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Oscillating Cylinders and the Stokes Paradox.

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OSCILLATING CYLINDERS AND THE STOKES PARADOX

A Dissertation

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in partial fulfillment of the
requirements for the degree of
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in

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ABSTRACT

Measurements are reported of the inertia coefficient k and damping coefficient k' for a circular cylinder (radius a) oscillating at angular frequency ω in a fluid of kinematic viscosity ν . Good agreement with the theory of Stokes is obtained throughout the range $0.201 < (2\sqrt{\omega a^2})^{\frac{1}{2}} < 4.11$ even when the Reynolds number R and the relative amplitude are not small compared to 1. The experiment employs a torsion pendulum whose period of oscillation is measured electronically with an optical lever and whose amplitude of oscillation is digitized with an optical vernier system. Stokes' theory is expressed in terms of modified Bessel functions, and values of k and k' are tabulated for $0.001 < (a^2\omega/4\nu)^{\frac{1}{2}} < 4.0$. The relation to Stokes paradox is considered by examining the damping force in the limit $(\omega a^2/\nu)^{\frac{1}{2}} \ll 1$. The damping force for a cylinder, unlike that for a sphere, does not reduce to the low R drag in this limit. However, comparison with Lamb's solution of the Oseen equations suggests an expression that agrees well with drag measurements up to $R = 2.5$.

CHAPTER I

INTRODUCTION

A. The Navier-Stokes Equation

The problem considered here is that of an infinite right circular cylinder executing sinusoidal oscillations normal to its longitudinal axis in an unbounded, incompressible, Newtonian viscous fluid. This problem is two dimensional because of the translational symmetry along the longitudinal axis. The equations governing the motion of an incompressible, Newtonian fluid in the presence of conservative (gravitational) body forces in an inertial reference system are the Navier-Stokes equations

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = - \nabla p + \mu \nabla^2 \vec{v} \quad (1)$$

and the continuity equation

$$\nabla \cdot \vec{v} = 0 \quad (2)$$

where $\vec{v} = \vec{v}(\vec{r}, t)$ and $p = p(\vec{r}, t)$ are respectively the velocity and pressure at the position $\vec{r}$ and time t , and ρ and μ are respectively the density and viscosity of the fluid. For a Newtonian fluid, ρ and μ are material constants, independent of the rate of shear. The pressure includes the potential of the body force. The Navier-Stokes equations are non-linear and have no known exact solution for this problem.

Two successful approximation procedures for solving the Navier-Stokes equations are boundary layer theory¹ and the low Reynolds

number approximation². The Reynolds number is defined as $R = UL/\nu$ where U is a characteristic velocity, L a characteristic length of the geometry and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The Reynolds number can be thought of as a ratio of inertial force to viscous force. For large Reynolds numbers, inertia dominates the flow and viscosity is unimportant far from solid boundaries. Throughout most of the fluid region, the inviscid potential flow described by Bernoulli's equation is applicable, and only in a thin boundary layer are viscous effects appreciable. Approximations based on the smallness of certain terms in the boundary layer reduce the complexity of the Navier-Stokes equations and result in the simpler boundary layer equations. Many practical problems are rendered solvable in the boundary layer approximation and when solutions are matched with the potential flow, good agreement with experiment is obtained for large Reynolds number. For low Reynolds number problems the non-linear inertial terms are small and if these terms are neglected completely the equations of creeping motion or the Stokes equations result. These equations are linear and the extensive techniques of linear differential equations yield solutions to many problems.

B. Previous Theoretical Results

1. Solution of Stokes

By neglecting the non-linear convective acceleration term $\mathbf{v} \cdot \nabla \mathbf{v}$, an approximate solution for the oscillating cylinder was obtained originally by Stokes³ and was stated in modern terms by Stuart⁴ and Segal⁵. The resulting solution for the velocity field

is used to find the force exerted on the cylinder by the fluid. The force F per unit length of the cylinder of radius a can be written

$$F = - M'k \frac{dv}{dt} - M'k'\omega v \quad (3)$$

where M' is the mass of the displaced fluid per unit length of the cylinder, k is the added mass or inertia coefficient, k' is the damping coefficient, and ω is the angular frequency of oscillation. The coefficients k and k' are given by the equation

$$k - 1 - ik' = 4 K_1(z)/zK_0(z) \quad (4)$$

where $K_n(z)$ is the modified Bessel function of order n , and $z = (1+i) a \sqrt{\omega/\nu}$. The essential results are first that the cylinder moves as though it had an additional mass kM' and second that the force exerted on the cylinder has a component $k'M'\omega v$ that is out of phase with the acceleration and accounts for the dissipation of the energy of the oscillating cylinder.

2. Conditions for Validity of Stokes' Theory

The mathematical conditions for which the above approximation is expected to be valid will be considered here. It is convenient to introduce the viscous penetration depth parameter $\lambda = (2\nu/\omega)^{\frac{1}{2}}$. In the problem of an infinite plane executing sinusoidal oscillations in its own plane with fluid occupying the semi-infinite region above the plane, this parameter arises naturally as the distance into the fluid at which the oscillating fluid velocity has decreased in amplitude to $1/e$ of its value at the plane. In other oscillating flows this parameter is used as a qualitative

measure of the distance, normal to the oscillating surface, over which the oscillating velocity components change appreciably.

Consider first the low Reynolds number case. Let a be a characteristic dimension of the cylinder and take as a characteristic time $1/\omega$ and as a characteristic velocity $A\omega$, where A is the amplitude of oscillation. Measure pressure relative to the constant pressure p_∞ far from the cylinder and consider it to have a characteristic value of $v_\infty A/a$. The following dimensionless variables result:

$$\hat{r} = r/a, \quad \hat{t} = \omega t, \quad \hat{v} = v/A\omega, \quad \hat{p} = (p-p_\infty) \frac{a}{A v_\infty}.$$

The gradient in the convective acceleration term is taken in the direction of the velocity, or equivalently along a streamline. Since velocities change appreciably along streamlines over distances characteristic of the body, this directional gradient may be non-dimensionalized by $\hat{\nabla} = a\nabla$. The other gradient terms will also be non-dimensionalized by $\hat{\nabla} = a\nabla$. Therefore, in dimensionless form the Navier-Stokes equations (1) may be written

$$2\left(\frac{a}{\lambda}\right)^2 \frac{\partial \hat{v}}{\partial \hat{t}} + 2\left(\frac{a}{\lambda}\right)^2 \frac{A}{a} (\hat{v} \cdot \hat{\nabla}) \hat{v} = -\hat{\nabla} \hat{p} + \hat{\nabla}^2 \hat{v}. \quad (5)$$

Since the dimensionless variables and their derivatives are of $O(1)$, the relative sizes of the terms are indicated by the size of the coefficients. Thus, it is clear that if amplitudes of oscillations are small compared to the dimensions of the body ($A \ll a$), and if $a/\lambda \sim O(1)$, the non-linear term may be neglected^{3,4,5}. It must be mentioned that it is implicitly assumed that separation of

the boundary layer does not occur. This requires that the Reynolds number $R = Ua/\nu = 2Aa/\lambda^2$ be small. For the case of very large penetration depths $\lambda \gg a$ the equations of motion are approximately those for a linearly translating body and should have solutions similar to the drag equations for a falling body. This situation is discussed more fully in Chapter IV.

