Suboptimal Linear Regulator by Frequency Domain Compensation.

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73-13,649

BORG, David Alan, 1943-
SUBOPTIMAL LINEAR REGULATOR BY FREQUENCY DOMAIN COMPENSATION.

The Louisiana State University and Agricultural and Mechanical College, Ph.D., 1972
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

SUBOPTIMAL LINEAR REGULATOR
BY FREQUENCY DOMAIN
COMPENSATION

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Electrical Engineering

by
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December, 1972

ACKNOWLEDGEMENT

This research was conducted under the direction of Dr. Paul M. Julich, Associate Professor of Electrical Engineering at Louisiana State University to whom the author expresses appreciation for his encouragement and guidance given throughout this work. The work contained in this paper was originally supported by a Themis Grant administered by the Air Force Office of Scientific Research, Contract No. F-44620-68-C-0021, and completed under a Dissertation Year Fellowship awarded by Louisiana State University. Thanks are expressed to each of these institutions.

Thanks are also expressed to Mrs. Suzzanne Bright who has grammatically reviewed this manuscript. A special note of appreciation is given to the author's wife, Janet, for her loving support and for her assistance as a systems and research programmer.

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ABSTRACT

Classical control has developed many methods for determining compensating filters for control systems. Most of these methods use a frequency domain approach and involve trial and error in order to determine filter parameters. These procedures usually involve increasing the order of the system.

Unlike classical control theory, the compensating procedures in modern control theory do not allow an increase in the number of system states. The optimal control approach involves feeding back all the system states. This has been recognized as unrealistic in many situations. Hence, work has been expanded into determining observers or Kalman filters which estimate unmeasurable states and into determining suboptimal control laws which feed back only measurable states. When the optimal solution is either impossible, or very difficult to compute, the question: "How close to optimal is a suboptimal control?" can be difficult to answer. This leads to the desirability of being able to estimate the ratio of the optimal cost to the cost when a suboptimal control law is used.

One method of doing this is to determine a lower bound on the ratio. A general theorem is presented that specifies such a lower bound, which is easily determined using only a frequency domain description of the system. The lower bound given by this theorem is then used to develop a compensation algorithm for a unity feedback single-input, single-output linear system. The algorithm can be entirely implemented on a digital computer, but as with many

other search routines, the results are sensitive to the initialization procedures. Hence it is necessary to try several starting points. Sufficient conditions for the lower bound to be a lower bound on the ratio of the cost of the optimal system to the cost using a suboptimal control law are established. When checked, these conditions offer a means of improving (increasing) the lower bound.

I. INTRODUCTION

Classical control has developed many methods for determining compensating filters for control systems. Most of these methods use a frequency domain approach and involve trial and error in order to determine filter parameters. These procedures usually involve increasing the order of the system.

Unlike classical control theory, the compensating procedures in modern control theory do not allow an increase in the number of system states. The optimal control approach involves feeding back all the system states. This has been recognized as unrealistic in many situations. Hence work has been expanded into determining observer or Kalman filters which estimate unmeasurable states and into determining suboptimal control laws which only feed back measurable states. When the optimal solution is either impossible or very difficult to compute, the question: "How close to optimal is a suboptimal control?" can be difficult to answer. This leads to the desirability of being able to estimate the ratio of the optimal cost to the cost of a suboptimal control law purely as function of the suboptimal control.

Classical control is concerned with the frequency domain and transfer functions while optimal control deals with the time domain and matrix manipulation. As might be expected this difference is somewhat artificial and recently much work has been devoted to joining the classical and optimal control approaches. Several researchers have attempted to determine universal closed-loop pole configurations for optimal single-loop control systems. Kalman [5], studying the

"inverse problem of control theory", determined a frequency domain condition on an optimal feedback system. From the results of Kalman's paper, specific procedures [9] have been developed to determine frequency domain feedback compensations. Continuing along these lines, Canales [3] has presented a lower bound on the suboptimality of a particular feedback control law from only frequency domain information. Canales' work might lead to a design procedure except for a severe restriction on the $GH(j\omega)$ plot. In order for the lower bound to be applicable, it is required that $Re\{GH(j\omega)\} \leq 0$, for all real ω which rules out most interesting problems.

In the sequel a general theorem is developed which includes Canales' results as a special case. The lower bound given by this theorem is then used to develop a compensation algorithm for single-input, single-output linear systems. The algorithm can be entirely implemented on a digital computer, but as with many other search routines, the results are sensitive to the initialization procedures. Hence it is necessary to try several starting points. Sufficient conditions for the lower bound to be a lower bound on the ratio of the optimal to the suboptimal control are established. When checked, these conditions offer a means of improving (increasing) the lower bound.

II. DERIVATION OF LOWER BOUND ON OPTIMAL CONTROL FROM COST OF HOMOGENEOUS SYSTEM

Before we deal with the suboptimal control problem, it will be helpful to develop several relationships between the cost of the optimal control problem and the solution of the homogeneous system. Toward this problem we will consider the following controllable linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{II-1}$$

where $\dot{x}(t) = \frac{dx(t)}{dt}$ and A and B are constant n*n and n*1 matrices, respectively, with the quadratic cost functional

$$J(x,u) = \int_0^\infty x'(t)Qx(t) + u'(t)u(t)dt \tag{II-2}$$

(the prime denotes transpose). The restriction that $\mathrm{Re}\{\lambda(A)\} < 0$ (this symbol denotes that the real part of each eigenvalue of the matrix A be less than zero) will be imposed throughout this section. However, no assumption is made on the definiteness of the matrix Q. The only condition imposed on Q at this time is that Q be a symmetric matrix.

The following theorem is useful for determining the cost of the control law $u(t) = 0$ for all real $t \geq 0$.

Theorem II-1

Let $x(t)$ be given by

$$\dot{x}(t) = Ax(t); \quad x(0) = x_0 \tag{II-3}$$

where A is a constant n*n matrix with $Re\{\lambda(A)\} < 0$. If H satisfies the matrix equation,

$$A'H + HA = -Q \tag{II-4}$$

then

$$J_h(x_0) = \int_0^\infty x'(t)Qx(t)dt = x_0'Hx_0 \tag{II-5}$$

Proof: Let M be any constant symmetric matrix. Then

$$\frac{d}{dt}[x'(t)Mx(t)] = \dot{x}'(t)Mx(t) + x'(t)M\dot{x}(t) \tag{II-6}$$

Taking the integral from 0 to T of both sides of equation II-6 and letting T approach infinity results in

$$\lim_{T\to\infty} \int_0^T \frac{d}{dt}[x'(t)Mx(t)]dt = \lim_{T\to\infty} \int_0^T [\dot{x}'(t)Mx(t) + x'(t)M\dot{x}(t)]dt \tag{II-7}$$

Integrating the left hand side and using the definition of $\dot{x}(t)$, equation II-3, in the right hand side of equation II-7 gives

$$\lim_{T\to\infty} x'(t)Mx(t)\Big|_0^T = \lim_{T\to\infty} \int_0^T [x'(t)A'Mx(t) + x'(t)MAx(t)]dt$$

$$\lim_{T\to\infty} x'(T)Mx(T) - x_0'Mx_0 = \lim_{T\to\infty} \int_0^T x'(t)[A'M + MA]x(t)dt \tag{II-8}$$

But $Re\{\lambda(A)\} < 0$, which requires $\lim_{T\to\infty} x(T) = 0$. Using this information equation II-8 requires

$$0 = x_0'Mx_0 + \int_0^\infty x'(t)[A'M + MA]x(t)dt \tag{II-9}$$

Now consider the quantity

$$J(x_0) = \int_0^\infty x'(t)Qx(t)dt \tag{II-10}$$

Adding equation II-9 to equation II-10 produces

$$J(x_0) = \int_0^\infty x'(t)Qx(t)dt + \int_0^\infty x'(t)[A'M + MA]x(t)dt + x_0'Mx_0$$

$$= \int_0^\infty x'(t)[A'M + MA + Q]x(t)dt + x_0'Mx_0 \tag{II-11}$$

Hence for M = H where H satisfies equation II-4 it is true that

$$J_h(x_0) = x_0'Hx_0$$

Since this is equation II-5 the theorem is established.

End of Proof

There is widespread knowledge that for positive semidefinite Q matrix in the quadratic cost function, equation II-2, a unique solution exists to the optimal control problem. The situation where Q is not restricted to be positive semidefinite is not as familiar to most control engineers. However, this case has been well documented. The following theorem provides necessary and sufficient conditions for the existence of a minimum of the cost functional.

Theorem II-2 (Willems [10])

Assume that the system described by the dynamical equation $\dot{x} = Ax + Bu$ is controllable. Then the following four conditions are equivalent.

1. $$\int_0^T x'(t)Qx(t) + u'(t)u(t)dt \geq 0 \tag{II-12}$$

for every pair (x,u) constrained by the dynamical equations

and $x(0) = x(T) = 0$.

2. $$V = \inf_{u \in L_2} \int_0^{\infty} x'(t)Qx(t) + u'(t)u(t)dt < -\infty \tag{II-13}$$

where $x(t)$ and $u(t)$ are constrained by the dynamical equation and $x(0) = x_0$ and $\lim_{t \to \infty} x(t) = 0$.

3. There exists a real symmetric solution, K, to the algebraic Riccati equation

$$A'K + KA - KBB'K + Q = 0 \tag{II-14}$$

4. The frequency domain inequality

$$I + W(j\omega) = I + B'(-j\omega I - A')^{-1}Q(Ij\omega - A)^{-1}B \geq 0 \tag{II-15}$$

holds for all real ω.

Each of the above conditions implies the other. If these conditions hold, then there is exactly one solution, denoted by K^+, of the matrix Riccati equation, equation II-14, which has the additional property that $\mathrm{Re}\{\lambda(A^+)\} \leq 0$, with $A^+ = A - BB'K^+$. Moreover, $V^+ = x_0'K^+x_0$. If it is true that $\mathrm{Re}\{\lambda(A^+)\} < 0$ then a minimum, $J_*(x_0)$, of $J(x,u)$ equation II-2 exists for all x_0, with $J_*(x_0) = x_0'K^+x_0$. This minimum is uniquely attained by the feedback control law

$$u_*(t) = -B'Kx_*(t) \tag{II-16}$$

where $x_*(t)$ is given by the differential equation

$$\begin{aligned} \dot{x}_*(t) &= Ax_*(t) + Bu_*(t) = A^+x_*(t) \\ x_*(0) &= x_0 \end{aligned} \tag{II-17}$$

Proof: For a complete exposition the reader is referred to Willems [10; Theorem 2, Theorem 4, Theorem 5, and Theorem 7].

End of Proof

The following theorem provides a relationship between the frequency domain characteristics and the time domain response of a linear system. A similiar theorem is presented by Brocket [2]. Though the proof of Theorem II-3 is essentially the same as Brocket's, the situation is slightly different; thus the complete proof is presented.

Theorem II-3

Let u(t) be a Fourier transformable function, and let the pair (x,u) be related by the following differential equation

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = 0 \tag{II-18}$$

If

$$-\alpha_1^2 I \overset{\dagger}{\leq} W(j\omega\ ) \leq \alpha_2^2 I \tag{II-19}$$

for all real ω where

$$W(j\omega) = B'(-j\omega I - A')^{-1} Q (j\omega I - A)^{-1} B \tag{II-20}$$

for some symmetric matrix Q then

$$-\alpha_1^2 \int_0^\infty u'(t)u(t)dt \leq \int_0^\infty x'(t)Qx(t)dt \leq \alpha_2^2 \int_0^\infty u'(t)u(t)dt \tag{II-21}$$

provided the indicated integrals exist.

† The notation M $\geq$N where M and N are n*n matrices means that the matrix M - N is positive semi-definite.

Proof: Since $x(0) = 0$ the Fourier transform of $x(t)$ is

$$X(j\omega) = (j\omega I - A)^{-1}B\, U(j\omega)$$

Applying Parseval's relation [7] results in

$$\int_0^\infty x'(t)Qx(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X'(-j\omega)\; QX(j\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} U'(-j\omega)B'(-j\omega I-A')^{-1}Q(j\omega I-A)^{-1}BU(j\omega)d\omega$$

From equation II-20 it is seen that

$$\int_0^\infty x'(t)Qx(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} U'(-j\omega)\; W(j\omega)\; U(j\omega)d\omega$$

But the use of equation II-19 results in

$$-\frac{\alpha_1^2}{2\pi}\int_{-\infty}^{\infty} U'(-j\omega)\; U(j\omega)d\omega \leq \int_0^\infty x'(t)Qx(t)dt \leq \frac{\alpha_2^2}{2\pi}\int_{-\infty}^{\infty} U'(-j\omega)U(j\omega)d\omega$$

Using Parseval's relation of this expression gives

$$-\alpha_1^2\int_0^\infty u'(t)u(t)dt \leq \int_0^\infty x'(t)Qx(t)dt \leq \alpha_2^2\int_0^\infty u'(t)u(t)dt$$

This is the same as equation II-21; thus the theorem is established.

<u>End of Proof</u>

Now that the background work has been completed, the major results of this section can be presented. Each of the preceeding theorems will be used in the proof of Lemma II-1 and Theorem II-4. However before presenting these theorems several conditions will be defined which will be beneficial in the statement of the following lemma and theorem.

These conditions are as follows:

C1. A matrix H satisfies the equation

$$A'H + HA = -Q \tag{II-22}$$

C2. A matrix K satisfies the algebraic Riccati equation

$$A'K + KA - KBB'K = -Q \tag{II-23}$$

and $\mathrm{Re}\{\lambda(A-BB'K)\} < 0$.

C3. The quasi-Schwarz inequality

$$\left(\int_0^\infty x_h'(t)Q[x_*(t)-x_h(t)]dt\right)^2 \le \left|\int_0^\infty x_h'(t)Qx_h(t)\,dt\right| \cdot \left|\int_0^\infty [x_*(t)-x_h(t)]'Q[x_*(t)-x_h(t)]dt\right| \tag{II-24}$$

where $x_*(t)$ is the solution to the differential equation

$$\dot{x}_*(t) = [A - BB'K]x_*(t) \;;\quad x_*(0) = x_0$$

for K satisfying condition C2 and $x_h(t)$ is the solution to the differential equation

$$\dot{x}_h(t) = Ax_h(t) \;;\quad x_h(0) = x_0$$

holds at x_0 (note that if the Q matrix is either non-negative or non-positive definite, this condition reduces to the Schwarz inequality and holds for any arbitrary x_0).

C4. The inequality in the frequency domain

$$-\alpha_1^2 I \le W(j\omega) \le \alpha_2^2 I \tag{II-25}$$

holds for all real ω where

$$W(j\omega) = B'(-j\omega I - A')^{-1}Q(j\omega I - A)^{-1}B \tag{II-26}$$

With the definition of these conditions completed, Lemma II-1 will now be stated.

Lemma II-1

Consider the following controllable time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \ ; \quad x(0) = x_0 \tag{II-27}$$

where $Re\{\lambda(A)\} < 0$ with the cost functional

$$J(x,u) = \int_0^\infty x'(t)Qx(t) + u'(t)u(t)dt \tag{II-28}$$

If conditions C1 and C2 hold, then the following two results are true.

R1.

$$J(x_h(t),u_h(t)=0) = J_h(x_0) = x_0'Hx_0 \tag{II-29}$$

where $x_h(t)$ is constrained by the equation

$$\dot{x}_h(t) = Ax_h(t) \ ; \quad x_h(0) = x_0 \tag{II-30}$$

R2.

$$\min_{u \in L_2} J(x,u) = J(x_*(t),u_*(t)) = J_*(x_0) = x_0'Kx_0 \tag{II-31}$$

where

$$u_*(t) = -B'Kx_*(t) \tag{II-32}$$

and $x_*(t)$ and $u_*(t)$ are related by the dynamical equation

$$\dot{x}_*(t) = Ax_*(t) + Bu_*(t) \ ; \quad x_*(0) = x_0 \tag{II-33}$$

Proof: The requirements that $Re\{\lambda(A)\} < 0$ and that condition C1 hold satisfies the hypothesis of Theorem II-1; hence result R1 is true. Condition C2 invokes all results of Theorem II-2 and hence, result R2.

End of Proof

Lemma II-1 and the conditions previously defined simplify the presentation of Theorem II-4.

Theorem II-4

For the system and cost functional presented in Lemma II-1, if in addition to conditions C1 and C2 being valid, conditions C3 and C4 hold, then the following result is implied:

The optimal cost $J_*(x\ )$ is bounded from below by the inequality

$$J_*(x\ ) \geq \begin{cases} \min\left[\dfrac{1}{1+\alpha_2^2}\ J_h(x_0),\ \dfrac{1-2\alpha_1^2}{1-\alpha_1^2}\ J_h(x_0)\right] & \text{for } J_h(x_0) \geq 0 \\ \min\left[\dfrac{1+2\alpha_2^2}{1+\alpha_2^2}\ J_h(x_0),\ \dfrac{1}{1-\alpha_1^2}\ J_h(x_0)\right] & \text{for } J_h(x_0) < 0 \end{cases} \tag{II-34}$$

for any $\alpha_1^2 \leq 1$ and α_2^2 satisfying equation II-25.

Proof: Theorem II-2 gives $W(j\omega) \geq -I$. Thus there exists $\alpha_1^2 \leq 1$ such that the equation II-25 is satisfied.

In order to show that the rest of the theorem is true, consider the scalar $x_*'(t)Kx_*(t)$. Its derivative is

$$\frac{d}{dt}\ (x_*'(t)Kx_*(t)) = \dot{x}_*'(t)Kx_*(t) + x_*'(t)K\dot{x}_*(t)$$

Taking the integral of both sides of this equation gives

$$\int_0^\infty \frac{d}{dt}\ x_*'(t)Kx_*(t)dt = \int_0^\infty [\dot{x}_*'(t)Kx_*(t) + x_*'(t)K\dot{x}_*(t)]\ dt \tag{II-35}$$

Integrating the left hand side of equation II-35, noting that $\lim_{T\to\infty} x_*(T) = 0$, gives the following

$$\int_0^{\infty} \frac{d}{dt} \; x_*'(t)Kx_*(t)\; dt = \lim_{T\to\infty} \int_0^{T} \frac{d}{dt}\; x_*'(t)Kx_*(t)\; dt$$

$$= \lim_{\to\infty} x_*'(t)Kx_*(t)\Big|_0^T$$

$$= \lim_{T\to\infty} x_*'(T)Kx_*(T) - x_*'(0)Kx_*(0)$$

$$= -x_0'Kx_0$$

Using this result and the definition of $\dot{x}_*(t)$, equation II-33, reduces equation II-35 to

$$-x_0'Kx_0 = \int_0^{\infty} [u_*'(t)B'Kx_*(t) + x_*'(t)A'Kx_*(t)$$

$$+ x_*'(t)KAx_*(t) + x_*'(t)KBu_*(t)]\; dt$$

$$= \int_0^{\infty} x_*'(t)[KA + A'K - KBB'K]\; x_*(t) - u_*'(t)u_*(t)\; dt$$

The last step makes use of the fact $u_*(t) = -KB'x_*(t)$, equation II-32. Noting how Q is defined in equation II-23, gives

$$x_0'Kx_0 = \int_0^{\infty} x_*'(t)Qx_*(t) + u_*'(t)u_*(t)\; dt$$

Letting $x_1(t) = x_*(t) - x_h(t)$ produces

$$x_0'Kx_0 = \int_0^{\infty} [x_1(t) + x_h(t)]'Q[x_1(t) + x_h(t)] + u_*'(t)u_*(t)\; dt$$

$$= \int_0^{\infty} x_h'(t)Qx_h(t)\; dt + \int_0^{\infty} [x_1'(t)Qx_h(t) + x_h'(t)Qx_1(t)]\; dt$$

$$+ \int_0^{\infty} x_1'(t)Qx_1(t) + u_*'(t)u_*(t)\; dt$$

$$= \int_0^\infty x_h'(t)Qx_h(t)\ dt + 2\int_0^\infty x_h'(t)Qx_1(t)\ dt$$

$$+ \int_0^\infty [x_1'(t)Qx_1(t) + u_*'(t)u_*(t)]\ dt$$

Applying the quasi-Schwarz inequality, equation II-24, yields

$$x_0'Kx_0 \geq \int_0^\infty x_h'(t)Qx_h(t)\ dt - 2\sqrt{\left|\int_0^\infty x_h'(t)Qx_h(t)\ dt\right| \cdot \left|\int_0^\infty x_1'(t)Qx_1(t)dt\right|}$$

$$+ \int_0^\infty x_1'(t)Qx_1(t)\ dt + \int_0^\infty u_*'(t)u_*(t)\ dt \qquad \text{(II-36)}$$

At this point the proof will be divided into four parts. For cases a and b it will be assumed that

$$\int_0^\infty x_1'(t)Qx_1(t)\ dt \geq 0 \qquad \text{(II-37)}$$

Condition C4 and Theorem II-3 imply

$$\int_0^\infty u_*'(t)u_*(t)\ dt \geq \frac{1}{\alpha_2^2} \int_0^\infty x_1'(t)Qx_1(t)\ dt$$

Hence equation II-36 can be rewritten, using this inequality, as

$$x_0'Kx_0 \geq \int_0^\infty x_h'(t)Qx_h(t)\ dt - 2\sqrt{\left|\int_0^\infty x_h'(t)Qx_h(t)\ dt\right| \cdot \left|\int_0^\infty x_1'(t)Qx_1(t)dt\right|}$$

$$+ \left(1 + \frac{1}{\alpha_2^2}\right) \int_0^\infty x_1'(t)Qx_1(t)\ dt \qquad \text{(II-38)}$$

In light of equation II-37 let

$$\gamma^2 = \int_0^\infty x_1'(t)Qx_1(t)dt$$

Case a: Assume

$$\rho^2 = \int_0^\infty x_h'(t)Qx_h(t)\,dt \geq 0$$

Now rewrite equation II-38 in terms of γ and ρ. The result is

$$x_0'Kx_0 \geq \rho^2 - 2\rho\gamma + \left[1 + \frac{1}{\alpha_2^2}\right]\gamma^2 = f_1(\gamma) \qquad \text{(II-39)}$$

The minimum of $f_1(\gamma)$ can be determined as follows

$$\frac{df_1(\gamma)}{d\gamma} = 2\left(\frac{1 + \alpha_2^2}{\alpha_2^2}\right)\gamma - 2\rho = 0$$

This requires

$$\gamma_{min} = \frac{\alpha_2^2}{1 + \alpha_2^2}\,\rho$$

Substituting into $f_1(\gamma)$ gives

$$f_1(\gamma_{min}) = \left(1 - \frac{2\alpha_2^2}{1 + \alpha_2^2} + \frac{\alpha_2^2}{1 + \alpha_2^2}\right)\rho^2$$

$$= \frac{1}{1 + \alpha_2^2}\,\rho^2 \qquad \text{(II-40)}$$

Since $J_*(x_0) = x_0'Kx_0$ and $J_h(x_0) = \rho^2$, then under these conditions equation II-39 and II-40 imply

$$J_*(x_0) \geq \frac{1}{1 + \alpha_2^2}\,J_h(x_0) \qquad \text{(II-41)}$$

for $J_h(x_0) \geq 0$ and $\int_0^\infty x_1'(t)Qx_1(t)dt \geq 0$.

