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Extremal Problems in Matroid Connectivity

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EXTREMAL PROBLEMS IN MATROID CONNECTIVITY

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Abstract

Matroid k-connectivity is typically defined in terms of a connectivity function. We can also say that a matroid is 2-connected if and only if for each pair of elements, there is a circuit containing both elements. Equivalently, a matroid is 2-connected if and only if each pair of elements is in a certain 2-element minor that is 2-connected. Similar results for higher connectivity had not been known. We determine a characterization of 3-connectivity that is based on the containment of small subsets in 3-connected minors from a given list of 3-connected matroids.

Bixby's Lemma is a well-known inductive tool in matroid theory that says that each element in a 3-connected matroid can be deleted or contracted to obtain a matroid that is 3-connected up to minimal 2-separations. We consider the binary matroids for which there is no element whose deletion and contraction are both 3-connected up to minimal 2-separations. In particular, we give a decomposition for such matroids to establish that any matroid of this type can be built from sequential matroids and matroids with many fans using a few natural operations.

Wagner defined biconnectivity to translate connectivity in a bicircular matroid to certain connectivity conditions in its underlying graph. We extend a characterization of biconnectivity to higher connectivity. Using these graphic connectivity conditions, we call upon unavoidable minor results for graphs to find unavoidable minors for large 4-connected bicircular matroids.

Chapter 1
Introduction

This dissertation focuses on matroid theory. Matroid-theoretic notation and terminology follow Oxley [10]. Readers familiar with matroid theory as described by Oxley's text may wish to skip the introductory chapter. In the tradition of Oxley, we shall often express the singleton set $\{e\}$ as e when there is no risk of confusion.

1.1 A Rank-Based Definition of a Matroid

Suppose E is a finite set, and $r : 2^E \to \mathbb{Z}$ satisfies the following:

(R1) $0 \leq r(X) \leq |X|$, for all $X \subseteq E$.

(R2) If $X_2 \subseteq X_1 \subseteq E$, then $r(X_2) \leq r(X_1)$.

(R3) If $X_1, X_2 \subseteq E$, then $r(X_1) + r(X_2) \geq r(X_1 \cup X_2) + r(X_1 \cap X_2)$.

Then we call $M = (E, r)$ a *matroid* on E having r as its *rank function*. The set E is called the *ground set* of M. If $X \subseteq E$ such that $|X| = r(X)$, then X is said to be an *independent set* of M. A maximal independent set is called a *basis*. Subsets of E that are not independent are said to be *dependent*. A minimal dependent set is called a *circuit*. The property expressed in (R3) above is called *submodularity*.

If M is a matroid, we denote the rank function of M by r_M. The ground set, set of independent sets, and set of circuits of M are denoted by $E(M)$, $\mathcal{I}(M)$, and $\mathcal{C}(M)$, respectively. For brevity, we refer to $r_M(E(M))$ by $r(M)$.

1.2 Duality

Given any matroid M, there is an associated *dual matroid* M^* whose rank function is given as follows:

$$r_{M^*}(X) = r_M(E - X) + |X| - r(M).$$

When we have a matroid $M = (E, r)$, we often use r^* to refer to the rank function of the dual matroid M^*. It is elementary to show that $(M^*)^* = M$.

1.3 Isomorphism

Suppose $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ are matroids. We say that M_1 and M_2 are *isomorphic* whenever there is a bijection $\phi : E_1 \to E_2$ such that $r_1(X) = r_2(\phi(X))$ for all $X \subseteq E_1$. We call ϕ an *isomorphism* of M_1 to M_2. If M_1 and M_2 are isomorphic matroids, we write $M_1 \cong M_2$.

1.4 Deletion, Contraction, and Minors

Suppose M is a matroid on the ground set E, and $X \subseteq E$. There is a matroid, denoted by $M\backslash X$, on the ground set $E - X$ whose rank function is given by $r_{M\backslash X}(Y) = r_M(Y)$ for all $Y \subseteq E - X$. We refer to $M\backslash X$ as the *deletion of* X *from* M or the *restriction of* M *to* $E - X$. The matroid $M\backslash X$ is also denoted by $M|(E - X)$. We also say that M is an *extension* of $M|(E - X)$.

We may form another matroid, denoted by M/X, on the ground set $E - X$ whose rank function is given by $r_{M/X}(Y) = r_M(Y \cup X) - r_M(X)$ for each $X \subseteq E$. We refer to M/X as the *contraction of* X *from* M or the *contraction of* M *onto* $E - X$. The contraction of M onto $E - X$ is also denoted by $M.(E - X)$.

