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Higher algebraic K-theory and tangent spaces to Chow groups

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HIGHER ALGEBRAIC K-THEORY AND TANGENT SPACES TO CHOW GROUPS

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Abstract

In this work, using higher algebraic K-theory, we provide an answer to the following question asked by Green-Griffiths in [13]:

> Can one define the Bloch-Gersten-Quillen sequence $\mathcal{G}_j$ on infinitesimal neighborhoods $X_j = X \times Spec(k[t]/(t^{j+1})$ so that
>
> $$ker(\mathcal{G}_1 \to \mathcal{G}_0) = T\mathcal{G}_0,$$

here $T\mathcal{G}_0$ should be the Cousin resolution of $TK_m(\mathcal{O}_X)$ and X is any n-dimensional smooth projective variety over a field k, $char k = 0$.

Our main results are as follows. The existence of $\mathcal{G}_j$ is discussed in chapter 3, following [8] and [18]. The main theorems are theorem5.2.5, theorem 5.2.6 and theorem 5.2.8.

The proof for the above theorems, given in chapter 5, requires non-trivial techniques from higher algebraic K-theory and negative cyclic homology. The main ingredients of the proof are: existence of Chern character at spectrum level, effacement theorem and Goodwillie-type and Cathelineau-type results.

Chapter 1
Introduction

1.1 Introduction

Beginning with Bloch, Gersten and Quillen, K-theory enters into the picture of studying higher codimensional algebraic cycles. In the following, we use points on a surface, $Ch^2(X)$, to explain the ideas. The well-known Bloch-Gersten-Quillen exact sequence

$$0 \to K_2(O_X) \to K_2(\mathbb{C}(X)) \to \bigoplus_{y \in X^{(1)}} K_1(\mathbb{C}(y)) \to \bigoplus_{x \in X^{(2)}} K_0(\mathbb{C}(x)) \to 0$$

leads to

$$CH^2(X) = H^2(X, K_2(O_X)).$$

Combining with Van der Kallen's isomorphism

$$T_{formal} K_2(X) = \Omega^1_{X/\mathbb{Q}},$$

one can get

$$T_{formal} CH^2(X) = H^2(X, \Omega^1_{X/\mathbb{Q}}).$$

Green and Griffiths would like to understand the geometric significance of the above isomorphism. The clue to do this is from the Cousin flasque resolution

$$0 \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \to \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) \xrightarrow{\partial} \bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}) \to 0$$

which gives rise to

$$H^2(X, \Omega^1_{X/\mathbb{Q}}) = \frac{\bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}})}{Im(\partial)}.$$

Following the above information and the classical result $CH^2(X) = \frac{Z^2(X)}{Z^2_{rat}(X)}$, Green and Griffiths define the tangent space $TZ^2(X)$ to the $0-cycles$ on X as

$$TZ^2(X) = \bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}),$$

and define a tangent subspace $TZ^2_{rat}(X)$ to the rational equivalence as

$$TZ^2_{rat}(X) = Im(\partial).$$

The question is to show that there is really a tangent map from "Arcs" to the local cohomology. In other words, Green and Griffiths ask for a Bloch-Gersten-Quillen type exact sequence which can fill in the middle in the following diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{X/\mathbb{Q}} & \xleftarrow{tan1} & K_2(X[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{\mathbb{C}(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_2(\mathbb{C}(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2(\mathbb{C}(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{?tan3} & Arcs^1(X) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(1)}} K_1(\mathbb{C}(y)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{?tan4} & Arcs^2(X) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0(\mathbb{C}(x)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $Arcs^1(X)$ and $Arcs^2(X)$ stand for the arc space associated with $\bigoplus_{y \in X^{(1)}} K_1(\mathbb{C}(y))$ and $\bigoplus_{x \in X^{(2)}} K_0(\mathbb{C}(x))$ respectively.

Green and Griffiths implicitly introduce groups of "Arcs". The idea is that, an element of $Arcs^1(X)$ should be a formal sum of the expression $(div(f+\varepsilon f_1), g+\varepsilon g_1)$, where $f = 0$ is a local expression for a divisor on X and $g \in \mathbb{C}(Y)^*$. We think of $f + \varepsilon f_1$ as the $1st$ order deformation of $Y = div(f)$ and $g + \varepsilon g_1$ is a deformation of g. The tangent to the $(div(f+\varepsilon f_1), g+\varepsilon g_1)$ is defined in the following way: First, the following diagram

$$\begin{cases} O_{X,y} \xrightarrow{f} O_{X,y} \longrightarrow O_{X,y}/(f) \longrightarrow 0 \\ O_{X,y} \xrightarrow{\frac{g_1 df}{g} - \frac{f_1 dg}{g}} \Omega^1_{X/\mathbb{Q},y} \end{cases} \tag{1.1.1}$$

gives an element α in $Ext^1_{O_{X,y}}(O_{X,y}/(f), \Omega^1_{X/\mathbb{Q},y})$. Noting that

$$H^1_y(\Omega^1_{X/\mathbb{Q}}) = \varinjlim_{n\to\infty} Ext^1_{O_{X,y}}(O_{X,y}/(f)^n, \Omega^1_{X/\mathbb{Q},y}),$$

the image $[\alpha]$ of α under the limit is in $H^1_y(\Omega^1_{X/\mathbb{Q}})$ and it is the tangent to $(div(f+\varepsilon f_1), g+\varepsilon g_1)$.

Similarly, an element of $Arcs^2(X)$ is a first order deformation of $0-cycles$ on X. To be precise, it is of the form $Z_\varepsilon = V(u+\varepsilon u_1, v+\varepsilon v_1)$, where $Z = V(u,v)$ is supported on x. The tangent to Z_ε is defined in the following way. First, the following diagram

$$\begin{cases} O_{X,x} \xrightarrow{(v,-u)} O^{\oplus 2}_{X,x} \xrightarrow{(u,v)} O_{X,x} \longrightarrow O_{X,x}/(u,v) \longrightarrow 0 \\ O_{X,x} \xrightarrow{v_1 du - u_1 dv} \Omega^1_{X/\mathbb{Q},x} \end{cases} \tag{1.1.2}$$

gives an element β in $Ext^2_{O_{X,x}}(O_{X,x}/(u,v), \Omega^1_{X/\mathbb{Q},x})$. Noting that

$$H^2_x(\Omega^1_{X/\mathbb{Q}}) = \varinjlim_{n\to\infty} Ext^2_{O_{X,x}}(O_{X,x}/(u,v)^n, \Omega^1_{X/\mathbb{Q},x}),$$

the image $[\beta]$ of β under the limit is in $H^2_x(\Omega^1_{X/\mathbb{Q}})$ and it is the tangent to $Z_\varepsilon = V(u+\varepsilon u_1, v+\varepsilon v_1)$.

By some heuristic arguments, Green and Griffiths conclude that

Theorem 1.1.1. The tangent sequence to the Bloch-Gersten-Quillen sequence

$$0 \to K_2(X) \to K_2(\mathbb{C}(X)) \to \bigoplus_{y\in X^{(1)}} K_1(\mathbb{C}(y)) \to \bigoplus_{x\in X^{(2)}} K_0(\mathbb{C}(x)) \to 0$$

is the Cousin flasque resolution of $\Omega^1_{X/\mathbb{Q}}$

$$0 \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \to \bigoplus_{y\in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) \to \bigoplus_{x\in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}) \to 0$$

More generally, Green and Griffths pose the following question:

Can one define the Bloch-Quillen-Gersten sequence $\mathcal{G}_i$ on infinitesimal neighborhoods $X_j = X \times Spec(k[t]/(t^{j+1})$ so that

$$ker(\mathcal{G}_1 \to \mathcal{G}_0) = \underline{\underline{T}}\mathcal{G}_0$$

What's meant here is that $\underline{\underline{T}}\mathcal{G}_0$ should be the Cousin resolution of $TK_m(\mathcal{O}_X)$. In more detail, if X is smooth of dimension n over a field k of characteristic 0, and if we denote by $\mathcal{G}_0$ the Bloch-Quillen-Gersten resolution

$$0 \to K_m(O_X) \to K_m(k(X)) \to \bigoplus_{d \in X^{(1)}} \underline{K}_{m-1}(O_{X,d} \text{ } on \text{ } d) \to \cdots \to \bigoplus_{x \in X^{(n)}} \underline{K}_{m-n}(O_{X,x} \text{ } on \text{ } x) \to 0.$$

They are asking for analogs of this with X replaced by infinitesimal thickenings X_i in such a way that $ker(\mathcal{G}_1 \to \mathcal{G}_0)$ is the Cousin complex of $TK_m(\mathcal{O}_X)$.

We will provide an answer to this question in the following sections.

Definition 1.1.2. definition-theorem[8,18]
Let T_j denote $Spec(k[t]/(t^{j+1})$, the Bloch-Quillen-Gersten sequence $\mathcal{G}_j$ is defined as the following flasque resolution :

$$0 \to K_m(O_{X_j}) \to K_m(k(X)_j) \to \bigoplus_{d_j \in X_j^{(1)}} \underline{K}_{m-1}(O_{X_j,d_j} \text{ } on \text{ } d_j) \to \cdots \to \bigoplus_{x_j \in X_j^{(n)}} \underline{K}_{m-n}(O_{X_j,x_j} \text{ } on \text{ } x_j) \to 0.$$

where $O_{X_j} = O_{X \times T_j}$, $k(X)_j = k(X) \times T_j$, $d_j = d \times T_j$ and etc.

Theorem 1.1.3. The formal tangent sequence to the Bloch-Gersten-Quillen sequence is the Cousin resolution. That is there exists the following commutative diagram(each column is a flasque resolution, m can be any integer, ε is a nilpotent with $\varepsilon^2 = 0.$):

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_m(O_{X[\varepsilon]}) & \longleftarrow & K_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_m(k(X)[\varepsilon]) & \longleftarrow & K_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d \in X^{(1)}} \underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{d[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{K}_{m-1}(O_{X,d}[\varepsilon] \textit{ on } d[\varepsilon]) & \longleftarrow & \oplus_{d \in X^{(1)}} \underline{K}_{m-1}(O_{X,d} \textit{ on } d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(2)}} \underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{K}_{m-2}(O_{X,y}[\varepsilon] \textit{ on } y[\varepsilon]) & \longleftarrow & \oplus_{y \in X^{(2)}} \underline{K}_{m-2}(O_{X,y} \textit{ on } y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(n)}} \underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan(n+2)} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(n)}} \underline{K}_{m-n}(O_{X,x}[\varepsilon] \textit{ on } x[\varepsilon]) & \longleftarrow & \oplus_{x \in X^{(n)}} \underline{K}_{m-n}(O_{X,x} \textit{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where

$$\Omega^{\bullet}_{O_X/\mathbb{Q}} = \Omega^{m-1}_{O_X/\mathbb{Q}} \oplus \Omega^{m-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$\Omega^{\bullet}_{k(X)/\mathbb{Q}} = \Omega^{m-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{m-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

Adams operations can decompose the above diagram into eigen-components. So we have the following finer result:

Theorem 1.1.4. There exists the following commutative diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_m^{(i)}(O_{X[\varepsilon]}) & \longleftarrow & K_m^{(i)}(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet,(i)}_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_m^{(i)}(k(X)[\varepsilon]) & \longleftarrow & K_m^{(i)}(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{H}^1_d(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{K}^{(i)}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \longleftarrow & \oplus_{d\in X^{(1)}}\underline{K}^{(i)}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{H}^2_y(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{K}^{(i)}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \longleftarrow & \oplus_{y\in X^{(2)}}\underline{K}^{(i)}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{H}^n_x(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan(n+2)} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{K}^{(i)}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \longleftarrow & \oplus_{x\in X^{(n)}}\underline{K}^{(i)}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

where

$$
\begin{cases}
\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = \Omega^{2i-m+1}_{O_X/\mathbb{Q}}, for \ \frac{m-1}{2} < \ i \leq m-1. \\
\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = 0, else.
\end{cases}
\tag{1.1.3}
$$

Moreover, we have

Theorem 1.1.5. There exists the following commutative diagram(each column is a flasque resolution, m can be any integer.):

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan1} & K_m(O_{X_j}) & \longleftarrow & K_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega^{\bullet}_{k(X)/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan2} & K_m(k(X)_j) & \longleftarrow & K_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{H}^1_d((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan3} & \oplus_{d_j\in X_j^{(1)}}\underline{K}_{m-1}(O_{X_j,d_j} \ on \ d_j) & \longleftarrow & \oplus_{d\in X^{(1)}}\underline{K}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{H}^2_y((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan4} & \oplus_{y_j\in X_j^{(2)}}\underline{K}_{m-2}(O_{X_j,y_j} \ on \ y_j) & \longleftarrow & \oplus_{y\in X^{(2)}}\underline{K}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{H}^n_x((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan(n+2)} & \oplus_{x_j\in X_j^{(n)}}\underline{K}_{m-n}(O_{X_j,x_j} \ on \ x_j) & \longleftarrow & \oplus_{x\in X^{(n)}}\underline{K}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

where

$$\Omega^{\bullet}_{O_X/\mathbb{Q}} = \Omega^{m-1}_{O_X/\mathbb{Q}} \oplus \Omega^{m-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$\Omega^{\bullet}_{k(X)/\mathbb{Q}} = \Omega^{m-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{m-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

The main points of the proof are outlined as follows.

- According to [5], there exists a Chern character from K-theory spectrum $\mathcal{K}$ to negative cyclic homology spectrum $\mathcal{HN}$,

  $$Ch : \mathcal{K} \to \mathcal{HN},$$

  here $\mathcal{K}$ is the Thomason-Trobaugh spectrum and $\mathcal{HN}$ is the spectrum associated to negative cyclic complex constructed by Keller[24,25]. This Chern character induces maps from the coniveau spectral sequence associated to $\mathcal{K}$ to the coniveau spectral sequence associated to $\mathcal{HN}$.

- Effacement theorem.

  In our approach, $\mathcal{K}$ and $\mathcal{HN}$ are considered as "cohomology theories with support" in the sense of [8]. Both $\mathcal{K}$ and $\mathcal{HN}$ satisfy étale excision and projective bundle formula. Therefore, according to [8], $\mathcal{K}$ and $\mathcal{HN}$ are effaceable functors. Effacement theorem gives us the exactness and universal exactness of sheafified Bloch-Gersten-Quillen sequence.

- Goodwillie-type and Cathelineau-type results

  Goodwillie -type and Cathelineau-type results enable us to compute relative K-groups(with support) in terms of relative negative cyclic groups(with support). Our computation is based on a recent version by Cortinas-Haesemeyer-Weibel[6].

This paper is organized as follows. We begin with an introduction of Green-Griffiths' work and their question in chapter 1. In chapter 2, we briefly survey background on K-theory, negative cyclic homology and descent property

In chapter 3, we discuss effacement theorem and Chern character which are the first two ingredients for proving our main result.

Lambda and Adams operations are discussed in chapter 4. We show Goodwillie-type and Cathelineau-type results, which are the third ingredient for proving our main results.

Our main theorem is proved in chapter 5. In order to give an intuitive picture to our audiences, we prove the theorem for surfaces firstly in 5.1. The general case is proved in section 5.2 by using the same ideas. Adams operation necessarily enter into the picture when we move to higher dimension:

Remark 1.1.6. In our setting, $Arcs^1(X)$ and $Arcs^2(X)$ on page 2 are defined as(theorem 5.1.6 on page 63):

$$Arcs^1(X) = \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} K_1(O_{X,y}[\varepsilon]),$$

$$Arcs^2(X) = \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} K_0(O_{X,x}[\varepsilon]).$$

On the other hand, our tangent maps are induced by Chern character(corollary 5.1.5 on page 63). It is a quite interesting question to compare our tangent maps(induced by Chern character) with Green-Griffiths'.