Consider next the case $\lambda \ll a$. This is analogous to a boundary layer approximation. As before, the length a , the time $1/\omega$, and the velocity $A\omega$ are used to construct dimensionless variables. However, now the pressure has a characteristic value $\rho A^2 \omega^2$ (the stagnation pressure), and spatial derivatives of the velocity taken normal to streamlines may be large. The directional gradient in the non-linear term is again non-dimensionalized as $\hat{\nabla} = a\nabla$, but the other gradient terms are non-dimensionalized by $\hat{\nabla} = \lambda\nabla$. Therefore, in dimensionless form the Navier-Stokes equation may be written

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + \frac{A}{a}(\hat{\mathbf{v}} \cdot \hat{\nabla})\hat{\mathbf{v}} = -\frac{A}{\lambda} \hat{\nabla} \hat{p} + \hat{\nabla}^2 \hat{\mathbf{v}}. \quad (6)$$

Again, the non-linear term will be small if the amplitude A is small compared to the radius a .

In summary, for the case of a body oscillating in a viscous fluid, the non-linear term in the Navier-Stokes equations will be small if either (1) the Reynolds number $R \ll 1$ and $A \ll a$, or (2) the penetration depth $\lambda \ll a$ and $A \ll a$. Since the Reynolds number can be written as $R = 2(a/\lambda)^2 A/a$, the two cases are not

completely independent.

3. Steady Streaming

More recent attempts to obtain approximate solutions for the oscillating cylinder problem have focused on long-range steady, or acoustic, streaming caused by the oscillations. Stuart⁶ and Riley⁷ have considered the case $\lambda \ll A \ll a$ which leads to an outer boundary layer. Riley⁸ and Wang⁹ have considered the case $A \ll \lambda \ll a$ for which the outer flow is governed by the equations of slow motion. Riley¹⁰ has recently reviewed both experimental and theoretical work on the steady streaming aspect of the problem. Riley's work uses the method of inner-outer expansions in which the non-linear term is suitably approximated near and far from the cylinder and the resulting solutions are matched in an intermediate region. In this procedure constant terms and terms of twice the driving frequency appear in the solution for the velocity. The constant terms imply a steady pumping of fluid toward or away from the cylinder depending on the conditions assumed. The terms of twice the driving frequency represent closed loops of circulating fluid. Since this work is directed toward the properties of the velocity field, no equations for the force on the cylinder are given.

C. Previous Experimental Results

Experimental observations of the forces exerted on the cylinder are in general agreement with Stokes' theory for λ/a values from 0.004 to 0.59 at small amplitudes of oscillation. The pioneering work of Baily and Coulomb was used by Stokes in

his original test of the theory. Because of uncertainties in the experimental conditions and large relative amplitudes, these experiments must be viewed with caution.

Baily¹¹ measured the periods of many different shapes of pendula. Among these were three long cylindrical rods (numbers 21, 43 and 44) which were suspended at one end and were swung in air at atmospheric pressure and at a lower pressure (about 1/30 atm). Experimental details were given only for number 21, but if we presume that the others were conducted under similar conditions, they also can be compared with Stokes' theory. Baily expressed his results as a factor n by which the previous pendulum correction (for buoyancy) is multiplied to include the effects of viscosity and inertia of the medium. Stokes has shown¹² that for Baily's experiment, this factor can be written

$$n-1 = (\rho_1 k_1 - \rho_0 k_0) / (\rho_1 - \rho_0) \quad (7)$$

where ρ_1 and ρ_0 are the densities of the air at high and low pressures, respectively, and k_1 and k_0 are the inertia coefficients of Stokes' theory at high and low pressures. Using modern values for the viscosity of air, we have calculated values of $n-1$ from Eq. (7) and the results are compared with Baily's experiment in Table I. The fourth and fifth columns of this table contain values of λ/a at high and low pressures, respectively. The experimental values of $n-1$ were corrected for boundary effects by Stokes. Although the agreement between theory and experiment is very good, there are several reasons for viewing this agreement with caution:

TABLE I
Bailey's Experiments with Cylindrical Rods

Experiment No.	Length cm	a cm	λ_1/a	λ_0/a	n-1 Experiment	n-1 Theory
21	153.3	0.521	0.581	2.98	1.923	1.960
43	147.2	0.235	1.31	7.17	3.081	3.175
44	147.2	0.091	3.37	18.4	6.530	6.735

(1) the experimental conditions for numbers 43 and 44 are not known, particularly the temperature of the air and the value of the lower pressure, (2) end effects are not taken into account, and (3) even though the angle of swing was less than one degree, the relative amplitude (A/a) at the end of the pendulum ranged up to 25.

Coulomb¹³ used a torsion pendulum to swing cylinders in a horizontal plane in water. Each cylinder was a thin rod of length 24.9 cm and was suspended from its midpoint with its longitudinal axis horizontal. According to Stokes' theory the logarithmic decrement $2\pi\delta$ for this experiment should be given by

$$2\pi\delta = \pi^2 a^2 \ell^3 \rho k' / 12I \quad (8)$$

where a is the radius, ℓ is the length, ρ is the fluid density, k' is Stokes' damping factor, and I is the moment of inertia of the system. As shown in Table II, the theoretical values of k' , calculated from λ/a , are much smaller than the experimental values calculated from Eq. (8). However, since Coulomb's observations were made at extremely high amplitude (angular displacements from 91 to 456 degrees), it is not surprising that Stokes' theory does not apply. The relatively good agreement obtained by Stokes in his analysis¹² of Coulomb's experiment is due to his using a value for the kinematic viscosity of water which is too large by a factor of 2.

For small penetration depths, $\lambda/a < 1$, several experiments have been performed using a variety of methods; the results agree well with Stokes' theory. As mentioned above the experiments of Bailly

TABLE II

Coulomb's Experiments with Cylinders

a cm	Period sec	λ/a	$2\pi\delta$ Experiment	k' Experiment	k' Theory
0.336	46	1.14	0.0818	5.25	3.43
0.179	45.5	2.13	0.0528	11.94	8.00
0.0138	45.5	27.6	0.0274	1043.	460.

on cylindrical pendula confirm Stokes theory. Stuart and Woodgate¹⁴ report agreement (within 2%) with Stokes' theory for damping at $\lambda/a = 0.102$ using a cylindrical pendulum vibrating in air at various pressures. An increase in damping was noted for amplitudes greater than one tenth of a diameter. The results of Stuart and Woodgate are uncertain because of a large (70%) nuisance damping correction.

Martin¹⁵ and Laird¹⁶ performed vibrating wire experiments in various fluids with results that agree with Stokes' theory. Martin measured frequency (related to added mass) and damping for wires vibrating in air and water. Values of λ/a were from 0.064 to 0.069. Measured frequencies of vibration of a wire by Laird agreed (within 4%) with Stokes' theory for λ/a from 0.26 to 0.33. Richardson and Tait¹⁷ measured the damping of a vibrating cylindrical rod clamped at one end and used Stokes' theory to calculate the viscosity of air and water; the results agree well with values obtained by other methods. For the air and water experiments λ/a was respectively 0.3 and 0.07.

Hussey, Good, and Reynolds¹⁸ measured added mass and damping for cylinders oscillating normal to their longitudinal axis in liquid helium and found good agreement with expressions based on Stokes' theory for the normal fluid and inviscid fluid theory for the superfluid. Values of λ/a ranged from 0.03 to 0.59 and no variation of the damping and inertia coefficients were observed for amplitudes less than 0.5 radii. Tough, McCormick and Dash¹⁹ reported measurement of the viscosity of liquid helium obtained

from the measured damping of a small diameter vibrating wire and the use of Stokes' theory. Values of λ/a ranged from 0.245 to 2.48. For values of λ/a near one and larger, reported errors were largest (28% at $\lambda/a = 2.48$). No change in damping was observed for amplitudes as high as eight diameters²⁰ which is remarkable in light of work by Stuart and Woodgate¹⁴, and Keulegan and Carpenter²¹. The presence of the superfluid and the existence of non-classical critical velocities in liquid helium obscure the interpretation of these results.

