx \\
= & \int_0^\pi \int_0^\pi \int_L \int_L \frac{\overline{\varphi(ug_y)}}{|\cos y|^\nu} \ \frac{\varphi(vg_x)}{|\cos x|^\nu} \\
& \qquad \sum_{n=0}^\infty \binom{n-1+\nu}{n} (\tan x \tan y \langle ue_{n-1}, ve_{n-1}\rangle)^n \ du \ dv \ dy \ dx.
\end{aligned}
$$

Here we used the fact that $-\frac{\pi}{4} \leq x, y \leq \frac{\pi}{4}$ and hence $-1 \leq \tan x, \tan y \leq 1$. But

$$\begin{aligned}
& \beta_\nu(\varphi, J\varphi) \\
= & \sum_{n=0}^{\infty} \binom{n-1+\nu}{n} \\
& \int_0^\pi \int_0^\pi \int_L \int_L \overline{\frac{\varphi(ug_y)\tan^n y}{|\cos y|^\nu}} \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu} \langle ue_{n-1}, ve_{n-1}\rangle^n \, du \, dv \, dy \, dx \\
= & \sum_{n=0}^{\infty} \binom{n-1+\nu}{n} \sum_{1\leq i_1,\cdots,i_n\leq n} \\
& \int_0^\pi \int_0^\pi \int_L \int_L \overline{\frac{\varphi(ug_y)\tan^n y}{|\cos y|^\nu}} \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu} f_{i_1,\cdots,i_n}(u) f_{i_1,\cdots,i_n}(v) \, du \, dv \, dy \, dx \\
= & \sum_{n=0}^{\infty} \binom{n-1+\nu}{n} \sum_{1\leq i_1,\cdots,i_n\leq n} \left| \int_0^\pi \int_L \frac{\varphi(vg_x)\tan^n x}{|\cos x|^\nu} f_{i_1,\cdots,i_n}(v) \, dv \, dx \right|^2 \\
\geq & \; 0.
\end{aligned}$$

This completes the proof. □

7.2 Integrability Without Positive Definitenes for Regular Representaions

To integrate a representation from a group to its dual, we use the Lüscher-Mack theorem or some integrability theorems. In this section, we generalize the integrability theorem given in Section 3.2.3 to the case of non positive definite distribution. By dropping the condition of positive definiteness in Definition 7.13 in [23], we get the following definition, which gives us the meaning of reflection positive distribution with respect to a manifold and some involutive diffeomorphism.

Definition 7.20. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $D \in \mathcal{D}'(\mathcal{B}\times\mathcal{B})$ be a distribution. Suppose further that $J : \mathcal{B} \to \mathcal{B}$ is an involutive diffeomorphism and that $\mathcal{B}_+ \subset \mathcal{B}$ is an open subset such that the distribution D_+ on $\mathcal{B}_+ \times \mathcal{B}_+$ defined by*

$$D_+(\varphi) := D(\varphi \circ (J \times \mathrm{id}_{\mathcal{B}}))$$

is positive definite. We say that D is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$.

Let $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$. For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, define $\varphi \otimes D(\psi) := D(\overline{\varphi} \otimes \psi)$. Let

$$\mathcal{H}_D := \mathcal{D}(\mathcal{B}) \otimes D = \{\varphi \otimes D \mid \varphi \in \mathcal{D}(\mathcal{B})\}$$

and

$$\mathcal{H}_{D_+} := \mathcal{D}(\mathcal{B}_+) \otimes D = \{\varphi \otimes D \mid \varphi \in \mathcal{D}(\mathcal{B}_+)\}.$$

We say that D is invariant under G if

$$D(\varphi \circ \lambda_g \otimes \psi \circ \lambda_g) = D(\varphi \otimes \psi).$$

Also, we say that D is invariant under J if

$$D(J\varphi \otimes J\psi) = D(\varphi \otimes \psi).$$

We need the following technical lemma to prove the next theorem.

Lemma 7.21. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $J : \mathcal{B} \to \mathcal{B}$ is an involutive diffeomorphism. Let $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ and $E \in \mathcal{D}'(\mathcal{B})$. For $\varphi \in \mathcal{D}(\mathcal{B})$ and $g \in G$, define $\pi_D(g)E(\varphi) := E(\varphi \circ \lambda_g)$ and $[JE](\psi) := E(\psi \circ J)$. Then*

(1) $J[\varphi \otimes D] = (J\varphi) \otimes D$.

(2) $\pi_D(g)[\varphi \otimes D] = (\varphi \circ \lambda_g^{-1}) \otimes D$.

Proof. For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, we have

$$\begin{aligned} J[\varphi \otimes D](\psi) &= [\varphi \otimes D](J\psi) \\ &= D(\overline{\varphi} \otimes J\psi) \\ &= D(\overline{J\varphi} \otimes \psi) \quad \text{because } D \text{ is } J \text{ invariant} \\ &= [J\varphi \otimes D](\psi). \end{aligned}$$

Therefore we have shown that Part 1.

Now, for $\varphi, \psi \in \mathcal{D}(\mathcal{B})$ and $g \in G$, we have

$$\begin{aligned}
\pi_D(g)[\varphi \otimes D](\psi) &= [\varphi \otimes D](\psi \circ \lambda_g) \\
&= D(\overline{\varphi} \otimes \circ\lambda_g) \\
&= D(\overline{\varphi \circ \lambda_g^{-1}} \otimes \psi) \quad \text{because } D \text{ is } G \text{ invariant} \\
&= [\varphi \circ \lambda_g^{-1} \otimes D](\psi).
\end{aligned}$$

Therefore we have shown that Part 2 and this completes the proof. □

The following theorem constructs a reflection positive space and a representation from the reflection positive distribution given in the above definition.

Theorem 7.22. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $D \in \mathcal{D}'(\mathcal{B}\times\mathcal{B})$ be a distribution which is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. Let (G, H, τ) be a symmetric Lie group acting on $\mathcal{B}$ such that $J(g \cdot m) = \tau(g) \cdot J(m)$ and $H \cdot \mathcal{B}_+ = \mathcal{B}_+$. We assume that D is invariant under G and J. Let $\mathcal{E} = \mathcal{H}_D := \mathcal{D}(\mathcal{B}) \otimes D$ and $\mathcal{E}_+ = \mathcal{H}_{D_+} := \mathcal{D}(\mathcal{B}_+) \otimes D$. For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, define the bilinear form $\beta(\varphi, \psi) = \langle \varphi, \psi \rangle := D(\overline{\varphi} \otimes \psi)$. Let π_D be the regular representation of G on $\mathcal{E}$, i.e., $\pi_D(g)E(\varphi) := E(\varphi \circ \lambda_g)$. Let $[JE](\psi) := E(\psi \circ J)$. Then*

(1) The quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ is a reflection positive vector space (as in Definition 7.1).

(2) The representation π_D is reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ (as in Definition 7.2).

Proof. For $\varphi \in \mathcal{D}(\mathcal{B}_+)$, we have

$$\begin{aligned}
\langle J[\varphi \otimes D], \varphi \otimes D \rangle &= \langle [J\varphi] \otimes D, \varphi \otimes D \rangle \quad \text{by Part 1 of Lemma 7.21} \\
&= D(\overline{J\varphi} \otimes \psi) \geq 0,
\end{aligned}$$

because D is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. This completes the proof of Part 1. Now, for $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, we have

$$
\begin{aligned}
\langle \pi_D(g)[\varphi \otimes D], \pi_D(g)[\psi \otimes D] \rangle &= \langle [\varphi \circ \lambda_g^{-1}] \otimes D, [\psi \circ \lambda_g^{-1}] \otimes D \rangle \\
&\quad \text{by Part 2 of Lemma 7.21} \\
&= D(\overline{\varphi \circ \lambda_g^{-1}} \otimes \psi \circ \lambda_g^{-1}) \\
&= D(\overline{\varphi} \otimes \psi) \\
&= \langle \varphi \otimes D, \psi \otimes D \rangle
\end{aligned}
$$

This completes the proof of Part RP0 of Definition 7.2.

Let $g \in G$ and $m \in \mathcal{B}$. Then

$$
\begin{aligned}
[J \circ \lambda_{\tau(g)} \circ J](m) &= J(\tau(g) \cdot J(m)) \\
&= \tau(\tau(g)) \cdot J(J(m)) \quad \text{by assumption} \\
&= g \cdot m \\
&= \lambda_g(m).
\end{aligned}
$$

Thus we have

$$
J \circ \lambda_{\tau(g)} \circ J = \lambda_g. \tag{7.2}
$$

For $\varphi \in \mathcal{D}(\mathcal{B})$ and $E \in \mathcal{D}'(\mathcal{B})$, we have

$$
\begin{aligned}
[J\pi_D(\tau(g))JE](\varphi) &= E(\varphi \circ J \circ \lambda_{\tau(g)} \circ J) \\
&= E(\varphi \circ \lambda_g) \quad \text{by} \quad (7.2) \\
&= [\pi_D(g)E](\varphi).
\end{aligned}
$$

This completes the proof of Part RP1 of Definition 7.2. Finally, we want to show that $\pi_D(H)\mathcal{E}_+ \subset \mathcal{E}_+$. Let $\varphi \in \mathcal{D}(\mathcal{B}_+)$, $h \in H$ and $m \in \mathcal{B}$. Then

$$\begin{aligned}
\varphi \circ \lambda_{h^{-1}}(m) \neq 0 \ &\Longrightarrow \ \varphi(h^{-1}m) \neq 0 \\
&\Longrightarrow \ h^{-1}m \in \operatorname{supp}(\varphi) \\
&\Longrightarrow \ m \in h \cdot \operatorname{supp}(\varphi).
\end{aligned}$$

Thus

$$\operatorname{supp}(\varphi \circ \lambda_{h^{-1}}) = \overline{\{m \in \mathcal{B} \mid \varphi(h^{-1}m) \neq 0\}} \ \subset \ \overline{h \cdot \operatorname{supp}(\varphi)} = h \cdot \operatorname{supp}(\varphi).$$

Hence

$$\begin{aligned}
\operatorname{supp}(\varphi \circ \lambda_{h^{-1}}) \ &\subset \ h \cdot \operatorname{supp}(\varphi) \\
&\subset \ h \cdot \mathcal{B}_+ \\
&\subset \ \mathcal{B}_+ \ \text{ because } H \cdot \mathcal{B}_+ = \mathcal{B}_+.
\end{aligned}$$