Chapter 2
Preliminaries

In this chapter, we recall background on K-theory in section 2.1 and negative cyclic homology in section 2.2. In section 2.3, we discuss descent properties, especially *cdh*-descent and *scdh*-descent followng[5][7][41][42]. We do not really need *cdh*-descent or *scdh*-descent in this paper. I put them here because of indepedent interest. One can accept theorem 2.35 and move to next section directly.

2.1 K-theory

2.1.1 Non-connective K-spectrum

In this subsection, we recall the non-connective K-theory spectrum constructed by Thomason-Trobaugh in [40]. It is based on Waldhausen's work and theory of perfect complexes of SGA6. This is the K-theory we will use in the following.

From now on, X is a quasi-compact and seperated scheme, Z is a closed suset of X with quasi-compact open complement $U = X - Z$.

Definition 2.1.1. A complex (A, d) of quasi-coherent O_X -modules is called perfect if there is a covering $X = \bigcup_{i \in I} U_i$ of X by affine open subschemes $U_i \subset X$ such that the restriction of the complex (A, d) to U_i is quasi-isomorphic to a bounded complex of vector bundles for $i \in I$. It is a fact that this is independent of the chosen affine cover.

We write $Perf_Z(X)$ for the exact category of perfect complexes on X which are acyclic over $X - Z$. $Perf_Z(X)$ can be considered as a biwaldhausen category by defining a cofibration to be a monomorphisms in $Perf_Z(X)$ and defining a weak equivalence to be an isomorphisms in $Perf_Z(X)$. Thomason-Trobaugh firstly define the K-theory spectra $K(X\ on\ Z)$ associated to $Perf_Z(X)$ by using Waldhuasen's

$S_\bullet$-construction.

$$K(X\ on\ Z) = (\Omega \mid wS_\bullet(Perf_Z(X)) \mid, \Omega \mid wS^2_\bullet(Perf_Z(X)) \mid, \Omega \mid wS^3_\bullet(Perf_Z(X)) \mid, \dots).$$

We note that this K-theory spectrum gives no negative K-groups.

Thomason-Trobaugh extend the above K-theory spectrum to non-connective spectrum by mimicking Bass' fundamental exact sequence. We let

$$F^0(X\ on\ Z) = K(X\ on\ Z),$$

where $K(X\ on\ Z)$ is the Waldhausen K-theory spectrum defined above. We look at the cone of the following map

$$F^0(X[t]\ on\ Z[t]) \bigcup_{F^0(X\ on\ Z)} F^0(X[t^{-1}]\ on\ Z[t^{-1}]) \xrightarrow{b} F^0(X[t,t^{-1}]\ on\ Z[t,t^{-1}]),$$

and define $F^{-1}(X\ on\ Z) = \Omega(cone(b))$. Then we define $F^{-k}(X\ on\ Z)$ inductively. Finally, the non-connective K-theory spectrum $K^B(X\ on\ Z)$ ("B" refers to Bass) is defined as the direct limit of the direct system:

$$F^0 \to F^{-1} \to F^{-2} \to \dots$$

Thomason-Trobaugh prove several useful theorems of these K-theory spectra, including Zariski excision, Mayer-Vietoris sequence, localization sequence and projective bundle formula etc. We will discuss them in the following.

With the development of triangulated category, we can rewrite and generalize the above definition and results. We mainly follow Schlichting's [36,37,38].

Definition 2.1.2. A Frobenius category is an exact category $\mathcal{A}$ which has enough projective and injective objects, and in which projectives coincide with injectives. We write $\mathcal{A}$-prinj for the full subcategory of projective-injective objects of a Frobenius category $\mathcal{A}$.

Definition 2.1.3. A Frobenius pair $(\mathcal{A}, \mathcal{A}_\prime)$ is a fully faithful inclusion $\mathcal{A}_\prime \to \mathcal{A}$ of small Frobenius categories.

We also recall the following definition.

Definition 2.1.4. A map of Frobenius categories is an exact functor preserving projective-injective objects.

Definition 2.1.5. A map of Frobenius pairs $(\mathcal{A}, \mathcal{A}_\prime) \to (\mathcal{B}, \mathcal{B}_\prime)$ is a map of Frobenius categories $\mathcal{A} \to \mathcal{B}$ such that $\mathcal{A}_\prime$ is mapped into $\mathcal{B}_\prime$.

Now we recall the following two facts.

Theorem 2.1.6. $Perf_Z(X)$ is a Frobenius category whose projective-injective objects are the contractible chain complexes in $Perf_Z(X)$.

Theorem 2.1.7. Let $quis$ be the full subcategory of chain complexes which are homotopy equivalent to acyclic chain complexes, $(Perf_Z(X), quis)$ is a Frobenius pair.

Schlichting constructed K-theory spectra associated to Frobenius pairs in a functorial way. In other words, he defined a functor from Frobenius pairs to spectra.

Definition 2.1.8. Following Schlichting, The K-theory spectrum of X with support in Z is the K-theory spectrum

$$\mathcal{K}(X \ on \ Z) = \mathcal{K}(Perf_Z(X), quis)$$

associated to the Frobenius pair $(Perf_Z(X), quis)$.

2.1.2 K-theory results

In this section, we recall the following theorems from K-theory which will be used later.

Theorem 2.1.9. Localization.

Let X be a quasi-compact and separated scheme. Let $U \subset X$ be a quasi-compact open subscheme with closed complement $Z = X - U$. Then there is a homotopy fibration of K-theory spectra

$$\mathcal{K}(X \text{ } on \text{ } Z) \to \mathcal{K}(X) \to \mathcal{K}(U).$$

In particular, there is a long exact sequence of K-groups : for $i \in \mathbb{Z}$,

$$\cdots \to K_{i+1}(U) \to K_i(X \text{ } on \text{ } Z) \to K_i(X) \to K_i(U) \to \dots$$

Theorem 2.1.10. Étale excision.

The K-theory spectrum $\mathcal{K}$ satisfy étale excision, that is for any given diagram:

$$\begin{array}{ccc} Z & \longrightarrow & X' \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{j} & X \end{array}$$

where $j : Z \to X$ is the closed immersion, f is étale and $Z \simeq f^{-1}(Z)$, the pullback

$$f^* : K_q(X \text{ } on \text{ } Z) \xrightarrow{\simeq} K_q(X' \text{ } on \text{ } Z)$$

is an isomorphism for any integer q.

Proof. (Sketch) It is a fact that the functors Lf^* and Rf_* induce quasi-inverse equivalences on derived categories

$$\mathbb{D}^b(X \text{ } on \text{ } Z) \simeq \mathbb{D}^b(X' \text{ } on \text{ } Z).$$

One can check this fact from [40] by Thomason-Trobaugh. By the invariance of K-theory under derived equivalence, we have a homotopy equivalence of K-theory spectra

$$\mathcal{K}(X \text{ } on \text{ } Z) \simeq \mathcal{K}(X' \text{ } on \text{ } Z).$$

In particular, there are isomorphisms of K-groups

$$K_q(X \text{ } on \text{ } Z) \xrightarrow{\simeq} K_q(X' \text{ } on \text{ } Z).$$

$\square$

Theorem 2.1.11. Zariski excision.

Let X be a quasi-compact and separated scheme. Let $V \subset X$ be a quasi-compact open subscheme and Z a closed subset with quasi-compact complement such that $Z \subset V$. The restriction of quasi-coherent sheaves induces a homotopy invariance of K-theory spectra.

$$\mathcal{K}(X \text{ } on \text{ } Z) \simeq \mathcal{K}(V \text{ } on \text{ } Z).$$

In particular, there are isomorphisms of K-groups

$$K_q(X \text{ } on \text{ } Z) \xrightarrow{\simeq} K_q(V \text{ } on \text{ } Z).$$

Proof. It is an obvious corollary of the previous étale excision. $\square$

Theorem 2.1.12. Mayer-Vietoris for Open Covers.

Let $X = U \cup V$ be a quasi-compact and separated scheme which is covered by two open quasi-compact subschemes U and V. Then restriction of quasi-coherent sheaves induces a homotopy cartesian square of K-theory spectra

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}(U) \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \longrightarrow & \mathcal{K}(U \cap V) \end{array}$$

In particular, we obtain a long exact sequence of K-groups: for $i \in \mathbb{Z}$,

$$\cdots \to K_{i+1}(U \cap V) \longrightarrow K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to \ldots$$

Proof. We note that the homotopy fiber $\mathcal{K}$ of

$$\mathcal{K}(X) \to \mathcal{K}(U)$$

is $\mathcal{K}(X \text{ } on \text{ } Z)$, where $Z = X - U = V - U \cap V$. And the homotopy fiber of

$$\mathcal{K}(V) \to \mathcal{K}(U \cap V)$$

is $\mathcal{K}(V\ on\ Z)$. Zariski excision tells us that

$$\mathcal{K}(X\ on\ Z) = \mathcal{K}(V\ on\ Z).$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{K}(X\ on\ Z) & \longrightarrow & \mathcal{K}(X) & \longrightarrow & \mathcal{K}(U) \\ \simeq\downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(V\ on\ Z) & \longrightarrow & \mathcal{K}(V) & \longrightarrow & \mathcal{K}(U \cap V) \end{array}$$

It results that we have the following homotopy cartesian square of K-theory spectra

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}(U) \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \longrightarrow & \mathcal{K}(U \cap V) \end{array}$$

□

So roughly speaking, Mayer-Vietoris sequences come from Zariski excision.

Theorem 2.1.13. Projective space bundle formula.

Let X be a quasi-compact and separated scheme. Let's write π for the natural projection

$$\pi : \mathbb{P}^j_X \to X.$$

K-theory spectra $\mathcal{K}$ satisfies the projective space bundle formula,

$$\prod_0^{j-1} O(-l) \otimes \pi^* : \bigoplus_0^{j-1} \mathcal{K}(X) \to \mathcal{K}(\mathbb{P}^j_X).$$

While the non-connective K-theory spectrum give us the above useful theorems, the price to pay for it is the appearance of the non-zero negative K-groups. We will show them by explicit computation in the following.

Theorem 2.1.14. Combining the Mayer-Vietoris property with projective space bundle formula (for $j = 1$) yields the fundamental theorem for K-theory spectrum, which states that there is a short exact sequence,

$$0 \to \mathcal{K}(X \times A^1) \bigcup_{\mathcal{K}(X)} \mathcal{K}(X \times A^1) \to \mathcal{K}(X \times (A^1 - 0)) \to \mathcal{K}(X)[1] \to 0.$$

In fact,this sequence is split up to homotopy.

2.2 Cyclic Homology and Its Variants

In this section, we recall the basic definitions and properties of cyclic, periodic cyclic, negative cyclic homology, from algebras, schemes, to category level. The main references are Loday[27], Weibel[48] and Keller[24,25].

2.2.1 Algebra level

We recall that for a commutative ring k, the Hochschild homology $HH_n(A)$ of a k-algebra A is the homology of the standard Hochschild complex $C^h_*(A)$ which has $C^h_n(A) = A^{\otimes n+1}$

$$C^h_* : \ldots \xrightarrow{b} A^{\otimes *+1} \xrightarrow{b} \ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \to 0.$$

We can associate cyclic, periodic cyclic, negative cyclic homology complexes with the standard Hochschild complex $C^h_*(A)$ by taking total complexes of the corresponding Connes(B, b) double complexes:

$$HC(C^h_*(A), b, B) = Tot(\cdots \to C^h_*(A)[+1] \xrightarrow{B} C^h_*(A) \to 0 \to 0 \to \ldots)$$

$$HP(C^h_*(A), b, B) = Tot(\cdots \to C^h_*(A)[+1] \xrightarrow{B} C^h_*(A) \xrightarrow{B} C^h_*(A)[-1] \xrightarrow{B} \ldots)$$

$$HN(C^h_*(A), b, B) = Tot(\cdots \to 0 \to 0 \to C^h_*(A) \xrightarrow{B} C^h_*(A)[-1] \xrightarrow{B} \ldots)$$

where $C^h_*(A)$ is placed in horizontal degree 0.

Taking homology of the above complexes gives us the corresponding cyclic, periodic cyclic, negative cyclic homology groups.

2.2.2 Scheme level

Weibel extends cyclic homology from algebras to all schemes over a ring k by using hypercohomology in the sense that the usual cyclic homology of any commutative algebra agrees with the cyclic homology of its corresponding affine scheme.

Let us write C^h_* for the sheafification of the corresponding complex of presheaves $U \to C^h_*(O_X(U))$:

$$C^h_* : \ldots \xrightarrow{b} O_X^{\otimes *+1} \xrightarrow{b} \ldots \xrightarrow{b} O_X \otimes O_X \xrightarrow{b} O_X \to 0.$$

The Hochschild homology of X over k is defined as the (Cartan-Eilenberg) hypercohomology of the unbounded cochain complex $C^n = C^h_{-n}$

$$HH_n(X) = \mathbb{H}^{-n}(C^h_*) = \mathbb{H}^{-n}(C^{-*}).$$

For an affine scheme $X = SpecA$, Weibel shows that $HH_n(X)$ agrees with $HH_n(A)$ defined above.

As we have seen above, we can associate cyclic, periodic cyclic, negative cyclic homology complexes with the standard Hochschild complex C^h_* by taking total complexes of the corresponding Connes(B, b) double complexes. In other word, we can sheafify Connes(B, b) double complexes and get a double complex of sheaves.

$$HC(C^h_*(X), b, B) = Tot(\cdots \to C^h_*(X)[+1] \xrightarrow{B} C^h_*(X) \to 0 \to 0 \to \ldots)$$

$$HP(C^h_*(X), b, B) = Tot(\cdots \to C^h_*(X)[+1] \xrightarrow{B} C^h_*(X) \xrightarrow{B} C^h_*(X)[-1] \xrightarrow{B} \ldots)$$

$$HN(C^h_*(X), b, B) = Tot(\ldots 0 \to 0 \to C^h_*(X) \xrightarrow{B} C^h_*(X)[-1] \xrightarrow{B} \ldots)$$

where $C^h_*(X)$ is placed in horizontal degree 0.

Taking hypercohomology of the above complexes of sheaves gives us the corresponding cyclic, periodic cyclic, negative cyclic homology groups. This means we define

$$HC_n(X) = \mathbb{H}^{-n}(HC(C^h_*(X), b, B))$$

$$HP_n(X) = \mathbb{H}^{-n}(HP(C^h_*(X), b, B))$$

$$HN_n(X) = \mathbb{H}^{-n}(HN(C^h_*(X), b, B))$$

where the (product) total chain complex regarded as a cochain complex by re-indexing in the usual way. Weibel proves that if $X = Spec(A)$ is affine, then the natural maps $HC_n(A) \to HC_n(X)$ are isomorphisms.

2.2.3 Category level

Let $Ch_{perf}(X)$ be the category of perfect complexes on X and let $Ac(X) \subset Ch_{perf}(X)$ be the full dg-subcategory of acyclic complexes. Considering $Ch_{perf}(X) = (Ch_{perf}(X), Ac(X))$ as a localization pair over k, Keller defines $C(X)$ to be the mixed complex (over k) associated to $Ch_{perf}(X)$ in [24,25]. To be more precise, $C(X)$ is the cone of $C(Ac(X)) \to C(Ch_{perf}(X))$.

We define $HC(X)$, $HP(X)$, $HN(X)$ to be the cyclic, periodic cyclic, negative cyclic homology complexes associated with the mixed complex $C(X)$ as before:

$$HC(C(X), b, B) = Tot(\cdots \to C(X)[+1] \xrightarrow{B} C(X) \to 0 \to 0 \to \dots)$$

$$HP(C(X), b, B) = Tot(\cdots \to C(X)[+1] \xrightarrow{B} C(X) \xrightarrow{B} C(X)[-1] \xrightarrow{B}\to \dots)$$

$$HN(C(X), b, B) = Tot(\cdots \to 0 \to C(X) \xrightarrow{B} C(X)[-1] \xrightarrow{B} C(X)[-2] \dots)$$

where $C(X)$ is placed in horizontal degree 0.