Other experiments at small λ/a with constant velocity along the cylinder have been performed in addition to those mentioned above. Ye-Tak Yu²² conducted torsional oscillation experiments with several kinds of objects including cylinders moving normal to their longitudinal axis. Hollow brass cylinders of 1.27 cm diameter and various lengths with a thin disk soldered in their midplane were oscillated in water and an empirical equation for the added mass was given. Values of λ/a in the range 0.05 to 0.08 may be inferred from the data. Stelson and Mavis²³ vibrated, among other objects, two inch diameter cylinders of various lengths normal to their longitudinal axis in water. The value of λ/a was 0.004. An advantage of these two experiments is the experimental determination of end effects.

High amplitude observations at small λ/a have also been made. As mentioned above Stuart and Woodgate¹⁴ found an increase in damping of a pendulum for amplitudes larger than 0.1 diameters. Keulegan and Carpenter²¹ conducted high amplitude experiments in a water bath in

which standing waves were established. Cylinders were immersed at the central node and the forces due to the nearly horizontally varying water velocity were measured. Observations were made at λ/a from 0.02 to 0.08 and relative amplitudes from 0.9 to 38 cylinder diameters. Essential results were separation of the boundary layer and accompanying increase in the drag force. A slight vertical velocity component existed at the node. Heinzer and Dalton²⁴ reported flow visualization studies for cylinders oscillating in water. Values of λ/a varied from 0.044 to 0.10 and relative amplitudes ranged from 0.8 to 9.6 cylinder radii. No separation was observed for relative amplitudes of 0.8 and 1.0, but above these amplitudes separation and vortex shedding were observed.

Experimental data thus confirm the approximation of Stokes for small values of λ/a and small amplitudes of oscillation. Few measurements have been made for large λ/a and these have suffered from various defects. Few of the experiments have measured both the inertia and damping coefficients.

The present work has systematically measured both the inertia and damping coefficients for $0.201 \leq \lambda/a \leq 4.11$. The present experiment has the desirable features of experimentally accounting for end effects and of maintaining a unique Reynolds number and relative amplitude for the entire length of the cylinder. Critical values of the relative amplitude and Reynolds number are reported; for amplitudes larger than the critical values damping was larger and the period was smaller than the respective constant low amplitude values. Stokes' theory compares well with the measurements throughout this range, even when the Reynolds number and the relative

amplitude are not extremely small.

The agreement with Stokes' theory for large values of λ/a leads to the speculation that the Stokes expression for the damping of a slowly oscillating cylinder is related to the drag experienced by a slowly translating cylinder. For an oscillating sphere the Stokes dissipative force reduces exactly to the low Reynolds drag in the limit of extremely long periods. The Stokes equations for the low Reynolds number translation of a cylinder has no solution for unbounded plane flow. This so called Stokes paradox makes the transition from slow oscillatory motion to low Reynolds number drag more difficult for the cylinder. However, the long penetration depth limit of the oscillating cylinder is compared to the low Reynolds number solution of the Oseen equation for a translating cylinder, and an empirical equation for the drag is obtained. This empirical equation reduces to Lamb's²⁵ solution for the Oseen equations in the limit of very small Reynolds number and agrees well with experimental measurements in the difficult region around $R = 1$.

CHAPTER II

EXPERIMENT

A. Apparatus

1. Oscillator Apparatus

Measurements were made with the torsion pendulum illustrated in Fig. 1. The two symmetrically placed cylinders (G) are made to oscillate about an axis parallel to their longitudinal axis in a cylindrical container (I) filled with a viscous liquid (H). Briefly, the method consists of measuring the period τ and logarithmic decrement $2\pi\delta$ as functions of immersion depth. From this data the damping coefficient k' and the added mass coefficient k are calculated as described below.

The cylinders (G) were accurately matched pairs cut from pieces of precision ground stainless steel shafting of radii 0.159, 0.238, 0.318, and 0.635 cm; they were mounted symmetrically on a diameter of a 0.64 cm thick by 15 cm diameter stainless steel or aluminum disk (F) in holes at radii d of 2.032, 4.572, or 7.112 cm. A short axle (D) with a mirror affixed coupled the oscillator to a quartz fiber (B). A pair of short stainless steel calibration weights (E) could be placed at either pair of unoccupied mounting holes for calibrating the quartz fiber. The total moment of inertia of the oscillator was typically $30,000 \text{ g-cm}^2$.

Quartz fibers were used in the torsion pendulum apparatus because of their long term stability, low internal damping, and linear restoring force at large amplitude in contrast to metal wires²⁶.

Figure 1. The Torsion Pendulum.

Quartz fibers approximately 66 cm long were drawn by hand from quartz rod to mean diameters of 0.2 to 1.3 mm. The ends of the fibers were epoxied to 0.64 cm diameter by 1.3 cm long brass rods, center drilled to the diameter of the fiber. The torsion constant of the fiber was obtained from the relation $\alpha = \Delta I / (\tau_c^2 - \tau_0^2)$, where ΔI is the calculated change in moment of inertia and τ_c and τ_0 are, respectively, the periods measured with and without the calibration weights. Such fibers supported more than 1.5 kg and had torsion constants in the range $100 < \alpha < 1000$ dyne-cm. Nuisance log decrements $2\pi\delta_0$ oscillations of the apparatus in air were less than 0.001. Air current shields (C) were placed around the fiber and oscillator.

Several methods of initiating the torsional oscillations were attempted. During initial testing phases the oscillator was turned by hand through an initial angle and released. This procedure introduces undesirable pendular motion. In an effort to eliminate the pendular motion a magnetic clutch was used as follows. A soft iron disk on a brass axle was placed between the mirror and oscillator plate. The solenoid of the magnetic clutch was placed concentric with the axle about 0.2 mm below the disk. When energized the iron disk was clamped to the magnetic clutch and the oscillator and magnetic clutch rotated together. After appropriately centering and leveling of the magnetic clutch assembly it could be rotated and then disengaged introducing very slight pendulum motion. This system was rejected because of too much damping and the tendency for the soft iron clutch plate to become magnetized.

Experiments 2 through 5 employed this starting device. Experiments 6 through 13 employed a modification of this by replacing the magnetic clamping of the oscillator to a rotatable base with a mechanical clamp.

The remaining experiments 14 through 28 employed the starting mechanism (A) shown in Fig. 1. A long axle passing through a long brass bushing has the fiber attached at one end by a set screw. A spring exerts a torque on the starting arm and forces it against the mechanical stop at the top. The bushing is attached to the wall by a suitable arm and clamp. The oscillator is set into motion by turning the starting arm by hand through an appropriate angle and gently returning to the rest position. As long as the fiber is straight and all components of the oscillator are coaxial no perceptible pendulum motion occurs. A torque in excess of 5×10^5 dyne-cm is required to rotate the starting mechanism and is more than 10 times larger than the torque exerted by the stiffest fiber at maximum angular displacement. The oscillator returned to the same zero position within 5×10^{-4} radians.

Liquids used were water, butyl phthalate, ethylene glycol, and a 3:1 water ethylene glycol mixture. For each liquid the kinematic viscosity ν and density ρ were determined as functions of temperature; viscosities were measured with Cannon-Fenske routine viscometers and densities were measured with a Westphal balance. During the experiment the viscosity and density were obtained indirectly from the fluid temperature. Kinematic viscosities ranged from 0.01 to $0.175 \text{ cm}^2/\text{sec}$.

