Surgery description of colored knots

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Abstract

By a knot, or link, we mean a circle, or a collection of circles, embedded in the three-sphere $S^3$. The study of knots is a very rich subject and plays a key role in the area of low-dimensional topology. In fact, a theorem of W.B.R. Lickorish and A.D. Wallace states that any three-dimensional manifold may be described by Dehn surgery along a link which is the process of removing the link from $S^3$ and then gluing it back in a way that possibly changes the resulting manifold.

In this dissertation, we will be interested in the pair $(K, \rho)$ consisting of a knot $K$ and a surjective map $\rho$ from the knot group onto a dihedral group of order $2p$ called a coloring. Such an object is said to be a $p$-colored knot. In Surgery untying of colored knots, D. Moskovich conjectures that for any odd prime $p$ there are exactly $p$ equivalence classes of $p$-colored knots up to surgery which preserves colorability. This is an analog to the classical result that every knot has a “surgery description” or equivalently that every knot is surgery equivalent to the unknot if we place fewer restrictions on the allowed surgery curves.

We show that there are at most $2p$ equivalence classes for $p$ any odd number. This is an improvement upon the previous results by Moskovich for $p = 3$, and 5, with no upper bound given in general. We do this by defining a new invariant, or an algebraic object associated to a $p$-colored knot, which is “complete” in the sense that two $p$-colored knots are surgery equivalent if and only if they both have the same value of this invariant. The complete invariant consists of Moskovich’s “colored untying invariant” redefined in the same way as the three-manifold invariants developed by T. Cochran, A. Gerges, and K. Orr, and another object we call the $\eta$ invariant. We also extend these methods to give similar results for “$A_4$-colored knots” which have representations onto the alternating group on four letters.
Chapter 1
Introduction

A link $L$ in a 3-manifold $Y$ is the image of an embedding of one or more circles into $Y$ up to ambient isotopy. Note that these circles are mapped into $Y$ disjointly and the image of a single circle is called a component of $L$. A link with a single component is called a knot. By a knot diagram we mean a planar projection of the knot with only double points called crossings together with information describing which arcs go “over” or “under” at each crossing. Knots and links are only uniquely determined by their diagrams up to Reidemeister moves (see Figure 1.1). By an oriented knot we mean a knot together with a choice of a direction which may be described by placing arrows on the arcs of a diagram.

FIGURE 1.1. Reidemeister moves.
for the knot. A \textit{framed knot} is an oriented knot together with a \textit{framing} or choice of generating set for the first homology of the torus boundary of a tubular neighborhood of the knot. The \textit{unknot} \( U \subset S^3 \) is characterized by the property that the fundamental group of its complement \( \pi_1(S^3 - U) \) is infinite cyclic. Not all knots are isotopic to the unknot and indeed the study of knots and links is as rich a subject as all of low-dimensional topology.

Given any 3-dimensional manifold \( Y \) and a \textit{framed link}, or a collection of framed knots, \( L \subset Y \), we may construct a new 3-manifold \( Y' \) by performing \textit{Dehn surgery on \( Y \) along \( L \)}. This is done by drilling out a tubular neighborhood of each component of \( L \) and gluing back in new solid tori according to the prescribed framings. More precisely, if \( K \subset Y \) is an oriented knot then a \textit{meridian} \( \mu \) of \( K \) is a positively oriented loop in the boundary torus of a tubular neighborhood of \( K \), \( N(K) \), which bounds a disk in that neighborhood. The \textit{preferred longitude} \( \lambda \) for \( K \) is a parallel of the knot in \( \partial N(K) \) which intersects \( \mu \) transversely exactly once and has linking number with the knot \( lk(\lambda, K) \) equal to zero. Then the \( \frac{p}{q} \)-\textit{framed Dehn surgery of \( Y \) along \( K \)} is obtained by gluing a solid torus \( T \) into \( Y - N(K) \) by the map described by a diffeomorphism of the torus which sends the meridian of \( T \) to a simple closed curve which goes \( p \) times around in the direction of \( \mu \) and \( q \) times around in the direction of \( \lambda \). Here we require that \( p \) and \( q \) be coprime. The fraction \( \frac{p}{q} \) is called the \textit{framing} of \( K \). In the early 1960's W.R.B. Lickorish and A. Wallace independently proved that any 3-manifold may be described as surgery on \( S^3 \) along some framed link. Moreover, the framings may be chosen to be either 1 or \(-1\). If \( Y' \) is described by a framed link \( L \subset S^3 \) we refer to \( L \) as a \textit{surgery description} for the 3-manifold \( Y' \).

It is well known that any knot \( K \subset S^3 \) may be unknotted by a sequence of \textit{crossing changes} or by changing the \textit{“over/under”} information at a crossing in a diagram for the knot. A crossing change may be obtained by performing \( \pm 1 \)-framed surgery on \( S^3 \) along an unknot, in the complement of the knot, which loops once around both strands of the crossing. The framing is determined by the sign of the crossing (see Figure 1.2). The
result of the surgery is once again the 3-sphere, however the knot $K$ has changed. By the same token any knot may be obtained from an unknot $U$ in the 3-sphere by $\pm 1$-framed surgery along null-homotopic unknots in the complement of $U$. This idea is called the surgery description of a knot. For two knots $K_1$, and $K_2$ in $S^3$ we have an equivalence relation defined by $K_1 \sim K_2$ if $K_2$ may be obtained from $K_1$ via a sequence of $\pm 1$ surgeries along unknots. Since every knot may be unknotted via this type of surgery we have that $K_1 \sim K_2$ for any two knots $K_i$ in the 3-sphere and hence there is only one equivalence class.

![Figure 1.2. Crossing change due to surgery.](image)

A knot invariant is an algebraic object which may be assigned to a knot which is unchanged by isotopy. Equivalently, it is an object which is assigned to a diagram for a knot which is unchanged by the Reidemeister moves. One elementary invariant of knots is unknotted number or the minimum number of crossing changes it takes to obtain a diagram of the unknot where the minimum is taken over all diagrams for the knot. If one investigates this invariant, it becomes clear that it is quite difficult to compute the unknotting number for many knots. Another example of a knot invariant is Fox colorability.

A knot is Fox $p$-colorable if the arcs of a diagram for the knot may be labeled with the “colors” $\{0, 1, \ldots, p-1\}$ so that at least two of the colors are used, and at each crossing the sum of the labels of the underarcs is twice the label of the overarc modulo $p$. See Figure 1.3 for an example of a 3-coloring of the trefoil knot. Existence of such a labeling
or coloring is detected by another invariant of knots called the determinant which is easily computable for $p$ an odd number without repeated prime factors given a diagram for the knot by taking the determinant of the Goeritz Matrix (see Section 3.4 or [GorLi, Liv]).

A $p$-colored knot, for any odd $p$, is a pair $(K, \rho)$ consisting of a knot together with a surjection $\rho : \pi_1(S^3 - K) \to D_{2p}$ from the knot group onto the dihedral group of order $2p$. An analog of the surgery description of knots for $p$-colored knots is given by restricting the surgeries to those which preserve the existence of a coloring $\rho$. It is natural, then, to ask what the equivalence classes are of $p$-colored knots modulo this surgery relation. This relation will be referred to as surgery equivalence in the kernel of $\rho$, or surgery equivalence of $p$-colored knots.

In [Mos], D. Moskovich proves that for $p = 3, 5$ there are exactly $p$ surgery equivalence classes of $p$-colored knots. Moskovich conjectures that this holds for all $p$ and although he has shown that $p$ is a lower bound on the number of equivalence classes in general, no upper bound is given. In this thesis, we will show that the number of surgery equivalence classes of $p$-colored knots is at most $2p$ for any odd number $p$. More precisely, we will prove the following theorem.

**Theorem 1.1. Main Theorem**

*There are at most $2p$ surgery equivalence classes of $p$-colored knots. Moreover, if $K_p$ de-
notes the left-handed \((p, 2)\)-torus knot and \(\rho\) is any coloring for \(K_p\), then

\[(K_p, \rho), (K_p, \rho) \# (K_p, \rho), \ldots, \#_{i=1}^{p} (K_p, \rho)\]

are \(p\) distinct surgery classes.

Moreover, in Chapter 7 we not only prove Theorem 1.1 we will also establish a complete invariant for surgery equivalence of \(p\)-colored knots. By “complete” we mean that two \(p\)-colored knots are surgery equivalent if and only if they both have the same value of this invariant.

Here \((K_p, \rho) \# (K_p, \rho)\) denotes the connected sum of the \(p\)-colored knots \((K_p, \rho)\) and itself. We will discuss what this means in Chapter 2. Note that the list of distinct classes is given in [Mos] but we will use a new definition for his “colored untangling invariant” to obtain the same result.

One way to attempt to establish an upper bound on the number of surgery equivalence classes is by using some basic moves on diagrams which preserve colorability and thereby perhaps reducing the complexity of the diagram or knot. For instance, one may try to find a set of moves on diagrams which reduce the crossing number or the minimum number of crossings over all diagrams of the resulting knot. This is a direct analog to the classical unknotting result where the basic move is a simple crossing change. Although a crossing change does not reduce the number of crossings of the diagram, it may reduce the minimum over all diagrams for the subsequent knot represented by that diagram. It was in this way that Moskovich proved his result for \(p = 3, 5\). These basic moves are called the \(RR\) and the \(R2G\)-moves shown in Figure 1.4 (a) and (b). The names of the moves suggest that to apply an \(RR\) move we must have the situation that two of the arcs at a crossing must be colored “Red” or for an \(R2G\) we have three parallel strands with one colored “Red” and the other two colored “Green.”
Another interesting question arises: Is there always a finite list of basic moves which are sufficient to describe surgery equivalence of colored knots as Reidemeister moves do for isotopy of knots? Although it is not proven directly in his paper, Moskovich’s result for $p = 3, 5$ gives a sufficient list of moves which may be used to untie a colored knot consisting of the $RR$, and $R2G$-moves, along with the “unlinking of bands.” So the answer is yes for $p = 3, 5$ but it is unknown otherwise. The following example shows a non-trivial relation between 3-colored knots.

**Example 1.2.** The right-handed trefoil knot $(3_1)$ and the $7_4$-knot are surgery equivalent $p$-colored knots.
Proof. Performing a single $RR$-move changes $7_4$ into the trefoil as in Figure 1.5. Note that this also shows that the mirror images of these knots are equivalent. However, neither of these knots is surgery equivalent to its mirror image. This may be seen by calculation of the colored untying invariant as in Section 4.4.

\[ a \in \mathbb{Z}_3 \]
\[ c = 2a - b \]
\[ a \quad b \quad c \]
\[ a \quad b \quad c = 2b - a \]

FIGURE 1.5. The $7_4$ knot is surgery equivalent to the trefoil knot.

We will not attempt a direct proof for $p > 5$ as Moskovich does for the first two cases. Instead we will show that an analog to the Lickorish-Wallace Theorem [Lic1, Wal] and some basic bordism theory suffices to show that there are no more than $2p$ classes. The Lickorish-Wallace Theorem states that any closed, oriented, connected 3-manifold may be obtained by performing Dehn surgery on a link in $S^3$ with $\pm 1$-framings on each component. Furthermore, each component may be assumed to be unknotted. In [CGO], T. Cochran, A. Gerges, and K. Orr ask what the equivalence classes of 3-manifolds would
be if we restrict the surgeries to a smaller class of links. Surgery equivalence of \( p \)-colored knots may be described in a similar way.

The proof of the Main Theorem then is outlined in four steps. Step 1 is to establish a 3-manifold bordism invariant which coincides with colored knot surgery. Step 2 is to show that if two colored knots have bordant knot exteriors with the property that the boundary of the bordism 4-manifold \( W \) is

\[
\partial W = (S^3 - K_1) \sqcup (S^3 - K_2) \cup_{T^2 \cup T^2} (T^2 \times [0, 1]),
\]

where \((K_i, \rho_i)\) are the colored knots, then the colored knots are surgery equivalent. That is, under these conditions, the bordism may be obtained by adding 2-handles to the 4-manifold \((S^3 - K_1) \times [0, 1]\). We may then “fill in” the boundary of the bordism 4-manifold by gluing in a solid torus crossed with \([0, 1]\). This new 4-manifold is a bordism between two copies of \( S^3 \) which corresponds to some surgery description for the 3-sphere. So Step 3 is to apply Kirby’s Theorem to unknot and unlink the surgery curves which may be done by only handle slides and blow-ups (see [GomSt]). This establishes a surgery equivalence for the knots that are “taken along for the ride” during the handle slides and are unchanged (up to surgery equivalence) by blow-ups. The final step is to show that if any three colored knots have bordant knot complements, then at least two of the colored knots must be surgery equivalent.

The thesis is organized as follows. First we will precisely state what is meant by \( p \)-colored knots and surgery equivalence. Then we will define some invariants of \( p \)-colored knot surgery equivalence in Chapters 4 and 5. There are three types of invariants: the colored untying invariant, and the closed and relative bordism invariants. The colored untying invariant may be computed using the Seifert matrix as in [Mos], but we show in Section 4.2 that it may be defined using the Goeritz matrix which allows for a simple and geometric proof of invariance under surgery. It turns out that each of these invariants give
the same information and thus they are all computable given a diagram for the colored knot. We will then show, by direct example, that there are at least $p$ surgery equivalence classes represented by connected sums of $(p, 2)$-torus knots with appropriate colorings. In Section 7, we will show that a relative bordism over the Eilenberg-Maclane space pair $(K(D_{2p}, 1), K(\mathbb{Z}_2, 1))$ between two colored knot exteriors $(M_i = S^3 - K_i, \rho_i)$ establishes a surgery equivalence between the colored knots $(K_i, \rho_i)$ at least half of the time. This gives an upper bound on the number of equivalence classes for any $p$ which is the main result of the paper. Then, in Chapters 8 and 9 we will extend our main result to “$A_4$-colored knots” and discuss some applications this theory has for low dimensional topology. Lastly, in the Appendix, we give a computation of the homology groups of the dihedral group $D_{2p}$. 
Chapter 2
Colored Knots

We will first introduce what is meant by a \( p \)-colored knot and surgery equivalence of \( p \)-colored knots.

2.1 Definitions
Throughout, let \( p \) denote an odd number. The following will be the definition of a \( p \)-colored knot:

**Definition 2.1.** The pair \((K, \rho)\) consisting of a knot \( K \subset S^3 \) and a surjective homomorphism, \( \rho : \pi_1(S^3 - K, x_0) \to D_{2p} \), from the knot group with basepoint \( x_0 \) onto the dihedral group of order \( 2p \), up to an inner automorphism of \( D_{2p} \), is said to be a \( p \)-colored knot.

The knot \( K \) is said to be \( p \)-colorable with coloring given by \( \rho \).

A coloring \( \rho \) is only considered up to an inner automorphism of the dihedral group. In particular, this means that two \( p \)-colored knots \((K_1, \rho_1)\) and \((K_2, \rho_2)\) are in the same coloring class if \( K_1 \) is ambient isotopic to \( K_2 \) and that the following diagram commutes:

\[
\pi_1(S^3 - K_1, x_1) \xrightarrow{\rho_1} D_{2p} \\
\downarrow \epsilon \quad \quad \quad \quad \quad \downarrow \sigma \\
\pi_1(S^3 - K_2, x_2) \xrightarrow{\rho_2} D_{2p}
\]

where, \( \sigma : D_{2p} \to D_{2p} \) is an inner automorphism and \( \epsilon : \pi_1(S^3 - K_1, x_1) \to \pi_1(S^3 - K_2, x_2) \) is the isomorphism given by

\[
[\alpha] \in \pi_1(S^3 - K_1, x_1) \mapsto [h^{-1} \alpha h] = [h]^{-1} [\alpha] [h] \in \pi_1(S^3 - K_2, x_2)
\]

where \( h \) is any fixed path from \( x_2 \) to \( x_1 \) in \( S^3 - K_1 \).

If we let \( K_1 = K_2 \) we see that the choice of a different basepoint results in an inner automorphism of the knot group and thus results in an inner automorphism of the dihedral group.
group. So the definition is well-defined for any choice of basepoint. We will then ignore basepoints from now on and denote a coloring simply by a surjection

$$\rho : \pi_1(S^3 - K) \to D_{2p}$$

from the knot group onto the dihedral group.

**Definition 2.2.** Two \(p\)-colored knots \((K_1, \rho_1)\) and \((K_2, \rho_2)\) are surgery equivalent in the kernel of \(\rho\) (or simply surgery equivalent) if \(K_2 \in S^3\) may be obtained from \(K_1\) via a sequence of \(\pm 1\)-framed surgeries of \(S^3\) along unknots in the kernel of \(\rho_1\). Furthermore, \(\rho_2\) must be compatible with the result on \(\rho_1\) after the surgeries. That is, if \(K(D_{2p}, 1)\) denotes an Eilenberg-Maclane space then

\[
\begin{array}{ccc}
S^3 - K_1 & \xrightarrow{\nu} & S^3 - K_2 \\
\downarrow f_1 & & \downarrow f_2 \\
& & K(D_{2p}, 1)
\end{array}
\]

is a commutative diagram where \(\rho_i\) are the induced maps of the \(f_i\) on \(\pi_1\) and \(\nu\) is the map resulting from surgery restricted to \(S^3 - K_1\).

So there are two conditions for surgery equivalence of \(p\)-colored knots: (1) the knots must be surgery equivalent in the classical sense with the restriction that the surgery curves are in the kernel of the coloring, and (2) the coloring of the second knot arises from the coloring of the first knot via surgery. Notice that (1) assumes that the surgery curves are unknotted with \(\pm 1\)-framings.

The condition that coloring maps are defined up to inner automorphism is important. We will use this condition to define what is called a “based” \(p\)-colored knot and allows for the connected sum operation to be well-defined.

### 2.2 Justifications

To illustrate the necessity of the condition that a coloring need only be defined up to conjugation by an element of the dihedral group, consider the connected sum operation
of knots. If \((K_1, \rho_1)\) and \((K_2, \rho_2)\) are \(p\)-colored knots, we wish to have \(K_1 \# K_2\) also be \(p\)-colorable with some coloring which is constructed from the \(\rho_i\). Since the usual connected sum of oriented knots is unique up to ambient isotopy, so should the connected sum of oriented colored knots be unique.

To do this, first note that a Fox coloring is classically described by a labeling of the arcs in a diagram for \(K\) with the “colors” \(\{0, \ldots, p - 1\}\) (see [CrFo, Chap IV, exercise 6]). At each crossing, the labeling must satisfy the coloring condition which requires that the sum of the labels of the underarcs must equal twice the label of the overarc modulo \(p\). We also require that the coloring be nontrivial, that is, we require that more than one color is used.