In particular, HC, HP and HN are presheaves of complexes on Sch/k. Keller's definitions agree with the definitions given by Weibel, with $HC_n(X) = H^{-n}HC(X)$, etc. If it is necessary, we can associate presheaves of spectra $\mathcal{HC}$,$\mathcal{HP}$ and $\mathcal{HN}$with these presheaves of complexes by applying Eilenber-MacLane functor respectively.

We collect the following theorems which will be used later.

Theorem 2.2.1. Cyclic homology and its variants satisfy the usual projective space bundle formula, for example

$$\mathcal{HN}(\mathbb{P}^j_X) = \bigoplus_0^{j-1} \mathcal{HN}(X).$$

Theorem 2.2.2. Combining the Mayer-Vietoris property with projective space bundle formula (for $j = 1$) yields the fundamental theorem for negative cyclic homology, which states that there is a short exact sequence,

$$0 \to \mathcal{HN}(X \times A^1) \bigcup_{\mathcal{HN}(X)} \mathcal{HN}(X \times A^1) \to \mathcal{HN}(X \times (A^1 - 0)) \to \mathcal{HN}(X)[1] \to 0.$$

In fact,this sequence is split up to homotopy.

Finally, we state the following fact(I learned this from M.Schlichting).

Theorem 2.2.3. Any cohomology theory on DG-category satisfying

1.) invariance under derived equivalence.

2.) localization sequence.

satisfies étale excision and projective bundle formula. For example, $\mathcal{K}$ and $\mathcal{HN}$.

2.3 Descent Property

2.3.1 cd-structure and cd-topology

In this section, following Voevodsky[41,42], Cortiñas-Haesemeyer-Schlichting-Weibel[5], Cortiñas-Haesemeyer-Weibel[4] and Cortiñas-Haesemeyer-Walker-Weibel[7], we give a brief discussion on descent property. These results are already known. The readers can accept theorem[2.35] and move to next section. We do not really need too much descent property in this paper. I am writing this section because of independent interest in descent, especially *cdh*-descent.

Let's recall the following definitions of *cd*-structure and *cd*-topology given by Voevodsky in [42].

Definition 2.3.1. *cd*-structure.

A *cd*-structure P on a small category C with initial object is a class of commutative squares in C that is closed under isomorphism. We call these commutative squares distinguished squares.

For example, let C be the category of open subsets of a Noetherian topological space, the set of squares of the form

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

is a *cd*-structure.

We are interested in the following *cd*-structures on the category Sch/F, where F is a field with characteristic 0.

Definition 2.3.2. Upper *cd*-structure(or Nisnevich *cd*-structure).

Upper *cd*-structure is a *cd*-structure consisting of Nisnevich squares. Here a Nisnevich square means a square of the form:

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{e} & X \end{array}$$

such that it is a pull-back square and p is étale, e is an open embedding and $p^{-1}(X - e(A)) \to X - e(A)$ is an isomorphism. Here $X - e(A)$ is considered with the reduced scheme structure.

Definition 2.3.3. Lower *cd*-structure(or proper *cd*-structure).

Lower *cd*-structure is a *cd*-structure which consists of abstract blow-up squares. Here an abstract blow-up square means a square of the form:

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{e} & X \end{array}$$

such that it is a pull-back square and p is proper, e is a closed embedding and $p^{-1}(X - e(A)) \to X - e(A)$ is an isomorphism.

Definition 2.3.4. Combined *cd*-structure.

Combined *cd*-structure consists of Nisnevich squares and abstract blow-up squares.

Definition 2.3.5. Combined *cd*-structure on Sm/F.

Combined *cd*-structure on Sm/F consists of Nisnevich squares and abstract blow-up squares of smooth schemes isomorphic to a blow-up of a smooth scheme along a smooth center.

Definition 2.3.6. Plain Lower *cd*-structure.

Plain Lower *cd*-structure is a *cd*-structure consists of the following distinguished squares:

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{e} & X, \end{array}$$

where both p and e are open embedding and $X = p(Y) \cup e(A)$.

A *cd*-structure defines a topology as follows.

Definition 2.3.7. *cd*-topology.

The *cd*-topology associated with a *cd*-structure P is defined as the Grothendieck topology generated by the following two kinds of coverings:

1. the empty covering is 0.

2. coverings of the form $\{A \to X, Y \to X\}$, where the morphisms $A \to X$ and $Y \to X$ are sides of an element of the *cd*-structure P of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X. \end{array}$$

The *cd*-topology associated with the *cd*-structure in the above example is the canonical topology on the category of open subsets of a Noetherian topological space.

Moreover, we have the following fact.

Theorem 2.3.8. 1. The *cd*-topology associated with Plain Lower *cd*-structure on the category Sch/F is the Zariski topology.

2. The *cd*-topology associated with Upper *cd*-structure on the category Sch/F is the Nisnevich topology.

3. The *cd*-topology associated with combined *cd*-structure on the category Sch/F is the *cdh*-topology.

4. The *cd*-topology associated with combined *cd*-structure on the category Sm/F is the *scdh*-topology. The *scdh*-topology coincides with the restriction of the *cdh*-topology to Sm/F.

In the following, we will focus on a *cd*-structure which is *complete*, *regular* and *bounded*. For precise definitions of *complete*, *regular* and *bounded*, we refer the readers to [41] by Voevodsky.

We need the following fact for our discussion:

Theorem 2.3.9. Zariski topology, Nisnevich topology, *cdh*-topology and *scdh*-topology are generated by *complete*, *regular* and *bounded* *cd*-structure.

We note that one useful result for these three topologies is that the cohomological dimension of a sheaf on a scheme X is at most equal to the dimension of X. This is very useful for arguments involving degeneration of spectral sequences.

2.3.2 Descent property

Now suppose $\mathcal{F}$ is a presheaf of spectra (or simplicial sets) on the category Sch/F, where F is a field of characteristic 0. We write $\mathbb{H}(-, \mathcal{F})$ for the Thomason's sheaf of hypercohomology spectra

$$U \to \mathbb{H}_{Zar}(U, \mathcal{F}).$$

Definition 2.3.10. $\mathcal{F}$ (as above) is said quasi-fibrant if $\mathcal{F}(U) \to \mathbb{H}(U, \mathcal{F})$ is a weak equivalence for all U in Sch/F.

Jardine showed that $\mathbb{H}(-,\mathcal{F})$ is the fibrant replacement for $\mathcal{F}$ in the local injective model structure. So we can generalize the above definition as follows, as Cortinas-Haesemeyer-Schlichting-Weibel did in [5,deinition 3.3].

Definition 2.3.11. Suppose $\mathcal{F}$ is a presheaf of spectra (or simplicial sets) on a site C, and write $\mathbb{H}(-,\mathcal{F})$ for its local injective fibrant replacement. $\mathcal{F}$ is said quasi-fibrant if $\mathcal{F}(U) \to \mathbb{H}(U,\mathcal{F})$ is a weak equivalence for each U in the site C.

We also need to definie MV-property, short for Mayer-Vietoris property.

Definition 2.3.12. Let C be a category with a *cd*-structure P. A presheaf of spectra (or simplicial sets) $\mathcal{F}$ on the category C satisfies MV-property, if, for any square $Q \in P$, the square of spectra $\mathcal{F}(Q)$ is a homotopy cartesian.

Let C be a category with a *cd*-structure P. A presheaf of cochain complex $A^\bullet$ on the category C satisfies MV-property, if, for any square $Q \in P$, the square of complexes $A^\bullet(Q)$ is a homotopy cartesian.

Established on the results of [41] by Voevodsky, Cortinas-Haesemeyer-Schlichting-Weibel proved the following theorem in [5].

Theorem 2.3.13. Let C be a category with a *complete regular bounded cd-structure* P. Then a presheaf of spectra (or simplicial sets) $\mathcal{F}$ is quasi-fibrant with respect to the topology induced by P if and only if $\mathcal{F}$ satisfies the MV-property for P.

Now we can give the definition of descent.

Definition 2.3.14. If a presheaf of spectra (or simplicial sets) $\mathcal{F}$ satisfies the above equivalent conditions in the above theorem, then we say $\mathcal{F}$ satisfies t-descent, where t is the topology generated by the *complete regular bounded cd*-structure P.

Keeping negative cyclic homology(complex) in mind, we also need the variants of the above definitions and theorem.

Definition 2.3.15. Let C be a category with a *cd*-structure P, $A^\bullet$ be a presheaf of cochain complex on C. $A^\bullet$ is said to be quasi-fibrant for the topology generated by *cd*-structure P provived the natural map

$$A^\bullet(U) \to R\Gamma(U, A^\bullet)$$

is an quasi-isomorphism for each object U in C.

For example, any cochain complex of flasque sheaves is quasi-fibrant.

The following result is a variant of the above theorem.

Theorem 2.3.16. Let C be a category with a *complete regular bounded cd-structure* P. Then a complex of presheaves $A^\bullet$ is quasi-fibrant with respect to the topology induced by P if and only if it satisfies the MV-property for P.

Definition 2.3.17. If a complex of presheaves $A^\bullet$ satisfies the above equivalent conditions in the above theorem, then we say $A^\bullet$ satisfies t-descent, where t is the topology generated by the *complete regular bounded cd*-structure P.

Now we can state the main theorem which will be used in the following.

Theorem 2.3.18. K-theory spectra $\mathcal{K}$ and negative cyclic homology $\mathcal{HN}$ satisfy Nisnevich descent. In particular, they satisfy Zariski descent.

This theorem tells us that

$$\mathcal{K}(X) \simeq \mathbb{H}_{Zar}(X, \mathcal{K}(O_X))$$

and

$$\mathcal{HN}(X) \simeq \mathbb{H}_{Zar}(X, \mathcal{HN}(O_X)),$$

or more generally, with support(still in Zariski topology),

$$\mathcal{K}(X \text{ } on \text{ } Y) \simeq \mathbb{H}_Y(X, \mathcal{K}(O_X))$$

and

$$\mathcal{HN}(X \text{ } on \text{ } Y) \simeq \mathbb{H}_Y(X, \mathcal{HN}(O_X)).$$

We will use them frequently in the following chapters. For example, we will compute the relative negative cyclic homology with supports by using hypercohomogy in section 4.3

2.3.3 cdh-descent and scdh-descent

In this section, let's focus on *cdh*-descent and *scdh*-descent. We write a for the change-of-topology morphism

$$a : (Sch/F)_{cdh} \to (Sch/F)_{Zar}.$$

If A is a Zariski sheaf, we will write A_{cdh} or a^*A for the *cdh*-sheafification of A.

If $A^\bullet$ is a complex of presheaves of abelian groups on $(Sch/F)_{Zar}$, then we write $\mathbb{H}_{cdh}(A^\bullet)$ for a *cdh*-fibrant replacement of $A^\bullet$. The hypercohomology $\mathbb{H}^n_{cdh}(X, A^\bullet)$ is $H^n\mathbb{H}_{cdh}(X, A^\bullet)$, where $\mathbb{H}_{cdh}(X, A^\bullet)$ is the sections of $\mathbb{H}_{cdh}(A^\bullet)$.

In particular, a presheaf A can be considered as a presheaf of complex concentrated at degree 0. The $\mathbb{H}_{cdh}(A)$ is an injective resolution of A_{cdh}. Then $\mathbb{H}^n_{cdh}(X, A)$ is the cohomology $H^n_{cdh}(X, A_{cdh})$.

In general, when $A^\bullet$ is unbounded complex, then we can use Cartan-Eilenberg flasque resolution to construct $\mathbb{H}_{cdh}(A^\bullet)$. (A fact of Suslin and Voevodsky in [34] says the columns of the Cartan-Eilenberg double complex are locally cohomologically bounded).

Now let's recall the following fact about *cdh* site.

Theorem 2.3.19. The *cdh* site is Noetherian, that is, every covering has a finite subcovering, so *cdh* cohomolgy commutes with filtered direct limits of sheaves.

Corollary 2.3.20. If M is a sheaf of F-modules and V a vector space, then

$$H^n_{cdh}(X, V \otimes_F M) \cong H^n_{cdh}(X, M) \otimes_F V.$$

The following fact says that the *cdh* fibrant replacement of Hochschild, cyclic, negative cyclic and period cyclic homology admit natural decompositions.

Theorem 2.3.21. Let X be a scheme over F, where F is a field with characteristic 0. We let H denote any of HH, HC, HN or HP and $H^{(i)}$ for the $i - th$ eigen-component of the lambda decomposition of H. Then

$$\mathbb{H}_{cdh}(H) \cong \prod \mathbb{H}_{cdh}(H^{(i)}).$$

Now we state some examples satisfying *cdh*-descent and *scdh*-descent. A fact about *scdh*-descent firstly.

Theorem 2.3.22. A presheaf of spectra (or complexes) $\mathcal{F}$ on Sm/F satisfies *scdh*-descent if and only if $\mathcal{F}$ satisfies Nisnevich descent and MV-property for smooth blow-up squares.

Theorem 2.3.23. Since K-theory spectra $\mathcal{K}$ and negative cyclic homology $\mathcal{HN}$ satisfy Nisnevich descent and MV-property for smooth blow-up squares, both of them satisfy *scdh*-descent.

Now, we come to the following fact focusing on smooth schemes.

Theorem 2.3.24. H and $H^{(i)}$ satisfy *scdh*-descent. In particular, when X is smooth over F, we have

$$H^{(i)}(X) \cong \mathbb{H}_{cdh}(X, H^{(i)}).$$

Theorem 2.3.25. When X is smooth over F, for all p and i, we have

$$H^p_{Zar}(X, \Omega^i_{X/F}) \cong H^p_{cdh}(X, \Omega^i_{X/F}).$$

In particular, lettting $i = 0$, we have

$$H^p_{Zar}(X, O) \cong H^p_{cdh}(X, O).$$

Letting $p = 0$, we have

$$\Omega^i_{X/F} = H^0_{cdh}(X, \Omega^i_{X/F}).$$

The following theorem provides 3 examples satisfying *cdh*-descent.

Theorem 2.3.26. 1. Homotopy K-theory KH satisfies *cdh*-descent on the category Sch/F, where F is a field of characteristic 0.

2. Periodic cyclic homology satisfies *cdh*-descent on the category Sch/F, where F is a field of characteristic 0.

3. Singular cohomology satisfies *cdh*-descent on the category $Sch/\mathbb{C}$.

Cortinas defines the infinitesimal K-theory, $\mathcal{K}^{inf}$, as the homotopy fiber of the Chern character

$$Ch : \mathcal{K} \to \mathcal{HN}.$$

Cortinas-Haesemeyer-Schlichting-Weibel proved that in [5]

Theorem 2.3.27. $\mathcal{K}^{inf}$ satisfies *cdh*-descent on the category Sch/F, where F is a field of characteristic 0.

This result plays an important role for their proof of K-dimension conjecture for Noetherian scheme over a field F of characteristic 0. We won't use this result in this paper. However, we will use the Chern character to prove the existence of tangent maps.

Finally, we would like to say something about a scheme which is not necessarliy smooth.

Theorem 2.3.28. If X is a d-dimensional scheme, essentially of finite type over F, where F is a field of characteristic 0. then we have

$$H^d_{Zar}(X, \Omega^i_{X/F}) \to H^d_{cdh}(X, \Omega^i_{X/F})$$

is surjective. If X is affine and $d > 0$, then $H^d_{cdh}(X, \Omega^i_{X/F}) = 0$.

