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Analytic Direct Integrals and Their Applications to Factorization Theory.

Alvin Edward Horne
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Doctor of Philosophy

in

The Department of Mathematics

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ABSTRACT

The analytic direct integrals of \( H^2(U^n) \) and \( H^\infty(U^n) \) are examined for \( n \geq 1 \) and their structure is utilized to study factorization theory in \( H^2(U^m) \) for \( m \geq 2 \).

In Chapter I notation is established and a few theorems are proved concerning functions in \( H^p \) spaces.

In Chapter II the analytic direct integral of \( H^2(U^n) \) is defined and shown to be a separable Hilbert space which in a natural way is isometrically isomorphic to \( H^2(U^{n+1}) \). In Chapter III the analytic direct integral of \( H^\infty(U^n) \) is defined and shown to be a subset of the analytic direct integral of \( H^2(U^n) \). Utilizing the structure that \( H^\infty(U^n) \) possesses as a ring of operators acting on \( H^2(U^n) \) it is shown that the analytic direct integral of \( H^\infty(U^n) \) is spacially isomorphic to \( H^\infty(U^{n+1}) \) under the aforementioned isometric isomorphism defined on the analytic direct integral of \( H^2(U^n) \).

In Chapter IV various definitions of inner and outer functions in \( H^2(U^n) \) are examined with respect to the
analytic direct integral. It is shown that with the definitions of inner and outer used it is not possible to factor each element of $H^2(U^n)$ into inner and outer factors if $n \geq 2$. 
The Banach algebra $H^\infty(U)$ has a standard representation as a ring of operators on the Hilbert space $H^2(U)$. This representation is given by the map $f \mapsto A_f$ where

$$A_f g = fg,$$

for all $g \in H^2(U)$.

Each algebra $H^\infty(U^n)$ has a similar representation as a ring of operators on $H^2(U^n)$, $n=2,3,\cdots$. These rings have been studied by Bourque [1].

In this paper we use the concept of analytic direct integral to study $H^\infty(U^n)$ and $H^2(U^n)$. That this turns out to be an effective technique is due largely to the fact that the analytic direct integral of $H^\infty(U^n)$ can be represented as a maximal commutative ring of operators acting on the analytic direct integral of $H^2(U^n)$. Although we do not use the terminology of rings of operators, many of the theorems can be readily seen to have been motivated by such considerations. For example, the proof that the analytic direct integral of $H^\infty(U^n)$ is
isometrically isomorphic to $H^\infty(\mathbb{U}^{n+1})$ is in essence a proof that they are spacially isomorphic rings of operators.

In Chapter I we present the notation and definitions connected with the study of $H^p$ spaces in several complex variables. For the case $n=1$ a complete discussion, including proofs of many of the results referred to here, can be found in Hoffman [2] or Porcelli [4]. Rudin [6] gives a brief discussion of $H^p$ spaces on polydiscs. Most of the notation and definitions found in Chapter I agree with those of Rudin.

In Chapters II and III we give definitions of the analytic direct integrals of $H^2(\mathbb{U}^n)$ and $H^\infty(\mathbb{U}^n)$, and investigate their properties. These concepts were first introduced by Bourque [1] for the case $n=1$.

In Chapter IV we utilize the analytic direct integral to study the factorization problem in $H^2(\mathbb{U}^n)$. It is shown that if $n \geq 2$, it is not possible to express each element of $H^2(\mathbb{U}^n)$ as a product of inner and outer functions as is the case when $n=1$. 
CHAPTER I
PRELIMINARIES

In this chapter we present the notation of several complex variables and review the necessary theory of analytic functions and $H^p$ spaces. We shall also prove several theorems that will be used in the construction of the analytic direct integrals.

If $I$ denotes the set of integers and $I_+$ the set of non-negative integers, then $I^n$ and $I_+^n$ are the cartesian products of $n$-copies of $I$ and $I_+$ respectively. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $I_+^n$ we define $\alpha!$ and $|\alpha|$ by the formulas

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$$

and

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$ 

If $\alpha$ and $\beta$ are in $I^n$ then $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for $i=1,2,\ldots,n$. In particular, $\alpha \geq 0$ means that each $\alpha_i$ is non-negative. We define the sum of two elements $\alpha$ and $\beta$ in $I^n$ by
\[
\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n).
\]

As usual, \(\mathbb{C}\) will represent the complex field, \(U\) and \(T\) the unit disc and unit circle in \(\mathbb{C}\). The sets \(\mathbb{C}^n, U^n\) and \(T^n\) are the cartesian products of \(n\)-copies of \(\mathbb{C}\), \(U\) and \(T\) respectively, with \(U^n\) referred to as the unit polydisc and \(T^n\) the \(n\)-dimensional torus. Normalized Lebesgue measure on \(T^n\) will be written \(m_n\) so that

\[
\int_{T^n} dm_n(w) = m_n(T^n) = 1.
\]

For \(z \in \mathbb{C}^n\),

\[
z = (z_1, z_2, \ldots, z_n),
\]

we write

\[
\|z\| = \max \{|z_i| \mid i = 1, 2, \ldots, n\},
\]

and if \(\alpha \in \mathbb{N}^n\), \(z^\alpha\) will represent the monomial

\[
z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.
\]

If \(0 < r < 1\) and \(z \in U^n\), we use the notation

\[
rz = (rz_1, rz_2, \ldots, rz_n)
\]

so that, for example,

\[
(rz)^\alpha = r^{\|\alpha\|} z^\alpha, \quad \alpha \in \mathbb{N}_+.
\]
A complex-valued function $f$ defined on $U^n$ is said to be analytic if $f$ is analytic in each variable separately. A function which is analytic on $U^n$ is also continuous on $U^n$. The polydisc algebra $A(U^n)$ is the set of all functions which are continuous on $\overline{U^n} = \overline{U^n}$, the closure of $U^n$ in $\mathbb{C}^n$, and analytic on $U^n$. With all operations defined pointwise, $A(U^n)$ is a Banach algebra under the sup norm

$$\|f\|_{A(U^n)} = \sup_{\|z\|\leq 1} |f(z)|.$$ 

If $f$ is an analytic function on $U^n$ and $0 < r < 1$, then the function $f_r$ is in $A(U^n)$ where $f_r$ is defined by

$$f_r(z) = f(rz), \quad z \in \overline{U^n}.$$ 

If $f$ is analytic on $U^n$ and $0 < r < 1$, we write

$$M_p(f;r) = \left( \int_{\mathbb{T}^n} |f(rw)|^p dm_n(w) \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$M_\infty(f;r) = \sup_{w \in \mathbb{T}^n} |f(rw)|.$$ 

For $1 \leq p \leq \infty$ it follows that $M_p(f;r)$ is a monotonically increasing function of $r$. The set of functions $f$ which are analytic on $U^n$ and for which $\lim_{r \to 1} M_p(f;r) < \infty$ is known as $H^p(U^n)$. A norm is de-
defined in $H^p(U^n)$ by

$$\|f\|_{H^p(U^n)} = \lim_{r \to 1} M_p(f;r),$$

so that $\|f\|_{H^p(U^n)} \geq M_p(f;r)$ for all $r \in [0,1)$. For $1 \leq p \leq \infty$, $H^p(U^n)$ is a Banach space, and in addition $H^2(U^n)$ is a Hilbert space and $H^\infty(U^n)$ is a Banach algebra. The following inclusions hold:

$$A(U^n) \subset H^\infty(U^n) \subset H^p(U^n) \subset H^q(U^n) \subset H^1(U^n)$$

where $1 < q < p < \infty$. Also, if $f \in H^p(U^n) \subset H^q(U^n)$, $1 \leq q \leq p < \infty$, then

$$\|f\|_{H^q(U^n)} \leq \|f\|_{H^p(U^n)}.$$  

A function $f$ is in $H^\infty(U^n)$ if and only if $f$ is in $H^p(U^n)$ for all $1 \leq p < \infty$, and in this case

$$\|f\|_{H^\infty(U^n)} = \lim_{p \to \infty} \|f\|_{H^p(U^n)}.$$  

The space $H^p(U^n)$ may be isometrically embedded in the space $L^p(T^n)$ by the mapping $f \to f^*$, where

$$f^*(w) = \lim_{r \to 1} f(rw)$$

is defined almost everywhere on $T^n$. The function $f^*$ is called the boundary value function of $f$, and the subspace of $L^p(T^n)$ which is identified with $H^p(U^n)$
will be denoted \( L^p_+(T^n) \). The Fourier coefficients of
a function \( f \in L^p_+(T^n) \) are given by
\[
c_\alpha = \int_{T^n} f(w) w^{-\alpha} dm_n(w), \quad \alpha \in I^n.
\]
Functions in \( L^p_+(T^n) \) are characterized by the property
that \( c_\alpha = 0 \) if \( \alpha \notin I^n_+ \), see [6], p. 18.