Deletion and contraction are dual operations; that is, $M\backslash X = (M^*/X)^*$ for each $X \subseteq E(M)$. Furthermore, these operations commute. Given distinct subsets X and Y of $E(M)$, we have that $(M\backslash X)/Y = (M/Y)\backslash X$. Likewise, $(M/X)/Y = (M/Y)/X$, and $(M\backslash X)\backslash Y = (M\backslash Y)\backslash X$.

If N and M are matroids and $N = M/Y\backslash X$ for some distinct subsets X and Y of $E(M)$, then we say that N is a *minor* of M. In the event that N is isomorphic to some minor of M, we say that M has an N*-minor*.

1.5 Simple and Cosimple Matroids

It is customary to use the prefix "co-" in discussing duality. For instance, a circuit of M^* is a *cocircuit* of M. A one-element circuit of M is called a *loop* of M. A loop of M^* is a *coloop* of M.

Elements in a two-element circuit are said to be in *parallel*. The ground set of a matroid is partitioned into *parallel classes* in which elements in each parallel class are pairwise parallel with each other. A two-element cocircuit is called a *series pair*. A *triangle* is a circuit of size three, and a *triad* is a cocircuit of size three.

A matroid that contains no loops or parallel pairs of elements is known as a *simple matroid*. It is often the case that we do not consider loops or parallel elements to be significant pieces of the structure of a matroid. In such cases, we sometimes focus on an underlying simple structure that is a minor of the original matroid M. Let $\mathrm{si}(M)$ be the *simplification* of M, which is the restriction of M to a set that meets each parallel class of non-loop elements in $E(M)$ in exactly one element. The matroid $\mathrm{si}(M)$ is well-defined up to isomorphism. We call $(\mathrm{si}(M^*))^*$ the *cosimplification* of M and denote it $\mathrm{co}(M)$.

1.6 Connectivity

For each matroid $M = (E, r)$, there is a *connectivity function* $\lambda_M : 2^E \to \mathbb{Z}$ such that $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Note that $\lambda_M(X) = \lambda_M(E - X)$, for all $X \subseteq E$. We also have that $\lambda_M(X) = r(X) + r^*(X) - |X|$, so $\lambda_M(X) = \lambda_{M^*}(X)$. Note that the connectivity function of a matroid, like the rank function, is submodular (see [10, Lemma 8.2.9]).

If $\lambda_M(X) < k$, we say that X is a *k-separating* subset of E. When $\lambda_M(X) = k - 1$, we say that X is *exactly k-separating*. If X is k-separating and $|X|, |E - X| \geq k$, then we say that $(X, E - X)$ is a *k-separation* of M. If one of $|X|$ and $|E - X|$ is k and $(X, E - X)$ is a k-separation, we say that $(X, E - X)$ is a *minimal k-separation*. When $(X, E - X)$ is a k-separation for which $\lambda_M(X) = k - 1$, we say that $(X, E - X)$ is an *exact k-separation*. If for all for $k < n$ the matroid M has no k-separation, then M is said to be *n-connected*. We shall use the terms 2-connected and *connected* interchangeably.

The following theorem of Tutte [14] (see also [10, Theorem 4.3.1]) is useful in many inductive arguments concerning connected matroids.

Theorem 1.1. *Let e be an element of a connected matroid M. Then $M\backslash e$ or M/e is connected.*

An analogous result of Tutte [14] (see also [10, Theorem 8.8.4]) for 3-connectivity is below. Let $\mathcal{W}_r$ be an edge-labeled r-spoked wheel graph. There is a matroid $M(\mathcal{W}_r)$ called the *rank-r wheel matroid* on the set E of edges of $\mathcal{W}_r$ whose circuits are precisely the edge sets of cycles of $\mathcal{W}_r$. We also call $M(\mathcal{W}_r)$ the cycle matroid of the r-spoked wheel graph (see Section 1.11.2 below). There is another matroid $\mathcal{W}^r$ on E, called the *rank-r whirl matroid*, whose circuits are precisely precisely the edge sets of cycles of $whorl_r$ with the exception of exactly one r-cycle forming the rim of the wheel graph. Note that this rim cycle is a circuit and a hyperplane of $M(\mathcal{W}_r)$ but is a basis of $\mathcal{W}^r$. We call the process of changing a circuit-hyperlane to a basis in this manner *relaxing* a circuit-hyperplane. We say that $\mathcal{W}^r$ is a *relaxation* of $M(\mathcal{W}_r)$.