In particular, letting $i = 0$, we have

$$H^d_{Zar}(X, O) \to H^i_{cdh}(X, O)$$

is surjective. If X is affine and $d > 0$, then $H^d_{cdh}(X, O) = 0$.

Chapter 3
Effacement Theorem And Chern Character

In this chapter, we discuss effeacement theorem and Chern character which are the first two ingredients for proving our main result.

In section 3.1, we discuss effeacement theorem, mainly following [8]. That is, we consider $\mathcal{K}$ and $\mathcal{HN}$ as "cohomology theories with support" in the sense of [8]. Both $\mathcal{K}$ and $\mathcal{HN}$ are effaceable functors, since they satisfy étale excision and projective bundle formula. Please see theorem 2.17 for a more general statement. I learned this from Professor Schlichting.

In section 3.2, I write out explicitly the existence of Chern character from K-theory spectra $\mathcal{K}$ to negative cyclic homology spectra $\mathcal{HN}$, following [5]. No orginality is claimed. This Chern character induces maps from the coniveau spectral sequence associated to $\mathcal{K}$ to the coniveau spectral sequence associated to $\mathcal{HN}$. The main results of this section are theorems 3.13, 3.14 and 3,15.

3.1 Effacement theorem

Now, we would like to discuss the effacement theorem which enables one to prove the Bloch-Ogus theorem. The following background is from [8]. The interested readers can check more detail of the effacement theorem in [8].

As an important result in algebraic geometry, the Bloch-Ogus theorem, briefly described, is as follows. Given a smooth algebraic variety X and a cohomology theory h^* satisfying étale excision and "Key lemma", filtration by codimension of support yields Cousin complexes which form the E_1-terms of the coniveau spectral sequence converging to $h^*(X)$. Restriction of the Cousin complexes to the open subsets of X defines complexes of flasque Zariski sheaves. The Bloch-Ogus theorem says that these complexes of sheaves are acyclic, except in degree 0 where their cohomology is the Zariski sheaf $\mathcal{H}^*$ associated to the presheaf $U \to h^*(U)$. This identifies the E_2-term of the coniveau spectral sequence to $H^*(X, \mathcal{H}^*)$.

Bloch-Ogus reduce their theorem to proving the "effacement theorem"(see below for precise statement) which is proved by using a geometric presentation lemma. Later, Gabber gave a different proof of effacement theorem for étale cohomology by essentially using the section at infinity (coming from an embedding of the affine line into the projective line) as well as a computation of the cohomology of the projective line.

Gabber's proof gives us more. In [8], Colliot-Thélène, Hoobler and Kahn axiomatize Gabber's argument and show that Gabber's argument applies to any "Cohomology theory with support" which satisfies étale excision and "Key lemma". The latter follows either from homotopy invariance or from projective bundle formula. For a list of such cohomology theory with support, we refer the readers to [8].

Now we adopt Colliot-Thélène, Hoobler and Kahn's discussion to our setting. The following expression is essentially following the lecture given by M.Schlichting.

Let's begin by defining a "cohomology theory with support" to a pair (X, Z), where Z is closed in a scheme X.

Definition 3.1.1. Let $\mathcal{A}$ be a functor from the category Sch^{op}/k to spectra or chain complexes:(The readers can take $\mathcal{A}$ to be K-theory for spectra and (negative)cyclic homology for complexes)

$$\mathcal{A} : Sch^{op}/k \to spectra$$

or

$$\mathcal{A} : Sch^{op}/k \to chain\ complexes$$

then we can extend $\mathcal{A}$ to a pair (X, Z), where Z is closed in X as follows.

For $\mathcal{A}$ spectrum-valued, $\mathcal{A}(X \ on \ Z)$ is defined as the homotopy fiber of $\mathcal{A}(X) \to \mathcal{A}(X-Z)$

$$\mathcal{A}(X \ on \ Z) \to \mathcal{A}(X) \to \mathcal{A}(X-Z)$$

and $\mathcal{A}^q(X \ on \ Z)$ is defined as $\pi_{-q}(\mathcal{A}(X \ on \ Z))$.

For $\mathcal{A}$ chain complexes-valued, if we write $C^\bullet$ to be the cone of $\mathcal{A}(X) \to \mathcal{A}(X-Z)$, $\mathcal{A}(X \ on \ Z)$ is defined as $C^\bullet[-1]$

$$\mathcal{A}(X \ on \ Z) \to \mathcal{A}(X) \to \mathcal{A}(X-Z)$$

and $\mathcal{A}^q(X \ on \ Z)$ is defined as $H_{-q}(\mathcal{A}(X \ on \ Z))$.

One can check that the above definition does define a "cohomology theory with support" in the sense of [8]. The naturality can be verified by Octahedral axioms.

Definition 3.1.2. Étale excision

The functor $\mathcal{A}$ is said to satisfy étale excision if for any given diagram:

$$\begin{array}{ccc} Z & \longrightarrow & X' \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{j} & X \end{array}$$

where $j : Z \to X$ is the closed immersion and f is étale, the pullback

$$f^* : \mathcal{A}^q(X \ on \ Z) \xrightarrow{\simeq} \mathcal{A}^q(X' \ on \ Z)$$

is an isomorphism for any integer q.

Definition 3.1.3. Zariski excision

The functor $\mathcal{A}$ is said to satisfy Zariski excision if the pullback

$$f^* : \mathcal{A}^q(X \ on \ Z) \xrightarrow{\simeq} \mathcal{A}^q(X' \ on \ Z)$$

is an isomorphism for any integer q, when f runs over all open immersions.

Definition 3.1.4. Projective bundle formula for $\mathbb{P}^1$

Let X be any scheme over k and $\mathbb{P}^1_X$ be the projective line over X. We write π for the natural projection

$$\pi : \mathbb{P}^1_X \to X.$$

The functor $\mathcal{A}$ is said to satisfy projective bundle formula for $\mathbb{P}^1$ if

$$(\pi^*, O_{\mathbb{P}^1}(-1) \otimes \pi^*) : \mathcal{A}^q(X) \oplus \mathcal{A}^q(X) \xrightarrow{\simeq} \mathcal{A}^q(\mathbb{P}^1_X)$$

is an isomorphism for any integer q.

Now, let's recall a fact.

Theorem 3.1.5. Let X be a noetherian scheme with finite dimension and $\mathcal{A}$ is a functor as above. If $\mathcal{A}$ satisfies $\mathcal{A}^q(\emptyset) = 0$ and Zariski excision. then there exist two strongly convergent spectral sequences:

1.Brown-Gersten spectral sequence (or Descent spectral sequence)

$$E_2^{p,q} = H^p_{Zar}(X, \underline{\mathcal{A}^q}) \Longrightarrow \mathcal{A}^{p+q}(X).$$

2.Coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p} \mathcal{A}^{p+q}(X \ on \ x) \Longrightarrow \mathcal{A}^{p+q}(X)$$

where $\mathcal{A}^{p+q}(X)(X \ on \ x) = \varinjlim_{x \in U} \mathcal{A}^{p+q}(U \ on \ \{x\}^- \cap U)$.

We explain a little bit about coniveau spectral sequence. Let

$$Z^\bullet : \ Z^d \subset Z^{d-1} \subset \cdots \subset Z^0 = X$$

be a chain of closed subsets of X, where $codim_X(Z^p) \geq p$. For a pair$(Z^{p+1} \subset Z^p)$, the homotopy fibration

$$\mathcal{A}(X \ on \ Z^{p+1}) \to \mathcal{A}(X \ on \ Z^p) \to \mathcal{A}(X - Z^{p+1} \ on \ Z^p - Z^{p+1})$$

induces a long exact sequence:

$$\cdots \to \mathcal{A}^q(X\ on\ Z^{p+1}) \to \mathcal{A}^q(X\ on\ Z^p) \to \mathcal{A}^q(X - Z^{p+1}\ on\ Z^p - Z^{p+1}) \to \mathcal{A}^{q+1}(X\ on\ Z^{p+1}) \to$$

We can construct an exact couple from the above by setting $D^{p,q} = \mathcal{A}^{p+q}(X\ on\ Z^p)$ and $E^{p,q} = \mathcal{A}^{p+q}(X - Z^{p+1}\ on\ Z^p - Z^{p+1})$.

Order the set of $(d+1)$-tuples $Z^\bullet$ by $Z^\bullet \leq Z'^\bullet$ if $Z^p \subset Z'^p$ for all p. Passing to direct limit, we get a new exact couple with (a direct limit of exact couples is still an exact couple)

$$D_1^{p,q} = \varinjlim_{Z^\bullet} \mathcal{A}^{p+q}(X\ on\ Z^p)$$

$$E_1^{p,q} = \varinjlim_{Z^\bullet} \mathcal{A}^{p+q}(X - Z^{p+1}\ on\ Z^p - Z^{p+1}) = \bigoplus_{x \in X^{(p)}} \mathcal{A}^{p+q}(X\ on\ x)$$

where $X^{(p)}$ denotes the set of points of codimension p in X and

$$\mathcal{A}^{p+q}(X)(X\ on\ x) = \varinjlim_{x \in U} \mathcal{A}^{p+q}(U\ on\ \{x\}^- \cap U).$$

.

The spectral sequence associated to this new exact couple is the coniveau spectral sequence:

$$E_1^{p,q} = \bigoplus_{x \in X^p} \mathcal{A}^{p+q}(X\ on\ x) \Longrightarrow \mathcal{A}^{p+q}(X).$$

Its E_1-terms give rise to Cousin complexes:

$$0 \to \bigoplus_{x \in X^{(0)}} \mathcal{A}^q(X\ on\ x) \xrightarrow{d_1^{0,q}} \bigoplus_{x \in X^{(1)}} \mathcal{A}^{q+1}(X\ on\ x) \xrightarrow{d_1^{1,q}} \bigoplus_{x \in X^{(2)}} \mathcal{A}^{q+2}(X\ on\ x) \to \dots$$

We have the following Gersten sequence:

$$0 \to D_1^{0,q} \to E_1^{0,q} \to E_1^{1,q} \to E_1^{2,q} \to \dots$$

which means

$$0 \to \mathcal{A}^q(X) \to \bigoplus_{x \in X^{(0)}} \mathcal{A}^q(X\ on\ x) \to \bigoplus_{x \in X^{(1)}} \mathcal{A}^{q+1}(X\ on\ x) \to \bigoplus_{x \in X^{(2)}} \mathcal{A}^{q+2}(X\ on\ x) \to \dots$$

We are interested in when the sheafified Gersten sequence

$$0 \to \mathcal{A}^q(X) \to \bigoplus_{x\in X^{(0)}} \underline{\mathcal{A}}^q(X\ on\ x) \to \bigoplus_{x\in X^{(1)}} \underline{\mathcal{A}}^{q+1}(X\ on\ x) \to \bigoplus_{x\in X^{(2)}} \underline{\mathcal{A}}^{q+2}(X\ on\ x) \to \dots$$

is exact, where $\bigoplus_{x\in X^{(p)}} \underline{\mathcal{A}}^n(X\ on\ x)$ is the sheaf associated to the presheaf, for all n and p,

$$U \to \bigoplus_{x\in U^{(p)}} \mathcal{A}^n(U\ on\ x).$$

In other words, $\bigoplus_{x\in X^{(p)}} \underline{\mathcal{A}}^n(X\ on\ x)$ is the flasque sheaf $\bigoplus_{x\in X^{(p)}} j_{x*}\mathcal{A}^n(X\ on\ x)$, where j_x is the immersion $x \to X$.

The following Effacement theorem tells us when the sheafified Gersten sequence is exact.

Definition 3.1.6. let X be a scheme over k and t is a point in X. A functor

$$\mathcal{A} :\ Sch^{op}/k \to Abelian\ groups$$

is called effaceable at (X, t), if the following conditions are satisfied:

Given any integer $p \geq 0$, a closed subvariety Z of codimension at least $p+1$, and $t \in Z$, there exists an open neighbourhood $U$of t, $U \subset X$, and a closed subset $Z^{'}$ containing Z such that

$$codim_X(Z^{'}) \geq p$$

and

$$\mathcal{A}^n(U\ on\ Z \cap U) \xrightarrow{0} \mathcal{A}^n(U\ on\ Z^{'} \cap U)$$

for all $n \geq 0$.

Now we state the effacement theorem due to Gabber.

Theorem 3.1.7. Effacement theorem

Let $\mathcal{A}$ be a functor from the category Sch^{op}/k , k is an infinite field, to spectra or chain complexes:

$$\mathcal{A} : Sch^{op}/k \to spectra$$

or

$$\mathcal{A} : Sch^{op}/k \to chain\ complexes$$

If $\mathcal{A}$ satisfies étale excision and projective bundle formula, then for any given q, $\mathcal{A}^q$ is effaceable at (X, t), where X/k is smooth at t.

And we have the following corollary due to the definition of effaceable functor.

Corollary 3.1.8. If $\mathcal{A}^n(X)$ is effaceable at (X, t), for arbitrary $t \in X$, then

$$a_{Zar} D_1^{p,q} \xrightarrow{i=0} a_{Zar} D_1^{p-1,q+1}$$

for any $p \geq 0$, where $a_{Zar} D_1^{p,q}$ is the Zariski sheafification of $D_1^{p,q}$. Hence, the sheafified Gersten complexes are exact because of the following fact.

Theorem 3.1.9. The following statements are equivalent:

1.The sheafified Gersten sequence is exact.

2.For any $p \geq 0$

$$a_{Zar} D_1^{p,q} \xrightarrow{i=0} a_{Zar} D_1^{p-1,q+1}.$$

3.

$$\varinjlim_{(Z^{p+1} \subset Z^p) \leq (X^{p+1} \subset X^p)} \mathcal{A}^{p+q}(X\ on\ Z^{p+1}) \xrightarrow{i=0} \varinjlim_{(Z^{p+1} \subset Z^p) \leq (X^{p+1} \subset X^p)} \mathcal{A}^{p+q}(X\ on\ Z^p).$$

Corollary 3.1.10. If the sheafified Gersten sequence is exact, then the E_2-term of he coniveau spectral sequence is

$$E_2^{p,q} = H^p(E^{\bullet,q}, d) = H^p_{Zar}(X, \underline{\mathcal{A}^q})$$

We recall that

$$E_2^{p,q} = H^p(E^{\bullet,q}, d) = \frac{ker(d_1^{p,q})}{Im(d_1^{p-1,q})}.$$

This corollary tells us that,under the above hypothesis, the E_2 pages of coniveau spectral sequence agree with those of Brown-Gersten spectral sequence.

Corollary 3.1.11. Universal exactness

Let $\mathcal{A}$ be a spectrum valued or complexes valued functor in above. For arbitrary scheme $T/k(T$ might be singular), we can define a new functor $\mathcal{A}_{\mathcal{T}}$

$$X \to \mathcal{A}(X \times T).$$

If $\mathcal{A}$ satisfies étale excision and projective bundle formula, then so does the new functor $\mathcal{A}(X \times T)$. This means that if X is smooth, then $\mathcal{A}_{\mathcal{T}}$ is effaceable.

3.2 Chern character

Now, we discuss the existence of Chern character at spectrum level. Recall that X is a smooth projective variety over k, where k is a field with characteristic 0. By applying Eilenberg- Maclane functor which sends complexes to spectrum, we get a spectrum associated to negative cyclic homology complex. Let's still call it $\mathcal{HN}$.

The following fact is pointed out in [5]. We sketch the proof following [5] because of its importance in our approach. As we will see later, the tangent maps are induced from Chern Character.

Theorem 3.2.1. There exists a Chern character from $\mathcal{K}$ spectrum to $\mathcal{HN}$ spectrum,

$$Ch : \mathcal{K} \to \mathcal{HN}.$$

Proof. We will prove it in 3 steps, following Cortinas-Haesemeyer-Schlichting-Weibel. We begin with affine case, extend to negative range by using fundamental exact sequence and then globalize it by using Zariski descent.