The proof of the following theorem is due to Dennis
Clayton. It is an extension of a theorem of Bourque [1].
If \( i+j = n \) we write
\[(\lambda, z) = (\lambda_1, \cdots, \lambda_i, z_1, \cdots, z_j)\]
for an element in \( U^i \times U^j = U^n \). The notation
\[D^\beta F(\lambda, z)\] represents
\[
\frac{\partial |\beta|}{\partial z_1^\beta_1 \partial z_2^\beta_2 \cdots \partial z_j^\beta_j} F(\lambda, z),
\]
where \( \beta \in I^j_+ \).

**Theorem 1.1.** Suppose \( i+j = n \) and \( 1 \leq p \leq \infty \).
If \( F \in H^p(U^n) \) with \( (\lambda, z) \in U^i \times U^j = U^n \), then
\[D^\beta F(\cdot, z) \in H^p(U^i)\]
for each \( z \in U^j \) and \( \beta \in I^j_+ \). Furthermore,
\[
\|D^\beta F(\cdot, z)\|_{H^p(U^i)} \leq \frac{\beta!}{(1-\|z\|)^{\beta+1}} \|F\|_{H^p(U^n)}.
\]
Proof. For $0 < r < 1$ the function $F_r$ defined by

$$F_r(\lambda, z) = F(r\lambda, rz), \quad z \in \overline{U}^j,$$

is in $A(U^j)$ for each $\lambda \in \overline{U}^i$. Applying the Cauchy integral formula we obtain

$$D_2^β F(r\lambda, rz) = \frac{β!}{(2\pi i)^{1-j}} \int_{T^j} \frac{F(r\lambda, r\xi)}{(r\xi - z)^{β+1}} \, d\xi,$$

for each $\lambda \in \overline{U}^i, \ z \in U^j$. Then

$$|D_2^β F(r\lambda, rz)| \leq \frac{β!}{(1-||z||)^{β+1}} \int_{T^j} |F(r\lambda, r\xi)| \, dm_j(\xi).$$

If $1 < p < \infty$ we apply Holder's inequality,

$$|D_2^β F(r\lambda, rz)| \leq \frac{β!}{(1-||z||)^{β+1}} \left[ \int_{T^j} |F(r\lambda, r\xi)|^p \, dm_j(\xi) \right]^{1/p},$$

for all $\lambda \in \overline{U}^i, \ z \in U^j$. Integrating over $T^i$ we see that

$$\left[ \int_{T^i} |D_2^β F(r\lambda, rz)|^p \, dm_1(\eta) \right]^{1/p} \leq \frac{β!}{(1-||z||)^{β+1}} \left[ \int_{T^i} |F(r\lambda, r\xi)|^p \, dm_1(\eta, \xi) \right]^{1/p}.$$

That is,

$$M_p(D_2^β F(\cdot, z); r) \leq \frac{β!}{(1-||z||)^{β+1}} \, M_p(F; r).$$
Therefore, $D_2^βF(\cdot, z)$ is in $H^p(U^i)$ with

$$\|D_2^βF(\cdot, z)\|_{H^p(U^i)} = \lim_{r \to 1} M_p(D_2^βF(\cdot, z); r)$$

$$\leq \frac{β!}{(1-\|z\|)^{β+1}} \|F\|_{H^p(U^n)}.$$ 

If $F \in H^∞(U^n)$, then $F \in H^p(U^n)$ for each $p$, $1 \leq p < ∞$, and by the above we have $D_2^βF(\cdot, z)$ in $H^p(U^i)$ for each $p$, $1 \leq p < ∞$, with

$$\|D_2^βF(\cdot, z)\|_{H^p(U^i)} \leq \frac{β!}{(1-\|z\|)^{β+1}} \|F\|_{H^∞(U^n)}.$$ 

It follows that $D_2^βF(\cdot, z)$ is in $H^∞(U^i)$ and

$$\|D_2^βF(\cdot, z)\|_{H^∞(U^i)} \leq \frac{β!}{(1-\|z\|)^{β+1}} \|F\|_{H^∞(U^n)}.$$ 

**Corollary 1.2.** Suppose $i + j = n$, $1 \leq p \leq ∞$, and $F \in H^p(U^n)$. If

$$F(λ, z) = \sum_{α \subseteq I^j_+} f_α(λ)z^α$$

where $(λ, z) ∈ U^i × U^j = U^n$, then $f_β ∈ H^p(U^i)$ for each $β ∈ I^j_+$, and
\[ \|f_\beta\|_{H^p(U^1)} \leq \|F\|_{H^p(U^n)}. \]

**Proof.** If \( z = 0 \in U^j \), then

\[ D_2^\beta F(\cdot,0) = \beta! f_\beta, \beta \in I^j_+. \]

It follows from Theorem 1.1 that \( f_\beta \in H^p(U^1) \) with

\[ \|f_\beta\|_{H^p(U^1)} \leq \frac{1}{\beta!} \frac{\beta!}{(1-\|0\|^{\beta+1})} \|F\|_{H^p(U^n)} \]

\[ = \|F\|_{H^p(U^n)}. \]

Next we present two forms of a test which can be used to decide if a function in \( H^2(U^n) \) is in \( H^\infty(U^n) \).

**Theorem 1.3.** A function \( f \) is in \( H^\infty(U^n) \) if and only if \( f^q \) is in \( H^2(U^n) \) with

\[ \|f^q\|_{H^2(U^n)} \leq K^q \]

for all \( q \in I_+ \), where \( K \) is a constant that is independent of \( q \). In this case,

\[ \|f\|_{H^\infty(U^n)} \leq K. \]

**Proof.** Suppose \( f \in H^\infty(U^n) \) and \( q \in I_+ \). Then

\[ f^q \in H^\infty(U^n) \subseteq H^2(U^n) \]
and
\[ \|f^q\|_{H^2(U^n)}^2 \leq \|f^q\|_{H^\infty(U^n)}^2 \leq \|f\|_{H^\infty(U^n)}^2. \]

Conversely, suppose \( f^q \) is in \( H^2(U^n) \) with
\[ \|f^q\|_{H^2(U^n)} \leq k^q \]
for each \( q \in I_+ \). If \( 0 < r < 1 \) and \( q \in I_+ \) we have
\[ \int_{T^n} |f(rw)|^{2q} \text{dm}_n(w) = \int_{T^n} |f^q(rw)|^{2q} \text{dm}_n(w) \]
\[ = M_2^2(f^q; r) \leq \|f^q\|_{H^2(U^n)}^2 \leq k^{2q}. \]

For \( \epsilon > 0 \), set
\[ N_\epsilon = \{ w \in T^n \mid |f(rw)| \geq k+\epsilon \}. \]

The continuity of \( f_r \) on \( T^n \) implies that \( N_\epsilon \) is open in \( T^n \). Then
\[ k^{2q} \geq \int_{T^n} |f(rw)|^{2q} \text{dm}_n(w) \]
\[ \geq \int_{N_\epsilon} |f(rw)|^{2q} \text{dm}_n(w) \]
\[ \geq (k+\epsilon)^{2q} \text{m}_n(N_\epsilon) \]
for every \( q \in I_+ \). This is a contradiction unless \( m_n(N_\epsilon) = 0 \) and since \( N_\epsilon \) is an open set we must have \( N_\epsilon = \emptyset \). Since \( \epsilon > 0 \) was arbitrary we have
\[ |f(rw)| \leq K \]

for all \( w \in T^n \). If \( z \in U^n \) choose \( r = \|z\| \) and by the maximum modulus theorem

\[ |f(z)| \leq \sup_{w \in T^n} |f(rw)| \leq K. \]

Since \( f \in \mathcal{H}^2(U^n) \) and \( |f(z)| \leq K \) for all \( z \in U^n \), we have \( f \in \mathcal{H}^\infty(U^n) \) with

\[ \|f\|_{\mathcal{H}^\infty(U^n)} \leq K. \]

**Corollary 1.4.** Suppose \( G \) is a subset of \( \mathcal{H}^2(U^n) \) which is dense in the \( \mathcal{H}^2(U^n) \) norm. A function \( f \) is in \( \mathcal{H}^\infty(U^n) \) if and only if \( f \cdot g \in \mathcal{H}^2(U^n) \) with

\[ \|f \cdot g\|_{\mathcal{H}^2(U^n)} \leq K \|g\|_{\mathcal{H}^2(U^n)} \]

for all \( g \in G \), where \( K \) is a constant that is independent of \( g \). In this case

\[ \|f\|_{\mathcal{H}^\infty(U^n)} \leq K. \]

