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Stability of stochastic pricing models under volatility fluctuations

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STABILITY OF STOCHASTIC PRICING MODELS UNDER VOLATILITY FLUCTUATIONS

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Abstract

The standard theory of the stochastic models used to value financial derivatives contracts involves models whose input parameters are deterministic functions and often constants. Because of the random nature of the changes in the market prices of the financial instruments, the coefficients of these models are inevitably susceptible to random perturbations from their initial estimates. In this paper we will investigate the behavior of some of the most widely used models when small changes are applied to their volatility component. Starting with the Black-Scholes model for the price of a European call option, we will continue our analysis of the traditional models for pricing American options, Asian options, Barrier options, as well as some of the models for the short term rate of interest. In addition to obtaining convergence results for all of the models, we will examine a method of controlling the deviations in the volatility parameter of the Black-Scholes model and the resulting estimate can be used for further extensions on the topic. Moreover, we will present an example on how to calculate probabilities of rare events, using a technique called importance sampling. We will concentrate only on the case of discrete random variables but the same algorithm can be applied to estimate the probabilities of the deviations of the pricing functions of the models under question. The latter is left for future research.

Introduction

The stochastic pricing models used today in the financial word for valuing various kinds of financial instruments are aimed at capturing the behavior of the fair prices of the securities with a minimal degree of deviation from the real world. Volatility of the prices is invariably a major component in all of the models and therefore getting precise estimates is crucial for the effectiveness of the model. However, it is rarely the case that volatility stay constant with time and its changes lead to deviations in the accuracy of the models. The goal of this paper is to examine the extent to which small changes in the volatility coefficient influences the probability of the events. We will start with the basic Black-Scholes option pricing model for European options, extend its results to the pricing of Asian options and Barrier options, continue with the pricing models for the term structure of the interest rates. The discussion of the paper will have several stages. In Chapter 1, the basic terminology and the necessary background results on Stochastic Analysis and Finance Theory are presented. Chapter 2 is a brief introduction to the large deviations theory and its applications to importance sampling. Stability of pricing models constitutes the discussion in Chapter 3. Finally, in Chapter 4 an application of the large deviations theory to the pricing models is described.

Chapter 1
Pricing Foundations

1.1 Key Mathematical Concepts

Before we begin our discussion of the practical aspects of stochastic analysis we need to go over some of the most common and widely used machinery in stochastic calculus. Let $(\Omega, \mathcal{F}, P)$ be a probability space, where $\mathcal{F}$ denotes a σ - *algebra* on Ω, and P, a probability measure on $(\Omega, \mathcal{F})$ and the space is assumed to be complete. Let the space $[0, \infty)$ be equipped with the Borel σ - *algebra* $\mathcal{B}[0, \infty)$. A *stochastic process* is defined to be a measurable function $X(t, \omega)$ on the product space $[0, \infty) \times \Omega$, equipped with the σ-*algebra* $\mathcal{B}[0.\infty) \otimes \mathcal{F}$. Thus a stochastic process can be expressed as $\{X(t)(\omega)\}$ or simply as $\{X(t)\}$.

Definition 1.1. *A stochastic process W is called a Wiener Process (or Brownian motion) if the following conditions hold:*

(i) $W(0) = 0$.

(ii) The process W has independent increments, that is if $0 \leq r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent random variables.

(iii) For $0 \leq s < t$ the stochastic variable $W(t) - W(s)$ has a Gaussian distribution with mean 0 and variance $t - s$.

(iv) W has continuous paths.

We will first state a helpful definition:

Definition 1.2. *Let T be an interval on $\mathbb{R}$. A filtration on T is an increasing family of σ - fields $\{\mathcal{F}_t | t \in T\}$. Assume that the filtration satisfies the usual hypotheses* *for completeness and right continuity. A stochastic process X_t, $t \in T$ is said to be adapted to filtration $\mathcal{F}_t$ if, for each t, the random variable X_t is $\mathcal{F}_t$- measurable.*

This definition logically leads to the concept of a martingale.

Definition 1.3. *Let $X(t)$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$. It is called a martingale with respect to $\mathcal{F}_t$ if $X(t) \in L^1(P)$ for each t, and, for any $s \leq t$,*

$$E[X_t|\mathcal{F}_s] = X_s \ P\text{- almost surely.}$$

We also need to state several theorems, starting with one of the foundations of stochastic differential equations introduced by Itô in 1944. We will note that Brownian motion is a martingale. Moreover, Itô has defined integration with Brownian motion as the integrator.

Theorem 1.4. *(A particular case of the Itô Lemma) Assume that the process X has a stochastic differential given by*

$$dX(t) \quad = \quad \mu(t)dt + \sigma(t)dW_t.$$

where $\mu(t)$ and $\sigma(t)$ are deterministic functions of time. Let f be a C^2 function in X and a C^1 function in t. Then $f(t, X(t))$ has a stochastic differential given by

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{\partial^2 f}{\partial x^2}[dX(t)]^2$$

where the following formal multiplication table applies

$$\begin{aligned} (dt)^2 &= 0 \\ dt.dW &= 0 \\ (dW)^2 &= dt. \end{aligned}$$

Let us start with a general overview of stochastic differential equations. Roughly speaking, what we are looking for is a stochastic process X, which satisfies the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned}$$

Let us look at geometric Brownian motion, which is given by the solution of the following SDE

$$\begin{aligned} dX &= \mu X dt + \sigma X dW \\ X(0) &= x_0 \ (x_0 > 0) \end{aligned}$$

where μ and σ are constants.

Proposition 1.5. *The solution of the above equation is*

$$X(t) = x_0 exp^{\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}}.$$

Proof. Assume $X > 0$ a.s. Let us consider the function $f(X) = \ln(X)$ and apply Ito's lemma to $f(X)$.

$$\begin{aligned} df &= \frac{1}{X}dX - \frac{1}{2}\sigma^2\frac{1}{X^2}(dX)^2dt \\ &= \frac{1}{X}(\mu X dt + \sigma X dW) - \frac{1}{2}\sigma^2\frac{1}{X^2}X^2dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW \end{aligned}$$

If we now integrate from 0 to t, we will get

$$f(t) = \ln X(t) = \ln x_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W.$$

This means that X is, indeed, a solution to the geometric brownian motion since clearly $X > 0$ thus our assumption at the beginning of the proof is realized. $\square$

Next, we will consider the linear SDE, which in the scalar case has the form

$$\begin{aligned} dX &= (aX + b)dt + \sigma dW \\ X(0) &= x_0 \end{aligned}$$

Proposition 1.6. *The solution of the above equation is*

$$X = e^{at}x_0 + b\int_0^t e^{a(t-s)}ds + \sigma\int_0^t e^{a(t-s)}dW.$$

Proof. If there were no randomness term in the form of the Wiener process, we would solve the equation as an ordinary differential equation. However, although there is a stochastic process with unbounded variation, we can still solve the equation in that manner. Regroup terms and multiply by e^{-at} to get

$$e^{-at}\{dX - aXdt\} = e^{-at}bdt + e^{-at}\sigma dW.$$

In the spirit of ordinary differential equations, our intuition would lead us to find the differential of the process

$$f(t, X(t)) = e^{-at}X(t).$$

Since we are dealing with stochastic differential equations, we will use Ito's formula in order to do so.

$$\begin{aligned} d\{e^{-at}X(t)\} &= -ae^{-at}Xdt + e^{-at}\{\sigma dW(t) + a(X + b)dt\} \\ &= e^{-at}bdt + \sigma e^{-at}dW \end{aligned}$$

Therefore, we can conclude that

$$e^{-at}\{dX - (aX + b)dt\} = e^{-at}bdt + e^{-at}\sigma dW = d\{e^{-at}X\}.$$

Now, we can integrate and divide both sides of the equation by e^{-at} to get

$$\begin{aligned} e^{-at}X &= x_0 + b\int_0^t e^{-as}ds + \sigma\int_0^t e^{-as}dW \\ X &= e^{at}x_0 + b\int_0^t e^{a(t-s)}ds + \sigma\int_0^t e^{a(t-s)}dW \end{aligned}$$

□

The following important theorem is due to Girsanov and it establishes a method to change drift by a change of the probability measures.

Theorem 1.7. *Let $W(t)$ be standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$. Let $\xi(t)$ be any adapted real-valued process such that*

$$\int_0^T \xi^2(t)ds < \infty$$

with probability 1. Define $\mathbb{Q}$ on $\mathcal{F}_T$ such that the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ on $\mathcal{F}_t$ is given by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^t \xi(s)dW(s) - \frac{1}{2}\int_0^t \xi^2(s)ds} \quad \forall 0 \leq t \leq T$$

Such a measure $\mathbb{Q}$ exists if $\mathbb{E}[e^{\int_0^T \xi^2(s)ds}] < \infty$. Moreover,

$$\hat{W}(t) = W(t) - \int_0^t \xi(s)ds$$

is a Brownian motion under the measure $\mathbb{Q}$.

1.2 Option Pricing Preliminaries

We will start with the general definition of one of the most important derivative contracts - the European call option.

Definition 1.8. *A European call option with a strike price K and exercise date T on an underlying asset S is contract between the buyer and the seller such that:*

- *It gives the buyer the right but not the obligation to buy from the seller one unit of asset S at a price K at time T*
- *It can be exercised precisely at time T*

Similarly, a European put option gives the right to sell one unit of an asset on a specified date at a specified price. Another important type of option is the

American option. The definition of that contract is almost the same as given above, except for the second condition. The holder of an American option can exercise it at any time t between $[0, T]$. In financial terms, derivative contracts are often referred as *contingent claims*(or T-claims) and we will denote them by means of a function, called a contract function, of the form $X = \Phi(S(T)$, where X is a $\mathcal{F}_T$-measurable stochastic variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the case of the European call option, the contract function has the form:

$$\Phi(S(T)) = \max[S(T) - K, 0]$$

The price process of such claims is denoted by a general function of the form $\Pi(t; X)$ and by the assumption $\Pi(t; X) = F(t; S(t))$, where F is some smooth function. Moreover, the value of F at time T is $F(T; S(T)) = \Phi(S(T))$. Before we commence with the general description of the option pricing, we have to state several very important assumptions:

(i) *All financial instruments can be bought and sold freely in the market.*

(ii) *The market is efficient i.e. it provides all the information to all of the participants.*

(iii) *There exists a risk-free bond on the market.*

(iv) *The market is free of arbitrage.*

(v) *The market is complete.*

The last assumption is a very important one and we will elaborate on it further. Let us first investigate the simplest case when the market is described only by $\{S(t), B(t), \Pi(t)\}$ where $S(t)$ is the price process of a given stock, $B(t)$ is the risk free bond (or often referred as the money market account) and as we defined earlier $\Pi(t)\}$ is the price process of the derivative instrument. In this type of setting we can introduce a *trading strategy* consisting of a unit of stock and the bond, by which we can replicate the derivative at any moment $t \in [0, T]$ just by changing the proportions of each other. Denote by $h_t = (\delta_t^1, \delta_t^2)$ the vector consisting of the weights of the replicating instruments and we can state the following definition:

Definition 1.9. *The wealth process given by $V_t(h) = \delta_t^1 S(t) + \delta_t^2 B(t)$ is said to be self-financing if for $\forall t \in [0, T]$ we have the following condition*

$$dV_t(h) = \delta_t^1 dS(t) + \delta_t^2 dB(t)$$

Now, we can define what an arbitrage means in mathematical terms.