Then such a labeling defines a surjection \(\rho : \pi_1(S^3 - K) \rightarrow D_{2p} = \langle s, t | s^2 = t^p = stst = 1 \rangle\) by the rule \(\rho(\mu) = ts^l\) where \(l\) is the label given to the arc corresponding to the meridian \(\mu\). Conversely, in a Fox coloring meridians are necessarily mapped by a coloring to elements of order two in the dihedral group so a coloring map determines a labeling of any diagram for the knot.

Since we may alter any coloring by an inner automorphism of \(D_{2p}\) we may assume that any one arc we choose in a diagram for \(K\) be labeled with the color 0. We may assume, then, that for any meridian \(m\) of \(K\) there is an equivalent coloring \(\rho\) which maps \(m\) to \(ts^0 \in D_{2p}\). We call the triple \((K, \rho, m)\) a based \(p\)-colored knot. Therefore, given \((K_1, \rho_1)\) and \((K_2, \rho_2)\) where the \(\rho_i\) are defined by a nontrivial labeling of a diagram for (oriented) knots \(K_i\) we may take \((K_1 \# K_2, \rho_3)\) to be the usual connected sum of oriented knots with \(\rho_3 = \tilde{\rho}_1 \# \tilde{\rho}_2\) (see Figure 2.1). Part (a) of the Figure illustrates that we may assume that the \((K_i, \rho_i)\) are actually the based \(p\)-colored knots \((K_i, \rho_i, m_i)\) where \(m_i\) is the meridian that corresponds to the chosen arc of the diagram for \(K_i\).

To verify that this process is well-defined for any choice of diagram, we must establish the existence and uniqueness of labelings for each Reidemeister move. This is done in Figure 2.2.

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(a) Relabeling colored knots

or (depending on orientation)

(b) Connected sum of colored knots after relabeling

FIGURE 2.1. Connected sum of colored knots.
We may now define what is meant by a prime $p$-colored knot.

**Definition 2.3.** A $p$-colored knot is said to be prime if it is not the connected sum of two nontrivial $p$-colored knots.

Unfortunately, the notion of prime $p$-colored knots is slightly different from the usual notion of a prime knot. For example let $K = K_1 \# K_2$ where $K_1$ is the left-handed trefoil...
and $K_2$ is the figure eight knot. Indeed, $K$ is 3-colorable since we can label $K$ using all 3 colors as in Figure 2.3. A knot is $p$-colorable if and only if its determinant is divisible by $p$ [Liv]. So since $\det(K_2) = 5$ is not divisible by 3, we have that no non-trivial coloring of $K_2$ exists. Thus, $(K, \rho) \neq (K_1, \rho_1) \# (K_2, \rho_2)$ for any 3-colorings $\rho_1$ and $\rho_2$.

As we have seen in Example 1.2, it possible to show that two $p$-colored knots are surgery equivalent directly in some cases. However, much like trying to distinguish knots by using Reidemeister moves, it is impossible to prove that two $p$-colored knots are not surgery equivalent by simply using a collection of moves on diagrams. In fact, it is often difficult to show that two knots are the same using Reidemeister moves, and surgery equivalence of $p$-colored knots faces the same type of difficulty. It is useful then to define algebraic invariants to help distinguish between knot types and the same is true for surgery equivalence. We will do so in the next several chapters.
Chapter 3
Preliminaries

To show something is a knot invariant one need only show that it is unchanged by any of the three Reidemeister moves. We do not have a complete list of moves to determine surgery equivalence of $p$-colored knots so surgery equivalence invariance requires a more abstract approach. First we must show that the value is unchanged under the choice of $p$-colored knot representative, in particular it must be an invariant of knots, and then we must show that it is invariant under $\pm 1$-surgery. We will do this for all of the three types of proposed surgery equivalence invariants. Then we will show that these invariants are in fact three different ways to define the same thing.

In this chapter we will introduce some of the background that will be needed in defining the three types of invariants for $p$-colored knots. We begin the discussion with maps arising from algebraic topology. The Bockstein homomorphisms appear in the definition of D. Moskovich’s colored untying invariant $cu(K, \rho)$ in [Mos]. We will then discuss what is meant by a handle structure for a manifold arising from Morse Theory. Later we will give a brief overview of the bordism theory needed to define the closed bordism invariants $\omega_2$, $\omega_0$, as well as the relative bordism invariant $\omega$. It will also be useful to recall the definition of the Goeritz matrix using the Gordon-Litherland form [GorLi].

3.1 Bockstein Homomorphisms

Definition 3.1. Let $\mathcal{C} = \{C_n, \partial_n\}$ be a free chain complex. Given an exact sequence of abelian groups

$$0 \rightarrow G \rightarrow G' \rightarrow G'' \rightarrow 0$$
consider the associated short exact sequences:

\[ 0 \to C_n \otimes G \to C_n \otimes G' \to C_n \otimes G'' \to 0, \]

and

\[ 0 \to Hom(C_n, G) \to Hom(C_n, G') \to Hom(C_n, G'') \to 0. \]

From the zig-zag lemma [Mun], we obtain homomorphisms

\[ \beta_* : H_k(C; G'') \to H_{k-1}(C; G), \]

and

\[ \beta^* : H^k(C; G'') \to H^{k+1}(C; G). \]

These are called the Bockstein homomorphisms associated with the coefficient sequence

\[ 0 \to G \to G' \to G'' \to 0. \]

The situation we will be concerned with will be the one in which the coefficient sequence is

\[ 0 \to \mathbb{Z} \overset{x}{\to} \mathbb{Z} \overset{\text{mod} \, p}{\to} \mathbb{Z}_p \to 0, \]

for \( p \) an odd number. Let \( C \) be the chain complex for the homology of a 3-manifold \( M \) which has a map, \( f : M \to K(\mathbb{Z}_p, 1) \), from itself to the Eilenberg-Maclane space \( K \). Then \( \beta^* : H^k(M; \mathbb{Z}_p) \to H^{k+1}(M, \mathbb{Z}) \). In particular,

\[ \beta^1 : H^1(M; \mathbb{Z}_p) \to H^2(M; \mathbb{Z}). \]

The colored untying invariant will arise from the cup product of a certain element \( a \in H^1(M; \mathbb{Z}_p) \) (depending only on the coloring class) with its image under the Bockstein homomorphism, \( \beta^1(a) \). In this way we obtain a \( \mathbb{Z}_p \)-valued invariant.

Now we turn our attention to Morse theory and the handle decomposition of manifolds, then we will briefly introduce bordism theory.

### 3.2 Morse Theory

*Morse Theory* was introduced by Marston Morse in the 1930’s as a way of applying calculus to the theory of manifolds. Most of the following background information on
Morse theory is explained in greater detail in [Mil2] and [Mor]. See [GomSt] as well as [Mil1, Mil2] for the details on handle decompositions.

Let $W$ be a differentiable $n$-manifold and $f : W \rightarrow \mathbb{R}$ a $C^\infty$ function from $W$ to the real line. A point $p \in W$ is a critical point if $df(p) = 0$, or, in local coordinates given by $\{x^1, \ldots, x^n\}$ for $W$ near $p$, the partial derivatives $\frac{\partial f}{\partial x^i}(p)$ all vanish. A critical point is non-degenerate if the Hessian matrix $H = \left( \frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right)$ has non-zero determinant.

**Definition 3.2.** A Morse function is a $C^\infty$ function $f : W \rightarrow \mathbb{R}$ from a smooth $n$-manifold $W$ to the real line so that all critical points of $f$ are non-degenerate.

The Morse Lemma states that if $p \in M$ is a critical point of a Morse function $f$, then there are local coordinates $\{x^1, \ldots, x^n\}$ for $W$ centered at $p$ so that

$$f = f(p) - (x^1)^2 - \cdots - (x^i)^2 + (x^{i+1})^2 + \cdots + (x^n)^2$$

where $i = i(p)$, which does not depend on the choice of coordinates, is called the index of the critical point.

Morse functions determine what is called a handle structure for a smooth manifold and vice versa. To see this first note that Morse functions on smooth manifolds exist and indeed it is a theorem that the set of Morse functions on $W$ is open and dense in the set of $C^\infty$ functions on $W$. A special kind of Morse function $f$ is called self-indexing if $f(p) = i$ for all index $i$ critical points.

**Theorem 3.3.** Every connected, closed, oriented, smooth $n$-manifold $W$ admits a self-indexing Morse function with one minimum and one maximum. If $W$ has boundary components a self-indexing Morse function may have no minimum nor maximum.

Although it follows from the content of [Mor] this theorem is not explicitly stated there. Instead, the proof of the above theorem is discussed in Propositions 4.2.7 and 4.2.13 of [GomSt] in the context of handle decompositions.
**Definition 3.4.** An \( n \)-dimensional \( k \)-handle for \( 0 \leq k \leq n \) is a copy of \( D^k \times D^{n-k} \), where \( D^i \) denotes an \( i \)-dimensional ball, attached to the boundary of an \( n \)-manifold \( W \) along \( \partial D^k \times D^{n-k} \) by an embedding \( \varphi : \partial D^k \times D^{n-k} \to \partial M \).

We call \( \varphi \) the *attaching map*, \( D^k \times \{0\} \) the *core* of the handle, \( \{0\} \times D^{n-k} \) the *cocore*, and \( k \) is called the *index* of the handle.

By a *handle* or (*handlebody*) *structure* of an \( n \)-manifold \( W \) we mean that \( W \) is constructed by attaching \( k \)-handles \( 0 \leq k \leq n \). Such a decomposition of \( M \) is called a *handle decomposition*. The critical points of index \( i(p) \) of a Morse function correspond to index \( i(p) \)-handles in a handle decomposition. See Figure 3.1 for an example of a handle decomposition of a genus 2 orientable surface consisting of one 0-handle, four 1-handles, and one 2-handle.

![Figure 3.1. Handle decomposition of a genus 2 surface.](image)

The fact that a Morse function determines a handle decomposition is a consequence of the following theorem. If \( f \) is a Morse function on \( W \) denote by \( W_b \) the preimage of the interval \( (-\infty, b] \).

**Theorem 3.5 (Morse).** The two submanifolds \( W_a \subset W \) and \( W_b \subset M \) are diffeomorphic if there are no critical points of \( f \) in \( f^{-1}([a, b]) \). If the preimage of the interval \( [a-\epsilon, a+\epsilon] \) contains a unique critical point \( p \) with \( f(p) = a \) then \( W_{a+\epsilon} \) is diffeomorphic to \( W_{a-\epsilon} \) with an \( i(p) \)-handle attached.
In dimensions 3 and 4 handle decompositions may be described by diagrams in the plane called Heegaard and Kirby diagrams respectively. These diagrams describe how to attach $k$-handles for $0 < k < n$ for $n = 3, 4$ to the unique 0-handle (if $\partial W = \emptyset$) or to $\partial W$ (if $\partial W \neq \emptyset$) as there is then no choice in where to “fill in” using the unique 4-handle if one is needed.

It may also be noted that a $k$-handle may be thought of as an “upside down” $(n-k)$-handle in dimension $n$. In particular, in dimension 4, let $f$ be a self-indexing Morse function, then a 3-handle attached to $f^{-1}([0,2])$ is the same as a 1-handle attached to $(-f)^{-1}([0,2])$.

In 1933, James Singer [Sin] proved that any two Heegaard diagrams for the same 3-manifold differ by isotopy and stabilization. In [Kir1], Kirby proved the 4-dimensional analog. Any two Kirby diagrams for the same 4-manifold differ by isotopy, handle slides, and blow-ups or blow-downs (see [GomSt, Chapter 5]). A handle slide is illustrated in Figure 3.2. In terms of handle decompositions, stabilization, and blow-ups and blow-downs are described by the addition or deletion of a “trivial” handle or a “cancelling pair” of handles.

FIGURE 3.2. Handle slide.
We will now consider the special case when $W$ is a 4-manifold and the boundary of $W$ is given by the disjoint union of two 3-manifolds. In this case, the two 3-manifolds are said to be bordant.

### 3.3 Bordism

**Definition 3.6.** Let $(X, A)$ be a pair of topological spaces with $A \subseteq X$. The $n$-dimensional oriented relative bordism group of the pair, denoted $\Omega_n(X, A)$, is defined to be the set of bordism classes of triples $(M, \partial M, \varphi)$ consisting of a compact, oriented $n$-manifold $M$ with boundary $\partial M$ and a continuous map $\varphi : (M, \partial M) \to (X, A)$. The triples $(M_1, \partial M_1, \varphi_1)$ and $(M_2, \partial M_2, \varphi_2)$ are in the same bordism class if there exists an $n$-manifold $N$ and a triple $(W, \partial W, \Phi)$ consisting of a compact, oriented $(n+1)$-manifold $W$ with boundary $\partial W = (M_1 \sqcup M_2) \cup_{\partial N} N$ and a continuous map $\Phi : (W, \partial W) \to (X, A)$ satisfying $\Phi|_{M_i} = \varphi_i$ and $\Phi(N) \subseteq A$. We also require that $M_1$ and $M_2$ are disjoint and $M_i \cap N = \partial M_i$ for $i = 1, 2$. In this case, we say that $(M_1, \partial M_1, \varphi_1)$ and $(M_2, \partial M_2, \varphi_2)$ are bordant over $(X, A)$ denoted $(M_1, \partial M_1, \varphi_1) \sim_{(X, A)} (M_2, \partial M_2, \varphi_2)$ (see Figure 3.3).

A triple $(M, \partial M, \varphi)$ is null-bordant, or bords, over $(X, A)$ if it bounds $(W, \partial W, \Phi)$. That is, it bords if it is bordant to the empty set $\emptyset$. The set $\Omega_n(X, A)$ forms a group with the operation of disjoint union and identity element $\emptyset$. We will denote $\Omega_n(X, \emptyset)$ by $\Omega_n(X)$ and so our definition makes sense for pairs $(M, \varphi) = (M, \emptyset, \varphi)$ with $M$ a closed $n$-manifold.

We will only be interested in the case when $n = 3$. In this case, the *Atiyah-Hirzebruch spectral sequence* (see [Whi] for the extraordinary homology theory made up of the bordism groups $\Omega_n(X, A)$) implies that $\Omega_3(X, A) \cong H_3(X, A; \Omega_0) \cong H_3(X, A)$ where $\Omega_0 \cong \mathbb{Z}$ is the 0-dimensional bordism group of a single point. The isomorphism is given by $(M, \partial M, \varphi) \mapsto \varphi_*([M, \partial M])$ where $[M, \partial M]$ is the fundamental class in $H_3(M, \partial M)$. Furthermore, if we assume that $X$ is an Eilenberg-Maclane space $K(G, 1)$ and $A$ is the subspace corresponding to a subgroup $H \subset G$, then the bordism group is isomorphic to the homology of the group $H_3(M, \partial M)$. [21]
$G$ relative the subgroup $H$, that is $\Omega_3(X, A) \cong H_3(G, H)$. When $K(H, 1)$ is a subspace of $K(G, 1)$ we will denote $\Omega_3(K(G, 1), K(H, 1))$ by $\Omega_3(G, H)$.

If $\partial W = M_1 \sqcup M_2$ and the bordism $W$ is built via attaching 2-handles to $M_1 \times [0, 1]$ then the two 3-manifolds bordant but they are also related by surgery along the link formed by the attaching circles.

We will need to recall the definition of one last object before we can define the surgery equivalence invariants for colored knots.

### 3.4 The Goeritz Matrix

Given a spanning surface $F$ for a link $K$ and a basis $x_i$ for its homology, the Goeritz matrix is given by evaluating the Gordon-Litherland form, $\mathfrak{G}_F : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$, on the basis elements (see [GorLi]). That is, $G = (g_{ij})$ is defined by

$$g_{ij} = \mathfrak{G}_F(x_i, x_j) = \text{lk}(x_i, \tau^{-1}(x_j))$$

where $\tau^{-1}(y)$ is $y$ pushed off in “both directions.” Precisely, $\tau : \tilde{F} \rightarrow F$ is the orientable double covering space of $F$ (see Chapter 7 of [Lic2]). Note that $\tilde{F}$ is a connected, orientable surface regardless of the orientability of $F$. 

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If $F$ is orientable and $y$ is a loop in $F$ then $\tau^{-1}(y)$ is comprised of two loops, the positive $y_+$ and negative $y_-$ push offs on either side of $F$. Figure 3.4 illustrates the non-orientable case. If $F$ is non-orientable then $\tau^{-1}(y)$ is a single loop which double covers $y$. In this case you can think of $\tau^{-1}(y)$ to be the loop that arises from pushing $y$ off to one side which then comes back around on the other side and vice versa.

![Figure 3.4. “Double push off” of a orientation reversing curve $y$.](image)

Note that the Goeritz matrix for a knot $K$ may also be calculated from a checkerboard coloring for a diagram for the knot (see Chapter 9 of [Lic2]). First we must pick a white region, the so-called infinite region $R_0$, and then we number the other white regions $R_1, \ldots, R_n$. We then define an incidence number $\iota(c) = \pm 1$ assigned to any crossing $c$ by the rule in Figure 3.5. We define a $(n + 1) \times (n + 1)$ matrix $(g_{ij})$ for $i \neq j$ by

$$g_{ij} = \sum \iota(c),$$

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where the sum is over all crossings which are incident with both $R_i$ and $R_j$. The diagonal terms are chosen so that the rows and columns sum to 0, namely

$$g_{ii} = -\sum_{l \neq k} g_{lk}.$$ 

The Goeritz matrix is then obtained from the “pre-Goeritz matrix” $(g_{ij})$ by deleting the row and column corresponding to the infinite region. The group that this matrix presents is independent of the choice of infinite region.

We will use this diagramatic way to calculate the Goeritz matrix in Section 4.4.
Chapter 4
The Colored Untying Invariant

We will now define precisely the $p$-colored knot invariant $cu$, and later define the bordism invariants $\omega_2$, $\omega_0$, and $\omega$.

4.1 Moskovich’s Definition

Throughout, let $(K, \rho)$ be a $p$-colored knot with coloring $\rho : \pi_1(S^3 - K) \to D_{2p}$ where $D_{2p} = \langle s, t \mid t^2 = s^p = tst = 1 \rangle$ is the dihedral group with $2p$ elements. Also let $\tilde{X}$ denote the 2-fold cover of $S^3$ branched over $K$. Let $X_0$ denote the manifold obtained from $S^3$ by performing 0-framed surgery along $K$.