Case b: Assume

$$-\rho^2 = \int_0^\infty x_h'(t)Qx_h(t)dt < 0$$

Rewriting equation II-38 in terms of γ and ρ gives

$$x_0'Kx_0 \geq -\rho^2 - 2\rho\gamma + \left(1 + \frac{1}{\alpha_2^2}\right)\gamma^2 = f_2(\gamma) \tag{II-42}$$

The minimum of this function occurs at

$$\gamma_{min} = \frac{\alpha_2^2}{1 + \alpha_2^2} \quad \rho$$

The value of $f_2(\gamma_{min})$ is

$$f_2(\gamma_{min}) = \left(-1 - \frac{2\alpha_2^2}{1 + \alpha_2^2} + \frac{\alpha_2^2}{1 + \alpha_2^2}\right) \rho^2$$

$$= \frac{1 + 2\alpha_2^2}{1 + \alpha_2^2} (-\rho^2)$$

This equation and equation II-42 establish the relation

$$J_*(x_0) \geq \frac{1 + 2\alpha_2^2}{1 + \alpha_2^2} J_h(x_0) \tag{II-43}$$

for $J_h(x_0) < 0$ and $\int_0^\infty x_1'(t)Qx_1(t)dt \geq 0$.

For cases c and d the condition

$$\int_0^\infty x_1'(t)Qx_1(t)dt < 0 \tag{II-44}$$

will apply. For this situation Condition C4 and Theorem II-3 give

$$-\alpha_1^2 \int_0^\infty u_*'(t)u_*(t)dt \leq \int_0^\infty x_1'(t)Qx_1(t)\ dt \tag{II-45}$$

which imples

$$\left| \int_0^\infty x_1'(t)Qx_1(t)\ dt \right| \leq \alpha_1^2 \int_0^\infty u_*'(t)u_*(t)\ dt \qquad \text{(II-46)}$$

Hence equation II-36, II-45, and II-46 require that

$$x_0'Kx_0 \geq \int_0^\infty x_h'(t)Qx_h(t)\ dt - 2\alpha_1\sqrt{\left|\int_0^\infty x_h'(t)Qx_h(t)\ dt\right| \cdot \left|\int_0^\infty u_*'(t)u_*(t)dt\right|}$$

$$+ (\ 1-\alpha_1^2\ )\int_0^\infty u_*'(t)u_*(t)\ dt \qquad \text{(II-47)}$$

Since

$$\int_0^\infty u_*'(t)u_*(t)\ dt \geq 0$$

for all real $u_*(t)$ let

$$\gamma^2 = \int_0^\infty u_*'(t)u_*(t)\ dt$$

Case c: Assume

$$\rho^2 = \int_0^\infty x_h'(t)Qx_h(t)\ dt \geq 0$$

Now rewrite equation II-47 in terms of ρ and γ.

$$x_0'Kx_0 \geq \rho^2 - 2\alpha_1\rho\gamma + (\ 1-\alpha_1^2\ )\gamma^2 = f_3(\gamma) \qquad \text{(II-48)}$$

The function $f_3(\gamma)$ is obviously a quadratic equation whose graph opens upward as long as $\alpha_1^2 < 1$. Note that the coefficient of γ^2 being greater than zero is consistant with Condition 1 of Theorem II-2 which requires not only that

$$\int_0^\infty x_1'(t)Qx_1(t) + u_*'(t)u_*(t)\ dt \geq 0$$

but that for every pair $(x(t), u(t))$ such that $x(0) = x(T) = 0$ the equation

$$\int_0^T x'(t)Qx(t) + u'(t)u(t)\ dt \geq 0$$

be true for all $T \geq 0$. The minimum of $f_3(\gamma)$ can be determined by setting the first derivative to zero which gives

$$\gamma_{min} = \frac{\alpha_1}{1-\alpha_1^2}\ \rho$$

At $\gamma = \gamma_{min}$ the value of $f_3(\gamma)$ is

$$f_3(\gamma_{min}) = \left[1 - \frac{2\alpha_1^2}{1-\alpha_1^2} + \frac{\alpha_1^2}{1-\alpha_1^2}\right] \rho^2$$

$$= \frac{1-2\alpha_1^2}{1-\alpha_1^2}\ \rho^2 \tag{II-49}$$

Again since $J_*(x_0) = x_0'Kx_0$ and $J_h(x_0) = \rho^2$, equation II-48 and II-49 imply that

$$J_*(x_0) \geq \frac{1-2\alpha_1^2}{1-\alpha_1^2} J_h(x_0) \tag{II-50}$$

for $J_h(x_0) \geq 0$ and $\int_0^\infty x_1'(t)Qx_1(t)\ dt < 0$.

Case d: Assume

$$-\rho^2 = \int_0^\infty x_h'(t)Qx_h(t)dt < 0$$

Rewriting equation II-47 in terms of γ and ρ gives

$$x_0'Kx_0 \geq -\rho^2 - 2\alpha_1\rho\gamma + (\ 1-\alpha_1^2\ )\gamma^2 = f_4(\gamma) \tag{II-51}$$

As in case c, the graph of this quadratic function, $f_4(\gamma)$, opens upward as long as $\alpha_1^2 < 1$ and the minimum value of the function occurs at

$$\gamma_{min} = \frac{\alpha_1}{1 - \alpha_1^2} \rho$$

The value of $f_4(\gamma)$ at $\gamma = \gamma_{min}$ is

$$f_4(\gamma_{min}) = \left[-1 - \frac{2\alpha_1^2}{1 - \alpha_1^2} + \frac{\alpha_1^2}{1 - \alpha_1^2} \right] \rho^2$$

$$= \frac{1}{1 - \alpha_1^2} (-\rho^2) \tag{II-52}$$

Equations II-51 and II-52 imply

$$J_*(x_0) \geq \frac{1}{1 - \alpha_1^2} J_h(x_0)$$

for $J_h(x_0) < 0$ and $\int_0^\infty x_1'(t)Qx_1(t)\, dt < 0$.

Combining case a and case c implies

$$J_*(x_0) \geq \min \left\{ \frac{1}{1 + \alpha_2^2} J_h(x\ ),\ \frac{1 - 2\alpha_1^2}{1 - \alpha_1^2} J_h(x_0) \right\}$$

for $J_h(x_0) \geq 0$ and combining case b and c gives

$$J_*(x_0) \geq \min \left\{ \frac{1 + 2\alpha_2^2}{1 + \alpha_2^2} J_h(x\ ),\ \frac{1}{1 - \alpha_1^2} J_h(x_0) \right\}$$

whenever $J_h(x_0) < 0$; thus the theorem is established.

<u>End of Proof</u>

Theorem II-4 places a lower bound on the optimal control in terms of the cost of the homogeneous system (control law $u(t) = 0$ for all $t \geq 0$). This result will prove to be invaluable in establishing a lower bound on the ratio of a suboptimal control to the optimal control. The suboptimal control will be considered in the following chapter.

III. LOWER BOUND ON OPTIMAL COST FROM COST OF SUBOPTIMAL CONTROL LAW

In this section the suboptimal feedback control law will be studied. Consider the controllable and observable linear time-invariant system

$$\Sigma: \quad \dot{x}(t) = Ax(t) + Bu(t) \;; \quad x(0) = x$$
$$y(t) = Cx(t) \qquad \text{(III-1)}$$

where A, B, and C are constant matrices of the following respective dimensions: n*n, n*r, and s*n. Thus x(t) is an n-vector; u(t) is an r-vector; and y(t) is an s-vector. The control law

$$u_D(t) = -Dx_D(t) \qquad \text{(III-2)}$$

where $x_D(t)$ is the state trajectory of system Σ when $u(t) = u_D(t)$ and D is a constant r*n matrix, will be referred to as the suboptimal feedback control law. In all cases considered it will be assumed that the application of the control law $u_D(t)$ results in a stable closed loop system. The system Σ, equation III-1, with the control law $u_D(t)$, is schematically displayed in the block diagram of Figure 1.

The problem is to compare the cost, $J_D(x_0)$, of system Σ when the control law is $u_D(t)$, equation III-2, to the cost, $J_*(x_0)$, of system Σ when the optimal control law is used for a cost functional given by

$$J(y,u) = \int_0^\infty y'(t)y(t) + u'(t)u(t)\, dt \qquad \text{(III-3)}$$

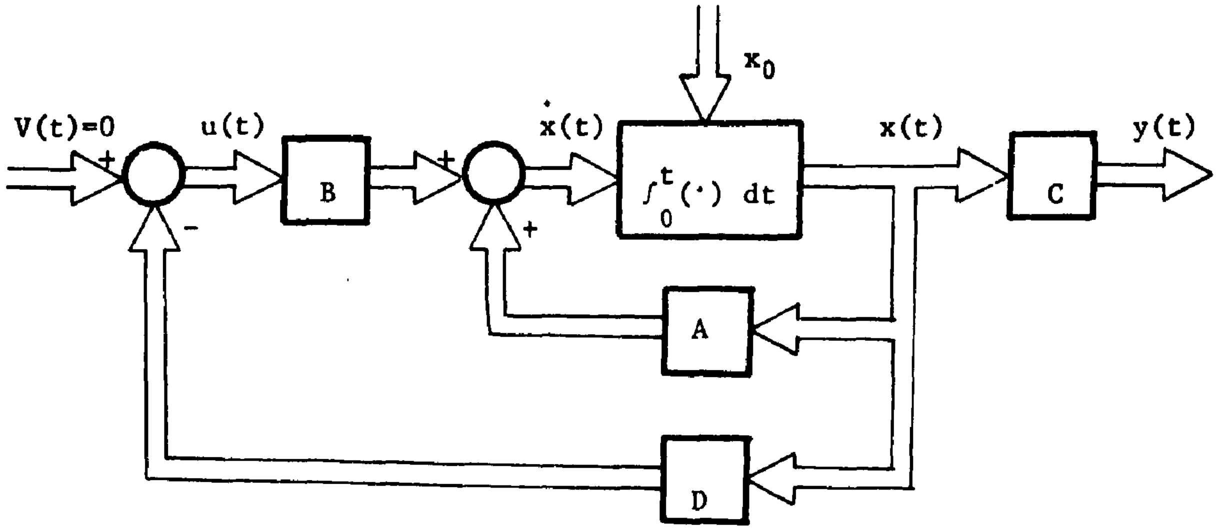


Figure 1. Block diagram of suboptimal feedback system

For system Σ and the above cost functional, the optimal control exists, is unique, and is given by

$$u_*(t) = -B'K_*x_*(t) \tag{III-4}$$

where K_* is the positive definite [2] solution of the matrix Riccati equation

$$A'K + KA - KBB'K = -C'C \tag{III-5}$$

and $u_*(t)$ and $x_*(t)$ are the trajectories of system Σ with the control law in equation III-4, as was indicated in Theorem II-2. Furthermore when $u(t) = u_*(t)$, then the cost for the optimal system is given by

$$J_*(x_0) = x_0'K_*x_0 \tag{III-6}$$

A. Derivation of Lower Bound

Before considering the main theorem of this chapter it will be useful to obtain an auxiliary result. The following development will be useful in establishing Theorem III-2.

<u>Theorem III-1</u> If the constant matrices A, B, and C and constant symmetric matrices Q and L, whose dimensions are n*n, n*r, s*n, n*n

and n*n, respectively, satisfy the matrix equation

$$A'L + LA - LBB'L + C'C = Q \tag{III-7}$$

then

$$B'\Phi_L'(-s)Q\Phi_L(s)B = [\ B'\Phi_L'(-s)LB - I]\ [\ B'L\Phi_L(s)B - I]$$
$$+ B'\Phi_L'(-s)C'C\Phi_L(s)B - I \tag{III-8}$$

where

$$\Phi_L(s) = (\ sI - A + BB'L)^{-1}$$

Proof: Consider the matrix identity

$$-sI + sI + LBB'L = LBB'L$$

Subtract equation III-7 from this identity. After collecting terms, the result is

$$(-sI - A' + LBB')L + L(sI - A + BB'L) = LBB'L + C'C - Q$$

Pre- and post-multiply this equation by $B'\Phi_L'(-s)$ and $\Phi_L(s)B$, respectively, to get

$$B'L\Phi_L(s)B + B'\Phi_L'(-s)LB = B'\Phi_L'(-s)LBB'L\Phi_L(s)B$$
$$+ B'\Phi_L'(-s)C'C\Phi_L(s)B - B'\Phi_L'(-s)\Phi_L(s)B$$

Rearranging terms and adding and subtracting I from this equation results in

$$B'\Phi_L'(-s)Q\Phi_L(s)B = [B'\Phi_L'(-s)LB-I]\ [B'L\Phi_L(s)B - I]$$
$$+ B'\Phi_L'(-s)C'C\Phi_L(s)B - I$$

which is the same as equation III-8.

<u>End of Proof</u>

The primary objective is to compare the cost of a suboptimal linear control law to the cost of the optimal control law for system Σ. Canales [3] derived an expression for a lower bound for ratio of the optimal to the suboptimal cost. Though the case he presented is more limited, his theorem suggested the proof of the following theorem. The difference between Canales' theorem and Theorem III-2 arises out of the general nature of Theorem II-4. Theorem II-4 expresses a relationship only between the optimal cost and the cost of the homogenous system. However, the following theorem,which relates the optimal cost and cost of suboptimal control law is a direct result of Theorem II-4.

Before stating Theorem III-2 it will be convenient to define several conditions which will be useful in the presentation of the theorem. These conditions are as follows:

CC1. Given the system Σ, a n*n symmetric matrix L where $D = B'L$, and x_0, the quasi-Schwarz inequality

$$\left(\int_0^\infty x_h'(t) Q x_1(t)\ dt\right)^2 \leq \left| \int_0^\infty x_h'(t) Q x_h(t)\ dt \right| \cdot \left| \int_0^\infty x_1'(t) Q x_1(t)\ dt \right| \tag{III-9}$$

where

$$Q = A'L + LA - LBB'L + C'C$$

$$x_h(t) = \Phi_D(t)\ x_0$$

$$x_1(t) = \Phi_D(t)\ R(t)\ x_0$$

$$R(t) = -BB'(\ K - L)\ \Phi_K(t)$$

K is the solution of the Riccati equation

$$A'K + KA - KBB'K = -C'C$$

$\Phi_K(t)$ is the transition matrix for the closed loop optimal system (i.e., $\Phi_K(t) = \exp[(A - BB'K)t]$),

and $\Phi_D(t)$ is the transition matrix for the closed loop suboptimal system (i.e., $\Phi_D(t) = \exp[(A-BD)]$).

hold at x_0.

CC2. Given the system Σ, the feedback control law in equation III-2, ε, x_0, and a n*n symmetric matrix L, it is true that

$$\text{if } x_0'(P - L)x_0 \geq 0, \text{ then } x_0'(L - \varepsilon P)x_0 \geq 0$$

$$\text{or} \tag{III-10}$$

$$\text{if } x_0'(P - L)x_0 < 0, \text{ then } x_0'[(2 - \varepsilon)P - L]x_0 \geq 0$$

where P is the solution to the linear matrix equation

$$PF + F'P = -C'C - D'D \tag{III-11}$$

for

$$F = A - BD \tag{III-12}$$

CC3. For the system Σ and the feedback control law in equation III-2, the inequality in the frequency domain

$$-\alpha_1^2 I \leq W(j\omega) \leq \alpha_2^2 I \tag{III-13}$$

holds for all ω where

$$W(s) = [B'\Phi_D'(-s)D' - I]\ [D\Phi_D(s)B - I]$$

$$+B'\Phi_D'(-s)C'C\Phi_D(s)B - I \tag{III-14}$$

with $\Phi_D(s) = (sI - A + BD)^{-1}$

With these condition defined, Theorem III-2 may now be concisely stated as:

<u>Theorem III-2</u> Given system Σ, the cost functional of equation III-3, and the suboptimal (stable) feedback control law of equation III-2,

if there exists some symmetric matrix L and some scalar ε such that $D = B'L$ and conditions CC1 and CC2 are satisfied, then the cost, $J_D(x_0)$ of the system using the suboptimal control law $u_D(t)$ bounds from below the optimal cost, $J_*(x_0)$ by the following equation

$$J_*(x_0) \geq \min\left[\frac{1 + \varepsilon\alpha_2^2}{1 + \alpha_2^2} J_D(x_0), \frac{1 - (2-\varepsilon)\alpha_1^2}{1 - \alpha_1^2} J_D(x_0)\right] \qquad \text{(III-15)}$$

for any $\alpha_1^2 < 1$ and α_2^2 such that condition CC3 is true.

Proof: Substituting the feedback control law $u_D(t)$, equation III-2, into the system equation, equation III-1, gives

$$\dot{x}_D(t) = (A - BD)x_D(t) = Fx_D(t) \; ; \; x_D(0) = x_0$$
$$y_D(t) = Cx_D(t) \qquad \text{(III-16)}$$

The cost functional, equation III-3, is revised as follows

$$J_D(x_D) = J(y_D(t), u_D(t))$$
$$= \int_0^\infty x_D'(t)C'Cx_D(t) + x_D'(t)D'Dx_D(t)\, dt$$
$$= \int_0^\infty x_D'(t)\, [C'C + D'D]\, x_D(t)\, dt$$

Since by hypothesis the eigenvalues of matrix F have negative real parts, Theorem II-1 indicates the cost associated with the system in equation III-16 and the cost functional given above is

$$J_D(x_0) = x_0'Px_0 \qquad \text{(III-17)}$$

where P satisfies the equation

$$PF + F'P = -C'C - D'D \qquad \text{(III-18)}$$

Note that P is positive definite since $C'C + D'D$ is positive definite [2].

Now define an $n*n$ matrix Q by the following equation

$$Q = LA + A'L - LBB'L + C'C$$

where L is some symmetric matrix such that $D = B'L$. Rearranging this equation and adding and subtracting $LBB'L$ results in

$$L(A - BB'L) + (A' - LBB')L + LBB'L + C'C = Q$$

or using the definition of F gives

$$LF + F'L = Q - LBB'L - C'C \tag{III-19}$$

Subtracting equation III-19 from equation III-18 and noting that $LBB'L = D'D$ gives

$$(P - L)F + F'(P - L) = -Q \tag{III-20}$$

Letting

$$N = P - L \tag{III-21}$$

in equation III-20 gives

$$NF + F'N = -Q \tag{III-22}$$

Now consider the optimal system for which the control law is $u_*(t) = B'Kx_*(t)$, where K is the solution to equation III-5. Add and subtract the terms $LBB'K$ and $KBB'L$ from the left hand side of this equation to get

$$K(A - BB'L) + KBB'L + (A' - LBB')K + LBB'K - KBB'K = -C'C$$

Again using the definition of F gives

$$\begin{aligned} KF + F'K &= - KBB'L - LBB'K + KBBK - C'C \\ &= (K - L)'BB'(K - L) - LBB'L - C'C \end{aligned}$$

Subtracting equation III-19 from this equation results in

$$(K - L)F + F'(K - L) = (K - L)'BB'(K - L) - Q$$

or letting

$$M = K - L \tag{III-23}$$

gives

$$MF + F'M - MBB'M = -Q \tag{III-24}$$

Note that $F - BB'M = A - BB'K$, hence $\mathrm{Re}\{\lambda(F - BB'M)\} < 0$ as was indicated in the opening of this chapter.

Equation III-22 and III-24 satisfy condition C1 and C2 of Theorem II-4. Since condition CC1 is assumed, if it can be shown that

$$-\alpha_1^2 I \leq B'(-j\omega I - F')^{-1}Q(j\omega I - F)B \leq \alpha_2^2 I \tag{III-25}$$

then the results of Theorem II-4 may be invoked. But by theorem III-1 it is evident that equation III-25 is equivalent to equation III-13, hence it is established that

$$x_0'Mx_0 \geq \begin{cases} \min\left(\dfrac{1}{1+\alpha_2^2}\, x_0'Nx_0,\ \dfrac{1-2\alpha_1^2}{1-\alpha_1^2}\, x_0'Nx_0\right) & \text{for } x_0'Nx_0 \geq 0 \\ \min\left(\dfrac{1+2\alpha_2^2}{1+\alpha_2^2}\, x_0'Nx_0,\ \dfrac{1}{1-\alpha_1^2}\, x_0'Nx_0\right) & \text{for } x_0'Nx_0 < 0 \end{cases} \tag{III-26}$$

for any $\alpha_1^2 < 1$ and α_2^2 satisfying equation III-13.

Consider equation III-26 for the case $x'Nx \geq 0$. Making the substitution $M = K - L$ and $N = P - L$ gives

$$x_0'(K - L)x_0 \geq \min\left(\frac{1}{1+\alpha_2^2}\, x_0'(P - L)x_0,\ \frac{1-2\alpha_1^2}{1-\alpha_1^2}\, x_0'(P - L)\, x_0\right)$$

which results in

$$x_0'Kx_0 \geq \min\left(\frac{1+\varepsilon\alpha_2^2}{1+\alpha_2^2}\, x_0'Px_0 + \frac{\alpha_2^2}{1+\alpha_2^2}\, x_0'(L-\varepsilon P)x_0,\ \frac{1-(2-\varepsilon)\alpha_1^2}{1-\alpha_1^2}\, x_0'Px_0 + \frac{\alpha_1^2}{1-\alpha_1^2}\, x_0'(L-\varepsilon P)x_0\right)$$

$$\geq \min\left(\frac{1+\varepsilon\alpha_2^2}{1+\alpha_2^2}\, x_0'Px_0,\ \frac{1-(2-\varepsilon)\alpha_1^2}{1-\alpha_1^2}\, x_0'Px_0\right) \tag{III-27}$$

for any ε such that $x_0'(L - \varepsilon P)x_0 \geq 0$ and $x_0'(P - L)x_0 \geq 0$. Making the substitutions $M = K - L$ and $N = P - L$ for the case $x_0'Nx_0 < 0$ gives

$$x_0'(K - L)x_0 \geq \min \left\{\frac{1 + 2\alpha_2^2}{1 + \alpha_2^2} x_0'(P - L)x_0, \frac{1}{1 - \alpha_1^2} x_0'(P - L)\, x_0\right\}$$

which implies

$$x_0'Kx_0 \geq \min \left\{\frac{1 + \varepsilon\alpha_2^2}{1 + \alpha_2^2} x_0'Px_0 + \frac{\alpha_2^2}{1 + \alpha_2^2} x_0'[(2 - \varepsilon)P - L]\, x_0 \,, \right.$$

$$\left. \frac{1 - (2 - \varepsilon)\alpha_1^2}{1 - \alpha_1^2} x_0'Px_0 + \frac{\alpha_1^2}{1 - \alpha_1^2} x_0'[(2 - \varepsilon)P - L]\, x_0\right\}$$

$$\geq \min \left\{\frac{1 + \varepsilon\alpha_2^2}{1 + \alpha_2^2} x_0'Px_0 \,, \frac{1 - (2 - \varepsilon)\alpha_1^2}{1 - \alpha_1^2} x_0'Px_0\right\} \tag{III-28}$$

for any ε such that $x_0'[(2 - \varepsilon)P - L]x_0 \geq 0$ and $x_0(P - L)x_0 < 0$. Equations III-27 and III-28 complete the proof of the theorem.

<u>End of Proof</u>

Note that Theorem III-2 as stated is dependent on the initial condition. If one can find a set of L matrices and some ε independent of x_0 such that conditions CC1 and CC2 hold for ε and some L in the set for every x_0, then the dependence on x_0 in equation III-15 can be removed. Theorem III-2 completes the theoretical development. However, there is one matrix identity which will often be helpful in understanding the results. That identity [10] is

$$C_2(sI - A + BB'L)^{-1}B = C_2(sI - A)^{-1}B\,[I + B'L(sI - A)^{-1}B]^{-1} \tag{III-29}$$

where the constant matrices A, B, C_2 and L are of the respective

dimensions n*n, n*r, s*n and n*n. This identity can be used to express the closed loop frequency inequality, equation III-13 in terms of an inequality on the open loop transfer function. Though the preceeding theorems can be used for a general multi-input, multi-output system, in the sequel only single-input single-output systems will be considered.

B. Suboptimal Single-input, Single-output Feedback Systems

Up until this point multi-input, multi-output systems have been considered. Unfortunately such systems do not lend themselves to easy manipulation and display. For this reason only single-input, single-output systems will be discussed in the sequel.