Definition 1.10. *A self-financing strategy is called an arbitrage if*

- $\mathbb{P}(V_t(h)) > 0) = 1$
- $V_0(h) = 0$

In order for us to find the fair price of any asset in our market, we have to be able to assign the correct probabilities to all of the events that happen within $[0, T]$. Since normally any stochastic process within the market is defined on a probability space with a different probability measure, we have to be able to switch to one unique measure that would give us the corresponding price processes. Thus, we will define an equivalent class of probability measures $\mathbb{Q}$ (also referred as a martingale measure) such that they have the same null sets as $\mathbb{P}$ but the discounted stock price $\hat{S}(t)$ is a martingale under $\mathbb{Q}$. If the discounted stock price is a martingale under $\mathbb{Q}$, then the replicating value process is a martingale under $\mathbb{Q}$ as well. We will state and prove this fact.

Lemma 1.11. *If the stock price follows a martingale measure under $\mathbb{Q}$, then the value process is a martingale under $\mathbb{Q}$.*

Proof. We have the discounted stock price given by $\hat{S}(t) = \frac{S(t)}{B(t)}$. Define $\hat{V}_t(h) = \frac{V_t(h)}{B(t)}$. Assume $V_t(h)$ is self-financing. Use stochastic integration by parts to find the process for $\hat{V}_t(h)$

$$\begin{aligned} d\hat{V}_t(h) &= \frac{1}{B(t)}dV_t(h) + V_t(h)d(\frac{1}{B(t)}) + d\langle V_t(h), \frac{1}{B(t)}\rangle_t \\ &= h_t[\frac{1}{B(t)}dS(t) + S(t)d(\frac{1}{B(t)})] \end{aligned}$$

We can now apply again stochastic integration by parts to the process $\hat{S}(t)$ to get

$$\hat{S}(t) = \frac{1}{B(t)}dS(t) + S(t)d(\frac{1}{B(t)})$$

Therefore, for the discounted value process we obtain:

$$\hat{V}_t(h) = h_t d\hat{S}(t)$$

Since by assumption $\hat{S}(t) = \frac{S(t)}{B(t)}$ is a martingale, then $d\hat{V}_t(h)$ is also a martingale under the same measure □

We have to note that in general, the Itô integral as defined above for any predictable process yields local martingales for which $\sup_{0\leq t\leq T} E(\hat{V}_t(h)) = \infty$, whereas for martingales $\sup_{0\leq t\leq T} E(\hat{V}_t(h)) < \infty$. Thus, as noted by Harrison and Pliska (1981), the economy can be free of arbitrage only if the value processes of self-financing trading strategies are martingales.

We are left to find the price process of $S(t)$ under $\mathbb{Q}$.

Theorem 1.12. *The unique martingale measure for the discounted stock price $\hat{S}(t)$ is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp^{-\frac{\alpha-r}{\sigma}\hat{W}(T) - \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2}T}$$

where $d\hat{W}(t) = dW(t) - \frac{\alpha - r}{\sigma}dt$, $\forall t \in [0, T]$ *is a standard Brownian motion on the probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$. *Moreover,* $\hat{S}(t)$ *satisfies*

$$d\hat{S}(t) = \sigma \hat{S}(t) d\hat{W}(t)$$

Proof. First, let us use the Itô Lemma to find the differential of the process $Z(t) = \frac{S(t)}{B(t)}$

$$\begin{aligned} dZ(t) &= \frac{dS(t)}{B(t)} - \frac{S(t)}{B^2(t)} = \frac{\alpha S(t)dt + \sigma S(t)dW(t)}{B(t)} - \frac{S(t)}{B(t)}rdt \\ &= (\alpha - r)Z(t)dt + \sigma Z(t)dW(t) \end{aligned}$$

If we set $\xi(t) = -\frac{\alpha - r}{\sigma}$ and apply directly the Girsanov's Theorem to get $dW(t) = -\frac{\alpha - r}{\sigma}dt + d\hat{W}(t)$. Substituting this into the equation above yields $dZ(t) = \sigma Z(t)d\hat{W}(t) = \sigma \hat{S}(t)d\hat{W}(t) = d\hat{S}(t)$ □

To get the dynamics of $S(t)$ under $\mathbb{Q}$, we have to only substitute the new Brownian motion under the martingale (risk-neutral) measure. This yields:

$$dS(t) = rS(t)dt + \sigma S(t)d\hat{W}(t) \tag{1.1}$$

1.2.1 Black-Scholes Pricing Equation

To derive the pricing equation for the Black-Scholes model we use the assumption that there exists a self-financing trading strategy that can replicate at any moment the price of the derivative. For that purpose we will use Definition 1.9 and substitute the corresponding price processes:

$$\begin{aligned} dV(h) &= \delta^1 dS(t) + \delta^2 dB(t) = \delta^1(\alpha S(t)dt + \sigma S(t)dW(t)) + \delta^2(rB(t)) \\ &= (r(V - \delta^1 S(t)) + \alpha\delta^1 S(t))dt + \sigma S(t)\delta^1 dW(t) \end{aligned}$$

Moreover, we also assume that the price process $\Pi(t; X) = F(t; S(t))$ for some smooth function F and since $F(t; S(t)) = V(h)$ we can use Ito's Lemma with respect to F to get:

$$\begin{aligned} dF(t;S(t)) &= (\frac{\partial F}{\partial t} + \alpha S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2})dt + \sigma S\frac{\partial F}{\partial S}dW(t) \\ &= (\frac{\partial F}{\partial t} - r(F - \delta^1 S) + \alpha S(\frac{\partial F}{\partial S} - \delta^1) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}dt \\ &\quad +\sigma S(\frac{\partial F}{\partial S} - \delta^1)dW(t) \end{aligned}$$

Since we want to have a risk-neutral valuation, we have to set the variance to zero by letting $\frac{\partial F}{\partial S} - \delta^1 = 0$. Then we will get the Pricing equation:

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} - rF = 0 \tag{1.2}$$

To solve this equation we need to refer to an important theorem. We will note that this is a particular case of the theorem, where α, σ and r are constants.

Theorem 1.13. *(Feynman-Kac) Let the process for S(t) be given by*

$$\begin{aligned} dS(t) &= \alpha(t, S(t))dt + \sigma(t, S(t))dW(t) \\ S(t) &= s \end{aligned}$$

The solution of the backward PDE

$$\begin{aligned} \frac{\partial F}{\partial t}(t,s) + \alpha\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial S^2} - rF(t,s) &= 0 \\ F(T,x) &= \Phi(x) \end{aligned}$$

admits a probabilistic representation given by:

$$F(t,s) = e^{-r(T-t)}\mathbb{E}_{t,s}\left[\Phi(\hat{S}(T))\right]$$

Proof. A proof of the Feynman-Kac theorem can be found in [13] □

It is not very difficult now to evaluate the expectation in the last equation. It is given by the formulae, where Ψ is the cdf of the standard normal distribution:

$$F(t,s)] = s\Psi[d_1(t,s)] - e^{-r(T-t)}K\Psi[d_2(t,s)] \tag{1.3}$$

where

$$
\begin{aligned}
d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}[\ln\frac{s}{K} + (r+\frac{1}{2}\sigma^2)(T-t)] \\
d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}
\end{aligned}
$$

1.2.2 Models for the Short Rate of Interest

We will start this section with a priori specification of the dynamics of the short rate of interest. This, in fact, has been the 'classical' approach to interest rate theory and so the model for the short rate under the probability measure $\mathbb{P}$ is the solution of the stochastic differential equation of arithmetic brownian motion:

$$
dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)
$$

In this case, the short rate of interest is assumed to be given along with a risk free asset, a riskless bond, whose price process B is defined by the dynamics:

$$
dB(t) = r(t)B(t)dt
$$

The dynamics of B can be interpreted as the dynamics of the value of a bank account with a short rate of interest r. In constructing the interest rate models we will use the following assumptions

(i) There exists risk free bond in the market.

(ii) There exists a market for zero coupon bonds for each time interval T.

(iii) The market is free of arbitrage. That is, there is no possibility of making a riskless profit at any time.

Similarly to the treatment of the stock price in the Black-Scholes model, we will view the interest rate as our underlying asset and all bonds will be regarded as derivatives of the short rate of interest r. It is natural to look for similarities between the Black-Scholes model and the models for the bond prices and thus as the logical question whether there is a unique martingale measure under which we can have the arbitrage-free prices of all interest rate products. Unfortunately, there is a major difference in the market structure as defined by Black-Scholes and the one defined for the interest rate models. While, as we saw earlier, in the former type of models we had a risk-free bond and an underlying asset, in the latter, there is no underlying asset since the short rate of interest is not a physical object openly traded in the market. Thus, there is no unique martingale measure for the market. To overcome this obstacle, we have to introduce such an asset, a particular bond, called a *benchmark*, whose price we will regard as given. In that manner, we will be able to establish our unique pricing measure and find the corresponding pricing equation for the zero coupon bond, whose price for every time interval $[t, T]$ by assumption will be given by

$$p(t, T) = F(t, r(t); T) \tag{1.4}$$

where F is a smooth function. Now we can adopt the same technique as before and assume that there exists a replicating self-financing portfolio consisting of two bonds with different maturities: $F(t, r(t); T) = F_T$ and $F(t, r(t); S) = F_S$ such that

$$V = F_T + \delta F_S$$

Let F_T be given by (1.4) and apply the Itô lemma to find its dynamics:

$$\begin{aligned} dF_T &= \left(\frac{\partial F_T}{\partial t} + \mu \frac{\partial F_T}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 F_T}{\partial r^2}\right)dt + \sigma \frac{\partial F_T}{\partial r} dW \\ dF_T &= \Lambda_T dt + \Sigma_T dW \end{aligned}$$

Since we have a self financing portfolio, we have the following result:

$$V = dF_T + \delta dF_S = (\Lambda_T - \delta \Lambda_S)dt + (\Sigma_T - \delta \Sigma_S)dW$$