Consider the following diagram:

\[
\begin{array}{ccc}
H_1(S^3 - F) & \xrightarrow{\varphi} & \mathbb{Z}_p \\
\downarrow \pi_1(S^3 - F) & \xrightarrow{\rho|_{S^3 - F}} & \mathbb{Z}_p \\
\downarrow \pi_1(S^3 - K) & \xrightarrow{\rho} & \mathbb{Z}_{2p} \\
\end{array}
\]

with the map $l$ defined by $l(x) = lk(x, K)$ (mod 2). Note that the coloring map sends meridians to elements of order 2, in particular, $\rho(\mu_i) = ts^k$ for some $k \in 0, \ldots, p - 1$ where $\mu_i$ are Wirtinger generators for $\pi_1(S^3 - K)$. Then the lower triangle of Diagram (4.1) commutes by construction. Furthermore, we see that if $x$ is a loop in $(S^3 - F)$ then $lk(x, K) \equiv 0$ (mod 2) which is enough to establish the commutativity of the rest of Diagram (4.1). Indeed, since $x$ is in the complement of the Seifert surface $F$ we may assume that $lk(x, K) = 0$. Notice that the commutativity of the upper triangle of the diagram is immediate since the image of $\rho|_{S^3 - F}$ is abelian.
Therefore we have established the existence of a map \( f : (S^3 - F) \to K(\mathbb{Z}_p, 1) \) from the complement of the surface to an Eilenberg-Maclane space over \( \mathbb{Z}_p \). The map \( f \) may be extended to the *unbranched 2-fold cyclic cover* of \( S^3 - K \) denoted \( \tilde{Y} \) which is obtained by gluing two copies of \( S^3 - F \) together along two copies of a bicollar \( (F - K) \times (-1, 1) \) of the interior of the surface. Call this “new” map \( f : \tilde{Y} \to K(\mathbb{Z}_p, 1) \). Now we can form the 2-fold branched cover \( \tilde{X} \) by gluing in a solid torus so that the meridian of the solid torus maps to twice the meridian of the torus boundary of \( \tilde{Y} \) (see [Rol, Chapters 5 and 10]). Since twice a meridian is mapped trivially by \( f \) we may extend this map to the 2-fold branched cover. Thus we have a map \( f : \tilde{X} \to K(\mathbb{Z}_p, 1) \) which will be used to associate \( \omega_2 \) with \( cu \) later in Section 6.1.

We have shown that

\[
\begin{align*}
H_1(\tilde{X}) & \xrightarrow{\rho'} H_1(\tilde{Y}) \\
\pi_1(\tilde{X}) & \xrightarrow{f_*} \pi_1(\tilde{Y}) \\
\pi_1(\tilde{Y}) & \xrightarrow{f_*} \mathbb{Z}_p \\
\pi_1(S^3 - K) & \xrightarrow{\rho} D_{2p} \\
D_{2p} & \xrightarrow{l} \mathbb{Z}_2
\end{align*}
\]

is a commutative diagram. So the coloring map \( \rho \) restricts in the double covering to a map

\[
\rho' : H_1(\tilde{X}; \mathbb{Z}) \to \mathbb{Z}_p
\]

which corresponds to a cohomology class

\[
a \in H^1(\tilde{X}; \mathbb{Z}_p) \cong Hom(H_1(\tilde{X}; \mathbb{Z}), \mathbb{Z}_p)
\]
by the Universal Coefficient Theorem for Cohomology. The colored untying invariant is defined to be the cup product of $a$ with its image under the Bockstein homomorphism $\beta^1 : H^1(\tilde{X}; \mathbb{Z}_p) \to H^2(\tilde{X}; \mathbb{Z})$.

**Definition 4.1.** Given a $p$-colored knot $(K, \rho)$ the colored untying invariant of $(K, \rho)$ is

$$cu(K, \rho) := a \cup \beta^1 a \in H^3(\tilde{X}; \mathbb{Z}_p)$$

which we may think of as an element of $\mathbb{Z}_p \cong H^3(\tilde{X}; \mathbb{Z}_p)$.

Note that the isomorphism $\mathbb{Z}_p \cong H^3(\tilde{X}; \mathbb{Z}_p)$ is given by evaluation on the fundamental class.

To show that this is actually an invariant of $p$-colored knots we must assert that it is well-defined for any choice of equivalent coloring. Invariance of the choice of coloring is clear since $cu$ is defined using homology and cohomology groups which are independent of basepoint and conjugacy class in $\pi_1(S^3 - K)$. To show that $cu$ is a non-trivial invariant we will introduce a way to compute $cu$ by using the Seifert matrix for a given Seifert surface. It turns out that there is a way to determine the invariant for any spanning surface (including perhaps a non-orientable surface) by using the Goeritz matrix. We will use this definition to establish non-triviality and invariance under $\pm 1$-framed surgery in the kernel of $\rho$. Note that Moskovich [Mos] gives an alternate proof of the surgery invariance and does not mention the Goeritz definition.

Let $F$ be a Seifert surface for $K$ with Seifert matrix $S$ with respect to a basis $x_1, \ldots, x_{2k}$ of $H_1(F)$. Let $\xi_1, \ldots, \xi_{2k}$ be a basis for $H_1(S^3 - F)$ with orientations so that $lk(x_i, \xi_j) = \delta_{ij}$. The proof of the following lemma can be found in [Mos] and will be omitted here.

**Lemma 4.2.** [Mos] Let $v := (v_1, \ldots, v_{2k})^T \in \mathbb{Z}^{2k}$ be a column vector such that

$$v_i \pmod{p} = \rho(\xi_i)$$
for all \( i \in 1, \ldots, 2k \). Then

\[
\text{cu}(K, \rho) = 2 \frac{v^T \cdot S \cdot v}{p} \pmod{p}.
\]

The vector \( v \) is called a \( p \)-coloring vector.

If \( K = (p, 2) \) torus knot, then, for a choice of \( p \)-colorings \( \rho_1 \) and \( \rho_2 \), the lemma may be used to show that \( \text{cu}(K, \rho_1) \neq \text{cu}(K, \rho_2) \). We will show this later in Section 4.4 using the Goeritz definition of the colored untying invariant defined below.

### 4.2 Goeritz Definition

We will now extend Lemma 4.2 to any spanning surface for the knot \( K \) including perhaps non-orientable surfaces. We will use this definition for the colored untying invariant to give a geometric proof that \( \text{cu} \) is a surgery equivalence invariant.

**Proposition 4.3.** The colored untying invariant \( \text{cu} \) may be calculated using the Goeritz matrix for a diagram for \( K \). That is

\[
\text{cu}(K, \rho) = \frac{v^T \cdot G \cdot v}{p} \pmod{p} \quad (4.3)
\]

where \( v \) is any \( p \)-coloring vector and \( G \) is the Goeritz matrix.

Before we prove this proposition we will proceed with a few lemmas which will show that the colored untying invariant when defined by Equation (4.3) is actually a well-defined invariant of \( p \)-colored knots.

**Lemma 4.4.** Let the coloring vector \( v = (v_1, \ldots, v_n)^T \) be defined by \( \overline{\rho}(\xi) \). Then the number

\[
\frac{v^T \cdot G \cdot v}{p} \pmod{p}
\]

is independent of the choice of basis \( \{\xi_i\} \) for \( H_1(S^3 - F) \).
Proof. Suppose \( \{x_i\} \), and \( \{y_i\} \) are two bases for \( H_1(F; \mathbb{Z}) \). Also let \( \{\xi_i\} \), and \( \{\eta_i\} \) be bases for \( H_1(S^3 - F; \mathbb{Z}) \) which are dual to \( \{x_i\} \), and \( \{y_i\} \) respectively in the sense that

\[
lk(x_i, \xi_i) = \delta_{ij}
\]

and

\[
lk(y_i, \eta_i) = \delta_{ij}
\]  
(4.4)

for \( i, j = 1, \ldots, n \). Then there is a change of basis matrix \( A = (a_{ij}) \) so that

\[
y_i = \sum_j a_{ji} x_j
\]  
(4.5)

and since \( A \) is invertible over \( \mathbb{Z} \) we have that \( \text{det}(A) = \pm 1 \). Likewise, there is an invertible, unimodular, change of basis matrix \( B = (b_{ij}) \) so that

\[
\eta_i = \sum_j b_{ji} \xi_j.
\]  
(4.6)

Using (4.5) and (4.6) we may substitute for \( y_i \) and \( \eta_j \) in Equation (4.4) to obtain

\[
\delta_{ij} = \sum_l \sum_k a_{li} b_{kj} lk(x_l, \xi_k)
= \sum_l a_{li} b_{lj}
\]

which implies that \( B = (A^T)^{-1} \).

Now let \( v_i \in \mathbb{Z} \) be so that \( v_i \pmod{p} = \overline{\rho}(\xi_i) \) and \( w_i \pmod{p} = \overline{\rho}(\eta_i) \) define coloring vectors associated to the different choice of bases. Then

\[
w = A^{-1} v
\]

because \( w_i = \sum_j b_{ji} v_j \). We will show that the Goeritz matrices for each choice of basis are related in such a way so that the colored untying invariants associated to those matrices and the corresponding coloring vectors are unchanged. So take \( G = (g_{ij}) \) defined by the
Gordon-Litherland form \((g_{ij} = lk(x_i, \tau^{-1}(x_j)))\) and \(G' = (lk(y_i, \tau^{-1}(y_j)))\). Then

\[
G' = lk \left( \sum_k a_{ki} x_k, \tau \sum_l a_{lj} x_l \right)
\]

\[
= \sum_{k,l} a_{ki} a_{lj} g_{kl}
\]

which implies that \(G' = A^T G A\). Hence

\[
w^T G' w = v^T (A^T)^{-1} A^T G A A^{-1} v = v^T G v
\]

as desired.

Now we will show the the colored untying invariant is well-defined for any choice of coloring vector.

**Lemma 4.5.** Let \(cu(v)\) denote the product \(v^T G v\) for a \(p\)-coloring vector \(v\) then \(cu(v) = cu(v + pw) \pmod{p}\) for any vector \(w\).

**Proof.** Let \(\bar{v} = v + pw\) for some vector \(w\). Then \(\bar{v}\) is a coloring vector since \(\rho(\bar{v}_i) = \rho(v_i) \pmod{p}\). Then \(cu(\bar{v}) \equiv cu(v) + pw^T G v + pv^T G w \pmod{p^2}\). Since \(G\) is symmetric and \(w^T G v\) is an integer we have that

\[
cu(\bar{v}) \equiv cu(v) + 2pv^T G w \pmod{p^2}
\]

but coloring vectors have the property that

\[
v^T G w \equiv 0 \pmod{p}
\]

for any vector \(w\) (see Lemma 4.6 below) which establishes the result.

Now we will prove the one loose end from the previous lemma.

**Lemma 4.6.** If \(v\) is a coloring vector, then \(v^T G w \equiv 0 \pmod{p}\) for any vector \(w = (w_1, \ldots, w_n)^T\).
Figure 4.1. The loop $\zeta_1$.

Proof. Let $F$ be a spanning surface for $K$. Consider again the “pushoff function” $\tau^{-1}(y) \in H_1(S^3 - F)$. We must treat the case when $y$ is orientation preserving separately from the orientation reversing case.

If $y$ is an orientation preserving curve, $\tau^{-1}(y)$ is represented by two loops $y_+$ and $y_-$ in the complement of the surface. Take the basepoint for the fundamental group of the complement of $F$ to be on $y_+$ (the positive pushoff). Then $[y_+]$ and $[y_-]$ when thought of as homotopy classes are represented by the loops $y_+$ and $\zeta y_- \zeta^{-1}$ where $\zeta$ is a path from the basepoint to $y_-$ which misses the surface. Now consider $[y_+ y_-]$ as an element of the fundamental group of the knot complement (with the same choice of basepoint). Then $y_+ y_- \in S^3 - K$ is homotopic to $y_+ \zeta_1 y_+ \zeta_1^{-1}$ where $\zeta_1$ is the loop arising from $\zeta$ during the homotopy from $y_-$ to $y_+$ in $S^3 - K$ as in Figure 4.1. Then $\rho([y_+ y_-])$ is trivial in $D_{2p}$.

In the case when $y$ is an orientation reversing curve, let us choose the basepoint for $\pi_1(S^3 - K)$, call it $\ast$, to coincide with the basepoint for $\pi_1(F)$ and let $y$ be a loop in $F$ based at $\ast$. Now $\tau^{-1}(y)$ is a single loop that double covers $y$ which is homotopic to the element $[y^2] \in \pi_1(S^3 - K)$. Furthermore $\tau^{-1}(y)$ must intersect the surface once (mod 2) because we can push it slightly off the surface everywhere except at $\ast$ suggested by the local picture in Figure 4.2. Therefore $\rho(\tau^{-1}(y))$ is the square of an element of order two and so it is trivial in $D_{2p}$.
We have shown that in both the orientable and non-orientable cases the Gordon-Litherland form has the property

$$\mathcal{G}_F(\kappa, \tau(y)) \equiv 0 \pmod{p}$$

for

$$\kappa = \sum_i v_i x_i$$

and $y$ is any homology class in $H_1(F; \mathbb{Z})$. In particular we have

$$v^T Gw \equiv 0 \pmod{p}$$

for a coloring vector $v$ and any vector $w$.

Thus, we have shown that

$$v^T Gv \equiv 0 \pmod{p}$$

for a coloring vector $v$. That is $\frac{v^T Gv}{p} \pmod{p}$ is an element of $\mathbb{Z}_p$ as is needed for the right hand side of Equation (4.3) to make sense.

We have shown that the colored untying invariant defined by the Goeritz matrix is well-defined for any coloring vector arising from any choice of basis for $H_1(F; \mathbb{Z})$ for a fixed spanning surface $F$. We will now show that we may choose any spanning surface.

**Lemma 4.7.** The colored untying invariant is independent of the choice of spanning surface.

**Proof.** Spanning surfaces are related by (i) $S$-equivalence in the usual sense (see [BFK]), or (ii) addition or deletion of a single twisted band (see Figure 4.3). Note that operation
(ii) may perhaps change the orientability of the resulting surface. We will now show that the right hand side of Equation (4.3) is unchanged by all three types of moves.

Let $F$ and $F'$ denote $S$-equivalent possibly non-orientable spanning surfaces for $K$ and let $(G, v)$ and $(G', v')$ be the corresponding pairs consisting of a Goeritz matrix and a coloring vector. Then $(G', v')$ may be obtained from $(G, v)$ by a finite number of the following operations:

$$
\Lambda_1 : (G, v) \mapsto (PGP^T, Pv \pmod{p})
$$

and

$$
\Lambda_2 : (G, v) \mapsto (G'', v'')
$$

where $P$ is an invertible, unimodular, integer matrix and

$$
G'' = \begin{pmatrix}
* & 0 \\
G & \vdots \\
* & 0 \\
* \cdots & * & 0 & 1 \\
0 \cdots & 0 & 1 & 0 \\
\end{pmatrix}
$$

and $v'' = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$. A straightforward calculation shows that $cu$ is unchanged by either of the $\Lambda$-moves.
The effect on \((G, v)\) when we add a single twisted band is

\[
G'' = \begin{pmatrix}
0 \\
G \\
\vdots \\
0 \\
0 & \cdots & 0 & \pm 1
\end{pmatrix}
\]

and \(v'' = \begin{pmatrix} v \\ 0 \end{pmatrix}\) (see Figure 4.4).

Thus, the colored untying invariant defined by the Goeritz matrix is unchanged by any of the moves.

The next lemma will be used exclusively in the proof of Theorem 4.10 below.

**Lemma 4.8.** If \(L \in S^3 - K\) is a link so that its homotopy class \([L]\) is in \(\ker(\rho)\) then \(L \in S^3 - F\) for some spanning surface \(F\) for \(K\). Notice that we do not need to assume that \(L\) has unlinked or unknotted components.

**Proof.** From Diagram (4.1) we have seen that if \([L]\) is in the kernel of \(\rho\) then \(lk(L, K) \equiv 0 \pmod{2}\). Then \(L\) intersects \(F\) an even number of times however two adjacent (innermost) intersections can have opposite or the same sign. If they have opposite sign then we may resolve them by “tubing off” these intersections with a tube which does not change the orientability of the surface as in Figure 4.5 (a). Otherwise we may resolve the intersections with a non-orientable tube as in Figure 4.5 (b). The resulting spanning surface is \(S\)-equivalent (in the non-orientable sense of \(S\)-equivalence) to \(F\) and has reduced the number of intersections with \(L\).
4.3 Surgery Equivalence

First we will prove Proposition 4.3, then we will show, via the Goeritz definition, that the colored untying invariant is an invariant of $\pm 1$ surgery in the kernel of $\rho$.

Proof of Proposition 4.3. The authors would like to thank Pat Gilmer for suggesting this method of proof.

We wish to relate $cu(K, \rho) = a \cup \beta^1(a) \in \mathbb{Z}_p$ to $cu(K, \rho)' = \frac{x_{\text{Go}}}{p}(\text{mod } p)$. We will show that the “bockstein definition” $cu(K, \rho)$ is given by the linking pairing on $H^1(\tilde{X}; \mathbb{Q}/\mathbb{Z})$ where $\tilde{X}$ is the double-branched cover along $K$ of the 3-sphere. On the other hand the Goeritz matrix gives an equivalent linking pairing on $\text{Hom}(H_1(\tilde{X}), \mathbb{Q}/\mathbb{Z})$. Moreover, given a presentation of the first homology of the double-branched cover, the two pairings give the same element of $\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z}$.

Consider the following commutative diagram consisting of coefficient groups.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times p} & \mathbb{Z} & \xrightarrow{(\text{mod } p)} & \mathbb{Z}_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \xrightarrow{\times 1/p} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array}
\]
where \( j \) is the natural inclusion of \( \mathbb{Z}_p \) into \( \mathbb{Q}/\mathbb{Z} \), more precisely \( \mathbb{Z}_p \cong (1/p)\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} \). In particular, if \( \hat{a} \) is the element of \( H^1(\tilde{X}, \mathbb{Q}/\mathbb{Z}) \) corresponding to \( a \in H^1(\tilde{X}; \mathbb{Z}_p) \) from the bockstein definition of the colored untying invariant then \( \hat{a} \) is determined by the vector \( \hat{v} = \frac{u}{p} \) with respect to a choice of basis for \( H_1(\tilde{X}; \mathbb{Z}) \). That is \( v \) is the coloring vector which describes where the “coloring” \( \rho' \) sends a generating set of \( H_1(\tilde{X}; \mathbb{Z}) \).