Step1. Let's assume $X = specA$ firstly, where A is any commutative ring. If $C_\bullet$ is a chain complexes, let $0\backslash C_\bullet$ denote the truncation:

$$\begin{cases} 0 & n < 0 \\ Ker(C_0 \to C_{-1}) & n = 0 \\ C_n & n > 0 \end{cases}$$

We have the following composition sending chain complexes to spaces:

$$chain\ complexes \xrightarrow{0\backslash} Ch_{\geq 0} \xrightarrow{Dold-Kan} Simplicial\ abelian\ groups \xrightarrow{|\bullet|} Spaces$$

where $|\bullet|$ stands for the realization.

We write $X(C_\bullet)$ for $\mid DK(0\backslash C_\bullet) \mid$, then $H_n(C_\bullet) = \pi_n X(C_\bullet)$, $n \geq 0$. In particular, $HN_n(A) = \pi_n X(tot(cyc^-(A,A)))$, where $cyc^-(A,A)$ is the negative cyclic complexes.

In[44], Weibel proved that the classical Chern character

$$Ch : K_n(A) \to HN_n(A), n > 0.$$

is obtained by applying π_n to the corresponding spaces:

$$Ch : BGL^+(A) \to X(tot(cyc^-(A,A))).$$

In order to extend the Chern character to spectrum level, we reformulate $BGL^+(A)$ and $tot(cyc^-(A,A))$ as follows:

$$BGL^+(A) = \Omega BQ(Perf(A)) = \Omega \mid wS_\bullet(Perf(A)) \mid$$

and

$$tot(cyc^-(A,A)) = tot\ C(Perf(A)),$$

where $C(Perf(A))$ is the complex associted to the category $Perf(A)$, constructed by Keller. Then we can write Chern character as:

$$Ch : \Omega \mid wS_\bullet(Perf(A)) \mid \to X(totC(Perf(A))).$$

Since both $\Omega \mid wS_\bullet(Perf(A)) \mid$ and $X(totC(Perf(A)))$ are functorial on $Perf(A)$, the above Chern character is also functorial on $Perf(A)$.

Applying Waldhausen's $S_\bullet$-construction, we have

$$Ch : \Omega \mid wS^2_\bullet(Perf(A)) \mid \to X(totC(S_\bullet Perf(A))).$$

A theorem of Keller says

$$totC(S_\bullet Perf(A)) = C(Perf(A))[1],$$

therefore,

$$X(totC(S_\bullet Perf(A))) = \Omega^{-1}X(totC(Perf(A))).$$

This means $X(totC(S_\bullet Perf(A))$ is really the delooping of $X(totC(Perf(A)))$.

In the following, we write $K(A)$ and $HN(A)$ for $BGL^+(A)$ and $X(tot(cyc^-(A,A)))$ respectively for simplicity. Let SA denotes the suspension ring of A, then we have

$$K(SA) = \Omega \mid wS_\bullet(Perf(SA)) \mid$$

and

$$HN(SA) = X(totC(S_\bullet Perf(A))).$$

The naturality of Chern character :

$$Ch : K \to HN.$$

says the following commutative diagram:

$$\begin{array}{ccc} K(A) & \xrightarrow{Ch} & HN(A) \\ \downarrow & & \downarrow \\ K(SA) & \xrightarrow{Ch} & HN(SA) \end{array}$$

Repeating this $S_\bullet$ construction, we can have a Chern character at the spectrum level(Chern character is a bonding map because of its naturality.) :

$$Ch : \mathcal{K}(X) \to \mathcal{HN}(X),$$

where $X = specA$, $\mathcal{K}(X)$ is the Waldhausen spectra

$$(\Omega \mid wS_{\bullet}(Perf(A)) \mid, \Omega \mid wS^2_{\bullet}(Perf(A)) \mid, \Omega \mid wS^3_{\bullet}(Perf(A)) \mid, \dots)$$

and $\mathcal{HN}(X)$ is the spectra

$$(X(totC(Perf(A))), X(totC(S_{\bullet}Perf(A)), X(totC(S^2_{\bullet}Perf(A)), \dots).$$

(Remark: The suspension ring construction can give us a non-connective K-spectra, i.e it can produce negative K-groups. However, for the negative cyclic homlogy, we can't see negative range in this step. Because the construction sending chain complexes to spaces involves Dold-Kan correspondence between non-negative complexes and Simplicial abelian groups. Then we come to the following step 2.)

Step2. Next, we can extend the Chern character to negative range by using fundamental exact sequences(they are splitting):

$$0 \to \mathcal{HN}(X \times A^1) \bigcup_{\mathcal{HN}(X)} \mathcal{HN}(X \times A^1) \to \mathcal{HN}(X \times (A^1 - 0)) \to \mathcal{HN}(X)[1] \to 0$$

and

$$0 \to \mathcal{K}(X \times A^1) \bigcup_{\mathcal{K}(X)} \mathcal{K}(X \times A^1) \to \mathcal{K}(X \times (A^1 - 0)) \to \mathcal{K}(X)[1] \to 0.$$

Step3. Lastly, we can use Zariski descent to globalize it. Since X is quasi-compact and seperated, it suffices to assume that $X = U \cup V$ with U and V being affine open schemes. We have the following square:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

Since $\mathcal{K}$ and $\mathcal{HN}$ satisfy Zariski excision, we have the following homotopy cartesian squares:

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}(V) \\ \downarrow & & \downarrow \\ \mathcal{K}(U) & \longrightarrow & \mathcal{K}(U \cap V) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{HN}(X) & \longrightarrow & \mathcal{HN}(V) \\ \downarrow & & \downarrow \\ \mathcal{HN}(U) & \longrightarrow & \mathcal{HN}(U \cap V) \end{array}$$

Therefore, there is a Chern character $\mathcal{K}(X) \to \mathcal{HN}(X)$ induced from Chern characters defined on affine ones.

□

Now we state the main theorem of this section.

Theorem 3.2.2. There exists the following commutative diagram between the sheafified Gersten sequences(m is any integer, both sequences are flasque resolutions):

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ HN_m(O_X) & \xleftarrow{Chern} & K_m(O_X) \\ \downarrow & & \downarrow \\ HN_m(k(X)) & \xleftarrow{Chern} & K_m(k(X)) \\ \downarrow & & \downarrow \\ \oplus_{d \in X^{(1)}} \underline{HN}_{m-1}(O_{X,d} \ on \ d) & \xleftarrow{Chern} & \oplus_{d \in X^{(1)}} \underline{K}_{m-1}(O_{X,d} \ on \ d) \\ \downarrow & & \downarrow \\ \oplus_{y \in X^{(2)}} \underline{HN}_{m-2}(O_{X,y} \ on \ y) & \xleftarrow{Chern} & \oplus_{y \in X^{(2)}} \underline{K}_{m-2}(O_{X,y} \ on \ y) \\ \downarrow & & \downarrow \\ \dots & \xleftarrow{Chern} & \dots \\ \downarrow & & \downarrow \\ \oplus_{x \in X^{(n)}} \underline{HN}_{m-n}(O_{X,x} \ on \ x) & \xleftarrow{Chern} & \oplus_{x \in X^{(n)}} \underline{K}_{m-n}(O_{X,x} \ on \ x) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Proof. The above Chern character at spectrum level induces maps on coniveau spectral sequences:

$$E_1^{p,q} = \bigoplus_{x \in X^p} \mathcal{K}^{p+q}(O_{X,x} \text{ } on \text{ } x) \longrightarrow E_1^{p,q} = \bigoplus_{x \in X^p} \mathcal{HN}^{p+q}(O_{X,x} \text{ } on \text{ } x),$$

where

$$\mathcal{K}^{p+q}(O_{X,x} \text{ } on \text{ } x) = \pi_{-p-q}(\mathcal{K}(O_{X,x} \text{ } on \text{ } x)) = K_{-p-q}(O_{X,x} \text{ } on \text{ } x)$$

and

$$\mathcal{HN}^{p+q}(O_{X,x} \text{ } on \text{ } x) = \pi_{-p-q}(\mathcal{HN}(O_{X,x} \text{ } on \text{ } x)) = HN_{-p-q}(O_{X,x} \text{ } on \text{ } x).$$

By the theorem 2.17, both the non-connective K-theory $\mathcal{K}$ and the negative cyclic homology $\mathcal{HN}$ satisfy étale excision and projective bundle formula. Thus, K^q and HN^q are effaceable, $\forall q$, for any smooth scheme X/k.

Combining with the Effacement theorem, we obtain the following commutative diagram between the sheafified Gersten sequences(for any integer m, both sequences are flasque resolutions):

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
HN_m(O_X) & \xleftarrow{Chern} & K_m(O_X) \\
\downarrow & & \downarrow \\
HN_m(k(X)) & \xleftarrow{Chern} & K_m(k(X)) \\
\downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{HN}_{m-1}(O_{X,d} \ on \ d) & \xleftarrow{Chern} & \oplus_{d\in X^{(1)}}\underline{K}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{HN}_{m-2}(O_{X,y} \ on \ y) & \xleftarrow{Chern} & \oplus_{y\in X^{(2)}}\underline{K}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{HN}_{m-n}(O_{X,x} \ on \ x) & \xleftarrow{Chern} & \oplus_{x\in X^{(n)}}\underline{K}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

□

We know that that for any integer q, $K^q(-\times T)$ and $HN^q(-\times T)$ are effaceable, for any $T/k(T$ might be singular). Hence, we also have the following result.

Theorem 3.2.3. There exists the following commutative diagram between the sheafified Gersten sequences(for any integer m, both sequences are flasque resolutions):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
HN_m(O_{X\times T}) & \xleftarrow{Chern} & K_m(O_{X\times T}) \\
\downarrow & & \downarrow \\
HN_m(k(X)\times T) & \xleftarrow{Chern} & K_m(k(X)\times T) \\
\downarrow & & \downarrow \\
\oplus_{d\times T\in X\times T^{(1)}}\underline{HN}_{m-1}(O_{X\times T,d\times T} \ on\ d\times T) & \xleftarrow{Chern} & \oplus_{d\times T\in X\times T^{(1)}}\underline{K}_{m-1}(O_{X\times T,d\times T} \ on\ d\times T) \\
\downarrow & & \downarrow \\
\oplus_{y\times T\in X\times T^{(2)}}\underline{HN}_{m-2}(O_{X\times T,y\times T} \ on\ y\times T) & \xleftarrow{Chern} & \oplus_{y\times T\in X\times T^{(2)}}\underline{K}_{m-2}(O_{X\times T,y\times T} \ on\ y\times T) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Chern} & \dots
\end{array}$$

For our purpose, we would like to take T to be the dual number, ie, $speck[\varepsilon]$.

Corollary 3.2.4. There exists the following commutative diagram between the exact sheafified Gersten sequences(for any integer m, both sequences are flasque resolutions):

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
HN_m(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_m(O_{X[\varepsilon]}) \\
\downarrow & & \downarrow \\
HN_m(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \xleftarrow{Chern} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow \\
\oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

Chapter 4
Lambda And Adams Operations

In this chapter, we discuss lambda and Adams operations for (negative) cyclic homology and K-theory. In 4.1, I do explicit computation on Adams' eigen-spaces of negative cyclic homology. In section 4.2, I recall lambda and Adams operaions on K-theory biefly and then extend Adams operations to negative K-groups by using a method of Weibel. In section 4.3, we prove Goodwillie-type and Cathelineau-type results by using a recent result of Cortiñas-Haesemeyer-Weibel[6].

4.1 Lambda and Adams operations on negative cyclic homology

In this subsection, all the variant of cyclic homology are taken over $\mathbb{Q}$. Let A be any commutative k-algebra, where k is a field of characteristic 0, and I be an ideal of A. We can associate a Hochschild complexes $C_*^h(A)$ to A as in section 2.2. The action of the symmetric groups on $C_*^h(A)$ gives the lambda operation

$$HH_n(A) = HH_n^{(1)}(A) \oplus \cdots \oplus HH_n^{(n)}(A),$$

and similarly

$$HC_n(A) = HC_n^{(1)}(A) \oplus \cdots \oplus HC_n^{(n)}(A),$$

$$HN_n(A) = HN_n^{(1)}(A) \oplus \cdots \oplus HN_n^{(n)}(A).$$

There is also a Hochschild complexes $C_*^h(A/I)$ associated to A/I. We use $C_*^h(A, I)$ to denote the kernel of the natural map

$$C_*^h(A) \to C_*^h(A/I).$$

Then the relative Hochschild module $HH_*(A, I)$ is the homology of the complex $C_*^h(A, I)$. Moreover, the action of the symmetric groups on $C_*^h(A, I)$ gives the lambda operation

$$HH_n(A, I) = HH_n^{(1)}(A, I) \oplus \cdots \oplus HH_n^{(n)}(A, I)$$

and similarly

$$HC_n(A, I) = HC_n^{(1)}(A, I) \oplus \cdots \oplus HC_n^{(n)}(A, I),$$

$$HN_n(A, I) = HN_n^{(1)}(A, I) \oplus \cdots \oplus HN_n^{(n)}(A, I).$$

Let's assume R is a regular noetherian ring and also a commutative $\mathbb{Q}$-algebra from now on, and ε is the dual number. We consider $R[\varepsilon] = R \oplus \varepsilon R$ as a graded $\mathbb{Q}$-algebra. The following SBI sequence is obtained from the corresponding eigen-piece of the relative Hochschild complex:

$$\to HC_{n+1}^{(i)}(R[\varepsilon], \varepsilon) \xrightarrow{S} HC_{n-1}^{(i-1)}(R[\varepsilon], \varepsilon) \xrightarrow{B} HH_n^{(i)}(R[\varepsilon], \varepsilon) \xrightarrow{I} HC_n^{(i)}(R[\varepsilon], \varepsilon) \to$$

According to a result of Geller-Weibel [10], the above S map is 0 on $HC(R[\varepsilon], \varepsilon)$. This enable us to break the SBI sequence up into short exact sequence:

$$0 \to HC_{n-1}^{(i-1)}(R[\varepsilon], \varepsilon) \xrightarrow{B} HH_n^{(i)}(R[\varepsilon], \varepsilon) \xrightarrow{I} HC_n^{(i)}(R[\varepsilon], \varepsilon) \to 0.$$

In the following, we will use this short exact sequence to compute $HC_n^{(i)}(R[\varepsilon], \varepsilon)$.

Theorem 4.1.1.

$$\begin{cases} HC_n^{(i)}(R[\varepsilon], \varepsilon) = \Omega_{R/\mathbb{Q}}^{2i-n}, for\ [\frac{n}{2}] \le i \le n. \\ HC_n^{(i)}(R[\varepsilon], \varepsilon) = 0, else. \end{cases} \tag{4.1.1}$$

Proof. Step 1. we will prove

$$HC_n^{(i)}(R[\varepsilon], \varepsilon) = 0, for\ i < \frac{n}{2}.$$

by showing $HH_n^{(i)}(R[\varepsilon], \varepsilon) = 0$. Noting that $HH_n^{(i)}(R) = 0$, it suffices to show $HH_n^{(i)}(R[\varepsilon]) = 0$, for $i < \frac{n}{2}$. By applying Kunneth formula to $R[\varepsilon] = R \otimes k[\varepsilon]$, we have

$$HH_n^{(i)}(R[\varepsilon]) = HH_0^{(0)}(R) \otimes HH_n^{(i)}(k[\varepsilon]) \oplus HH_1^{(1)}(R) \otimes HH_{n-1}^{(i-1)}(k[\varepsilon]) \oplus \cdots \oplus HH_i^{(i)}(R) \otimes HH_{n-i}^{(0)}(k[\varepsilon])$$

According to proposition 5.4.15 of [27], the only possibilities for $HH_{n-j}^{(i-j)}(k[\varepsilon])$ being nonzero are the followings:

$$\begin{cases} n-j \ is \ even, \ n-j = 2(i-j). \\ n-j \ is \ odd, \ n-j+1 = 2(i-j). \end{cases} \tag{4.1.2}$$

Neither of them will occur, since $i < \frac{n}{2}$. Therefore, $HH_n^{(i)}(R[\varepsilon]) = 0$.