Thus $\varphi \circ \lambda_{h^{-1}} \in \mathcal{D}(\mathcal{B}_+)$. Using Part 2 of Lemma 7.21, we have $\pi_D(h)[\varphi \otimes D] = [\varphi \circ \lambda_{\sigma(h^{-1})}] \otimes D$. Hence $\pi_D(h)[\varphi \otimes D] \in \mathcal{E}_+$. This completes the proof of Part RP2 of Definition 7.2. Hence the representation π_D is reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ and that completes the proof of Part 2. $\square$

Now, we are ready to state and prove the integrability theorem without the positive definiteness. Let $\beta : \mathfrak{g} \longrightarrow \mathcal{V}(\mathcal{B}_+)$ be the homomorphism given by

$$\begin{aligned}
[\beta(x)\varphi](m) = [\beta(x)]_m \varphi \ &= \ \frac{d}{dt}\Big|_{t=0} \varphi(\Phi_t^{\beta(x)}(m)) \\
&:= \ \frac{d}{dt}\Big|_{t=0} \varphi(e^{-tx} \cdot m).
\end{aligned}$$

Theorem 7.23. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ be a distribution which is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. Let (G, H, τ) be a symmetric Lie group acting on $\mathcal{B}$ such that $J(g \cdot m) = \tau(g) \cdot J(m)$ and*

$H \cdot \mathcal{B}_+ = \mathcal{B}_+$. We assume that D is invariant under G and τ. Let G^c be a simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ and define $\mathcal{L}_x$, $x \in \mathfrak{g}$, on maximal domain in the Hilbert subspace $\mathcal{H}_{D_+} \subset \mathcal{D}'(\mathcal{B}_+)$. Then there exists a unique smooth unitary representation $(\pi^c, \mathcal{H}_{D_+})$ of G^c such that

(i) $\overline{\mathrm{d}\,\pi^c}(x) = \mathcal{L}_x$ *for* $x \in \mathfrak{h}$.

(ii) $\overline{\mathrm{d}\,\pi^c}(iy) = i\mathcal{L}_y$ *for* $y \in \mathfrak{q}$.

Proof. The proof follows directly from Theorem 3.10 and Theorem 7.22. The proof is similar to the proof given in [23, p. 34]. The only difference is that we don't assume that our distribution D to be positive definite. For the sake of completeness we provide the proof.

First of all, for $\varphi \in \mathcal{D}(\mathcal{B}_+)$, we have $D_+(\overline{\varphi} \otimes \varphi) = D(\overline{J\varphi} \otimes \varphi) \geq 0$ by the assumption. Thus the distribution D_+ is positive definite. Let $\sigma : H \times \mathcal{B}_+ \longrightarrow \mathcal{B}_+$ be the action given by $\sigma(h, m) = h \cdot m$. It is well define because of the assumption that $H \cdot \mathcal{B}_+ = \mathcal{B}_+$.

It is clear that the pair (σ, β) is a smooth action of $(\mathfrak{g}, H)$ on $\mathcal{B}_+$ as in Definition 3.9. Now we are to show that the distribution D_+ is compatible with the smooth action (σ, β), i.e., D_+ is β-compatible as in Definition 3.8. More precisely, we are to show that for $x \in \mathfrak{g}$, we have

$$\mathcal{L}^1_{\beta(x)} D_+ = \mathcal{L}^2_{\beta(-\tau(x))} D_+ = -\mathcal{L}^2_{\beta(\tau(x))} D_+.$$

Let $\varphi, \psi \in \mathcal{D}(\mathcal{B}_+)$ and let $x \in \mathfrak{g}$. Then we have

$$
\begin{aligned}
[\mathcal{L}^1_{\beta(x)} D_+](\varphi \otimes \psi) &= -D_+(\mathcal{L}^1_{\beta(x)}[\varphi \otimes \psi]) \\
&= -D_+([\mathcal{L}_{\beta(x)}\varphi] \otimes \psi) \\
&= -D_+\left(\left[\frac{d}{dt}\bigg|_{t=0} \varphi \circ \lambda_{e^{-tx}}\right] \otimes \psi\right) \\
&= -\frac{d}{dt}\bigg|_{t=0} D_+\left([\varphi \circ \lambda_{e^{-tx}}] \otimes \psi\right) \\
&= -\frac{d}{dt}\bigg|_{t=0} D\left(\overline{J\left[\varphi \circ \lambda_{e^{-tx}}\right]} \otimes \psi\right) \\
&= -\frac{d}{dt}\bigg|_{t=0} D\left(\overline{[\varphi \circ \lambda_{e^{-tx}} \circ J]} \otimes \psi\right).
\end{aligned}
$$

But we have

$$
\begin{aligned}
-\frac{d}{dt}\bigg|_{t=0} D\left(\overline{[\varphi \circ \lambda_{e^{-tx}} \circ J]} \otimes \psi\right) &= -\frac{d}{dt}\bigg|_{t=0} D\left(\overline{[\varphi \circ J \circ \lambda_{e^{-t\tau(x)}}]} \otimes \psi\right) \\
&\qquad \text{by the assumption that } J(g \cdot m) = \tau(g) \cdot J(m) \\
&= -\frac{d}{dt}\bigg|_{t=0} D\left(\overline{\varphi \circ J} \otimes [\psi \circ \lambda_{e^{-t\tau(-x)}}]\right) \\
&\qquad \text{by the assumption that } D \text{ is } G \text{ invariant} \\
&= -D_+\left(\varphi \otimes \left[\frac{d}{dt}\bigg|_{t=0} \psi \circ \lambda_{e^{-t\tau(-x)}}\right]\right) \\
&= -D_+\left(\varphi \otimes \left[\mathcal{L}_{\beta(\tau(-x))}\psi\right]\right) \\
&= -D_+\left(\mathcal{L}^2_{\beta(\tau(-x))}\left[\varphi \otimes \psi\right]\right) \\
&= \left[\mathcal{L}^2_{\beta(\tau(-x))} D_+\right](\varphi \otimes \psi).
\end{aligned}
$$

Hence we have $\mathcal{L}^1_{\beta(x)} D_+ = \mathcal{L}^2_{\beta(-\tau(x))} D_+$ and so the distribution D_+ is compatible with the smooth action (σ, β). Now by Theorem 7.22 the proof is complete. $\square$

Corollary 7.24. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ be a distribution which is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. Let (G, H, τ) be a symmetric Lie group acting on $\mathcal{B}$ such that $J(g \cdot m) = \tau(g) \cdot J(m)$ and $H \cdot \mathcal{B}_+ = \mathcal{B}_+$. We assume that D is invariant under G and J. Let G^c be a simply*

connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. Let $\mathcal{E} = \mathcal{H}_D := \mathcal{D}(\mathcal{B}) \otimes D$ and $\mathcal{E}_+ = \mathcal{H}_{D_+} := \mathcal{D}(\mathcal{B}_+) \otimes D$. For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, define the bilinear form $\beta(\varphi \otimes D, \psi \otimes D) = \langle \varphi \otimes D, \psi \otimes D \rangle := D(\overline{\varphi} \otimes \psi)$. Let π_D be the regular representation of G on $\mathcal{E}$, i.e., $\pi_D(g)E(\varphi) := E(\varphi \circ \lambda_g)$. Let $[JE](\psi) := E(\psi \circ J)$. Then

(1) The quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ is a reflection positive vector space (as in Definition 7.1).

(2) The representation $(\pi_D, \mathcal{E})$ of G is reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ (as in Definition 7.2).

(3) There exists a unique smooth unitary representation $(\pi^c, \mathcal{E}_+)$ of G^c such that

(i) $\overline{\mathrm{d}\,\pi^c}(x) = \mathrm{d}\,\pi_D(x)$ for $x \in \mathfrak{h}$.

(ii) $\overline{\mathrm{d}\,\pi^c}(iy) = i\,\mathrm{d}\,\pi_D(y)$ for $y \in \mathfrak{q}$.

Proof. Parts 1 and 2 are nothing but Theorem 3.10. Part 3 follows form Theorem 7.27 and the fact that for $x \in \mathfrak{g}$, we have

$$\mathrm{d}\,\pi_D(x) = \mathcal{L}_{\beta(x)}. \tag{7.3}$$

Let $\varphi \in \mathcal{D}(\mathcal{B}_+)$ and let $x \in \mathfrak{g}$. Let $E \in \mathcal{E}_+$. Then we have

$$\begin{aligned}
[\mathcal{L}_{\beta(x)}E](\varphi) &= -E(\mathcal{L}_{\beta(x)}\varphi) \\
&= -E\left(\frac{d}{dt}\Big|_{t=0}\varphi \circ \lambda_{e^{-tx}}\right) \\
&= -\frac{d}{dt}\Big|_{t=0} E\left(\varphi \circ \lambda_{e^{-tx}}\right) \\
&= -\frac{d}{dt}\Big|_{t=0}\left[\pi_D(e^{-tx})E\left(\varphi\right)\right] \\
&= \left[\frac{d}{dt}\Big|_{t=0}\pi_D(e^{tx})E\right](\varphi) \\
&= \left[\mathrm{d}\,\pi_D(x)E\right](\varphi).
\end{aligned}$$

Hence we proved 7.3. This completes the proof. □

7.3 Integrability Without Positive Definiteness for Representations With a Cocycle Condition

In this section, we generalize the integrability theorem given in the previous section to the case of non positive definite distribution for representations with a cocycle condition. This cocycle condition shows up in the generalized principle series representation which will be discuss in the upcoming sections. Therefore this section helps us to integrate some representations of non-compact groups. The idea of this section is based on Example 5.17 in [23]. We add some σ-twist conditions.

We need the following technical lemma to prove the next theorem.

Lemma 7.25. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $J : \mathcal{B} \to \mathcal{B}$ is an involutive diffeomorphism. Let $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ and $E \in \mathcal{D}'(\mathcal{B})$. Let σ be an involution on a group G. For $\varphi \in \mathcal{D}(\mathcal{B})$, define $J\varphi := \varphi \circ J$. Assume that G acts on $\mathcal{D}(\mathcal{B})$ by π such that the following hold:*

(i) $D(J\varphi \otimes J\psi) = D(\varphi \otimes \psi)$ *for any* $\varphi, \psi \in \mathcal{D}(\mathcal{B})$.

(ii) $D(\pi(\sigma(g))\varphi \otimes \pi(g)\psi) = D(\varphi \otimes \psi)$ *for any* $\varphi, \psi \in \mathcal{D}(\mathcal{B})$.

For $\varphi \in \mathcal{D}(\mathcal{B})$ and $g \in G$, define $\pi_D(g)E(\varphi) := E(\pi(g^{-1})\varphi)$ and $[JE](\psi) := E(J\psi)$. Then

(1) $J[\varphi \otimes D] = (J\varphi) \otimes D$.