**Proof.** Suppose \( f \in \mathcal{H}^\infty(U^n) \). Then \( f \cdot g \in \mathcal{H}^2(U^n) \) for all \( g \in \mathcal{H}^2(U^n) \) and

\[ \|f \cdot g\|_{\mathcal{H}^2(U^n)} \leq \|f\|_{\mathcal{H}^\infty(U^n)} \|g\|_{\mathcal{H}^2(U^n)}. \]
Conversely, suppose \( f \cdot g \in H^2(U^n) \) with

\[
\|f \cdot g\|_{H^2(U^n)} \leq k \|g\|_{H^2(U^n)}
\]

for all \( g \) in a dense subset \( G \) of \( H^2(U^n) \). If \( h \in H^2(U^n) \) and \( \{g_k\}_{k=1}^\infty \) is a sequence of functions in \( G \) that converges to \( h \) in the \( H^2(U^n) \) norm, then for \( 0 \leq r < 1 \), the sequence \( \{g_k(rw)\}_{k=1}^\infty \) converges to \( h(rw) \) for almost all \( w \) in \( T^n \). Therefore,

\[
M^2_2(fh;r) = \int_{T^n} |f(rw)h(rw)|^2 \, dm_n(w)
\]

\[
= \int_{T^n} \lim_{k \to \infty} |f(rw)g_k(rw)|^2 \, dm_n(w) .
\]

By Fatou's lemma

\[
\int_{T^n} \lim_{k \to \infty} |f(rw)g_k(rw)|^2 \, dm_n(w)
\]

\[
\leq \lim_{k \to \infty} \int_{T^n} |f(rw)g_k(rw)|^2 \, dm_n(w) ,
\]

so that

\[
M^2_2(fh;r) \leq \lim_{k \to \infty} M^2_2(fg_k;r) \leq \lim_{k \to \infty} \|fg_k\|_{H^2(U^n)}^2 .
\]

Since \( g_k \in G \) we have

\[
\|fg_k\|_{H^2(U^n)} \leq K \|g_k\|_{H^2(U^n)}
\]
and hence

\[ M_2^2(fh; r) \leq \lim_{k \to \infty} K^2 \| g_k \|_{L^2(U^n)}^2 = K^2 \| h \|_{L^2(U^n)}^2. \]

Therefore \( f \cdot h \in H^2(U^n) \) with

\[ \| fh \|_{H^2(U^n)} \leq K \| h \|_{H^2(U^n)} \]

for every \( h \in H^2(U^n) \) and we may assume that \( G = H^2(U^n) \).

It follows that \( f = f \cdot 1 \in H^2(U^n) \) with

\[ \| f \|_{H^2(U^n)} \leq K \| 1 \|_{H^2(U^n)} = K. \]

And \( f \cdot f \in H^2(U^n) \) with

\[ \| f^2 \|_{H^2(U^n)} \leq K \| f \|_{H^2(U^n)} \leq K^2. \]

By induction \( f^q \in H^2(U^n) \) with

\[ \| f^q \|_{H^2(U^n)} \leq K^q \]

for all \( q \in I_+ \). According to Theorem 1.3 we have \( f \in H^\infty(U^n) \) with

\[ \| f \|_{H^\infty(U^n)} \leq K. \]

We conclude this chapter with an approximation theorem.
Theorem 1.5. Suppose \( h \) is a non-negative function in \( L^1(T^n) \). Then there is a sequence \( \{p_k\}_{k=1}^\infty \) of analytic trigonometric polynomials on \( T^n \) such that:

i) \( \lim_{k \to \infty} \int_{T^n} |h(w) - |p_k(w)|^2|dm_n(w) = 0 \);

ii) \( \lim_{k \to \infty} \int_{T^n}|p_k(w)|^2 dm_n(w) = \int_{T^n} h(w) dm_n(w) \);

and

iii) \( \lim_{k \to \infty} |p_k(w)|^2 = h(w) \) for almost all \( w \) in \( T^n \).

Proof. Condition ii) follows from i) by the triangle inequality. Also condition iii) follows from i) since convergence in the \( L^1(T^n) \) norm implies pointwise convergence a.e. Therefore we need only prove that there exists a sequence of analytic trigonometric polynomials such that i) holds.

The set of continuous functions is dense in \( L^1(T^n) \) so for each \( k=1,2,\cdots \), there is a continuous function \( g_k \) on \( T^n \) such that

\[
\int_{T^n} |h(w) - g_k(w)| dm_n(w) \leq \frac{1}{k} .
\]

If \( f_k(w) = \max (g_k(w),0) \), then \( f_k \) is a non-negative continuous function and
\[
\int_{T^n} |h(w) - f_k(w)| \, dm_n(w)
\]
\[
\leq \int_{T^n} |h(w) - g_k(w)| \, dm_n(w)
\]
\[
\leq \frac{1}{k}.
\]

By the Stone-Weierstrass theorem the set of trigonometric polynomials is dense in the set of continuous functions, hence there exist trigonometric polynomials \( q_k \) such that

\[
\|q_k - (f_k)^{1/2}\|_\infty \leq \frac{1}{k} (2\|(f_k)^{1/2}\|_\infty + 1)^{-1}
\]

for each \( k = 1, 2, \cdots \). Then

\[
|q_k(w)|^2 - f_k(w)| \leq |q_k^2(w) - f_k(w)|
\]

\[
= |q_k(w) - (f_k)^{1/2}(w)| \cdot |q_k(w) + (f_k)^{1/2}(w)|
\]

for each \( w \) in \( T^n \). Since

\[
|q_k(w) + (f_k)^{1/2}(w)|
\]

\[
\leq |q_k(w) - (f_k)^{1/2}(w)| + 2(f_k)^{1/2}(w)
\]

\[
\leq 1 + 2\|(f_k)^{1/2}\|_\infty,
\]

we have
\[
|q_k(w)|^2 - f_k(w) - \frac{1}{k} (1 + 2\|f_k\|_\infty)^{-1} \left(1 + 2\|f_k\|_\infty\right) = \frac{1}{k}.
\]

Then
\[
\int_{T^n} |h(w) - |q_k(w)|^2| dm_n(w) \leq \int_{T^n} |h(w) - f_k(w)| dm_n(w) + \int_{T^n} |f_k(w) - |q_k(w)|^2| dm_n(w) \leq \frac{1}{k} + \frac{1}{k},
\]
for each \(k=1,2,\cdots\). Suppose
\[
q_k(w) = \sum_{\alpha \in J} a_\alpha w^\alpha
\]
where \(J\) is a finite set in \(I^n\). There is a \(\beta \in I^n_+\) such that \(\alpha + \beta \geq 0\) for all \(\alpha \in J\). Set
\[
p_k(w) = w^\beta q_k(w)
\]
so that \(p_k\) is an analytic trigonometric polynomial on \(T^n\) and \(|p_k(w)| = |q_k(w)|\) for all \(w\) in \(T^n\). Then
\[
\int_{\mathcal{T}_n} |h(w) - |p_k(w)|^2| \, dm_n(w)
\]

\[
= \int_{\mathcal{T}_n} |h(w) - |q_k(w)|^2| \, dm_n(w)
\]

\[
\leq \frac{1}{k}.
\]
CHAPTER II  
THE ANALYTIC DIRECT INTEGRAL OF $H^2(U^n)$

Since $H^2(U^n)$ is a separable Hilbert space we could try using the usual direct integral technique (cf. [3], p. 350) to formulate an acceptable theory for $H^2(U^n)$. These techniques always lead to some type of theory, however, such a theory has little to do with analytic function theory when $n > 1$. In order to relate analytic function theory with direct integral theory, we impose an extra condition of analyticity and define the analytic direct integral. This space is shown to be a separable Hilbert space which is isometrically isomorphic to $H^2(U^{n+1})$ in a natural and useful way.

Definition 2.1. The analytic direct integral of $H^2(U^n)$ with respect to normalized Lebesgue measure, written $\int_T H^2(U^n) dm(w)$, is the set of all vector-valued functions $B = \{B^w\}$ defined for almost all $w$ in $T$ such that:

(i) $B^w$ is in $H^2(U^n)$ for almost all $w$ in $T$;
ii) if $B^w(z) = \sum_{\alpha} b_\alpha(w) z^\alpha$ is the series expansion for $B^w$ whenever $B^w \in H^2(U^n)$, then $b_\beta \in L^2_T$ for each $\beta \in I^n_+$;

iii) $\int_T \|B^w\|_{H^2(U^n)}^2 \, dm(w) < \infty$, where $\int_T$ represents the upper Lebesgue integral.

Note. Each function $b_\alpha \in L^2_T$ is measurable, and since

$$\|B^w\|_{H^2(U^n)}^2 = \sum_{\alpha} |b_\alpha(w)|^2 < \infty,$$

it follows that $\|B^w\|_{H^2(U^n)}^2$ is a measurable function of $w$. We may thus replace the upper Lebesgue integral $\int_T$ in iii) with the ordinary Lebesgue integral $\int_T$.