Theorem 1.2 (Tutte's Wheels-and-Whirls Theorem)**.** *The following are equivalent for a 3-connected matroid M having at least one element.*

(i) For every element e of M, neither $M\backslash e$ nor M/e is 3-connected.

(ii) M has rank at least three and is isomorphic to a wheel or a whirl.

The next lemma of Bixby [3] (see also [10, Lemma 8.7.3]) is also widely used when dealing with 3-connected matroids. This lemma motivates the work done in Chapter 3.

Lemma 1.3 (Bixby's Lemma)**.** *Let e be an element of a 3-connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case,* $\mathrm{co}(M\backslash e)$ *is 3-connected, while, in the second case,* $\mathrm{si}(M/e)$ *is 3-connected.*

Other forms of connectivity appear in Chapter 4. For $k \geq 2$, a k-connected matroid that has no non-minimal k-separations is called *internally* $(k+1)$*-connected*. If M is a matroid and (X, Y) is a partition of $E(M)$ such that $\lambda_M(X) < k$ and $r(X), r(Y) \geq k$, then (X, Y) is called a *vertical* k*-separation of* M. If M has two disjoint cocircuits, we say that the *vertical connectivity of* M is the least positive integer j such that M has a vertical j-separation. Otherwise, we say that the vertical connectivity of M is $r(M)$. We say that M is *vertically* n*-connected* if $n \geq 2$ and n does not exceed the vertical connectivity of M.

1.7 Closure

Each matroid $M = (E, r)$ has an associated *closure function* or *closure operator* $\mathrm{cl} : 2^E \to 2^E$ given as follows:

$$\mathrm{cl}(X) = \{x \in E \mid r(X \cup x) = r(X)\}.$$

Given a matroid with closure function cl, we often use cl^* to refer to the closure function in the dual matroid. We call cl^* the *coclosure function* of the original matroid. We denote the closure function of a matroid M by cl_M. We often use cl_M^* as an alias for cl_{M^*}.

The closure operator is of particular interest to us when considering k-separations. For instance, suppose (X, Y) is a k-separation of a k-connected matroid M with $|X| > k$. If $e \in X$ and $e \in \mathrm{cl}_M(Y)$, then $(X - e, Y \cup e)$ is a k-separation of M. A similar effect occurs when an element is in the coclosure of each side of a k-separation. This leads to a natural notion of equivalence of k-separations.

Let M be a matroid, and let $X \subseteq E(M)$. We define $\mathrm{fcl}(X)$, the *full closure* of X, to be $X \cup \{e_1, e_2, \ldots, e_n\}$, where $(e_1, e_2, \ldots, e_n)$ is a maximal sequence such that $e_1 \in \mathrm{cl}_M(X) \cup \mathrm{cl}_M^*(X)$ and, for each $1 < i \leq n$

$$e_i \in \mathrm{cl}_M(X \cup \{e_1, e_2, \ldots, e_i\}) \cup \mathrm{cl}_M^*(X \cup \{e_1, e_2, \ldots, e_i\}).$$

If $X \subseteq E(M)$ and $\mathrm{cl}(X) = X$, then X is a *closed set* or a *flat*. If $T \subseteq \mathrm{cl}(X)$, then we say that T is *spanned* by X. If $\mathrm{fcl}(X) = X$, then X is said to be *fully closed*. A rank-$(r-1)$ flat of a rank-r matroid is called a *hyperplane*.

1.8 The Generalized Parallel Connection

The generalized parallel connection is employed in Chapter 3. It is sufficient to define the generalized parallel connection for simple matroids for our purposes. Refer to [10, 11.4] for a thorough description of the generalized parallel connection.

If X is a flat of a matroid M such that for each flat Y of M, we have $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$, then X is said to be a *modular flat* of M.