(2) $\pi_D(g)[\varphi \otimes D] = [\pi(\sigma(g))\varphi] \otimes D$.

Proof. For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, we have

$$\begin{aligned} J[\varphi \otimes D](\psi) &= [\varphi \otimes D](J\psi) \\ &= D(\overline{\varphi} \otimes J\psi) \\ &= D(\overline{J\varphi} \otimes \psi) \quad \text{because } D \text{ is } J \text{ invariant} \\ &= [J\varphi \otimes D](\psi). \end{aligned}$$

Therefore we have shown that Part 1.

For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$ and $g \in G$, we have

$$\begin{aligned} \pi_D(g)[\varphi \otimes D](\psi) &= [\varphi \otimes D](\pi(g^{-1})\psi) \\ &= D(\overline{\varphi} \otimes \pi(g^{-1})\psi) \\ &= D(\overline{\pi(\sigma(g))\varphi} \otimes \psi) \quad \text{by (ii)} \\ &= [\pi(\sigma(g))\varphi] \otimes D(\psi). \end{aligned}$$

Therefore we have shown that Part 2 and this completes the proof. □

The following theorem constructs a reflection positive vector space and a σ-twisted representation from a reflection positive distribution.

Theorem 7.26. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ be a distribution which is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. Let (G, H, τ) be a symmetric Lie group and let σ be an involution on G such that $\sigma\tau = \tau\sigma$ and $\sigma(H) \subset H$. For $\varphi \in \mathcal{D}(\mathcal{B})$, define $J\varphi := \varphi \circ J$. Assume that (G, H, τ) acts on $\mathcal{D}(\mathcal{B})$ by π such that the following hold:*

(i) $J\pi(\tau(g))J = \pi(\sigma(g))$.

(ii) $\pi(H)[\mathcal{D}(\mathcal{B}_+)] = \mathcal{D}(\mathcal{B}_+)$.

(iii) $D(\pi(\sigma(g))\varphi \otimes \pi(g)\psi) = D(\varphi \otimes \psi)$ *for any* $\varphi, \psi \in \mathcal{D}(\mathcal{B})$*.*

(iv) $D(J\varphi \otimes J\psi) = D(\varphi \otimes \psi)$ *for any* $\varphi, \psi \in \mathcal{D}(\mathcal{B})$*.*

Let $\mathcal{E} = \mathcal{H}_D := \mathcal{D}(\mathcal{B}) \otimes D$ *and* $\mathcal{E}_+ = \mathcal{H}_{D_+} := \mathcal{D}(\mathcal{B}_+) \otimes D$*. For* $\varphi, \psi \in \mathcal{D}(\mathcal{B})$*, define the bilinear form* $\beta(\varphi \otimes D, \psi \otimes D) = \langle \varphi \otimes D, \psi \otimes D\rangle := D(\overline{\varphi} \otimes \psi)$*. Let* $\pi_D(g)E(\varphi) := E(\pi(g^{-1})\varphi)$*. Let* $[JE](\psi) := E(\psi \circ J)$*. Then*

(1) The quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ *is a reflection positive vector space (as in Definition 7.1).*

(2) The representation π_D *is a* σ*-twisted reflection positive on* $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ *(as in Definition 7.2).*

Proof. For $\varphi \in \mathcal{D}(\mathcal{B}_+)$, we have

$$\begin{aligned}
\langle J[\varphi \otimes D], \varphi \otimes D\rangle &= \langle [J\varphi] \otimes D, \varphi \otimes D\rangle \quad \text{by Part 1 of Lemma 7.25} \\
&= D(\overline{J\varphi} \otimes \psi) \geq 0,
\end{aligned}$$

because D is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$. This completes the proof of Part 1.

Now, for $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, we have

$$\begin{aligned}
\langle \pi_D(\sigma(g))[\varphi \otimes D], \pi_D(g)[\psi \otimes D]\rangle &= \langle [\pi(g)\varphi] \otimes D, [\pi(\sigma(g))\psi] \otimes D\rangle \\
&\qquad \text{by Part 2 of Lemma 7.21} \\
&= D(\overline{\pi(g)\varphi} \otimes \pi(\sigma(g))\psi) \\
&= D(\overline{\varphi} \otimes \psi) \\
&= \langle \varphi \otimes D, \psi \otimes D\rangle.
\end{aligned}$$

This completes the proof of Part RP0 of Definition 7.2.

For $\varphi \in \mathcal{D}(\mathcal{B})$ and $E \in \mathcal{D}'(\mathcal{B})$, we have

$$\begin{aligned}[J\pi_D(\tau(g))JE](\varphi) &= E(J\pi(\tau(g^{-1}))J\varphi) \\ &= E(\pi(\sigma(g^{-1}))\varphi) \ \text{ by } \ \text{(i)} \\ &= [\pi_D(\sigma(g))E](\varphi).\end{aligned}$$

This completes the proof of Part RP1 of Definition 7.2.

Finally, we want to show that $\pi_D(H)\mathcal{E}_+ \subset \mathcal{E}_+$. Let $\varphi \in \mathcal{D}(\mathcal{B}_+)$, $h \in H$. Since $\sigma(H) \subset H$ and Part (ii), it follows that $\pi(\sigma(h))\varphi \in \mathcal{D}(\mathcal{B}_+)$. Using Part 2 of Lemma 7.25, we have $\pi_D(h)[\varphi \otimes D] = [\pi(\sigma(h))\varphi] \otimes D$. Hence $\pi_D(h)[\varphi \otimes D] \in \mathcal{E}_+$. This completes the proof of Part RP2 of Definition 7.2. Hence the representation π_D is reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, J)$ and that completes the proof of Part 2. $\square$

Now, we are ready to state and prove the integrability theorem without the positive definiteness for representations with a cocycle condition. Let $\mathcal{B}$ be a smooth finite dimensional manifold and let G be a group acting on $\mathcal{B}$. Let $j : G \times \mathcal{B} \to \mathbb{R}$ be a function satisfying the cocycle condition

$$j(g_1g_2, m) = j(g_1, g_2 \cdot m)j(g_2, m),$$

for all $g_1, g_2 \in G, m \in \mathcal{B}$. For $g \in G$ and $\varphi \in \mathcal{D}(\mathcal{B})$, define the representation π of G by

$$[\pi(g)\varphi](m) := j(g^{-1}, m)^{-1}\varphi(g^{-1} \cdot m).$$

Let $\mathcal{B}_+ \subset \mathcal{B}$ be an open subset and let $\widetilde{\mathcal{B}_+} := \mathcal{B}_+ \times \mathbb{R}$. Then we obtain a G-left action on $\widetilde{\mathcal{B}_+}$ given by

$$g \cdot (m, z) := (g.m, j(g, m)z).$$

Let $\widetilde{\beta} : \mathfrak{g} \longrightarrow \mathcal{V}(\widetilde{\mathcal{B}_+})$ be the homomorphism given by

$$\begin{aligned}
\left(\widetilde{\beta}(x)[\varphi \otimes f]\right)(m, z) &= \left(\widetilde{\beta}(x)\right)_{(m,z)}[\varphi \otimes f] \\
&= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](\Phi_t^{\widetilde{\beta}}(m, z)) \\
&:= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](e^{-tx} \cdot (m, z)) \\
&= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](e^{-tx} \cdot m, j(e^{-tx}, m)z) \\
&= \frac{d}{dt}\Big|_{t=0}\left[\varphi(e^{-tx} \cdot m) f\left(j(e^{-tx}, m)z\right)\right].
\end{aligned}$$

Theorem 7.27. *Let $\mathcal{B}$ be a smooth finite dimensional manifold and let (G, H, τ) be a symmetric Lie group acting on $\mathcal{B}$. Let σ be an involution on G such that $\sigma\tau = \tau\sigma$. For $g \in G$ and $\varphi \in \mathcal{D}(\mathcal{B})$, define the representation π of G by $[\pi(g)\varphi](m) := j(g^{-1}, m)^{-1}\varphi(g^{-1} \cdot m)$. Let $D \in \mathcal{D}'(\mathcal{B} \times \mathcal{B})$ be a distribution which is reflection positive with respect to $(\mathcal{B}, \mathcal{B}_+, J)$ and let $K \in \mathcal{D}'(\mathbb{R})$ be a distribution. Suppose that the following hold.*

(1) $D(\pi(\sigma(g))\varphi \otimes \pi(g)\psi) = D(\varphi \otimes \psi)$ for all $\varphi, \psi \in \mathcal{D}(\mathcal{B})$.

(2) $rK(f \circ \lambda_r) = K(f)$ for all $r \in \mathbb{R}$ and $f \in \mathcal{D}(\mathbb{R})$.

(3) $j(\tau(g), J(m)) = j(\sigma(g), m)$ for all $g \in G$, $m \in \mathcal{B}$.

(4) $J(g \cdot m) = \sigma(\tau(g)) \cdot J(m)$ for all $g \in G$, $m \in \mathcal{B}$.

Let $\widetilde{\mathcal{B}_+} := \mathcal{B}_+ \times \mathbb{R}$. Let $\widetilde{D_+} \in \mathcal{D}'(\widetilde{\mathcal{B}_+} \times \widetilde{\mathcal{B}_+})$ be the distribution defined by

$$\widetilde{D_+}([\varphi \otimes f] \otimes [\psi \otimes g]) := K(f)D_+(\varphi \otimes \psi)K(g),$$

where $f, g \in \mathcal{D}(\mathbb{R})$, $\varphi, \psi \in \mathcal{D}(\mathcal{B}_+)$. Assume further that the symmetric Lie group (G, H, τ) acts on $\mathcal{B}$ such that $H \cdot \mathcal{B}_+ = \mathcal{B}_+$. We assume that D is invariant under J. Let G^c be a simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ and define $\mathcal{L}_{\widetilde{\beta}(x)}$, $x \in \mathfrak{g}$, on maximal domain in the Hilbert subspace $\mathcal{H}_{\widetilde{D_+}} := \mathcal{D}(\widetilde{\mathcal{B}_+}) \otimes \widetilde{D_+} \subset$

$\mathcal{D}'(\widetilde{\mathcal{B}}_+)$. Then there exists a unique smooth unitary representation $(\pi^c, \mathcal{H}_{\widetilde{D_+}})$ of G^c such that

(i) $\overline{\mathrm{d}\,\pi^c}(x) = \mathcal{L}_{\widetilde{\beta}(x)}$ *for* $x \in \mathfrak{h}$.

(ii) $\overline{\mathrm{d}\,\pi^c}(iy) = i\mathcal{L}_{\widetilde{\beta}(y)}$ *for* $y \in \mathfrak{q}$.