To get this portfolio to be riskless portfolio we need to set as before $\Sigma_T - \delta\Sigma_S = 0$ or $\delta = \frac{\Sigma_T}{\Sigma_S}$ Moreover, for V to be an arbitrage-free portfolio, it must have a mean rate of return equal to the riskless rate of return - r. Thus,

$$\begin{aligned} dV &= rV dt \\ (\Lambda_T - \frac{\Sigma_T}{\Sigma_S}\Lambda_S)dt &= r(F_T + \frac{\Sigma_T}{\Sigma_S}F_S)dt \end{aligned}$$

After rearranging terms we get:

$$\frac{\Lambda_T - rF_T}{\Sigma_T} = \frac{\Lambda_S - rF_S}{\Sigma_S} = \lambda \tag{1.5}$$

Equation (1.5) is valid for any choice of F_T and F_S and in the financial theory (CAPM) λ is called the market price, which can be rewritten as $\lambda = \frac{\Lambda - rF}{\Sigma}$. We can now substitute Λ and Σ as defined above to obtain the pricing equation for $F(t, r(t); T)$

$$\frac{\partial F}{\partial t} + [\mu(t,r) - \lambda(t,r)\sigma(t,r)]\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 F}{\partial r^2} - rF = 0 \tag{1.6}$$

The solution of this equation can be calculated again with the help of the Feynman-Kac formula. This time, however, the rate of interest is not constant but rather stochastic thus the formula will have the following representation:

$$F(t,r;T) = \mathbb{E}_{t,\hat{r}}\left(e^{-\int_t^T \hat{r}(u)du}F(T,\hat{r}(T))\right)$$

where $\hat{r} = [\mu(t,r) - \lambda(t,r)\sigma(t,r)]dt + \sigma(t,r)dW(t)$. Furthermore, when we deal with zero-coupon bonds we have $F(T, \hat{r}(T)) = 1$ and then the solution to the pricing PDE takes the form:

$$F(t,r;T) = \mathbb{E}_{t,\hat{r}}\left(e^{-\int_t^T \hat{r}(u)du}\right) \tag{1.7}$$

A possible solution to the last equation involves the assumption of the existence of the so called affine term structure. In this type of structure of the interest rates, the price function has the form

$$F(t,r;T) = e^{A(t,T)-B(t,T)r} \tag{1.8}$$

where A and B are deterministic functions of time. To find these function we have to substitute (1.8) into equation (1.6) and apply the boundary condition $F(T, \hat{r}(T)) = 1$, which corresponds to

$$\begin{aligned} A(T,T) &= 0 \\ B(T,T) &= 0 \end{aligned}$$

Thus, we will get a result, whose derivation can be found in [3].

Proposition 1.14. *Let μ and σ are specified as follows:*

$$\begin{aligned} \mu(t,r) &= \alpha(t) + \beta(t)r \\ \sigma(t,r) &= \sqrt{\gamma(t)r + \delta(t)} \end{aligned}$$

Then, the corresponding expressions for A and B are given by

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} - \frac{1}{2}\gamma(t)B^2(t,T) + \alpha(t)B(t,T) + 1 &= 0 \\ B(T,T) &= 0 \end{aligned} \tag{1.9}$$

$$\begin{aligned} \frac{\partial A(t,T)}{\partial t} - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) &= 0 \\ A(T,T) &= 0 \end{aligned} \tag{1.10}$$

Equation (1.9) is a Riccatti equation and for a specific choice of α and γ, given by the dynamics of r we will later see that it produces a continuous solution. From equation (1.10) it can inferred that if $B(t,T)$ is continuous, then $A(t,T)$ is continuous too.

Chapter 2
Large Deviations and Applications

2.1 Introduction to The Theory of Large Deviations

Essentially, the theory of large deviations studies the behavior of random processes when small perturbations are applied to some of their arguments. Before we start with the formal treatment of the pricing models within that theoretical framework, we need to give some sort of a formulation of certain class of large deviation problems. Let X be complete separable metric space, and $\mathbb{P}_\epsilon$ a family of probability measures on Borel subsets of X. Then, for many sets A such that $A \subset X$, we have $\mathbb{P}_\epsilon(A) \overset{Distr.}{\to} \delta_{x_0}(A)$ as $\epsilon \to 0$. If $\bar{A} \bigcap \{x_0\} = \emptyset$, then $\mathbb{P}_\epsilon(A) \overset{Distr.}{\to} 0$, as $\epsilon \to 0$. In this context we can ask the question: "what is the rate of the convergence?" and to answer this we need to define the large deviation principle:

Definition 2.1. *If there exists a function $I : X \to [0, \infty)$ such that:*

- *(i) $0 \leq I(x) \leq \infty$ for all $x \in X$.*
- *(ii) I is lower semicontinuous.*
- *(iii) For each $l < \infty$ the set $\{x : I(x) \leq l\}$ is a compact set in X.*
- *(iv) For each closed set $C \subset X$*

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}_\epsilon(C) \leq - \inf_{x \in C} I(x)$$

- *(v) For each open set $O \subset X$*

$$\liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}_\epsilon(O) \geq - \inf_{x \in O} I(x)$$

then $\{P_\epsilon\}$ follows the large deviation principle with a rate function I.

Consequently, if we have a Borel set A such that:

$$\inf_{x \in A^\circ} I(x) = \inf_{x \in A} I(x) = \inf_{x \in \bar{A}} I(x),$$

then

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}_\epsilon(A) = - \inf_{x \in A} I(x)$$

The simplest case of an application of large deviations is to look at the sample mean $S_n = \frac{X_1+X_2+\cdots+X_n}{n}$ of i.i.d. random variables drawn from a common distribution. Let the moment generating function of X_1 exist. From the strong law of large numbers we know that $S_n \to \mathbb{E}X_1$ a.s as $n \to \infty$. If we take the distribution of S_n on the real line to be our $\mathbb{P}_\epsilon = \mathbb{P}_{1/n}$ (for simplicity we will denote that by $\mathbb{P}_n$), then we might be interested in finding how fast $\mathbb{P}_n \overset{Distr.}{\to} \delta_{\mathbb{E}X_1}$. The answer is given by the following fact, whose proof can be found in [18].

Proposition 2.2. *(Cramér's Theorem) The sequence $\mathbb{P}_n$ satisfies the large deviation principle with a rate function given by*

$$I(x) = \sup_\theta [\theta x - \log M(\theta)]$$

where $M(\theta)$ is the moment generating function of the random variable, that is $M(\theta) = \mathbb{E}e^{\theta X_1}$.

To illustrate how this proposition can be used in practice, we will look at a specific example.

Example 2.3. *Let $X_1, \ldots, X_n$ be $\mathcal{N}(0,1)$ i.i.d. random variables. We know that $M(\theta) = e^{\frac{\theta^2}{2}}$. Then, the rate function of S_n is given by:*

$$I(x) = \sup_\theta (\theta x - \log e^{\frac{\theta^2}{2}}) = \frac{x^2}{2}$$

Thus,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}(S_n > x) = -\frac{x^2}{2}$$

A consequence of the definition above is the following result:

Proposition 2.4. *If h is a bounded continuous function on X and if $\{\mathbb{P}_n\}$ satisfies the large deviation principle, then*

$$\lim_{n \to \infty} \epsilon \log \mathbb{E}\left(e^{-\frac{1}{\epsilon}h(X_n)}\right) = -\inf_x \{h(x) + I(x)\}$$

The converse holds as well.

Since we are introducing Brownian motion as the disturbance term in all of pricing models, we need to examine the convergence results for the Wiener measure. Let $C_0[0,1]$ be the space of continuous functions on the closed interval $[0,1]$ which vanish at 0. Let $\mathbb{P}^W$ be the Wiener measure on the space corresponding to the standard Brownian motion $W(t)$, $(0 \leq t \leq 1)$. Let $\mathbb{P}_\epsilon$ be the distribution of $\sqrt{\epsilon}W(t)$. Clearly, $\mathbb{P}_\epsilon \to \delta_0$ as $\epsilon \to 0$ where δ_0 is the measure with unit mass ot 0. Now, we can establish the following theorem as stated and proved in [18]:

Theorem 2.5. *For the measure P_ϵ the large deviation principle holds with a rate function $I(f)$, defined for $f \in C_0[0,1]$ as follows:*

$$I(f) = \frac{1}{2}\int_0^1 [\dot{f}(t)]^2 dt$$

if $f(t)$ is absolutely continuous with square integrable derivative $\dot{f}(t)$; otherwise $I(f) = \infty$.

Proof. We will give an outline of the proof of the theorem, which will help us later with the analysis of the pricing functions. Let us start by defining $Z_n(t) = \frac{1}{n}\sum_{j=1}^{[nt]} X_j$, for $t \in [0,1]$ where $[\cdot]$ is the greatest integer function and X_j's are i.i.d. Let μ_n be the law of $Z_n(\cdot)$ in $L^\infty[0,1]$. Define $\Lambda(x) = \sup[< \lambda, x > -H(\lambda)]$, where

$H(\lambda) = \log \mathbb{E}(e^{\lambda, x})$ and $\lambda \in \mathbb{R}^n$. Then, applying the Mogulski theorem, we get that μ_n satisfy the Large Deviation Principle with a rate function $I(f)$ defined as:

$$I(f) = \begin{cases} \int_0^1 \Lambda(\dot{f}(t))dt & \text{if } f \text{ is absolutely continuous, and } f(0) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Now, we can apply this reasoning to the Brownian motion case. Let $W_\epsilon(t) = \sqrt{\epsilon}W(t)$ and ν_ϵ be the Law of W_ϵ. Here we may assume that $\epsilon = \frac{1}{n}$. Define $\widehat{W_\epsilon}(t) = W_\epsilon(\epsilon[\frac{t}{\epsilon}])$. If we partition the interval $[0, t]$ in k number of segments each of length ϵ such that $k\epsilon \leq t < (k+1)\epsilon$ or alternatively $k \leq \frac{t}{\epsilon} < k+1$, then we see that we can approximate $\sqrt{\epsilon}W(t)$ by

$$\sqrt{\epsilon}\sum_{j=0}^{[\frac{t}{\epsilon}]}[W(\epsilon(j+1)) - W(\epsilon j)] = \epsilon \sum_{j=0}^{[\frac{t}{\epsilon}]} \frac{[W(\epsilon(j+1)) - W(\epsilon j)]}{\sqrt{\epsilon}} = \epsilon \sum_{j=0}^{[\frac{t}{\epsilon}]} X_j$$

Since the Brownian motion has independent increments, it follows that X_j's are standard normal i.i.d. random variables. Therefore, $\log \mathbb{E}e^{\lambda X_1} = \frac{1}{2}\lambda^2 = H(\lambda)$. Let $\lambda \in \mathbb{R}^1$ and assume that $\widehat{W_\epsilon}(t)$ is a "good approximation" of $W_\epsilon(t)$, then ν_n satisfies the Large Deviation Principle with a rate function

$$I(f) = \int_0^1 \sup_\lambda[\lambda \dot{f}(t) - \frac{1}{2}\lambda^2]dt = \frac{1}{2}\int_0^1 |\dot{f}(t)|^2$$

for any f that is absolutely continuous.