The fluid bath was approximately 20 cm deep and was 15, 20, and 26 cm in diameter for experiments 1 through 13, 14 through 20 and 21 through 28, respectively. The cylinder immersion depth (h) was adjusted by moving the fluid container (I) vertically with a scissors jack (J). The elevation of the container was changed in fixed increments determined by four sets of three accurately measured spacers placed between the top and bottom plates of the jack. The immersion depth h_i was measured from the fluid level to the bottom of the cylinders. The measured depths were fitted to the spacer heights adjusted for fluid displacement using the following equations:

$$H_i = \sigma(L_i - L_0) \quad (9a)$$

$$L_0 = \left(\frac{1}{n}\right) \sum_{i=1}^n [L_i - h_i/\sigma] \quad (9b)$$

where H_i is the fitted immersion depth, L_0 is the calculated spacer height for zero immersion, L_i is the i th spacer height, $\sigma = [1 - 2(\frac{a}{b})^2]^{-1}$ is the adjustment for fluid displacement, n is the number of depths and h_i is the measured immersion depth.

2. Apparatus for Period Measurement

The electronics for this experiment are shown in Fig. 2 and the signal wave forms and their temporal relationship are shown in Fig.

3. Period measurements were made with a one meter optical lever and a photodetector. With the oscillator at rest, the image of a line filament bulb reflected from the mirror (D) is focused on a slit in front of the photodetector. Two photodetectors were used and the electronic circuits are shown in Fig. 4. A 931A photo-multiplier tube with a slit placed at the photocathode was used for

Figure 2. Electronics Block Diagram.

Figure 3. Waveforms and Timing Diagram.

a) 931A PHOTOMULTIPLIER AND PREAMP

b) IN2175 PHOTODIODE AND PREAMP

Figure 4. Photodetector Electronics.

experiments 1 through 20; a 1N2175 photodiode, outside diameter 0.21 cm, without a slit was used for experiments 21 through 28. With the oscillator in motion voltage pulses from the photodetector were processed and divided by two with a suitable electronic circuit.

For experiments 1 through 20 this circuit consisted of a Schmitt trigger circuit²⁷, adjusted to give a fast risetime square pulse when the photodetector voltage reached a predetermined level (usually one half of the maximum pulse height), and a flip-flop driven by the Schmitt trigger pulse. For experiments 21 through 28 the shaping circuit consisted of a Schmitt trigger, as above, in coincidence with a zero crossing detector²⁸ for the first derivative of the photodetector pulse. The zero crossing circuit gave a fast risetime square pulse when the photodetector had reached its maximum voltage, which corresponds more closely to the zero position of the oscillator than the pulse from the Schmitt trigger. The resulting coincidence pulses were divided by two with a flip-flop. The circuit diagram for the latter circuitry is shown in Fig. 5. The circuitry for the former case is obtained by deleting the zero crossing detector for the first derivative and the coincidence circuit.

The period of the square wave from the flip-flop was measured by a CMC Model 400C vacuum tube timer-counter for experiments 1 through 8. Every other period was measured by this method. For experiments 9 through 14 each successive period was measured with the CMC counter and a solid state Hewlett-Packard

5623B timer-counter. Either of these two methods is sensitive to off center errors in the positioning of the photodetector. Consider in Fig. 6, the damped sinusoidal wave form that represents the position of the light beam as a function of time. If the photodetector is at the position $+e$ from the zero position, note that the times labeled τ_+ increase as the amplitude of oscillation decreases and the times τ_- decrease. Thus, an off center error will cause alternately measured and successively measured periods to either increase or decrease continually as the amplitude of oscillation decays.

To reduce off center errors, staggered periods were measured in experiments 15 through 28 by measuring the period with one timer as the light beam crossed the zero position from the left and by measuring with the other timer the period as the light beam crossed the photodetector from the right. When the resulting staggered periods are averaged the off center error is eliminated. Periods were measured to 0.01 ms and typical standard deviations were from 1 to 5 ms.

3. Apparatus for Damping Measurement

Damping measurements were made with a conventional galvanometer light and scale at one meter for experiments 1 through 20. Amplitudes were measured on a straight ground glass scale from 25 cm to 1 cm. The moment of inertia of the oscillator and the radius at which the cylinder was placed were chosen so that the log decrement would be less than 0.1. The tangent correction for the straight scale was made and the log decrements were determined

Figure 6. Period Error Due to Off-Centering.

from semi-logarithmic plots of the amplitude versus the number of periods. The log decrements were measured with a precision of 0.6%.

In order for large amplitude measurements to be made conveniently, a vernier slit amplitude measurement system was constructed as shown in Fig. 7. This system was constructed by making a reduced photographic negative of accurately spaced vertical lines drawn by the Cal Comp peripheral plotter of the Louisiana State University Computer Center. Vernier lines were drawn and photographed at the same time for a one half least count. The main scale photographic slits were wrapped around the diameter of a 15.2 cm diameter disk mounted on the axle coupling the oscillator plate to the quartz fiber. A collimated light beam of about 1 cm diameter illuminated the main slits from the inside. The vernier slits were placed on the outside close to the main scale slits. A lense focused an image of the vernier slits on a photodiode. As the oscillator rotated the vernier slits were alternately illuminated by the light passing through the main scale slits.

The resulting modulated light intensity is detected by the photodiode, each maximum of the intensity corresponding to a rotation of the oscillator through an angle $\Delta\theta = \ell/r$ where ℓ is the distance corresponding to the least count of the vernier and r is the radius of the main scale circle. The photodiode pulses are counted during a full period of oscillation. The resulting number corresponds to four times the average amplitude during the period. Two main scales were used with a least count distance of 0.635 mm and 0.3175 mm. One count equals 0.00867 radians and 0.00417

Figure 7. Vernier Slit Amplitude Apparatus.

radians, respectively. The average amplitude θ for the period is given by $\theta = \frac{1}{4}N\ell/r$ where N is the total count for the period. The vernier slit system thus measures the angle with an accuracy of ± 0.00214 radians and 0.00104 radians, respectively, which corresponds to reading the straight scale of the optical lever to about ± 4 mm and 2 mm, respectively. Calibration with the optical lever confirmed this agreement.

B. Analysis

The Stokes force exerted by a fluid on a length h of an infinite oscillating cylinder of radius a was given in Eq. (3) and is rewritten here for convenience

$$F = - M'k \frac{dv}{dt} - M'k'\omega v \quad . \quad (10)$$

The torque N exerted by the fluid on two cylinders at a radius d is given by

$$N = - 2M'd(k \frac{dv}{dt} + k'\omega v) \quad . \quad (11)$$

The equation of motion for the torsional pendulum of this experiment is

$$I_0 \ddot{\theta} = - (2\pi)^2 \alpha \theta - 2M'd^2(k\ddot{\theta} + k'\omega\dot{\theta}) \quad (12)$$

where $d \dot{\theta}$ has been substituted for v , I_0 is the moment of inertia of the oscillator in air, $4\pi^2 \alpha \theta$ is the restoring torque for a fiber with torsion constant α , and θ , $\dot{\theta}$, and $\ddot{\theta}$ are the angular displacement, velocity and acceleration, respectively.

A solution of the form $\theta = \theta_0 e^{(i-\delta)\omega t}$ is assumed; the resulting damped harmonic motion has a period $\tau = 2\pi/\omega$ and log decrement $2\pi\delta$ given (to order δ) by:

$$\tau^2 = \tau_0^2 + (2\pi a^2 d^2 k/\alpha)h \quad (13a)$$

and

$$2\pi\delta\tau^2 = (2\pi^2 a^2 d^2 \rho k'/\alpha)h \quad (13b)$$

where τ_0 is the period of the apparatus in air, $\alpha\tau_0^2$ has been substituted for I_0 , and $\pi a^2 h$ has been substituted for M' . Since $2\pi\delta$ was always less than 0.1, the neglect of δ^2 in these equations introduces errors less than 0.1%.