Under the isomorphisms

\[
\begin{align*}
\text{Hom}(H_1(\tilde{X}), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\Gamma} H^1(\tilde{X}; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta^1} H^2(\tilde{X}; \mathbb{Z}) \xrightarrow{\cong} H_1(\tilde{X}; \mathbb{Z})
\end{align*}
\]

arising from the universal coefficient theorem, the definition of the bockstein homomorphism \( \beta^1 \), and Poincare’ duality there is a correspondence between the bilinear pairing on \( H^1(\tilde{X}; \mathbb{Q}/\mathbb{Z}) \) defined by \( (a, b) \mapsto [\tilde{X}] \cap (a \cup \beta^1(b)) \) and the linking form on \( H_1(\tilde{X}; \mathbb{Z}) \). Here \( [M] \in H_3(\tilde{X}) \) denotes the fundamental class of the 3-manifold. Furthermore, under the isomorphism \( \Gamma \), the pairing corresponds to the form \( \lambda \) given in [Gil, page 8] on \( \text{Hom}(H_1(\tilde{X}), \mathbb{Q}/\mathbb{Z}) \) relative to the generators \( \{x_i\} \) for \( H_1(F) \), for some spanning surface \( F \), and their duals \( \{\xi_i\} \) which generate \( H_1(S^3 - F) \). Now by [GorLi], this matrix is the Goeritz matrix \( G \). Thus

\[
p \cdot \lambda(\Gamma^{-1}(\hat{a}), \Gamma^{-1}(\hat{a})) = \hat{a} \cup \beta(\hat{a}) = cu(K, \rho) \in \mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z}.
\]

And so

\[
\frac{cu(K, \rho)}{p} = \frac{\nu'G\nu}{p^2} = \frac{cu(K\rho)'}{p}
\]

as desired.

\[ \Box \]

**Remark 4.9.** Notice that the above proof is independant Moskovich’s Seifert matrix definition ([Mos, Lemma 18]) altogether. The following is an alternate proof which assumes that the colored untying invariant is defined by the Seifert matrix.

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Alternate Proof of Proposition 4.3. Let $F$ be a spanning surface for the knot $K$. If $F$ is orientable, then the Goeritz matrix associated to $F$ is exactly the symmeterized Seifert matrix, $G = S + S^T$, for $F$ with Seifert matrix $S$ and there is nothing to show.

In both the orientable and non-orientable cases, the Goeritz matrix $G$ is a presentation matrix for $H_1(\tilde{X})$ where $\tilde{X}$ is the brached double cover of the 3-sphere as before (see [GorLi] or Chapter 9 of [Lic2]). So $G$ is just an extension of the symmeterized Seifert matrix to include non-orientable spanning surfaces.

Lemma 4.6 shows that $v^T \cdot G \cdot v \equiv 0 \pmod{p}$ as required for Equation (4.3) to make sense. The equality of the Goeritz definition of $cu$ and Moskovich’s definition then follows from the fact that $H_1(\tilde{X})$ is presented by some symmeterized Seifert matrix $S + S^T$ as well as $G$. So we have the equality $v^T G v = v^T (S + S^T) v$. 

We will now show that the colored untying invariant is a surgery equivalence invariant for $p$-colored knots. Note that Moskovich gives an alternate algebraic proof in [Mos].

**Theorem 4.10.** The colored untying invariant $cu(K, \rho)$ is invariant under $\pm 1$-framed surgery in the kernel of $\rho$.

**Proof.** From Proposition 4.3 we may assume that $cu(K, \rho) = \frac{v^T G x}{p} \pmod{p}$ for some coloring vector $v = (v_1, \ldots, v_n)^T$ and Goeritz matrix $G$ corresponding to a spanning surface for $K$. Let $[L]$ be in the kernel of the coloring for $K$ represented by an unlink $L$ in the complement of the knot. Lemmas 4.7, and 4.8 imply that the spanning surface $F$ may be chosen so that $L \cap F = \emptyset$. Furthermore, let $K$ be in disk-band form (see [BuZi, Chapter 8]).

Under these conditions, $\pm 1$-surgery along one component of $L$ adds a single full twist in $k$ parallel bands of $K$ corresponding to generators (after renumbering perhaps) $x_1, \ldots, x_k$.
for $H_1(F)$ with $v_1 + \cdots + v_k \equiv 0 \pmod{p}$. Then the pair $(G, v)$ changes as follows:

$$G \mapsto G + \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} = G' \text{ and } v \mapsto v$$

where $N$ is a $k \times k$ matrix whose entries are all 2. Thus,

$$v^T G' v = p \cdot cu(K, \rho) + v^T \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} v$$

$$= p \cdot cu(K, \rho) + (v_1 \cdots v_k) \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

$$= p \cdot cu(K, \rho) + 2(v_1 + \cdots + v_k)^2$$

$$\equiv p \cdot cu(K, \rho) \pmod{p^2}$$

and so the colored untying invariant is unchanged by $\pm 1$-surgery along $L$. □

We will now show by explicit example that $cu$ is non-trivial for all $p$. We will also show that there are at least $p$ surgery classes of $p$-colored knots and that connected sums of $(p,2)$-torus knots give a representative of each of these $p$ classes.

### 4.4 Examples

Since we may pick any spanning surface for the knot regardless of orientation, we shall always use the spanning surface corresponding to a checkerboard coloring for a diagram for $K$.

**Example 4.11. 7-colorable knots of genus 1 with at most 12 crossings and the $7_1$ knot.**

From the table of knots given by KnotInfo [Knot], the only 7-colorable knots of genus 1 with at most 12 crossings are $5_2$, $11_{n141}$, and $12_n0803$. The $7_1$ knot is not of genus 1 but is an interesting example with a similar calculation. By genus here, I mean the
minimum genus over all possible orientable (Seifert) spanning surfaces for the knot. We will show that the colors of two arcs at any crossing in the diagrams given in Figure 4.6 determine the coloring as well as the colored unknotting invariants. Note that Figure 4.6 (a) shows the coloring which is forced by the choice of $a$ and $b$ in $\mathbb{Z}_p$ as well as the choice of generators $\{x_i\}$ and $\{\xi_i\}$ for $H_1(F;\mathbb{Z})$ and $H_1(S^3 - F;\mathbb{Z})$ respectively. However, in (b)-(d), the redundant labels are omitted. The infinite region is labeled by * and the other white regions are understood to be numbered to coincide with the numbering of the $\xi$’s.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{knots}
\caption{The (a) $5_2$, (b) $11_{n141}$, (c) $12_{a0803}$, and (d) $7_1$ knots.}
\end{figure}
Proposition 4.12. The colored untying invariants for the 7-colorable knots 5_2, 11_{n141},
12_{a0803}, and 7_1 are non-zero multiples of squares for any non-trivial coloring. In particular,
there are three distinct values of cu, one for each square modulo 7, for each of the four
knots depending on the coloring class.

Proof. First we must pick a white region in a checkerboard coloring for the diagram to be
the so-called “infinite region.” If F is the spanning surface described by the black regions
of the checkerboard coloring, then a basis for $H_1(F)$ is represented by loops $\{x_1, \ldots, x_n\}$
which are parallel to the boundary of each white region excluding the infinite region.
Then, the coloring vector is

$$v = (\overline{p}(\xi_1), \ldots, \overline{p}(\xi_n))^T$$

where $\overline{p} : H_1(S^3 - F) \to \mathbb{Z}_p$ is the map at the top of Diagram 4.1, and $q : \mathbb{Z}_p \to \mathbb{Z}$ is the
forgetful map as in Proposition 4.3. Here $\{\xi_i\}$ is a basis for $H_1(S^3 - F)$ represented by
loops in the complement of the surface that pass through the infinite region and the i-th
white region exactly once each so that $lk(x_i, \xi_i) = \delta_{ij}$.

Then the Goeritz matrices in question are:

$$G(5_2) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix},$$

$$G(11_{n141}) = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 & -5 \end{pmatrix},$$

$$G(12_{a0803}) = \begin{pmatrix} -11 & 1 \\ 1 & -2 \end{pmatrix}.$$
and

\[ G(7_1) = (-7). \]

And so the colored untying invariants are: \( cu(5_2) = 5(b - a)^2 \), \( cu(11_{n141}) = 5(b - a)^2 \), \( cu(12_{a0808}) = (b - a)^2 \), and \( cu(7_1) = 6(b - a)^2 \) where each is understood to be modulo 7.

Notice that the above construction for \( cu(7_1) \) easily generalizes for all odd integers \( p \).

**Example 4.13. The \((p, 2)\)-torus knots for any \( p \).**

The \( 7_1 \) knot is also known as the \((7, 2)\)-torus knot. As an extension of the construction used to calculate \( cu(7_1) \), Figure 4.7 gives the general result. Note that the \( p \) in the figure denotes \( p \) “positive” half twists.

So \( cu((p, 2), \rho) = -(b - a)^2 \) which implies that there is one colored untying class for each square modulo \( p \) for the “left-handed” \((p, 2)\)-torus knot. Notice that if we changed all the crossings from “positive” to “negative” half-twists then the resulting colored untying invariant would just be \( +(b - a)^2 \). Therefore, if \(-1\) is not a square modulo \( p \), the left-handed \((p, 2)\)-torus knot is not surgery equivalent to the “right-handed” one for any choice of coloring. For example, if \( p = 7 \) then this is the case, however, if \( p = 5 \) then there are choices of \( a \) and \( b \) so that the colored untying invariants give the same value of \( \mathbb{Z}_p \).
We will now show that the colored untying invariant is additive under the operation of the connected sum of \( p \)-colored knots. As an immediate corollary of this we see that the connected sum of \( k \) \((p, 2)\)-torus knots for \( k = 1, \ldots, p \), with the appropriate choices of colorings, give a complete list of representatives of the colored untying invariant classes.

**Proposition 4.14.** The colored untying invariant is additive under the operation of the connected sum of \( p \)-colored knot.

**Proof.** Let \((K_1, \rho_1)\) and \((K_2, \rho_2)\) be \( p \)-colored knots with Goeritz matrix-coloring vector pairs \((G_1, w)\) and \((G_2, z)\). Then the pair, \((G, v)\), for the connected sum \((K_1 \# K_2, \rho_1 \# \rho_2)\) is given by

\[
G = \begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\]

and

\[
v = \begin{pmatrix}
w \\
w
\end{pmatrix}
\]

the block diagonal of the Goeritz matrices and the concatenation of the coloring vectors of the summands. It is clear then that

\[
cu(K_1 \# K_2, \rho_1 \# \rho_2) = cu(K_1, \rho_1) + cu(K_2, \rho_2)
\]

holds. Note that, as Figure 4.8 suggests, it is easy to see that the Goeritz matrix and coloring vectors have the properties above if we pick an appropriate checkerboard coloring. Namely, we wish to pick the infinite regions for the checkerboard colorings for the summands so that the checkerboard coloring for the connected sum is determined. \( \square \)

We have shown that \( cu(K, \rho) \) is a non-trivial, additive, surgery equivalence invariant of \( p \)-colored knots. We will now define the bordism invariants which exhibit the same properties. They are all, in fact, the same invariant. We used the Goeritz definition of the colored untying invariant to establish a lower bound on the number of surgery equivalence
classes. To obtain an upper bound we will need a definition of $cu$ in the context of bordism theory.
Chapter 5
The Bordism Invariants

Once again, let $\tilde{X}_0$ and $X_0$ be the manifolds obtained by performing 0-framed surgery along $K$ to the manifolds $\tilde{X}$ (the 2-fold brached cover of $S^3$) and $S^3$ itself respectively. If we have a map $f : M^3 \rightarrow K(G)$ where $K(G)$ denotes the Eilenberg-Maclane space $K(G, 1)$ then the image of the fundamental class under the induced map $f_* : H_3(M; \mathbb{Z}) \rightarrow H_3(K(G); \mathbb{Z})$ is an 3-manifold invariant. The construction is exactly the same as the invariants described by T. Cochran, A. Gerges, and K. Orr in [CGO]. We will divide the bordism invariants into two categories closed and relative.

5.1 The Closed Bordism Invariants

As mentioned earlier in the “Preliminaries” $H_3(K(G); \mathbb{Z}) \cong \Omega_3(G)$ and it is in this context that the bordism invariants arise. So to define the closed bordism invariants, denoted by $\omega_2(K, \rho)$ and $\omega_0(K, \rho)$, we must find maps from $\tilde{X}_0$ and $X_0$ to Eilenberg-Maclane spaces over the appropriate groups.

We wish to have maps which arise naturally from the coloring $\rho$. Recall that the second derived group of $G$, denoted $G^{(2)}$, is defined to be the commutator subgroup of the commutator subgroup of $G$. That is $G^{(2)} = [G_2, G_2]$ where $G_2 = [G, G]$. Since a preferred longitude of the knot $K$ is in the second derived group of $\pi_1(S^3 - K)$ it must be mapped trivially by $\rho$. Hence the map $\rho'$ from Diagram 4.1 factors through

$$
\pi_1(\tilde{X}_0) \quad \pi_1(X - \tilde{K}) \quad \pi_1(\tilde{X}) \quad \mathbb{Z}_p
$$
which establishes the existence of a map $\tilde{f} : \tilde{X}_0 \to K(\mathbb{Z}_p)$ as desired. Likewise, and perhaps even easier to see, we have that $\rho$ factors through

$$ \pi_1(\tilde{X}_0) \xrightarrow{\pi_1(S^3 - K)} \pi_1(S^3 - K) \xrightarrow{\rho} D_{2p} $$

which gives a natural map

$$ f : X_0 \to K \left( \frac{\pi_1(S^3 - K)}{\ker \rho} \cong D_{2p}, 1 \right). $$

We will show that the induced maps on homology of $\tilde{f}$ and $f$ define invariants of not only the 3-manifolds $\tilde{X}_0$ and $X_0$ but they are also surgery equivalence invariants for the $p$-colored knot $(K, \rho)$.

**Definition 5.1.** Suppose $\tilde{f} : \tilde{X}_0 \to K(\mathbb{Z}_p)$ and $f : X_0 \to K(D_{2p})$ are the maps obtained via the coloring $\rho$ as above. Then define the closed bordism invariants to be

$$ \omega_2(K, \rho) := \tilde{f}_*([\tilde{X}_0]) \in H_3(\mathbb{Z}_p; \mathbb{Z}) $$

and

$$ \omega_0(K, \rho) := f_*([X_0]) \in H_3(D_{2p}; \mathbb{Z}) $$

where $[M] \in H_3(M; \mathbb{Z})$ denotes the fundamental class of $M$.

Notice that the invariants depend on the bordism classes of the (closed) 3-manifolds over $\mathbb{Z}_p$ and $D_{2p}$ respectively which is the motivation for the names. It is also clear that $\tilde{X}$ and $\tilde{X}_0$ are in the same bordism class over $\mathbb{Z}_p$. The bordism is constructed from $\tilde{X} \times [0, 1]$ by attaching a 2-handle along the lift of the preferred longitude.

The final bordism invariant, denoted simply by $\omega$, arises from the manifold $M = (S^3 - K)$ which is not closed so it will be defined separately. Also note that since $\pm 1$-framed surgery along links in the kernel of the coloring $\rho$ defines a bordism between the resulting
manifolds then $\omega_2$ and $\omega_0$ are actually surgery equivalence invariants. The bordism is obtained by attaching a 2-handle along each component of the surgery link to $M \times [0, 1]$ (for $M = \tilde{X}_0, X_0$).

### 5.2 The Relative Bordism Invariant

Recall the definition of a based $p$-colored knot which is a $p$-colored knot with a chosen meridian $m$ so that $\rho(m) = ts^0$. That is, if the coloring $\rho$ is defined by a labeling of a diagram for $K$ then the arc corresponding to $m$ would have the label 0. We may assume this because $p$-colored knots are only defined up to an inner automorphism of the dihedral group. This allows, in particular, for any chosen arc to have the label 0. We will now define the last of the three bordism invariants.

**Definition 5.2.** Let $(K, \rho, m)$ be a based $p$-colored knot. If $K(\mathbb{Z}_2)$ is the subspace of $K(D_{2p})$ corresponding to the image of $m$ under the coloring, then define

$$\omega(K, \rho) := \rho([M, \partial M]) \in H_3(K(D_{2p}), K(\mathbb{Z}_2); \mathbb{Z})$$

where $[M, \partial M]$ denotes the fundamental class of $M = (S^3 - K)$ relative to the boundary and $f : (M, \partial M) \rightarrow (D_{2p}, \mathbb{Z}_2)$ arises directly from the coloring.

Indeed, we may think of $K(\mathbb{Z}_2)$ as a subspace of $K(D_{2p})$ because we may construct a $K(D_{2p})$ from a $K(\mathbb{Z}_2)$ by adding $k$-cells, $k = 1, 2, \ldots$, to obtain the correct homotopy groups. Furthermore, since we can assume that the fundamental group of the boundary torus is generated by the classes represented by the preferred longitude and our chosen meridian $m$, it is clear that $\partial M$ is mapped into the correct subspace.

We will now prove a few special properties of the bordism invariants.

### 5.3 Properties

Consider the Bordism Long Exact Sequence of the pair $(X, A)$

$$\cdots \rightarrow \Omega_n(A) \overset{i_*}{\rightarrow} \Omega_n(X) \overset{j_*}{\rightarrow} \Omega_n(X, A) \rightarrow \Omega_{n-1}(A) \rightarrow \cdots$$  \hspace{1cm} (5.1)
for \( i_\ast \) and \( j_\ast \) induced by inclusion (see Section 5 of [CoFl]). We will be concerned with
the pairs \((X, A) = (K(D_{2p}), K(\mathbb{Z}_p))\) and \((X, A) = (K(D_{2p}), K(\mathbb{Z}_2))\) which will relate \( \omega_2 \)
to \( \omega_0 \), and \( \omega_0 \) to \( \omega \) respectively.

In these cases, we may compute the bordism groups since

\[
\Omega_n(K(G, 1)) \cong H_n(G; \mathbb{Z}).
\]

The cohomology groups of cyclic groups are well-known and may be computed using a
spectral sequence for the fibration

\[
K(\mathbb{Z}, 1) \rightarrow K(\mathbb{Z}_p, 1) \rightarrow K(\mathbb{Z}, 2)
\]

with fiber \( K(\mathbb{Z}, 1) \) being a circle (see Chapter 9 [DaKi]). The homology groups are obtained
from the cohomology groups by using the Universal Coefficient Theorem. We have

\[
H_n(\mathbb{Z}_p) \cong \begin{cases} 
  \mathbb{Z} & \text{if } n = 0, \\
  \mathbb{Z}_p & \text{if } n \text{ is odd, and} \\
  0 & \text{if } n > 0 \text{ is even}
\end{cases}
\]

for \( p \) any odd number. The following proposition follows from a spectral sequence found
in [AdMi]; another calculation for this is given in the Appendix.