Note: If the support of $W_\epsilon(\cdot)$ is in $C[0, T]$ and if the domain of $I(\cdot) \subset C[0, T]$, then the Large Deviation Principle for W_ϵ holds in $C[0, T]$ with the same rate function.

□

To be able to prove several of the results in this paper, we need a special theorem, which is also known as the "contraction principle".

Theorem 2.6. *Let $\mathbb{P}_\epsilon$ satisfy the large deviation principle with a rate function I. Let $\{F_\epsilon\}$ be continuous maps from X to Y, where Y is another complete separable* *metric space. Define* $\mathbb{Q}_\epsilon$ *on* Y *such that* $\mathbb{Q}_\epsilon = \mathbb{P}_\epsilon F_\epsilon^{-1}$*. If,* $F_\epsilon \to F$ *uniformly over compact subsets of* X *as* $\epsilon \to 0$*, then* Q_ϵ *satisfies the large deviation principle with a rate function* J *defined by*

$$J(y) = \inf_{x:F(x)=y} I(x)$$

Proof. We will follow the proof presented in [5]. For a given M we can define the level sets:

$$L_J = \{y \in Y : J(y) \leq M\} \text{ and } L_I = \{x \in X : I(x) \leq M\}$$

From the definition it follows that $L_J \supset F(L_I)$. However, since I is a rate function for each $y \in F(X)$the infimum is attained at some x in the closed set $F^-1(y)$. Thus, $L_J \subset F(L_I)$. This means that $L_J = F(L_I)$. Since F is continuous and the level sets of I are compact, then the level sets of J are also compact. Clearly, J is nonnegative, hence it follows that it is a rate function. Next, define the composition $h \circ F$, where $h : Y \to \mathbb{R}$. Since $h \circ F$ is a continuous bounded map from X to $\mathbb{R}$, applying proposition 2.4 we get:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{-nh(F(X_n))}] &= -\inf_{x\in X}\{h(F(x)) + I(x)\} \\
&= -\inf_{y\in Y}\{h(y) + J(y)\}
\end{aligned}$$

The last equation completes the proof since we have already checked that J is a rate function. □

We can now continue investigating diffusion processes associated with the Brownian motion and extend the Large Deviation Principle to the case of the class of stochastic differential equations that admit strong solutions. This is often described as the Freidlin-Wentzell theory. In general, these stochastic differential equations do not possess independent increments. However, some underlying independence exists via the Brownian motion. To derive the Large Deviations Principle we have to use the contraction principle. We will state a very useful result, whose proof can be found in [5].

Lemma 2.7. *Let $X(t)$ be the process defined as:*

$$\begin{aligned} dX &= \mu(X)dt + \sigma(X)dW \\ X(0) &= x_0 \ (x_0 > 0) \end{aligned}$$

If $x_0(t) \in \mathbb{R}^d$, $\mu : \mathbb{R}^d \to \mathbb{R}^d$ is uniformly Lipschitz continuous function, all of the elements of the diffusion matrix σ are bounded, uniformly Lipschitz continuous functions, and $W \in \mathbb{R}^d$, then the rate function of $X(t)$ is given by:

$$I(f) = \inf_{\{g \in H_1 : f(t) = x_0 + \int_0^t \mu(f(s))ds + \int_0^t \sigma(f(s))\dot{g}(t)ds\}} \frac{1}{2} \int_0^1 ||\dot{g}(t)||^2 dt$$

where $||\cdot||$ denotes both the Euclidean norm and the corresponding operator norm of matrices and the infimum over the empty set is taken as ∞.

2.2 Importance Sampling

Normally, when we need to simulate occurances of rare events (that is, estimating the probability of a random variable being in a set with a very small chance of occurring), we need a very large number of samples so that we can make sure that we obtain the desired level of precision. However, the large number of simulations runs can impose significant difficulties on our computation devices. To overcome that problem we will utilize a variance reduction technique known as importance sampling. Since we will be dealing of such probability estimates, to illustrate how this works, consider a random vector $\{X_j\}_{j=1}^n$ with density $p_n(x_1, x_2, \ldots, x_n)$. Suppose we are interested in estimating $p_n = \mathbb{P}\{\frac{1}{n}\sum_{j=1}^n X_j \in E\}$ for some $E \subset \mathbb{R}$. To do that we can simulate another vector Y. If we let $(Y_1, \ldots, Y_n)$ have a pdf

$q_n(y_1, y_2, \ldots, y_n)$, then our estimate is:

$$\hat{p}_n = \frac{1}{k}\sum_{r=1}^{k} \frac{p_n(Y_1^r, Y_2^r, \ldots, Y_n^r)}{q_n(Y_1^r, Y_2^r, \ldots, Y_n^r)} 1_{\{\frac{1}{n}\sum_{j=1}^n Y_j \in E\}}$$

where $\{Y_j^r\}_{j=1}^n$ for $r = 1, 2\ldots, k$ denotes k i.i.d. samples, each of size n. We will also adopt the following assumptions:

A.1. Let $\phi(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(e^{\theta Y_n})$ exists, where $'\infty'$ is allowed as a value of ϕ.

A.2. Assume that the origin belongs to the interior of the effective domain of ϕ, that is, $0 \in D_\phi^0 = \text{Int}\{\theta : \phi(\theta) < \infty\}$ and ϕ itself is lower semi-continuous.

A.3. Let $\text{Int}(D_\phi) \neq$ empty set. Let ϕ be differentiable in D_ϕ^0 and $|\bigtriangledown \phi(x_n)| \to \infty$ as $x_n \to x \in \partial D_\phi$.

To get an efficient unbiased estimator $\hat{p}_n$ we need to minimize the variance of the estimate.

$$\begin{aligned}
\text{Var}(\hat{p}_n) &= \frac{1}{k^2} k \left[\mathbb{E}\left\{ \frac{p_n(Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_n^{(1)})}{q_n(Y_1^{(r)}, Y_2^{(1)}, \ldots, Y_n^{(1)})} 1_{\{\frac{1}{n}\sum_{j=1}^n Y_j \in E\}} \right\}^2 - p_n^2 \right] \\
&= \frac{1}{k} \int \left(\frac{dP_n}{dQ_n}(Z) \right)^2 1_{\{\frac{\sum_1^n z_j}{n} \in E\}} dQ_n(Z) - \frac{1}{k} p_n^2
\end{aligned}$$

Since k is fixed, the variance is minimized if

$$F_n = \frac{1}{k} \int \left(\frac{dP_n}{dQ_n}(Z) \right)^2 1_{\{\frac{\sum_1^n z_j}{n} \in E\}} dQ_n(Z)$$

is minimized. Hence, we need to know the behavior of F_n as a function of n.

$$\begin{aligned}
F_n &= \frac{1}{k} \int \left(\frac{dP_n}{dQ_n}(Z) \right)^2 1_{\{\frac{\sum_1^n z_j}{n} \in E\}} dP_n(Z) \\
&= \kappa_n \int 1_{\{\frac{\sum_1^n z_j}{n} \in E\}} d\mu_n(Z)
\end{aligned}$$

where

$$d\mu_n = \frac{\frac{dP_n}{dQ_n} dP_n}{c_n}$$

and κ_n is the normalizing constant so that μ_n is a probability measure, that is, $\kappa_n = \int_{\mathbb{R}^n} \frac{dP_n}{dQ_n} dP_n$. Let us define $C_n(\theta) = \frac{1}{n} \log \frac{dP_n}{dQ_n} e^{\theta \cdot \sum Z_j} dP_n(Z)$. Then, $\kappa_n = e^{nC_n(0)}$. Moreover, $(Z_1, Z_2, \ldots, Z_n)$ is a random vector with distribution μ_n and then

$$\begin{aligned}
\phi_n(\theta) &= \frac{1}{n} \log \mathbb{E}(e^{\theta \cdot \sum_{j=1}^{n} Z_j}) \\
&= \frac{1}{n} \log \int \frac{dP_n}{dQ_n}(Z) e^{\theta \cdot \sum_{j=1}^{n} Z_j} e^{-nC_n(0)} dP_n(Z) \\
&= C_n(\theta) - C_n(0)
\end{aligned}$$

Now, if we assume that $\{C_n(\theta)\}_{n \geq 1}$ satisfies A.1, A.2 and A.3, and let $\lim_{n\to\infty} \phi_n(\theta) = \phi(\theta)$ exist, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log F_n &= \lim_{n\to\infty} [C_n(0) + \frac{1}{n} \log \int 1_{\{\frac{\sum_1^n Z_j}{n} \in E\}} d\mu_n(Z)] \\
&= C(0) - \inf_{x \in E} I(x)
\end{aligned}$$

where $I(x) = \sup_{\theta \in \mathbb{R}^d} \{< \theta, x > -C(\theta) - C(0)\}$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log F_n = - \inf_{x \in E} \sup_{\theta \in \mathbb{R}^d} \{< \theta, x > -C(\theta)\} = -R(E)$$

The meaning of this result is easily understood if we go back to the variance of the estimator $\hat{p}_n$.

$$k\mathrm{Var}(\hat{p}_n) = F_n - p_n^2 \geq 0$$

Consequently, as we have just shown

$$F_n \approx e^{-nR(E)}$$

and also by assumption

$$p_n^2 \approx e^{-2nJ(E)}$$

where J is the rate function of $\frac{1}{n} \sum_{j=1}^{n} X_j$. Therefore,

$$e^{-n(R-2J)} \geq 1$$

or

$$R - 2J \leq 0$$

The last inequality logically brings up the question of how to determine q such that $R = 2J$. For such a distribution, the estimate $\hat{p}_n$ is said to be efficient.

It is important to understand why we are looking for efficiency. First of all, we should note that $p_n \approx 0$ since it is the probability of having our random variables in a deviant set. Suppose we want to estimate p_n with $100(1-\alpha)\%$ confidence. Then, we want

$$\mathbb{P}\{|\hat{p_n} - p_n| < p_n x\} = 100(1-\alpha)$$

which is equivalent to

$$\mathbb{P}\{\sqrt{k}\frac{|\hat{p_n} - p_n|}{\sqrt{F_n - p_n^2}} < \frac{p_n x\sqrt{k}}{\sqrt{F_n - p_n^2}}\} = 100(1-\alpha)$$

where $\sqrt{k}\frac{|\hat{p_n}-p_n|}{\sqrt{F_n-p_n^2}}$ is a standard normal variable. Let $z_{\frac{\alpha}{2}} = \frac{p_n x\sqrt{k}}{\sqrt{F_n-p_n^2}}$. Then,

$$k = \left(\frac{z_{\frac{\alpha}{2}}}{x}\right)\left(\frac{F_n}{p_n^2} - 1\right)$$

Now, since $\frac{F_n}{p_n^2} \approx e^{n(2J-R)}$, k, the number of simulations, will grow exponentially if $2J > R$. Thus, we would be really efficient if we can make $R = 2J$.