In particular, if we assume that $\pi_{\widetilde{D_+}}$ is the regular representation of G on $\mathcal{H}_{\widetilde{D_+}}$, i.e., $[\pi_{\widetilde{D_+}}(g)E](\varphi \otimes f) := E([\varphi \otimes f] \circ \lambda_g)$. Then

(a) $\overline{\mathrm{d}\,\pi^c}(x) = \mathrm{d}\,\pi_{\widetilde{D_+}}(x)$ *for* $x \in \mathfrak{h}$.

(b) $\overline{\mathrm{d}\,\pi^c}(iy) = i\,\mathrm{d}\,\pi_{\widetilde{D_+}}(y)$ *for* $y \in \mathfrak{q}$.

Proof. First of all, for $f \in \mathcal{D}(\mathbb{R})$, $\varphi \in \mathcal{D}(\mathcal{B}_+)$, we have $\widetilde{D_+}(\overline{[\varphi \otimes f]} \otimes [\varphi \otimes f]) := \overline{K(f)} D_+(\overline{\varphi} \otimes \varphi) K(f) = |K(f)|^2 D(\overline{J\varphi} \otimes \varphi) \geq 0$ by the assumption. Thus the distribution $\widetilde{D_+}$ is positive definite. Let $\widetilde{\sigma} : H \times \widetilde{\mathcal{B}_+} \longrightarrow \widetilde{\mathcal{B}_+}$ be the action given by $\widetilde{\sigma}(h,(m,z)) = (h \cdot m, j(h,m)z)$. It is well define because of the assumption that $H \cdot \mathcal{B}_+ = \mathcal{B}_+$. Let $\widetilde{\beta} : \mathfrak{g} \longrightarrow \mathcal{V}(\widetilde{\mathcal{B}_+})$ be the homomorphism as above given by

$$\begin{aligned} \left(\widetilde{\beta}(x)[\varphi \otimes f]\right)(m,z) &:= \frac{d}{dt}\Big|_{t=0} [\varphi \otimes f](e^{-tx} \cdot (m,z)) \\ &= \frac{d}{dt}\Big|_{t=0} \left[\varphi(e^{-tx} \cdot m) f\left(j(e^{-tx}, m)z\right)\right]. \end{aligned}$$

It is clear that the pair $(\widetilde{\sigma}, \widetilde{\beta})$ is a smooth action of $(\mathfrak{g}, H)$ on $\widetilde{\mathcal{B}_+}$ as in Definition 3.9. Now we are to show that the distribution $\widetilde{D_+}$ is compatible with the smooth action $(\widetilde{\sigma}, \widetilde{\beta})$, i.e., $\widetilde{D_+}$ is $\widetilde{\beta}$-compatible as in Definition 3.8. More precisely, we are to show that for $x \in \mathfrak{g}$, we have

$$\mathcal{L}^1_{\widetilde{\beta}(x)} \widetilde{D_+} = \mathcal{L}^2_{\widetilde{\beta}(-\tau(x))} \widetilde{D_+} = -\mathcal{L}^2_{\widetilde{\beta}(\tau(x))} \widetilde{D_+}.$$

Let $f, g \in \mathcal{D}(\mathbb{R})$, $\varphi, \psi \in \mathcal{D}(\mathcal{B}_+)$. Let $a \in G$ and let

$$I := \widetilde{D_+}\left(([\varphi \otimes f] \circ \lambda_a) \otimes [\psi \otimes g]\right).$$

Then we have

$$\begin{aligned}
I &= \int\int\int [\varphi\otimes f]\circ\lambda_a(m,z)\ [J\psi\otimes g](n,w)\ dK_z dD_{(m,n)} dK_w \\
&\qquad \text{by the definition of } \widetilde{D_+} \text{ and using the fact that } D \text{ is } J \text{ invariant} \\
&= \int\int\int [\varphi\otimes f](a\cdot m, j(a,m)z)\ [J\psi\otimes g](n,w)\ dK_z dD_{(m,n)} dK_w \\
&\qquad \text{by the definition of the action on } \widetilde{\mathcal{B}_+} \\
&= \int\int\int \varphi(a\cdot m) f(j(a,m)z)\ J\psi(n)\ g(w)\ dK_z dD_{(m,n)} dK_w
\end{aligned}$$

Now using assumption (2) we get

$$\begin{aligned}
I &= \int\int\int j(a,m)^{-1}\varphi(a\cdot m)\ f(z)\ J\psi(n)\ g(w)\ dK_z dD_{(m,n)} dK_w. \\
&= \int\int\int [\pi(a)\varphi](m)\ f(z)\ J\psi(n)\ g(w)\ dK_z dD_{(m,n)} dK_w.
\end{aligned}$$

By assumption (1), we have

$$\begin{aligned}
I &= \int\int\int \varphi(m)\ f(z)\ [\pi(\sigma(a^{-1}))J\psi](n)\ g(w)\ dK_z dD_{(m,n)} dK_w \\
&= \int\int\int \varphi(m)\ f(z)\ j(\sigma(a^{-1}),n)^{-1} J\psi(\sigma(a^{-1})\cdot n)\ g(w)\ dK_z dD_{(m,n)} dK_w.
\end{aligned}$$

Now by assumptions (3) and (4), we have

$$\begin{aligned}
I &= \int\int\int \varphi(m)\ f(z)\ j(\tau(a^{-1}),J(n))^{-1}\psi(\tau(a^{-1})\cdot J(n))\ g(w)\ dK_z dD_{(m,n)} dK_w \\
&= \int\int\int \varphi(m)\ f(z)\ \psi(\tau(a^{-1})\cdot J(n))\ g\left(j(\tau(a^{-1}),J(n))w\right)\ dK_z dD_{(m,n)} dK_w \\
&\qquad \text{by assumption (2)} \\
&= \int\int\int \varphi(m)\ f(z)\ [\psi\otimes g]\circ\lambda_{\tau(a^{-1})}(J(n),w)\ \ dK_z dD_{(m,n)} dK_w \\
&= \widetilde{D_+}\left([\varphi\otimes f]\otimes\left([\psi\otimes g]\circ\lambda_{\tau(a^{-1})}\right)\right).
\end{aligned}$$

Thus we have

$$\widetilde{D_+}\left(([\varphi\otimes f]\circ\lambda_a)\otimes[\psi\otimes g]\right) = \widetilde{D_+}\left([\varphi\otimes f]\otimes\left([\psi\otimes g]\circ\lambda_{\tau(a^{-1})}\right)\right). \tag{7.4}$$

Now for $f, g \in \mathcal{D}(\mathbb{R})$, $\varphi, \psi \in \mathcal{D}(\mathcal{B}_+)$ and $x \in \mathfrak{g}$, we have

$$\begin{aligned}
\mathcal{L}^1_{\widetilde{\beta}(x)} \widetilde{D_+}([\varphi \otimes f] \otimes [\psi \otimes g]) &= -\widetilde{D_+}(\mathcal{L}^1_{\widetilde{\beta}(x)}([\varphi \otimes f] \otimes [\psi \otimes g])) \\
&= -\widetilde{D_+}\left(\left(\mathcal{L}_{\widetilde{\beta}(x)}[\varphi \otimes f]\right) \otimes [\psi \otimes g]\right) \\
&= -\widetilde{D_+}\left(\left[\frac{d}{dt}\Big|_{t=0}[\varphi \otimes f] \circ \lambda_{e^{-tx}}\right] \otimes [\psi \otimes g]\right) \\
&= -\frac{d}{dt}\Big|_{t=0} \widetilde{D_+}\left(([\varphi \otimes f] \circ \lambda_{e^{-tx}}) \otimes [\psi \otimes g]\right) \\
&= -\frac{d}{dt}\Big|_{t=0} \widetilde{D_+}\left([\varphi \otimes f] \otimes ([\psi \otimes g] \circ \lambda_{e^{t\tau(x)}})\right) \text{ by (7.4)} \\
&= -\widetilde{D_+}\left([\varphi \otimes f] \otimes \left(\mathcal{L}_{\widetilde{\beta}(-\tau(x))}[\psi \otimes g]\right)\right) \\
&= \mathcal{L}^2_{\widetilde{\beta}(-\tau(x))} \widetilde{D_+}([\varphi \otimes f] \otimes [\psi \otimes g]).
\end{aligned}$$

Hence we have $\mathcal{L}^1_{\widetilde{\beta}(x)} \widetilde{D_+} = \mathcal{L}^2_{\widetilde{\beta}(-\tau(x))} \widetilde{D_+}$ and so the distribution $\widetilde{D_+}$ is compatible with the smooth action $(\widetilde{\sigma}, \widetilde{\beta})$. Now by Theorem 7.22 we are done.

The last statement of this theorem follows from the fact that for $x \in \mathfrak{g}$, we have

$$\mathrm{d}\,\pi_{\widetilde{D_+}}(x) = \mathcal{L}_{\widetilde{\beta}(x)}. \tag{7.5}$$

Let $f \in \mathcal{D}(\mathbb{R})$ and let $\varphi \in \mathcal{D}(\mathcal{B}_+)$. Let $x \in \mathfrak{g}$ and let $E \in \mathcal{H}_{\widetilde{D_+}} = \mathcal{D}(\widetilde{\mathcal{B}_+}) \otimes \widetilde{D_+}$. Then we have

$$\begin{aligned}
[\mathcal{L}_{\widetilde{\beta}(x)} E](\varphi \otimes f) &= -E(\mathcal{L}_{\widetilde{\beta}(x)}[\varphi \otimes f]) \\
&= -E\left(\frac{d}{dt}\Big|_{t=0}[\varphi \otimes f] \circ \lambda_{e^{-tx}}\right) \\
&= -\frac{d}{dt}\Big|_{t=0} E([\varphi \otimes f] \circ \lambda_{e^{-tx}}) \\
&= -\frac{d}{dt}\Big|_{t=0}\left[\pi_{\widetilde{D_+}}(e^{-tx})E\right](\varphi \otimes f) \\
&= \left[\frac{d}{dt}\Big|_{t=0}\pi_{\widetilde{D_+}}(e^{tx})E\right](\varphi \otimes f) \\
&= \left[\mathrm{d}\,\pi_{\widetilde{D_+}}(x)E\right](\varphi \otimes f).
\end{aligned}$$

Hence we proved (7.5) and this completes the proof. □

7.4 Non-Compactly Causal Symmetric Spaces and Reflection Positive Distribution on $\mathcal{D}(K/L)$

This section gives the relation between the non-compactly causal symmetric spaces and the reflection positive distributions. In other words, it builds up a reflection positive representation for those spaces using the language of the distributions.