**Proposition 5.3.** The homology groups of the dihedral group \( D_{2p} \) are as follows

\[
H_n(D_{2p}) \cong \begin{cases} 
  \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{4}, \\
  \mathbb{Z}_2 \oplus \mathbb{Z}_p & \text{if } n \equiv 3 \pmod{4} \\
  0 & \text{otherwise}
\end{cases}
\]

if \( p \) is an odd integer, and

\[
H_n(D_{2p}) \cong \begin{cases} 
  \mathbb{Z}_2^{(n+3)/2} & \text{if } n \equiv 1 \pmod{4}, \\
  \mathbb{Z}_2^{(n+1)/2} \oplus \mathbb{Z}_p & \text{if } n \equiv 3 \pmod{4} \\
  0 & \text{otherwise}
\end{cases}
\]

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if \( p \) is even.

So the closed bordism invariants \( \omega_2 \) and \( \omega_0 \) may be thought of as elements of \( \mathbb{Z}_p \) and \( \mathbb{Z}_{2p} \) respectively. The proof of this proposition is given in the Appendix.

We will use the Bordism Long Exact Sequence (5.1) to determine the group in which the relative bordism invariant \( \omega \) resides. Consider

\[
\cdots \longrightarrow \Omega_3(K(\mathbb{Z}_2)) \overset{i_*}{\longrightarrow} \Omega_3(K(D_{2p})) \overset{j_*}{\longrightarrow} \Omega_3(K(D_{2p}), K(\mathbb{Z}_2)) \longrightarrow \Omega_2(K(\mathbb{Z}_2)) \longrightarrow \cdots
\]

where the Eilenberg-Maclane space \( K(\mathbb{Z}_2) \) is the subspace of \( K(D_{2p}) \) arising from the subgroup \( \mathbb{Z}_2 \cong \langle t \rangle \in D_{2p} \). In this case \( i_* \) is injective since any singular manifold \( (M, \varphi) \) that is null-bordant over \( D_{2p} \) is null-bordant over \( \mathbb{Z}_2 \) via the same 4-manifold. As \( \Omega_2(K(\mathbb{Z}_2)) \) is trivial we have

\[
0 \longrightarrow \Omega_3(K(\mathbb{Z}_2)) \cong \mathbb{Z}_2 \longleftarrow \Omega_3(K(D_{2p})) \cong \mathbb{Z}_{2p} \longrightarrow \Omega_3(K(D_{2p}), K(\mathbb{Z}_2))) \longrightarrow 0
\]

is exact. In particular \( \Omega_3(K(D_{2p}), K(\mathbb{Z}_2))) \cong \mathbb{Z}_p \). So the relative bordism invariant \( \omega \) may be regarded as an element of \( \mathbb{Z}_p \). We will later show, in the proof of Theorem 6.3, that the closed bordism invariant \( \omega_0 \in \Omega_3(K(D_{2p})) \cong \mathbb{Z}_{2p} \) only takes values in the \( \mathbb{Z}_p \) part of \( \mathbb{Z}_{2p} \) which will establish an equivalence between all three bordism invariants.

We have already seen that the colored untangling invariant is additive under the operation of the connected sum of \( p \)-colored knots. The same is true for the bordism invariants. Of course, once we have established the equivalence of all the invariants, then the additivity of \( cu \) is enough to show this. However, we will show this for the closed bordism invariants directly here.

**Proposition 5.4.** The closed bordism invariants are additive.

**Proof.** Let \((K_1, \rho_1)\) and \((K_2, \rho_2)\) be \( p \)-colored knots and denote by \((K_\#, \rho_\#)\) their connected sum. Let \( \tilde{X}_0(K) \) denote the manifold obtained by 0-surgery along \( K \) of the 2-fold branched cover \( \tilde{X} \). Also let \( X_0(K) \) denote 0-surgery of \( S^3 \) along \( K \). Let \( \tilde{f}_K : \tilde{X}_0(K) \rightarrow \)
$K(\mathbb{Z}_p)$ and $f_K : X_0(K) \to D_{2p}$ be the maps arising from the colorings as above in Section 5.1. As addition in the bordism groups is by disjoint union and addition of $p$-colored knots is by connected sum, we must show that $(\tilde{X}_0(K\#), \tilde{f}_\# = \tilde{f}_{K_1}\#\tilde{f}_{K_2})$ is bordant to the disjoint union $(\tilde{X}_0(K_1) \sqcup \tilde{X}_0(K_2), \tilde{f}_{K_1} \sqcup \tilde{f}_{K_2})$ over $\mathbb{Z}_p$. We must also show a similar result for the 0-surgered manifolds over the dihedral group.

The former of the two conclusions follows from the fact that the 2-fold branched cover of $S^3$ along the connected sum of knots is homeomorphic to the connected sum of the 2-fold branched covers along each summand. Thus

$$(\tilde{X}_0(K\#), \tilde{f}_\#) \sim_{\mathbb{Z}_p} (\tilde{X}_0(K_1)\#\tilde{X}_0(K_2), \tilde{f}_\#)$$

$$(\tilde{X}_0(K_1) \sqcup \tilde{X}_0(K_2), \tilde{f}_{K_1} \sqcup \tilde{f}_{K_2})$$

where the final bordism is constructed by attaching a trivial 0-handle to $(\tilde{X}_0(K_1) \sqcup \tilde{X}_0(K_2)) \times [0, 1]$. To see the bordism between $(X_0(K\#), f_\# = f_{K_1}\#f_{K_2})$ and $(X_0(K_1) \sqcup X_0(K_2), f_{K_1} \sqcup f_{K_2})$ we will attach a 2-handle to $X_0(K\#) \times [0, 1]$ along an appropriate curve $\alpha$, and then perform a handle slide.

Let $\gamma$ represent the preferred longitude for $K\#$ so that $lk(\gamma, K_i) = 0$ for $i = 1, 2$. Then $X_0(K_1\#K_2)$ is defined by 0-surgery along $\gamma$ which we represent by a surgery diagram consisting of $\gamma$ labeled with a 0. Likewise, represent $X_0(K_i)$ by 0-surgery along $\gamma_i$ with $lk(\gamma_i, K_i) = 0$ and $[\gamma_1] + [\gamma_2] = [l] \in \pi_1(S^3 - K\#)$. If $S^2$ is a two-sphere separating $K_1$ from $K_2$ then it becomes a torus after the 0-surgery along $\gamma$. Let $\alpha$ be a curve representing $[\gamma_2]$ which lies on the boundary of the “separating torus” so that 0-surgery along $\alpha$ and $\gamma$ may be represented by the surgery diagram given in Figure 5.1 (a). Note that each curve is understood to have framing 0.

Notice that the two arcs joining $K_1$ and $K_2$ necessarily have the same label defined by the colorings $\rho_i$ (from the definition of the connected sum of $p$-colored knots). Moreover, we have $lk(\alpha, K\#) = 0$. Then $\alpha$ must be in the kernel of the coloring $\rho\#$ for the connected
sum. Thus the map $f_\#$ extends over the new manifold obtained by surgery along $\gamma$ and $\alpha$. We may now perform a handle slide to obtain a surgery diagram for $X_0(K_1)\#X_0(K_2)$ (see Figure 5.1 (b)). Thus, we have shown that $(X_0(K_\#), f_\#)$ is bordant to the connected sum of the manifolds obtained by 0-surgery along each of $K_i$. The bordism

$$(X_0(K_1)\#X_0(K_1), f_{K_1}\#f_{K_2}) \sim_{D_{2p}} (X_0(K_1) \sqcup X_0(K_2), f_{K_1} \sqcup f_{K_2})$$

is obtained by adding a 0-handle to

$$(X_0(K_1) \sqcup X_0(K_2), f_{K_1} \sqcup f_{K_2}) \times [0, 1]$$

analogously as before and this completes the proof.

As a corollary to Proposition 5.4 we see that if $\omega_0 \in \Omega_3(K(D_{2p})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_p$ is a $p$-valued invariant of $p$-colored knots, then $\omega_0(K, \rho) = (0, n)$ for any colored knot $(K, \rho)$. This is because if $\omega_0$ is $p$-valued then every $p$-colored knot must have the same value in the first coordinate of $\mathbb{Z}_2 \oplus \mathbb{Z}_p \cong \mathbb{Z}_{2p}$. Since $\omega_0$ is additive we have that

$$\omega_0(K\#K, \rho\#\rho) = (2k, 2n),$$

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and so the first coordinate value must be equal to 0 in \( \mathbb{Z}_2 \). We will show that \( \omega_2(K, \rho) = 2 \omega_0(K, \rho) \) which will establish an equivalence between \( \omega_0 \) and \( \omega_2 \) once we show that \( \omega_0 \) is \( p \)-valued.
Chapter 6
Proof of Equivalence

We will now show that all of the \( p \)-colored knot invariants defined above are the same.

6.1 Equivalence of \( cu \) and \( \omega_2 \)

**Proposition 6.1.** The colored untying invariant \( cu(K, \rho) \) is equivalent to the \((2\text{-fold branched cover})\) closed bordism invariant \( \omega_2(K, \rho) \) for any \( p \)-colored knot \( (K, \rho) \).

**Proof.** Again, denote by \( \tilde{X} \) the 2-fold branched cover of \( S^3 \). Then by the commutativity of Diagram (4.1), there is a map \( \tilde{f} : \tilde{X} \to K(Z_p,1) \) which corresponds to the coloring \( \rho \). Let \( \beta^1 : H^1(\tilde{X};Z_p) \to H^2(\tilde{X};Z) \) be a Bockstein homomorphism associated with the coefficient sequence

\[
0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{Z}_p \to 0.
\]

Recall that if \( a \in H^1(\tilde{X};Z_p) \) is the cohomology class corresponding to

\[
\rho' : H_1(\tilde{X};\mathbb{Z}) \to \mathbb{Z}_p
\]

then

\[
cu(K, \rho) = a \cup \beta^1(a) \in H^3(\tilde{X};Z_p) \cong \mathbb{Z}_p
\]

by Moskovich’s definition of the colored untying invariant. Notice that the identification of \( cu(K, \rho) \) with an element of \( \mathbb{Z}_p \) is via evaluation on the fundamental class.

Consider the maps \( \tilde{X} \xrightarrow{\tilde{f}} K(\mathbb{Z}_p) \xrightarrow{id} K(\mathbb{Z}_p) \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
H_1(\tilde{X}) & \xrightarrow{\rho'} & H_1(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
H_1(\tilde{X}) & \xrightarrow{f_*} & H_1(K(\mathbb{Z}_p)) \\
\end{array}
\]

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where $i : H_1(K(Z_p)) \to \mathbb{Z}_p$ corresponds to the cohomology class in $H^1(K(Z_p); \mathbb{Z}_p)$ induced by the identity $id : K(Z_p) \to K(Z_p)$. Notice that $a$ corresponds with the homomorphism $\rho'$ by construction, while $\rho'$ corresponds with the cohomology class $\tilde{f}^*1 \in H^1(\tilde{X}; \mathbb{Z}_p)$. The correspondence of $\tilde{f}^*1$ and $a$ is exactly $\tilde{f}^*1(i) = a$.

Then, by the properties of cup products we have
\[
\tilde{f}^*3(i \cup \beta^1(i)) = a \cup \beta^1(a)
\]
which gives the element of $\mathbb{Z}_p$ given by $\left[\tilde{X}\right] \cap (a \cup \beta^1(a))$. On the other hand, if we think of $(i \cup \beta^1(i))$ as a chosen fixed generator of $H^3(K(Z_p); \mathbb{Z}_p)$, then this is the same as
\[
\tilde{f}^*3\left([\tilde{X}]\right) \cap (i \cup \beta^1(i))
\]
which is the identification of $\omega_2(K, \rho)$ with an element of $\mathbb{Z}_p$. Note that the non-triviality of the colored untying invariant implies that $(i \cup \beta^1(i))$ is a generator of $H^3(K(Z_p); \mathbb{Z}_p)$.

Hence, with these identifications of $H^3(\tilde{X}; \mathbb{Z}_p)$ and $H_3(K(Z_p); \mathbb{Z})$ with $\mathbb{Z}_p$, the elements $cu(K, \rho) \in H^3(\tilde{X}; \mathbb{Z}_p)$ and $\omega(K, \rho) \in H_3(K(Z_p); \mathbb{Z})$ are the same as elements of $\mathbb{Z}_p$.

We will now prove that all of the colored surgery equivalence invariants are equivalent.

## 6.2 Equivalence of the Bordism Invariants

To show that the closed bordism invariants $\omega_0$ and $\omega_2$ are equivalent it suffices to show two facts. First we must show that $\omega_2(K, \rho)$ is roughly speaking “twice” $\omega_0(K, \rho)$. Then we must show that $\omega_0$ is a $p$-valued invariant. This, in turn, will show that all of the bordism invariants are equivalent to each other and to the colored untying invariant.

**Lemma 6.2.** The closed bordism invariants have the property that $\omega_2(K, \rho) = 2n$ if $\omega_0(K, \rho) = (m, n) \in \mathbb{Z}_2 \oplus \mathbb{Z}_p$.

**Proof.** Recall the Bordism Long Exact Sequence (5.1)
\[
\cdots \to \Omega_n(A) \xrightarrow{i_*} \Omega_n(X) \xrightarrow{j_*} \Omega_n(X, A) \to \Omega_{n-1}(A) \to \cdots
\]
with $X$ and $A$ the Eilenberg-Maclane spaces over $D_{2p}$, $\mathbb{Z}_p$, and $\mathbb{Z}_2$ where appropriate. We have

$$
\begin{array}{c}
0 \rightarrow \Omega_3(\mathbb{Z}_2) \xrightarrow{i_*} \Omega_3(D_{2p}) \rightarrow \Omega_3(D_{2p}, \mathbb{Z}_2) \rightarrow 0
\end{array}
$$

so we must show that $i_*[\tilde{X}_0] = 2[X_0]$. But $\tilde{X}_0$ is a branched double cover of $X_0$ so the result follows.

We will now show that all of the $p$-colored knot invariants give the same information. In particular, this shows that computation of the bordism invariants may be done by computing the colored untangling invariant using the Goeritz matrix.

**Theorem 6.3.** All of the $p$-colored knot invariants are equivalent.

*Proof.* By Proposition 6.1 we have that for an appropriate choice of generator for $\mathbb{Z}_p$, the elements $\omega_2(K, \rho)$ and $\text{cu}(K, \rho)$ are equal. By Lemma 6.2 above we need only show that $\omega_0(K, \rho)$ lies in the $\mathbb{Z}_p$ part of $\mathbb{Z}_{2p}$ to show that both of the closed bordism invariants are the same. The final equivalence between $\omega$ and $\omega_0$ will follow from the Bordism Long Exact Sequence.

There is a canonical short exact sequence

$$
0 \rightarrow \mathbb{Z}_p = \langle s \rangle \xrightarrow{\Phi} D_{2p} = \langle s, t \mid t^2 = s^p = tsts = 1 \rangle \xrightarrow{\Psi} \mathbb{Z}_2 \rightarrow 0
$$

where $\mathbb{Z}_2$ is the cokernel of the map $\Phi$. As a result, we may construct the commutative diagram

\[
\begin{array}{ccc}
\pi_1(X_0) & \xrightarrow{\rho} & D_{2p} \\
\downarrow{\alpha} & & \downarrow{\psi} \\
\mathbb{Z} & \xrightarrow{l} & \mathbb{Z}_2
\end{array}
\]

where $\rho$ is the coloring applied to the 0-surgery manifold, $\alpha$ is the abelianization, and $l(x) = \text{lk}(x, K) \pmod{2}$. Hence, we have a commutative diagram of the corresponding
spaces

\[
\begin{array}{c}
K(\mathbb{Z}_p) \\
\downarrow \\
X_0 \xrightarrow{f} K(D_{2p}) \xrightarrow{g} \mathbb{R}P^\infty = K(\mathbb{Z}_2)
\end{array}
\]

which induces

\[
\begin{array}{c}
\mathbb{Z}_p \\
\downarrow \\
\mathbb{Z} = \langle \Lambda \rangle \xrightarrow{f^*} \mathbb{Z}_2 \oplus \mathbb{Z}_p \\
\downarrow \\
0 \xrightarrow{A_*} \mathbb{Z}_2
\end{array}
\]

on the third homology groups. From this, we see that \(\omega_0(K, \rho) = f_*(\Lambda) = (0, n) \in \mathbb{Z}_2 \oplus \mathbb{Z}_p\) for some \(n \in \mathbb{Z}_p\) since \(A_* = 0\). Note that \(g^* \neq 0\) since

\[
\begin{array}{c}
\mathbb{Z}_2 \\
\downarrow \\
\mathbb{Z}_2 \oplus \mathbb{Z}_p \\
\downarrow \\
\mathbb{Z}_2
\end{array}
\xrightarrow{id} \xrightarrow{\psi}
\]

commutes. So the closed bordism invariants are equivalent \(p\)-valued invariants of \(p\)-colored knots. This also implies, in particular, that \(\omega_0\) and the relative bordism invariant \(\omega\) must be the same.

For any based \(p\)-colored knot, the Bordism Long Exact Sequence gives the exact sequence

\[
\cdots \longrightarrow \Omega_3(K(\mathbb{Z}_2)) \xrightarrow{i_*} \Omega_3(K(D_{2p})) \xrightarrow{j_*} \Omega_3(K(D_{2p}), K(\mathbb{Z}_2)) \longrightarrow \Omega_2(K(\mathbb{Z}_2)) \longrightarrow \cdots
\]

that is, we have the short exact sequence

\[
0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z}_2 \oplus \mathbb{Z}_p \xrightarrow{j_*} \mathbb{Z}_p \longrightarrow 0
\]

which gives an isomorphism between the order \(p\) subgroup of \(\Omega_3(D_{2p}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_p\) and \(\Omega_3(K(D_{2p}), K(\mathbb{Z}_2))\) and the result follows. \(\square\)
Incidentally, as a corollary to the proof of the Theorem we have the following result. A detailed proof will not be given here but the result follows from the Bordism Long Exact Sequence and the fact that $[\mathbb{R} P^3, \varphi] \neq 0$ in $\mathbb{Z}_2 \cong \Omega_3(K(\mathbb{Z}_2))$.