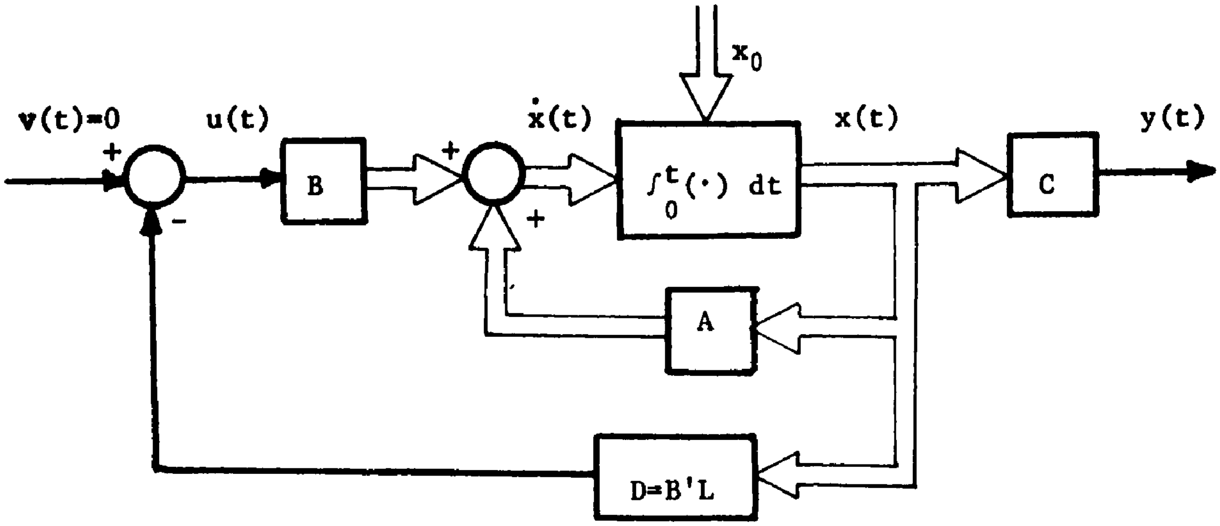
For a single-input, single-output system several interesting relationships exist between the matrix description, Figure 2a, and the frequency domain representation, Figure 2b, of the system. Some of the relationships are [9] these:

$$G(s) = C(sI - A)^{-1}B \tag{III-30}$$

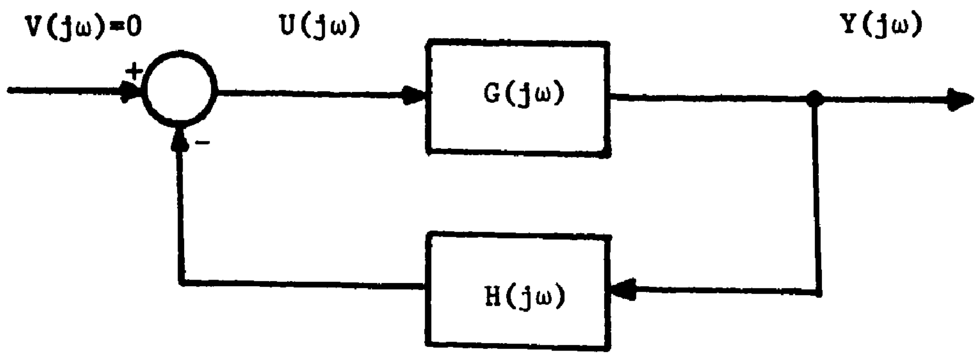
$$G(s)H(s) = D(sI - A)^{-1}B = B'L(sI - A)^{-1}B \tag{III-31}$$

and for the closed loop system this relationship is:

$$G_c(s) = \frac{G(s)}{1 + G(s)H(s)} = C(sI - A + BD)^{-1}B \tag{III-32}$$



(a)



(b)

Figure 2. Block diagram of single-input, single-output system

(a) Time domain representation

(b) Frequency domain representation.

Consider W(s) given in equation III-14 as

$$W(s) = [B'\Phi_D(-s)D' - I]\,[D\Phi_D(s)B - I] + B'\Phi_D(-s)C'C\Phi_D(s)B - I \tag{III-33}$$

where

$$\Phi_D(s) = (sI - A + BD)^{-1} \tag{III-34}$$

From equation III-32 it is seen that

$$G_c(s) = C\Phi_D(s)B \tag{III-35}$$

and combining equation III-31 and equation III-29, assuming $C_2 = B'L = D$, results in

$$D\Phi_D(s)B = \frac{G(s)H(s)}{1 + G(s)H(s)} \tag{III-36}$$

Hence for the single-input, single-output system, W(s) can be written using equation III-32 thru equation III-36 as

$$W(s) = \left(\frac{G(-s)H(-s)}{1 + G(-s)H(-s)} - 1\right)\left(\frac{G(s)H(s)}{1 + G(s)H(s)} - 1\right)$$

$$+\left(\frac{G(-s)}{1 + G(-s)H(-s)}\right)\left(\frac{G(s)}{1 + G(s)H(s)}\right) - 1$$

$$= \left(\frac{-1}{1 + G(-s)H(-s)}\right)\left(\frac{-1}{1 + G(s)H(s)}\right)$$

$$+ \frac{|G(s)|^2}{|1 + G(s)H(s)|^2} - 1$$

$$W(s) = \frac{1 + |G(s)|^2}{|1 + G(s)H(s)|^2} - 1 \tag{III-37}$$

Several interesting results can be derived using equation III-37. Consider equation III-13

$$-\alpha_1^2 \leq W(j\omega) \leq \alpha_2^2 \tag{III-38}$$

for all real ω. If equation III-38 holds for some control law $u_D(t)$ with $\alpha_1 = \alpha_2 = 0$, then this implies (equation III-37) that

$$\frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)H(j\omega)|^2} = 1 \qquad \text{(III-39)}$$

But if all the conditions of the theorem are satisfied, Theorem III-2 implies the optimal cost is related to the cost of the system using the control law $u_D(t)$ by equation III-15, which in this case implies

$$J_*(x_0) \geq J_D(x_0)$$

But since $J_*(x_0)$ is the minimum, it is true that

$$J_*(x_0) \leq J_D(x_0)$$

Hence

$$J_*(x_0) = J_D(x_0)$$

This is explained by noting that equation III-39 is Kalman's equation for optimality [5] as indeed it should.

One additional point should be mentioned about W(s). Note that

$$\frac{1}{|1 + G(s)H(s)|^2} = \frac{1}{|T_D(s)|^2}$$

where $T_D(s)$ in "classical control" language is the "return difference" for control law $u_D(t)$, and

$$\left|\frac{G(s)}{1 + G(s)H(s)}\right|^2 = |G_c(s)|^2$$

where, as above, $G_c(s)$ is the closed loop transfer function for control law $u_D(t)$. Hence W(s) can be written as

$$W(s) = |G_c(s)|^2 + \frac{1}{|T_D(s)|^2} - 1$$

Considering this equation for the case of equation III-39 it is easy to show that for an optimal system

$$|T_D(j\omega)|^2 \geq 1$$

for all real ω. This is another one of Kalman's results [5]. In "classical control" it has long been known that this is a necessary condition in order for the sensitivity to changes in plant parameters to be decreased by a feedback control.

C. Constant Cost Ratio Loci in the Frequency Domain

In classical control work frequency domain plots play a large role in the compensation of a control system. Such plots as Bode plots and Nyquist plots are used to estimate time domain behavior of a system from general time domain-frequency domain relationships. A common starting point in the frequency domain analysis is the open loop transfer function for a unity-feedback control system.

The lower bound developed in Theorem III-2 provides a means of estimating how optimal is a unity feedback system from a Nyquist plot. Consider the locus of all points in the Nyquist plane which result in a given ratio of $J_*(x_0) / J_D(x_0)$. These loci are determined from Theorem III-2 by equality for the inequality (equation III-15),

$$-\alpha_1^2 \leq W(j\omega) \leq \alpha_2^2 \quad \text{for all real } \omega \;, \tag{III-41}$$

where $W(j\omega)$ is given as in equation III-37,

$$W(j\omega) = \frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2} - 1 \tag{III-42}$$

$(H(j\omega) = 1)$. The loci of constant ratio will be determined in two parts. First assume that the lower bound is determined by the equality on the α_2 side of equation III-41,

$$W(j\omega) = \alpha_2^2 \tag{III-43}$$

and the ratio $J_*(x_0) \;/\; J_D(x_0) = k_2$ where k_2 is a scalar constant. This requires that

$$\frac{1 + \varepsilon\alpha_2^2}{1 + \alpha_2^2} = k_2$$

$$1 + \varepsilon\alpha_2^2 = k_2(1 + \alpha_2^2)$$

$$(1 - \varepsilon) = (k_2 - \varepsilon)\;(1 + \alpha_2^2) \tag{III-44}$$

Letting

$$\begin{aligned} a &= (1 - \varepsilon) \\ b &= (k_2 - \varepsilon) \end{aligned} \tag{III-45}$$

and noting from equation III-43 and equation III-42 that

$$1 + \alpha_2^2 = \frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2}$$

equation III-44 reduces to

$$a = b\left(\frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2}\right) \tag{III-46}$$

Let $x = \mathrm{Re}\{G(j\omega)\}$ and $y = \mathrm{Im}\{G(j\omega)\}$, then

$$a = b\left[\frac{1 + x^2 + y^2}{(1 + x)^2 + y^2}\right]$$

$$a + 2ax + ax^2 + ay^2 = b + bx^2 + by^2$$

$$(a - b)x^2 + (a - b)y^2 + 2ax = b - a$$

$$x^2 + \frac{2a}{a-b}\,x + \frac{a^2}{(a-b)^2} + y^2 = \frac{a^2}{(a-b)^2} - 1$$

$$\left(x + \frac{a}{a-b}\right)^2 + y^2 = \frac{a^2}{(a-b)^2} - 1 \tag{III-47}$$

The use of the definition of a and b given in equation III-45 and equation III-47 gives

$$\left(x + \frac{1-\varepsilon}{1-k_2}\right)^2 + y^2 = \frac{(1-\varepsilon)^2}{(1-k_2)^2} - 1 \tag{III-48}$$

Now assume that the lower bound is determined by the equality on the α_1 side of equation III-41,

$$-\alpha_1^2 = W(j\omega) \tag{III-49}$$

and the ratio $J_*(x_0) \,/\, J_D(x_0) = k_1$, where k_1 is a scalar constant. The problem requires that

$$\frac{1 - (2 - \varepsilon)\alpha_1^2}{1 - \alpha_1^2} = k_1$$

$$1 - (2 - \varepsilon)\alpha_1^2 = k_1(1 - \alpha_1^2)$$

$$(1 - \varepsilon) = (2 - k_1 - \varepsilon)(1 - \alpha_1^2) \qquad \text{(III-50)}$$

Letting

$$\begin{aligned} a &= (1 - \varepsilon) \\ b &= (2 - k \quad - \varepsilon) \end{aligned} \qquad \text{(III-51)}$$

and noting from equation III-49 and equation III-42 that

$$1 - \alpha_1^2 = \frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2}$$

equation III-50 reduces to

$$a = b\left(\frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2}\right)$$

Since this is the same as equation III-46, the loci for a constant ratio are given by equation III-47. Substituting the definition of a and b, equation III-51, into equation III-47 results in

$$\left(x - \frac{1 - \varepsilon}{1 - k_1}\right)^2 + y^2 = \left(\frac{1 - \varepsilon}{1 - k_1}\right)^2 - 1 \qquad \text{(III-52)}$$

If the right hand sides of both equations are positive, equations III-48 and III-52 represent acentric circles in the Nyquist plane.

For the ratio k_2 the circles lie in the left half plane, approaching a center at -1.0 with a radius of zero as k_2 approaches zero and approaching a center at $-\infty$ with an infinite radius (the imaginary axis) as k_2 approaches 1. In the case of ratio k_1, the circles are a mirror image of the results for k_2 . The case $\varepsilon = 0$ is shown in Figure 3. The effect of ε, for $1 > \varepsilon > 0$, is to move the center of each circle closer to the imaginary axis and to decrease the radius.

D. Second Order Example

In order to illustrate some of the preceeding concepts, consider the following example. This example was first used by Canales [3] as an application of the lower bound derived by him.

Problem 1: Given a system described by the differential equations (Figure 4)

$$\ddot{x}(t) + 4x(t) = u(t) \tag{III-53}$$

$$y(t) = 2\dot{x}(t) - x(t) \tag{III-54}$$

the control law

$$u(t) = -\alpha y(t) \tag{III-55}$$

and the cost functional

$$J = \int_0^\infty [u^2(t) + y^2(t)]\, dt \tag{III-56}$$

choose a value of α such that the performance of the system is near the optimal performance (when all the state variables are fed back).

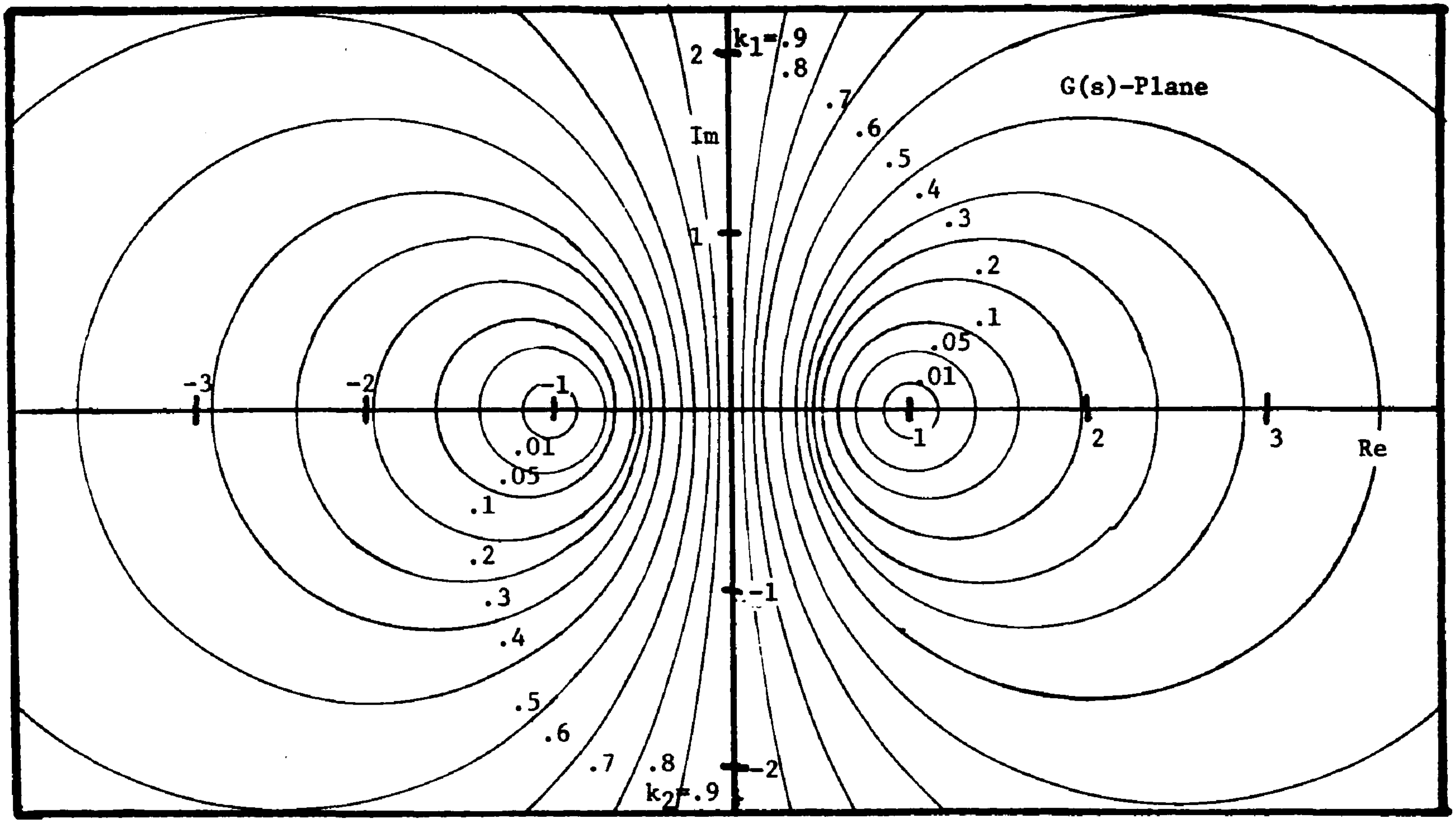


Figure 3. Constant cost ratio loci in the frequency domain

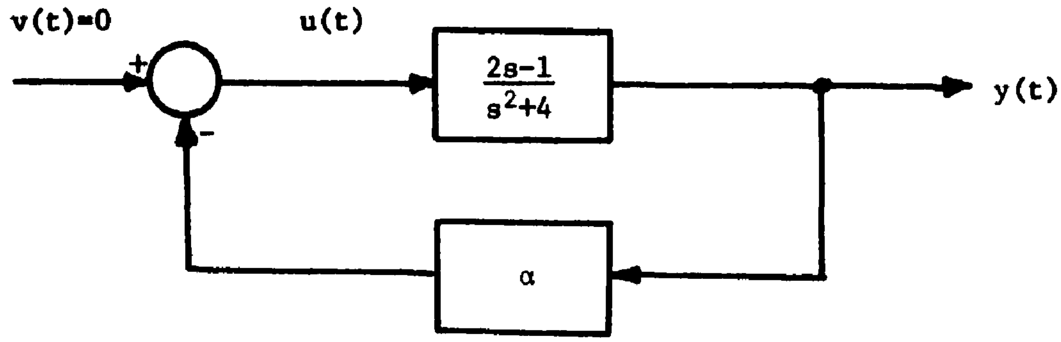


Figure 4. System for Problem 1

Solution: For this problem it is obviously desirable to determine a lower bound on the ratio of the optimal cost to the cost of control law in equation III-55. Such a lower bound is given by Theorem III-2. This lower bound is determined from the following equation (III-15)

$$J_*(x_0) \geq \min\left[\frac{1 + \varepsilon\alpha_2^2}{1 + \alpha_2^2} J_D(x_0)\ ,\ \frac{1 - (2 - \varepsilon)\alpha_1^2}{1 - \alpha_1^2} J_D(x_0)\right] \tag{III-57}$$

where $J_*(x_0)$ is the cost for the optimal system, $J_D(x_0)$ is the cost for the suboptimal control, α_1^2 and α_2^2 obey (equation III-13)

$$-\alpha_1^2 \leq W(j\omega) \leq \alpha_1^2 \tag{III-58}$$

and $W(j\omega)$ is defined in equation III-37 by

$$W(s) = \frac{1 + |G(s)|^2}{|1 + G(s)H(s)|^2} - 1 \tag{III-59}$$

For the system in equations III-53 thru III-55, $G(s)$ and $H(s)$ are given by

$$G(s) = \frac{2s - 1}{s^2 + 4} \tag{III-60}$$

$$H(s) = \alpha \tag{III-61}$$

Substituting equation III-60 and III-61 into equation III-59 and evaluating at $s = j\omega$, results in

$$W(j\omega) = \frac{4\omega^2 + 1 + (4 - \omega^2)^2}{(4 - \omega^2 - \alpha)^2 + 4\alpha^2\omega^2} - 1$$

$$= \frac{\omega^4 - 4\omega^2 + 17}{\omega^4 + (4\alpha^2 + 2\alpha - 8)\omega^2 + (\alpha^2 - 8\alpha + 16)} - 1 \tag{III-62}$$

In order to determine an α_1^2 and α_2^2 which satisfy equation III-58, it is desirable to determine the extrema with respect to ω of $W(j\omega)$, equation III-62. This can be accomplished by determining the extrema of

$$W_E = \frac{\omega^4 + a\omega^2 + b}{\omega^4 + c\omega^2 + d} - 1 \tag{III-63}$$

The extrema are characterized by the first derivative's being zero, or

$$\frac{dW_E}{d\omega} = \frac{(4\omega^3 + 2a\omega)(\omega^4 + c\omega^2 + d) - (\omega^4 + a\omega^2 + b)(4\omega^3 + 2c\omega)}{(\omega^4 + c\omega^2 + d)^2} = 0$$

Simplifying and collecting terms gives that any extremum is located at $\omega^2 = 0$ or at the solution of the quadratic equation in ω^2,

$$(c - a)\omega^4 + 2(d - b)\omega^2 + (ad - bc) = 0 \tag{III-64}$$

Note that since ω is a real number, any solution ω^2 of equation III-64 that is negative is an extraneous root. Equations III-63 and III-64 imply that the extrema of equation III-62 are located at $\omega^2 = 0$ or

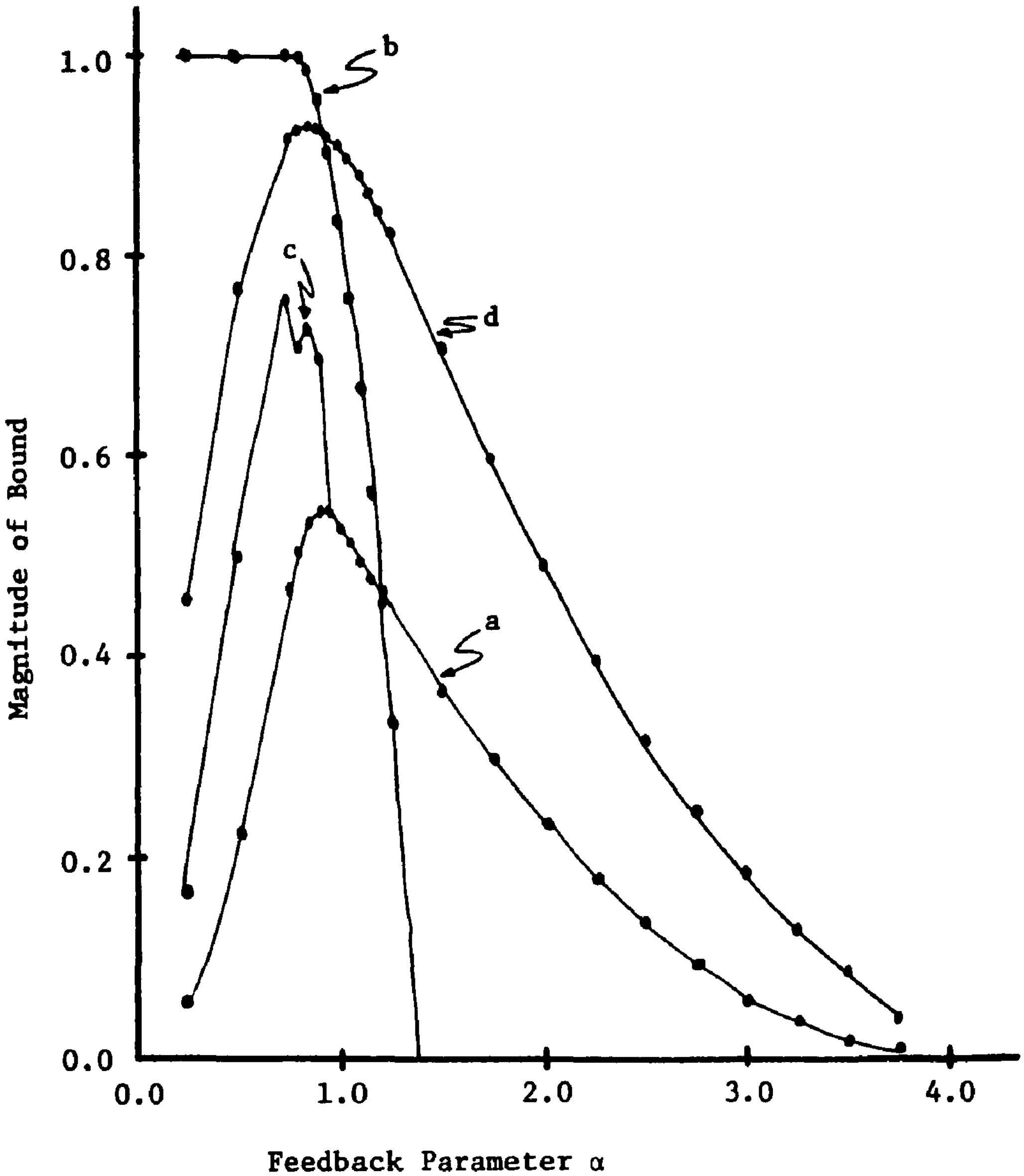
at the positive solutions for ω^2 of the equation

$$(4\alpha^2 + 2\alpha - 4)\omega^4 + 2(\alpha^2 - 8\alpha - 1)\omega^2 + (-72\alpha^2 - 2\alpha + 72) = 0 \tag{III-65}$$

The value of these maximum and minimum are found by evaluating equation III-62 at each of the extrema. Letting $-\alpha_1^2$ equal to the smallest of 0 or the minimum of $W(j\omega)$ and α_2^2 equal to the largest of 0 or the maximum of $W(j\omega)$, a program was written to evaluate $\frac{1}{1 + \alpha_2^2}$ and $\frac{1 - 2\alpha_1^2}{1 - \alpha_1^2}$ over the range of α for which the closed loop system, equations III-53 thru III-55, is stable ($0 < \alpha < 4$).