Assume that G/H is non-compactly causal symmetric space and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ.

Let $K = G^\theta$ and $L = H \cap K$. Let $\lambda - \rho \leq L_{pos}$ and let A^λ be the intertwining operator defined in (7.1). Then define the distribution D^λ on $\mathcal{D}(K/L)$ by

$$\begin{aligned} D^\lambda(\varphi) := [A^\lambda \varphi](b) &= \int_{K/L} \varphi(xb)\ \alpha(x)^{\lambda-\rho}\ d(xb) \\ &= \int_K \varphi(x)\ \alpha(x)^{\lambda-\rho}\ dx. \end{aligned}$$

Let $Jf(xb) := f(\tau(x)b)$. Recall that the bilinear form $\langle \cdot, \cdot \rangle_\lambda : L^2(K/L) \times L^2(K/L) \to \mathbb{C}$ is defined by

$$\begin{aligned} \beta_\lambda(f,h) = \langle f, h \rangle_\lambda &:= \left\langle A^\lambda f, h \right\rangle_{L^2} \\ &= \int_{K/L} A^\lambda f(xb) \overline{h(xb)}\ \ d(xb) \\ &= \int_{K/L} \int_{K/L} f(xb) \overline{h(yb)}\ \alpha(y^{-1}x)^{\lambda-\rho}\ \ d(xb)\ d(yb). \end{aligned}$$

The following theorem states that the bilinear form given in the previous section is nothing but the above distribution.

Theorem 7.28. *Let $f, g \in \mathcal{D}(K)^L$. Then $D^\lambda(f^* * h) = \beta_\lambda(f,h) = \langle f, h\rangle_\lambda$.*

Proof. For $f, g \in \mathcal{D}(K/L)$, we have

$$\begin{aligned} D^\lambda(f^* * h) &= \int_K \overline{f^* * h}(x)\ \alpha(x)^{\lambda-\rho}\ dx \\ &= \int_K \int_K \overline{f^*(y)}\ \overline{h(y^{-1}x)}\ \alpha(x)^{\lambda-\rho}\ dy\ dx \\ &= \int_K \int_K f(y^{-1})\ \overline{h(y^{-1}x)}\ \alpha(x)^{\lambda-\rho}\ dy\ dx \\ &= \int_K \int_K f(y)\ \overline{h(x)}\ \alpha(y^{-1}x)^{\lambda-\rho}\ dy\ dx \\ &= \beta_\lambda(f,h). \end{aligned}$$

This completes the proof. □

The following lemma is a technical lemma. It proves that α is τ-invariant when we are talking about the compact group K. This lemma plays an important role in proving that the above distribution is τ-invariant.

Lemma 7.29. *Let $k \in K \cap \overline{N}MAN$. Then*

$$\alpha(\tau(k)) = \alpha(k).$$

Proof. We know that if $a \in A$ and $m \in M$, then $aN = Na$, $mN = Nm$, $a\overline{N} = \overline{N}a$, $m\overline{N} = \overline{N}m$, $\theta\tau(\overline{N}) \subset \overline{N}$, $\theta\tau(N) \subset N$ and $\theta\tau(M) \subset M$. Also, if $a \in A$, then $\theta\tau(a) = a$. Let $k = \overline{n}m\alpha(k)n$. Then

$$\begin{aligned} \tau(k) &= \theta(\tau(k)) \qquad \text{because } \tau(k) \in K = G^\theta \\ &= \theta(\tau(\overline{n}m\alpha(k)n)) \\ &= \theta\tau(\overline{n})\theta\tau(m)\theta\tau(\alpha(k))\theta\tau(n) \\ &= \overline{n}'m'\alpha(k)n' \\ &\qquad \text{for some } \overline{n}' \in \overline{N}, m' \in M \text{ and } n' \in N. \end{aligned}$$

But we have

$$\begin{aligned} \tau(k) &= \overline{n}''m''\alpha(\tau(k))n'' \\ &\qquad \text{for some } \overline{n}'' \in \overline{N}, m'' \in M \text{ and } n'' \in N. \end{aligned}$$

Thus $\alpha(\tau(k)) = \alpha(k)$. This completes the proof. $\square$

Now, we are ready to prove that the above distribution is τ-invariant.

Lemma 7.30. *Let $J{\cdot}D^\lambda := D^\lambda \circ J$. Then D^λ is invariant under J, i.e., $J{\cdot}D^\lambda = D^\lambda$.*

Proof. Let $f \in \mathcal{D}(K)^L$. Then

$$\begin{aligned} D^\lambda(Jf) &= \int_K Jf(x)\ \alpha(x)^{\lambda-\rho}\ dx \\ &= \int_K f(\tau(x))\ \alpha(x)^{\lambda-\rho}\ dx \\ &= \int_K f(x)\ \alpha(\tau(x))^{\lambda-\rho}\ dx \\ &= \int_K f(x)\ \alpha(x)^{\lambda-\rho}\ dx \qquad \text{by Lemma 7.29} \\ &= D^\lambda(f). \end{aligned}$$

This completes the proof. $\square$

We need the following technical lemma to prove the next theorem.

Lemma 7.31. *Let $f, h \in \mathcal{D}(K)^L$. Then*

(1) $J \circ (f^* * h) = (Jf)^* * Jh$,

(2) $\beta_\lambda(Jf, Jh) = \beta_\lambda(f, h)$.

Proof. Let $f, h \in \mathcal{D}(K)^L$. Then

$$\begin{aligned}
((Jf)^* * Jh)(x) &= \int_K \overline{(Jf)^*(y)}\ Jh(y^{-1}x)\ \ dy \\
&= \int_K \overline{f^*(\tau(y))}\ h(\tau(y)^{-1}\tau(x))\ \ dy \\
&= \int_K \overline{f^*(y)}\ h(y^{-1}\tau(x))\ \ dy \\
&= (f^* * h)(\tau(x)) \\
&= J(f^* * h)(x).
\end{aligned}$$

This completes the proof of Part 1.

Let $f, h \in \mathcal{D}(K)^L$. Then

$$\begin{aligned}
\beta_\lambda(Jf, Jh) &= D^\lambda((Jf)^* * Jh) \qquad \text{by Theorem 7.28} \\
&= D^\lambda(J \circ (f^* * h)) \qquad \text{by Part 1} \\
&= D^\lambda(f^* * h) \qquad \text{by Lemma 7.30} \\
&= \beta_\lambda(f, h).
\end{aligned}$$

This completes the proof of Part 2. □

The following theorem proves the main conditions that are important in the main theorem of this section.

Theorem 7.32. *Assume that G/H is non-compactly causal and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ and assume that $\lambda - \rho \leqslant L_{pos}$. Let $K := G^\theta$ and $L := K \cap H$. Let $k(H) := \{k(h) \mid h \in H\}$. Let $\mathcal{B} = K/L$ and $\mathcal{B}_+ := k(H)/L$. Let $J(kL) := \tau(k)L$ and $Jf(kL) := f(J(kL)) = f(\tau(k)L)$. Define the distribution D^λ on $\mathcal{D}(K)^L$ by*

$$
\begin{aligned}
D^\lambda(\varphi) &:= \int_K \varphi(x)\ \alpha(x)^{\lambda-\rho}\ dx, \\
&= \int_{K/L} \varphi(xb)\alpha(x)^{\lambda-\rho}\ d(xb).
\end{aligned}
$$

Then the following hold:

(1) $D^\lambda(\varphi^* * J\varphi) \geq 0$ *for all* $\varphi \in \mathcal{D}(\mathcal{B}_+)$. *In other words,* D^λ *is a reflection positive distribution with respect to* $(\mathcal{B}, \mathcal{B}_+, J)$ *as in Definition 7.20.*

(2) D^λ *is invariant under* J.

(3) $(\varphi \circ \lambda_k)^* * (\psi \circ \lambda_k) = \varphi^* * \psi\ \forall\ k \in K,\ \forall\ \varphi, \psi \in \mathcal{B}$.

(4) $J(g \cdot m) = \tau(\theta(g)) \cdot J(m)\ \forall\ g \in K,\ \forall\ m \in \mathcal{B}$. *In particular,* $J(k \cdot m) = \tau(k) \cdot J(m)\ \forall\ k \in K,\ \forall\ m \in \mathcal{B}$.

(5) $H.\mathcal{B}_+ = \mathcal{B}_+$. *In particular,* $L.\mathcal{B}_+ \subset \mathcal{B}_+$.

Proof. Let $\varphi \in \mathcal{D}(\mathcal{B}_+)$. Then $D^\lambda(\varphi^* * J\varphi) = \langle \varphi, J\varphi \rangle_\lambda$ by Theorem 7.28. Using Theorem 7.13, we have $D^\lambda(\varphi^* * J\varphi) \geq 0$. This completes the proof of Part 1. Part 2 is nothing but Lemma 7.30.

Let $\varphi, \psi \in \mathcal{B}$ and $k, x \in K$. Then

$$
\begin{aligned}
[(\varphi \circ \lambda_k)^* * (\psi \circ \lambda_k)](x) &= \int_K \overline{(\varphi \circ \lambda_k)^*(y)}\ (\psi \circ \lambda_k)(y^{-1}x)\ \ dy \\
&= \int_K \varphi(ky^{-1})\ \psi(ky^{-1}x)\ \ dy \\
&= \int_K \varphi(ky)\ \psi(kyx)\ \ dy \\
&= \int_K \varphi(y)\ \psi(yx)\ \ dy \\
&= \int_K \varphi(y^{-1})\ \psi(y^{-1}x)\ \ dy \\
&= [\varphi^* * \psi](x).
\end{aligned}
$$

This completes the proof of Part 3. Let $m = xL \in K/L = \mathcal{B}$. Then

$$\begin{aligned} J(g \cdot m) &= J(g \cdot xL) = J(k(gx)L) = \tau(k(gx))L = k(\theta(\tau(gx)))L \\ &= \theta(\tau(g)) \cdot \tau(x)L = \theta(\tau(g)) \cdot J(xL) = \theta(\tau(g)) \cdot J(m). \end{aligned}$$

We used Part 1 of Lemma 7.4. This completes the proof of Part 4. Let $m = k(h)L \in k(H)/L = \mathcal{B}_+$ and let $x \in H$. Then

$$x \cdot m = x \cdot k(h)L = x \cdot (h \cdot L) = xh \cdot L = k(xh)L = m' \in \mathcal{B}_+.$$

This completes the proof of Part 5. $\square$

Now, this is the time to present the first main theorem of this section. This theorem constructs a reflection positive representation for the non-compactly causal symmetric spaces using the language of distributions. This theorem deals with the case of regular representations on compact groups.