**Theorem 6.4.** The bordism group $\Omega_3(D_{2^p}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_p$ is generated by the bordism class represented by the disjoint union of the singular manifolds $(\mathbb{R} P^3, \varphi)$ and $(X_0, f)$ where $X_0$ is the manifold obtained via 0-surgery along some prime $p$-colored knot $(K, \rho)$ with non-zero bordism invariant (a $(p, 2)$-torus knot for example). The maps $f$ and $\varphi$ correspond to the coloring

$$\rho : \pi_1(X_0) \to D_{2^p},$$

and the inclusion

$$\phi : \pi_1(\mathbb{R} P^3) \cong \mathbb{Z}_2 \to D_{2^p}$$

on the fundamental groups respectively.

We have shown that there are at least $p$ surgery equivalence classes of $p$-colored knots, we will now show that twice that is an upper bound on the number of equivalence classes.
Chapter 7
Main Result

We would like to show that the colored untying invariant is a complete invariant for \( p \)-colored knot surgery type. This is Moskovich’s conjecture, since as we have seen, \( cu(K, \rho) \) is \( p \)-valued. To show that \( cu \) is complete we must show that if \( cu(K_1, \rho_1) = cu(K_2, \rho_2) \) then \( (K_1, \rho_1) \) and \( (K_2, \rho_2) \) are surgery equivalent. The main result of this thesis is that this indeed is the case at least half of the time. We will also introduce a complete invariant for surgery equivalence of \( p \)-colored knots.

Let \( P_a \) denote the set of all based \( p \)-colored knots \( (K, \rho) \) with \( \omega(K, \rho) = a \in \mathbb{Z}_p \). If \( (K_1, \rho_1) \) and \( (K_2, \rho_2) \) are in the set \( P_a \) and \( M_i = S^3 - K_i \) then \( (M_1, \partial M_1, f_1) \) is bordant to \( (M_2, \partial M_2, f_2) \) over \( (K(D_{2p}, 1), K(\mathbb{Z}_2, 1)) = (X, A) \) by the definition of the bordism invariant \( \omega \). Here, the \( f_i : (M_i, \partial M_i) \rightarrow (K(D_{2p}, 1), K(\mathbb{Z}_2, 1)) \) are maps which induce the colorings on \( \pi_1 \). We have the existence of a 4-manifold \( W_{12} \) and a map

\[
\Phi : (W_{12}, \partial W_{12}) \rightarrow (K(D_{2p}, 1), K(\mathbb{Z}_2, 1))
\]

so that \( \partial W_{12} = (M_1 \coprod M_2) \cup_{\partial N_{12}} N_{12} \) and \( \Phi|_{M_i} = f_i \) as in Figure 7.1. The “connecting” 3-manifold in the boundary of the bordism \( W_{12} \) between \( M_1 \) and \( M_2 \) is denoted by \( N_{12} \).

Note that the boundary of \( N_{12} \) consists of two disjoint copies of the torus \( T^2 \), one for each boundary torus of the \( M_i \)'s. We would like to show that \( N_{12} \) is the product space \( T^2 \times [0, 1] \). We will show that this is necessarily the case at least half of the time. More precisely, we will construct a map \( \eta : P_a \times P_a \rightarrow \mathbb{Z}_2 \) that satisfies a certain “triangle equality” (Proposition 7.2 below). The map is defined by

\[
\eta(K_1, K_2) = \Phi_{12}(N_{12} \cup_{T^2 \times \{0, 1\}} (T^2 \times [0, 1]))
\]
where \([N_{12} \cup_{T^2 \times (0,1)} (T^2 \times [0,1])] = [\overline{N}]\) denotes the fundamental class of \(\overline{N}\) and \(\Phi_{12}\) is the obvious extension of the map \(\Phi|_{N_{12}} : N_{12} \to K(\mathbb{Z}_2, 1)\) coming from the bordism. So \(\eta(K_1, K_2)\) is an element of the bordism group \(\Omega_3(\mathbb{Z}_2) \cong \mathbb{Z}_2\).

**Proposition 7.1.** The function

\[
\eta(K_1, K_2) = 0
\]

if and only if there is a bordism \((W', \partial W', \Phi')\) between \((S^3 - K_1, f_1)\) and \((S^3 - K_2, f_2)\) with the connecting manifold consisting of the product space \(T^2 \times [0,1]\).

**Proof.** Assume that \(\eta(K_1, K_2) = 0\), note that we must also assume that both knots lie in the set \(P_a\) in order for the function \(\eta\) to make sense. Then we have bordisms \((W_0, \Phi_0)\) over \(\mathbb{Z}_2\) with boundary \(\overline{N_{12}}\) and \((W, \partial W, \Phi)\) over \((D_{2p}, \mathbb{Z}_2)\) with boundary \((S^3 - K_1) \cup N_{12} \cup (S^3 - K_2)\). So sufficiency is seen by glueing the bordism \((W_0, \Phi_0)\) to the bordism \((W, \partial W, \Phi)\) along the 3-manifold \(N_{ij}\). The result is a new bordism \((W', \partial W', \Phi')\) over \((D_{2p}, \mathbb{Z}_2)\) defined by

\[
W' = W \cup_{\psi} W_0
\]

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where $\psi : N_{ij} \to N_{ij}$ is an orientation reversing diffeomorphism. The map $\Phi'$ is defined by

$$
\Phi'(x) = \begin{cases} 
\Phi_0(x) & \text{if } x \in W_0, \\
\Phi(x) & \text{if } x \in W
\end{cases}
$$

and since the manifolds are glued by a diffeomorphism, it follows that

$$
\Phi' : (W', \partial W') \to (D_{2p}, \mathbb{Z}_2)
$$

is a differentiable map as required. We have shown that if $\eta(K_i, K_j)$ is trivial then there is a bordism $(W', \partial W', \varphi')$ over $(D_{2p}, \mathbb{Z}_2)$ with $\partial W' = (M_i \sqcup -M_j) \cup_{T^2 \times [0,1]} (T^2 \times [0, 1])$.

And so we may assume that $N_{ij} = T^2 \times [0,1]$ only in the case that $\eta$ is trivial. Necessity of this condition follows from the fact that $T^2 \times [0,1] \times [0, 1]$ has boundary homeomorphic to $(T^2 \times [0, 1]) \cup_{T^2 \times \{0,1\}} (T^2 \times [0, 1])$. \qed

Appealing to the proof of Theorem 1.1 below, we have that the colored surgery untying conjecture in [Mos, Conjecture 1] is equivalent to the property that $\eta$ is always trivial.

We will now show that the map $\eta$ is well-defined and satisfies the “triangle equality” property mentioned above.

**Proposition 7.2.** The map $\eta : P_a \times P_a \to \mathbb{Z}_2$ is well-defined and satisfies

$$
\eta(K_1, K_2) = \eta(K_1, K_3) + \eta(K_3, K_2)
$$

for any $(K_3, \rho_3) \in P_a$.

**Proof.** Let $W_{ij}$, and $N_{ij}$ denote the bordism and connecting 3-manifolds between $S^3 - K_i$ and $S^3 - K_j$ for $1 \leq i, j \leq 3$ as above. Also let $\overrightarrow{N_{ij}}$ and $\Phi_{ij}$ be as in the definition of $\eta(K_i, K_j)$.

To prove well-definedness we must show that $\eta(K_1, K_2)$ is unchanged by any choice of connecting manifold. Suppose there are two bordisms $(W_{12}, \partial W_{12}, \Phi)$ and $(W'_{12}, \partial W'_{12}, \Phi')$
\[ T^2 \times [0, 1] \]

**FIGURE 7.2.** A bordism over \( A = K(\mathbb{Z}_2) \).

**FIGURE 7.3.** “Triangle equality” and well-definedness of \( \eta \).
over \((D_{2p}, \mathbb{Z}_2)\) with connecting manifolds \(N_{12}\) and \(N'_{12}\). Gluing \(W_{12}\) together with \(W'_{12}\) along their common boundaries \(M_1 = S^3 - K_1\) and \(M_2 = S^3 - K_2\) we see that \(N_{12} \cup_{T^2 \times \{0,1\}} N'_{12}\) bords over \(\mathbb{Z}_2\). Call this bordism \(W\). Up to bordism over \(\mathbb{Z}_2\) we may assume that \(\partial W = N_{12} \coprod N'_{12} \cup_{T^2 \times \{0,1\} \times \{0,1\}} [(T^2 \times [0,1]) \times \{0,1\}]\) (see the top of Figure 7.3). We may glue in a copy of \(T^2 \times \left[\frac{1}{3}, \frac{2}{3}\right] \times [0,1]\) which shows that the disjoint union of \((\overline{N_{12}}, \Phi_{12})\) and \((\overline{N_{21}}, \Phi_{21})\) must also bord over \(\mathbb{Z}_2\). That is \(\Phi_{12,*}(\overline{N_{12}}) + \Phi'_{21,*}(\overline{N_{21}}) = 0\). Of course Figure 7.3 is just a rough diagram of this construction when thought of as a 5-manifold. Notice that \(\overline{N_{21}}\) is just \(\overline{N_{12}}\) with the reverse orientation but since we are working over \(\mathbb{Z}_2\) the order does not matter. That is,

\[
\Phi'_{21,*}(\overline{N_{21}}) = -\Phi'_{12,*}(\overline{N_{12}}) = \Phi'_{12,*}(\overline{N_{12}}) \pmod{2}
\]

and thus \(\eta\) is invariant under the choice of bordism class \(W_{12}\).

For the proof of the “triangle equality” we first obtain a bordism \(W\) as in Figure 7.4 by gluing all three \(W_{ij}\)’s along their common knot exterior boundaries. In particular, the 3-manifold obtained by gluing \(N_{12}, N_{23},\) and \(N_{31}\) together along their torus boundaries must bord over \(\mathbb{Z}_2\). But with a slight modification to the proof of well-definedness we
obtain the relation
\[ \eta(K_1, K_2) + \eta(K_2, K_3) + \eta(K_3, K_1) = 0 \]
in \( \Omega_3(Z_2) \cong Z_2 \) as desired.

So the bordism invariant \( \omega(K, \rho) \) which is \( Z_p \)-valued may not be a complete invariant for surgery equivalence classes of \( p \)-colored knots. However, if \( (K_1, \rho_1) \) and \( (K_2, \rho_2) \) are surgery equivalent based \( p \)-colored knots, then it is clear that \( \omega(K_1, \rho_1) = \omega(K_2, \rho_2) \). Recall that two \( p \)-colored knots are surgery equivalent if one may be obtained from the other by \( \pm \)-framed surgery on \( S^3 \) along an unlink \( L = L_1 \cup L_2 \) with \( [L_i] \in \text{ker}(\rho_i) \) for \( i = 1, 2 \). So the bordism over \( (D_{2p}, Z_2) \) is constructed by attaching 2-handles along the components of \( L_1 \) and dual 2-handles along the components of \( L_2 \) to the 4-manifold \( (S^3 - K_1) \times [0, 1] \).

Notice that the connecting manifold for this bordism is \( T^2 \times [0, 1] \). We have shown that surgery equivalent \( p \)-colored knots have the same bordism invariant. The difficulty with the converse is indeed the connecting manifold.

We will now prove the main result.

Proof of Theorem 1.1. By the discussion above, if two based \( p \)-colored knots are surgery equivalent then they have bordant complements over \( (K(D_{2p}, 1), K(Z_2, 1)) \) where the \( Z_2 = \langle t \rangle \subset D_{2p} = \langle s, t | s^p = t^2 = stst = 1 \rangle \). If we assume that two “\( p \)-colored knot exteriors” are bordant so that the connecting manifold is just the product space \( T^2 \times [0, 1] \) then the converse is true.

Assume that
\[ (M_1 = S^3 - K_1, f_1) \sim_{(D_{2p}, Z_2)} (M_2 = S^3 - K_2, f_2) \]
where the \( f_i \) correspond to the coloring maps \( \rho_i : \pi_1(S^3 - K_i) \to D_{2p} \) with bordism \( (W^4, \Omega) \). Suppose further that \( \partial W = (M_1 \coprod -M_2) \cup_{\partial T^2 \times [0,1]} (T^2 \times [0,1]) \). Take a smooth
handle decomposition of $W$ relative to the boundary with no 0 or 4-handles and proceed in a similar way to the proof of Theorem 4.2 in [CGO].

We may “trade” 1-handles for 2-handles (see [Kir1, pages 6-7] or [GomSt, Section 5.4]). Since $(f_1)_* : \pi_1(S^3 - K_1) \to D_{2p}$, is an epimorphism we may alter the attaching maps $c_i$ so that $\Phi_*(c_i) = 1$. Thus the map $\Phi$ extends to the “new” 4-manifold $W$ with no 1-handles. Since the 3-handles may be thought of as upside down 1-handles we may assume that $W$ is obtained from $(S^3 - K_1) \times [0, 1]$ by attaching 2-handles. This implies that $M_1$ and $M_2$ are related by surgery along links in the kernel of $\rho_i$. Now we must show that these links have $\pm 1$-framing and are unknotted.

Assume $\eta(K_1, K_2) = 0$. Then the connecting manifold is the product space $T^2 \times [0, 1]$. So we may glue in a solid torus crossed with an interval to the boundary tori of $W$ and “fill in” the $M_i$ and the connecting manifold. The result is a bordism between $S^3 = (S^3 - K_1) \cup (S^1 \times D^2)$ and $S^3 = (S^3 - K_2) \cup (S^1 \times D^2)$. That is we have a surgery description of $S^3 = (S^3 - K_1) \cup (S^1 \times D^2)$ consisting of a link $L$ in the complement of $K_1$. We now appeal to Kirby’s Theorem to obtain the standard surgery description for $S^3$ by using only blow ups and handle slides and no blow downs, consisting of a $\pm 1$-framed unlink. Notice that by taking $K_1 \subset S^3$ “along for the ride” when we do a handle slide we have only changed $K_1$ by an isotopy and so the resulting knot is surgery equivalent vacuously. By a blow up we mean the addition of a single $\pm 1$-framed unknot away from the rest of the surgery diagram. Since this unknot may be assumed to be in the kernel of $\rho_1$ this move is a surgery equivalence. Hence we have shown that $(K_1, \rho_1)$ is surgery equivalent to $(K_2, \rho_2)$ if we assume that $\eta(K_1, K_2) = 0$.

If $\eta(K_1, K_2) \neq 0$, then Proposition 7.2 implies that there are at most 2 surgery classes of $p$-colored knots which have the same value of $\omega$. As $\omega$ is $\mathbb{Z}_p$-valued we have that there are no more than $2p$ possible equivalence classes. Note that we have already seen that the
connected sum of $k$ $(p, 2)$-torus knots for $k = 1, \ldots, p$ give a complete list of representatives for the $\mathbb{Z}_p$-valued invariant $\omega$ and so the second statement of the proof follows from this.

As a corollary to Theorem 1.1 and Proposition 7.1 we have the following result.

**Corollary 7.3.** The pair $(\omega, \eta)$ consisting of the relative bordism and the $\eta$ invariants is a complete invariant of $p$-colored surgery equivalence for any odd integer $p$.

This result may be extended with a slight modification to “$A_4$-colored knots” which are pairs $(K, \rho)$ with $K$ a knot in the 3-sphere and $\rho$ a representation of the knot group onto the Alternating group on 4 letters. This will complete the theory if we are only considering representations onto finite subgroups of $SO(3)$ [Har].
Chapter 8
A₄-colored Surgery Equivalence

In this chapter we will extend the above main results for p-colored knots to A₄-colored knots.

8.1 A₄-colored Knots

A knot $K$ is said to be an $A₄$-colorable if there is a representation of the knot group onto the alternating group on four letters. That is, there is a surjective map $\rho : \pi_1(S^3 - K) \rightarrow A₄$ called the $A₄$-coloring. In this case, the pair $(K, \rho)$ is called an $A₄$-colored knot. It can be shown that a knot is $A₄$-colorable if and only if $t^3 + t + 1$ is a factor of its Alexander polynomial $\Delta(t)$ modulo 2 (see [Har]).

This means that there are many such knots. For example both the trefoil knot 3₁ and the figure eight knot 4₁ are $A₄$-colorable. Consider the commutative diagram:

\[
\begin{array}{ccc}
H_1(S^3 - F) & \xrightarrow{\eta} & \pi_1(S^3 - F) \\
\downarrow & & \downarrow \\
\pi_1(S^3 - F) & \xrightarrow{\rho_{|_{S^3 - F}}} & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\downarrow & & \downarrow l \\
\pi_1(S^3 - K) & \xrightarrow{\rho} & A₄ \\
\downarrow & & \downarrow \\
& & \mathbb{Z}_3
\end{array}
\]

with the map $l$ defined by $l(x) = lk(x, K) \ (mod \ 3)$. The diagram shows that a surjection from a knot group onto $A₄ = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ sends meridians to 3-cycles. So, in the same way as the classical Fox colorings for a knot, an $A₄$ coloring may be described uniquely by labeling each arc in a diagram with 3-cycles in $A₄$ so that the coloring condition in Figure 8.1 holds. Notice that the knot must be oriented for this to make sense. Indeed,
Figures 8.2 and 8.3 show two possible colorings for the trefoil and two possible colorings for the figure eight.

\[ c = b^{-1}ab \]

\[ bab^{-1} \]

\[ a \in \mathbb{Z}_3 \subset A_4 \]

**FIGURE 8.1.** The coloring condition.

\[ \begin{array}{c}
\text{(123)} \\
\text{(134)} \\
\text{(142)} \\
\text{(123)} \\
\end{array} \quad \begin{array}{c}
\text{(143)} \\
\text{(124)} \\
\text{(132)} \\
\end{array} \]

**FIGURE 8.2.** “Dual” or “conjugate” $A_4$-colored Trefoils.

We will now establish a notion of “duality” of $A_4$-colorings. Notice that the labelings for the two colorings for the trefoil differ by an inner automorphism of the symmetric group $S_4$, namely

\[ \rho'([\mu]) \mapsto (12)\rho([\mu])(12). \]

On the other hand the labelings for the colorings for the figure eight differ by an inner automorphism of the alternating group itself defined by

\[ \rho'([\mu]) = (12)(34)\rho([\mu])(12)(34). \]
We will call the colorings as in the former case *conjugate*, while in the latter case these colorings will be called *equivalent*.