In the sequel the functions $\frac{1}{1 + \alpha_2^2}$ and $\frac{1 - 2\alpha_1^2}{1 - \alpha_1^2}$ will be referred to as α_2-bound and α_1-bound, respectively. Note that the α_2-bound and the α_1-bound correspond to the two parts of equation III-57 for $\varepsilon = 0$. Hence if there exist any $\varepsilon \geq 0$ such that all the conditions of Theorem III-2 are satisfied, then the minimum of the α_2-bound and α_1-bound is less than the ratio $J_*(x_0) / J_D(x_0)$. The results of the program are plotted in Figure 5 and tabulated in Table 1 in the Appendix. Line a in Figure 5 represents the α_2-bound, while line b is the α_1-bound.

Before it can be stated that the min of α_1-bound and α_2-bound is less that the ratio of $J_*(x_0) / J_D(x_0)$ for a feedback law α, it must be shown that there exist an $\varepsilon \geq 0$ such that all the hypotheses of Theorem III-2 are satisfied for that value of α. The needed conditions in Theorem III-2 are in the form of inequality constraints on the initial conditions, and hence, best handled in a state



a - α_2-bound

b - α_1-bound

c - Improved bound (terminates at α = 0.95)

d - Riccati bound

Figure 5. Bounds determined for Problem 1

variable format. One state variable representation of the system in equations III-53 thru III-55 is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \qquad \text{(III-66)}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = [-1 \;,\; 2] \qquad \text{(III-67)}$$

with the feedback control law given by

$$u(t) = -Dx(t) \qquad \text{(III-68)}$$

where

$$D = \alpha C \qquad \text{(III-69)}$$

There are two conditions which need to be established. These are that there exist a symmetric matrix L satisfying $D = B'L$ such that 1) the quasi-Schwarz inequality is satisfied for each initial condition and 2) equation III-10,

$$x_0'(L - \varepsilon P)x_0 \geq 0 \quad \text{if } x_0'(P - L)x_0 \geq 0$$

or (III-70)

$$x_0'[(2 - \varepsilon)P - L] \geq 0 \quad \text{if } x_0'(P - L)x_0 < 0$$

where P is the solution to the linear matrix equation

$$PF + F'P = -C'C - D'D \qquad \text{(III-71)}$$

for

$$F = A - BD \qquad \text{(III-72)}$$

is satisfied for each initial condition. Since D = B'L and D also satisfies equation III-69, it is true that

$$B'L = C$$

or

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix} = \begin{bmatrix} -1 & 2 \end{bmatrix}$$

This equation requires that

$$L = \begin{bmatrix} \beta & -\alpha \\ -\alpha & 2\alpha \end{bmatrix} \qquad \text{(III-73)}$$

where β may be any real number. The matrix P,

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \qquad \text{(III-74)}$$

is determined by using equation III-67, III-69, III-71, and III-72 as follows

$$F = A - BD$$

$$= \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 , & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -(4 -\alpha) & -2\alpha \end{bmatrix}$$

$$F'P + PF = - \quad C'C - C'C$$
$$= -(1 + \alpha^2)C'C \tag{III-75}$$

$$\begin{bmatrix} -2p_{12}(4 - \alpha) & p_{11} - 2\alpha p_{12} - p_{22}(4 - \alpha) \\ p_{11} - 2\alpha p_{12} - p_{22}(4 - \alpha) & 2p_{12} - 4\alpha p_{22} \end{bmatrix} = -(1+\alpha^2)\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

This matrix equation implies that p_{11}, p_{12}, and p_{22} are given by the solution to the simultaneous equations

$$-2(4 - \alpha)p_{12} = -(1 + \alpha^2)$$

$$2p_{12} - 4\alpha p_{22} = -4(1 + \alpha^2)$$

$$p_{11} - 2\alpha p_{12} - p_{22}(4 - \alpha) = 2(1 + \alpha^2)$$

Hence

$$p_{12} = \frac{1}{2}\,\frac{(1 + \alpha^2)}{(4 - \alpha)}$$

$$p_{11} = \frac{68 - \alpha}{4\alpha(4 - \alpha)}\,(1 + \alpha^2) \tag{III-76}$$

and

$$p_{22} = \frac{17 - 4\alpha}{4\alpha(4 - \alpha)}\,(1 + \alpha^2)$$

The quasi-Schwarz inequality as given for Theorem III-2 (condition CC1) is

$$\left(\int_0^\infty x_h'(t)Qx_1(t)\,dt\right)^2 \leq \left|\int_0^\infty x_h'(t)Qx_h(t)\,dt\right| \cdot \left|\int_0^\infty x_1'(t)Qx_1(t)\,dt\right| \tag{III-77}$$

where

$$Q = A'L + LA - LBB'L + C'C \tag{III-78}$$

$$x_h(t) = \Phi_D(t)x_0 \tag{III-79}$$

$$x_1(t) = \Phi_D(t)R(t)x_0 \tag{III-80}$$

$$R(t) = -BB'(K-L)\Phi_K(t)$$

K is the solution of the matrix Riccati equation

$$A'K + KA - KBB'K = -C'C \tag{III-81}$$

$\Phi_K(t)$ is the transition matrix for the optimal closed loop system (i.e. $\Phi_K(t) = \exp[(A - BB'K)t]$), and $\Phi_D(t)$ is the transition matrix for the closed loop suboptimal system (i.e. $\Phi_D(t) = \exp[(A - BD)t]$). This condition will be established in parts.

First determine $\Phi_D(t)$.

$$\Phi_D(s) = (sI - A + BD)^{-1} = (sI - F)^{-1}$$

$$= \begin{bmatrix} s & -1 \\ 4-\alpha & s+2\alpha \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} s+2\alpha & 1 \\ -4+\alpha & s \end{bmatrix}}{s^2 + 2\alpha s + 4 - \alpha}$$

$$= \begin{bmatrix} \dfrac{s-a-b}{(s-a)(s-b)} & \dfrac{1}{(s-a)(s-b)} \\ \\ \dfrac{-ab}{(s-a)(s-b)} & \dfrac{s}{(s-a)(s-b)} \end{bmatrix} \tag{III-82}$$

where a and b are the roots of the charateristic equation (i.e., $(s-a)(s-b) = s^2 + 2\alpha s + 4 - \alpha$). If a is not equal to b, then

$$\phi_D(t) = \mathcal{L}^{-1}[\phi_D(s)]$$

$$= \frac{1}{a-b} \begin{bmatrix} -be^{at} + ae^{bt} & e^{at} - e^{bt} \\ -abe^{at} + abe^{bt} & ae^{at} - be^{bt} \end{bmatrix} \qquad \text{(III-83)}$$

Now consider

$$M(t) = \Phi_D'(t)Q\Phi_D(t)$$

$$= \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \qquad \text{(III-84)}$$

where

$$m_{11} = q_{11}\phi_{11}^2 + 2q_{12}\phi_{11}\phi_{21} + q_{22}\phi_{21}^2$$

$$m_{12} = q_{11}\phi_{11}\phi_{12} + q_{12}[\phi_{11}\phi_{22} + \phi_{12}\phi_{21}] + q_{22}\phi_{21}\phi_{22} \qquad \text{(III-85)}$$

$$m_{22} = q_{11}\phi_{12}^2 + 2q_{12}\phi_{12}\phi_{22} + q_{22}\phi_{22}^2$$

Using the values for ϕ_{11}, ϕ_{12} and ϕ_{22} given in equation III-83, equation III-85 expands to

$$m_{11} = \frac{1}{(a-b)^2} \{b^2[q_{11} + 2q_{12}a + q_{22}a\ ]\ e^{2at}$$

$$-2ab[q_{11} + q_{12}(a+b) + q_{22}ab]\ e^{(a+b)}$$

$$+ a^2[q_{11} + 2q_{12}b + q_{22}b^2]\ e^{2bt}\ \}$$

$$m_{12} = \frac{1}{(a-b)^2} \{-b[q_{11} + 2q_{12}a + q_{22}a^2]\, e^{2at}$$

$$+ (a+b)\, [q_{11} + q_{12}(a+b) + q_{22}ab]\, e^{(a+b)t}$$

$$-a\, [q_{11} + 2q_{12}b + q_{22}b^2]\, e^{2bt} \} \qquad \text{(III-86)}$$

$$m_{22} = \frac{1}{(a-b)^2} \quad [q_{11} + 2q_{12}a + q_{22}a^2]\, e^{2at}$$

$$-2[q_{11} + q_{12}(a+b) + q_{22}ab]\, e^{(a+b)t}$$

$$+[q_{11} + 2q_{12}b + q_{22}b^2]\, e^{2bt} \}$$

In order to evaluate equation III-80, $\Phi_K(t)$ must be determined. Assume the solution to the Riccati equation, equation III-81, is

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \qquad \text{(III-87)}$$

then

$$\Phi_K(s) = (sI - A + BB'K)^{-1}$$

$$= \begin{bmatrix} s & -1 \\ 4 + k_{12} & s + k_{22} \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} s + k_{22} & 1 \\ -4 - k_{12} & s \end{bmatrix}}{s^2 + k_{22}s + 4 + k_{12}}$$

$$= \begin{bmatrix} \dfrac{s-c-d}{(s-c)(s-d)} & \dfrac{1}{(s-c)(s-d)} \\[2ex] \dfrac{-cd}{(s-c)(s-d)} & \dfrac{1}{(s-d)(s-d)} \end{bmatrix}$$

where c and d are the roots of the closed loop characteristic equation for the optimal system (i.e., $(s-c)(s-d) = s^2 + k_{22}s + 4 + k_{12}$).

If c is not equal to d, then

$$\Phi_K(t) = \mathcal{L}^{-1}[\Phi_K(s)]$$

$$= \frac{1}{c-d}\begin{bmatrix} -de^{ct} + ce^{dt} & e^{ct} - e^{dt} \\ -cde^{ct} + cde^{dt} & ce^{ct} - de^{dt} \end{bmatrix} \qquad \text{(III-88)}$$

For B as given in equation III-67, K as given in equation III-87, L as given in equation III-73, and $\Phi_K(t)$ given as

$$\Phi_K(t) = \begin{bmatrix} \phi_{K11} & \phi_{K12} \\ \phi_{K21} & \phi_{K22} \end{bmatrix}$$

the evaluation of equation III-80 proceeds as follows;

$$R(t) = BB'(K - L)\Phi_K(t)$$

$$= \begin{bmatrix} 0 & 0 \\ (k_{12}-l_{12})\Phi_{K11}+(k_{22}-l_{22})\Phi_{K21} & (k_{12}-l_{12})\Phi_{K12}+(k_{22}-l_{22})\Phi_{K22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ r_{21} & r_{22} \end{bmatrix} \qquad \text{(III-89)}$$

For $\Phi_K(t)$ given in equation III-88, r_{21} and r_{22} can be written as follows:

$$r_{21} = \frac{1}{c-d} \quad [-d(k_{12} - l_{12}) - cd(k_{22} - l_{22})]\ e^{ct}$$

$$+\ [c(k_{12} - l_{12}) + cd(k_{22} - l_{22})]\ e^{dt}$$

$$\text{(III-90)}$$

$$r_{22} = \frac{1}{c-d} \quad [k_{12} - l_{12} + c(k_{22} - l_{22})]\ e^{ct}$$

$$-\ [k_{12} - l_{12} + d(k_{22} - l_{22})]\ e^{dt}$$

Let M_{2a11}, M_{ab11}, M_{2b11}, M_{2a12}, M_{ab12}, M_{2b12}, M_{2a22}, M_{ab22}, and M_{2b22}

be defined by

$$m_{11} = M_{2a11}e^{2at} + M_{ab11}e^{(a+b)t} + M_{2b11}e^{2bt}$$

$$m_{12} = M_{2a12}e^{2at} + M_{ab12}e^{(a+b)t} + M_{2b12}e^{2bt}$$

$$m_{22} = M_{2a22}e^{2at} + M_{ab22}e^{(a+b)t} + M_{2b22}e^{2bt}$$

and equation III-86. Let R_{c21}, R_{d21}, R_{c22}, and R_{d22} be defined by

$$r_{21} = R_{c21}e^{ct} + R_{d21}e^{dt}$$

$$\text{(III-92)}$$

$$r_{22} = R_{c22}e^{ct} + R_{d22}e^{dt}$$

and equation III-90. Then, the 2-1 element of the matrix product

$$M(t)R(t) = \begin{bmatrix} m_{12}r_{21} & m_{12}r_{22} \\ m_{22}r_{21} & m_{22}r_{22} \end{bmatrix} \qquad \text{(III-93)}$$

is given by

$$m_{22}r_{21} = M_{2a22}R_{c21}e^{(2a+c)t} + M_{ab22}R_{c21}e^{(a+b+c)t}$$
$$+ M_{2b22}R_{c21}e^{(2b+c)t} + M_{2a22}R_{d21}e^{(2a+d)t}$$
$$+ M_{ab22}R_{d21}e^{(a+b+d)} + M_{2b22}R_{d21}e^{(2b+d)t} \qquad \text{(III-94)}$$

with the other elements given by the same equation with an appropriate change of subscript, and the 2-1 element of the matrix product

$$R'(t)M(t)R(t) = \begin{bmatrix} m_{22}r_{21}^2 & m_{22}r_{21}r_{22} \\ m_{22}r_{21}r_{22} & m_{22}r_{22}^2 \end{bmatrix} \qquad \text{(III-95)}$$

is given by

$$m_{22}r_{21}r_{22} = M_{2a22}R_{c21}R_{c22}e^{(2a+2c)t} + M_{ab22}R_{c21}R_{c22}e^{(a+b+c)t}$$
$$+ M_{2b22}R_{c21}R_{c22}e^{(2b+2c)t} + M_{2a22}R_{d21}R_{c22}e^{(2a+c+d)t}$$
$$+ M_{ab22}R_{d21}R_{c22}e^{(a+b+c+d)} + M_{2b22}R_{d21}R_{c22}e^{(2b+c+d)t}$$
$$+ M_{2a22}R_{c21}R_{d22}e^{(2a+c+d)t} + M_{2a22}R_{c21}R_{d22}e^{(a+b+c+d)t}$$
$$+ M_{2b22}R_{c21}R_{d22}e^{(2b+c+d)t} + M_{2a22}R_{d21}R_{d22}e^{(2a+2b)t}$$
$$+ M_{ab22}R_{d21}R_{d22}e^{(a+b+2d)t} + M_{2b22}R_{d21}R_{d22}e^{(2b+2d)t}$$

with the other elements in equation III-95 given by appropriate change of subscript.

Now determine the matrix Q. The matrix Q is given by equation III-78 with the values of A, B, and C as in equation III-67, the

control law in equation III-68, and L as given by equation III-73.

Hence

$$Q = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta & -\alpha \\ -\alpha & 2\alpha \end{bmatrix} + \begin{bmatrix} \beta & -\alpha \\ -\alpha & 2\alpha \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} -\alpha \\ 2\alpha \end{bmatrix} [-\alpha \quad 2\alpha] + \begin{bmatrix} -1 \\ 2 \end{bmatrix} [-1 \quad 2]$$

or

$$Q = \begin{bmatrix} -\alpha^2 + 8\alpha + 1 & 2\alpha^2 - 8\alpha - 2 + \beta \\ 2\alpha^2 - 8\alpha - 2 + \beta & -4\alpha^2 - 2\alpha + 4 \end{bmatrix} \qquad \text{(III-97)}$$

Define a matrix T by

$$\begin{aligned} x_0' T x_0 &= \int_0^\infty x_h'(t) Q x_1(t) dt \\ &= x_0' \left(\int_0^\infty M(t) R(t) \, dt \right) x_0 \end{aligned}$$

a matrix U by

$$\begin{aligned} x_0' U x_0 &= \int_0^\infty x_h'(t) Q x_h(t) dt \qquad \text{(III-98)} \\ &= x_0' \left(\int_0^\infty M(t) \, dt \right) x_0 \end{aligned}$$

and a matrix V by

$$\begin{aligned} x_0' V x_0 &= \int_0^\infty x_h'(t) Q x_1(t) dt \\ &= x_0' \left(\int_0^\infty R'(t) M(t) R(t) \, dt \right) x_0 \end{aligned}$$

The solution of the Riccati equation was algebraically determined as

$$K = \begin{bmatrix} 10.4960 & 0.1231 \\ 0.1231 & 2.0606 \end{bmatrix} \qquad \text{(III-99)}$$

and a program was written to determine the elements of the matrices T, U, and V for any α and β. Hence the quasi-Schwarz inequality is reduced to the condition that

$$\left(x_0'Tx_0\right)^2 \leq \left| x_0'Ux_0 \right| \left| x_0'Vx_0 \right| \tag{III-100}$$

holds for all initial conditions of interest. The magnitude sign on the right hand side of equation III-100 makes the condition difficult to check. This problem is removed by squaring each side of equation III-100. The equation which is used to determine the truth of the quasi-Schwarz inequality is

$$\left(x_0'Tx_0\right)^4 \leq \left(x_0'Ux_0\right)^2 \left(x_0'Vx_0\right)^2$$

or, equivalently, the polynomial

$$T(x_0) \geq 0 \tag{III-101}$$

where

$$T(x_0) = \left(u_{11}x_1^2 + 2u_{12}x_1x_2 + u_{22}x_2^2\right)^2 \left(v_{11}x_1^2 + 2v_{12}x_1x_2 + v_{22}x_2^2\right)^2$$
$$-\left(t_{11}x_1^2 + [t_{12} + t_{21}]x_1x_2 + t_{22}x_2^2\right)^4 \tag{III-102}$$

for

$$x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Equation III-101 uses the facts that $u_{12} = u_{21}$ and $v_{12} = v_{21}$ (note that $t_{12} \neq t_{21}$).

Now the conditions needed to establish that the minimum of the α_1-bound and the α_2-bound does indeed constitute a lower bound on the ratio of the optimal cost to the cost for the control law $u(t) = -Dx(t)$ can be restated as follows:

<u>Sufficient Condition</u>

There exists a <u>set</u> of matrices, S_L, such that $B'L = D$ for every $L \in S_L$, and that for each initial condition, x_0, there exists some $L \in S_L$ such that equation III-70 is satisfied for some $\varepsilon \geq 0$, and that equation III-101 simultaneously holds for that L and x_0.

It is important to note that the above condition indicates it is not necessary that for all initial conditions a single L exists for which equation III-70 and III-101 simultaneously hold, but that for each initial condition, some L exist for which both equations hold. However, if such an L exists, then the Sufficient Condition is indeed satisfied.

In Figure 5 the peak in the α_2-bound occurs near $\alpha = 0.92$ and as can be seen this is the largest value of the minimum of the α_1-bound and α_2-bound. Hence $\alpha = 0.92$ is the best value predicted by the bound of Theorem III-2 for the feedback coefficient in Problem 1. It is hence desirable to show *a priori* that the α_2-bound is indeed a lower bound on the ratio of the optimal cost to the cost of the control law $u(t)=-\alpha y(t)$ when $\alpha = 0.92$. This will be established by verifying the Sufficient Condition for a set of α up to $\alpha = 0.95$. The Verification Procedure is outlined as follows:

Verification Procedure

For a given value of α

1. Choose a set of matrices, S, such that if L is an element of S, then L satisfies D=B'L(i.e., equation III-73,

$$L = \begin{bmatrix} \beta & -\alpha \\ -\alpha & 2\alpha \end{bmatrix} \tag{III-103}$$

deines the matrix L).

2. For each L ε S and for each $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ elements of some choosen set, S_ε, determine the range of inital conditions such that whenever $x_0'(P - L)x_0 \geq 0$ then

$$x_0'(L-\varepsilon_1 P)x_0 \geq 0$$

or if $x_0'(P - L)x_0 < 0$ then (III-104)

$$x_0'[(2 - \varepsilon_2)P - L]x_0 \geq 0$$

3. For each L ε S calculcate the matrices T, U, and V defined in equation III-98 and determine the range of initial conditions for which equation III-101 is satisfied.

4. If there is a set $S_L(\varepsilon_1, \varepsilon_2)$ contained in S such that for every x_0 there exist some L an element of S_L for which steps 2 and 3 simultaneously hold, then the Sufficient Condition is satisfied. Furthermore if ε_* is the minimum value of ε_1 or ε_2 used to obtain set $S_L(\varepsilon_1, \varepsilon_2)$, then the lower bound may be improved by letting $\varepsilon = \varepsilon_*$ in equation III-57.

The above procedure is easily stated, but the means of determining the desired set is by no means clear. First a means of determining the set of initial conditions in step 3 will be presented. Consider the expanded form of equation III-102,

$$T(x_0) = \tau_0 x_1^8 + \tau_1 x_1^7 x_2 + \tau_2 x_1^6 x_2^2 + \tau_3 x_1^5 x_2^3 + \tau_4 x_1^4 x_2^4$$

$$+ \tau_5 x_1^3 x_2^5 + \tau_6 x_1^2 x_2^6 + \tau_7 x_1 x_2^7 + \tau_8 x_2^8$$

where the τ_i are defined by equivalence with equation III-102. In factored form this equation is

$$T(x_0) = \tau_8 \prod_{i=1}^{8} (x_2 - r_i x_1) \qquad \text{(III-105)}$$

where the r_i are the roots of the polynomial

$$\tau_8 x^8 + \tau_7 x^7 + \tau_6 x^6 + \tau_5 x^5 + \tau_4 x^4 + \tau_3 x^3 + \tau_2 x^2 + \tau_1 x^1 + \tau_0 = 0$$

Now for any initial condition such that $x_2 = x \cdot x_1$, equation III-105 can be written as

$$T(x_0) = \tau_8 x_1^8 \prod_{i=1}^{8} (x - r_i)$$

Hence it is seen that $T(x_0) \geq 0$ for every x_1 and x_2 such that $x_2 = x \cdot x_1$ if

$$T_x = \tau_8 \prod_{i=1}^{8} (x - r_i) \geq 0 \qquad \text{(III-106)}$$

For $\tau_8 > 0$, if there are no real r_i then equation III-106 is satisfied; if there are two real r_i, say r_1 and r_2 with $r_1 < r_2$, then equation III-106 is satisfied for $x \leq r_1$ and $x \geq r_2$; if there are four real r_i, say r_1 thru r_4, with $r_1 < r_2 < r_3 < r_4$, then equation III-106 is satisfied for $x \leq r_1$, $r_2 \leq x \leq r_3$, and $x \geq r_4$; etc. For $\tau_8 < 0$ the set of x's for which equation III-106 is satisfied will be the closure of the complement of the above set of x's for an equivalent number of real roots. Now consider the determination of the sets in step 2. A similar procedure with the expanded matrix equations listed in step 2 will yield the desired set of initial conditions for each ε_1 and ε_2 in that step. The only difference is that the involved equations are quadratic.