Theorem 7.33. *Assume that G/H is non-compactly causal symmetric space and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ and assume that $\lambda - \rho \leqslant L_{pos}$. Let $K := G^\theta$ and $L := K \cap H$. Let $k(H) := \{k(h) \mid h \in H\}$. Let $\mathcal{B} := K/L$ and $\mathcal{B}_+ := k(H)/L$. Let $J(kL) := \tau(k)L$ and $Jf(kL) := f(J(kL)) = f(\tau(k)L)$. Define the distribution D^λ on $\mathcal{D}(K)^L$ by*

$$\begin{aligned} D^\lambda(\varphi) := [A^\lambda \varphi](b) &= \int_K \varphi(x)\ \alpha(x)^{\lambda-\rho}\ dx \\ &= \int_{K/L} \varphi(xb)\ \alpha(x)^{\lambda-\rho}\ d(xb). \end{aligned}$$

Let

$$\mathcal{E} = \mathcal{H}_{D^\lambda} := \mathcal{D}(\mathcal{B}) * D^\lambda = \{\varphi * D^\lambda \mid \varphi \in \mathcal{D}(\mathcal{B})\}$$

and

$$\mathcal{E}_+ = \mathcal{H}_{D_+^\lambda} := \mathcal{D}(\mathcal{B}_+) * D_+^\lambda = \{\varphi * D^\lambda \mid \varphi \in \mathcal{D}(\mathcal{B}_+)\}.$$

Let π_{D^λ} be the regular representation of K on $\mathcal{E}$, i.e., $\pi_{D^\lambda}(k)E(\varphi) := E(\varphi \circ \lambda_k)$. Let $[JE](\varphi) := E(J\varphi)$. Then the following hold:

(i) The triple $(\mathcal{E}, \mathcal{E}_+, J)$ is a reflection positive vector space.

(ii) The representation π_{D^λ} of K is reflection positive on $(\mathcal{E}, \mathcal{E}_+, J)$.

(i) There exists a unique smooth unitary representation $(\pi^c, \mathcal{E}_+)$ of K^c such that

(1) $\overline{\mathrm{d}\,\pi^c}(x) = \mathrm{d}\,\pi_{D^\lambda}(x)$ for $x \in \mathfrak{h} \cap \mathfrak{k}$.

(2) $\overline{\mathrm{d}\,\pi^c}(iy) = i\,\mathrm{d}\,\pi_{D^\lambda}(y)$ for $y \in \mathfrak{q} \cap \mathfrak{k}$.

Proof. The proof follows directly from Theorem 7.32 and Corollary 7.24. □

Let us present the second main theorem of this section. This theorem constructs a reflection positive representation for non-compactly causal symmetric spaces using the language of distributions. It deals with representations of non-compact groups with cocycle conditions.

Assume that G/H is non-compactly causal symmetric space and assume that $\lambda - \rho \leqslant L_{pos}$. Let $K := G^\theta$ and $L := K \cap H$. Let $k(H) := \{k(h) \mid h \in H\}$. Let $\mathcal{B} := K/L$ and $\mathcal{B}_+ := k(H)/L$ and let $\widetilde{\mathcal{B}_+} := \mathcal{B}_+ \times \mathbb{R}$. Then we obtain a G-left action on $\widetilde{\mathcal{B}_+}$ given by

$$\lambda_g(kb, z) := (g.m, a(gk)^{\rho-\lambda}z).$$

Let $j(g, kb) := a(gk)^{\rho-\lambda}$ and let $\widetilde{\beta} : \mathfrak{g} \longrightarrow \mathcal{V}(\widetilde{\mathcal{B}_+})$ be the homomorphism given by

$$\begin{aligned}
\left(\widetilde{\beta}(x)[\varphi \otimes f]\right)(m, z) &= \left(\widetilde{\beta}(x)\right)_{(m,z)}[\varphi \otimes f] \\
&= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](\Phi_t^{\widetilde{\beta}}(m, z)) \\
&:= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](e^{-tx} \cdot (m, z)) \\
&= \frac{d}{dt}\Big|_{t=0}[\varphi \otimes f](e^{-tx} \cdot m, j(e^{-tx}, m)z) \\
&= \frac{d}{dt}\Big|_{t=0}\left[\varphi(e^{-tx} \cdot m) f\left(j(e^{-tx}, m)z\right)\right].
\end{aligned}$$

Theorem 7.34. *Assume that G/H is non-compactly causal symmetric space and let τ be the corresponding involution. Let θ be the Cartan involution on G commuting with τ and assume that $\lambda - \rho \leqslant L_{pos}$. Let $K := G^\theta$ and $L := K \cap H$. Let $k(H) := \{k(h) \mid h \in H\}$. Let $\mathcal{B} := K/L$ and $\mathcal{B}_+ := k(H)/L$. Let $J(kL) := \tau(k)L$ and $Jf(kL) := f(J(kL)) = f(\tau(k)L)$. For $g \in G$ and $\varphi \in \mathcal{D}(\mathcal{B})$, define the representation π_λ of G by $[\pi_\lambda(g)\varphi](kb) := a(g^{-1}k)^{\rho-\lambda}\varphi(g^{-1} \cdot kb)$. Define the distribution D^λ on $\mathcal{D}(K)^L$ by*

$$D^\lambda(\varphi) := [A^\lambda\varphi](b) = \int_K \varphi(x)\ \alpha(x)^{\lambda-\rho}\ dx = \int_{K/L} \varphi(xb)\ \alpha(x)^{\lambda-\rho}\ d(xb).$$

Let $Q \in \mathcal{D}'(\mathbb{R})$ be given by

$$Q(f) = \int_{\mathbb{R}} f(x)dx.$$

Let $\widetilde{\mathcal{B}_+} := \mathcal{B}_+ \times \mathbb{R}$. Let $\widetilde{D}_+^\lambda \in \mathcal{D}'(\widetilde{\mathcal{B}_+} \times \widetilde{\mathcal{B}_+})$ be the distribution defined by

$$\widetilde{D}_+^\lambda([\varphi \otimes f] \otimes [\psi \otimes g]) := Q(f) D_+^\lambda(\varphi^\vee * \psi) Q(g),$$

where $f, g \in \mathcal{D}(\mathbb{R})$, $\varphi, \psi \in \mathcal{D}(\mathcal{B}_+)$. Let G^c be a simply connected Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ and define $\mathcal{L}_{\widetilde{\beta}(x)}$, $x \in \mathfrak{g}$, on maximal domain in the Hilbert subspace $\mathcal{H}_{\widetilde{D}_+^\lambda} := \mathcal{D}(\widetilde{\mathcal{B}_+}) \otimes \widetilde{D}_+^\lambda \subset \mathcal{D}'(\widetilde{\mathcal{B}}_+)$. Assume that $\pi_{\widetilde{D}_+^\lambda}$ is the regular representation of G on $\mathcal{H}_{\widetilde{D}_+^\lambda}$, i.e., $[\pi_{\widetilde{D}_+^\lambda}(g)E](\varphi \otimes f) := E([\varphi \otimes f] \circ \lambda_g)$. Then there exists a unique smooth unitary representation $(\pi^c, \mathcal{H}_{\widetilde{D}_+^\lambda})$ of G^c such that

(a) $\overline{\mathrm{d}\,\pi^c}(x) = \mathrm{d}\,\pi_{\widetilde{D}^\lambda_+}(x)$ *for* $x \in \mathfrak{h}$.

(b) $\overline{\mathrm{d}\,\pi^c}(iy) = i\,\mathrm{d}\,\pi_{\widetilde{D}^\lambda_+}(y)$ *for* $y \in \mathfrak{q}$.

Proof. The proof follows directly from Theorem 7.27 and the following discussions.

For $\varphi, \psi \in \mathcal{D}(\mathcal{B})$, we have

$$\begin{aligned} D^\lambda\left((\pi_\lambda(\theta(g))\varphi)^* * \pi_\lambda(g)\psi\right) &= \langle \pi_\lambda(\theta(g))\varphi, \pi_\lambda(g)\psi\rangle_\lambda \text{ by Theorem 7.28} \\ &= \langle \varphi, \psi\rangle_\lambda \text{ by Part 3 of Theorem 7.3} \\ &= D^\lambda\left(\varphi^* * \psi\right). \end{aligned}$$

Thus we have

$$D^\lambda\left((\pi_\lambda(\theta(g))\varphi)^* * \pi_\lambda(g)\psi\right) = D^\lambda\left(\varphi^* * \psi\right). \tag{7.6}$$

This completes the proof of Assumption 1 of Theorem 7.27.

For $f \in \mathcal{D}(\mathbb{R})$, $r \in \mathbb{R}$, we have

$$Q(f \circ \lambda_r) = \int_{\mathbb{R}} f(rx)\ dx = \frac{1}{r}\int_{\mathbb{R}} f(x)\ dx = \frac{1}{r}Q(f).$$

This completes the proof of Assumption 2 of Theorem 7.27.

For $g \in G$, $m = kb \in \mathcal{B}$, we have

$$\begin{aligned} j(\tau(g), J(m)) &= a(\tau(gk))^{\rho-\lambda} \\ &= a(\theta(gk))^{\rho-\lambda} \text{ by Part 2 of Lemma 7.4} \\ &= j(\theta(g), m). \end{aligned}$$

This completes the proof of Assumption 3 of Theorem 7.27.

By Part 4 of Theorem 7.32, we have $J(g \cdot m) = \theta(\tau(g)) \cdot J(m)$ for all $g \in G$, $m \in \mathcal{B}$. Thus we complete the proof of Assumption 4 of Theorem 7.27.

Parts 5 and 1 of Theorem 7.32 give us $H.\mathcal{B}_+ = \mathcal{B}_+$ and $D^\lambda(\varphi^* * J\varphi) \geq 0$ for all $\varphi \in \mathcal{D}(\mathcal{B}_+)$. In other words, D^λ is a reflection positive distribution with respect to $(\mathcal{B}, \mathcal{B}_+, J)$ as in Definition 7.20. Part 2 of Theorem 7.32 says that D^λ is J invariant and so this completes the proof. $\square$

Remark 7.35. *It is worth mentioning that the kernel of the distribution*

$$\widetilde{D}_+^\lambda([\varphi \otimes f] \otimes [\psi \otimes g]) := Q(f) D_+^\lambda(\varphi^\vee * \psi) Q(g),$$

is given by

$$\widetilde{D}_+^\lambda((xb, z), (yb, w)) := zw \ \alpha(\tau(x)^{-1} y)^{\lambda - \rho}.$$

It is also worth mentioning that the Condition (7.6) can be translated into the kernel language as

$$j(\theta(g), m) D^\lambda \left(\theta(g) \cdot m, g \cdot n\right) j(g, n) = D^\lambda(m, n) \ \forall \ m, n \in \mathcal{B}, g \in G.$$

This is equivalent to

$$a(\theta(g)m) \ \alpha(k(\theta(g)m)^{-1} k(gn)) \ a(gn) = \alpha(m^{-1} n) \ \forall \ m, n \in K, g \in G.$$

The above condition follows from Corollary 7.10.