Chapter 3
Stability of Pricing Models

3.1 Option Pricing with the Black-Scholes Model

To begin our discussion of the pricing models, let us first consider one of the primary models - The Black-Scholes model for the price of an European call option on a stock, whose process follows geometric Brownian motion of the kind:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \tag{3.1}$$

where W(s) is a standard one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$, $s \in [t, T]$ and α and σ are constants and are called respectively the drift and the volatility coefficients. For the solution of this equation, following the statement and the proof of Proposition 1.5, we will establish the following proposition:

Proposition 3.1. *The solution of the price process (3.1) is given by*

$$S(T) = S(t)e^{(\alpha - \frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}$$

Proof. Let us consider the function $f(S(t)) = \ln(S(t))$ and apply Ito's lemma to $f(S(t))$.

$$\begin{aligned} df(S(t)) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\sigma^2\frac{1}{S(t)^2}(dS(t))^2dt \\ &= \frac{1}{S(t)}(\alpha S(t)dt + \sigma S(t)dW) - \frac{1}{2}\sigma^2\frac{1}{S(t)^2}S(t)^2dt \\ &= (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW \end{aligned}$$

If we now integrate from t to T, we will get

$$\ln S(T) = \ln S(t) + (\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))$$

This means that $S(T)$ is a solution to equation 3.1.

$\square$

There are several questions that arise when we apply a small additive change $\sqrt{\epsilon}$ in the volatility parameter in the pricing equations. First of all, we need to address the question weather the price processes of the underlying asset under the new family of probability measures $\mathbb{P}_\epsilon$ (defined by the dynamics of the underlying stock price) converges to the probability measure $\mathbb{P}$ described by the model. We will assume form this point on that $\epsilon = o(\frac{1}{n})$. The second question is weather the price process of the derivative under $\mathbb{P}_\epsilon$, in turn, converges to the price process under $\mathbb{P}$.

Starting with the European call option price under the world-famous Black-Scholes model, we will show in what it follows of this that

(i) the price process of the underlying stock price $S_\epsilon(t)$ under $\mathbb{P}_\epsilon$ converges uniformly to $S_\epsilon(t)$

(ii) the price of the contingent claim $F_\epsilon(t, S_\epsilon(t))$ converges a.s. to $F(t, S(t))$.

Let us first define the price process $S_\epsilon(t)$. Its dynamics under $\mathbb{P}_\epsilon$ is given by:

$$dS_\epsilon(t) = \alpha(t)S_\epsilon(t)dt + (\sigma(t) + \sqrt{\epsilon})S_\epsilon(t)dW(t)$$

where $W(t)$ is a Wiener process, ϵ is a converging sequence of positive numbers and $\alpha(t)$ and $\sigma(t)$ are functions of time such that $\alpha \in L^1[0,T]$ and $\sigma \in L^2[0,T]$. The solution of this equation is as follows:

$$S_\epsilon(t) = s_0 e^{\int_0^t [\alpha(u) - \frac{1}{2}(\sigma(u)+\sqrt{\epsilon})^2]du + \int_0^t (\sigma(u)+\sqrt{\epsilon})dW(u)} \tag{3.2}$$

We can establish the following result:

Lemma 3.2. *For any $t \in [0,T]$ the stock price process $S_\epsilon(t)$ converges to $S(t)$ uniformly in t.*

Proof. We have the difference

$$\begin{aligned}\sup_{0\le t\le T} |\ln S_\epsilon(t) - \ln S(t)| &= \sup_{0\le t\le T} |-\sqrt{\epsilon}\int_0^t \sigma(u)du - \frac{\epsilon t}{2} + \sqrt{\epsilon}W(t)| \\ &\le \sqrt{\epsilon}\int_0^T |\sigma(u)|du + \epsilon t + \sqrt{\epsilon}\sup_{0\le t\le T}|W(t)|\end{aligned}$$

Since $E\sup_{0\le t\le T}(W(t))^2 \le 4T$, it follows that $\sup_{0\le t\le T}|W(t)| < \infty$ a.s. We also have that $\sigma \in L^1[0,T]$. Now, as $\epsilon \to 0$,

$$\begin{aligned}\lim_{\epsilon\to 0}\sup_{0\le t\le T} |\ln S_\epsilon(t) - \ln S(t)| &= \lim_{\epsilon\to 0}\sup_{0\le t\le T} \ln\frac{S_\epsilon(t)}{S(t)} \\ &= \ln\lim_{\epsilon\to 0}\sup_{0\le t\le T}\frac{S_\epsilon(t)}{S(t)} = 0\end{aligned}$$

This is equivalent to $\lim_{\epsilon\to 0}\sup_{0\le t\le T}\frac{S_\epsilon(t)}{S(t)} = 1$ a.s. TTherefore, $S_\epsilon(t) \longrightarrow S(t)$ uniformly. □

Now, let us consider a simple contingent claim of the form $X = \Phi(S(T))$. Its price process is defined as $\Pi(t;\Phi) = \Pi(t) = F(t, S(t))$ for some smooth function F. Applying the assumptions in the Black-Scholes model, we consider a market free of arbitrage given by $\{S(t), B(t), \Pi(t)\}$. As we saw earlier, for any given $t \in [0,T]$ we have the Black-Scholes pricing formula:

$$F(t, S(t)) = s(t)\Psi[d_1(t, S(t))] - e^{-r(T-t)}K\Psi[d_2(t, S(t))] \tag{3.3}$$

where $\Psi(\cdot)$is the cumulative density function of the standard normal distribution. For any fixed ω, $d_1(t, S(t))$ and $d_2(t, S(t))$ are functions of t in the interval $[0,T]$. This formula is the result of the evaluation of the following expectation:

$$F(t, S(t)) = e^{-r(T-t)}\mathbb{E}[\Phi(S(t)e^{\sigma(W(T)-W(t))-(\frac{\sigma^2}{2}-r)(T-t)})]$$

Consequently, we can state the following proposition:

Proposition 3.3. *For any $t \in [0,T]$, $F(t,S(t))$ defines a continuous functional.*

Proof. Let, $|y-x| = \sup_{0\le t\le T}|S(t,\omega_2) - S(t,\omega_1)|$. Define

$$M^t(T) = e^{\sigma(W(T)-W(t))-\frac{\sigma^2}{2}(T-t)}$$

Clearly, $M^t(T)$ is the evaluation of a martingale at time T whose starting time is t so that $M^t(t) = 1$. Then, using the fact that for any two numbers a and b, $|a^+ - b^+| \le |a-b|$, we have:

$$\begin{aligned} |F(t,y) - F(t,x)| &= e^{-r(T-t)}\mathbb{E}|\Phi(yM^t(T)e^{r(T-t)}) - \Phi(xM^t(T)e^{r(T-t)})| \\ &\le \mathbb{E}|yM^t(T) - xM^t(T)| = \mathbb{E}(M^t(T)|y-x|) = |y-x| \end{aligned}$$

This shows that $F(t,S(t))$ is Lipschitz continous therefore it is continous. □

Since we have established the convergence result for $S_\epsilon(t)$, we can now proceed with the examination of the price function. If we denote by $F_\epsilon(t,S_\epsilon(t))$ the random variable that corresponds to $\mathbb{P}_\epsilon$, then we can state and prove the following fact:

Lemma 3.4. *For any $t \in$ [0,T] the price process of the contingent claim $F_\epsilon(t,S_\epsilon(t))$ converges to $F(t,S(t))$ a.s.*

Proof. For the pricing function F we have:

$$F(t,s) = e^{-r(T-t)}\mathbb{E}^Q\left[\Phi(S(T))\right]$$

Alternatively,

$$F_\epsilon(t,s) = e^{-r(T-t)}\mathbb{E}^{Q_\epsilon}\left[\Phi(S_\epsilon(T))\right]$$

Now,

$$\begin{aligned} \mathbb{E}\left[\Phi(S_\epsilon(T))\right]^2 &\le \mathbb{E}([S_\epsilon(T) - K]^+)^2 \\ &\le \mathbb{E}\left[S_\epsilon^2(T)\right] \le C \end{aligned}$$

which is true for any $\epsilon > 0$ and some $C \in \mathbb{R}$. Thus, $\sup_\epsilon \mathbb{E}\left[\Phi(S_\epsilon(T))\right]^2 < \infty$ This means that $\{\Phi(S_\epsilon(T))\}_{\epsilon>0}$ is uniformly integrable and therefore L^1-convergent. Besides, $\mathbb{Q}_\epsilon \to \mathbb{Q}$ weakly. Consequently,

$$F_\epsilon(t,x) \to F(t,x)$$

a.s. $\forall x$ as $\epsilon \to 0$. Moreover, since $F(t,x)$ is continuous and $S_\epsilon(t) \to S(t)$ uniformly, it follows that

$$F(t, S_\epsilon(t)) \to F(t, S(t))$$

a.s. as $\epsilon \to 0$. Then, we have the result:

$$\begin{aligned} &\lim_{\epsilon \to 0} |F_\epsilon(t, S_\epsilon(t)) - F(t, S(t))| \\ &\leq \lim_{\epsilon \to 0} |F_\epsilon(t, S_\epsilon(t)) - F(t, S_\epsilon(t))| + |F(t, S_\epsilon(t)) - F(t, S(t))| = 0 \end{aligned}$$

$\square$

3.2 Pricing of Barrier Options

There are various kinds of contracts defined under the common name Barrier options. We will look the major ones and the results for all of the other contracts can inferred from the former.