In Fig. 8 typical data are plotted as $\tau^2 - \tau_0^2$ versus h and $2\pi\delta\tau^2$ versus h . The slopes s and s' of the straight lines in Fig. 8 determine the experimental values of k and k' :

$$k = s\alpha/2\pi a^2 d^2 \rho \quad (14a)$$

and

$$k' = s'\alpha/2\pi^2 a^2 d^2 \rho. \quad (14b)$$

The experimental penetration depth is given by $\lambda = (v_{\tau_{av}}/\pi)^{\frac{1}{2}}$, where τ_{av} is the average of the periods measured for different depths of immersion. The difference between the period at maximum immersion (about 10 cm) and minimum immersion (about 1 cm) was never more than 0.33 sec; since the periods ranged from 6 to 16 sec, the use of an average period here causes an uncertainty in λ

Figure 8. Typical Data Plot

no greater than 3% (and typically 1%).

The end effects and surface effects are evident from Fig. 8 as an effective shortening of the cylinder by an amount h_0 and as an additional damping $2\pi\delta_0$ (including losses in the fiber).

For the geometry of this experiment, the effect of the two cylinders on each other and the effect of the fixed cylindrical boundary on the moving cylinders have been calculated (for an inviscid fluid) by Hussey and Reynolds³⁸. The result is that the inviscid part of the inertia coefficient is multiplied by a factor $1 - \frac{1}{2}(a/d)^2 + 8a^2b^4d^2(b^4-d^4)^{-2}$. In the present experiment this factor varied from 0.9513 to 0.9998. The distance between cylinders ranged from 5.0λ to 79.7λ , and the distance from cylinder to outer boundary ranged from 3.5λ to 73.0λ , so the viscous interaction was presumed to be negligible. The long range steady streaming, generated by Reynolds stress, is strongly dependent on amplitude; since the amplitude was always small, and both period and log decrement were independent of amplitude over a large range, it was concluded that the present measurements were unaffected by steady streaming.

In writing the equation for the torque, we have neglected the small torque due to the rotation of the cylinders about their own axes. This neglect introduces an error of size $\frac{1}{2}a^2/d^2$ compared to 1; this error is never larger than 2% and is typically 0.6%.

C. Data

Experimental data for experiments 1 through 28 are found in Tables III and IV. Table III contains the measured immersion depths

TABLE III
Experimental Data

No.	h, cm	H, cm	τ , sec	$2\pi\delta$
1	0.0	0.0	5.332	-
	2.0	2.0	5.334	0.00672
	8.0	8.0	5.374	0.0210
2	0.0	0.0	10.599	-
	3.335	3.354	10.640	0.0148
	6.386	6.367	10.683	0.0256
3	0.0	0.0	5.912	-
	4.841	4.549	5.923	0.0212
	7.249	7.541	5.928	0.0320
4	0.0	0.0	13.821	-
	5.979	5.967	13.894	0.0480
	8.954	8.966	13.945	0.0810
5	0.0	0.0	16.317	-
	5.979	5.967	16.390	0.378
	8.954	8.966	16.433	0.650
6	0.0	0.0	14.751	-
	0.178	0.115	14.760	0.0180
	3.175	3.147	14.898	0.0824
	-	6.218	15.076	0.187
	9.170	9.259	15.175	0.260
7	0.0	0.0	13.797	-
	0.635	0.427	13.803	0.00450
	3.404	3.458	13.832	0.0191
	6.375	6.529	13.862	0.0350
	-	9.570	13.898	0.0515
8	0.0	0.0	13.890	-
	0.381	0.306	13.892	-
	3.302	3.306	13.956	0.0615
	6.274	3.444	14.044	0.118
	-	9.354	14.123	0.178
9	0.0	0.0	6.403	-
	3.531	3.668	6.422	0.0356
	6.604	6.706	6.446	0.0684
	9.955	9.716	6.472	0.100

TABLE III continued

No.	h , cm	H , cm	τ , sec	$2\pi\delta$
10	0.0	0.0	6.776	-
	0.660	0.568	6.782	0.0122
	3.581	3.599	6.834	0.0554
	6.655	6.670	6.891	0.105
	9.652	9.711	6.946	0.177
11	0.0	0.0	6.306	-
	3.505	3.498	6.318	0.0225
	6.528	6.529	6.330	0.0428
	9.525	9.531	6.341	0.0647
12	0.0	0.0	6.030	-
	0.787	0.670	6.045	0.0266
	3.658	3.701	6.143	0.106
	6.731	6.772	6.259	0.203
	9.779	9.813	6.368	0.306
13	0.0	0.0	8.373	-
	0.508	0.685	8.375	0.0066
	3.734	3.676	8.390	0.0279
	6.731	6.707	8.408	0.0530
	9.804	9.709	8.426	0.0784
14	0.0	0.0	6.355	-
	4.699	4.696	6.379	0.0365
	7.747	7.730	6.394	0.0594
	10.744	10.735	6.412	0.0800
15	0.0	0.0	7.509	-
	7.747	7.743	7.551	0.0517
	10.744	10.748	7.568	0.0744
16	0.0	0.0	7.158	0.000778
	1.702	1.670	7.160	0.00366
	4.648	4.666	7.165	0.0104
	7.722	7.699	7.169	0.0148
	10.668	10.704	7.173	0.0209
17	0.0	0.0	7.922	-
	1.397	1.382	7.924	0.00388
	4.369	4.374	7.930	0.0102
	7.391	7.406	7.935	0.0168
	10.414	10.408	7.941	0.0234
18	0.0	0.0	9.327	-
	7.544	7.554	9.339	0.0137
	10.566	10.556	9.344	0.0190

TABLE III continued

No.	h, cm	H, cm	τ , sec	$2\pi\delta$
19	0.0	0.0	9.326	-
	1.651	1.668	9.328	0.00352
	4.648	4.660	9.333	0.00792
	7.696	7.692	9.337	0.0128
	10.719	10.694	9.342	0.0172
20	0.0	0.0	9.351	0.000932
	3.378	3.356	9.367	0.0224
	6.375	6.385	9.382	0.0407
	9.373	9.385	9.397	0.0592
21	0.0	0.0	7.951	-
	0.533	0.445	7.953	0.00527
	3.480	3.436	7.968	0.0264
	6.325	6.465	7.982	0.0468
	9.474	9.466	7.999	0.0673
22	0.0	0.0	7.627	-
	2.108	2.120	7.682	0.0241
	5.156	5.149	7.774	0.0529
	8.153	8.148	7.864	0.0791
23	0.0	0.0	8.312	0.00263
	1.372	1.401	8.343	0.0164
	4.420	4.430	8.428	0.0409
	7.468	7.430	8.515	0.0629
24	0.0	0.0	5.782	0.00362
	1.194	1.214	5.784	0.00465
	4.242	4.243	5.794	0.00767
	7.264	7.243	5.803	0.0106
25	0.0	0.0	6.509	0.00300
	2.870	2.899	6.545	0.0172
	5.944	5.928	6.589	0.0302
	8.941	8.928	6.632	0.0452
26	0.0	0.0	5.782	-
	5.867	5.930	5.806	0.0301
	8.992	8.929	5.820	0.0428
27	0.0	0.0	6.791	0.00584
	3.886	3.884	6.809	0.0260
	6.883	6.884	6.827	0.0419
28	0.0	0.0	6.679	-
	0.762	0.745	6.680	0.00644
	6.756	6.773	6.695	0.0195