For the system given in equations III-66 and III-67, L given by equation III-73, Q as determined in equation III-97, and K given in equation III-99, a digital computer program was written to perform the indicated task in the Verification Procedure. The results of step 4, as indicated by this program, are accumulated in the appendix, Table 2, and show that at least up to the peak at $\alpha = 0.92$, the α_2-bound does indeed constitute the desired lower bound. The improved bound resulting from considering equation III-57 for non-zero ε is presented in Figure 5 as curve c. One should not attach any significance to the shape of this curve since no attempt was made to find a best ε. But curve c does show that by considering the matrix information, it is often possible to improve the frequency domain bound.

The Verification Procedure is an involved task--especially the verification of the quasi-Schwarz inequality. It is hence worthwhile

to note that for the case where the Q matrix, equation III-78, is either positive or negative semi-definite, the quasi-Schwarz inequality becomes the Schwarz inequality and holds in general for all initial conditions. Hence for any such Q matrix it is only necessary to find an ε_1 and ε_2 which satisfy equation III-104 for all x_0. From equation III-97 it can be seen that for $0 \leq \alpha \leq 0.78$ the Q matrix of Problem 1 can be made positive definite for several choices of β. For any such β note that it is not necessary to calculate the solution to the Riccati equation, or to obtain any transistion matricies in order to verify the lower bound or determine a value of ε to improve the lower bound. The result will be improved as long as an $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ exist, which satisfies equation III-104.

The foregoing procedure established the validity of the bound predicted by Theorem III-2 for all values of α up to $\alpha = 0.95$. It would be interesting to see how these values compare to the minimum of the actual ratio of the cost of the optimal system to the cost of the system using the feedback control law $u_D(t) = -Dx(t)$ where $D = B'L$. For a second order system such a comparision is easily made. First of all, note that the cost of the system using the control law $u_D(t)$ is given in the proof of Theorem III-2 as (equation III-17)

$$J_D(x_0) = x_0'Px_0$$

where P satisfies equation III-18. For Problem 1 the matrix P as a function of α is defined by equation III-76. Now note that the cost of the optimal system is given by equation III-6 as

$$J_*(x_0) = x_0'K_*x_0$$

where K_* is the solution of the algebraic Riccati equation, equation III-5. Hence the ratio $J_*(x_0)/J_D(x_0)$ can be written as

$$\frac{J_*(x_0)}{J_D(x_0)} = \frac{x_0'K_*x_0}{x_0'P\ x_0}$$

$$= \frac{k_{11}x_1^2 + 2k_{12}x_1x_2 + k_{22}x_2^2}{p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2} \qquad \text{(III-107)}$$

where

$$x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ K_* = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix},\ \text{and } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \qquad \text{(III-108)}$$

Setting the partial derivative with respect to x_1 of equation III-107 equal to zero characterizes the extremum points with respect to x_1 to obey

$$(k_{11}p_{12} - p_{11}k_{12})x_1^2 + (k_{11}p_{22} - p_{11}k_{22})x_1x_2 + (k_{12}p_{22} - p_{12}k_{22})x_2^2 = 0$$

if x_2 is not equal to zero. Solving this equation for x_1 in terms of x_2 results in

$$x_1 = \left[\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right] x_2 \qquad \text{(III-109)}$$

where

$$a = k_{11}p_{12} - p_{11}k_{12}$$

$$b = k_{11}p_{22} - p_{11}k_{22}$$

$$c = k_{12}p_{22} - p_{12}k_{22}$$

Defining d by

$$d = \frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \qquad \text{(III-110)}$$

then substituting equation III-109 into equation III-107 results in the extremum with respect to x_1 being given by

$$\frac{J_*(x_0)}{J_D(x_0)} = \frac{k_{11}d^2 + 2k_{12}d + k_{22}}{p_{11}d^2 + 2p_{12}d + p_{22}} \qquad \text{(III-111)}$$

Following the same procedure for the partial derivative of equation III-107 with respect to x_2 results in exactly the same equation for the extremum with respect to x_2 as equation III-111. Hence the two solutions (equation III-110) of equation III-111 correspond to the maximum and the minimum of ratio $J_*(x_0)/J_D(x_0)$. In the sequel the minimum of this ratio will be referred to as the Riccati bound. The Riccati bound for the system in Problem 1 is shown as curve d in Figure 5. The data used to construct curve d is tabulated in Table 1 of the appendix.

Now consider a second problem. The system to be considered is a unity feedback system whose closed loop transfer function is the same as that of Problem 1.

Problem 2: Given a system described by the differential equations (Figure 6)

$$\ddot{x}(t) + 2(\alpha - 1)\dot{x}(t) + (5 - \alpha)x(t) = u(t) \qquad \text{(III-112)}$$

$$y(t) = 2\dot{x}(t) - x(t) \qquad \text{(III-113)}$$

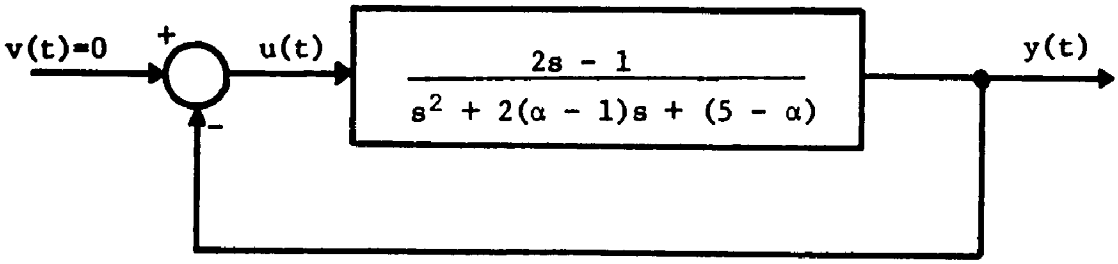


Figure 6. System for Problem 2

the control law

$$u(t) = -y(t) \tag{III-114}$$

and the cost functional

$$J = \int_0^\infty [u^2(t) + y^2(t)]\, dt$$

choose a value of α such that the performance of the system is near the optimal performance (when all the state variables are fed back).

Solution: This problem can be solved in a manner exactly analogous to Problem 1. In order to apply Theorem III-2, the function W(s) must be determined. For this system the associated transfer functions are

$$G(s) = \frac{2s - 1}{s^2 + 2(\alpha - 1)s + (5 - \alpha)} \tag{III-115}$$

and

$$H(s) = 1 \tag{III-116}$$

Substituting equations III-115 and III-116 into equation III-59 and evaluating at $s = j\omega$ gives

$$W(j\omega) = \frac{\omega^4 + (4\alpha^2 - 6\alpha - 2)^2 + (\alpha^2 - 10\alpha + 26)}{\omega^4 + (4\alpha^2 + 2\alpha - 8)^2 + (\alpha^2 - 8\alpha + 16)} - 1 \tag{III-117}$$

In order to choose a value of α_1 and α_2 in equation III-58, it is necessary to determine the extrema over ω of equation III-117. By redefining the values of a, b, c, and d in equation III-63, equation III-64 implies that the extrema are all located at $\omega^2 = 0$ or at the positive solutions for ω^2 of the equation

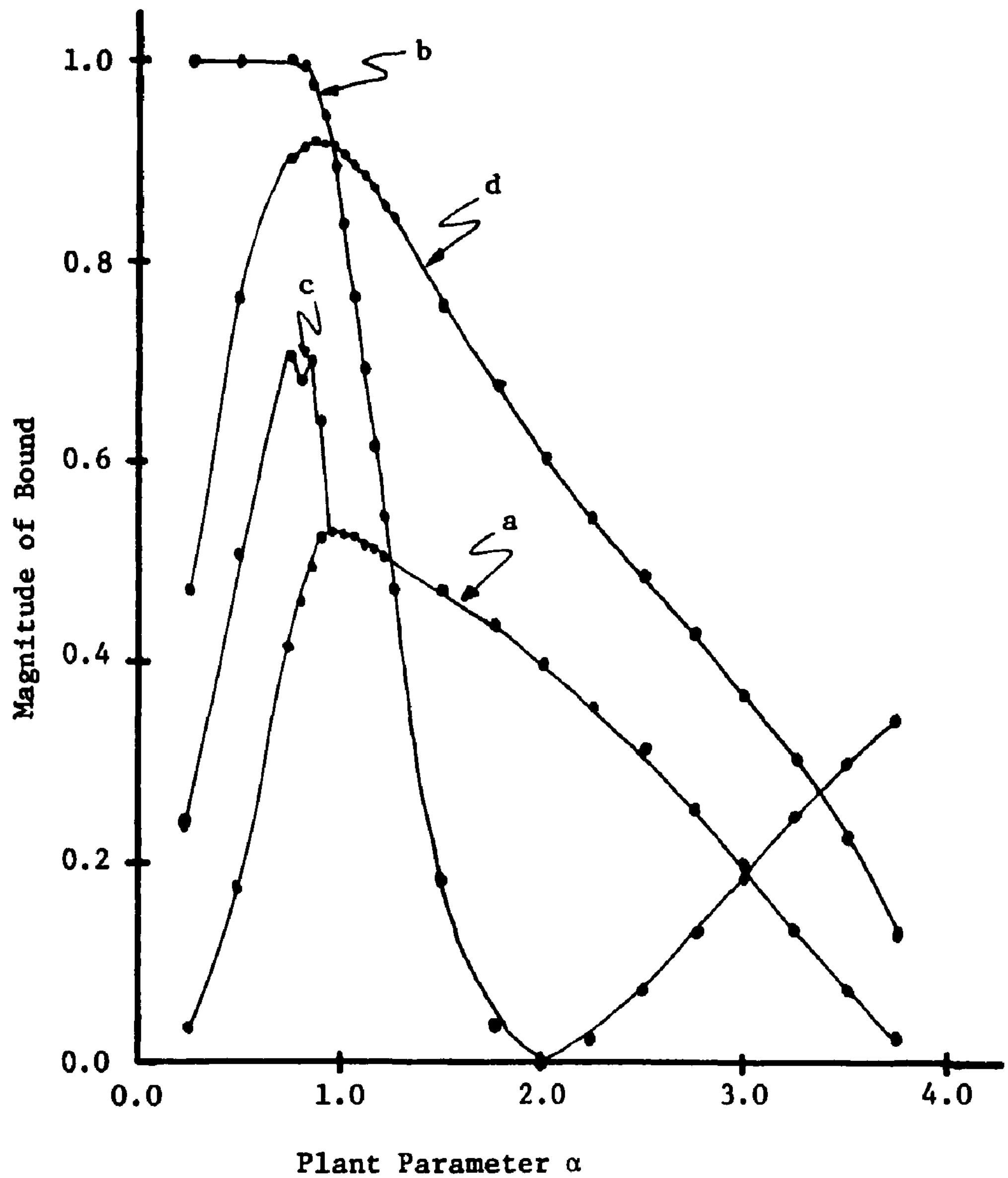
$$(4\alpha - 3)\omega^4 + 2(\alpha - 5)\omega^2 + (17\alpha^2 - 106\alpha + 88) = 0 \qquad \text{(III-118)}$$

The maximum value and minimum value are found by evaluating equation III-117 at each of these extremum. Again let $-\alpha_1^2$ equal the smallest of 0 or the minimum of $W(j\omega)$ and α_2^2 equal the largest of 0 or the maximum of $W(j\omega)$; a program was written to evaluate $\dfrac{1}{1 + \alpha_2^2}$ (α_2-bound) and $\dfrac{1 - 2\alpha_1^2}{1 - \alpha^2}$ (α_1-bound) over the range of α, for which the closed loop system, equation III-112 thru equation III-114, is stable ($0 < \alpha < 4$). The α_1-bound (curve b) and the α_2-bound (curve a), as calculated by the FORTRAN routine, are plotted versus α in Figure 7, and the data is tabulated in Table 3 in the appendix.

Since the open loop system is different from that of Problem 1, it must again be verified that the sufficient conditions of Theorem III-2 are satisfied for all values of α of interest. A state variable representation of the system given in equations III-112 thru III-114 is:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad \text{(III-119)}$$

where



a - α_2-bound

b - α_1-bound

c - Improved bound (terminates at α = 0.95)

d - Riccati bound

Figure 7. Bounds determined for Problem 2

$$A = \begin{bmatrix} 0 & 1 \\ -(5-\alpha) & -2(\alpha-1) \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = [-1 \quad 2] \qquad \text{(III-120)}$$

with the feedback control law given by

$$u(t) = -Dx(t) \qquad \text{(III-121)}$$

where

$$D = C \qquad \text{(III-122)}$$

As with Problem 1, in order to verify that the minimum of the α_1-bound and α_2-bound is the desired lower bound, it is necessary to show that: there exists a set of L matrices, where D = B'L, such that at each initial condition for some L in the set 1) the quasi-Schwarz inequality holds, and 2) equation III-70 is satisfied for $\epsilon \geq 0$. In order to show the second part, the matrices L and P are needed. The matrix L satisfies B'L = D and equation III-122 gives that D = C. Hence

$$L = \begin{bmatrix} \beta & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{(III-123)}$$

where β may be any real number. The matrix P is specified by equation III-71 or in this case

$$F'P + PF = -2C'C \qquad \text{(III-124)}$$

where F = A - BD. If P is given by

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \qquad \text{(III-125)}$$

then equation III-124 requires that

$$
\begin{aligned}
p_{12} &= \frac{1}{4 - \alpha} \\
p_{11} &= \frac{68 - \alpha}{2\alpha(4 - \alpha)} \qquad\qquad \text{(III-126)} \\
p_{22} &= \frac{17 - 4\alpha}{2\alpha(4 - \alpha)}
\end{aligned}
$$

The equations used to determine the quasi-Schwarz inequality in Problem 1 can also be used for this case. Since the closed loop system for Problem 1 and Problem 2 have the same equation, the transition matrix $\Phi_D(t)$ in Problem 2 is unchanged from that of equation III-83. However, the solutions to the Riccati equation and the Q matrix are different from those used in Problem 1. The Q matrix is given by the operations in equation III-78 using the values of A, B, C, and L given in equation III-120 and equation III-123 or

$$
Q = \begin{bmatrix} 2(5 - \alpha) & 4\alpha + \beta - 12 \\ 4\alpha + \beta - 12 & 2(3 - 4\alpha) \end{bmatrix} \qquad \text{(III-127)}
$$

If

$$
K_* = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \qquad \text{(III-128)}
$$

is the solution of the algebraic Riccati equation (equation III-81) for the values of A, B, and C given in equation III-120, then the elements of K_* are the solution of

$$k_{12}^2 + 2(5 - \alpha)k_{12} - 1 = 0$$

$$k_{11} - 2(\alpha - 1)k_{12} - k_{12}k_{22} - k_{22}(5 - \alpha) - 2 = 0$$

$$k_{22}^2 + 4(\alpha - 1)k_{22} - 4 - 2k_{12} = 0 \qquad \text{(III-129)}$$

which result in a positive-definite K_*.

For the values of A, B, and C given in equation III-120, the value of D in equation III-122, the value of L in equation III-123, the Q matrix in equation III-127, and the value of K_* determined from equation III-129, equations III-77 thru III-96 were used in a digital program to determine the matrices T, U, and V defined in equation III-97. These values were then used to obtain T_x given in equation III-106. The Verification Procedure was again carried out for α up to a value of 0.95. The results of step 4 of the Verification Procedure are tabulated in Table 4 in the appendix. The improved bound resulting in this process is plotted as curve c in Figure 7. The final curve, curve d, in Figure 7 represents the Riccati bound as calculated from equation III-110 and III-111 for K_* determined from equation III-129 and P given in equation III-126. This information is also presented in Table 3 in the appendix.

Several interesting results are noted by considering Figure 5 and Figure 7. First of all, note that the minimums of the α_1-bound and α_2-bound lie well below the actual lower bound (Riccati bound) for all stable values of α. In addition it can be seen that the peak of the predicted lower bound occurs near the peak of the actual lower bound. Hence at least for these problems the bound of Theorem III-2 seems to be a reasonable means of choosing the parameter α.

One should be cautious to note that the two problems are different, even though the closed loop transfer function of the two problems are the same. As a result there is a slight difference in the predicted value of the best α in the two problems. For Problems 1 the determined value of α was $\alpha \simeq 0.92$, while in Problem 2 the result is $\alpha \simeq 0.96$. For the two problems there is a different point of view. In Problem 1 the plant is held constant (hence the optimal system is fixed) while the control law $u_D(t) = -\alpha y(t)$ is varied. In Problem 2 the control law $u_D(t) = -y(t)$ is held constant while a plant parameter is varied. This difference affects the shape of the curves in Figure 5 and Figure 7. In each case the α_1-bound imposes the lower bound for values of α larger than about 1.2. However in Problem 2 (Figure 7) the α_1-bound starts to increase after $\alpha = 2$. This can be understood in light of Figure 3, as the frequency response does not enclose the point (1,0) in the Nyquist plane for values of $\alpha < 2$. It crosses the (1,0) point at $\alpha = 2$. and encloses the (1,0) point for values of $\alpha > 2$. It appears from the available data that for $\alpha > 2$ the verification of the lower bound will require a negative ε (equation III-57), resulting in a decrease in the predicted lower bound (possibly to zero or less) over this range of α.

IV. APPLICATION OF LOWER BOUND TO SERIES COMPENSATION OF A UNITY FEEDBACK CONTROL SYSTEM

The calculation of the lower bound determined in Theorem III-2 is easily accomplished for any linear control system from only frequency-domain information about the open loop system. For a single-input, single-output unity feedback system, the bound can be determined simply by using an overlay (Figure 3) on the $G(j\omega)$-plane plot of the system. The problem arises when one wishes to verify that the suboptimal system satisfies the necessary conditions of Theorem III-2. The Verification Procedure involves determining a state variable representation, calculating the L and Q matrices, and determining for what values of x_0 the quasi-Schwarz hold (condition CC1, Theorem III-2). This process approaches the impossible (unless the Q matrix is either positive or negative semi-definite) as the order of the system increases. Unfortunately the $G(j\omega)$ plot of many useful systems lies in both the left half and right half of the $G(j\omega)$ plane (this requires an indefinite Q matrix), resulting in a difficult Verification Procedure. However one should not let this problem prohibit him from obtaining useful results from the predicted bound.

By considering Figure 3, one can see that improving the predicted bound for a well-behaved unity feedback system is synonymous with improving the gain margin and/or the phase margin. In addition, unlike the gain and phase margins, the predicted bound of Theorem III-2 takes into consideration the entire locus of $G(j\omega)$ in the complex plane. One way in which the predicted bound might be used is

for series compensation of unity feedback systems. An algorithm for doing this is presented in the following section.

A. Presentation of the Compensation Algorithm

It has been suggested [9] that the time response and relative stability of the linear regulator is desirable as a solution to many problems. Much work has been devoted to the design of systems whose response approximates that of the optimal system. Suboptimal systems which estimate unmeasurable states and systems whose response is optimal for some subset of the state variables of the problem have been designed with no means of determining how chose the final result is to the optimal system. The bound determined by Theorem III-2 suggests a measure of the suboptimality of the designed procedure but unfortunately the sufficient conditions are difficult to verify. However, for a unity feedback system, we are assured that for a well-behaved system the higher the lower bound, the better the gain and phase margin. Hence for a unity feedback system the following procedure for determining a series compensation is proposed. A block diagram of the compensated system is shown in Figure 8 where $G_p(s)$ is assumed to be the plant and $G_c(s)$ is the compensator which is to be determined.

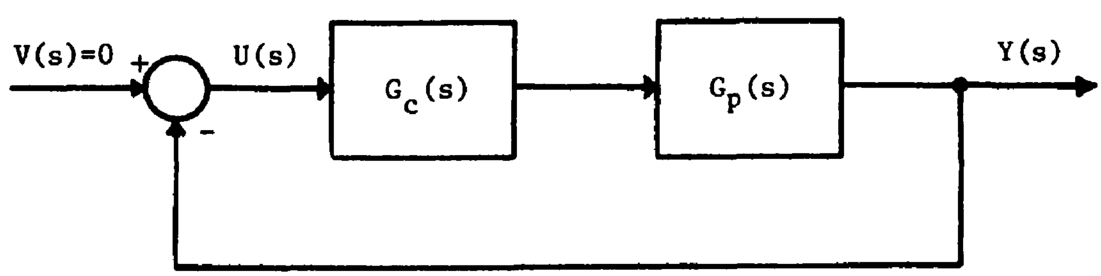


Figure 8. Series compensated unity feedback system

In the rest of this chapter, the term "predicted lower bound " is used to indicate the bound predicted by Theorem III-2 for $\varepsilon = 0$.

Compensation Procedure

1. Choose a structure for the compensator, and initialize all compensator parameters.

2. Determine the predicted lower bound.

3. Vary each of the compensator parameters in some orderly fashion; and determine the predicted lower bound at each step.

4. Redefine compensator parameters to be the values which obtain the highest predicted lower bound, and repeat step 3 until the precedure converges.

There are of course several problems which can develop with the above process. Two of these problems might be local minimum (or sensitivity to initial parameters of the compensator) and non-convergence in step 4. The first problem can usually be handled by considering several starting points. The second can often be solved by considering system constraints. In fact if there exist some constraint (such as bandwidth) , it would be desirable to incorporate it into the determination of a compensator.

The equation needed to determine the predicted lower bound are equation III-115 (for $\varepsilon = 0$)

$$J_*(x_0) \geq \min \left[\frac{1}{1-\alpha_1^2} J_D(x_0), \quad \frac{1-2\alpha_2^2}{1-\alpha_2^2} J_D(x_0) \right] \qquad \text{(IV-1)}$$

equation III-113

$$-\alpha_1^2 \leq W(j\omega) \leq \alpha_2^2 \tag{IV-2}$$

and equation III-42 ($H(j\omega) = 1$ for all ω)

$$W(j\omega) = \frac{1 + |G(j\omega)|^2}{|1 + G(j\omega)|^2} - 1 \tag{IV-3}$$

As was done in section D of the last chapter, the quanity $\dfrac{1}{1 + \alpha_2^2}$ will be referred to as the α_2-bound, while the function $\dfrac{1 - 2\alpha_1^2}{1 + \alpha_1^2}$ will be called the α_1-bound.