1. **Down-and-out Contracts**

Let $\mathcal{Z} = \Phi(S(T))$ and $\mathcal{Z}^{DO}$ denote the down-and-out contract. Then, by definition, we have:

$$\mathcal{Z}^{DO} = \begin{cases} \Phi(S(T)) & \text{if } S(t) > L, \forall T \in [0,T] \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we define:

$$\Phi_L(x) = \begin{cases} \Phi(x) & \text{if } x > L, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tau(x)$ be a hitting time defined as follows:

$$\tau(y) = \inf\{t \geq 0 : X(t) = y\}$$

where $\{X(t)\}_{t\geq 0}$ is a process with continous paths taking values on the real line. Moreover, the process $\{X(t)\}_{t\geq 0}$ is called absorbed at y if

$$X_y(t) = X(t \wedge \tau) \tag{3.4}$$

For the price process $S(t)$ we have:

$$\ln S(T) = \ln s + (r - \frac{1}{2}\sigma^2)T + \sigma W(T) = X(T) \text{ for } t \in [0, T]$$

Using equation 3.4, we can write the price process absorbed at the point L as: $S_L(t) = e^{X_{\ln L}(t)}$, where $S_L(t)$ is the price process $S(t)$ with absorption at point L. For the process $X_{\ln L}(t)$, where $X(t) = \ln s + (r - \frac{1}{2}\sigma^2)T + \sigma W(T)$, the density function can be found [3], which will allow us to define the density of $S_L(t)$. Thus, the pricing function is given by By definition [3], the price of the down-and-out contract is given by:

$$F^{DO}(t, s, \Phi) = \mathbb{E}^Q_{0,s}[\Phi_L(S_L(T))] = \int_{\ln L}^{\infty} \Phi_L(e^x) f(x) dx$$

where f is the density function of the stochastic variable $X_{\ln L}(t)$ and the expectation is evaluated at time $t = 0$ and $S(0) = s$. The evaluation of the integral leads to the following formula:

$$F^{DO}(t, s, \Phi) = e^{-rt}\mathbb{E}^Q_{0,s}\left[\Phi_L(S(T))\right] - \left(\frac{L^2}{s}\right)^{\frac{2(r-\frac{1}{2}\sigma^2)}{\sigma^2}} \mathbb{E}^Q_{0,\frac{L^2}{s}}\left[\Phi_L(S(T))\right]$$

where, the subscript of the second expectation means that the starting point is $S(0) = \frac{L^2}{s}$.

Let $\Phi = \max\{S - K, 0\}$. Then, we have two cases:

1. $L < K$

In this case it is clear that $\Phi_L(S(T)) \equiv \Phi(S(T))$. Consequently, we can use the results from Black-Scholes model as discussed earlier.

2. $L > K$

In this case we can write the contingent claim as:

$$\Phi_L(S(T)) = \Phi(S(T)) - \Phi(S(T))1_{\{S(T) \leq L\}}$$

This means that $\Phi_L(S(T) \leq \Phi(S(T))$, which in turn implies that

$$\mathbb{E}[\Phi_L(S(T)] \leq \mathbb{E}[\Phi(S(T))] < \infty$$

Following the results from the Black-Scholes model we can claim that

$$F_\epsilon^{DO}(t, S_\epsilon(t)) \to F^{DO}(t, S(t)) \text{ a.s. as } \epsilon \to 0$$

For all of the other type of contracts: up-and-out, down-and-in, up-and-in, we can follow the same lines of reasoning.

1. **Lookbacks**

Lookbacks are contracts that allow the holder to take advantage of the realized minimum or maximum of the price process of the underlying asset over the period of the contract. First, let us look at the lookback call, which is defined as $\Phi_{LBC}(S(T)) = S(T) - \min_{0 \leq t \leq T} S(t)$. From our discussion of the Black-Scholes model we had $S_\epsilon(t) \to S(t)$ uniformly, therefore $\min_{0 \leq t \leq T} S_\epsilon(t) \to \min_{0 \leq t \leq T} S(t)$ as $\epsilon \to 0$. The price function under $\mathbb{P}_\epsilon$ is defined as:

$$F_\epsilon^{LBC}(t, S_\epsilon(t)) = e^{-rT} \left[\mathbb{E}^Q(S_\epsilon(T)) - E^Q[\min_{0 \leq t \leq T} S_\epsilon(t)] \right]$$

Now, as before, we would like to look for uniform integrability:

$$\sup_{\epsilon} \mathbb{E}[S_\epsilon(T)]^2 \leq \sup_{\epsilon} \mathbb{E}[s^2 e^{2(r-\frac{1}{2}\sigma^2)T+2(\sigma+\sqrt{\epsilon})W(T)}]$$
$$\leq \mathbb{E}[s^2 e^{2(r-\frac{1}{2}\sigma^2)T+2(\sigma+1)\sup_{\leq t\leq T} W(T)}] < \infty$$

This means that $\mathbb{E}[S_\epsilon(T)]$ is L^2-bounded and therefore uniformly integrable. Moreover, $\sup_\epsilon \mathbb{E}[\min_{0\leq t\leq T} S_\epsilon(t)] \leq \sup_\epsilon \mathbb{E}[S_\epsilon(t)] < \infty$. All this shows that $F_\epsilon^{LBC}(t, S_\epsilon(t))$ is uniformly integrable. If we adopt the same arguments as in the case of the Black-Scholes model, we conclude that

$$F_\epsilon^{LBC}(t, S_\epsilon(t)) \to F^{LBC}(t, S(t)) \text{ a.s.}$$

Let us now investigate the lookback put. The price function under $\mathbb{P}_\epsilon$ is defined as:

$$F_\epsilon^{LBP}(t, S_\epsilon(t)) = e^{-rT}\left[E^Q[\max_{0\leq t\leq T} S_\epsilon(t)] - \mathbb{E}^Q(S_\epsilon(T))\right]$$

Now, we have:

$$\sup_{\epsilon} \mathbb{E}[\max_{t} S_\epsilon(t)] \leq \sup_{\epsilon} \mathbb{E}[s \max_{t}\{e^{(r-\frac{1}{2}\sigma^2)T+(\sigma+\epsilon)\sup_t |W(t)|}\}] < \infty$$

We have shown that the price process for the lookback put also L^2- bounded and following the same reasoning as before, we can establish the convergence and the continuity properties of the F^{LBP}.

3.3 Pricing of Asian Options

Asian options by definition have a payout at maturity equal to the difference between the average stock price over the period and the strike price. In mathematical terms, this function, denoted by $\Phi(x)$, is given by:

$$\Phi(\Delta_s) = \max\{\frac{1}{T}\int_0^T S(t)dt - K, 0\}$$

If we let $Z(t, S(t)) = \int_0^T S(t)dt$, then the pricing function under $\mathbb{P}_\epsilon$, has the form:

$$F_\epsilon(t, s, Z) = e^{-rT}\mathbb{E}_{t,s,Z}\left[\max\{\frac{1}{T}\int_0^T e^{(\sigma+\sqrt{\epsilon})W(t)-\frac{(\sigma+\sqrt{\epsilon})}{2}t+rt}dt - K, 0\}\right]$$

It turns out that in the case Asian options we can still claim that price process defined above converges.

Lemma 3.5. *For* $\forall t \in [0, T]$ $\lim_{\epsilon\to 0} F_\epsilon(t, S_\epsilon(t), Z(t)) = F(t, S(t), Z(t))$

Proof. From the proof of the convergence of the Black-Scholes model we can infer that it is sufficient to show that $X_\epsilon(T) = \left\{\frac{1}{T}\int_0^T e^{(\sigma+\sqrt{\epsilon})W(t)-\frac{(\sigma+\sqrt{\epsilon})}{2}t+rt}dt\right\}_{\epsilon>0}$ is uniformly integrable. Thus, we will be done if we are able to show $\sup_\epsilon \mathbb{E}(X_\epsilon^2(T)) < \infty$. After applying Cauchy-Schwartz inequality we have:

$$\begin{aligned}
\mathbb{E}(X_\epsilon^2(T)) &\leq \frac{e^{rT}}{T^2}\mathbb{E}\left[\int_0^T e^{(\sigma+\sqrt{\epsilon})W(t)-\frac{(\sigma+\sqrt{\epsilon})^2}{2}t}dt\right]^2 \\
&\leq \frac{e^{rT}}{T^2}T\mathbb{E}\left[\int_0^T e^{2(\sigma+\sqrt{\epsilon})W(t)-(\sigma+\sqrt{\epsilon})^2 t}dt\right] \\
&\leq \frac{e^{rT}}{T}\mathbb{E}\left[\int_0^T e^{2(\sigma+\sqrt{\epsilon})W(t)-2(\sigma+\sqrt{\epsilon})^2 t}e^{(\sigma+\sqrt{\epsilon})^2 t}dt\right]
\end{aligned}$$

Now, let us look at the term $M(t) = e^{2(\sigma+\sqrt{\epsilon})W(t)-2(\sigma+\sqrt{\epsilon})^2 t}$. If we refer back to Proposition 3.1, we see that $dM(t) = 2(\sigma + \sqrt{\epsilon})M(t)dW(t)$ or $M(t) = 2(\sigma + \sqrt{\epsilon})\int_0^t M(t)dW(t)$, which is a stochastic integral hence a martingale and $\mathbb{E}[M(t)] = 1$. This leads to:

$$\mathbb{E}(X_\epsilon^2(T)) \leq \frac{e^{rT}}{T}\int_0^T e^{(\sigma+\sqrt{\epsilon})^2 t}dt < \infty$$

for any fixed ϵ such that $0 < \epsilon < \infty$. □

We also need to check whether $F(t, s, Z)$ is continuous. We need to prove the following fact:

Lemma 3.6. *For any* $t \in [0, T]$, $F(t, s, Z)$ *is continuous.*

Proof. Let $\sup_{0\le s\le t}|w(s)-\hat{w}(s)|<\delta$ for $\delta>0$. For the price process we have:

$$\begin{aligned} F(t,s,z) &= e^{-r(T-t)}\mathbb{E}\left[\max\{\frac{1}{T}\int_0^T S(t)dt-K,0\}\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[\max\{\frac{1}{T}\left(z+\int_t^T S(u)du\right)-K,0\}\right] \end{aligned}$$

where $z=\int_0^t s(0)e^{\sigma w(s)-(\frac{1}{2}\sigma^2-\alpha)s}ds$. Hence:

$$\begin{aligned} |F(t,s,z)-F(t,\hat{s},\hat{z})| &\le \mathbb{E}\frac{1}{T}|z+\int_t^T s(t)e^{\sigma w(u)-(\frac{1}{2}\sigma^2-r)u}du-\hat{z}-\int_t^T \hat{s}(t)e^{\sigma w(u)-(\frac{1}{2}\sigma^2-r)u}du| \\ &\le \mathbb{E}\frac{1}{T}|z-\hat{z}|+\mathbb{E}\frac{1}{T}\left(\int_t^T |s-\hat{s}|e^{\sigma w(u)-(\frac{1}{2}u^2-u)t}du\right) \end{aligned}$$

where

$$\begin{aligned} |z-\hat{z}| &\le s(0)\int_0^t |e^{\sigma w(s)-(\frac{1}{2}\sigma^2-\alpha)s}-e^{\sigma\hat{w}(s)-(\frac{1}{2}\sigma^2-\alpha)s}ds \\ &\le s(0)e^{(\frac{1}{2}\sigma^2-\alpha)t}t\int_0^t |e^{\sigma w(s)}-e^{\sigma\hat{w}(s)}|ds\le C\sup_{0\le s\le t}|w(s)-\hat{w}(s)| \end{aligned}$$

for some $C\in\mathbb{R}$ since e^x is a continuous function.