7.5 σ-Twisted Reflection Positive Distribution

This section introduces the definition of the σ-twisted reflection positive distribution and it gives us a way to construct σ-twisted reflection positive representation. At the end of this section, we give some examples of σ-twisted reflection positive representations on $\mathbb{R}^n$.

Let σ be an involution on G such that $\sigma\tau = \tau\sigma$, $g^\bigstar := \sigma(g^{-1})$ and $\varphi^\bigstar(g) := \overline{\varphi(g^\bigstar)}\Delta(g^\bigstar)$. Let $D \in \mathcal{D}^{'}(G)$ be a distribution on the Lie group G and $(\varphi \bigstar D)(\psi) := D(\varphi^\bigstar * \psi)$. Let

$$\mathcal{D}(G)\bigstar D \;\; := \;\; \{\varphi\bigstar D \mid \varphi \in \mathcal{D}(G)\}.$$

Let us start with the definition of σ-twisted reflection positive distribution.

Definition 7.36. *If (G, τ) is a symmetric Lie group and S is open and $\sharp$-invariant semigroup, then we call a distribution $D \in \mathcal{D}^{'}(G)$ σ-twisted reflection positive for (G, S, τ) if the following conditions are satisfied:*

(RP0) S is σ-invariant, i.e., $\sigma(S) \subset S$,

(RP1) $\tau\sigma D = D$, i.e., $D(\varphi \circ \tau\sigma) = D(\varphi)$ for $\varphi \in \mathcal{D}(G)$, and

(RP2) $D|_S$ is positive definite as a distribution on the involution semigroup $(S, \sharp)$, i.e., $D(\varphi^\sharp * \varphi) \geq 0$ for $\varphi \in \mathcal{D}(S)$.

Lemma 7.37. *For $E \in \mathcal{D}^{'}(G)$, define $\theta E(\psi) := E(\psi \circ \sigma\tau)$. Then the following assertions hold:*

(1) $\varphi^\bigstar * (\psi \circ \sigma\tau) = (\varphi^\sharp * \psi) \circ \sigma\tau$.

(2) $(\varphi \circ \sigma\tau)^\bigstar = \varphi^\sharp$.

(3) If $\tau\sigma D = D$, then $\theta(\varphi\bigstar D) = (\varphi \circ \sigma\tau)\bigstar D$.

Proof. Let $\varphi \in \mathcal{D}(G)$. Then

$$\begin{aligned}
[\varphi^{\bigstar} * (\psi \circ \sigma\tau)](x) &= \int_G \varphi^{\bigstar}(y)\ \psi \circ \sigma\tau(y^{-1}x)\ \ dy \\
&= \int_G \overline{\varphi((\sigma(y))^{-1})}\Delta(y^{-1})\ \psi((\sigma\tau(y))^{-1}\sigma\tau(x))\ \ dy \\
&= \int_G \overline{\varphi((\tau(y))^{-1})}\Delta(y^{-1})\ \psi(y^{-1}\sigma\tau(x))\ \ dy \\
&= \int_G \varphi^{\sharp}(y)\ \psi(y^{-1}\sigma\tau(x))\ \ dy \\
&= [(\varphi^{\sharp} * \psi) \circ \sigma\tau](x).
\end{aligned}$$

This completes the proof of Part 1. Let $\varphi \in \mathcal{D}(G)$. Then

$$\begin{aligned}
(\varphi \circ \sigma\tau)^{\bigstar}(x) &= \overline{\varphi \circ \sigma\tau(x^{\bigstar})}\Delta(x^{\bigstar}) \\
&= \overline{\varphi \circ \sigma\tau(\sigma(x^{-1}))}\Delta(x^{-1}) \\
&= \overline{\varphi(\tau(x^{-1}))}\Delta(x^{-1}) \\
&= \varphi^{\sharp}(x).
\end{aligned}$$

This completes the proof of Part 2.

Let $\varphi, \psi \in \mathcal{D}(G)$ and assume that $\sigma\tau D = D$. Then

$$\begin{aligned}
\theta(\varphi{\bigstar}D)(\psi) &= (\varphi{\bigstar}D)(\psi \circ \sigma\tau) \\
&= D(\varphi^{\bigstar} * (\psi \circ \sigma\tau)) \\
&= D((\varphi^{\sharp} * \psi) \circ \sigma\tau) \quad \text{by Part 1} \\
&= D(\varphi^{\sharp} * \psi) \quad \text{because } D \text{ is } \sigma\tau \text{ invariant} \\
&= D((\varphi \circ \sigma\tau)^{\bigstar} * \psi) \quad \text{by Part 2} \\
&= (\varphi \circ \sigma\tau){\bigstar}D(\psi).
\end{aligned}$$

This completes the proof of Part 3. $\square$

Lemma 7.38. *For $E \in \mathcal{D}^{'}(G)$, define $\pi_D(g)E(\psi) := E(\psi \circ \lambda_g)$. Then the following assertions hold:*

1. $(\varphi \circ \lambda_g)^* * (\psi \circ \lambda_g) = \varphi^* * \psi$,

2. $\varphi^{\bigstar} = (\varphi \circ \sigma)^*$,

3. $\lambda_{\sigma(g)} \circ \sigma = \sigma \circ \lambda_g$,

4. $\pi_D(g)[\varphi \bigstar D] = [\varphi \circ \lambda_{\sigma(g^{-1})}] \bigstar D$.

Proof. Let $\varphi \in \mathcal{D}(G)$. Then

$$\begin{aligned}
[(\varphi \circ \lambda_g)^* * (\psi \circ \lambda_g)](x) &= \int_G (\varphi \circ \lambda_g)^*(y)\ (\psi \circ \lambda_g)(y^{-1}x) \ \ dy \\
&= \int_G \overline{\varphi(gy^{-1})}\Delta(y^{-1})\ \psi(gy^{-1}x) \ \ dy \\
&= \int_G \overline{\varphi(gy)}\ \psi(gyx) \ \ dy \\
&= \int_G \overline{\varphi(y)}\ \psi(yx) \ \ dy \\
&= \int_G \overline{\varphi(y^{-1})}\ \Delta(y^{-1})\ \psi(y^{-1}x) \ \ dy \\
&= [\varphi^* * \psi](x).
\end{aligned}$$

This completes the proof of Part 1.

Let $\varphi \in \mathcal{D}(G)$. Then

$$\begin{aligned}
(\varphi \circ \sigma)^*(x) &= \overline{\varphi(\sigma(x^{-1}))}\Delta(x^{-1}) \\
&= \varphi^{\bigstar}(x).
\end{aligned}$$

This completes the proof of Part 2. Let $g, x \in G$. Then

$$\begin{aligned}
[\lambda_{\sigma(g)} \circ \sigma](x) &= \sigma(g)\sigma(x) \\
&= \sigma(gx) \\
&= [\sigma \circ \lambda_g](x).
\end{aligned}$$

This completes the proof of Part 3. Let $\varphi, \psi \in \mathcal{D}(G)$. Then

$$\begin{aligned}
\pi_D(g)[\varphi \bigstar D](\psi) &= (\varphi \bigstar D)(\psi \circ \lambda_g) \\
&= D(\varphi^{\bigstar} * (\psi \circ \lambda_g)) \\
&= D((\varphi \circ \sigma)^* * (\psi \circ \lambda_g)) \text{ by Part 2} \\
&= D((\varphi \circ \sigma \circ \lambda_{g^{-1}})^* * \psi) \text{ by Part 1} \\
&= D((\varphi \circ \lambda_{\sigma(g^{-1})} \circ \sigma)^* * \psi) \text{ by Part 3} \\
&= D((\varphi \circ \lambda_{\sigma(g^{-1})})^{\bigstar} * \psi) \text{ by Part 2} \\
&= [\varphi \circ \lambda_{\sigma(g^{-1})}] \bigstar D(\psi).
\end{aligned}$$

This completes the proof of Part 4. □

Theorem 7.39. *Let (G, τ) be a symmetric Lie group. Let σ be an involution on G such that $\sigma\tau = \tau\sigma$ and let S be an open and $\sharp$-invariant semigroup. Let $D \in \mathcal{D}'(G)$ be a σ-twisted reflection positive for (G, S, τ). Let $\mathcal{E} := \mathcal{D}(G) \bigstar D$ and $\mathcal{E}_+ := \mathcal{D}(S) \bigstar D$. Define the hermitian form $\beta(\psi \bigstar D, \varphi \bigstar D) = \langle \psi \bigstar D, \varphi \bigstar D \rangle := D(\psi^{\bigstar} * \varphi)$. Let $\pi_D(g)E(\psi) := E(\psi \circ \lambda_g)$ and let $\theta E(\psi) := E(\psi \circ \sigma\tau)$. Then the following assertions hold:*

1. *The quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, \theta)$ is reflection positive vector space.*

2. *The representation π_D is σ-twisted reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, \theta)$.*

Proof. Let $\varphi \in \mathcal{D}(S)$. Then

$$\begin{aligned}
\langle \theta(\varphi \bigstar D), \varphi \bigstar D \rangle &= \langle (\varphi \circ \sigma\tau) \bigstar D, \varphi \bigstar D \rangle \text{ by Part 3 of Lemma 7.37} \\
&= D((\varphi \circ \sigma\tau)^{\bigstar} * \varphi) \\
&= D(\varphi^{\sharp} * \varphi) \text{ by Part 2 of Lemma 7.37} \\
&\geq 0 \text{ by Part RP2 of Definition 7.36.}
\end{aligned}$$

This completes the proof of Part 1. To prove Part 2, first we need to show that

$$\langle \pi_D(\sigma(g))[\varphi\bigstar D], \pi_D(g)[\psi\bigstar D]\rangle = \langle \varphi\bigstar D, \psi\bigstar D\rangle.$$