Two functions $B$ and $D$ in $\int_T H^2(U^n) \, dm(w)$ will be considered equal if $B^w = D^w$ for almost all $w$ in $T$. That is, $B = D$ if and only if

$$\int_T \|B^w - D^w\|_{H^2(U^n)}^2 \, dm(w) = 0.$$ 

We also define vector addition and scalar multiplication coordinate-wise,

$$B + D = \{B^w + D^w\},$$

and

$$\alpha B = \{\alpha B^w\}, \quad \alpha \in \mathbb{C}.$$
By the polarization identity

\[ 4(B^w, D^w)_{H^2(U^n)} = \|B^w + D^w\|_{H^2(U^n)}^2 - \|B^w - D^w\|_{H^2(U^n)}^2 + i\|B^w + iD^w\|_{H^2(U^n)}^2 - i\|B^w - iD^w\|_{H^2(U^n)}^2, \]

the measurability of \((B^w, D^w)\) as a function of \(w\) follows from the measurability of \(\|B^w\|_{H^2(U^n)}^2\). We define an inner product in \(\int_T H^2(U^n)dm(w)\) by

\[ (B, D)_2 = \int_T (B^w, D^w)dm(w) \]

and it can be verified that the analytic direct integral of \(H^2(U^n)\) is an inner product space. As usual we define a norm by

\[ \|B\|_2 = (B, B)^{1/2}_2 \]

or,

\[ \|B\|_2^2 = \int_T \|B^w\|_{H^2(U^n)}^2 dm(w). \]

**Theorem 2.2.** The space \(\int_T H^2(U^n)dm(w)\) is complete in the norm \(\|\cdot\|_2\), and hence is a Hilbert space.

**Proof.** Suppose \(\{B_k\}_{k=1}^\infty\) is a Cauchy sequence in \(\int_T H^2(U^n)dm(w)\). There is a subsequence \(\{B_{k_1}\}_{i=1}^\infty\), \(k_1 \leq k_2 \leq \cdots\), such that
Using Holder's inequality we have

\[
\int_{T} \|B_{k_{i+1}}^{w} - B_{k_{i}}^{w}\|_{H^{2}(U^{n})} \, dm(w)
\leq \left( \int_{T} \|B_{k_{i+1}}^{w} - B_{k_{i}}^{w}\|_{H^{2}(U^{n})}^{2} \, dm(w) \right)^{1/2}
\]

\[
= \|B_{k_{i+1}}^{w} - B_{k_{i}}^{w}\|_{2} \leq \frac{1}{2^{i}}.
\]

Then

\[
\int_{T} \sum_{i=1}^{\infty} \|B_{k_{i+1}}^{w} - B_{k_{i}}^{w}\|_{H^{2}(U^{n})} \, dm(w)
\leq \sum_{i=1}^{\infty} \frac{1}{2^{i}} < 2,
\]

and it follows that for almost all \( w \)

\[
\sum_{i=1}^{\infty} \|B_{k_{i+1}}^{w} - B_{k_{i}}^{w}\|_{H^{2}(U^{n})} < \infty.
\]

The sequence \( \{B_{k_{i}}^{w}\}_{i=1}^{\infty} \) is a Cauchy sequence in the complete space \( H^{2}(U^{n}) \) for almost all \( w \). If we set

\[
B_{w}^{\infty} = \lim_{i \to \infty} B_{k_{i}}^{w},
\]

where the limit is taken in the \( H^{2}(U^{n}) \) norm, then
$B^w \subset H^2(U^n)$ for almost all $w$. Since

$$\|B^w\|_{H^2(U^n)} - \|B^w_{k_1}\|_{H^2(U^n)} \leq \|B^w - B^w_{k_1}\|_{H^2(U^n)},$$

we see that

$$\lim_{i \to \infty} \|B^w\|_{H^2(U^n)} = \|B^w\|_{H^2(U^n)}$$

almost everywhere. We may write

$$\|B^w\|_{H^2(U^n)} - \|B^w_{k_1}\|_{H^2(U^n)} = \sum_{i=1}^{j} \left( \|B^w_{k_i+1}\|_{H^2(U^n)} - \|B^w_{k_i}\|_{H^2(U^n)} \right) \leq \sum_{i=1}^{j} \|B^w_{k_i+1} - B^w_{k_i}\|_{H^2(U^n)} \leq 2,$$

and therefore,

$$\|B^w\|_{H^2(U^n)} = \lim_{j \to \infty} \|B^w_{k_j+1}\|_{H^2(U^n)} \leq 2 + \|B^w_{k_1}\|_{H^2(U^n)}.$$

Consequently,

$$\left\{ \int_{T} \|B^w\|_{H^2(U^n)}^2 \, dm(w) \right\}^{1/2} \leq \left\{ \int_{T} 4 \, dm(w) \right\}^{1/2} + \left\{ \int_{T} \|B^w_{k_1}\|_{H^2(U^n)}^2 \, dm(w) \right\}^{1/2} = 2 + \|B^w_{k_1}\|_2 < \infty.$$
If the series expansions for $B_{k_1}^w$, $i=1,2,\ldots$, and $B^w$ are given by

$$B_{k_1}^w(z) = \sum_{i=1}^{k_1} b_i^\omega(z)z^\alpha$$

and

$$B^w(z) = \sum_{i=1}^\omega b_i^\omega(z)z^\alpha,$$

then for each $\beta \in I_+^n$, $b_{k_1}^\beta \in L^2(T)$, and $k_i$, $b_{k_i}^\beta(b_{k_i}^\beta - b_{k_i}^\beta) - \int_T^n([B^w]^*(z) - [B_{k_1}^w]^*(z))z^{-\beta}dm_n(z)$.

Then

$$|b_{k_1}^\beta(w) - b_{k_i}^\beta(w)| \leq \int_T^n|[B^w]^*(z) - [B_{k_1}^w]^*(z)|dm_n(z)$$

$$\leq \left(\int_T^n|[B^w]^*(z) - [B_{k_1}^w]^*(z)|^2dm_n(z)\right)^{1/2}$$

$$= \|B^w - B_{k_1}^w\|_{H^2(U^n)},$$

and taking limits

$$\lim_{i \to \infty} |b_{k_i}^\beta(w) - b_{k_i}^\beta(w)|$$

$$\leq \lim_{i \to \infty} \|B^w - B_{k_1}^w\|_{H^2(U^n)} = 0 \text{ for almost all } w.$$
Consequently, each $b_\beta$ is a measurable function of $w$ and

$$|b_\beta(w)|^2 \leq \sum_{I_+} |b_\alpha(w)|^2$$

$$= \|B^w\|^2_{H^2(U^n)} ,$$

so that

$$\left\{ \int_T |b_\beta(w)|^2 dm(w) \right\}^{1/2} \leq 2 + \|B_{k_1}\|_2 .$$

For $\beta \in I_+^n$, $b_\beta \in L_+^2(T)$ and we have shown that $B \in \int_T H^2(U^n) dm(w)$, where $B$ is defined by

$$B = \{B^w\} .$$

In order to finish our proof we need to show that

$$B = \lim_{k \to \infty} B_k ,$$

in the $\| \cdot \|_2$ norm.

Suppose $\epsilon > 0$. Choose $N$ so that

$$\|B_k - B_p\|_2 < \epsilon$$

for $k, p > N$. If $p > N$ then

$$\|B - B_p\|_2^2 = \int_T \|B^w - B_p^w\|^2_{H^2(U^n)} dm(w)$$

$$= \int_T \lim_{k_1 \to \infty} \|B_{k_1}^w - B_p^w\|^2_{H^2(U^n)} dm(w)$$

$$\leq \lim_{k_1 \to \infty} \int_T \|B_{k_1}^w - B_p^w\|^2_{H^2(U^n)} dm(w)$$

$$= \lim_{k_1 \to \infty} \|B_{k_1} - B_p\|_2^2 < \epsilon^2 .$$
The space $H^2(U^{n+1})$ can be expressed as the direct sum of a countable number of copies of $H^2(U^n)$,

$$H^2(U^{n+1}) = H^2(U^n) \oplus H^2(U^n) \oplus \cdots.$$  

Each element $F \in H^2(U^{n+1})$ has an expansion of the form

$$F(\lambda, z) = \sum_{j=0}^{\infty} f_j(z) \lambda^j,$$

where $\lambda \in U$, $z \in U^n$, and $f_k \in H^2(U^n)$, $k=0,1,2,\cdots$, with

$$\sum_{j=0}^{\infty} \|f_j\|^2_{H^2(U^n)} = \|F\|^2_{H^2(U^{n+1})} < \infty.$$

**Theorem 2.3.** The analytic direct integral of $H^2(U^n)$ is isometrically isomorphic to the direct sum of a countable number of copies of $H^2(U^n)$ and therefore is isometrically isomorphic to $H^2(U^{n+1})$.

**Proof.** For each $j=0,1,2,\cdots$, define a map $\sigma_j$ from $H^2(U^n)$ into $\int_T H^2(U^n) dm(w)$ by

$$\sigma_j(f) = w^j f.$$

Then $\sigma_j$ is an isomorphism and

$$\|\sigma_j(f)\|_2 = \{ \int_T \|w^j f\|^2_{H^2(U^n)} dm(w) \}^{1/2} = \|f\|_{H^2(U^n)}.$$

Consequently, for $j \in I_+$, the set
\[ S_j = \sigma_j(H^2(U^n)) \]

is an isomorphic isometric copy of \( H^2(U^n) \) in \( \int_T H^2(U^n) \, dm(w) \). Suppose \( j \neq k \), then

\[
(\sigma_j(f), \sigma_k(g)) = \int_T (w^j f, w^k g) \, dm(w) = (f, g) \int_T w^{j-k} \, dm(w) = 0
\]
for all \( f, g \in H^2(U^n) \). Hence \( S_j \perp S_k \) for \( j \neq k \).