TABLE IV

Constants of the Experiment

No.	a, cm	d, cm	2b, cm	α , g-cm/sec ²	ν , cs	ρ g/cm ³
1	0.634	4.572	15.2	1080	0.957	0.9978
2	0.634	4.572	15.2	270	0.967	0.9988
3	0.158	2.032	15.2	270	16.6	1.1121
4	0.317	4.572	15.2	140	7.16	1.0905
5	0.317	4.572	15.2	140	7.16	1.0905
6	0.634	4.572	15.2	140	15.6	1.1110
7	0.634	2.032	15.2	140	14.9	1.1100
8	0.317	4.572	15.2	140	16.3	1.1116
9	0.317	4.572	15.2	661	15.0	1.1101
10	0.317	4.572	15.2	661	14.75	1.1098
11	0.158	4.572	15.2	661	14.7	1.1097
12	0.634	4.572	15.2	374	15.3	1.1105
13	0.158	4.572	15.2	374	14.7	1.1097
14	0.237	4.572	15.2	661	13.8	1.1085
15	0.237	4.572	15.2	661	13.8	1.1085
16	0.237	2.032	15.2	508	14.25	1.1090
17	0.237	2.032	20.3	413	16.0	1.1113
18	0.237	2.032	20.3	413	15.85	1.1110
19	0.237	2.032	20.3	413	14.0	1.1087
20	0.158	4.572	20.3	413	14.15	1.1087
21	0.158	4.572	20.3	413	14.2	1.1090
22	0.634	7.112	25.7	393	0.904	0.9972
23	0.634	7.112	25.7	393	0.902	0.9972
24	0.634	2.032	25.7	393	0.890	0.9970
25	0.634	4.572	25.7	393	0.886	0.9970
26	0.158	7.112	25.7	393	0.886	0.9970
27	0.158	7.112	25.7	286	0.873	0.9968
28	0.158	4.572	25.7	282	0.945	0.9976

TABLE V
End Effects and Relative Distances

No.	h_0 , cm	$2\pi\delta_0$	$(b-d)/\lambda$	$2d/\lambda$
1	1.70	0.0019	24.6	72.0
2	0.49	0.0026	16.5	50.5
3	-2.02	0.0047	10.0	7.3
4	1.69	-0.019	5.3	16.2
5	0.89	-0.017	4.9	14.9
6	-0.15	0.0075	3.5	10.6
7	-0.01	0.017	6.9	5.0
8	0.44	-0.0054	3.5	10.7
9	1.31	-0.0044	5.4	16.5
10	0.30	-0.0058	5.3	15.1
11	0.32	-0.0026	5.5	16.8
12	0.39	-0.0066	5.5	16.6
13	0.49	-0.00021	4.8	14.6
14	0.42	0.0023	5.7	17.2
15	0.35	-0.0077	5.2	15.8
16	0.05	0.00090	9.8	7.1
17	0.25	0.00079	12.7	6.4
18	0.36	0.00031	11.8	5.9
19	0.29	0.00092	12.5	6.3
20	0.15	0.00162	8.5	14.0
21	0.11	0.0023	9.2	15.2
22	0.32	0.0036	38.0	68.0
23	0.35	0.0052	36.5	65.4
24	0.53	0.0034	73.0	26.0
25	0.42	0.0028	60.2	61.7
26	0.80	0.0046	44.5	79.7
27	0.89	0.0052	41.3	73.9
28	0.34	0.0048	57.7	64.4

h in the second column, and the fitted immersion depths H in the third column. The period τ and damping $2\pi\delta$ are found in the fourth and fifth columns, respectively.

The constant parameters for each experiment are found in Table IV. The cylinder radius a , the distance d from the center of the oscillator and the diameter of the fluid bath are found in the second, third and fourth columns, respectively. The torsion fiber constant α is found in the fifth column. The kinematic viscosity ν and density ρ of the fluid is found in the sixth and seventh columns, respectively.

Table V contains the graphically determined end effects h_0 and $2\pi\delta_0$ in columns two and three, respectively. The distance from the cylinder to the container wall $(b-d)/\lambda$ and the distance between the cylinders $2d/\lambda$, expressed as multiples of the penetration depth, are found in columns four and five, respectively.

CHAPTER III

RESULTS

A. Stokes' Solution

Stokes' solution for the coefficients k and k' of Eq. (10) was given in Eq. (4) and is rewritten here for convenience

$$k - 1 - ik' = 4K_1(z)/[zK_0(z)] \quad (15)$$

where $K_n(z)$ is the modified Bessel function of order n , and $z = (1 + i)a/\lambda$. Tables of k and k' were given in the original paper of Stokes¹ and in the more recent work of Hussey and Vujacic²⁹. We have extended the calculations to smaller values of a/λ and the combined results are presented in Table VI. These results are most easily expressed in terms of the parameter m used by Stokes:

$$m = a/(\lambda\sqrt{2}) = \frac{1}{2} a(\omega/\nu)^{\frac{1}{2}} \quad (16)$$

The values of k and k' for $m < 0.1$ in Table VI were calculated from tables of the modulus N_n and phase ϕ_n of the Kelvin function $\ker_n x + i\operatorname{kei}_n x$, where $x = 2m$;

$$k = 1 - (N_1/xN_0)\cos(\phi_1 - \phi_0 - \frac{3}{4}\pi) \quad (17a)$$

$$k' = (4N_1/xN_0)\sin(\phi_1 - \phi_0 - \frac{3}{4}\pi) \quad (17b)$$

TABLE VI

Values of the Inertia Coefficient k and the DampingCoefficient k' . For $m < 0.001$ see Eqs. (18); for $m > 4.0$ see Eqs. (19).

m	$m^2 k'$	$m^2(k - 1)$	k'	k
0.001	0.15557	0.019298	155,570.	19,299.
0.005	0.20611	0.034227	8,244.5	1,370.1
0.010	0.23918	0.046382	2,391.8	464.82
0.020	0.28418	0.065854	710.44	165.635
0.040	0.34825	0.098916	217.65	62.823
0.060	0.39946	0.12924	110.96	36.900
0.080	0.44466	0.15842	69.478	25.753
0.100	0.48630	0.18700	48.630	19.700
0.120	0.52555	0.21522	36.497	15.946
0.140	0.56310	0.24324	28.730	13.410
0.160	0.59936	0.27111	23.412	11.590
0.180	0.63461	0.29889	19.587	10.2251
0.200	0.66905	0.32662	16.726	9.1655
0.250	0.75252	0.39581	12.040	7.3329
0.300	0.83326	0.46494	9.2584	6.1660
0.350	0.91211	0.53410	7.4458	5.3600
0.400	0.98958	0.60333	6.1849	4.7708
0.450	1.0660	0.67262	5.2643	4.3216
0.500	1.1416	0.74200	4.5666	3.9680
0.600	1.2911	0.88099	3.5863	3.4472
0.700	1.4388	1.0203	2.9363	3.0822
0.800	1.5852	1.1598	2.4769	2.8122
0.900	1.7308	1.2996	2.1368	2.6044
1.00	1.8757	1.4395	1.8757	2.4395

TABLE VI continued

m	$m^2 k'$	$m^2(k-1)$	k'	k
1.00	1.8757	1.4395	1.8757	2.4395
1.10	2.0201	1.5795	1.6695	2.3054
1.20	2.1640	1.7198	1.5028	2.1943
1.30	2.3077	1.8602	1.3655	2.1007
1.40	2.4510	2.0006	1.2505	2.0207
1.50	2.5940	2.1411	1.1529	1.9516
1.60	2.7372	2.2817	1.0692	1.8913
1.70	2.8798	2.4224	0.99649	1.8382
1.80	3.0225	2.5628	0.93287	1.7910
1.90	3.1650	2.7039	0.87674	1.7490
2.00	3.3075	2.8444	0.82687	1.7111
2.10	3.4498	2.9856	0.78227	1.6770
2.20	3.5921	3.1262	0.74217	1.6459
2.30	3.7343	3.2671	0.70591	1.6176
2.40	3.8764	3.4082	0.67299	1.5917
2.50	4.0185	3.5494	0.64296	1.5679
2.60	4.1606	3.6903	0.61547	1.5459
2.70	4.3026	3.8309	0.59020	1.5255
2.80	4.4445	3.9725	0.56690	1.5067
2.90	4.5864	4.1133	0.54535	1.4891
3.00	4.7283	4.2543	0.52537	1.4727
3.10	4.8702	4.3956	0.50678	1.4574
3.20	5.0120	4.5363	0.48945	1.4430
3.30	5.1538	4.6783	0.47326	1.4296
3.40	5.2956	4.8194	0.45810	1.4169
3.50	5.4374	4.9600	0.44387	1.4049
3.60	5.5792	5.1010	0.43049	1.3936
3.70	5.7209	5.2419	0.41789	1.3829
3.80	5.8626	5.3832	0.40600	1.3728
3.90	6.0043	5.5243	0.39476	1.3632
4.00	6.1459	5.6656	0.38412	1.3541