Using equation IV-1. IV-2, and IV-3, a subroutine can be written to calculate the predicted lower bound, given the open loop poles and zeroes of the transfer function

$$G(j\omega) = G_c(j\omega)\ G_p(j\omega) \tag{IV-4}$$

A generalized flow chart of such a routine is given in Figure 9 (a, b, c). Notice that two provisions in addition to the determination of the predicted lower bound are provided by this routine. One is a section to determine if the close loop system would be stable. This is accomplished by examining the minus one crossings of the $Re\{G(j\omega)\}$ to see if the Nyquist plot encloses the minus one point. If the system proves to be unstable, the bound is returned as a negative number. A second section was provided in order to penalize the predicted lower bound when some prespecified inequality constraint is not met. This second provision may or may not be used depending on the particular problem at hand. No provision was made to consider

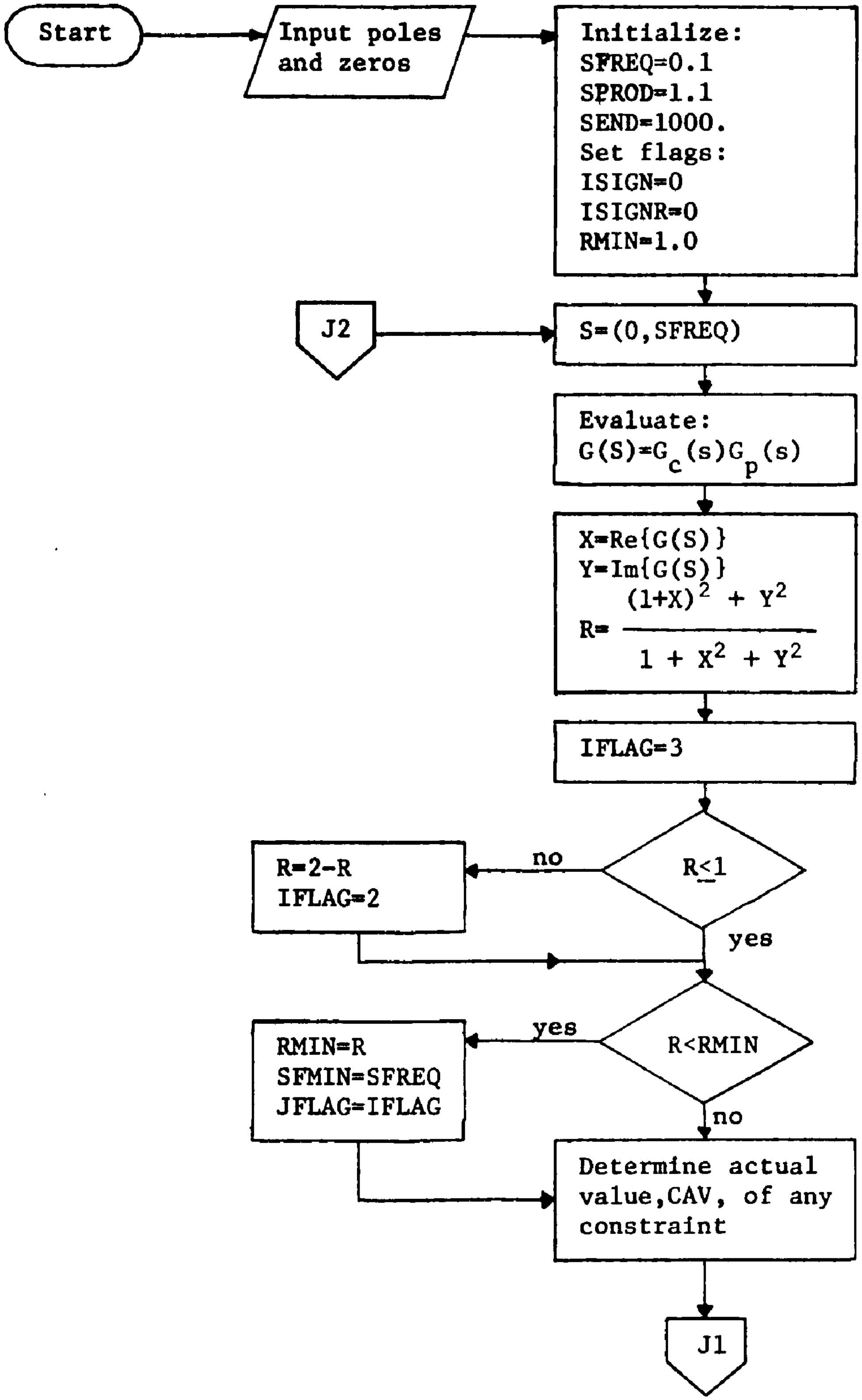


Figure 9a. Generalized flowchart of subroutine to determine the predicted lower bound

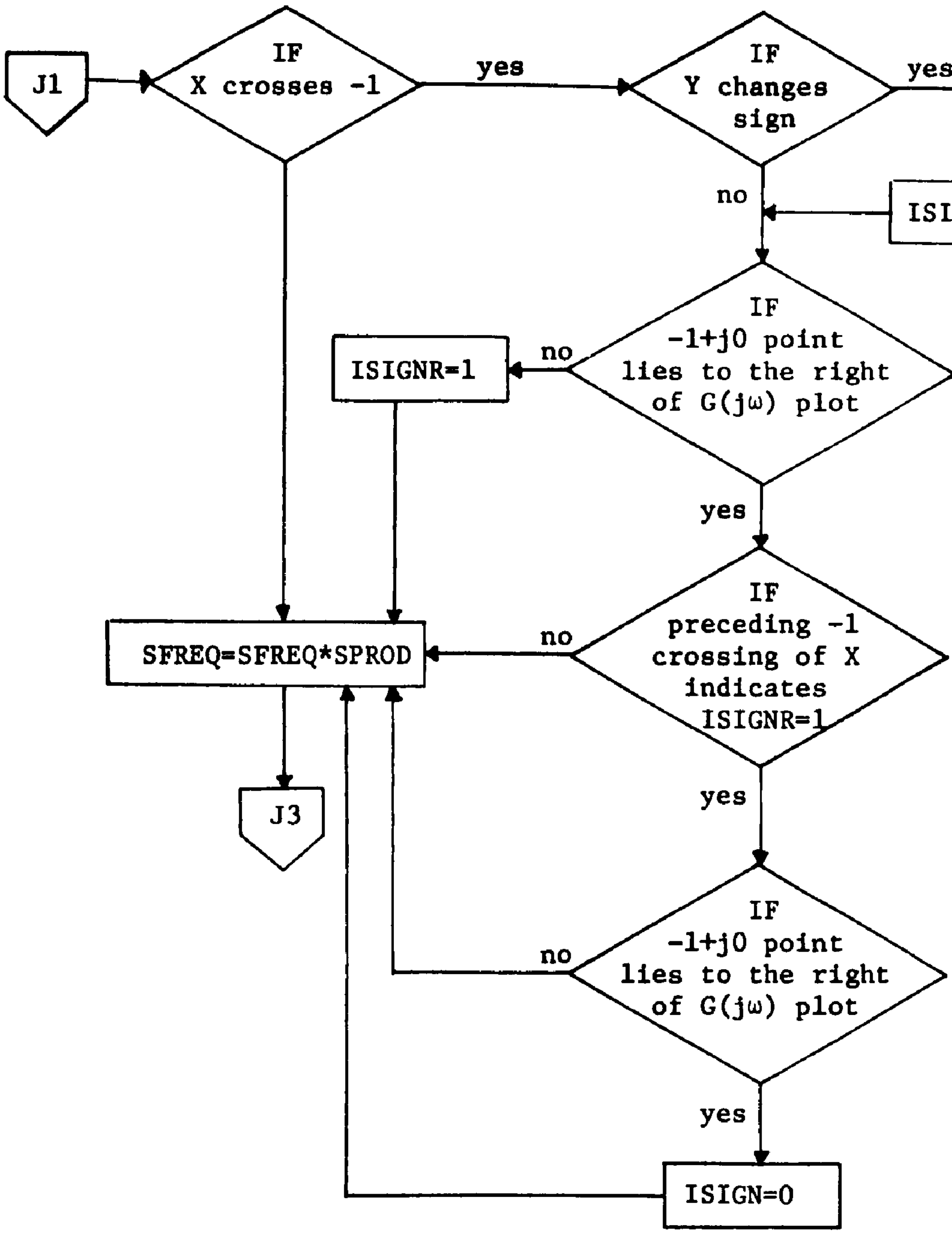
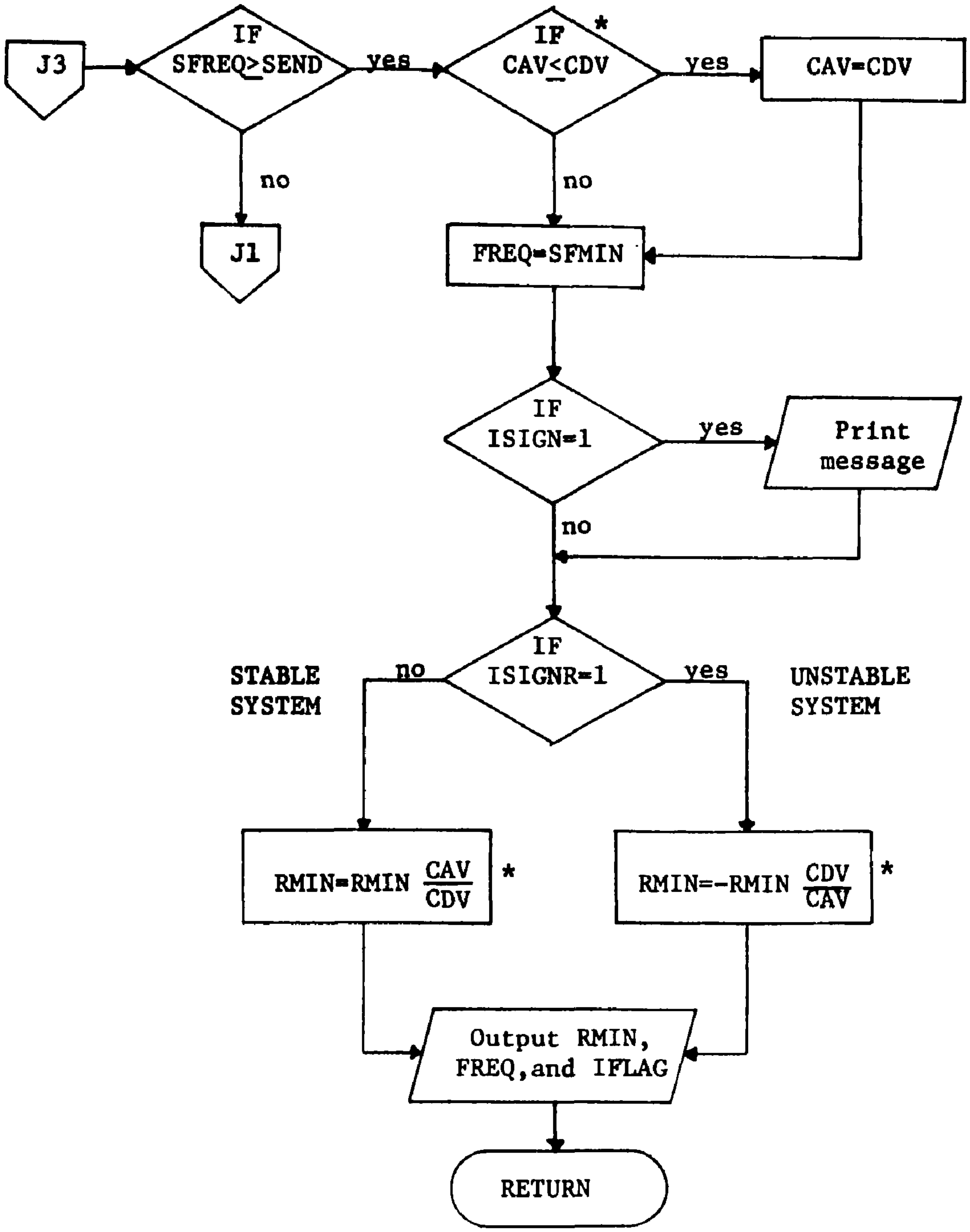


Figure 9b. Generalized flowchart of subroutine to determine the predicted lower bound (continued)



Note, CDV is desired value of constraint. As shown here, the actual value is constrained to be less than the desired value; if the opposite inequality is needed, it is necessary only to take the reciprocal of the * statements.

Figure 9c. Generalized flowchart of subroutine to determine the predicted lower bound (continued)

the folding of the α_1-bound that can occur when the plus one point is enclosed by G(jω) plane plot. This was deemed unnecessary since this condition reflects itself as local minimum. This subroutine can be used to evaluate the predicted lower bound of step 2 and step 3 in the Compensation Procedure.

There have been many search routines developed which would choose the best set of compensator parameters in order to obtain the maximum predicted lower bound. However, a straight-forward approach was taken in the examples in the following section. Each parameter or set of parameters were varied in turn by some determined increment. Step 4 then simply sets the variable parameters to those which gave the largest predicted lower bound. Convergence is accomplished by decreasing the increment of change each time no improvement is made in the predicted lower bound.

B. Application of the Compensation Procedure

The procedure presented in part A can be used to compensate control systems. For each problem in which the subroutine of Figure 9 is to be used, it will be necessary to choose these parameters: initial starting frequency, SFIRST, the terminating frequency, SEND, and the multiple step size, SPROD. For some problems, system constraints need to be specified and incorporated into the program. After the above program parameters have been selected, one then needs to choose the structure of the compensator and initialize all the structural parameters. The parameter variation procedure can then be programmed and the problem run on a computer. The above process is probably best illustrated by a few examples.

Example 1

Problem statement for Example 1: Consider the unity feedback system of Problem 2 given in section III-D. Choose the parameter α using the algorithm outlined in section IV-A.

Solution: The form of Problem 2 is not quite that shown in Figure 8. However if $G(j\omega)$ in equation IV-3 is taken to be the plant transfer function, the algorithm in section IV-A may still be used. The plant transfer function for Problem 2 is given in equation III-115 as

$$G(s) = \frac{2s - 1}{s^2 + 2(\alpha - 1)s + (5 - \alpha)}$$

Using this equation and the subroutine of Figure 9 (without any constraints), the algorithm of section IV-A was programmed as a one-parameter search. The FORTRAN results indicated that the best choice of α was $\alpha = 0.96422$, for which the predicted lower bound was 0.53199. The program was written in double precision and executed in about 13 seconds CPU time on the IBM 360/65. These results agree with those in section III-D. A Nyquist plot of the transfer function is shown in Figure 10 (a and b). The frequency 0.49 is marked. This is the frequency at which the α_2-bound is equal to 0.53199. Further insight can be obtained by comparing Figure 10a to Figure 3 in Chapter III. In this case the predicted lower bound is obtained by determining the largest constant ratio circle which can be contained inside the frequency response of Figure 10a.

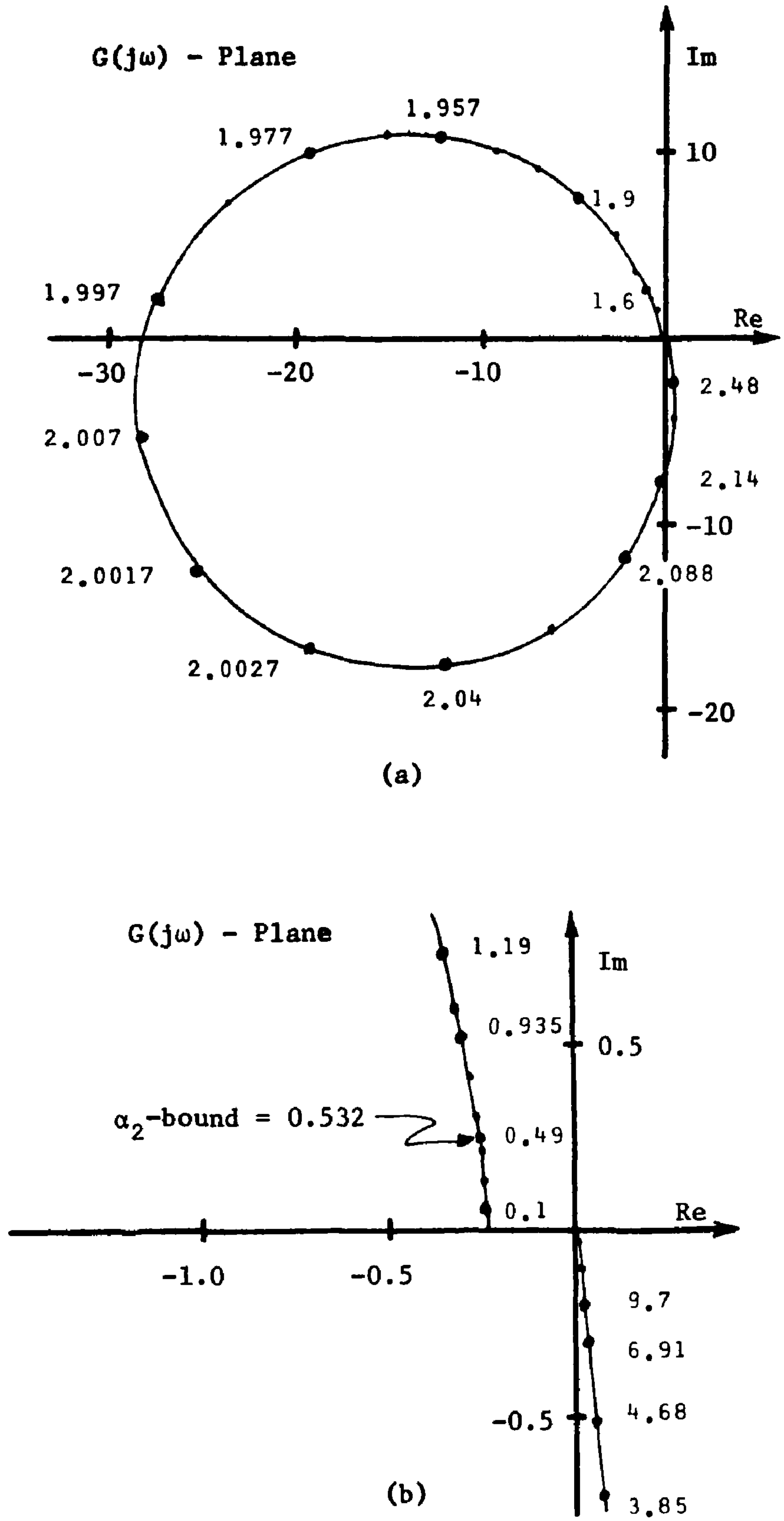


Figure 10. Nyquist plot of system in Example 1 for $\alpha = 0.96422$

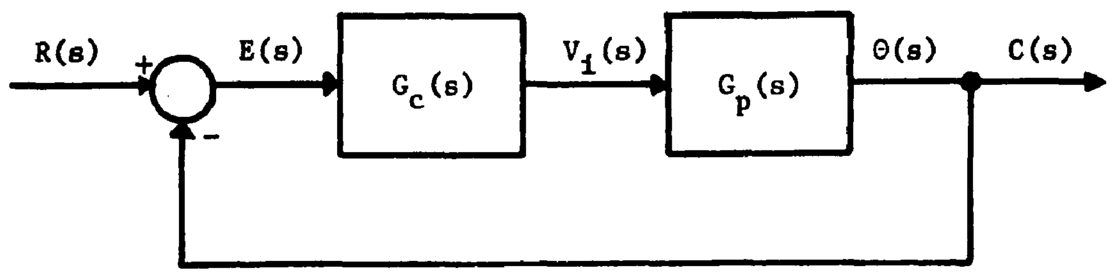


Figure 11. Block diagram of the system considered in Example 2

Example 2

The transfer function of a given power amplifier and motor combination, which is to be used for shaft position control in the configuration shown in Figure 11, is found to be

$$G_p(s) = \frac{\Theta(s)}{V_i(s)} = \frac{250}{s(1 + s/10)} \tag{IV-5}$$

This specific example was first used by Dr. V. W. Eveleigh [4] for which he determined using classical techniques the lag compensator

$$G_c(s) = \frac{1 + s/1.25}{1 + s/0.02} \tag{IV-6}$$

and a lead compensator

$$G_c(s) = \frac{1 + s/30}{1 + s/225} \tag{IV-7}$$

in order to obtain a 50° phase margin.

Problem statement for Example 2: Given the plant transfer function in equation IV-5 and the compensator structure

$$G_c(s) = \frac{p_c}{z_c} \frac{s + z_c}{s + p_c} \tag{IV-8}$$

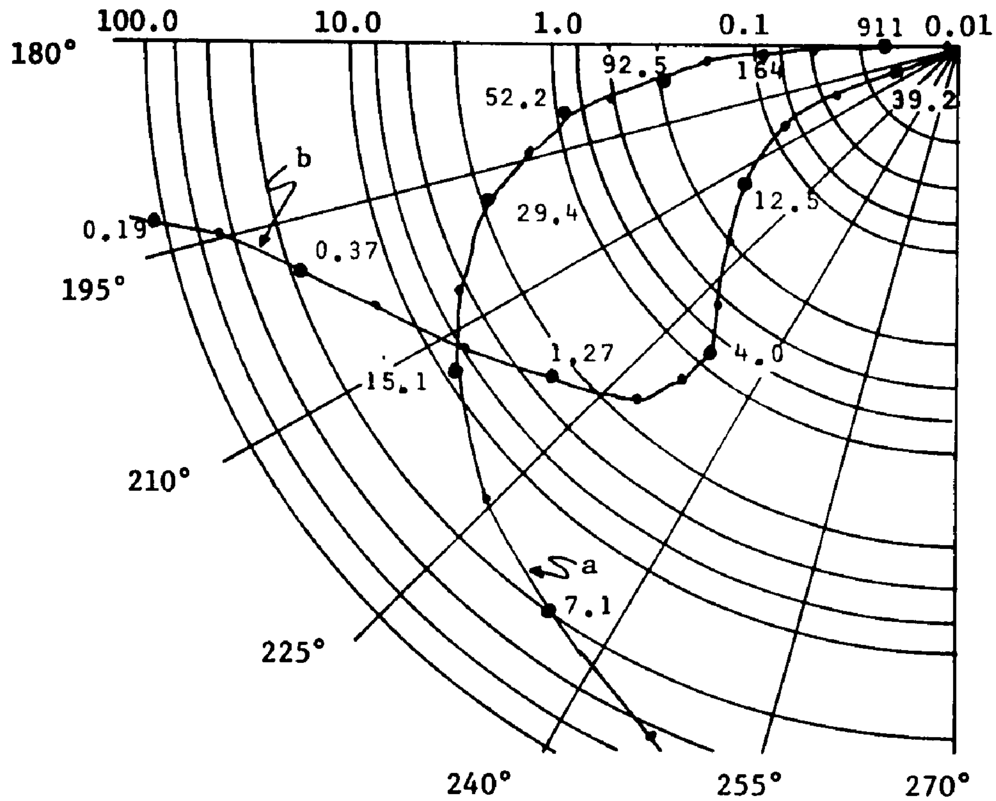
choose the compensator pole, p_c, and zero, z_c, by using the algorithm of section IV-A.

Solution: This problem is of the same form as that in Figure 8. Using the algorithm of section IV-A and a subroutine (with no constraints) such as that presented in Figure 9 ($G_p(s)$ is given in equation IV-5 and $G_c(s)$ is given by equation IV-8), a two-parameter search routine was programmed in order to find an appropriate choice for z_c and p_c. Several interesting aspects occur in this problem.

For some initial choices of p_c and z_c the algorithm converged to a local minimum at

$$G_c(s) = \frac{1 + s/1.2375}{1 + s/0.016875} \tag{IV-9}$$

Note that this compensation is very close to the lag compensation in equation IV-6. For the uncompensated system the predicted lower bound was 0.02321, while the compensation in equation IV-9 gave 0.3778 for the predicted lower bound. Hence this local minimum indeed offers considerable improvement (in the sense of predicted lower bound) over the uncompensated system. However there is very little difference between the result for the compensator of equation IV-9 and the classically determined compensation of equation IV-6, whose predicted lower bound is 0.3725. The frequency response of the uncompensated system and the system compensated by the compensator in equation IV-9 are shown in Figure 12. From this plot it is seen that for the compensated system the gain cross over frequency (frequency at which gain



a - Uncompensated system

b - System compensated by compensator in equation IV-9

Figure 12. Polar plot for local minimum of unconstrained system in Example 2

equal 1) occurs near ω =3.4, at which the phase margin is about 51°. The frequency response of the system, using the compensator of equation IV-6, is approximately the same as curve b in Figure 12.