If \( B \in \int_T H^2(U^n) \, dm(w) \) and \( B \perp S_j \) for each \( j \in I_+ \), then

\[
C = (B, \sigma_j(f)) = \int_T (B^w, w^j f) \, dm(w)
\]
for each \( f \in H^2(U^n) \). In particular, for \( e_\alpha(z) = z^\alpha \), \( \alpha \in I_+^n \), we have

\[
0 = \int_T w^{-j} (B^w, e_\alpha) \, dm(w)
= \int_T w^{-j} b_\alpha(w) \, dm(w)
= b_j,\alpha \quad \text{where}
\]

\[
b_\alpha(w) = \sum_{k=0}^{\infty} b_{k,\alpha} w^k \quad \text{is the series expansion for } b_\alpha.
\]

Since \( b_\alpha \in L^2_+(T) \) we know that \( b_{j,\alpha} = 0 \) for \( j < 0 \) and from the above we have \( b_{j,\alpha} = 0 \) for \( j=0,1,2,\cdots \). Consequently \( b_\alpha(w) = 0 \) for each \( \alpha \in I_+^n \) and \( B^w = 0 \) for almost all \( w \). The function \( B = 0 \) is the only element of \( \int_T H^2(U^n) \, dm(w) \) which is orthogonal
to every $S_j$. The direct sum $\bigoplus_{j=0}^{\infty} S_j$ of this countable collection of mutually orthogonal subspaces is therefore dense in $\bigoplus_j T^{-1} H^2(U^n) dm(w)$. Since $\bigoplus_{j=0}^{\infty} S_j$ is closed and $\bigoplus_j T^{-1} H^2(U^n) dm(w)$ is complete we have

$$\bigoplus_j T^{-1} H^2(U^n) dm(w) = \bigoplus_{j=0}^{\infty} S_j.$$ 

We shall define a mapping from $\bigoplus_j T^{-1} H^2(U^n) dm(w)$ into $H^2(U^{n+1})$ and show that it is an isometric isomorphism. Suppose $B \in \bigoplus_j T^{-1} H^2(U^n) dm(w)$, $B = \{B^w\}$ and whenever $B^w \in H^2(U^n)$, the series expansion of $B^w$ is given by

$$B^w(z) = \sum_{\alpha \in I^*_+} b_{\alpha}(w) z^\alpha.$$

Then for each $\beta \in I^*_+$, $b_{\beta} \in L^2(T)$ so there is a function $f^*_\beta \in H^2(U)$ such that $f^*_\beta = b_{\beta}$. Each $f^*_\beta$ has a series expansion

$$f^*_\beta(\lambda) = \sum_{j=0}^{\infty} b_{j,\beta} \lambda^j, \lambda \in U.$$

For each $\gamma \in I^*_{n+1}$ we define

$$a_{\gamma} = b_{\gamma_1,\ldots,\gamma_{n+1}}.$$

Then

$$\sum_{\gamma \in I^*_{n+1}} |a_{\gamma}|^2 = \sum_{\alpha \in I^n_+} \sum_{j=0}^{\infty} |b_{j,\alpha}|^2.$$
\[ = \sum_{I+} \| f_{a} \|_{H^2(U)}^2 \]
\[ = \sum_{I+} \| b_{a} \|_{L^2(T)}^2 \cdot \]

Also,
\[ \| B \|_2^2 = \int_{T} \| B^w \|_{H^2(U^n)}^2 \, dm(w) \]
\[ = \int_{T} \sum_{I+} | b_{a}(w) |^2 \, dm(w) \]
\[ = \sum_{I+} \int_{T} | b_{a}(w) |^2 \, dm(w) \]
\[ = \sum_{I+} \| b_{a} \|_{L^2(T)}^2 \cdot \]

Consequently,
\[ \sum_{\gamma \in I^{n+1}+} | a_{\gamma} |^2 = \| B \|_2^2 < \infty . \]

There is a function \( F \in H^2(U^{n+1}) \) such that
\[ F(z) = \sum_{i^{n+1}+} a_{\gamma} z^\gamma , \quad z \in U^{n+1} . \]

Define the map \( V \) by
\[ VB = F . \]

\( V \) is a linear mapping with
\[ \| F \|_{H^2(U^{n+1})}^2 = \sum_{I^{n+1}+} | a_{\gamma} |^2 = \| B \|_2^2 , \]
so that $V$ is an isometry. To show that $V$ maps 
\[ \int_T H^2(U^n)dm(w) \] onto $H^2(U^{n+1})$ we reverse the above construction. Suppose $F \in H^2(U^{n+1})$ with 
\[ F(z) = \sum_{\alpha \in I_+^n} a_\alpha z^\alpha, \quad z \in U^{n+1}. \]

Define $C_{k,\beta} = a(k, \beta_1, \ldots, \beta_n)$ where $k \in I_+$ and $\beta \in I_+^n$. Then for each $\beta \in I_+^n$,
\[ \sum_{j=0}^{\infty} |C_{j,\beta}|^2 \leq \sum_{\alpha \in I_+^n} \sum_{j=0}^{\infty} |C_{j,\alpha}|^2 = \sum_{\alpha \in I_+^n} |a_\alpha|^2 = ||F||^2_{H^2(U^{n+1})} < \infty. \]

There is a function $f_\beta \in H^2(U)$ such that 
\[ f_\beta(\lambda) = \sum_{j=0}^{\infty} C_{j,\beta} \lambda^j. \]

Set $b_\beta = f_\beta^*$ so that $b_\beta \in L^2(T)$. Finally, define $B^w$ for almost all $w$ by 
\[ B^w(z) = \sum_{\alpha \in I_+^n} b_\alpha(w)z^\alpha. \]
Since

\[ \int_T \sum_{I^+} |b_\alpha(w)|^2 dm(w) \]

\[ = \sum_{I^+} \int_T |b_\alpha(w)|^2 dm(w) \]

\[ = \sum_{I^+} \|b_\alpha\|_{L^2(T)}^2 \]

\[ = \sum_{\alpha \in I^+} \sum_{j=0}^{\infty} |b_j, \alpha|^2 \]

\[ = \sum_{I^{n+1}} |a_\gamma|^2 < \infty, \]

we have

\[ \sum_{I^+} |b_\alpha(w)|^2 < \infty \]

for almost all \( w \). Consequently, \( B^w \in H^2(U^n) \) for almost all \( w \) and \( B \in \int_T H^2(U^n) dm(w) \) where \( B = \{B^w\} \).

Clearly \( \|B\|_2 = \|F\|_{H^2(U^{n+1})} \) and \( VB = F \).
CHAPTER III

THE ANALYTIC DIRECT INTEGRAL OF $H^\infty(U^n)$

The properties of the analytic direct integral of $H^2(U^n)$ depended heavily on the Hilbert space structure of $H^2(U^n)$. On the other hand, the proofs and techniques used in this chapter have been strongly influenced by the representation of the Banach algebra $H^\infty(U^n)$ as a maximal commutative ring of operators on $H^2(U^n)$. We begin with the definition of the analytic direct integral of $H^\infty(U^n)$.

Definition 3.1. The analytic direct integral of $H^\infty(U^n)$ with respect to normalized Lebesgue measure, written $\int_T H^\infty(U^n)dm(w)$, is the set of all vector-valued functions $B = \{B^w\}$ defined for almost all $w$ in $T$ such that:

1) $B^w$ is in $H^\infty(U^n)$ for almost all $w$ in $T$;

2) if $B^w(z) = \sum_{I_+} b^w_\alpha(w) z^\alpha$ is the series expansion for $B^w$ whenever $B^w \in H^\infty(U^n)$, then for each $\beta \in I_+^n$, $b^w_\beta \in L^\infty(T)$;
and

\[ \text{iii) ess sup}_{w \in T} \|B^w\|_{H^\infty(U^n)} < \infty. \]

We define a norm in \( \int_T H^\infty(U^n) \, dm(w) \) by

\[ \|B\|_\infty = \text{ess sup}_{w \in T} \|B^w\|_{H^\infty(U^n)}. \]

Suppose \( B \in \int_T H^\infty(U^n) \, dm(w) \), so that for almost all \( w \)

\[ B^w \in H^\infty(U^n) \subset H^2(U^n). \]

If the series expansion for \( B^w \) is given by

\[ B^w(z) = \sum_{\alpha} b_\alpha(w) \, z^\alpha, \quad z \in U^n, \]

then for each \( \beta \in I^n_+ \)

\[ b_\beta \in L^\infty_+(T) \subset L^2_+(T). \]