**Definition 8.1.** We say that two $A_4$-colorings of a knot $K$ are conjugate (or dual) if they are conjugate as representations into the symmetric group $S^4$. On the other hand, two $A_4$-colorings are equivalent if they are conjugate as representations onto $A_4$.

More precisely, if $\rho_1$ and $\rho_2$ are two colorings for an $A_4$-colorable knot $K$, then $\rho_1$ is conjugate to $\rho_2$ if the following diagram commutes.

\[
\begin{array}{c}
\pi_1(S^3 - K)^{\rho_1} \xrightarrow{A_4 \text{ incl.}} S_4 \\
\rho_2 \downarrow \\
A_4 \xrightarrow{i \text{ incl.}} S_4 \\
\end{array}
\]

where $i$ is an inner automorphism. The two colorings are equivalent if

\[
\begin{array}{c}
\pi_1(S^3 - K)^{\rho_1} \xrightarrow{A_4} \\
\rho_2 \downarrow \\
A_4 \xrightarrow{j} \\
\end{array}
\]

commutes so that $j$ is an inner automorphism.
Notice that equivalent colorings are indeed conjugate however because there are two conjugacy classes of 3-cycles in $A_4$ it is possible that there are colorings which are conjugate but not equivalent. It turns out that colorings which are conjugate but not equivalent are in some sense dual to one another which is a motivation for the alternate name.

**Proposition 8.2.** If $(K, \rho)$ is an $A_4$-colored knot then there is exactly one, up to equivalence of colorings, $A_4$-colored knot $(K, \rho^*)$ which $\rho^*$ conjugate but not equivalent to $\rho$. We will call $(K, \rho^*)$ the dual $A_4$-colored knot to $(K, \rho)$.

**Proof.** Let $(K, \rho)$ be an $A_4$-colored knot. We would like to show that there is another coloring, $\rho^*$, satisfying (i) $\rho^*$ is conjugate but not equivalent to $\rho$ and (ii) $\rho^*$ is unique up to equivalence. In other words, $\rho^*$ is equivalent to all colorings which are conjugate but not equivalent to $\rho$.

Let $D$ be a diagram for the knot and $\mu$ be a meridian corresponding to a chose arc of the diagram. Up to equivalence of colorings, we may assume that $\rho([\mu]) = (123)$ or $(132)$. Then the first assertion is immediate as $\rho^*(x) = (23)\rho(x)(23)$ defines a coloring which is conjugate but not equivalent. Furthermore, $\rho^*([\mu]) = \rho([\mu])^{-1}$.

For the second assertion notice that if $\rho_2$ is conjugate but not equivalent to $\rho$, then

$$\rho_2(x) = t^{-1}\rho(x)t$$

for some even permutation $t$. Then

$$\rho^*(x) = (23)tt^{-1}\rho(x)tt^{-1}(23)$$

$$= (t^{-1}(23))^{-1}\rho_2(x)(t^{-1}(23))$$

$$= s^{-1}\rho_2(x)s$$

for $s$ an odd permutation. \qed

In Section 5.2, we saw that for any based $p$-colored knot we may assign a well-defined element of the relative bordism group $\Omega_3(K(D_{2p}, 1), K(\mathbb{Z}_2, 1))$ of the Eilenberg-Maclane
space over the dihedral group relative to the subspace corresponding to a specified \( \mathbb{Z}_2 \) subgroup of \( D_{2p} \). As a direct analog of this we define the triple \((K, \rho, \mu)\) consisting of a knot \( K \subset S^3 \), an \( A_4 \)-coloring \( \rho \), and a chosen meridian representing a generator of \( H_1(S^3 - K) \), to be a based \( A_4 \)-colored knot.

As conjugation in the alternating group is transitive on the \( \mathbb{Z}_3 \)-subgroups, we may assume that \( \rho([\mu]) = (123) \) or \( (132) \). In other words, up to an inner automorphism of the alternating group, we may assume that \([\mu]\) maps into the subgroup of \( A_4 \) generated by the 3-cycle \((123)\). However, since the element \((123)^{-1} = (132)\) is in a different conjugacy class as \((123)\), if \( \rho([\mu]) = (123) \) then up to equivalence the dual coloring \( \rho^* \) sends \([\mu]\) to \((132)\) and vice versa.

**Remark 8.3.** The fact that there are two conjugacy classes of 3-cycles in \( A_4 \) makes for a strictly well-defined notion of a connected sum of \( A_4 \)-colored knots to be impossible. Instead, if we look at \( A_4 \)-colored knots up to conjugacy of colorings, then we may define the connected sum as in Section 2.2. If \((K_1, \rho_1, \mu_1)\) and \((K_2, \rho_2, \mu_2)\) are two based \( A_4 \)-colored knots then up to equivalence (which is also conjugacy) we may assume that \( \rho_1([\mu_1]) = \rho_2([\mu_2]) \) or \( \rho_2([\mu_2])^{-1} \). In the first case, we may form the usual connected sum of oriented knots and the coloring for the new knot is induced by the colorings of the summands. In the latter case we may take \((K_1, \rho_1, \mu_1) \# (K_2, \rho_2, \mu_2)\) to be \((K_1 \# K_2, \rho_1 \# \rho_2)\) or \((K_1 \# K_2, \rho_1^* \# \rho_2)\) which is well-defined up to conjugacy of colorings.

### 8.2 Surgery Equivalence

**Definition 8.4.** Two \( A_4 \)-colored knots are surgery equivalent if one may be obtained from the other via surgeries of the three sphere along \( \pm 1 \) framed unknots in the kernel of the colorings.

As with dihedral colored knots, not all \( A_4 \)-colored knots are surgery equivalent. In fact any \( A_4 \)-colored knot is not surgery equivalent to its dual. This establishes a lower bound
for the number of equivalence classes. We will use a bordism approach to give an upper bound by constructing an “η invariant” as in Chapter 7.

**Proposition 8.5.** The number of $A_4$-colored surgery equivalence classes is at least 2.

**Proof.** We will show that $(K, \rho)$ is not surgery equivalent with $(K, \rho^*)$.

Suppose $(K_1, \rho_1)$ and $(K_2, \rho_2)$ are surgery equivalent $A_4$-colored knots via a single surgery along $C$ in the kernel of $\rho_1$. Then we have the following commutative diagram:

$$
\begin{array}{c}
\pi_1(S^3 - K_1) \\
\pi_1(S^3 - (K_1 \cup C)) \\
\pi_1(S^3 - K_2)
\end{array}
\xrightarrow{i} 
\xrightarrow{k}
\xrightarrow{\rho_1}
\xrightarrow{\rho_2}
A_4
$$

where $i$ is the inclusion map and $k$ is the map defined by killing the meridian of $C$. Hence we have that a meridian of $K_1$ when thought of as sitting inside $(S^3 - (K_1 \cup C))$ maps to a meridian in $\pi_1(S^3 - K_2)$ by $k$. Therefore, since all meridians are conjugate in $\pi_1$, the conjugacy class of the 3-cycle in $A_4$ to which any meridian in $\pi_1(S^3 - K_1)$ maps to by $\rho_1$ is the same as the conjugacy class of the corresponding meridian in $\pi_1(S^3 - K_2)$ by $\rho_2$.

We have shown that a surgery equivalence must preserve the conjugacy class of the image of the meridional generators of the fundamental group of the $A_4$-colored knots. In particular, since $\rho^*([\mu]) = \rho([\mu])^{-1}$ we cannot have a surgery equivalence between an $A_4$-colored knot and its dual.

\[\square\]

### 8.3 The Bordism and $\eta$ Invariants

In [CGO], T. Cochran, A. Gerges, and K. Orr define surgery equivalence invariants of 3-manifolds. These invariants are used to answer the question, “What are the classes of 3-manifolds up to ±1 surgery if we restrict the surgery curves to lie in some normal subgroup of the fundamental group of the manifold?” This is in the spirit of the Lickorish-
Wallace Theorem. The relative bordism invariant, in Section 5.2, is defined like the “CGO-invariants” [CGO] in the case where the manifolds are knot exteriors so they are manifolds with boundary and has a clear analog in the \( A_4 \) case. If two \( A_4 \)-colored knots have the same bordism invariant denoted \( \omega \in \Omega_3(K(A_4, 1), K(Z_3, 1)) \approx H_3(A_4, Z_3; Z) \) then there is a bordism between the two knot exteriors. If the “connecting manifold” is the product space \( T^2 \times [0, 1] \), then the colored knots are indeed surgery equivalent. Otherwise they are not surgery equivalent and the bordism invariant is not a complete invariant. We will show that \( 6 |H_3(A_4, Z_3)| \) is an upper bound for the number of surgery equivalence classes. However, we will also show that the number of classes may be \( 4 |H_3(A_4, Z_3)| \) or \( 2 |H_3(A_4, Z_3)| \). This is because there is no obvious obstruction to the number of distinct values of the \( \eta \) invariant is either 3, 2, or 1. We have already seen in Proposition 8.5 that 2 is a lower bound for the number of surgery equivalence classes.

**Proposition 8.6.** Let \((K_i, \rho_i)\) for \( i = 1, 2, 3 \), be \( A_4 \)-colored knots with the same value of the \( A_4 \)-colored bordism invariant, \( \omega_{A_4} \). Then,

\[
\eta(K_i, K_j) = -\eta(K_j, K_i),
\]

and

\[
\eta(K_1, K_2) + \eta(K_2, K_3) + \eta(K_3, K_1) \equiv 0 \mod 3.
\]

**Proof.** The proof is the same as Proposition 7.2 if special attention is paid to orientations.

As a corollary to this we have the following result.

**Theorem 8.7.** (i) If we let \( E_{A_4} \) denote the number of \( A_4 \)-colored surgery equivalence classes, then

\[
2 \leq E_{A_4} \leq 6|H_3(A_4, Z_3; Z)|.
\]

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(ii) Let \( \mathcal{C}_i(K, \rho) \) for \( i = 1, 2 \) denote the conjugacy class in \( A_4 \) of the image of a meridian for \( K \) under \( \rho \). Then the triple of \( A_4 \)-colored knot invariants \( (\omega, \eta, \mathcal{C}_i) \) is a complete invariant of surgery equivalence.

**Proof.** The proof mirrors the proof of Theorem 1.1 and Corollary 7.3. \( \square \)
Chapter 9
Applications

In this chapter, we will discuss some applications of the theory of colored surgery equivalence to low dimensional topology. Note that these applications are outlined in [Mos] in the case where the knot group representations are onto the dihedral group.

9.1 $D_{2p}$-periodic 3-manifolds

First we will introduce a result of J. Przytycki and M. Sokolov [PrSo] which is an analog of the Lickorish-Wallace Theorem [Lic1]Wal for periodic 3-manifolds admitting a $\mathbb{Z}_p$ action. Then we will give the statement of an analogous result which would hold as a corollary to the conjecture that there are exactly $p$ surgery equivalence classes of $p$-colored knots.

**Definition 9.1.** A (framed) link $L \subset S^3$ is called $p$-periodic if there is a $\mathbb{Z}_p$-action of $S^3$, with fixed point set a circle, which takes $L$ onto itself. By a $p$-periodic 3-manifold, we mean a 3-manifold which admits a $\mathbb{Z}_p$-action so that the fixed point set of the action is a circle and that the action is free outside the fixed circle.

Przytycki and Sokolov establish that $p$-periodic 3-manifolds are surgery on the 3-sphere along $p$-periodic links. More precisely, they prove the following theorem.

**Theorem 9.2** (Przytyki-Sokolov). *Let $p$ be a prime integer and $M$ be a closed orientable 3-manifold. The manifold $M$ admits a $\mathbb{Z}_p$-action with fixed point set equal to a circle if and only if there is a framed $p$-periodic link $L \subset S^3$ so that $M$ is the result of surgery along $L$ and that $\mathbb{Z}_p$ acts freely on the components of $L$.*

A key element of the proof of this theorem is the fact that all knots may be untied using surgery. That is to say, the main obstruction to obtaining a dihedral analog to this theorem is the existence of a surgery description for $p$-colored knots. If there are exactly
$p$ equivalence classes of $p$-colored knots, for $p$ an odd integer, then all colored knots may be described as a framed link in the complement of a connected sum of $(p,2)$-torus knots sitting inside the 3-sphere. Precisely stated, if the $p$-colored surgery conjecture were true then we would have:

**Conjecture 9.3.** Let $p$ be an odd integer and $M$ be a closed orientable 3-manifold. The manifold $M$ admits a $D_{2p}$-action with fixed point set equal to a connected sum of $(p,2)$-torus knots if and only if there is a framed $D_{2p}$-periodic link $L \subset S^3$ so that $M$ is the result of surgery along $L$ and that $D_{2p}$ acts freely on the components of $L$.

Here, a $D_{2p}$-periodic link turns out to be a link contained in the kernel of a coloring for the corresponding connected sum of torus knots. Indeed, if we have a coloring of a connected sum of $(p,2)$-torus knots, then there is a corresponding branched covering space which is diffeomorphic to itself [Har]. The dihedral action on $S^3$ with the correct fixed point set is given by the covering translations when we regard $S^3$ as this branched covering space. We will explore this relationship between $p$-colored knots and their dihedral branched covers more in the next section.

In the alternating group $A_4$ case, to obtain a similar result we would have to discover a complete set of knots which give all possible values of the invariant $(\omega, \eta, C_i)$ from Theorem 8.7.

### 9.2 Irregular Dihedral Branched Coverings of Knots

If $(K \subset S^3, \rho)$ is a $p$-colored knot then by [BuZi, Theorem 11.11] we may construct the $p$-fold covering space of $S^3$ with monodromy given by the coloring. Such a manifold is called an *irregular dihedral covering space* of $S^3$ branched along $K$. It is an “irregular” covering space because it corresponds to the preimage under $\rho$ of the subgroup of $D_{2p}$ generated by an element of order 2 which is not a normal subgroup.
The construction is outlined in two steps. The first step is to take the $p$-fold unbranched covering of the knot complement $S^3 - K$ with the monodromy given by $\rho$. This is done by cutting the complement up into cells and gluing the correct number of copies of these cells back together as prescribed by the monodromy. Recall that the monodromy of a covering space $p r : \tilde{X} \to X$ is described by lifts of loops based as some point $x$ which become directed paths between the $n$ lifts of the base point. The second step is to glue in solid tori to fill in the $\frac{p+1}{2}$ boundary tori of the unbranched cover. This gluing is done so that the longitude of the solid tori are mapped to the lifts of the preferred longitude of the boundary component while the meridians are mapped to some power of the lifts of a meridian for $K$.

Suppose $M$ is constructed as an irregular dihedral covering space defined by a $p$-colored knot $(K, \rho)$. Then once again by the Lickorish-Wallace Theorem we have that $M$ has a surgery description as a framed link in $S^3$. If Moskovich’s colored surgery equivalence conjecture were true, then $(K, \rho)$ would be given as a $\pm 1$-framed link in the complement of a connected sum of $(p, 2)$-torus knots. This would imply that a surgery description of $M$ would arise from the lifts of this link via the $p$-fold covering.
References


[Knot] *KnotInfo Table of Knots*, URL http://www.indiana.edu/~knotinfo/.


Appendix A: Homology of $D_{2p}$

We now give a complete calculation for the homology of the dihedral group $D_{2p}$ for $p$ any integer. For some of the homological algebra we refer the reader to [DaKi]. Note that this result follows from Theorem 2.4 and Proposition 2.5 in [AdMi] but the “heavy machinery” of spectral sequences must be applied. However, to compute the homology groups we will construct a free resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}[D_{2p}/\mathbb{Z}_p] = \mathbb{Z}[C_2]$ then tensor with $\mathbb{Z}$ and take the homology.

**Definition 9.4.** Let $M$ be an $R$-module for some commutative ring $R$ with 1. A free resolution of $M$ over $R$ is a chain complex $\{B_n,d\}$ with $B_n = 0$ if $n < 0$ consisting of $R$-modules $B_n$ and differentials $d$, together with a map $\epsilon$ called the augmentation map so the sequence

$$ \cdots \xrightarrow{d} B_n \xrightarrow{d} B_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{\epsilon} M \xrightarrow{0} $$

is exact.

We first prove a general result for any group with a normal subgroup. The case we will ultimately be interested in is case when the group is $D_{2p}$ and the normal subgroup is $\mathbb{Z}_p \subset D_{2p}$. The commutative ring that we will be working over will be the group ring $\mathbb{Z}[D_{2p}/\mathbb{Z}_p] = \mathbb{Z}[C_2]$ and the $\mathbb{Z}[C_2]$-module will be $\mathbb{Z}$ with trivial action.

Let $H \triangleleft G$ be a normal subgroup of a group $G$. Let

$$ \cdots \xrightarrow{\delta_3} B_2 \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} $$

be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G/H]$, and for $i = 0, 1, \ldots$, let

$$ \cdots \xrightarrow{d_{0i}} C_i \xrightarrow{d_{1i}} C_{i-1} \xrightarrow{d_{0i-1}} \cdots \xrightarrow{d_{01}} C_1 \xrightarrow{d_{00}} C_0 \xrightarrow{\epsilon_i} B_i $$

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be a free resolution of $B_i$ over $\mathbb{Z}[G]$. We set $C_{ij} = 0$ for $i < 0$ or $j < 0$, and $d_{0ij} = 0$: $C_{ij} \to C_{i,j-1}$ for $i < 0$ or $j \leq 0$. We claim that there are maps $d_{rij} : C_{ij} \to C_{i-r,j+r-1}$ for $r = 1, 2, \ldots$ such that

1. $\epsilon_{i-1}d_{1i0} = \delta_{i}\epsilon_{i}$ for $i > 0$;

2. $\sum_{m=0}^{m} d_{m-s,i-j-s+1}d_{sij} = 0$ for $m = 0, 1, \ldots$.