Moreover,

$$|s-\hat{s}|\le s(0)|e^{\sigma w(t)-(\frac{1}{2}\sigma^2-\alpha)t}-e^{\sigma\hat{w}(t)-(\frac{1}{2}\sigma^2-\alpha)t}|\le D\sup_{0\le s\le t}|w(s)-\hat{w}(s)|$$

Consequently,

$$\sup_{0\le t\le T}|F(t,s,z)-F(t,\hat{s},\hat{z})|\le A\sup_{0\le s\le t}|w(s)-\hat{w}(s)|<\epsilon$$

□

3.4 Pricing of Zero-Coupon Bonds

We will concentrate on the models which assume that the interest rate process follows an Ornstein-Uhlenbeck type of stochastic process, namely:

$$dr(t)=(a(t)-b(t)r(t))dt+\sigma(t)dW(t) \tag{3.5}$$

There are several less popular models that assume that this process is a geometric Brownian motion i.e. $dr(t) = \alpha(t)r(t)dt + \beta(t)r(t)dW(t)$. In this case the analysis of these models is equivalent to the one of options on non-dividend paying stock.

Let us start our discussion by finding the solution to 3.5 .

$$
\begin{aligned}
dr(t) &= (a(t) - b(t)r(t))dt + \sigma(t)dW(t) \\
d\left(r(t)e^{\int_0^t b(s)ds}\right) &= a(t)e^{\int_0^t b(s)ds}dt + \sigma(t)e^{\int_0^t b(s)ds}dW(t) \\
r(t) &= e^{-\int_0^t b(s)ds}\int_0^T a(s)e^{\int_0^s b(u)du}ds + \int_0^T \sigma(s)e^{\int_0^s b(u)du}dW(s)
\end{aligned}
$$

Now, $r_\epsilon(t)$ is given by

$$
\begin{aligned}
r_\epsilon(t) &= e^{-\int_0^t b(s)ds}\int_0^T a(s)e^{\int_0^s b(u)du}ds + \int_0^T (\sigma(s) + \sqrt{\epsilon})e^{\int_0^s b(u)du}dW(s) \\
&= r(t) + \sqrt{\epsilon}\int_0^T e^{\int_0^s b(u)du}dW(s)
\end{aligned}
$$

We see that the last term in the equation above is a stochastic integral ($e^{\int_0^s b(u)du} \in L^1[0,T]$) and therefore using Doob's L^p inequality:

$$
\sup_{0\le s\le T} \mathbb{E}\left|\int_0^T e^{\int_0^s b(u)du}dW(s)\right| \le 4\int_0^T \mathbb{E}\left(e^{\int_0^s b(u)du}\right)^2 ds \le 4T\int_0^T e^{2\int_0^s b(u)du}ds < \infty
$$

This is enough to conclude that as $\epsilon \to 0$, we have:

$$
\sup_{0\le t\le T} |r_\epsilon(t) - r(t)| = \sup_{0\le s\le T} |\sqrt{\epsilon}\int_0^T e^{\int_0^s b(u)du}dW(s)| \to 0
$$

To summarize these facts, we can formulate the following Lemma:

Lemma 3.7. *For any* $\epsilon > 0$, $\lim_{\epsilon\to 0} r_\epsilon(t) = r(t)\ \forall t \in [0,T]$

As we have already established the convergence of the underlying process, we can look at the price of the zero coupon bond, which is given by:

$$
F(t, r(t); T) = \mathbb{E}^Q_{r,t}\left[e^{\int_0^T r(t)dt}\right] = e^{A(t,T)-B(t,T)r(t)}
$$

Lemma 3.8. *For every* $t \in [0,T]$, $\lim_{\epsilon \to 0} F_\epsilon(t, r_\epsilon(t); T) = F(t, r(t); T)$

Proof. By Lemma 3.7 we have

$$\sup_{0 \le t \le T} |r_\epsilon(t) - r(t)| \to 0$$

Thus,

$$e^{\sup_{0 \le t \le T} |r_\epsilon(t) - r(t)|} \longrightarrow 1$$

However,

$$\frac{e^{|r_\epsilon(t)|}}{e^{|r(t)|}} = e^{|r_\epsilon(t) - r(t)|} \le e^{\sup_{0 \le t \le T} |r_\epsilon(t) - r(t)|} \longrightarrow 1$$

$\forall t \in [0,T]$. Hence, $e^{|r_\epsilon(t)}| \to e^{|r(t)|}$, $\forall t \in [0,T]$. Since, by definition $A(t,T)$ and $B(t,T)$ are continuous, it follow that:

$$|F_\epsilon(t, r_\epsilon(t); T) - F(t, r(t); T)| \to 0$$

□

As in the cases of all of the previous instruments, we have also to show the following fact:

Lemma 3.9. $F(t, r(t); T)$ *is continuous for all* $t \in [0,T]$

Proof. Let w and $\hat{w}$ be two members of $C_0([0,T])$. Let $\sup_{0 \le t \le T} |w(t) - \hat{w}(t)| < \delta$, $\delta > 0$, ω fixed. Then,

$$\sup_{0 \le t \le T} |\ln F(t, r(t); T) - \ln \hat{F}(t, r(t); T)| = \sup_{0 \le t \le T} |B(t,T)(r(t) - \hat{r}(t))|$$
$$= \sup_{0 \le t \le T} |B(t,T) e^{-\int_0^t b(s)ds} \left(\int_0^t \sigma(s) e^{\int_0^s b(r)dr} dW(s)(w) - \int_0^t \sigma(s) e^{\int_0^s b(r)dr} d\hat{W}(s)(\hat{w}) \right) |$$

Now, $\int_0^t e^{\int_0^s b(r)dr} dW(s) = e^{\int_0^t b(s)ds} W(t) - \int_0^t W(s) b(t) e^{\int_0^s b(r)dr} ds$.

Therefore:

$$
\begin{aligned}
&\sup_{0\le t\le T} |\ln F(t,r(t);T) - \ln \hat{F}(t,r(t);T)| \\
&\le \sup_{0\le t\le T} |\max\{B(t,T)\} e^{-b(s).0} b(t) \left(\int_0^t e^{\int_0^s b(r)dr} (\hat{W}(s) - W(s)) ds \right) | \\
&\le \sup_{0\le t\le T} |\max\{B(t,T)\} b(t)| \int_0^t e^{\int_0^s b(r)dr} |\hat{w}(s) - w(s)| ds \\
&\le \sup_{0\le t\le T} |\max\{B(t,T)\} b(t)| \int_0^t e^{\int_0^s b(r)dr} \delta ds \\
&\le \sup_{0\le t\le T} |\max\{B(t,T)\} b(t)| (e^{\sup_{0\le t\le T} |b(t)| t} t) \delta < \epsilon
\end{aligned}
$$

for some $\epsilon > 0$ □

3.5 The General Gaussian Model

We will generalize the pricing of contingent claims whose underlying asset i.e. the short-term rate of interest, has a normal distribution. Examples of such models are the Hull-White and Vasicek models. In this setting the equation for the short-term rate follows an Ornstein-Uhlenbeck type of process, where the stochastic term is only a function of time. In general the Gaussian model can be described by:

$$dr = \Lambda(t, r(t))dt + \sigma(t)dW(t)$$

where $\Lambda(t, r(t))$ is some linear combination of $r(t)$ and a deterministic function of time. We will look at the pricing of claims of the kind $X = \Phi(r(T))$ and using the standard theory, we have the price process given by:

$$\Pi(t, X) = \mathbb{E}^Q \left[e^{-\int_0^T r(s)ds} \Phi(r(T)) \right] = p(t,T)\mathbb{E}^*[\Phi(r(T))] \tag{3.6}$$

where $\mathbb{E}^*$ is expectation taken with respect to forward measure. To find the price process, we need to find the distribution of $r(T)$ under the forward measure Q^*. Under Q we have the following process for $p(t,T)$:

$$dp(t,T) = m(t,T)p(t,T)dt + \nu(t,T)p(t,T)dW(t) \tag{3.7}$$

where $\nu(t,T) = -\sigma(t)B(t,T)$. Now, we have to change the measure to get the forward price process for $r(t)$. To do that we use the relationship:

$$\frac{dQ^*}{dQ} = \frac{e^{-\int_0^T r(s)ds}}{\mathbb{E}^Q[e^{-\int_0^T r(s)ds}]} = \frac{1}{e^{\int_0^T r(s)ds}p(0,T)} \text{ Q - a.s.}$$

If we condition on the σ-field $\mathcal{F}_t$ for $t \in [0,T]$. we get:

$$\frac{dQ^*}{dQ} = \mathbb{E}^Q\left(\frac{1}{e^{\int_0^T r(s)ds}p(0,T)}|\mathcal{F}_t\right) = \frac{p(t,T)}{e^{\int_0^t r(s)ds}p(0,T)} = \frac{1}{e^{\int_0^t r(s)ds}p(0,t)}$$

which shows that $\frac{dQ^*}{dQ}$ is a martingale. Therefore, we can apply the Girsanov theorem to equation 3.7 to get:

$$\frac{dQ^*}{dQ} = e^{\int_0^t \nu(u,T)dW - \frac{1}{2}\int_0^t |\nu(u,T)|^2 du}$$

Thus, we have the following representation for $W^*(t)$

$$W^*(t) = W(t) - \int_0^t \nu(u,T)dt$$

or

$$dW^*(t) = dW(t) - \nu(t,T)$$

This leads to the following process for $r(t)$ under the T-forward measure:

$$\begin{aligned} dr &= \Lambda(t,r(t))dt + \sigma(t)[dW^*(t) - \nu(t,T)]dt \\ &= [\Lambda(t,r(t)) - \sigma(t)\nu(t,T)]dt - \sigma(t)dW^*(t) \end{aligned} \tag{3.8}$$

Now, we have the following representation of the forward rates:

$$f(0,T) = -\frac{\partial}{\partial T}\ln p(0,T) = -\frac{\frac{\partial p(0,T)}{\partial T}}{p(0,T)}$$

On the other hand:

$$p(0,T) = \mathbb{E}^Q[e^{-\int_0^T r(s)ds}]$$

or

$$\frac{\partial p(0,T)}{\partial T} = -\mathbb{E}^Q[r(T)e^{-\int_0^T r(s)ds}]$$

Hence,

$$f(0,T) = \mathbb{E}^Q\left[r(T)\frac{e^{-\int_0^T r(s)ds}}{p(0,T)}\right] = E^*[r(T)]$$

As we have just found the expectation of $r(t)$ under the forward measure, we need to find its variance in order to be able to write the density function. To do this, we need to solve the equation for $r(t)$ under the forward measure. Since by definition $r(t)$ has a normal distribution we get the following solution to equation 3.6:

$$\Pi(t,X) = F(t,r(t),\Phi(r)) = \frac{p(t,T)}{\sqrt{2\pi \mathrm{VAR}(r(T))}}\int_{-\infty}^{\infty}\Phi(z)\{e^{-\frac{(z-f(t,T))^2}{2\mathrm{VAR}(r(T))}}\}dz$$

Now, for any L^1-bounded function $\Phi(r(T))$ such that $\Phi(r_\epsilon(T)) \to \Phi(r(T))$ a.s. we can apply the Lebesgue Dominated Convergence theorem to get:

$$|F_\epsilon(t,r_\epsilon,\Phi) - F(t,r,\Phi)| \to 0 \text{ a.s. as } \epsilon \to 0$$

where $F_\epsilon(t,r_\epsilon,\Phi)$ is the random variable under $\mathbb{P}_\epsilon$ as defined earlier. As far as the continuity of $F(t,r(t),\Phi(r))$ is concerned, it follows straight from the pricing formula.