For other initial choices of p_c and z_c the algorithm did not converge. For each step in the search procedure, changes in the values of p_c and z_c were able to improve the predicted lower bound. Some of the frequency response plots of these steps are shown in Figure 13. In this figure, curve a is the frequency response of the uncompensated system, curve b is determined for the compensator

$$G_c(s) = \frac{1 + s/25.5}{1 + s/156} \tag{IV-10}$$

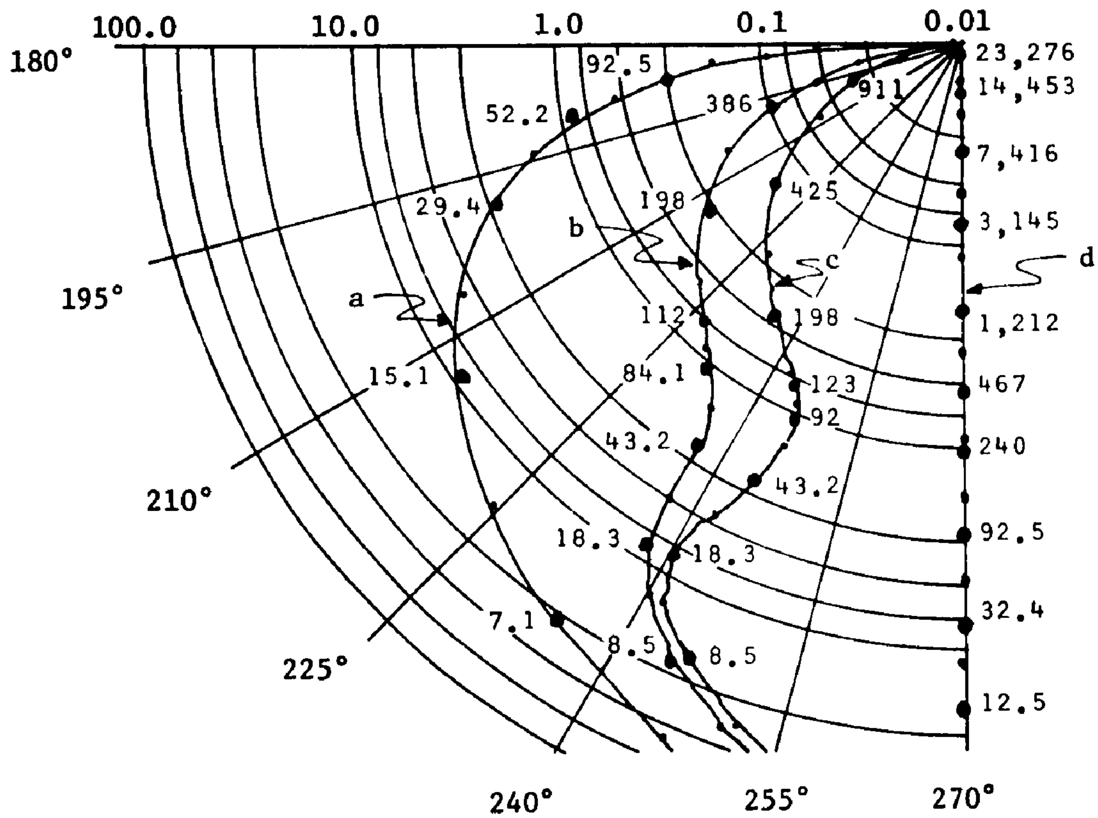
curve c results when the compensator is

$$G_c(s) = \frac{1 + s/26.5}{1 + s/357} \tag{IV-11}$$

and curve d becomes the frequency response when

$$G_c(s) = \frac{1 + s/10.03}{1 + s/1,123,818.5} \tag{IV-12}$$

The predicted lower bounds are 0.0232, 0.3474, 0.5734, and 0.9996, respectively. The gain cross over frequencies are approximately 52 rad/sec, 86 rad/sec, 92 rad/sec, and 241 rad/sec, respectively, while the phase margins are approximately 11°, 51°, 75°, and 90°, respectively. It is seen that the increasing improvement in the predicted lower bound results in increasing the system bandwidth and phase margin. Because of noise or some system limitation this



a - Uncompensated system

b - System compensated by compensator in equation IV-10

c - System compensated by compensator in equation IV-11

d - System compensated by compensator in equation IV-12

Figure 13. Polar plot of intermediate compensators in search routine for the unconstrained system of Example 2

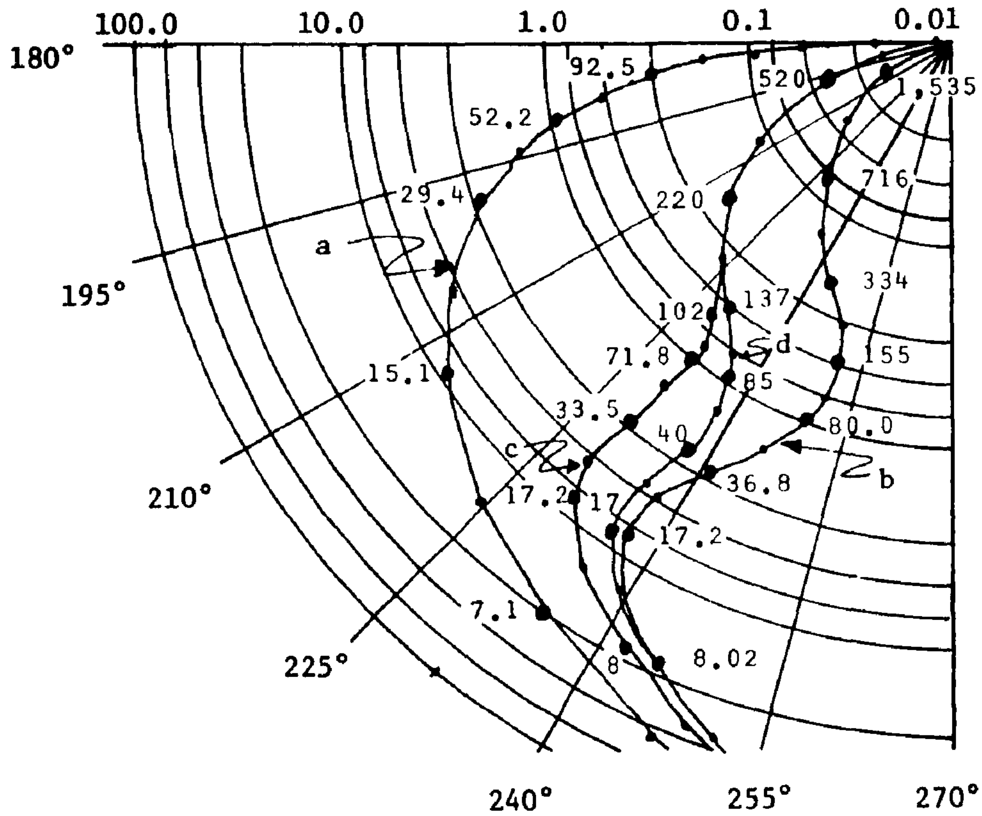
is unacceptable for many systems. One is thus led to seek some means of imposing system constraints into the search procedure. One means of doing this, if the bound is exceeded, is to multiply the predicted lower bound by the ratio of the desired value of the constraint to the actual value of the constraint before the choice is made as to the best parameter values in step 4 of the Compensation Procedure of section IV-4. If the bound is not exceeded no change is made in the predicted lower bound.

Using the procedure outlined above, the gain cross over frequency was constrained to occur at 80 rad/sec. The algorithm then converged for all initial values of p_c and z_c tried. The best compensator found was

$$G_c(s) = \frac{1 + s/33.44}{1 + s/800.7} \qquad \text{(IV-13)}$$

for which the predicted lower bound was 0.61623. A frequency-domain polar plot for the plant in equation IV-5 with the compensator in equation IV-13 is given as curve b in Figure 14. Curve a again represents the uncompensated system.

Instead of constraining bandwidth it might be desirable to constrain the phase margin to be less than some value, say 50°. Using the same procedure as for the bandwidth constraint, a program was written and executed. However the resulting system had a phase margin of nearly 70° and was unacceptable. The constraint was strengthened by multiplying the predicted lower bound by the ratio of the desired constraint minus half the actual phase margin to half the actual phase margin. The resulting compensator then was



a - Uncompensated system

b - System compensated by compensator in equation IV-13

c - System compensated by compensator in equation IV-14

d - System compensated by compensator in equation IV-7

Figure 14. Polar plot of compensators for constrained system of Example 2

$$G_c(s) = \frac{1 + s/36.45}{1 + s/202.3} \qquad \text{(IV-14)}$$

The frequency response is shown as curve c of Figure 14, from which it can be seen that the phase margin is approximately 51°. The frequency response of the classically determined lead compensator, equation IV-7, is shown as curve d in Figure 14.

Example 3

As a final example, consider the 18 order stiff sample data system shown in block diagram form in Figure 15 (a,b, and c). This block diagram represents a highly accurate positioning system for which it is desired for the gain to be greater than 10^6 for all frequencies less than 0.2 rad/sec in order that the steady state error will be small. The high values of gain cause difficult problems in determining a stable system. Compensation of such a system by any method is no simple task.

Problem statement for Example 3: For the system whose block diagram is shown in Figure 15, determine a digital compensator using the algorithm outlined in section IV-A.

Solution: The system as shown in Figure 15 is not in a form that can be easily handled by the prescribed algorithm. In order to get the system into more of a tactful form, it is reduced using block diagram reduction techniques to that shown in Figure 16 where

$$G(s) = \frac{K \prod_{i=1}^{5}(1 - s/z_i)}{s^2 \prod_{i=1}^{15}(1 - s/p_i)} \qquad \text{(IV-15)}$$

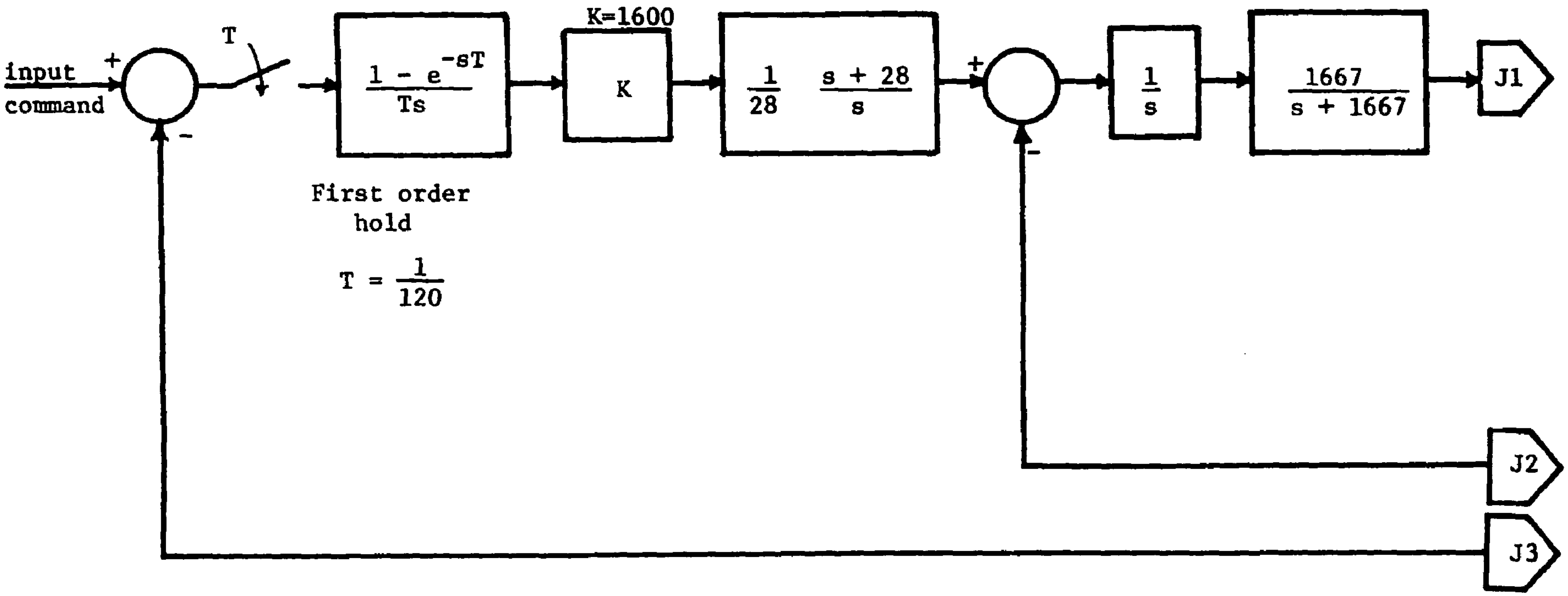


Figure 15(a). Block diagram of system for Example 3

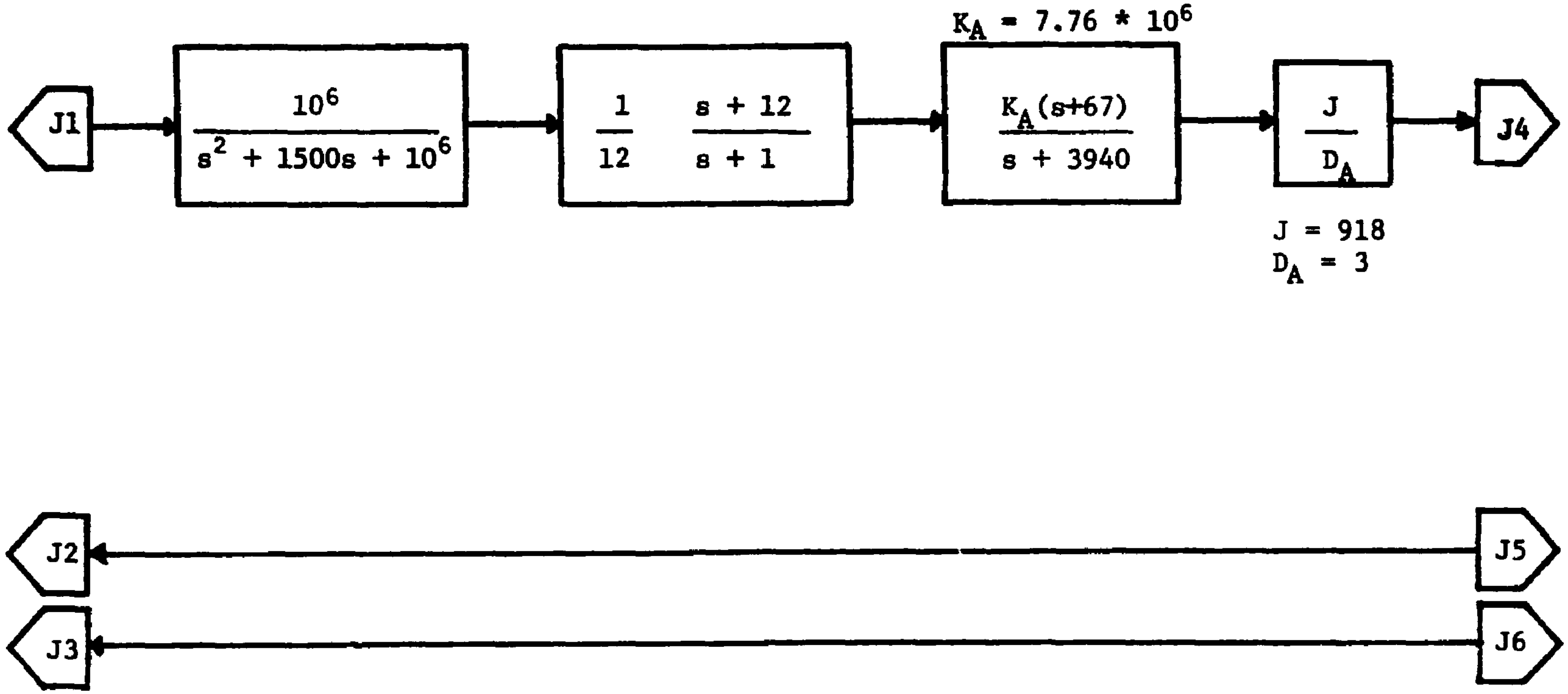
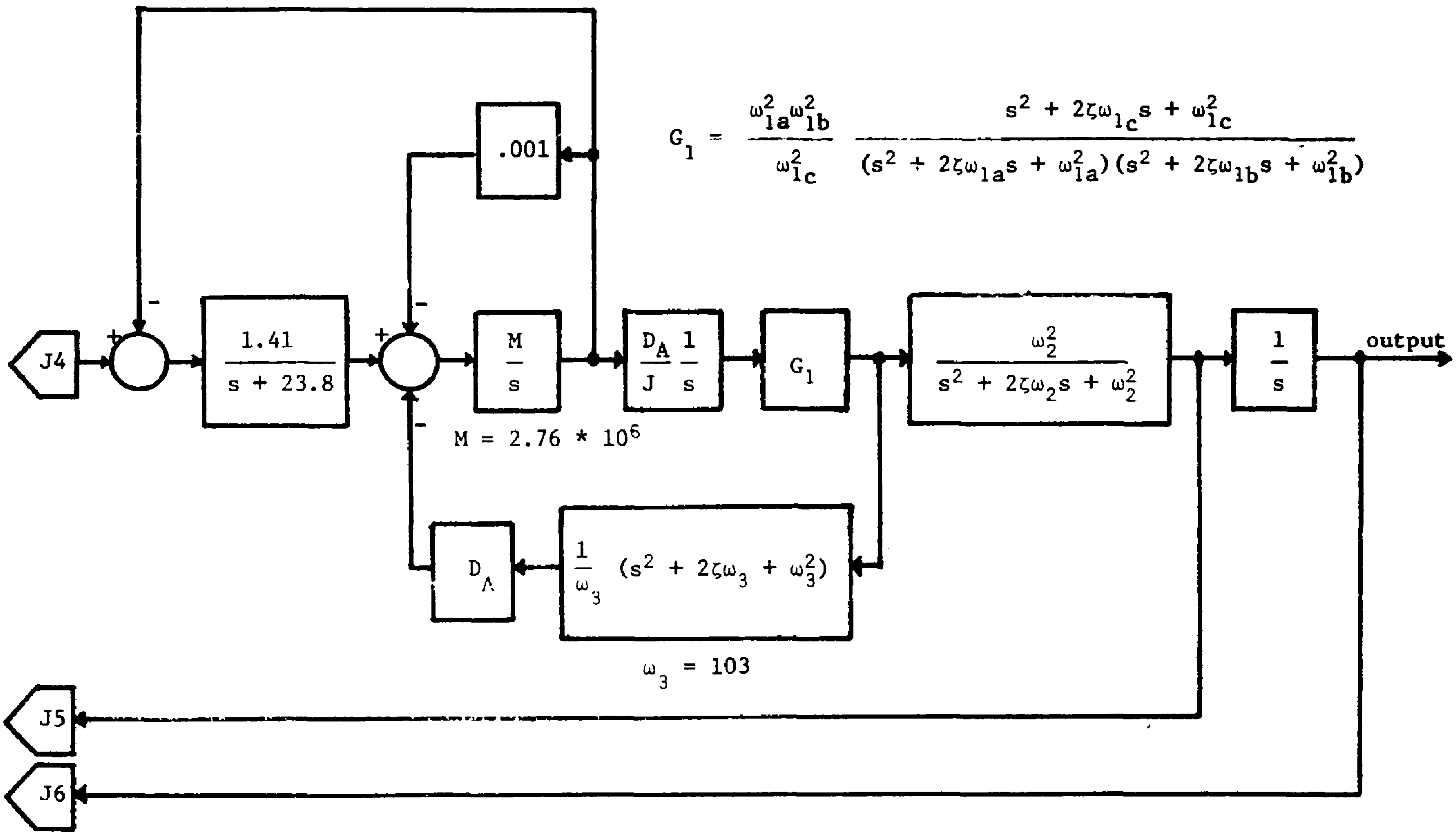


Figure 15(b). Block diagram of system for Example 3 (continued)

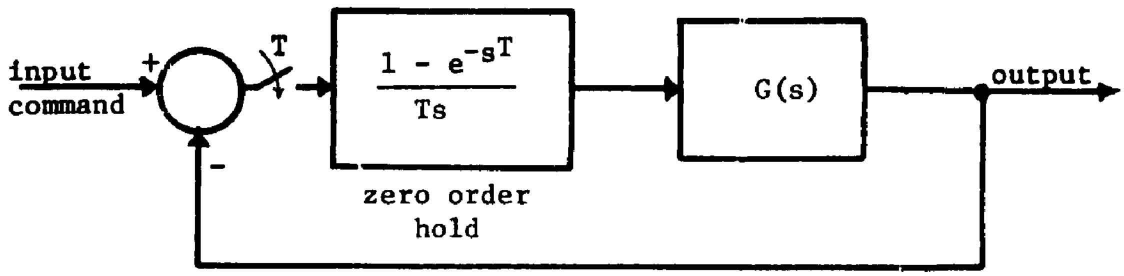


$$G_1 = \frac{\omega_{1a}^2 \omega_{1b}^2}{\omega_{1c}^2} \frac{s^2 + 2\zeta\omega_{1c}s + \omega_{1c}^2}{(s^2 + 2\zeta\omega_{1a}s + \omega_{1a}^2)(s^2 + 2\zeta\omega_{1b}s + \omega_{1b}^2)}$$

$\zeta = 0.05$ $\quad \omega_{1a} = 2{,}070$ $\quad \omega_{1c} = 1{,}695$

$\omega_{1b} = 39{,}000$ $\quad \omega_2 = 3{,}520$

Figure 15(c). Block diagram of system for Example 3. (continued)



$$T = \frac{1}{120}$$

Figure 16. Reduced block diagram for Example 3

with

$$p_1 = -1{,}991.4 + j\ 45{,}786.3$$

$$p_2 = -1{,}991.4 - j\ 45{,}786.3$$

$$p_3 = -176.01 + j\ 3{,}515.51$$

$$p_4 = -176.01 - j\ 3{,}515.51$$

$$p_5 = -88.1 + j\ 1{,}965.13$$

$$p_6 = -88.1 - j\ 1{,}965.13$$

$$p_7 = -4{,}020.12$$

$$p_8 = -1{,}641.60$$

$$p_9 = -655.9 + j\ 637.9$$

$$p_{10} = -655.9 - j\ 637.9$$

$$p_{11} = -99.67 + j\ 401.96$$

$$p_{12} = -99.67 - j\ 401.96$$

$$p_{13} = -125.55 + j\ 94.4$$

$$p_{14} = -125.55 - j\ 94.4$$

$$p_{15} = -11.84$$

$z_1 = -12.$

$z_2 = -67.$

$z_3 = -169.5 + j\ 1686.5$

$z_4 = -169.5 - j\ 1686.5$

$z_5 = -28.$

and

$K = 1600$

The z-transform of the system shown in Figure 16 can be taken using the standard z-transform technique [6, 8] and results in the system shown in Figure 17a where

$$D(z) = \frac{Kg(T) \prod_{i=1}^{16} (z - \beta_i)}{(z-1)^2 \prod_{i=1}^{15} (z - \alpha_i)} \tag{IV-16}$$

where

$\alpha_1 = -0.9170 * 10^{-8} + j\ 0.6140 * 10^{-7}$

$\alpha_2 = -0.9170 * 10^{-8} - j\ 0.6140 * 10^{-7}$

$\alpha_3 = -0.1204 + j\ 0.1967$

$\alpha_4 = -0.1240 - j\ 0.1967$

$\alpha_5 = -0.3767 + j\ 0.2973$

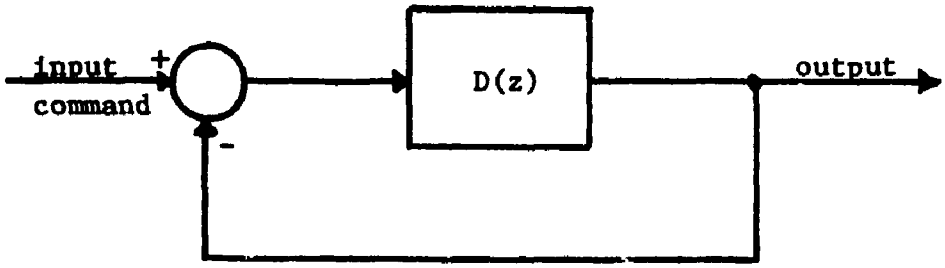
$\alpha_6 = -0.3767 - j\ 0.2973$

$\alpha_7 = 0.2823 * 10^{-14}$

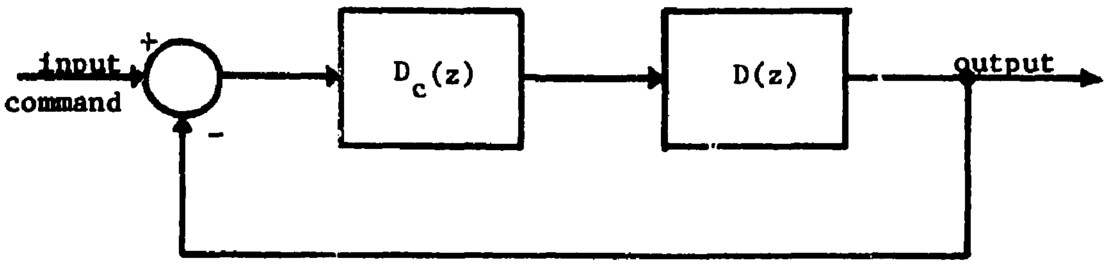
$\alpha_8 = 0.1145 * 10^{-5}$

$\alpha_9 = 0.2401 * 10^{-2} + j\ 0.3481 * 10^{-2}$

$\alpha_{10} = 0.2401 * 10^{-2} - j\ 0.3481 * 10^{-2}$



a. Uncompensated system



b. Compensated system

Figure 17. Block diagram of z-plane system of Example 3

$$\alpha_{11} = -0.4264 + j\ 0.09003$$

$$\alpha_{12} = -0.4264 - j\ 0.09003$$

$$\alpha_{13} = 0.2481 + j\ 0.2487$$

$$\alpha_{14} = 0.2481 - j\ 0.2487$$

$$\alpha_{15} = 0.9061$$

$$\beta_1 = -0.9170 * 10^{-8} + j\ 0.6140 * 10^{-7}$$

$$\beta_2 = -0.9170 * 10^{-8} - j\ 0.6140 * 10^{-7}$$

$$\beta_3 = -0.8753 * 10^{-9}$$

$$\beta_4 = -0.1030 * 10^{-3}$$

$$\beta_5 = 0.2473 * 10^{-2} + j\ 0.6013 * 10^{-2}$$

$$\beta_6 = 0.2473 * 10^{-2} - j\ 0.6013 * 10^{-2}$$

$$\beta_7 = -0.1204 + j\ 0.1967$$

$$\beta_8 = -0.1204 - j\ 0.1967$$

$$\beta_9 = -0.3567 + j\ 0.1635$$

$$\beta_{10} = -0.3567 - j\ 0.1635$$

$$\beta_{11} = -0.3767 + j\ 0.2973$$

$$\beta_{12} = -0.3767 - j\ 0.2973$$

$$\beta_{13} = 0.5727$$

$$\beta_{14} = 0.9048$$

$$\beta_{15} = 0.9048$$

$$\beta_{16} = -6.8364$$

$$g(T) = 0.6747 * 10^{-4}$$

and

$$K = 1600.$$

Numerical problems occur in the determination of the z-plane zeros. These problems were eliminated by writing FORTRAN routines for extended precision arithmetic [1]. By this means it was possible to carry all calculations to 45 significant figures of accuracy.