Also,

\[ \|B^w\|_{H^2(U^n)} \leq \|B^w\|_{H^\infty(U^n)} \leq \|B\|_\infty \]

for almost all \( w \) in \( T \). Consequently,

\[ \int_T \|B^w\|_{H^2(U^n)}^2 \, dm(w) \leq \int_T \|B\|_{H^\infty(U^n)}^2 \, dm(w) = \|B\|_\infty^2 < \infty, \]

and we see that \( B \in \int_T H^2(U^n) \, dm(w) \) with

\[ \|B\|_2 \leq \|B\|_\infty. \]
Since $\int T H^{\infty}(U^n) dm(w)$ is a subset of $\int T H^{2}(U^n) dm(w)$ we use the same definitions of scalar multiplication and vector addition but we also define an operation of vector multiplication. If $B, D \in \int T H^{\infty}(U^n) dm(w)$ with $B = \{B^w\}$ and $D = \{D^w\}$, then

$$BD = \{B^w D^w\}.$$ 

Since $B^w$ and $D^w$ are elements of $H^{\infty}(U^n)$ for almost all $w$ it follows that $B^w D^w$ is in $H^{\infty}(U^n)$ for almost all $w$, with

$$\|B^w D^w\|_{H^{\infty}(U^n)} \leq \|B^w\|_{H^{\infty}(U^n)} \|D^w\|_{H^{\infty}(U^n)} \leq \|B\|_{\infty} \|D\|_{\infty} < \infty$$

almost everywhere. If

$$B^w(z) = \sum_{I^n}\Sigma I^{n+} b_{\alpha}(w) z^\alpha$$

and

$$D^w(z) = \sum_{I^n}\Sigma I^{n+} d_{\alpha}(w) z^\alpha,$$

then

$$B^w D^w(z) = \sum_{\alpha I^n} b_{\alpha}(w) d_{\alpha}(w) z^\alpha$$

whenever both $B^w$ and $D^w$ are in $H^{\infty}(U^n)$. For $\alpha \in I^n_+, 0 \leq \beta \leq \alpha$, we have

$$b_{\beta} d_{\alpha-\beta} \in L^{\infty}(T)$$
because both $b_{\beta}$ and $d_{\alpha-\beta}$ are elements of $L_+(T)$. Therefore, for each $\alpha \in \Gamma^+_+$

$$\sum_{0 \leq \beta \leq \alpha} b_{\beta} d_{\alpha-\beta} \in L_+(T).$$

We have shown that if $B$ and $D$ are elements of $\int_T H^\infty(U^n)dm(w)$, then $BD$ is also in $\int_T H^\infty(U^n)dm(w)$ and furthermore

$$\|BD\|_\infty \leq \|B\|_\infty \|D\|_\infty.$$

With the operations we have defined, $\int_T H^\infty(U^n)dm(w)$ is a commutative normed algebra with identity. We will show that the map $V$ defined on $\int_T H^2(U^n)dm(w)$ is an isometric algebra isomorphism from $\int_T H^\infty(U^n)dm(w)$ onto the Banach algebra $H^\infty(U^{n+1})$, and that as a result $\int_T H^\infty(U^n)dm(w)$ is also a Banach algebra. In order to do this we will need the following theorem in which we extend the definition of multiplication to products of the form $BD$ with $B \in \int_T H^\infty(U^n)dm(w)$ and $D \in \int_T H^2(U^n)dm(w)$.

**Theorem 3.2.** Suppose $B \in \int_T H^\infty(U^n)dm(w)$ and $D \in \int_T H^2(U^n)dm(w)$. Then $BD \in \int_T H^2(U^n)dm(w)$ with

$$\|BD\|_2 \leq \|B\|_\infty \|D\|_2.$$
Proof. For almost all \( w \), \( B^w \in H^\infty(U^n) \) and \( D^w \in H^2(U^n) \), and as a result

\[
B^w D^w \in H^2(U^n)
\]

with

\[
\|B^w D^w\|_{H^2(U^n)} \leq \|B^w\|_{H^\infty(U^n)} \|D^w\|_{H^2(U^n)} \leq \|B\|_{\infty} \|D^w\|_{H^2(U^n)} .
\]

Then

\[
\int_T \|B^w D^w\|_{H^2(U^n)}^2 \, dm(w)
\]

\[
\leq \|B\|_{\infty}^2 \int_T \|D^w\|_{H^2(U^n)}^2 \, dm(w)
\]

\[
= \|B\|_{\infty}^2 \|D\|_{2}^2 < \infty .
\]

As in an earlier calculation,

\[
B^w D^w(z) = \sum_{\alpha \in I^n_+} \sum_{0 \leq \beta \leq \alpha} b_\beta(w) d_{\alpha-\beta}(w) z^\alpha .
\]

Since \( b_\gamma \in L^\infty(T) \) and \( d_{\alpha-\gamma} \in L^2_+(T) \) for \( \alpha \in I^n_+ \), \( 0 \leq \gamma \leq \alpha \), we have

\[
\sum_{0 \leq \beta \leq \alpha} b_\beta d_{\alpha-\beta} \in L^2_+(T) ,
\]

for each \( \alpha \in I^n_+ \). Therefore,

\[
BD \in \int_T H^2(U^n) \, dm(w)
\]

and

\[
\|BD\|_{2}^2 = \int_T \|B^w D^w\|_{H^2(U^n)}^2 \, dm(w) \leq \|B\|_{\infty}^2 \|D\|_{2}^2 .
\]
Next we shall show that the mapping $V$, defined on all of $\int_T H^2(U^n) dm(w)$, is multiplicative on elements of the form $BD$ with $B \in \int_T H^\infty(U^n) dm(w)$ and $D \in \int_T H^2(U^n) dm(w)$.

**Theorem 3.3.** If $B \in \int_T H^\infty(U^n) dm(w)$ and $D \in \int_T H^2(U^n) dm(w)$, then

$$V(BD) = V(B)V(D).$$

**Proof.** We have seen that the series expansion for $B^D$ is given by

$$B^D(z) = \sum_{\alpha \in \Gamma^+_n} \sum_{0 \leq \mu \leq \alpha} b_{\mu}(w)d_{\alpha-\mu}(w)z^\alpha$$

for almost all $w$ in $T$, where $z \in U^n$.

Suppose that

$$f_\alpha(\lambda) = \sum_{j=0}^g b_j,_{\alpha} \lambda^j, \lambda \in U,$$

is the series expansion for the function $f_\alpha \in H^\infty(U)$ such that $f_\alpha^* = b_\alpha$, and that

$$g_\alpha(\lambda) = \sum_{j=0}^\infty d_j,_{\alpha} \lambda^j, \lambda \in U,$$

is the series expansion for the function $g_\alpha \in H^2(U)$ such that $g_\alpha^* = d_\alpha$. Then for each $\alpha \in \Gamma^+_n$,
\[
\sum_{0 \leq \beta \leq \alpha} \sum_{j=0}^{\infty} \sum_{0 \leq k \leq j} b_k,\beta \cdot d_{j-k,\alpha-\beta} \cdot \lambda^j
\]

is the series expansion for the function in \( H^2(U) \)
whose boundary value function is \( \sum_{0 \leq \beta \leq \alpha} b_\beta d_{\alpha-\beta} \). It follows from the definition of the map \( V \) that for \((\lambda,z) \in U^1 \times U^n = U^{n+1}\),

\[
V(BD)(\lambda,z) = \sum_{\alpha \in \mathbb{I}_+^n} \sum_{j=0}^{\infty} \left( \sum_{0 \leq \beta \leq \alpha} \sum_{k=0}^{j} b_k,\beta \cdot d_{j-k,\alpha-\beta} \right) \lambda^j z^\alpha
\]
is the series expansion for the function \( V(BD) \in H^2(U^{n+1}) \). Consequently,

\[
V(BD) = \sum_{\alpha \in \mathbb{I}_+^n} \sum_{j=0}^{\infty} \left( \sum_{0 \leq \beta \leq \alpha} \sum_{k=0}^{j} b_k,\beta \cdot k z^\beta \cdot d_{j-k,\alpha-\beta} \right) \lambda^j z^\alpha
\]

We are now able to show that \( V \) maps
\[
\mathcal{H}^\infty(U^n) \text{d}m(w) \text{ into } \mathcal{H}^\infty(U^{n+1}) .
\]
**Theorem 3.4.** Suppose that $B \in \int_T H^\infty(U^n)dm(w)$. Then $VB \in H^\infty(U^{n+1})$ with

$$\|VB\|_{H^\infty(U^{n+1})} \leq \|B\|_\infty.$$

**Proof.** For each $q = 0, 1, 2, \cdots$, we have $B^q \in \int_T H^\infty(U^n)dm(w)$ and therefore,

$$V(B^q) \in H^2(U^{n+1}).$$

Since $V(B^q) = (VB)^q$ we have $(VB)^q \in H^2(U^{n+1})$ for each $q = 0, 1, \cdots$. Also,

$$\|B^q\|_2 \leq \|B^q\|_\infty \leq \|B\|_q$$

so that

$$\|(VB)^q\|_{H^2(U^{n+1})} = \|V(B^q)\|_{H^2(U^{n+1})} = \|B^q\|_2 \leq \|B\|^q.$$

By Theorem 1.3,

$$VB \in H^\infty(U^{n+1})$$

with

$$\|VB\|_{H^\infty(U^{n+1})} \leq \|B\|_\infty.$$

**Theorem 3.5.** $V$ maps $\int_T H^\infty(U^n)dm(w)$ onto $H^\infty(U^{n+1})$ and furthermore,
\[
\|V B\|_{H^\infty(U^{n+1})} = \|B\|_\infty
\]

for each \( B \in \int_T H^\infty(U^n)\,dm(w) \).