Suppose M_1 and M_2 are matroids on ground sets E_1 and E_2, respectively, for which $E_1 \cap E_2 = T$, and $M_1|T = M_2|T$. Furthermore, suppose that T is a modular flat of M_1. Then there is a matroid $P_T(M_1, M_2)$ on the ground set $E_1 \cup E_2$ called the *generalized parallel connection of M_1 and M_2 across T* for which a set F is a flat of $P_T(M_1, M_2)$ if and only if $F \cap E_i$ is a flat of M_i for each i in $\{1, 2\}$. The rank function r of $P_T(M_1, M_2)$ has the property that for each flat F of $P_T(M_1, M_2)$, we have $r(F) = r_{M_1}(F \cap E_1) + r_{M_2}(F \cap E_2) - r(M_1|T)$. The generalized parallel connection is a natural way to glue together the matroids M_1 and M_2 along T so that $P_T(M_1, M_2)|E_i = M_i$ for each i in $\{1, 2\}$. The matroid $P_T(M_1, M_2)$ is also denoted as $P_{M_1|T}(M_1, M_2)$.

1.9 Geometric Representations

We can capture dependencies using geometric representations. The non-loop elements of the matroid are placed as points in space. Parallel elements are thought of as copunctual points. A circuit of size three can be represented as three distinct points on a line. A 4-element circuit can be represented as four distinct points in the plane, no three of which are collinear. This idea extends to matroids of higher rank. This leads to a natural way to visualize matroids. For matroids of rank at least four, we often focus on particular low-rank flats of the matroid or choose to project to a lower dimension to aid in visualization. This geometric concept of a matroid inspires much of the matroid-theoretic terminology. Elements of the ground set are sometimes called *points* of the matroid. A rank-2 flat is often called a *line*, and a rank-3 flat is called a *plane*.

Figure 1.1 gives an example of a geometric representation of a rank-3 matroid we call Q_6. The dots mark points that correspond to elements of the matroid. The matroid Q_6 has two rank-2 flats of size three, which are indicated by the intersecting line segments in the drawing. Let $X \subseteq E(Q_6)$. If $|X| \leq 2$, then $r(X) = |X|$. If $|X| = 3$ and the points corresponding to X are all collinear in the drawing, then $r(X) = 2$. If $|X| \geq 3$ and the points corresponding to X do not all fall on one of the two marked lines, we have that $r(X) = 3$.

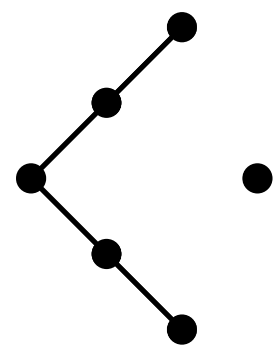


FIGURE 1.1: A geometric representation of the matroid Q_6.

1.10 Fans

A set X in a matroid M is a *fan* if $|X| = n \geq 3$ and there is an ordering $(x_1, x_2, \ldots, x_n)$, called a *fan ordering of* X, such that, up to duality, $\{x_i, x_{i+1}, x_{i+2}\}$ is a triangle for all odd $i < n - 2$ and a triad for all even $i < n - 2$. When we refer to a sequence as a fan, it is understood that the sequence is a fan ordering of some fan. A fan X is *maximal* if no fan ordering of X is a proper subsequence of a fan ordering of another fan in M.

1.11 Some Examples of Matroids

Below are four classes of matroids that are of particular interest to this dissertation.

1.11.1 Vector Matroids

Suppose A is a matrix over a field $\mathbb{F}$. If the columns of A are labeled distinctly by all of the elements of a finite set E, then there is a matroid on E with rank function r such that $r(X)$ is the dimension of the vector space spanned by the column vectors of A that are labeled by the elements of X. We call this matroid the *vector matroid of* A and denote it $M[A]$.

Whenever a matroid M is isomorphic to the vector matroid of some matrix A over some field $\mathbb{F}$, we say that M is a *representable matroid* or that M is an $\mathbb{F}$*-representable matroid*. We call A a *representation* of M. If M is a $GF(2)$-representable matroid, we say that M is *binary*. Some matroids are $\mathbb{F}$-representable for every field $\mathbb{F}$. We call these the *regular* matroids.

1.11.2 Graphic Matroids

Given a finite graph G with edge set E, there is a matroid on E with rank function r such that, for each $X \subseteq E$, the value of $r(X)$ is the size of a largest subset of the edges of X that does not contain the edge set of any cycle of G. We denote this matroid by $M(G)$ and call $M(G)$ the *cycle matroid* of G. If a matroid M is isomorphic to $M(G)$ for some graph G, we say that M is a *graphic matroid*.

Every graphic matroid is a regular matroid. Given any finite graph $G = (V, E)$ and field $\mathbb{F}$, construct a $|V| \times |E|$ matrix A over $\mathbb{F}$ as follows:

(i) Set A to be the vertex-edge incidence matrix for G.