The case $m = 0$ of (2) is trivially satisfied, while (1) and the case $m = 1$ of (2) assert that in the diagram

\[
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \cdots \\
C_{02} & \xrightarrow{d_{112}} & C_{12} & \xrightarrow{d_{122}} & C_{22} & \xrightarrow{d_{132}} & \cdots \\
C_{01} & \xrightarrow{d_{111}} & C_{11} & \xrightarrow{d_{121}} & C_{21} & \xrightarrow{d_{131}} & \cdots \\
C_{00} & \xrightarrow{d_{110}} & C_{10} & \xrightarrow{d_{120}} & C_{20} & \xrightarrow{d_{130}} & \cdots \\
\mathbb{Z} & \xleftarrow{\epsilon} & B_0 & \xleftarrow{\delta_1} & B_1 & \xleftarrow{\delta_2} & B_2 & \xleftarrow{\delta_3} & \cdots
\end{array}
\]

the squares in the bottom row commute and all other squares anti-commute, so the $d_{1ij}$ exist. We now prove the existence of the $d_{rij}$ for $r > 1$ assuming that the $d_{r'ij}$ have already been constructed for $r' < r$, satisfying (2) for $m < r$. We have $d_{rij} = 0$ for $i < r$ and $j < 0$. We now do a double induction on $i$ and $j$. Suppose that for some $k \geq r$ and $l \geq 0$, $d_{rij}$ has been defined for $i < k$ and any $j$, and for $i = k$ and $j < l$, and that (2) holds for $m = r$ and those values of $i$, $j$ for which $d_{rij}$ is defined (in which case all the terms in the sum have been defined). Then $f = -\sum_{s=0}^{r-1} d_{r-s,k-s,l+s-1}d_{srl}$ is defined, and we need to show that there exists $d_{rkl} : C_{kl} \to C_{k-r,l+r-1}$ so that $d_{0,k-r,l+r-1}d_{rkl} = f$. To do this we use exactness of the columns. The target of $f$ is $C_{k-r,l+r-2}$ so if $r = 2$ and $l = 0$ we must show that $\epsilon_{k-2}f = 0$, and in all other cases we need $d_{0,k-r,l+r-2}f = 0$. Suppose first that $r = 2$ and $l = 0$. Then $f = -d_{2,k-1}d_{0k0} - d_{1,k-1}d_{1k0} = -d_{1,k-1}d_{1k0}$,
so $\epsilon_{k-2}f = -\epsilon_{k-2}d_{1,k-1,0}d_{1k0} = -\delta_{k-1}\delta_k \epsilon_k = 0$. Now consider the general case. We have
\[
d_{0,k-r,l+r-2}f = -\sum_{s=0}^{r-1} (d_{0,k-r,l+i+r-2}d_{r-s,k-s,l+s-1}) d_{skl}
\]
\[
= \sum_{s=0}^{r-1} \sum_{t=0}^{r-s-1} d_{r-s-t,k-s-t,l+s+t-2}d_{t,k-s,l+s-1} d_{skl}
\]
\[
= \sum_{s=0}^{r-1} \sum_{u=1}^{r-s} d_{u,k-r+u,l+r-u-2}d_{r-s-u,k-s,l+s-1} d_{skl}
\]
\[
= \sum_{u=1}^{r} d_{u,k-r+u,l+r-u-2} \sum_{s=0}^{r-u} d_{r-u-s,k-s,l+s-1} d_{skl}
\]
\[
= 0,
\]
which completes the inductive step.

We now set $D_{n} = \bigoplus_{i=0}^{n} C_{i,n-i}$, and for $n > 0$ let $\partial_{n}: D_{n} \rightarrow D_{n-1}$ be the map with components $d_{r,i,n-i}: C_{i,n-i} \rightarrow C_{i-r,n-i+r-1}$. The equations (2) ensure that this defines a chain complex. We write an element $a$ of $D_{n}$ as $(a_{0}, \ldots, a_{n})$ with $a_{i} \in C_{i,n-i}$. Then $\partial_{n}(a) = (b_{0}, \ldots, b_{n-1})$ where $b_{i} = \sum_{r=0}^{n-i} d_{r,i+r,n-i-r}(a_{i+r})$. We also define $\eta = \epsilon \epsilon_{0}: D_{0} = C_{00} \rightarrow Z$. The only non-zero components of $\partial_{1}$ are $d_{001}$ and $d_{110}$; since $\epsilon_{0}d_{001} = 0$ and $\epsilon_{0}d_{110} - \epsilon_{1}\epsilon_{1} = 0$, $\eta \partial_{1} = 0$. We wish to prove that
\[
\cdots \partial_{n} \partial_{2} \partial_{1} \partial_{0} \eta Z
\]
is a resolution of $Z$. Clearly $\eta$ is onto. Suppose $a \in ker(\eta)$. Because $\epsilon_{0}(a) \in ker(\epsilon) = \text{im}(\partial_{1})$ and $\epsilon_{1}$ is onto, there exists $b_{1} \in C_{10}$ with $\delta_{1}\epsilon_{1}(b_{1}) = \epsilon_{0}(a)$. Now $\epsilon_{0}(a - d_{110}(b_{1})) = \epsilon_{0}(a) - \delta_{1}\epsilon_{1}(b_{1}) = 0$, so there exists $b_{0} \in C_{01}$ with $d_{001}(b_{0}) = a - d_{110}(b_{1})$. Now $(b_{0}, b_{1}) \in D_{1}$ and $\partial_{1}(b_{0}, b_{1}) = a$, so we have exactness at $D_{0}$. Now let $n > 0$ and suppose $a = (a_{0}, \ldots, a_{n}) \in ker(\partial_{n})$, so that $b_{i} = \sum_{r=0}^{n-i} d_{r,i+r,n-i-r}(a_{i+r}) = 0$ for $0 \leq i < n$. Then
\[
0 = \epsilon_{n-1}(b_{n-1}) = \epsilon_{n-1}(d_{0,n-1,1}(a_{n-1}) + d_{1,n,0}(a_{n})) = \delta_{n}\epsilon_{n}(a_{n}),
\]
so there exists $c_{n+1} \in C_{n+1,0}$ with $\delta_{n+1}\epsilon_{n+1}(c_{n+1}) = \epsilon_{n}(a_{n})$. Now
\[
\epsilon_{n}(a_{n} - d_{1,n+1,0}(c_{n+1})) = \epsilon_{n}(a_{n}) - \delta_{n+1}\epsilon_{n+1}(c_{n+1}) = 0,
\]
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so there exists $c_n \in C_{n,1}$ with $d_{0n1}(c_n) = a_n - d_{1,n+1,0}(c_{n+1})$. Then

$$c = (0, \ldots, 0, c_n, c_{n+1}) \in D_{n+1}$$

and $a - \partial_{n+1}(c)$ has last component 0. Replacing $a$ by $a - \partial_{n+1}(c)$, we may assume that $a_n = 0$. Suppose there is some $i$, $0 \leq i < n$, such that $a_j = 0$ for $i < j \leq n$. Then

$$0 = b_i = \sum_{r=0}^{n-i} d_{r,i+r,n-i-r}(a_{i+r}) = d_{0,i,n-i}(a_i),$$

so there is some $c_i \in C_{i,n-i+1}$ such that $d_{0,i,n-i+1}(c_i) = a_i$. Now $c = (0, \ldots, 0, c_i, 0, \ldots, 0) \in D_{n+1}$ and $a - \partial_{n+1}(c)$ has its $j$ entry 0 for $i \leq j \leq n$. Continuing in this way gives exactness at $D_n$.

Note the following special case: if we can choose the $d_{ij}$ so that $d_{1,i-1,j}d_{ij} = 0$, taking $d_{2ij} = 0$ satisfies (2) for $m = 2$, and then taking $d_{rij} = 0$ for $r \geq 3$ satisfies (2) for all $m \geq 3$ since there in each term we have either $s \geq 2$ or $m-s \geq 2$. Hence $\partial_n(a_0, \ldots, a_n) = (b_0, \ldots, b_{n-1})$ where $b_i = d_{0,i,n-i}(a_i) + d_{1,i+1,n-i-1}(a_{i+1})$. Also recall that when we say that a homomorphism of $\mathbb{Z}[G]$ to itself “is” an element of $\mathbb{Z}[G]$, we mean that it is right multiplication by the element.

We now specialize to the dihedral group $D_{2p}$, with notation as in [AdMi]. We have each $B_t$ equal to $\mathbb{Z}[C_2]$, $\epsilon$ is the augmentation map, and

$$\delta_i = \begin{cases} 
T + 1 & \text{if } i \text{ is even;} \\
T - 1 & \text{if } i \text{ is odd.}
\end{cases}$$

Further, each $C_{ij}$ is $\mathbb{Z}[D_{2p}]$, $\epsilon_i$ is the natural map, and

$$d_{0ij} = \begin{cases} 
\Sigma_\tau & \text{if } j \text{ is even;} \\
\tau - 1 & \text{if } j \text{ is odd.}
\end{cases}$$

To get commutativity in the bottom row, we take

$$d_{1i0} = \begin{cases} 
T + 1 & \text{if } i \text{ is even;} \\
T - 1 & \text{if } i \text{ is odd.}
\end{cases}$$
To get anticommutativity elsewhere, we can take
\[
\begin{cases}
(-1)^{j/2}T + (-1)^i & \text{if } j \text{ is even}; \\
(-1)^{(j-1)/2}\tau T + (-1)^{i-1} & \text{if } j \text{ is odd}.
\end{cases}
\]

Part of the diagram is shown below.

\[
\begin{array}{c}
\vdots \\
C_{02} \rightarrow C_{12} \rightarrow C_{22} \rightarrow \cdots \\
\Sigma \tau \rightarrow \Sigma \tau \rightarrow \Sigma \tau \\
C_{01} \rightarrow C_{11} \rightarrow C_{21} \rightarrow \cdots \\
\tau \rightarrow \tau \rightarrow \tau \\
C_{00} \rightarrow C_{10} \rightarrow C_{20} \rightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
Z \quad B_0 \quad T \quad \ldots \\
\vdots \\
\end{array}
\]

Let’s check the anticommutativity. For \( j \) even, we have
\[
d_{1,i,j-1}d_{0ij} = \Sigma \tau((-1)^{j/2-1}\tau T + (-1)^{i-1}) = (-1)^{j/2-1}\Sigma \tau T + (-1)^{i-1}\Sigma \tau;
\]
\[
d_{0,i-1,j}d_{1ij} = ((-1)^{j/2}T + (-1)^i)\Sigma \tau = (-1)^{j/2}\Sigma \tau T + (-1)^i\Sigma \tau.
\]

For \( j \) odd, we have
\[
d_{1,i,j-1}d_{0ij} = (\tau - 1)((-1)^{(j-1)/2}\tau + (-1)^i)
\]
\[
= (-1)^{(j-1)/2}\tau T + (-1)^{j+1} + (-1)^{(j+1)/2}T + (-1)^{i+1};
\]
\[
d_{0,i-1,j}d_{1ij} = ((-1)^{(j-1)/2}\tau T + (-1)^{i-1})(\tau - 1)
\]
\[
= (-1)^{(j-1)/2}\tau T \tau + (-1)^{i-1}\tau + (-1)^{(j+1)/2}\tau T + (-1)^i,
\]
and the result follows in this case since \( \tau T \tau = T \). Note that we do have \( d_{1,i-1,j}d_{1ij} = 0 \).
Now we tensor everything with \( \mathbb{Z} \), and denote the resulting objects by the same symbols with a hat. We have

\[
\hat{d}_{0ij} = \begin{cases} 
  p & \text{if } j \text{ is even;} \\
  0 & \text{if } j \text{ is odd.}
\end{cases}
\]

and

\[
\hat{d}_{1ij} = \begin{cases} 
  (1-j)^{i/2} + (-1)^i & \text{if } j \text{ is even;}
  
  (1)^{(j-1)/2} + (-1)^{i-1} & \text{if } j \text{ is odd;}
  
  2 & \text{if } i \equiv 0 \pmod{2} \text{ and } j \equiv 0 \pmod{4},
  
  \text{or } i \equiv 1 \pmod{2} \text{ and } j \equiv 1 \pmod{4};
  
  -2 & \text{if } i \equiv 1 \pmod{2} \text{ and } j \equiv 2 \pmod{4},
  
  \text{or } i \equiv 0 \pmod{2} \text{ and } j \equiv 3 \pmod{4};
  
  0 & \text{otherwise.}
\end{cases}
\]

Let \( e_0, \ldots, e_n \) be a basis of \( \hat{D}_n \), with \( e_i \in \hat{C}_{i,n-i} \), and let \( f_0, \ldots, f_{n+1} \) be a basis of \( \hat{D}_{n+1} \), with \( f_i \in \hat{C}_{i,n+1-i} \). Suppose that \( n \) is even. Then \( \partial_{n+1}(f_0) = 0 \), \( \partial_{n+1}(f_{n+1}) = 0 \), and for \( 0 < i \leq n \),

\[
\partial_{n+1}(f_i) = \begin{cases} 
  0 & \text{if } n - i \equiv 0 \pmod{4};
  
  pe_i - 2e_{i-1} & \text{if } n - i \equiv 1 \pmod{4};
  
  -2e_{i-1} & \text{if } n - i \equiv 2 \pmod{4};
  
  pe_i & \text{if } n - i \equiv 3 \pmod{4}.
\end{cases}
\]

Thus \( \text{im}(\partial_{n+1}) \) is generated by

\[
2e_i - pe_{i+1} \quad \text{for } 0 \leq i < n, i \equiv n - 2 \pmod{4};
\]

\[
\langle 2, p \rangle e_i \quad \text{for } 0 \leq i < n, i \equiv n - 3 \pmod{4}.
\]
In particular, $\text{im} (\hat{\partial}_1 = 0)$, so $H_0(D_{2p}) \cong \mathbb{Z}$ as expected. Now, for $n > 0$ and $a = (a_0, \ldots, a_n) \in \hat{D}_n$, $\hat{\partial}_n(a) = (b_0, \ldots, b_{n-1})$, where

$$b_i = \hat{d}_{0,i,n-i}(a_i) + \hat{d}_{1,i+1,n-i-1}(a_{i+1})$$

$$= \begin{cases} 
pa_i & \text{if } n - i \equiv 0 \pmod{4}; \\
2a_{i+1} & \text{if } n - i \equiv 1 \pmod{4}; \\
pa_i + 2a_{i+1} & \text{if } n - i \equiv 2 \pmod{4}; \\
0 & \text{if } n - i \equiv 3 \pmod{4}.
\end{cases}$$

Hence $a \in \text{ker}(\hat{\partial}_n)$ iff, for $0 \leq i \leq n$, $a_i = 0$ if $i \equiv n \pmod{4}$, and $pa_i + 2a_{i+1} = 0$ if $i \equiv n - 2 \pmod{4}$, so $\text{ker}(\hat{\partial}_n)$ is generated by

$$\langle 2e_i - pe_{i+1}, 2, p \rangle$$

for $0 \leq i < n$, $i \equiv n - 2 \pmod{4}$;

$$e_i$$

for $0 \leq i < n$, $i \equiv n - 3 \pmod{4}$.

Therefore (for $n$ even and non-zero) $H_n(D_{2p})$ is trivial if $p$ is odd, and $\mathbb{Z}_2^n$ if $p$ is even.

Now suppose that $n$ is odd. Then $\hat{\partial}_{n+1}(f_0) = pe_0$, $\hat{\partial}_{n+1}(f_{n+1}) = 2e_n$, and for $0 < i \leq n$,

$$\hat{\partial}_{n+1}(f_i) = \begin{cases} 
2e_{i-1} & \text{if } n - i \equiv 0 \pmod{4}; \\
p e_i & \text{if } n - i \equiv 1 \pmod{4}; \\
0 & \text{if } n - i \equiv 2 \pmod{4}; \\
p e_i + 2e_{i-1} & \text{if } n - i \equiv 3 \pmod{4}.
\end{cases}$$

Thus $\text{im}(\hat{\partial}_{n+1})$ is generated by

$$2e_i + pe_{i+1}$$

for $0 < i < n$, $i \equiv n \pmod{4}$,

$$\langle 2, p \rangle e_i$$

for $0 < i < n$, $i \equiv n - 1 \pmod{4}$.

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$2e_n$, and $(2,p)e_0$ if $n \equiv 1 \pmod{4}$, or $pe_0$ if $n \equiv 3 \pmod{4}$. Next, for $a = (a_0, \ldots, a_n) \in \hat{D}_n$, $\hat{\partial}_n(a) = (b_0, \ldots, b_{n-1})$, where

\[
b_i = \hat{d}_{0,i,n-i}(a_i) + \hat{d}_{1,i+1,n-i-1}(a_{i+1}) = \begin{cases} 
  pa_i - 2a_{i+1} & \text{if } n - i \equiv 0 \pmod{4}; \\
  0 & \text{if } n - i \equiv 1 \pmod{4}; \\
  pa_i & \text{if } n - i \equiv 2 \pmod{4}; \\
  -2a_{i+1} & \text{if } n - i \equiv 3 \pmod{4}.
\end{cases}
\]

Hence $a \in \ker(\hat{\partial}_n)$ iff, for $0 < i < n$, $a_i = 0$ if $i \equiv n - 2 \pmod{4}$, and $pa_i - 2a_{i+1} = 0$ if $i \equiv n \pmod{4}$, so $\ker(\hat{\partial}_n)$ is generated by

\[
(2e_i + pe_{i+1})/(2,p) \quad \text{for } 0 < i < n, \ i \equiv n \pmod{4}, \\
\]

$e_i \quad \text{for } 0 < i < n, \ i \equiv n - 1 \pmod{4},$

$e_0$, and $e_n$. Hence $H_n(D_{2p})$ is the sum of: $\mathbb{Z}_{(2,p)}$ for each $i$ with $0 < i < n$ and $i \equiv n$ or $n - 1 \pmod{4}$; $\mathbb{Z}_{(2,p)}$ if $n \equiv 1 \pmod{4}$ or $\mathbb{Z}_p$ if $n \equiv 3 \pmod{4}$; and $\mathbb{Z}_2$. The number of $\mathbb{Z}_{(2,p)}$ summands is $(n + 1)/2$ if $n \equiv 1 \pmod{4}$ and $(n - 1)/2$ if $n \equiv 3 \pmod{4}$. Thus we have

\[
H_n(D_{2p}) \cong \begin{cases} 
  \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{4}; \\
  \mathbb{Z}_p \oplus \mathbb{Z}_2 & \text{if } n \equiv 3 \pmod{4}
\end{cases}
\]

if $p$ is odd, and

\[
H_n(D_{2p}) \cong \begin{cases} 
  \mathbb{Z}_2^{(n+3)/2} & \text{if } n \equiv 1 \pmod{4}; \\
  \mathbb{Z}_p \oplus \mathbb{Z}_2^{(n+1)/2} & \text{if } n \equiv 3 \pmod{4}
\end{cases}
\]

if $p$ is even.

So we have shown the proof of Proposition 5.3.
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Steven Daniel Wallace was born in March 1979, in Houston, Texas. He finished his undergraduate studies at the University of California at Los Angeles June 2001; he graduated Magna Cum Laude, majoring in pure mathematics with a specialization in computing. He earned a Master of the Arts degree in mathematics from Rice University in May 2004. In August 2004 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2008.