For $m < 0.001$, the form given by Stokes is more convenient:

$$m^2(k - 1) = [\frac{1}{4}\pi - 2m^2(B - L + \frac{1}{2})]/B \quad (18a)$$

$$m^2k' = [-L + \frac{1}{4}\pi m^2(2 + L)]/B \quad (18b)$$

where $L = \gamma + \ln m$, $B = L^2 + (\frac{1}{4}\pi)^2$, and γ is Euler's constant.

For $m > 4.0$, values of k and k' are most easily obtained from series derived from the asymptotic expansion of the Hankel function. The first three terms in each series are²⁹,

$$k = 1 + 2\lambda/a + \lambda^3/8a^3 \quad (19a)$$

$$k' = 2\lambda/a + \lambda^2/a^2 - \lambda^3/8a^3. \quad (19b)$$

B. Comparison with Experiment

Stokes' theory is compared with our experimental results in Figs. 9 and 10. Figure 9 shows the results for the added mass coefficient k . Clearly, there is good agreement between theory and experiment over the entire range of λ/a . The same conclusion is drawn from the measurements of the damping coefficient k' , shown in Fig. 10.

Most of these measurements were made with the angular displacement of the torsion pendulum in the range 0.125 to 0.01 rad. At the upper end of this range, it was frequently observed that the logarithmic decrement was higher and the period was lower than their constant low amplitude values. If we designate by A_m the maximum linear amplitude at which the period and decrement are equal to their low amplitude values, then the maximum relative

Figure 9. Experimental Inertia Coefficient and Stokes Theory.

Figure 10. Experimental Damping Coefficient and Stokes Theory.

amplitude is A_m/a and the maximum Reynolds number R_m is $2aA_m\omega/\nu$. This Reynolds number is based on the cylinder diameter $2a$ and on the largest velocity $A\omega$ during one cycle. Approximate values ($\pm 40\%$) of A_m/a and R_m for the 28 experimental points are given in Table VII along with the experimental values of k and k' and the boundary correction $1 - \frac{1}{2}(a/d)^2 + 8a^2b^4d^2(b^4-d^4)^{-2}$.

It is apparent that the restrictions on amplitude and Reynolds number are not nearly as severe as expected. Consider, for example, the three experimental points for which $\lambda/a \approx 2$; for these points $A_m/a \approx 1.1$ and $R_m \approx 1.4$. None of these parameters can be considered to be very small compared to 1. Therefore, it appears that the approximation made by Stokes in neglecting the non-linear term is less restricted than the mathematical analysis³⁰ implies.

Since the range of amplitudes employed in our experiment is rather limited, it occurred to us that perhaps the agreement with Stokes' theory at large λ/a is fortuitous, and that measurement at lower velocities would reveal a different region of flow. To investigate this question, we extended the measurements to lower amplitudes as follows: a laser was used as the light source, and by means of a spatial filter and cylindrical lenses, the beam was focused to a narrow line (about 2 mm wide) on a scale 9 m from the mirror. Observations were made down to angular displacements of 1×10^{-4} rad. The result is shown in Fig. 5: the log decrement is constant over more than two decades of amplitude. Therefore, we conclude that the agreement with Stokes' theory at $\lambda/a = 4.12$

TABLE VII

Experimental Values of the Force Coefficients,
Critical Amplitude and Reynolds number.

λ/a	k	k'	A_m/a	R_m	Boundary Correction
0.201	1.469	0.4511	> 0.90	> 89.	0.9933
0.215	1.427	0.4926	0.60	52.	0.9917
0.234	1.428	0.4014	0.23	17.	0.9513
0.236	1.449	0.5645	0.58	42.	0.9969
0.245	1.485	0.5562	0.45	30.	0.9969
0.286	1.563	0.6733	0.76	37.	0.9933
0.809	2.613	2.288	1.8	11.	0.9998
0.866	2.820	2.544	0.50	2.7	0.9933
0.870	2.946	2.856	2.6	14.	0.9998
0.895	2.791	3.097	.65	3.1	0.9933
0.896	2.868	2.669	2.0	9.7	0.9994
1.28	3.504	3.825	.34	0.80	0.9518
1.34	3.337	4.779	1.4	3.7	0.9933
1.75	4.808	6.456	1.0	1.2	0.9983
1.78	4.604	6.691	1.1	1.4	0.9983
1.93	4.577	7.634	1.3	1.4	0.9983
2.23	5.633	7.663	0.42	0.34	0.9991
2.40	6.416	9.565	1.0	0.71	0.9932
2.43	6.894	11.21	> 2.4	> 1.6	0.9991
2.68	7.504	11.05	0.89	0.49	0.9932
2.69	6.916	11.81	0.90	0.48	0.9983
2.72	7.276	10.81	1.0	0.55	0.9932
2.89	7.922	12.51	1.1	0.51	0.9932
3.44	8.746	16.28	0.80	0.26	0.9996
3.54	7.408	15.15	0.62	0.49	0.9970
3.80	9.130	15.85	3.7	1.0	0.9995
3.96	9.786	18.53	1.1	0.27	0.9996
4.11	10.58	19.52	2.5	0.58	0.9995

Figure 11. Constant Log Decrement over Wide Relative Amplitude Range.

is not fortuitous, and that the Reynolds number is not an important parameter, at least for $A < a$.

CHAPTER IV

RELATION TO STOKES' PARADOX

If the period of oscillation becomes extremely long, one would expect that the dissipative part of the force should approach the low Reynolds number drag. This is indeed the case for the sphere, as shown by Lamb³¹:

$$\begin{aligned} F_D &= k'M\omega U = \frac{9}{4} \frac{\lambda^2}{a^2} \left(1 + \frac{a}{\lambda}\right) \left(\frac{4}{3} \pi a^3 \rho\right) \omega U \\ &= 6\pi a \mu U [1 + (a/\lambda)] \quad . \end{aligned} \quad (20)$$

The term $6\pi a \mu U$ is the well-known Stokes drag, where μ is the viscosity. The term a/λ is small in the limit of long period.

For the circular cylinder, the transition to low Reynolds number is more difficult because of the Stokes paradox³², i.e., the lack of a solution to the Stokes equations for unbounded plane flow. In this case the dissipative part of the force (per unit length) is

$$F_D = k'M\omega U = 4\pi \mu U k'm^2 \quad . \quad (21)$$

For $\lambda \gg a$,

$$F_D = -4\pi \mu U (\gamma + \ln m)^{-1} \quad (22)$$

where γ is Euler's constant. We can compare this with Lamb's solution²⁵ of the Oseen equations

$$F_D = -4\pi \mu U \left(\gamma - \frac{1}{2} + \ln \frac{1}{8} R\right)^{-1} \quad . \quad (23)$$

The logarithmic terms in Eqs. (22) and (23) are slowly varying but dominate the equations for small m and R , so the two equations, though similar in form, are not identical. Equation (22) results from the limit $a/\lambda \rightarrow 0$, with $A/a \ll 1$; Equation (23) results from the limit $a/\lambda \rightarrow 0$, with $A/a \sim O(1)$. In the case of the sphere, these two limit processes give the same result; for the cylinder, they do not. This is perhaps another aspect of the Stokes paradox.