(ii) If a column of A corresponds to a loop edge, set that column to be the zero vector.

(iii) If a column of A corresponds to a non-loop edge, set one of its non-zero entries to 1 and the other to -1.

Then the circuits of $M[A]$ are precisely the edge sets of cycles of G. Thus, $M[A] = M(G)$.

1.11.3 Bicircular Matroids

Cycle matroids are not the only matroids that can arise from graphs. Let G be a finite graph. The *bicircular matroid* of G, denoted by $B(G)$, is the matroid with ground set $E(G)$ whose rank function r is such that $r(X)$ is the number of edges in a maximal subgraph H of the subgraph induced by X such that each component of H contains at most one cycle. A subset of $E(G)$ is a circuit in $B(G)$ if and only if it is the edge set of a minimal connected subgraph of G that contains at least two cycles.

A subgraph of G is called a Θ-*graph* if it consists of two distinct vertices and three internally disjoint paths connecting them; a subgraph is called a *tight handcuff* if it consists of two cycles having just one vertex in common; and a subgraph is called a *loose handcuff* if it consists of two vertex-disjoint cycles and a minimal path meeting each cycle. The circuits of $B(G)$ are the edge sets of Θ-graphs, tight handcuffs, or a loose handcuffs in G. Figure 1.2 contains illustrations of each of the types of graphs underlying circuits in a bicircular matroid. We also call these three types of graphs *bicycles*.

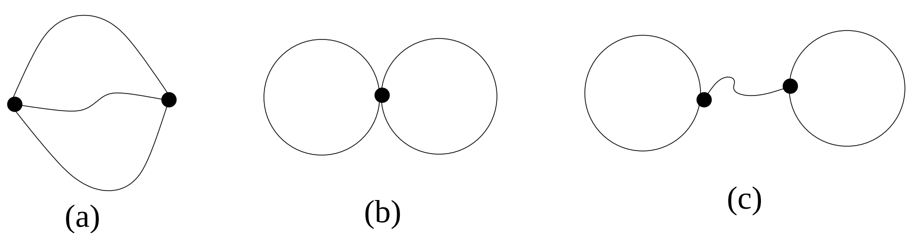


FIGURE 1.2: Three types of bicycles.

If M is isormorphic to $B(G)$ for some graph G, we say that M is *bicircular*. Chapter 4 focuses on bicircular matroids.

1.11.4 Spikes

Let $E = \{t, x_1, y_1, x_2, y_2, \dots, x_r, y_r\}$ for some $r \geq 3$. Let $\mathcal{C}_1 = \{\{t, x_i, y_i\} \mid 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j\} \mid 1 \leq i < j \leq r\}$. Let $\mathcal{C}_3$ be a possibly empty collection of sets of the form $\{z_1, z_2, \dots, z_r\}$ such that z_i is in $\{x_i, y_i\}$ for all i and no two members of $\mathcal{C}_3$ have more than $r-2$ common elements. Let $\mathcal{C}_4$ be the collection of all $(r+1)$-element subsets of E that do not contain a member of C_1, C_2, or C_3. There is a matroid called the *rank-r spike with tip t* on E whose set of circuits is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Spikes appear in each of Chapters 2 and 4.

1.12 Overview

In Chapter 2, we give an alternate characterization of 3-connectivity in matroids. Rather than use the connectivity function, this characterization is based on the containment of small subsets in certain 3-connected minors. The work in that chapter has appeared in publication [8].

Chapter 3 considers a natural extremal problem related to Bixby's Lemma in the case of binary matroids. Specifically, we look at the binary matroids for which every ground set element can be either deleted or contracted to maintain 3-connectivity up to small separations, but not both.

An unavoidable matroid minors result gives a list of matroids such that any sufficiently large matroid of a certain type is guaranteed to contain a minor from the given list. Chapter 4 relates connectivity in a bicircular matroid to special connectivity conditions in its underlying graph. Using this relation, we use known unavoidable minor results for graphs to find unavoidable minors of large 4-connected bicircular matroids. Chapter 4 is joint work that will appear in publication [5].

Chapter 2

A Minor-Based Characterization of Matroid 3-connectivity[1]

2.1 Introduction

Matroid k-connectivity is typically defined in terms of a connectivity function and the absence of small separations. There is a familiar alternative characterization of 2-connectivity that can be stated quite plainly.