Chapter 4

Applications of the Large Deviations Theory

4.1 The Large Deviation Principle for the Price Process in the Black-Scholes Model

In this section we will use the results from the previous sections to establish the large deviation results for a general price function $F_\epsilon(\Psi_\epsilon(\cdot))$. We have already shown that $F_\epsilon(\Psi_\epsilon(\cdot))$ is a continuous functional of the underlying price process defined under the family of probability measures $\mathbb{P}_\epsilon$ and also $F(\Psi(\cdot))$ is a continuous functional of the price function process under $\mathbb{Q}$ defined for any of the financial instruments discussed earlier. Moreover, $F(\Psi_\epsilon(\cdot))$ is a continuous function of $S_\epsilon(t)$ and also $\sup_{0\leq t\leq T}|F_\epsilon(\Psi_\epsilon(\cdot)) - F(\Psi(\cdot))| \to 0$ a.s. as $\epsilon \to 0$. We will concentrate our discussion only on the Black-Scholes model and once we establish the framework, it can easily be extended to all of the other models, which we have already discussed. Although $|F_\epsilon(\Psi_\epsilon(\cdot)) - F(\Psi(\cdot))|$ defines a random variable, it is still possible to define a Large Deviation Principle for the family of random measures $\mathbb{P}_\epsilon$ defined by $F_\epsilon(t, S_\epsilon(t))$. Assume μ_ϵ satisfies the Large Deviation Principle with a rate function $I : \mathcal{X} \to [0, \infty]$ and the map $f : \mathcal{X} \to \mathcal{Y}$ is continuous and also that $f_\epsilon : \mathcal{X} \to \mathcal{Y}$ are measurable such that $\forall\delta > 0$ the set $\mathcal{A} = \{x \in X : d(f(x), f_\epsilon(x)) > \delta)$ is measurable. Then, if $\lim_{\epsilon\to 0} \sup \epsilon \log \mu_\epsilon(\mathcal{A}) = \infty$, the Large Deviation Principle holds for the measures $\mu_\epsilon f^{-1}$ on $\mathcal{Y}$. Further discussion on the topic can be found in [5].

We will start by checking initially whether such a principle can be derived for the underlying stock price precess. We will consider, as before, the price processes:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \text{ and}$$

$$dS_\epsilon(t) = \alpha S_\epsilon(t)dt + \sigma(1+\sqrt{\epsilon})S_\epsilon(t)dW(t)$$

Define $Y_\epsilon(t) = \frac{S_\epsilon(t)}{S(t)}$ and apply Itô's lemma to $Y_\epsilon(t)$:

$$\begin{aligned}
Y_\epsilon(t) &= 1 + \int_0^t \frac{1}{S(t)} dS_\epsilon(t) - \int_0^t \frac{S_\epsilon(t)}{S^2(t)} dS(t) + \frac{1}{2}\int_0^t 2\frac{S_\epsilon(t)}{S^3(t)} d[S]_t - \int_0^t \frac{1}{S^2(t)} d < S_\epsilon, S >_t \\
&= 1 + \int_0^t \sigma(1+\sqrt{\epsilon})\frac{S_\epsilon(t)}{S(t)} dW + \int_0^t \alpha\frac{S_\epsilon(t)}{S(t)} ds - \int_0^t \frac{S_\epsilon(t)}{S(t)} dW - \int_0^t \alpha\frac{S_\epsilon(t)}{S(t)} \\
&\quad + \int_0^t \frac{S_\epsilon(t)}{S(t)}\sigma^2 ds - \int_0^t \frac{S_\epsilon(t)}{S(t)}\sigma^2(1+\sqrt{\epsilon})ds \\
&= 1 + \int_0^t \sqrt{\epsilon}\sigma\frac{S_\epsilon(t)}{S(t)} dW - \int_0^t \sqrt{\epsilon}\sigma^2\frac{S_\epsilon(t)}{S(t)} ds
\end{aligned}$$

This translates into the following process:

$$\begin{aligned}
dY_\epsilon(t) &= \sigma Y_\epsilon(t)\sqrt{\epsilon}dW(t) - \sqrt{\epsilon}\sigma^2 Y_\epsilon(t)dt \\
Y_\epsilon(0) &= 1
\end{aligned}$$

Now, following the principles of the Wentzell-Freidlin Theory, we can claim that $\{Y_\epsilon\}$ satisfies the Large Deviations Principle with a rate function J for all $f \in C([0,T] : \mathbb{R})$ defined as follows:

$$J(f) = \inf_{\{\dot{g} \in H_1[0,T]: \forall 0 \le t \le T,\ f(t) = 1 + \int_0^t \sigma f(s)\dot{g}(s)ds\}} \frac{1}{2}\int_0^T ||\dot{g}(t)||^2 dt$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^d$. If we call the set over which the infimum is taken $\mathcal{A}$, then rate function can be defined as:

$$J(f) = \begin{cases} \inf \frac{1}{2}\int_0^T ||\dot{g}(t)||^2 dt & \text{if } \mathcal{A} \neq \oslash \\ \infty & \text{otherwise.} \end{cases}$$

To explore the meaning of this result, let us set $Z_\epsilon(\cdot) = \ln Y_\epsilon(\cdot)$. Then, define

$$I'(g) = \inf_{\{f : \ln f = g\}} J(f)$$

Now, we can use the "contraction principle" outlined earlier and claim that $\{Z_\epsilon\}$ satisfies the Large Deviation Principle with a rate function I'. In other words,

$\{\ln S_\epsilon - \ln S\}$ satisfies the Large Deviation Principle with a rate function I'. Now, if $f = e^g$, and if $f(t) = 1 + \int_0^t \sigma f(s)v(s)ds$, then it must be that $g = \int_0^t \sigma v(s)ds$. Therefore, $\dot{g}(t) = \sigma v(t)$ or the rate function of Z_ϵ equals

$$I'(g) = \frac{1}{2}\int_0^T \frac{(\dot{g}(t))^2}{\sigma^2}dt$$

In other words, $\{\ln S_\epsilon - \ln S\}$ behaves like the process $\{\sigma\sqrt{\epsilon}W(t)\}$. Now we can proceed and find the rate function for the price process. For the Black-Scholes model the price function is defined as:

$$F(t,x) = e^{-r(T-t)}\mathbb{E}[\Phi(xe^{\sigma(W(T)-W(t))-(\frac{\sigma^2}{2}-r)(T-t)})$$

and consequently

$$F_\epsilon(t,y) = e^{-r(T-t)}\mathbb{E}[\Phi(xe^{\sigma(1+\sqrt{\epsilon})(W(T)-W(t))-(\frac{\sigma^2(1+\sqrt{\epsilon})^2}{2}-r)(T-t)})$$

where $y = S_\epsilon(t)$ and $x = S(t)$. Let, as in the proof of proposition 3.3, $M^t(T) = e^{\sigma(W(T)-W(t))-\frac{\sigma^2}{2}(T-t)}$. Then, $\{M^t(t)\}$ is a martingale that starts at time is t and $M^t(t) = 1$. Let $M^t_\epsilon(T) = e^{\sigma(1+\sqrt{\epsilon})(W(T)-W(t))-\frac{\sigma^2(1+\sqrt{\epsilon})^2}{2}(T-t)}$. Then,

$$\begin{aligned} |F_\epsilon(t,y) - F(t,x)| &= e^{-r(T-t)}\mathbb{E}|\Phi(yM^t_\epsilon(T)e^{r(T-t)}) - \Phi(xM^t(T)e^{r(T-t)})| \\ &\leq \mathbb{E}|yM^t_\epsilon(T) - xM^t(T)| \end{aligned}$$

using the fact that for any two numbers a and b, $|a^+ - b^+| \leq |a-b|$. Then:

$$|F_\epsilon(t,y)-F(t,x)| \leq \mathbb{E}|y-x|M^t_\epsilon(T)+|x|\mathbb{E}|M^t_\epsilon(T)-M^t(T)| = |y-x|+|x|\mathbb{E}|M^t_\epsilon(T)-M^t(T)|$$

Letting $\epsilon \to 0$ we obtain:

$$|F_\epsilon(t,y) - F(t,x)| = o|y-x|$$

Define a stopping time τ_N such that $\tau_N = \inf\{t : |S(t)| > N\} \wedge T$. We observe that as $N \uparrow \infty$, $\tau_N \uparrow \infty$. To define a Large Deviation Principle for $F_\epsilon(t,y) - F(t,x)$ we need first to find $\mathbb{P}\{\sup_{0\le t\le\tau_N}|F_\epsilon(t,y)-F(t,x)|>\delta\}$. We will not be able to get an exact rate function but instead we can obtain an upper bound.

$$\begin{aligned}
\mathbb{P}\{\sup_{0\le t\le\tau_N}|F_\epsilon(t,S_\epsilon(t))-F(t,S(t))|>\delta\} &\le \mathbb{P}\{\sup_{0\le t\le\tau_N}|S_\epsilon(t)-S(t)|>\delta\}\\
&= \mathbb{P}\{\sup_{0\le t\le\tau_N}|S(t)||\frac{S_\epsilon(t)}{S(t)}-1|>\delta\}\\
&\le \mathbb{P}\{\sup_{0\le t\le\tau_N}|\frac{S_\epsilon(t)}{S(t)}-1|>\frac{\delta}{N}\}
\end{aligned}$$

This result means that:

$$\overline{\lim_{\epsilon\to 0}}\epsilon\log\mathbb{P}\{\sup_{0\le t\le\tau_N}|F_\epsilon(t,S_\epsilon(t))-F(t,S(t))|>\delta\}\le -J(E)$$

where $E=\{f\in C([0,T]:\mathbb{R}):||f-1||>\frac{\delta}{N}\}$. It is important to note that such a bound holds till a stopping time τ_N for a fixed large number N. Moreover, the process $\Gamma_\epsilon=\frac{F_\epsilon(t,S_\epsilon(t))}{F(t,S(t))}$ solves a Stochastic Differential Equation that can be written as a linear equation with random coefficients. Such equations are difficult to analyze using Wentzell-Freidlin type deviation results.

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Vita

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