The similarity in the roles played by the Stokes parameter m in Eq. (22) and the Reynolds number R in Eq. (23) is suggestive and leads us to make the following speculation: we know that Eq. (22) is the limiting form of Eq. (21) for $a/\lambda \ll 1$. We presume that Eq. (23) is the limiting form of a more general function of R for $R \ll 1$. We speculate that the role played by R in this more general function is identical to the role played by m in Eq. (21). More precisely, we speculate that the function is

$$F_D = 4\pi\mu U k' R^2 / 173.98 \quad (24)$$

where k' , given by Eq. (17b), is now a function of the real variable $x = 0.15163R$. Using Eq. (17b) or Table VI (with $m = R/13.190$), we can calculate values of the drag from Eq. (24) and compare with measurements made at low Reynolds number. This is done in Fig. 12, where the full Oseen solution of Tomotika and Aoi³³, the matched asymptotic expansion solution of Kaplun³⁴, and the truncated series solution of Underwood³⁵ are also shown. Equation (24) agrees very well with the measurements of Tritton³⁶ and of Jayaweera and Mason³⁷ up to $R = 2.5$. Admittedly, this speculation

has no rigorous justification, but it provides a relatively simple mathematical expression which reduces to Lamb's solution for $R \ll 1$ and agrees well with measurement in the difficult region around $R = 1$.

It is interesting to compare Eq. (24) with Kaplun's solution. If we put $\epsilon = -(\gamma - \frac{1}{2} + i\frac{1}{8}R)^{-1}$, then for small R , Eq. (24) can be written

$$F_D/4\pi\mu U = \epsilon - (\pi^2/16)\epsilon^3. \quad (25)$$

The coefficient $\pi^2/16 = 0.62$ is to be compared with Kaplun's value of 0.87.

Figure 12. Low Reynolds Number Drag Measurements and Eq. (24).

CHAPTER V

CONCLUSIONS

Experimentally measured values of the inertial mass coefficient and the damping coefficient are found to be in good agreement with the theory of Stokes [Eq. (4)] throughout the range $0.286 < \lambda/a < 4.13$, even when the Reynolds number R and the relative amplitude A/a are $O(1)$. Order of magnitude analysis [Eqs. (5) and (6)] indicates that neglect of the non-linear convective acceleration in the Navier-Stokes equation, as done in the Stokes solution, is justified when (1) $R \ll 1$ and $A \ll a$, or (2) when $\lambda \ll a$ and $A \ll a$. Thus, our experiments indicate that the range of validity of the Stokes solution for the forces on an oscillating cylinder is not as restricted as implied by order of magnitude analysis.

The values of the inertial mass coefficient and the damping coefficient have been calculated from Eqs. (17) for $m = \frac{1}{\sqrt{2}}(a/\lambda)$ in the range $.001 \leq m \leq .06$ and are presented in Table VI, along with the values calculated by Stokes for $0.01 \leq m \leq 4.0$.

The relation to Stokes paradox for a linearly translating cylinder is shown by considering the large penetration depth limit of the Stokes solution for the damping force on an oscillating cylinder. The resulting limit for the damping force [Eq. (22)] is compared to the Lamb solution [Eq. (23)] for the drag on a translating cylinder. An empirical equation [Eq. (24)] for the drag force is given which reduces to Lamb's solution in the limit

of small Reynolds number and which agrees well with the experimentally measured drag force for Reynolds number up to 2.5. The empirical equation is considered useful because of its computational simplicity, and perhaps may give insight to a better theoretical solution for the drag on a cylinder.

REFERENCES

1. H. Schlichting, Boundary Layer Theory (McGraw Hill: New York, New York, 1960) 4th ed., Chaps. 9, 10, and 11.
2. J. Happel and H. Brenner, Low Reynolds Number Hydrodynamics (Prentice-Hall: Englewood Cliffs, N. J., 1965), Chaps. 3 and 4.
3. G. G. Stokes, Trans. Cambridge Phil. Soc. 9, 8 (1851); Mathematical and Physical Papers (Cambridge University Press: London, 1880-1905), Vol. 3, p. 1. (hereafter referred to as Papers).
4. J. T. Stuart, in Laminar Boundary Layers, edited by L. Rosenhead, (Clarendon Press: Oxford, 1963), Chap. 7, p. 390.
5. L. A. Segal, J. Math. and Phys. 39, 189 (1960).
6. J. T. Stuart, J. Fluid Mech. 24, 673 (1966).
7. N. Riley, Mathematika 12, 161 (1965).
8. N. Riley, Quart. J. Mech. Appl. Math. 19, 461 (1966).
9. Chang-Yi Wang, J. Fluid Mech. 32, 55 (1968).
10. N. Riley, J. Inst. Math. Appl. 3, 419 (1967).
11. F. Baily, Phil. Trans. Roy. Soc. London 122, 399 (1832).
12. G. G. Stokes, Papers, Vol. 3, p. 138.
13. C. A. Coulomb, Acad. des Sciences, Memoires, ser. 1, 3, 246 (1800).
14. J. T. Stuart and L. Woodgate, Phil. Mag. 46, 40 (1955).
15. H. Martin, Ann. Phys., Leipzig 77, 627 (1925).
16. L. R. Laird, Phys. Rev. 7, 102 (1898).

17. E. G. Richardson and R. I. Tait, Oesterr. Ing.-Arch. 8, 200 (1954).
18. R. G. Hussey, B. J. Good, and J. M. Reynolds, Phys. Fluids 10, 89 (1967).
19. J. T. Tough, W. D. McCormick, and J. G. Dash, Phys. Rev. 132, 2373 (1963).
20. J. T. Tough, private communication (1963).
21. G. H. Keulegan and L. H. Carpenter, J. Res. Natl. Bur. Std. (U.S.) 60, 423 (1958).
22. Ye-Tak Yu, J. Appl. Phys. 13, 66 (1942).
23. T. E. Stelson and F. T. Mavis, Proc. Am. Soc. Civ. Eng., Eng. Mech. Div. 81, no. 670 (April, 1955).
24. A. Heinzer and C. Dalton, Trans. Am. Soc. Mech. Eng., Ser. D, J. Basic Eng. 91, 850 (1969).
25. H. Lamb, Phil. Mag. 21, 112 (1911).
26. J. Strong, Procedures in Experimental Physics (Prentice-Hall: Englewood Cliffs, N. J., 1938), p. 188.
27. J. N. Giles, Fairchild Semiconductor Linear Integrated Circuits Applications Handbook (Fairchild Semiconductor: Mountain View, Cal.), p. 165 (1967).
28. Ibid., p. 166.
29. R. G. Hussey and P. Vujacic, Phys. Fluids 10, 96 (1967).
30. G. G. Stokes, Papers, p. 1.
31. H. Lamb, Hydrodynamics (Cambridge University Press, Cambridge, 1940) 6th ed., p. 644.
32. M. van Dyke, Perturbation Method in Fluid Mechanics (Academic Press: New York, 1964), p. 152.

- 33. S. Tomotika and T. Aoi, Quant. J. Mech. Appl. Math. 3,
140 (1950).
- 34. S. Kaplun, J. Math. Mech. 6, 595 (1957).
- 35. R. L. Underwood, J. Fluid Mech. 37, 95 (1969).
- 36. D. J. Tritton, J. Fluid Mech. 6, 547 (1959).
- 37. K. O. L. F. Jayaweera and B. J. Mason, J. Fluid Mech. 22,
709 (1965).
- 38. R. G. Hussey and J. M. Reynolds, Phys. Fluids 8, 1213 (1965).

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Date of Examination:

July 18, 1972