Step 2. we will show that

$$HC_n^{(i)}(R[\varepsilon], \varepsilon) = \Omega_{R/\mathbb{Q}}^{2i-n}, for \ [\frac{n}{2}] \leq i < n.$$

by computing $HH_n^{(i)}(R[\varepsilon], \varepsilon)$ directly and using induction on $HC_{n-1}^{(i-1)}(R[\varepsilon], \varepsilon)$.

Firstly, we have $HH_n^{(i)}(R) = 0$ and $HH_n^{(i)}(R[\varepsilon])$ can be expressed as

$$HH_n^{(i)}(R[\varepsilon]) = HH_0^{(0)}(R) \otimes HH_n^{(i)}(k[\varepsilon]) \oplus HH_1^{(1)}(R) \otimes HH_{n-1}^{(i-1)}(k[\varepsilon]) \oplus \cdots \oplus HH_i^{(i)}(R) \otimes HH_{n-i}^{(0)}(k[\varepsilon]).$$

The only possiblities for $HH_{n-j}^{(i-j)}(k[\varepsilon])$ being nonzero are the followings:

$$\begin{cases} n-j \ is \ even, \ n-j = 2(i-j), \ then \ j = 2i-n. \\ n-j \ is \ odd, \ n-j+1 = 2(i-j), \ then \ j = 2i-n-1. \end{cases} \tag{4.1.3}$$

Therefore,

$$HH_n^{(i)}(R[\varepsilon]) = HH_{2i-n}^{(2i-n)}(R) \otimes HH_{2n-2i}^{(n-i)}(k[\varepsilon]) \oplus HH_{2i-n-1}^{(2i-n-1)}(R) \otimes HH_{2n-2i+1}^{(n-i+1)}(k[\varepsilon]).$$

$$HH_n^{(i)}(R[\varepsilon]) = \Omega_{R/\mathbb{Q}}^{2i-n} \oplus \Omega_{R/\mathbb{Q}}^{2i-n-1}.$$

By induction,

$$HC_{n-1}^{(i-1)}(R[\varepsilon], \varepsilon) = \Omega_{R/\mathbb{Q}}^{2i-n-1}.$$

thus,

$$HC_n^{(i)}(R[\varepsilon], \varepsilon) = \Omega_{R/\mathbb{Q}}^{2i-n}, for \ [\frac{n}{2}] \leq i < n.$$

Step 3. We prove the formula for $i = n$. It is known that

$$HH_n^{(n)}(R) = \Omega_{R/\mathbb{Q}}^n.$$

and

$HH_n^{(n)}(R[\varepsilon]) = HH_0^{(0)}(R) \otimes HH_n^{(n)}(k[\varepsilon]) \oplus HH_1^{(1)}(R) \otimes HH_{n-1}^{(n-1)}(k[\varepsilon]) \oplus \cdots \oplus HH_n^{(n)}(R) \otimes HH_0^{(0)}(k[\varepsilon])$.

Since $HH_i^{(i)}(k[\varepsilon]) = 0$,unless $i = 0, 1$, we have

$$HH_n^{(n)}(R[\varepsilon]) = HH_n^{(n)}(R) \otimes HH_0^{(0)}(k[\varepsilon]) \oplus HH_{n-1}^{(n-1)}(R) \otimes HH_1^{(1)}(k[\varepsilon]),$$

which can be simplified as

$$HH_n^{(n)}(R[\varepsilon]) = \Omega^n_{R/\mathbb{Q}} \otimes k[\varepsilon] \oplus \Omega^{n-1}_{R/\mathbb{Q}} \otimes k.$$

Therefore, we have

$$HH_n^{(n)}(R[\varepsilon], \varepsilon) = \Omega^n_{R/\mathbb{Q}} \oplus \Omega^{n-1}_{R/\mathbb{Q}}.$$

Once again, we still have

$$HC_n^{(n)}(R[\varepsilon], \varepsilon) = \Omega^n_{R/\mathbb{Q}}.$$

□

The above result tells us that

Theorem 4.1.2.

$$HC_n(R[\varepsilon], \varepsilon) = \Omega^n_{R/\mathbb{Q}} \oplus \Omega^{n-2}_{R/\mathbb{Q}} \oplus \dots$$

the last term is $\Omega^1_{R/\mathbb{Q}}$ or R, depending on n odd or even.

The following corollaries are obvious from the fact that for any commutative k-algebra A , where k is a field of characteristoc 0, and I be an ideal of A,

$$HN_n(A, I) = HC_{n-1}(A, I).$$

$$HN_n^{(i)}(A, I) = HC_{n-1}^{(i-1)}(A, I).$$

Corollary 4.1.3.

$$\begin{cases} HN_n^{(i)}(R[\varepsilon], \varepsilon) = \Omega^{2i-n-1}_{R/\mathbb{Q}}, for\ [\frac{n}{2}] < i \le n. \\ HN_n^{(i)}(R[\varepsilon], \varepsilon) = 0, else. \end{cases} \tag{4.1.4}$$

Corollary 4.1.4.

$$HN_n(R[\varepsilon],\varepsilon) = \Omega^{n-1}_{R/\mathbb{Q}} \oplus \Omega^{n-3}_{R/\mathbb{Q}} \oplus \dots$$

the last term is $\Omega^1_{R/\mathbb{Q}}$ or R, depending on n odd or even.

We can also generalize the above results to the sheaf level.

Theorem 4.1.5. Let X be a smooth scheme over a field k, $char k = 0$. we have the following

$$\begin{cases} HN^{(i)}_n(O_X[\varepsilon],\varepsilon) = \Omega^{2i-n-1}_{O_X/\mathbb{Q}}, for\ [\frac{n}{2}] < i \leq n. \\ HN^{(i)}_n(O_X[\varepsilon],\varepsilon) = 0, else. \end{cases} \tag{4.1.5}$$

It follows that

$$HN_n(O_X[\varepsilon],\varepsilon) = \Omega^{n-1}_{O_X/\mathbb{Q}} \oplus \Omega^{n-3}_{O_X/\mathbb{Q}} \oplus \dots$$

the last term is $\Omega^1_{O_X/\mathbb{Q}}$ or O_X, depending on n odd or even.

4.2 Lambda and Adams operations on K-groups

4.2.1 Background on Lambda and Adams operations

In this section, we recall the history of lambda and Adams operations on K-groups briefly. We assume X to be a noetherian scheme with finite Krull dimension. It is well known that the Grothendieck group of X has a λ-ring structure given by exterior power,namely,$\lambda^k(E) = \Lambda^k E$ for any given vector bundle E over X. There are also several ways extending the Adams operations on Grothendieck groups to higher K-groups, [31] by Soulé, [14,15]by Gillet-Soulé and [16,17] by Grayson.

Soulé defines lambda operations on higher K-groups(with support) and shows that there is a λ-ring structure for the higher K-groups. This result is further generalized by Gillet-Soulé in [15], where they define lambda operations for all K-coherent spaces in any locally ringed topos and also discuss the filtrations on K-groups.

The key of their approach is to consider K-theory as a generalized cohomology theory:

$$K = \mathbb{Z} \times BGL^+.$$

Hence, we have

$$K(X) = \mathbb{H}(X, \mathbb{Z} \times BGL^+)$$

and

$$K(X \ on \ Y) = \mathbb{H}_Y(X, \mathbb{Z} \times BGL^+).$$

where Y is closed in X and and $K(X \ on \ Y)$, K-theory of X with support in Y, is defined as the homotopy fibre of

$$BQP(X) \to BQP(X \ - \ Y)$$

here $P(X)$ is the category of locally free sheaves of finite rank on X and Q stands for Quillen's Q-construction.

Now, we let $R_{\mathbb{Z}}(GL_N)$ be the Grothendieck group of representations of the general linear group scheme of GL_N. Then it is well known that $R_{\mathbb{Z}}(GL_N)$ has a λ-ring structure. And moreover, an element of $R_{\mathbb{Z}}(GL_N)$ induces a morphism

$$\mathbb{Z} \times BGL_N^+ \to \mathbb{Z} \times BGL^+.$$

In other word, there is a morphism between abelian groups:

$$R_{\mathbb{Z}}(GL_N) \to [\mathbb{Z} \times BGL_N^+, \mathbb{Z} \times BGL^+].$$

Passing to limit, we have

$$R_{\mathbb{Z}}(GL) \to [\mathbb{Z} \times BGL^+, \mathbb{Z} \times BGL^+].$$

Furthermore, we have the following morphism by taking hypercohomology:

$$R_{\mathbb{Z}}(GL) \to \{\mathbb{H}_Y(X, \mathbb{Z} \times BGL^+), \mathbb{H}_Y(X, \mathbb{Z} \times BGL^+)\}.$$

And finally we arrive at group level:

$$R_{\mathbb{Z}}(GL) \to \{K_m(X\ on\ Y), K_m(X\ on\ Y)\}.$$

In other word, the λ-operations on $K_m(X\ on\ Y)$ are induced from the λ-operations of $R_{\mathbb{Z}}(GL_N)$. In fact, this is exact the point to prove $K_m(X\ on\ Y)$ carries a λ-ring structure.

4.2.2 Adams operations on negative K-groups

Since the appearance of the non-zero negative non-connective K-groups in our study, we need to extend the above Adams operations ψ^k to negative range. This can be done by descending induction, which was already pointed out by Weibel in[45].

For every integer $n \in \mathbb{Z}$, we have the following Bass fundamental exact sequence.

$$... \to K_n(X[t,t^{-1}]\ on\ Y[t,t^{-1}]) \to K_{n-1}(X\ on\ Y) \to 0.$$

In particular, for any $x \in K_{-1}(X\ on\ Y)$, we have $x \cdot t \in K_0(X[t,t^{-1}]\ on\ Y[t,t^{-1}])$, where $t \in K_1(k[t,t^{-1}])$. We have

$$\psi^k(x \cdot t) = \psi^k(x)\psi^k(t) = \psi^k(x)k \cdot t.$$

Tensoring with $\mathbb{Q}$, we have obtained Adams operations ψ^k on $K_{-1}(X\ on\ Y)$:

$$\psi^k(x) = \frac{\psi^k(x \cdot t)}{k \cdot t}.$$

Continuing this procedure, we obtain Adams operations on all the negative K-groups.

4.3 Goodwillie-type result and Cathelineau-type result

In this section, we will show Goodwillie-type and Cathelineau-type results for non-connective K-groups. All the variant of cyclic homology are taken over $\mathbb{Q}$. Let's recall that in [9] Goodwillie shows the relative Chern character is an isomorphism between the relative K-group $K_n(A,I)$ and negative cyclic homology $HN_n(A,I)$, where A is a commutative $\mathbb{Q}$-algebra and I is a nilpotent ideal in A.

Theorem 4.3.1. Goodwillie's Isomorphism.

Let I be a nilpotent ideal in a commutative $\mathbb{Q}$-algebra A, the relative Chern character

$$Ch : K_n(A,I) \to HN_n(A,I)$$

is an isomorphism.

Remark: In fact, Goodwillie provided two isomorphisms, the relative Chern character Ch and the rational homotopy character ρ. Cortiñas-Weibel identify the relative Chern character Ch with the rational homotopy character ρ by showing they are induced by maps which are naturally homotopic.

This result is further generalized by Cathelineau in [3]

Theorem 4.3.2. The Goodwillie's isomorphism

$$K_n(A,I) = HN_n(A,I)$$

is an isomorphism of trivial γ-rings. That is,

$$K_n^{(i)}(A,I) = HN_n^{(i)}(A,I).$$

In [6], Cortinas-Haesemeyer-Weibel show a space level version of Goodwillie's theorems in appendix B.

For every nilpotent sheaf of ideal I, we define $K(O,I)$ and $HN(O,I)$ as the following presheaves respectively:

$$U \to K(O(U), I(U))$$

and

$$U \to HN(O(U), I(U)).$$

We write $\mathcal{K}(O, I)$ and $\mathcal{HN}(O, I)$ for the presheaves of spectrum whose initial spaces are $K(O, I)$ and $HN(O, I)$ respectively. Moreover, one define $\mathcal{K}^{(i)}(O, I)$ as the homotopy fiber of $\mathcal{K}(O, I)$ on which $\psi^k - k^i$ acts acyclicly. And we define $\mathcal{HN}^{(i)}(O, I)$ similarly. Goodwillie's theorem and Cathelineau's isomorphism can be generalized in the following way.

Theorem 4.3.3. Cortinas-Haesemeyer-Weibel[6]

The relative Chern character induces homotopy equivalence of spectra:

$$Ch : \mathcal{K}(O, I) \simeq \mathcal{HN}(O, I)$$

and

$$Ch : \mathcal{K}^{(i)}(O, I) \simeq \mathcal{HN}^{(i)}(O, I).$$

Now, let X be a scheme essenially finite type over a field k, where $Chark = 0$. Let Y be a closed subset in a scheme X and $U = X - Y$.

Let $\mathbb{H}(X, \bullet)$ denote Thomason's hypercohomology of spectra. We have the following Nine-diagrams(each column and row are homotopy fibration):

$$\begin{array}{ccccc}
\mathbb{H}_Y(X, \mathcal{K}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O, \varepsilon)) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O, \varepsilon)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{K}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O_U[\varepsilon])) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X, \mathcal{K}(O_X)) & \longrightarrow & \mathbb{H}(X, \mathcal{K}(O_X)) & \longrightarrow & \mathbb{H}(U, \mathcal{K}(O_U))
\end{array}$$

and

$$\begin{array}{ccccc}
\mathbb{H}_Y(X,\mathcal{HN}(O,\varepsilon)) & \longrightarrow & \mathbb{H}(X,\mathcal{HN}(O,\varepsilon)) & \longrightarrow & \mathbb{H}(U,\mathcal{HN}(O,\varepsilon)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X,\mathcal{HN}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(X,\mathcal{HN}(O_X[\varepsilon])) & \longrightarrow & \mathbb{H}(U,\mathcal{HN}(O_U[\varepsilon])) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_Y(X,\mathcal{HN}(O_X)) & \longrightarrow & \mathbb{H}(X,\mathcal{HN}(O_X)) & \longrightarrow & \mathbb{H}(U,\mathcal{HN}(O_U))
\end{array}$$

The above diagrams result in the following result

Theorem 4.3.4. $\mathbb{H}_Y(X,\mathcal{K}(O,\varepsilon))$ is the homotpy fibre of

$$\mathbb{H}_Y(X,\mathcal{K}(O_X[\varepsilon])) \to \mathbb{H}_Y(X,\mathcal{K}(O_X)),$$

and $\mathbb{H}_Y(X,\mathcal{HN}(O,\varepsilon))$ is the homotopy fibre of

$$\mathbb{H}_Y(X,\mathcal{HN}(O_X[\varepsilon])) \to \mathbb{H}_Y(X,\mathcal{HN}(O_X)).$$

Combining Goodwillie's isomorphism(space version) with the above result, we have proved the following theorem, which can be considered as a Goodwillie-type isomorphism for relative K-groups with support.