But

$$\begin{aligned}
\langle \pi_D(\sigma(g))[\varphi\bigstar D], \pi_D(g)[\psi\bigstar D]\rangle &= \langle [\varphi\circ\lambda_{g^{-1}}]\bigstar D, [\psi\circ\lambda_{\sigma(g^{-1})}]\bigstar D\rangle \\
&\quad\text{by Part 4 of Lemma 7.38} \\
&= D((\varphi\circ\lambda_{g^{-1}})^{\bigstar} * (\psi\circ\lambda_{\sigma(g^{-1})})) \\
&= D((\varphi\circ\lambda_{g^{-1}}\circ\sigma)^{*} * (\psi\circ\lambda_{\sigma(g^{-1})})) \\
&\quad\text{by Part 2 of Lemma 7.38}
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \pi_D(\sigma(g))[\varphi\bigstar D], \pi_D(g)[\psi\bigstar D]\rangle &= D((\varphi\circ\sigma\circ\lambda_{\sigma(g^{-1})})^{*} * (\psi\circ\lambda_{\sigma(g^{-1})})) \\
&\quad\text{by Part 3 of Lemma 7.38} \\
&= D((\varphi\circ\sigma)^{*} * \psi) \quad\text{by Part 1 of Lemma 7.38} \\
&= D(\varphi^{\bigstar} * \psi) \quad\text{by Part 2 of Lemma 7.38} \\
&= \langle \varphi\bigstar D, \psi\bigstar D\rangle.
\end{aligned}$$

Secondly, we need to show that

$$\theta\pi_D(g)\theta = \pi_D(\sigma(\tau(g))).$$

But we have

$$\begin{aligned}
\theta\pi_D(g)\theta[\varphi\bigstar D] &= \theta\pi_D(g)[(\varphi\circ\sigma\tau)\bigstar D] \text{ by Part 3 of Lemma 7.37}\\
&= \theta[(\varphi\circ\sigma\tau\circ\lambda_{\sigma(g^{-1})})\bigstar D] \text{ by Part 4 of Lemma 7.38}\\
&= (\varphi\circ\sigma\tau\circ\lambda_{\sigma(g^{-1})}\circ\sigma\tau)\bigstar D \text{ by Part 3 of Lemma 7.37}\\
&= (\varphi\circ\lambda_{\tau(g^{-1})})\bigstar D\\
&= \pi_D(\sigma(\tau(g)))[\varphi\bigstar D] \text{ by Part 4 of Lemma 7.38.}
\end{aligned}$$

Finally, we want to show that $\pi_D(S)\mathcal{E}_+ \subset \mathcal{E}_+$. Let $\varphi \in \mathcal{D}(S)$. Then

$$\begin{aligned}
\varphi\circ\lambda_{\sigma(s^{-1})}(x) \neq 0 &\implies \varphi(\sigma(s^{-1})x) \neq 0\\
&\implies \sigma(s^{-1})x \in \operatorname{supp}(\varphi)\\
&\implies x \in \sigma(s)\cdot\operatorname{supp}(\varphi).
\end{aligned}$$

Thus

$$\operatorname{supp}(\varphi\circ\lambda_{\sigma(s^{-1})}) = \overline{\{x\in G \mid \varphi(\sigma(s^{-1})x)\neq 0\}} \subset \overline{\sigma(s)\cdot\operatorname{supp}(\varphi)} = \sigma(s)\cdot\operatorname{supp}(\varphi).$$

Hence

$$\begin{aligned}
\operatorname{supp}(\varphi\circ\lambda_{\sigma(s^{-1})}) &\subset \sigma(s)\cdot\operatorname{supp}(\varphi)\\
&\subset S \text{ because } S \text{ is } \sigma\text{-invariant.}
\end{aligned}$$

Thus $\varphi\circ\lambda_{\sigma(s^{-1})} \in \mathcal{D}(S)$. Using Part 4 of Lemma 7.38, we have $\pi_D(s)[\varphi\bigstar D] = [\varphi\circ\lambda_{\sigma(s^{-1})}]\bigstar D$. Hence $\pi_D(s)[\varphi\bigstar D] \in \mathcal{E}_+$. This completes the proof of Part 2 $\square$

Let us now give a family of examples of σ-twisted reflection positive representations on $\mathbb{R}^n$.

Theorem 7.40. *Let $G = \mathbb{R}^n$ and $S = \mathbb{R}^n_+$. For a subset $A \subset \{1, 2, \cdots, n-1\}$, define the involution σ_A as the following: If $x \in \mathbb{R}^n$ then the i-th coordinate of the the image is given by*

$$(\sigma_A(x))_i = \begin{cases} -x_i, & i \in A, \\ x_i, & i \notin A. \end{cases}$$

Let

$$\tau(x_0, x_1, \cdots, x_n) \quad = \quad (-x_0, x_1, \cdots, x_n).$$

For $\lambda = 0$ or $\lambda \geqslant \max(0, n-2)$, define

$$D^\lambda(\varphi) := \int_{\mathbb{R}^n} \varphi(x) \ \|x\|^{-\lambda} \ dx \ .$$

Let $\mathcal{E} := \mathcal{D}(G) \bigstar D^\lambda$ and $\mathcal{E}_+ := \mathcal{D}(S) \bigstar D^\lambda$. Define the hermitian form

$$\beta(\psi \bigstar D, \varphi \bigstar D) = \langle \psi \bigstar D^\lambda, \varphi \bigstar D^\lambda \rangle := D^\lambda(\psi^\bigstar * \varphi).$$

Let $\pi_{D^\lambda}(g)E(\psi) := E(\psi \circ \lambda_g)$ and let $\theta E(\psi) := E(\psi \circ \sigma\tau)$. Then the following assertions hold:

1. *The distribution D^λ is σ_A-twisted reflection positive for (G, S, τ).*

2. *The quadruple $(\mathcal{E}, \beta, \mathcal{E}_+, \theta)$ is reflection positive vector space.*

3. *The representation π_{D^λ} is σ_A-twisted reflection positive on $(\mathcal{E}, \beta, \mathcal{E}_+, \theta)$.*

Proof. It is easy to see that $\sigma_A(\mathbb{R}^n_+) \subset \mathbb{R}^n_+$.

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned} D^\lambda(\varphi \circ \tau\sigma) &= \int_{\mathbb{R}^n} \varphi(\tau\sigma(x)) \ \|x\|^{-\lambda} \ dx \\ &= \int_{\mathbb{R}^n} \varphi(x) \ \|\tau\sigma(x)\|^{-\lambda} \ dx \\ &= \int_{\mathbb{R}^n} \varphi(x) \ \|x\|^{-\lambda} \ dx \\ &= D^\lambda(\varphi). \end{aligned}$$

Using Theorem 4.21, we have $D^\lambda(\varphi^\sharp * \varphi) \geq 0$ for $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$. These complete the proof of Part 1. Parts 1 and 3 follow from Part 1 and Theorem 7.39. $\square$

7.6 Reflection Positive Distribution Vector on the Circle Group

We will prove that this distribution vector generates the well known reflection positive function, see [19] and [25], given by

$$g_\lambda(x) = e^{-x\lambda} + e^{-(\beta - x)\lambda}.$$

In this section, we discover a reflection positive distribution vector for the well-known reflection positive function g_λ given in [25].

Let $G = \mathbb{T}_\beta := \mathbb{R}/\beta\mathbb{Z}$ and let

$$g_\lambda(x) = e^{-x\lambda} + e^{-(\beta - x)\lambda}.$$

For $\varphi \in \mathcal{D}(\mathbb{T}_\beta)$, define

$$\alpha(\varphi) = \int_0^\beta \overline{\varphi(x)}\ e^{\lambda x} dx,$$

and define

$$D_\alpha(\varphi) = \langle \alpha, \pi^{-\infty}(\varphi)\alpha \rangle.$$

Theorem 7.41. *Let $\varphi \in \mathcal{D}(\mathbb{T}_\beta)$. Then*

$$D_\alpha(\varphi) = \frac{e^{\lambda\beta}(e^{\lambda\beta} - 1)}{2\lambda} \int_0^\beta \overline{\varphi(x)}\ g_\lambda(x)\ dx.$$

Proof. Recall that

$$\pi^{-\infty}(\varphi)\alpha = \langle \alpha_y, \overline{\varphi(x-y)} \rangle = \int \overline{\varphi(x-y)}\ d\alpha(y).$$

Hence

$$D_\alpha(\phi) = \langle \alpha_x, \langle \alpha_y, \overline{\varphi(x-y)} \rangle \rangle = \int \int \overline{\varphi(x-y)}\ d\alpha(y)\ d\alpha(x).$$

Let

$$k_0(x) = \begin{cases} e^{\lambda(x+\beta)}, & -\beta \le x \le 0 \\ e^{\lambda x}, & 0 \le x \le \beta \\ e^{\lambda(x-\beta)}, & \beta \le x \le 2\beta, \end{cases}$$

and then define $k : \mathbb{R} \to \mathbb{R}$, to be its periodic extension on $\mathbb{R}$. It is easy to see that

$$D_\alpha(\varphi) = \int_0^\beta \overline{\varphi(x)} \int_0^\beta k(y)k(x+y)\ dy\ dx = \int_0^\beta \overline{\varphi(x)}\ k\bigstar k(x)\ dx,$$

where $k\bigstar k(x)$ is the cross-corolation of k. Now, it is enough to show that

$$k\bigstar k(x) = \frac{e^{\lambda\beta}(e^{\lambda\beta}-1)}{2\lambda}\ g_\lambda(x).$$

In other words

$$k\bigstar k(x) = \frac{e^{\lambda\beta}(e^{\lambda\beta}-1)}{2\lambda}\ \left(e^{-x\lambda} + e^{-(\beta-x)\lambda}\right).$$

Finally,

$$\begin{aligned} \int_0^\beta k(y)k(x+y)\ dy &= \int_0^{\beta-x} k(y)k(x+y)\ dy + \int_{\beta-x}^\beta k(y)k(x+y)\ dy \\ &= \int_0^{\beta-x} e^{\lambda y} e^{\lambda(x+y)}\ dy + \int_{\beta-x}^\beta e^{\lambda y} e^{\lambda(x+y-\beta)}\ dy \\ &= e^{\lambda x}\ \frac{e^{2\lambda y}}{2}\Big|_0^{\beta-x} + e^{\lambda(x-\beta)}\ \frac{e^{2\lambda y}}{2}\Big|_{\beta-x}^{\beta} \\ &= \frac{e^{\lambda\beta}(e^{\lambda\beta}-1)}{2\lambda}\ \left(e^{-x\lambda} + e^{-(\beta-x)\lambda}\right). \end{aligned}$$

This completes the proof. $\square$

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Vita

Mostafa Hayajneh was born in 1986 in Dubai City, United Arab Emirates. He finished his undergraduate studies at Yarmouk University in 2007. He earned a master of science degree in mathematics from Yarmouk University in 2010. In August 2012 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2016.