**Proof.** Suppose \( F \in H^\infty(U^{n+1}) \). Since \( V \) maps 
\[ \int_T H^2(U^n)\,dm(w) \]
on to \( H^2(U^{n+1}) \supset H^\infty(U^{n+1}) \), there is a 
\( B \in \int_T H^2(U^n)\,dm(w) \) such that \( V B = F \). We will show that 
\( B \in \int_T H^\infty(U^n)\,dm(w) \). If

\[
F(\lambda, z) = \sum_{I_n^+} f_\alpha(\lambda) z^\alpha,
\]

for \((\lambda, z) \in U \times U^n = U^{n+1}\), then by Corollary 1.2, we have \( f_\beta \in H^\infty(U) \), for each \( \beta \in I_n^+ \).

If the series expansion for \( B^w \) is given by

\[
B^w(z) = \sum_{I_n^+} b_\alpha(w) z^\alpha
\]

for almost all \( w \) in \( T \), then we have seen that 
\( b_\beta = f_\beta^* \) and hence that \( b_\beta \in L^\infty(T) \) for each \( \beta \in I_n^+ \).

Suppose that \( G \) is a countable subset of \( H^\infty(U^n) \)
which is dense in \( H^2(U^n) \) in the \( H^2(U^n) \) norm. If \( h \)
is any non-negative function in \( L^1(T) \), let \( \{p_k\}_{k=1}^\infty \)
be a sequence of analytic trigonometric polynomials
satisfying the conclusions of Theorem 1.5. For a function \( g \in G \), we define an element \( h_k \in \int_T H^\infty(U^n)\,dm(w) \),
for each \( k=1, 2, \ldots \), by
\[ h_k = \{p_k(w)g\} . \]

Then
\[
\int_T |p_k(w)|^2 \|B^w g\|_{H^2(U^n)}^2 \, dm(w)
\]
\[
= \int_T \|B^w h_K^w\|_{H^2(U^n)}^2 \, dm(w)
\]
\[
= \|B h_k\|_2^2 .
\]

We have shown that \( V \) is multiplicative on products of the form \( Bh_k \), since \( B \in \int T H^2(U^n) dm(w) \) and \( h_k \in \int T H^\infty(U^n) dm(w) \), so that

\[
\|B h_k\|_2 = \|V(Bh_k)\|_{H^2(U^{n+1})}
\]
\[
\leq \|V B\|_{H^\infty(U^{n+1})} \|V h_k\|_{H^2(U^{n+1})}
\]
\[
= \|V B\|_{H^\infty(U^{n+1})} \|h_k\|_2
\]
\[
= \|V B\|_{H^\infty(U^{n+1})} \|g\|_{H^2(U^n)} \|p_k\|_{L^2(T)} .
\]

Consequently,
\[
\int_T |p_k(w)|^2 \|B^w g\|_{H^2(U^n)}^2 \, dm(w)
\]
\[
\leq \|V B\|_{H^\infty(U^n)}^2 \|g\|_{H^2(U^n)}^2 \int_T |p_k(w)|^2 dm(w) .
\]

Using Fatou's lemma,
\[ \int_{T} h(w) \|B^w g\|_{H^2(U^n)}^2 \, dm(w) \leq \lim_{k \to \infty} \int_{T} p_k(w) \|B^w g\|_{H^2(U^n)}^2 \, dm(w) \leq \|V_B\|_{H^\infty(U^{n+1})}^2 \|g\|_{H^2(U^n)}^2 \int_{T} h(w) \, dm(w). \]

Suppose that
\[ \|B^w g\|_{H^2(U^n)}^2 > \|V_B\|_{H^\infty(U^{n+1})}^2 \|g\|_{H^2(U^n)}^2 \]
on a set \(E_g\) of positive measure. With \(h(w) = \frac{1}{m(E_g)} \chi_{E_g}\) we would have
\[ \int_{T} h(w) \|B^w g\|_{H^2(U^n)}^2 \, dm(w) \geq \|V_B\|_{H^\infty(U^{n+1})}^2 \|g\|_{H^2(U^n)}^2 \int_{T} h(w) \, dm(w). \]

This contradiction shows that
\[ \|B^w g\|_{H^2(U^n)} \leq \|V_B\|_{H^\infty(U^{n+1})} \|g\|_{H^2(U^n)} \]
except possibly on a set \(E_g\) of measure zero. Set \(E = \bigcup_{E_g} E_g\). Then \(E\) has measure zero since \(G\) is countable, and for \(w \notin E\)
\[ \|B^w g\|_{H^2(U^n)} \leq \|V_B\|_{H^\infty(U^{n+1})} \|g\|_{H^2(U^n)} \]

for all \( g \in G \). By Corollary 1.4,

\[ B^w \in H^\infty(U^n) \]

and

\[ \|B^w\|_{H^\infty(U^n)} \leq \|V_B\|_{H^\infty(U^{n+1})} \]

for almost all \( w \) in \( T \). Therefore,

\[ B \in \int_T H^\infty(U^n)dm(w) \]

and

\[ \|B\|_{H^\infty(U^n)} \leq \|V_B\|_{H^\infty(U^{n+1})} . \]

The reverse inequality has already been established so that

\[ \|B\|_{H^\infty(U^n)} = \|V_B\|_{H^\infty(U^{n+1})} . \]
In the case of analytic functions of one complex variable there is a complete theory of factorization of $H^p$ functions into inner and outer factors. We shall first generalize the definitions of inner and outer to functions in $H^p(U^n)$ and then show that with these definitions no such general factorization exists.

**Definition 4.1.** A function $G$ in $H^\infty(U^n)$ is said to be an inner function if $|G^\ast(w)| = 1$ for almost all $w$ in $T^n$.

There are a number of equivalent ways to characterize outer functions in $H^p(U)$. For $n \geq 2$ these conditions are no longer equivalent, and as a result lead to different definitions of outer functions in $H^p(U^n)$.

**Definition 4.2.** (Porcelli) A function $K$ in $H^p(U^n)$ is said to be an outer function if $K \cdot H^\infty(U^n)$ is dense in $H^p(U^n)$ in the $H^p(U^n)$ norm.
**Definition 4.3.** (Rudin) A function \( K \) in \( H^p(U^n) \) is said to be an outer function if

\[
\log |K(0)| = \int_{T^n} \log|K^*(w)| \, dm_n(w) .
\]

We note again that these definitions agree when \( n=1 \). On polydiscs, each function which is outer in the sense of Definition 4.2 is also outer in the sense of Definition 4.3, see [6], pp. 74-75.

**Theorem 4.4.** Suppose \( f \neq 0 \) is an element of \( H^1(U) \). Then \( f \) is uniquely expressible in the form \( f = gk \), where \( g \) is an inner function and \( k \) is an outer function. Furthermore, \( f \in H^p(U) \) if and only if \( k \in H^p(U) \), \( 1 \leq p \leq \infty \).

**Proof.** See [2], pp. 67-69 for a proof of this factorization theorem.

**Theorem 4.5.** Suppose \( G \) is an inner function in \( H^\infty(U^{n+1}) \) and \( B \in \int_T H^\infty(U^n) \, dm(w) \) with \( VB = G \). Then \( B^w \) is an inner function in \( H^\infty(U^n) \) for almost all \( w \) in \( T \).

**Proof.** Since \( G \) is inner we have \( |G^*(z)| = 1 \) for almost all \( z \in T^{n+1} \). Therefore,
\[ \|G\|_{H^2(U^{n+1})}^2 = \int_{T^{n+1}} |G^*(z)| \, dm_n(w) = 1. \]

Also,

\[ \|B\|_2 = \|VB\|_{H^2(U^{n+1})} \]

\[ = \|G\|_{H^2(U^{n+1})} = 1. \]

With

\[ \int_T \|B^w\|_{H^2(U^n)}^2 \, dm(w) = 1 \]

and

\[ \|B^w\|_{H^2(U^n)}^2 \leq \|B^w\|_{H^\infty(U^n)}^2 \]

\[ = \|VB^w\|_{H^\infty(U^{n+1})}^2 \leq 1 \]

for almost all \( w \), it follows that

\[ \|B^w\|_{H^2(U^n)}^2 = 1 \]

almost everywhere. Consequently,

\[ \|B^w\|_{H^\infty(U^n)} \geq 1 \]

for almost all \( w \), and since we already have

\[ \|B^w\|_{H^\infty(U^n)} \leq 1 \]

almost everywhere,
\[ \|B^w\|_{H^\infty(U^n)} = 1 \]

for almost all \( w \).