Theorem 4.3.5. Let $K_n(X[\varepsilon] \text{ on } Y[\varepsilon],\varepsilon)$ denote the kernel of

$$K_n(X[\varepsilon] \text{ on } Y[\varepsilon]) \to K_n(X \text{ on } Y)$$

and $HN_n(X[\varepsilon] \text{ on } Y[\varepsilon],\varepsilon)$ denote the kernel of

$$HN_n(X[\varepsilon] \text{ on } Y[\varepsilon]) \to HN_n(X \text{ on } Y),$$

we have

$$K_n(X[\varepsilon] \text{ on } Y[\varepsilon],\varepsilon) = HN_n(X[\varepsilon] \text{ on } Y[\varepsilon],\varepsilon).$$

There exists the following two splitting fibrations:

$$\mathcal{K}^{(i)}(O,\varepsilon) \to \mathcal{K}(O,\varepsilon) \to \prod_{j\neq i}\mathcal{K}^{(j)}(O,\varepsilon),$$

and

$$\mathcal{HN}^{(i)}(O,\varepsilon) \to \mathcal{HN}(O,\varepsilon) \to \prod_{j\neq i} \mathcal{HN}^{(j)}(O,\varepsilon).$$

Sine taking $\mathbb{H}_Y(X,-)$ perserves homotopy fibrations, there exists the following two splitting fibrations:

$$\mathbb{H}_Y(X,\mathcal{K}^{(i)}(O,\varepsilon)) \to \mathbb{H}_Y(X,\mathcal{K}(O,\varepsilon)) \xrightarrow{\psi^k - k^i} \mathbb{H}_Y(X,\prod_{j\neq i} \mathcal{K}^{(j)}(O,\varepsilon)),$$

$$\mathbb{H}_Y(X,\mathcal{HN}^{(i)}(O,\varepsilon)) \to \mathbb{H}_Y(X,\mathcal{HN}(O,\varepsilon)) \xrightarrow{\psi^k - k^{i+1}} \mathbb{H}_Y(X,\prod_{j\neq i} \mathcal{HN}^{(j)}(O,\varepsilon)).$$

Passing to group level, we obtain the following results:

Theorem 4.3.6.

$$\mathbb{H}_Y^{-n}(X,\mathcal{K}^{(i)}(O,\varepsilon)) = \{x \in \mathbb{H}_Y^{-n}(X,\mathcal{K}(O,\varepsilon)) | \psi^k(x) - k^i(x) = 0\}.$$

$$\mathbb{H}_Y^{-n}(X,\mathcal{HN}^{(i)}(O,\varepsilon)) = \{x \in \mathbb{H}_Y^{-n}(X,\mathcal{HN}(O,\varepsilon)) | \psi^k(x) - k^{i+1}(x) = 0\}.$$

We have shown that

$$\mathbb{H}_Y^{-n}(X,\mathcal{K}(O,\varepsilon)) = K_n(X[\varepsilon] \text{ } on \text{ } Y[\varepsilon],\varepsilon),$$

and

$$\mathbb{H}_Y^{-n}(X,\mathcal{HN}(O,\varepsilon)) = HN_n(X[\varepsilon] \text{ } on \text{ } Y[\varepsilon],\varepsilon).$$

Therefore, the homotopy equivalences

$$\mathcal{K}(O,\varepsilon) \simeq \mathcal{HN}(O,\varepsilon)$$

and

$$\mathcal{K}^{(i)}(O,\varepsilon) \simeq \mathcal{HN}^{(i)}(O,\varepsilon),$$

give us the following finer result:

Theorem 4.3.7.

$$K_n^{(i)}(X[\varepsilon] \text{ } on \text{ } Y[\varepsilon],\varepsilon) = HN_n^{(i)}(X[\varepsilon] \text{ } on \text{ } Y[\varepsilon],\varepsilon).$$

This result enables us to compute the relative K-groups with support in terms of the relative negative cyclic groups with support, since the relative negative cyclic groups with support are more computable than the relative K-groups with support. Now, we show an explicit computation on relative negative cyclic groups with support which will be used later.

Theorem 4.3.8. Suppose X is a d-dimensional smooth projective variety over a field k, where $Chark = 0$ and $y \in X^{(j)}$. For any integer m, we have

$$HN_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}),$$

where $\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}} = \Omega^{m+j-1}_{O_{X,y}/\mathbb{Q}} \oplus \Omega^{m+j-3}_{O_{X,y}/\mathbb{Q}} \oplus \dots$

Proof. $O_{X,y}$ is a regular local ring with dimension j, so the depth of $O_{X,y}$ is j. For each $n \in \mathbb{Z}$, $\Omega^n_{O_{X,y}/\mathbb{Q}}$ can be written as a direct limit of $O'_{X,y}s$. Therefore, $\Omega^n_{O_{X,y}/\mathbb{Q}}$ has depth j.

Let's write $HN_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon)$ for the kernel of the projection:

$$HN_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \xrightarrow{\varepsilon=0} HN_m(O_{X,y} \ on \ y).$$

Then $HN_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon)$ can be identified with $\mathbb{H}^{-m}_y(O_{X,y}, HN(O_{X,y}[\varepsilon], \varepsilon))$, where $HN(O_{X,y}[\varepsilon], \varepsilon)$ is the relative negative cyclic complex, that is the kernel of

$$HN(O_{X,y}[\varepsilon]) \xrightarrow{\varepsilon=0} HN(O_{X,y}).$$

There is a spectral sequence :

$$H^p_y(O_{X,y}, H^q(HN(O_{X,y}[\varepsilon], \varepsilon))) \Longrightarrow \mathbb{H}^{-m}_y(HN(O_{X,y}[\varepsilon], \varepsilon)).$$

By corollary 4.1.4, we have

$$H^q(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_{-q}(O_{X,y}[\varepsilon], \varepsilon) = \Omega^{-q-1}_{O_{X,y}/\mathbb{Q}} \oplus \Omega^{-q-3}_{O_{X,y}/\mathbb{Q}} \oplus \dots$$

As each $\Omega^n_{O_{X,y}/\mathbb{Q}}$ has depth j, only $H^j_y(X, H^q(HN(O_{X,y}[\varepsilon], \varepsilon)))$ can survive because of the depth condition. This means $q = -m-j$ and

$$H^{-m-j}(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_{m+j}(O_{X,y}[\varepsilon], \varepsilon) = \Omega^{m+j-1}_{O_{X,y}/\mathbb{Q}} \oplus \Omega^{m+j-3}_{O_{X,y}/\mathbb{Q}} \oplus \dots$$

Let's write

$$\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}} = \Omega^{m+j-1}_{O_{X,y}/\mathbb{Q}} \oplus \Omega^{m+j-3}_{O_{X,y}/\mathbb{Q}} \oplus \dots$$

Thus

$$\mathbb{H}^{-m}_y(HN(O_{X,y}[\varepsilon], \varepsilon)) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}).$$

this means

$$HN_m(O_{X,y}[\varepsilon] \ on \ y_\varepsilon, \varepsilon) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}).$$

□

Repeating the above proof and noting corollary 4.1.3, we have the following finer result:

Theorem 4.3.9. Suppose X is a d-dimensional smooth projective variety over a fielf k, where $Chark = 0$ and $y \in X^{(j)}$. For any integer m, we have

$$HN^{(i)}_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon) = H^j_y(\Omega^{\bullet,(i)}_{O_{X,y}/\mathbb{Q}}),$$

where

$$\begin{cases} \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = \Omega^{2i-(m+j)-1}_{O_X/\mathbb{Q}}, for \ \frac{m+j}{2} < \ i \leq m+j. \\ \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = 0, else. \end{cases} \tag{4.3.1}$$

Combining with theorem 4.3.5 and 4.3.7, we have the following corollary

Corollary 4.3.10. Under the same assumption as above, we have

$$K_m(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}),$$

where $\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}} = \Omega^{m+j-1}_{O_{X,y}/\mathbb{Q}} \oplus \Omega^{m+j-3}_{O_{X,y}/\mathbb{Q}} \oplus \dots$

$$K_m^{(i)}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon) = H_y^j(\Omega^{\bullet,(i)}_{O_{X,y}/\mathbb{Q}}),$$

where

$$\begin{cases} \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = \Omega^{2i-(m+j)-1}_{O_X/\mathbb{Q}}, for \ \frac{m+j}{2} < \ i \leq m+j. \\ \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = 0, else. \end{cases} \tag{4.3.2}$$

Chapter 5
Tangent Sequence To Bloch-Gersten-Quillen Sequence Is Cousin Resolution

The aim of this chapter is to prove the existence of formal tangent maps from Bloch-Gersten-Quillen sequence to Cousin resolution and also to prove the tangent sequence to Bloch-Gersten-Quillen sequence to Cousin resolution. In other word, we will see "arrows" from the Bloch-Gersten-Quillen sequence to the Cousin resolution in a functorial way. We shall show that the formal tangent maps can be obtained as compositions of the Chern character and natural projections. In order to give an intuitive picture to our audiences, we will discuss smooth projective surfaces firstly, then we will move onto smooth projective varieties with dimension n.

5.1 On surfaces

Suppose X is a smooth projective surface over a field k, $chark = 0$. We will show

Theorem 5.1.1. There exists the following commutative diagram of exact sequences of sheaves (all the squares are commutative):

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_2(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_2(O_X[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_2(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_2(k(X)[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(1)}} \underline{H}^1_y(\Omega^1_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{HN}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{K}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(2)}} \underline{H}^2_x(\Omega^1_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{HN}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{K}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

This means there exist maps from the Bloch-Gersten-Quillen sequence to the Cousin resolution.

We will show the following lemma first.

Lemma 5.1.2. There exists the following commutative splitting diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_2(O_{X[\varepsilon]}) & \xrightarrow{\varepsilon=0} & HN_2(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_2(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & HN_2(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(1)}}\underline{H}^1_y(\Omega^1_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{HN}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \oplus_{y\in X^{(1)}}\underline{HN}_1(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(2)}}\underline{H}^2_x(\Omega^1_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{HN}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \oplus_{x\in X^{(2)}}\underline{HN}_0(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Recall that HN is the negative cyclic homology defined by Keller. Since negative cyclic homology HN satisfies Zariski Excision, Keller's definition agrees with classical definition in terms of hypercohomology More precisely, for any scheme X/k and any integer m,

$$HN_m(X \ on \ Z) = \mathbb{H}^{-m}_Z(HN(X)),$$

where $HN(X)$ is the negative cyclic complex associated to X. We have a very similiar identity after putting ε into X:

$$HN_m(X[\varepsilon] \ on \ Z[\varepsilon]) = \mathbb{H}^{-m}_{Z[\varepsilon]}(HN(X[\varepsilon])),$$

where $HN(X[\varepsilon])$ is the negative cyclic complex associated to $X[\varepsilon]$.

We define the relative negative cyclic bicomplex $HN(X[\varepsilon], \varepsilon)$ as the kernel of the projection

$$HN(X[\varepsilon]) \xrightarrow{\varepsilon=0} HN(X).$$

In other words, we have a direct sum decomposition:

$$HN(X[\varepsilon]) = HN(X) \oplus HN(X[\varepsilon], \varepsilon).$$

It results that

$$\mathbb{H}^{-m}_{Z[\varepsilon]}(X[\varepsilon], HN(X[\varepsilon])) = \mathbb{H}^{-m}_{Z}(X, HN(X)) \oplus \mathbb{H}^{-m}_{Z}(X, HN(X[\varepsilon], \varepsilon)).$$

Then we have the following fact.

Lemma 5.1.3. Let $HN_m(X[\varepsilon] \ on \ Z[\varepsilon], \varepsilon)$ denote the kernel of the projection:

$$HN_m(X[\varepsilon] \ on \ Z[\varepsilon]) \xrightarrow{\varepsilon=0} HN_m(X \ on \ Z),$$

then we have

$$\mathbb{H}^{-m}_{Z}(X, HN(X[\varepsilon], \varepsilon)) = HN_m(X[\varepsilon] \ on \ Z[\varepsilon], \varepsilon).$$

We give a proof of lemma 5.1.2 now.

Proof. It is classical that

$$HN_2(O_{X[\varepsilon]}, \varepsilon]) = \Omega^1_{X/\mathbb{Q}},$$

and

$$HN_2(k(X)[\varepsilon], \varepsilon]) = \Omega^1_{k(X)/\mathbb{Q}}.$$

We need to show that

$$HN_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) = HN_1(O_{X,y} \ on \ y) \bigoplus H^1_y(\Omega^1_{O_{X,y}/\mathbb{Q}}),$$

and

$$HN_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) = HN_0(O_{X,x} \ on \ x) \bigoplus H^2_x(\Omega^1_{O_{X,x}/\mathbb{Q}}).$$

Step1. According to the above lemma, $HN_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon], \varepsilon)$, which is defined as the kernel of the projection:

$$HN_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \xrightarrow{\varepsilon=0} HN_1(O_{X,y} \ on \ y),$$

can be identified with $\mathbb{H}^{-1}_y(HN(O_{X,y}[\varepsilon], \varepsilon))$.

There is a spectral sequence :

$$H_y^p(O_{X,y}, H^q(HN(O_{X,y}[\varepsilon], \varepsilon))) \Longrightarrow \mathbb{H}_y^{-1}(HN(O_{X,y}[\varepsilon], \varepsilon)).$$

By theorem 4.1.4, we have

$$HN_n(X[\varepsilon], \varepsilon) = \Omega^{n-1}_{X/\mathbb{Q}} \oplus \Omega^{n-3}_{X/\mathbb{Q}} \oplus \dots$$

This means that

$$H^q(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_{-q}(X[\varepsilon], \varepsilon) = \Omega^{-q-1}_{X/\mathbb{Q}} \oplus \Omega^{-q-3}_{X/\mathbb{Q}} \oplus \dots$$

Since y is the generic point of a curve Y on the surface X, each $\Omega^i_{O_{X,y}/\mathbb{Q}}$ has depth 1, only $H_y^1(X, H^q(HN(O_{X,y}[\varepsilon], \varepsilon))$ can survive because of the depth condition. This means $q = -2$ and

$$H^{-2}(HN(O_{X,y}[\varepsilon], \varepsilon)) = HN_2(O_{X,y}[\varepsilon], \varepsilon) = \Omega^1_{O_{X,y}/\mathbb{Q}}.$$

Thus

$$\mathbb{H}_y^{-1}(HN(O_{X,y}[\varepsilon], \varepsilon)) = H_y^1(\Omega^1_{O_{X,y}/\mathbb{Q}}),$$

this means

$$HN_1(O_{X,y}[\varepsilon] \text{ } on \text{ } y[\varepsilon], \varepsilon) = H_y^1(\Omega^1_{O_{X,y}/\mathbb{Q}}).$$

Step2. Now for x a point the surface X, $HN_0(O_{X,x}[\varepsilon] \text{ } on \text{ } x[\varepsilon], \varepsilon)$, which is defined as the kernel of the projection:

$$HN_0(O_{X,x}[\varepsilon] \text{ } on \text{ } x[\varepsilon]) \xrightarrow{\varepsilon=0} HN_0(O_{X,x} \text{ } on \text{ } x),$$

can be identified with $\mathbb{H}_x^0(HN(O_{X,x}[\varepsilon], \varepsilon))$.

There is a spectral sequence :

$$H_x^p(O_{X,x}, H^q(HN(O_{X,x}[\varepsilon], \varepsilon))) \Longrightarrow \mathbb{H}_x^0(HN(O_{X,x}[\varepsilon], \varepsilon)).$$

Noting each $\Omega^i_{O_{X,x}/\mathbb{Q}}$ has depth 2, only $H^2_x(X, H^q(HN(O_{X,x}[\varepsilon], \varepsilon)))$ can survive because of the depth condition. This means $q = -2$ and

$$H^{-2}(HN(O_{X,x}[\varepsilon], \varepsilon)) = HN_2(O_{X,x}[\varepsilon], \varepsilon) = \Omega^1_{O_{X,x}/\mathbb{Q}}.$$

Thus

$$\mathbb{H}^0_x(HN(O_{X,x}[\varepsilon], \varepsilon)) = H^2_x(\Omega^1_{O_{X,x}/\mathbb{Q}}).$$

This means

$$HN_0(O_{X,x}[\varepsilon] \textit{ on } x[\varepsilon], \varepsilon) = H^2_x(\Omega^1_{O_{X,x}/\mathbb{Q}}).$$

□

Now we give a proof of theorem 5.1.1.