The system shown in Figure 17a is similar to that for which the compensation algorithm of section IV-A was derived. Hence a series digital compensator was added (see Figure 17b), and the compensation procedure of section IV-A was used. The structure of the digital compensator was chosen as

$$D_c(z) = \frac{(1 - \alpha_a)(1 - \alpha_b)}{(1 - \beta_a)(1 - \beta_b)} \quad \frac{(z - \beta_a)(z - \beta_b)}{(z - \alpha_a)(z - \alpha_b)} \qquad \text{(IV-17)}$$

The above redefines the problem of Example 3 as follows: For the system shown in Figure 17b ($D(z)$ is given by equation IV-16), choose a set of parameters (β_a, β_b, α_a, α_b) for $D_c(z)$ given in equation IV-17 by using the algorithm of section IV-A. If the digital transfer function

$$D(z) = D_c(z)\, D_p(z)$$

is used in place of $G(s)$ in equation IV-4, the algorithm of section IV-A can be used directly as long as $D(z)$ is evaluated for values of z on the unit circle, or if the transformation

$$z = \frac{1 + w}{1 - w}$$

is used; then the transfer function is evaluated for w-plane values along the imaginary axis [6, 8]. Using the more stringent constraint that the gain remain greater than 10^6 for all w-plane frequencies below 0.001, the problem was programmed using double precision and executed for several different starting values of α_a, α_b, β_a, and β_b.

In order to decrease the number of arbitrary parameters to be selected, the starting compensator was chosen so that initially $\alpha_a = \alpha_b = \beta_a = \beta_b = \eta$, where η is some real number. From equation IV-17, it is seen that this initial compensator is then unity regardless of the value of η. Some means needs to be determined in order to specify this initial value. One method of selecting an initial compensator is to start with a parameter value, which is near a critical frequency such as: the frequency which determines the predicted lower bound, or where some constraint is imposed (for this problem at w = 0.001). There is, of course, no set rule which will work for all systems, and one must

use the experience gained as work progresses on a problem.

The search routine constrained the z-plane poles of the compensator to lie inside the unit circle in order that the system would be stable. The manner in which the search was conducted is sketched in the sequel. Two convergence factors Δ_p and Δ_z were chosen, and the magnitude of each pole determined. If the poles are real, then the predicted lower bound is calculated by using for α_a and α_b in equation IV-17 each of four possible sets for α_a and α_b determined by $\alpha_a = \alpha_a' \pm |\alpha_a'|^2 \Delta_p$ and $\alpha_b = \alpha_b' \pm |\alpha_b'|^2 \Delta_p$, where α_a' and α_b' are the last values of α_a and α_b. The largest predicted lower bound is chosen. The term, predicted lower bound, as used here indicates only the number given by using the subroutine outlined in Figure 9. Without further investigation no other meaning can be given to this term for the system being discussed. If the poles are complex, then the four points which are used are the angle, increased and decreased by an amount Δ_p for constant magnitude of the two poles, and the magnitude of α_a and α_b, increased and decreased by an amount $|\alpha_a|^2 \Delta_p$. A means is provided which allows the poles to change from real to complex or complex to real. If none of the four points provides any improvement in the predicted lower bound, the convergence factor Δ_p is divided by 2 and the process repeated. After the poles have completed one step, the zeros are then varied in exactly the same manner as above, except Δ_z, β_a, and β_b are used instead of Δ_p, β_a, and β_b. The entire process is continued until both convergence factors Δ_p and Δ_z are less than some chosen value.

The best value for the predicted lower bound for all initial values of poles and zeros considered resulted when

$$\alpha_a = 0.9985 + j\ 0.1092 * 10^{-3}$$

$$\alpha_b = 0.9985 - j\ 0.1092 * 10^{-3}$$

$$\beta_a = 0.9980 + j\ 0.4074 * 10^{-3}$$

$$\beta_b = 0.9980 - j\ 0.4074 * 10^{-3} \qquad \text{(IV-18)}$$

The CPU time for the problem to converge varied depending on the starting point and nearest local minimum, but a typical value was between one and two minutes on the IBM 360/65. The value of the predicted lower bound using the compensator given by equations IV-17 and IV-18 is 0.2806 while the predicted lower bound for the uncompensated system was only 0.0456. A Nyquist plot of both the uncompensated system (curve a) and the system compensated as indicated above (curve b) is given in Figure 18.

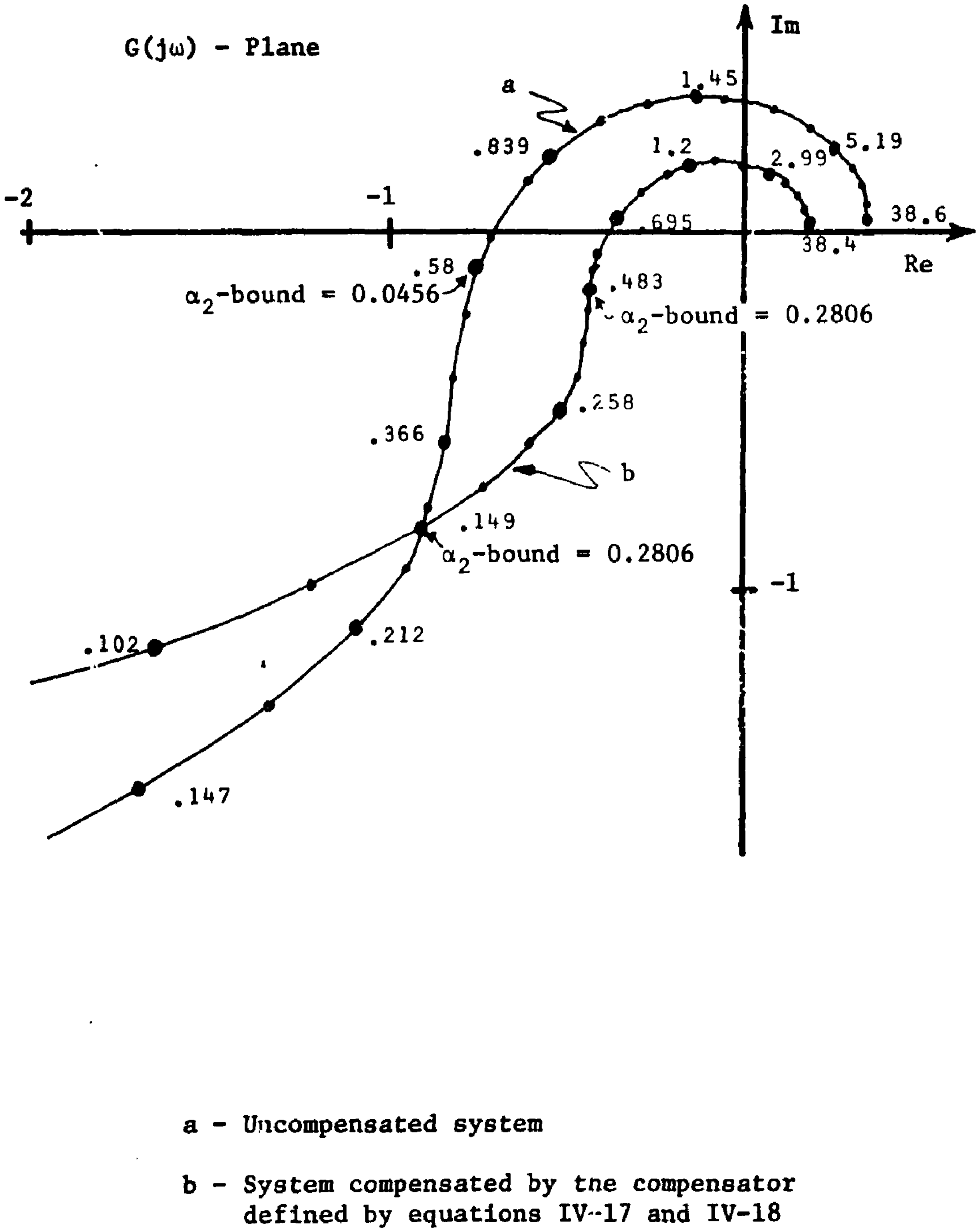


Figure 18. Nyquist plot of system given in Example 3

V. CONCLUSIONS

The work up to Theorem III-2 is devoted to determining a lower bound on the ratio of the optimal cost to the cost of a suboptimal control law. This bound is easily determined using only a frequency domain description of the system. However, the verification that the bound is actually a lower bound on the ratio of the optimal cost to the cost using a suboptimal control law can be a difficult task.

For a class of system determined by the positive or negative semi-definiteness of a certain matrix (Q-matrix, see condition CC1, page 23), it is relatively easy to verify the validity of the determined lower bound. This class of problems has the characteristic that the Nyquist plot lies completely in either the left half or the right half of the $G(j\omega)$-plane. The lower bound can then be verified and even improved by testing the positive definiteness of some easily determined matrices. The main problem for this case may be the determination of the state space representation of the open loop system.

In the case where the Nyquist plot crosses the imaginary axis, it becomes necessary to examine a condition (the quasi-Schwarz condition, equation III-9) which is difficult to verify for arbitrary initial values of the state variables. Regardless of this fact, several interesting results can be obtained by considering the derived lower bound.

The work after Theorem III-2 presents conclusions and applications derived by considering the lower bound. It is demonstrated that for

some systems, even though the Nyquist plot crosses the imaginary axis, it is possible to verify all sufficient conditions to insure that the derived lower bound represents a lower bound on the ratio of the optimal cost to the cost for a system using a suboptimal control law. Most of the work presented after III-2 is confined to single-input, single-output unity feedback systems. For such systems, the lower bound can be determined simply by using an overlay (Figure 3) on the Nyquist plot of the system.

An algorithm is presented which will determine a series compensator for a unity feedback system. The compensator is chosen to give the largest predicted lower bound. Under certain circumstances the algorithm terminates with a compensator which is close to that which might be designed using classical control techniques.

There are many other possible uses for the lower bound derived in Theorem III-2. One such use is as an estimate of how optimal is a suboptimal control law. This could be helpful in determining the desirability of using an observer or adding physical equipment in some process in order to determine unmeasurable or difficult-to-measure states. The lower bound would provide a measure of how much was to be gained by increasing the complexity of the system.

The obvious problem is the difficulty in verifying the validity of the lower bound. Additional work is needed in this area to determine if there exists a large class of problems for which the sufficient conditions of Theorem III-2 hold in general. Many other areas of investigation are open, ranging from application to interpretation of the results implied by Theorem III-2.

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APPENDIX

Table 1. Calculated Bounds for Problem 1 - Section III-D

[illegible]	α_1-bound	α_2-bound	Riccati bound
0.25	1.0	0.05833	0.4541
0.5	1.0	0.2274	0.7692
0.75	1.0	0.4641	0.9122
0.8	0.999	0.5036	0.9222
0.85	0.9857	0.5320	0.9263
0.9	0.9542	0.5457	0.9250
0.95	0.9038	0.5440	0.9188
1.0	0.8367	0.5294	0.9085
1.05	0.7559	0.5119	0.8950
1.1	0.6641	0.4947	0.8788
1.15	0.5631	0.4773	0.8606
1.2	0.4545	0.4612	0.8409
1.25	0.3389	0.4449	0.8200
1.5	-0.3295	0.3676	0.7075
1.75	-1.1337	0.2978	0.5951
2.0	-2.0668	0.2353	0.4916
2.25	-3.1267	0.1801	0.3994
2.5	-4.3125	0.1324	0.3182
2.75	-5.6238	0.09191	0.2470
3.0	-7.0605	0.05882	0.1846
3.25	-8.6224	0.03309	0.1297
3.5	-10.3095	0.01471	0.08131
3.75	-12.1216	0.003676	0.03832

Table 2. Step 4 of Verification Procedure for Problem 1-Section III-D (page 55)

$\alpha = 0.25$

If $\beta = 11.5950$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.1087$ and $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.1607.

$\alpha = 0.5$

If $\beta = 9.7254$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are posi-positive semi-definite for $\varepsilon_1 = 0.3518$ and $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.4992.

$\alpha = 0.75$

If $\beta = 8.6440$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.5442$ and $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.7557.

$\alpha = 0.80$

For $\beta_1 = 5.5285$ if $\beta = \beta_1$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.4057$ and $T_x \geq 0$ for $-42.60 \leq x \leq -0.1566$ and $-0.03268 \leq x \leq 20.23$. For $\beta_2 = 7.58$ if $\beta = \beta_2$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.5224$ and $T_x \geq 0$ for $x \leq 0.4952$ and $x \geq 1.078$. The union of these sets covers all x. The improved bound using $\varepsilon = 0.4057$ in equation III-57 is 0.7050.

$\alpha = 0.85$

For $\beta_1 = 5.5363$ if $\beta = \beta_1$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.4070$ and $T_x \geq 0$ for $x \leq -16.20$, for $-12.63 \leq x \leq -0.9844$, and for $-0.9824 \leq x$. For slight perturbation from β_1 a repeated real root for equation III-105 occurs near -0.983. For $\beta_2 = 6.559$ if $\beta = \beta_2$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.4735$ and $T_x \geq 0$ for $x \leq -0.9964$, for $-0.9050 \leq x \leq 3.84$, and $134.6 \leq x$. The improved bound using $\varepsilon = 0.4070$ in equation III-57 is 0.7225.

$\alpha = 0.90$

For $\beta_1 = 4.5401$ if $\beta = \beta_1$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.3324$ and $T_x \geq 0$ for $x \leq -11.33$, for $-11.31 \leq x \leq -1.642$ and for $-1.634 \leq x$. For slight perturbation from β_1 a repeated real root for equation III-105 occurs near -1.64. For $\beta_2 = 5.5627$ if $\beta = \beta_2$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-definite for $\varepsilon_1 = 0.4070$ and $T_x \geq 0$ for $x \leq -55.12$, for $-24.94 \leq x \leq -1.699$, and for $-1.588 \leq x$. The improved bound using $\varepsilon = 0.3324$ in equation III-57 is 0.6967.

Table 2.(continued)

$\alpha = 0.95$

For $\beta_1 = -0.5510$ if $\beta = \beta_1$ the matrix $P - L$ is positive semi-definite and if $x_0 = x_1[\ x\ \ 1]'$ then $x_0'(L - \varepsilon_1 P)x_0 \geq 0$ for $-4.258 \leq x \leq 0.8098$ whenever $\varepsilon_1 = 0.0$. For this value of β $T_x \geq 0$ for $x \leq -16.71$, for $-8.313 \leq x \leq -2.264$; and for $-2.261 \leq x$. A slight perturbation in β_1 results in a repeated root of equation III-105 near -2.263. For $\beta_2 = 3.553$ if $\beta = \beta_2$ the matrix $P - L$ and the matrix $L - \varepsilon_1 P$ are positive semi-sefinite for $\varepsilon_1 = 0.2518$ and $T_x \geq 0$ for $x \leq -31.59$, for $-30.63 \leq x \leq -2.439$, and for $-2.209 \leq x$. For $\beta_3 = 5.605$ if $\beta = \beta_3$ the matrix $L - \varepsilon_1 P$ and the matrix $(2 - \varepsilon_2)P - L$ are positive semi-definite for $\varepsilon_1 = 0.4060$ and $\varepsilon_2 = 0.9892$ and $T_x \geq 0$ for $x \leq -9.17$, for $-1.8044 \leq x \leq 13.14$, and for $56.62 \leq x$. The improved bound using $\varepsilon = 0.0$ in equation III-57 is 0.5440.

Table 3. Calculated Bounds for Problem 2 - Section III-D

α	α_1-bound	α_2-bound	Riccati bound
0.25	1.0	0.03677	0.4736
0.5	1.0	0.1760	0.7652
0.75	1.0	0.4157	0.9016
0.8	0.9948	0.4609	0.9124
0.85	0.9769	0.4972	0.9180
0.9	0.9442	0.5212	0.9190
0.95	0.8967	0.5315	0.9156
1.0	0.8367	0.5294	0.9085
1.05	0.7680	0.5242	0.8986
1.1	0.6943	0.5188	0.8865
1.15	0.6187	0.5134	0.8726
1.2	0.5438	0.5078	0.8575
1.125	0.4714	0.5021	0.8416
1.5	0.1827	0.4717	0.7580
1.75	0.03717	0.4378	0.6790
2.0	0.0	0.4	0.6088
2.25	0.02318	0.3577	0.5463
2.5	0.07362	0.3103	0.4882
2.75	0.1332	0.2577	0.4314
3.0	0.1932	0.2000	0.3721
3.25	0.2498	0.1385	0.3063
3.50	0.3014	0.07692	0.2286
3.75	0.3480	0.02439	0.1307

Table 4. Step 4 of Verification Procedure for Problem 2-Section III-D (page 55)

$\alpha = 0.25$

If $\beta = 14.82$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.2157$ and since the Q matrix is positive definite $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.2646.

$\alpha = 0.5$

If $\beta = 11.81$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.3970$ and since the Q matrix is positive definite $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.5031.

$\alpha = 0.75$

If $\beta = 9.00$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.4903$ and since the Q matrix is positive definite $T_x \geq 0$ for all x. The improved bound using $\varepsilon = \varepsilon_1$ in equation III-57 is 0.7021

$\alpha = 0.8$

For $\beta_1 = 6.7428$ if $\beta = \beta_1$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.4067$ and $T_x \geq 0$ for $-11.12 \geq x$, for $-9.034 \leq x \leq -1.0122$, and for $-0.9701 \leq x$. For $\beta_2 = 7.9914$ if $\beta = \beta_2$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.4714$ and $T_x \geq 0$ for $-0.9186 \geq x$, for $-0.8906 \leq x \leq 1.165$, and for $29.37 \leq x$. The improved bound using $\varepsilon = 0.4067$ in equation III-57 is 0.6801.

$\alpha = 0.85$

For $\beta_1 = 6.4278$ if $\beta = \beta_1$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.4073$ and $T_x \geq 0$ for $-11.11 \geq x$, for $-10.43 \leq x \leq -1.4592$, and for $-1.4591 \leq x$. For $\beta_2 = 7.6134$ if $\beta = \beta_2$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.4744$ and $T_x \geq 0$ for $-49.61 \leq x \leq -14.92$ and for $-13.85 \leq x \leq 5.6$. The improved bound using $\varepsilon = 0.4073$ in equation III-57 is 0.7020.

$\alpha = 0.9$

For $\beta_1 = 3.8877$ if $\beta = \beta_1$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.2534$ and $T_x \geq 0$ for $-8.799 \geq x$, for $-7.626 \leq x \leq -1.9279$ and for $-1.9274 \leq x$. For a slight perturbation from β_1 a repeated real root for equation III-105 occurs near -1.9276. For $\beta_2 = 5.0169$ if $\beta = \beta_2$ the matrix P - L and the matrix L - ε_1P are positive semi-definite for $\varepsilon_1 = 0.3324$ and $T_x \geq 0$ for $x \leq -11.61$, for $-11.57 \leq x \leq -1.94$, and for $-1.94 \leq x$. The improved bound using $\varepsilon = 0.2534$ in equation III-57 is 0.6425.

Table 4. (continued)

$\alpha = 0.95$

For $\beta_1 = 0.5$ if $\beta = \beta_1$ the matrix $P - L$ and the matrix $L - \epsilon_1 P$ are positive semi-definite for $\epsilon_1 = 0.0$ and $T_x \geq 0$ for $x \leq -15.02$, for $-10.06 \leq x \leq -2.427$, and for $-2.423 \leq x$. For a slight perturbation from β_1 a repeated root for equation II III-105 occurs near -2.425. For $\beta_2 = 5.892$ the matrix $P - L$ and the matrix $L - \epsilon_1 P$ are positive semi-definite for $\epsilon_1 = 0.4060$ and $T_x \geq 0$ for $x \leq -9.571$, for $-1.890 \leq x \leq 12.80$, and for $52.53 \leq x$. The improved bound using $\epsilon = 0.0$ in equation III-57 is 0.5315.

VITA

David Alan Borg was born on August 14, 1943, in Corsicana, Texas. He received his early education in that city and was graduated from Corsicana High School in June, 1961. For the next four years he attended Texas A. & M. University at College Station, Texas, and received a Bachelor of Science degree in Electrical Engineering in June, 1965.

Between June, 1965, and September, 1967, he was employed as a plant engineer at the Aluminum Company of America's Point Comfort operations. During this period he married Janet Joyce Zrubek (they presently have two children, Thel Angela and Davin Arno). Since September, 1967, the author has been awarded graduate assistantships and fellowships from the Electrical Engineering Department of Louisiana State University from which he received a Masters of Science degree in June, 1969.

He is a member of IEEE and the honor societies Eta Kappa Nu and Phi Eta Sigma. Mr. Borg is now a candidate for the degree of Doctor of Philosophy in Electrical Engineering.

EXAMINATION AND THESIS REPORT

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Major Field: Electrical Engineering

Title of Thesis: Suboptimal Linear Regulator by Frequency Domain Compensation

Approved:

Paul M Julich
Major Professor and Chairman

Max Goodrich
Dean of the Graduate School

EXAMINING COMMITTEE:

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Date of Examination:

December 4, 1972