Whenever \( B^w \in H^\infty(U^n) \),

\[ |B^w(z)| \leq \|B^w\|_{H^\infty(U^n)} = 1 \]

for almost all \( z \in U^n \). Since

\[ \int_{T^n} |(B^w)^*(z)|^2 dm_n(z) = \|B^w\|_{H^2(U^n)}^2 = 1, \]

we must have \( |(B^w)^*(z)| = 1 \) almost everywhere on \( T^n \), for almost all \( w \in T \). That is, \( B^w \) is an inner function for almost all \( w \).

We next prove a similar result for each of the two definitions of outer functions.

**Theorem 4.6.** Suppose that \( F \) is an outer (P) function in \( H^2(U^{n+1}) \) and that \( B \in \int_T H^2(U^n) dm(w) \) with \( VB = F \). Then for almost all \( w \), \( B^w \) is an outer (P) function in \( H^2(U^n) \).

**Proof.** If \( p \in H^2(U^n) \), then the function \( P \), defined by

\[ P(z_1, z_2, \ldots, z_{n+1}) = p(z_2, z_3, \ldots, z_{n+1}), \]
is in $H^2(U^{n+1})$. Suppose $\{Q_j\}_{j=1}^{\infty}$ is a sequence of functions in $H^\infty(U^{n+1})$ such that

$$\lim_{j \to \infty} \|Q_j - P\|_{H^2(U^{n+1})} = 0.$$ 

Choose a subsequence $\{Q_{j_1}\}_{i=1}^{\infty}$, $j_1 \leq j_2 \leq \cdots$, such that

$$\|Q_{j_1} - P\|_{H^2(U^{n+1})} \leq \frac{1}{2^i}$$

for $i = 1, 2, \cdots$. The map $V$ is onto $H^\infty(U^{n+1})$, so there are functions

$$D_1 \in \int_T H^\infty(U^n) dm(w)$$

with $VD_1 = Q_{j_1}$, $i = 1, 2, \cdots$.

The constant function $p = \{p\}$ is in $\int_T H^2(U^n) dm(w)$ and $Vp = P$, so we have

$$\int_T \|D_1 - p\|_{H^2(U^n)} dm(w)$$

$$\leq \|D_1 - p\|_2$$

$$= \|V(D_1 - p)\|_{H^2(U^{n+1})}$$

$$= \|Q_{j_1} - P\|_{H^2(U^{n+1})} \leq \frac{1}{2^i}.$$ 

Consequently,
and thus
\[
\sum_{i=1}^{\infty} \frac{1}{2^i} < \infty,
\]
and thus
\[
\sum_{i=1}^{\infty} \|D_i B^w - p\|_{H^2(U^n)} < \infty
\]
for almost all \(w\). Since \(p\) was arbitrary in \(H^2(U^n)\), we see that \(B^w\) is outer \((P)\) for almost all \(w\) in \(T\).

**Theorem 4.7.** Suppose \(F \in H^2(U^{n+1})\), \(B \in \int_T H^2(U^n)dm(w)\), and \(VB = F\). Then \(F\) is an outer \((R)\) function if and only if \(B^w\) is an outer \((R)\) function in \(H^2(U^n)\) for almost all \(w\) and \(f_0^w\) is an outer function in \(H^2(U)\), where \(f_0^* = b_\beta\),

\[B^w(z) = \Sigma_{I_+} b_\alpha(w)z^\alpha.\]

**Proof.** If
\[F(\lambda,z) = \Sigma_{i=1}^{\infty} \Sigma_{j=0}^{\infty} b_j, a^j, \lambda^j z^\alpha,\]
where \((\lambda,z) \in U \times U^n = U^{n+1}\), is the series expansion for \(F\), then \(F(0) = b_{0,0}\). The function \(f_0^* \in H^2(U)\)
is given by
\[ f_\beta(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j, \quad \lambda \in U, \]
so that \( b_{0,0} = f_0(0) \). By Jensen's inequality
\[
\log |f_0(0)| \leq \int_T \log |b_0(w)| \, dm(w)
= \int_T \log |B^W(0)| \, dm(w).
\]

Using Jensen's inequality again
\[
\log |B^W(0)| \leq \int_T n \log |(B^W)^*(z)| \, dm_n(z),
\]
for almost all \( w \), and consequently,
\[
\log |f_0(0)| \leq \int_{T^{n+1}} \log |(B^W)^*(z)| \, dm_{n+1}(w,z).
\]

For emphasis we rewrite the above steps
\[
\log |F(0)| = \log |f_0(0)|
\leq \int_T \log |f_0^*(w)| \, dm(w)
= \int_T \log |B^W(0)| \, dm(w)
\leq \int_{T^{n+1}} \log |(B^W)^*(z)| \, dm_{n+1}(w,z)
= \int_{T^{n+1}} \log |F^*(w,z)| \, dm_{n+1}(w,z).
\]
If \( f_0 \) is outer the first inequality becomes an equality, and if \( B^w \) is outer (R) for almost all \( w \), the second inequality also becomes an equality, and therefore \( F \) is outer (R). Clearly if \( F \) is outer (R) we see that the converse holds.

We now show that it is not in general possible to factor elements of \( H^2(U^1) \) into inner and outer factors. We do not distinguish between the alternate definitions of outer since by use of the analytic direct integral we arrive at a contradiction in \( H^2(U) \) where the definitions are equivalent.

**Theorem 4.8.** There is a function in \( H^2(U^2) \) that cannot be factored in the form \( F = GK \), where \( G \) is an inner function and \( K \) is an outer function.

**Proof.** Consider the function

\[
F(z_1, z_2) = \frac{1}{3} + \frac{1}{3} z_1 + z_2.
\]

Then

\[
f_0(z_1) = \frac{1}{3} + \frac{1}{3} z_1, \quad f_1(z_1) = 1
\]

and \( f_2 = f_3 = \ldots = 0 \). For \( b_\alpha = f_\alpha^* \) we have

\[
b_0(w) = \frac{1}{3} + \frac{1}{3} w, \quad b_1(w) = 1
\]

and \( b_2 = b_3 = \ldots = 0 \). The function \( B \in \int_T H^2(U)dm(w) \)
such that $VB = F$ is given by

$$B^w(z) = \left(\frac{1}{3} + \frac{1}{3}w\right) + z,$$

for all $w$ in $T$. Each $B^w$ is an element of $H^2(U)$ and by Theorem 4.4, has a unique inner and outer factorization. In this case,

$$B^w(z) = \frac{-\frac{1+w}{3} - z}{1 + \left(-\frac{1+w}{3}\right)z} \cdot (-1 - \frac{1+w}{3}z).$$

Suppose $F = GK$ is a factorization of $F$ into inner and outer functions. Then for almost all $w$ in $T$ the functions $g^w \in H^\infty(U)$ are inner and the functions $k^w \in H^2(U)$ are outer, where

$$g = V^{-1}G, \quad k = V^{-1}K,$$

$$g \in \int_T H^\infty(U)dm(w)$$

and

$$k \in \int_T H^2(U)dm(w).$$

The map $V$ is an isomorphism and each $B^w$ factors uniquely so it follows that

$$g^w(z) = \frac{-\frac{1+w}{3} - z}{1 + \left(-\frac{1+w}{3}\right)z}$$

and
\[ k^w(z) = -1 - \left(\frac{1+w}{3}\right)z \]
for almost all \( w \).

It is clear however, that \( k = \{k^w\} \) could not be an element of \( \int_T \mathcal{H}^2(U)dm(w) \) since in particular \( \frac{1+w}{3} \) is not a function in \( L^2(T) \). The assumption that \( F \) factors must be false.

To see that there is a function in \( \mathcal{H}^2(U^3) \) that can not be factored, we need only consider

\[ F(z_1, z_2, z_3) = \frac{1}{3} + \frac{1}{3} z_2 + z_3. \]

Set

\[ B = \{B^w\} \in \int_T \mathcal{H}^2(U^2)dm(w) \]

where

\[ B^w(z_1, z_2) = \frac{1}{3} + \frac{1}{3} z_1 + z_2 \]

for each \( w \). Then \( VB = F \), and if \( F \) could be factored, it follows that each \( B^w \) would have to factor in \( \mathcal{H}^2(U^2) \).

By induction we conclude that for each \( n \geq 2 \) there is a function \( F \in \mathcal{H}^2(U^n) \) such that \( F \) can not be factored into a product of inner and outer functions.
BIBLIOGRAPHY


VITA

Alvin Edward Horne was born in Devils Lake, North Dakota on September 11, 1939. He graduated from Central High School in Devils Lake in May, 1957 and attended Valley City State College from September, 1957 to August, 1959. He received a B.S. degree from the University of North Dakota which he attended from September, 1959 to June, 1961. He served as a graduate teaching assistant at the University of North Dakota from September, 1961 to August, 1963 and as a graduate teaching assistant at the University of Minnesota from September, 1963 to June, 1964. From September, 1964 to June, 1967 he taught as an instructor at the University of North Dakota, receiving an M.S. degree in August, 1966.

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