Proof. (Proof of theorem 5.1.1)

The above result tells us there are natural projections from Gersten sequence involving negative cyclic homology to Cousin resolution of $\Omega^1_{X/\mathbb{Q}}$:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_2(O_{X[\varepsilon]}) \\
\downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_2(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y \in X^{(1)}} \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{HN}_1(O_{X,y}[\varepsilon] \textit{ on } y[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{x \in X^{(2)}} \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{HN}_0(O_{X,x}[\varepsilon] \textit{ on } x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

We have shown the following commutative diagram induced by Chern character in section 3(theorem 3,15, taking $m = 2$);

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
HN_2(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_2(O_X[\varepsilon]) \\
\downarrow & & \downarrow \\
HN_2(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_2(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{HN}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{k}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{HN}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{K}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Combining the above two commutative diagrams, we see there exists the following commutative diagrams.

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_2(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_2(O_X[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_2(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_2(k(X)[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y\in X^{(1)}} \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{HN}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \bigoplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{K}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(2)}} \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{HN}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{K}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

□

Definition 5.1.4. The formal tangent maps from the Bloch-Gersten-Quillen sequence to the Cousin resolution are defined as compositions of Chern character and natural projections as above.

Corollary 5.1.5. There exists formal tangent maps from the Bloch-Gersten-Quillen sequence to the Cousin resolution:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_2(O_X[\varepsilon]) \\
\downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_2(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y\in X^{(1)}}\underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{K}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{x\in X^{(2)}}\underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{K}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where $tani$ is defined as $Pr_i \circ Ch$, for $i = 1, 2, 3, 4$.

Combining the above diagram with results on computing non-connective K-groups in subsection 4.3, theorem 4.3.10, we get the following theorem which says that the formal tangent sequence to the Bloch-Gersten-Quillen sequence is the Cousin resolution.

Theorem 5.1.6. The formal tangent sequence to the Bloch-Gersten-Quillen sequence is the Cousin resolution. That is there exists the following splitting commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_2(O_{X[\varepsilon]}) & \xrightarrow{\varepsilon=0} & K_2(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_2(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(1)}} \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{K}_1(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \oplus_{y \in X^{(1)}} \underline{K}_1(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(2)}} \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{K}_0(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \oplus_{x \in X^{(2)}} \underline{K}_0(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $tani$ is defined above.

5.2 On varieties

Now suppose X is a smooth projective n-dimensional variety over a field k, $char k = 0$. We will show

Theorem 5.2.1. There exists the following commutative diagram,for any integer m:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^\bullet_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_m(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_m(O_X[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^\bullet_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_m(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d \in X^{(1)}} \underline{H}^1_d(\Omega^\bullet_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{d[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \xleftarrow{Chern} & \oplus_{d[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(2)}} \underline{H}^2_y(\Omega^\bullet_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{Pr} & \dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(n)}} \underline{H}^n_x(\Omega^\bullet_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_{n+2}} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(n)}} \underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(n)}} \underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where

$$\Omega^{\bullet}_{O_X/\mathbb{Q}} = \Omega^{m-1}_{O_X/\mathbb{Q}} \oplus \Omega^{m-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$\Omega^{\bullet}_{k(X)/\mathbb{Q}} = \Omega^{m-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{m-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

We will show the following lemma first.

Lemma 5.2.2. There exists the following commutative splitting diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_m(O_{X[\varepsilon]}) & \longleftarrow & HN_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_m(k(X)[\varepsilon]) & \longleftarrow & HN_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}} \underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{HN}_{m-1}(O_{X,d}[\varepsilon] \text{ on } d[\varepsilon]) & \longleftarrow & \oplus_{d\in X^{(1)}} \underline{HN}_{m-1}(O_{X,d} \text{ on } d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}} \underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{HN}_{m-2}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \longleftarrow & \oplus_{y\in X^{(2)}} \underline{HN}_{m-2}(O_{X,y} \text{ on } y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{Pr} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}} \underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_{n+2}} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{HN}_{m-n}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \longleftarrow & \oplus_{x\in X^{(n)}} \underline{HN}_{m-n}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where

$$\Omega^{\bullet}_{O_X/\mathbb{Q}} = \Omega^{m-1}_{O_X/\mathbb{Q}} \oplus \Omega^{m-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$\Omega^{\bullet}_{k(X)/\mathbb{Q}} = \Omega^{m-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{m-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

Proof. Since X is smooth, we have the following identities for any integer i(section 4.1, theorem 4.1.5):

$$HN_i(O_X[\varepsilon], \varepsilon) = \Omega^{i-1}_{O_X/\mathbb{Q}} \oplus \Omega^{i-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$HN_i(k(X)[\varepsilon], \varepsilon) = \Omega^{i-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{i-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

Now suppose z is a generic point of a closed subset with codimension j in X. This is equivalent to say $dimO_{X,z} = j$. In fact, $O_{X,z}$ is a regular local ring with dimension j.

Repeating the procedures in last section , let $HN_i(O_{X,z}[\varepsilon] \; on \; z[\varepsilon], \varepsilon)$ denote the kernel of the projection:

$$HN_i(O_{X,z}[\varepsilon] \; on \; z[\varepsilon]) \xrightarrow{\varepsilon=0} HN_i(O_{X,z} \; on \; z).$$

Then $HN_i(O_{X,z}[\varepsilon] \; on \; z[\varepsilon], \varepsilon)$ can be identified with $\mathbb{H}^{-i}_z(HN(O_{X,z}[\varepsilon], \varepsilon)$.

There is a spectral sequence :

$$H^p_z(O_{X,z}, H^q(HN(O_{X,z}[\varepsilon], \varepsilon))) \Longrightarrow \mathbb{H}^{-i}_z(HN(O_{X,z}[\varepsilon], \varepsilon)).$$

Noting that each $\Omega^i_{O_{X,z}/\mathbb{Q}}$ has depth j, only $H^j_z(X, H^q(HN(O_{X,z}[\varepsilon], \varepsilon))$ can survive because of the depth condition. This means $q = -i - j$ and

$$H^{-i-j}(HN(O_{X,z}[\varepsilon], \varepsilon)) = HN_{i+j}(O_{X,z}[\varepsilon], \varepsilon) = \Omega^{i+j-1}_{O_{X,z}/\mathbb{Q}} \oplus \Omega^{i+j-3}_{O_{X,z}/\mathbb{Q}} \oplus \dots$$

Let's write

$$\Omega^{\bullet}_{O_{X,z}/\mathbb{Q}} = \Omega^{i+j-1}_{O_{X,z}/\mathbb{Q}} \oplus \Omega^{i+j-3}_{O_{X,z}/\mathbb{Q}} \oplus \dots$$

Thus

$$\mathbb{H}^{-i}_y(HN(O_{X,z}[\varepsilon], \varepsilon)) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}).$$

This means

$$HN_i(O_{X,z}[\varepsilon] \; on \; z[\varepsilon], \varepsilon) = H^j_y(\Omega^{\bullet}_{O_{X,y}/\mathbb{Q}}).$$

For our purpose, taking $i = m - j$, we have the following commutative splitting diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_m(O_{X[\varepsilon]}) & \longleftarrow & HN_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_m(k(X)[\varepsilon]) & \longleftarrow & HN_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d \in X^{(1)}} \underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{d[\varepsilon] \in X[\varepsilon]^{(1)}} \underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \longleftarrow & \oplus_{d \in X^{(1)}} \underline{HN}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(2)}} \underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} \underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \longleftarrow & \oplus_{y \in X^{(2)}} \underline{HN}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{Pr} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(n)}} \underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_{n+2}} & \oplus_{x[\varepsilon] \in X[\varepsilon]^{(n)}} \underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \longleftarrow & \oplus_{x \in X^{(n)}} \underline{HN}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

□

Now we give a proof of theorem 5.2.1.

Proof. (Proof of theorem 5.2.1)

The above result tells us there are natural projections from Gersten sequence involving negative cyclic homology to Cousin resolution of $\Omega^{\bullet}_{O_X/\mathbb{Q}}$:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_m(O_{X[\varepsilon]}) \\
\downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Pr} & \dots \\
\downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_{n+2}} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

We have shown the following commutative diagram induced by Chern character in section 3, corollary 3.15;

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
HN_m(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_m(O_X[\varepsilon]) \\
\downarrow & & \downarrow \\
HN_m(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \xleftarrow{Chern} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow \\
\oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Combining the above two commutative diagrams, we see there exists the following commutative diagrams.

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{Pr_1} & HN_m(O_{X[\varepsilon]}) & \xleftarrow{Chern} & K_m(O_X[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{Pr_2} & HN_m(k(X)[\varepsilon]) & \xleftarrow{Chern} & K_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}} \underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{HN}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \xleftarrow{Chern} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}} \underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{HN}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \xleftarrow{Chern} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{Pr} & \dots & \xleftarrow{Chern} & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}} \underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{Pr_{n+2}} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{HN}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \xleftarrow{Chern} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

$\square$

Definition 5.2.3. The formal tangent maps from the Bloch-Gersten-Quillen sequence to the Cousin resolution are defined as compositions of Chern character and natural projections as above.

Corollary 5.2.4. There exists formal tangent maps from the Bloch-Gersten-Quillen sequence to the Cousin resolution:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_m(O_X[\varepsilon]) \\
\downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_m(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}} \underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}} \underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}} \underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}} \underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) \\
\downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots \\
\downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}} \underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan(n+2)} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}} \underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where $tani$ is defined as $Pr_i \circ Ch$.

Combining the above diagram with results on computing relative K-groups with support,theorem 4.3.10, we get the following theorem which says that the formal tangent sequence to the Bloch-Gersten-Quillen sequence is the Cousin resolution.

Theorem 5.2.5. The formal tangent sequence to the Bloch-Gersten-Quillen sequence is the Cousin resolution. That is there exists the following splitting commutative diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K_m(O_{X[\varepsilon]}) & \longleftarrow & K_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet}_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K_m(k(X)[\varepsilon]) & \longleftarrow & K_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{H}^1_d(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{K}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \longleftarrow & \oplus_{d\in X^{(1)}}\underline{K}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{H}^2_y(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{K}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \longleftarrow & \oplus_{y\in X^{(2)}}\underline{K}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{H}^n_x(\Omega^{\bullet}_{O_X/\mathbb{Q}}) & \xleftarrow{tan(n+2)} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{K}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \longleftarrow & \oplus_{x\in X^{(n)}}\underline{K}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

where $tani$ is defined above.

Based on theorem 4.3.10, we also have the following commutative diagram which roughly says Adams operations ψ^k on K-theory can decompose the above diagram into eigen-components.. We have the following result:

Theorem 5.2.6. There exists the following splitting commutative diagram:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} & \xleftarrow{tan1} & K^{(i)}_m(O_{X[\varepsilon]}) & \longleftarrow & K^{(i)}_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\bullet,(i)}_{k(X)/\mathbb{Q}} & \xleftarrow{tan2} & K^{(i)}_m(k(X)[\varepsilon]) & \longleftarrow & K^{(i)}_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d\in X^{(1)}}\underline{H}^1_d(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan3} & \oplus_{d[\varepsilon]\in X[\varepsilon]^{(1)}}\underline{K}^{(i)}_{m-1}(O_{X,d}[\varepsilon] \ on \ d[\varepsilon]) & \longleftarrow & \oplus_{d\in X^{(1)}}\underline{K}^{(i)}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y\in X^{(2)}}\underline{H}^2_y(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan4} & \oplus_{y[\varepsilon]\in X[\varepsilon]^{(2)}}\underline{K}^{(i)}_{m-2}(O_{X,y}[\varepsilon] \ on \ y[\varepsilon]) & \longleftarrow & \oplus_{y\in X^{(2)}}\underline{K}^{(i)}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x\in X^{(n)}}\underline{H}^n_x(\Omega^{\bullet,(i)}_{O_X/\mathbb{Q}}) & \xleftarrow{tan(n+2)} & \oplus_{x[\varepsilon]\in X[\varepsilon]^{(n)}}\underline{K}^{(i)}_{m-n}(O_{X,x}[\varepsilon] \ on \ x[\varepsilon]) & \longleftarrow & \oplus_{x\in X^{(n)}}\underline{K}^{(i)}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

where

$$\begin{cases} \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = \Omega^{2i-m+1}_{O_X/\mathbb{Q}}, for \ \frac{m-1}{2} < \ i \leq m-1. \\ \Omega^{\bullet,(i)}_{O_X/\mathbb{Q}} = 0, else. \end{cases} \tag{5.2.1}$$

We recall the following definition given in the introduction.

Definition 5.2.7. Let T_j denote $Spec(k[t]/(t^{j+1})$, the Bloch-Quillen-Gersten sequence $\mathcal{G}_j$ is defined as the following flasque resolution :

$$0 \to K_m(O_{X_j}) \to K_m(k(X)_j) \to \bigoplus_{d_j \in X_j^{(1)}} \underline{K}_{m-1}(O_{X_j,d_j} \ on \ d_j) \to \cdots \to \bigoplus_{x_j \in X_j^{(n)}} \underline{K}_{m-n}(O_{X_j,x_j} \ on \ x_j) \to 0.$$

where $O_{X_j} = O_{X \times T_j}$, $k(X)_j = k(X) \times T_j$, $d_j = d \times T_j$ and etc.

By repeating the proof of theorem 5.2.5, we have

Theorem 5.2.8. There exists the following commutative diagram(each column is a flasque resolution, m and j can be any integer.):

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan1} & K_m(O_{X_j}) & \longleftarrow & K_m(O_X) \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega^{\bullet}_{k(X)/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan2} & K_m(k(X)_j) & \longleftarrow & K_m(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{d \in X^{(1)}} \underline{H}^1_d((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan3} & \oplus_{d_j \in X_j^{(1)}} \underline{K}_{m-1}(O_{X_j,d_j} \ on \ d_j) & \longleftarrow & \oplus_{d \in X^{(1)}} \underline{K}_{m-1}(O_{X,d} \ on \ d) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{y \in X^{(2)}} \underline{H}^2_y((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan4} & \oplus_{y_j \in X_j^{(2)}} \underline{K}_{m-2}(O_{X_j,y_j} \ on \ y_j) & \longleftarrow & \oplus_{y \in X^{(2)}} \underline{K}_{m-2}(O_{X,y} \ on \ y) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \xleftarrow{tan} & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{x \in X^{(n)}} \underline{H}^n_x((\Omega^{\bullet}_{O_X/\mathbb{Q}})^{\oplus j} & \xleftarrow{tan(n+2)} & \oplus_{x_j \in X_j^{(n)}} \underline{K}_{m-n}(O_{X_j,x_j} \ on \ x_j) & \longleftarrow & \oplus_{x \in X^{(n)}} \underline{K}_{m-n}(O_{X,x} \ on \ x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where

$$\Omega^{\bullet}_{O_X/\mathbb{Q}} = \Omega^{m-1}_{O_X/\mathbb{Q}} \oplus \Omega^{m-3}_{O_X/\mathbb{Q}} \oplus \dots$$

and

$$\Omega^{\bullet}_{k(X)/\mathbb{Q}} = \Omega^{m-1}_{k(X)/\mathbb{Q}} \oplus \Omega^{m-3}_{k(X)/\mathbb{Q}} \oplus \dots$$

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