Proposition 2.1. *A matroid M is 2-connected if and only if every 2-element subset of $E(M)$ is contained in a circuit of M.*

This characterization can also be expressed in terms of minors.

Proposition 2.2. *A matroid M is 2-connected if and only if every 2-element subset of $E(M)$ is contained in a $U_{1,2}$-minor of M.*

These characterizations of 2-connectivity are succinctly written in terms of well-understood containment relations. However, no characterizations of this type for higher connectivity had been known. The following is the main result of this chapter. Recall that $\mathcal{W}_r$ denotes the r-spoked wheel graph, and $\mathcal{W}^r$ denotes the rank-r whirl. Up to isomorphism, there is a unique relaxation of a circuit-hyperplane in the rank-3 whirl. We call this relaxation Q_6. Refer to Figure 1.1 for a geometric representation of Q_6.

Theorem 2.3. *A matroid M having at least four elements is 3-connected if and only if, for each 4-element subset X of $E(M)$, there is a minor N of M such that $X \subseteq E(N)$, and N is isomorphic to one of $\mathcal{W}^2$, $\mathcal{W}^3$, $\mathcal{W}^4$, $M(\mathcal{W}_3)$, $M(\mathcal{W}_4)$, or Q_6.*

[1]Reprinted from [8] with permission from Elsevier as given in Appendix: Copyright Agreement.

There is a similar characterization of 3-connectivity in terms of 5-element sets. The next theorem specializes that result to binary matroids. Here $K_5 - e$ denotes the single-edge deletion of the graph K_5, and S_8 is the unique deletion of a non-tip element from the tipped rank-4 binary spike.

Theorem 2.4. *A binary matroid M having at least five elements is* 3*-connected if and only if, for each* 5*-element subset X of $E(M)$, there is a minor N of M such that $X \subseteq E(N)$, and N is isomorphic to one of $M(\mathcal{W}_3)$, $M(\mathcal{W}_4)$, $M(\mathcal{W}_5)$, $M(K_5 - e)$, $M^*(K_5 - e)$, $M(K_{1,2,3})$, $M^*(K_{1,2,3})$, or S_8.*

Section 2.2 contains some basic results that are needed in the proofs of Theorems 2.3 and 2.4. These proofs appear in Sections 2.3 and 2.4. In Section 2.5, there is a discussion of extending this type of characterization to a result in terms of k-element sets for any fixed $k \geq 4$. Explicit lists of matroids characterizing 3-connectivity in the k-subset case are not given, but a description of the largest matroids in these lists is provided.

The concluding remarks in Section 2.6 note a variation on the results proved in this chapter that guarantees 3-connectivity in terms of a much weaker condition. Difficulties in obtaining characterizations of higher connectivity via minor containment are also discussed.

2.2 Preliminaries

In this chapter, we say that a matroid M *uses* a set X of elements if $X \subseteq E(M)$. For a positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

The next lemma follows from Bixby's Lemma and is used in the proofs of Theorems 2.3 and 2.4.

Lemma 2.5. *Let M be a* 3*-connected matroid having more than k elements for some fixed $k \geq 2$. Let X be a k-element subset of $E(M)$. If no* 3*-connected proper minor of M uses X, then, for each $e \in E(M) - X$, there is a pair $\{x, y\} \subseteq X$ such that $\{e, x, y\}$ is a triangle or a triad of M.*

Proof. Suppose some element e of $E(M)$ is not in a triangle or a triad containing two members of X. By switching to the dual if necessary, we may assume, by Bixby's Lemma, that M/e has no non-minimal 2-separations. Each parallel class of M/e contains at most one member of X, so there is a proper minor of M isomorphic to $\mathrm{si}(M/e)$ that uses X. $\square$

Section 2.5 contains a proof that relies on the following result of Bixby and Coullard [4] (see also [10, Theorem 12.3.6]).

Theorem 2.6. *Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geq 4$. Suppose that $e \in E(M) - E(N)$ and M has no 3-connected proper minor that both uses e and has N as a minor. Then, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of the following holds where $|E(M) - E(N)| = n$:*

(i) $n = 1$ *and* $N_1 = M_1\backslash e$;

(ii) $n = 2$ *and* $N_1 = M_1\backslash e/f$ *for some element* f; *and* N_1 *has an element* x *such that* $\{e, f, x\}$ *is a triangle of* M_1;

(iii) $n = 3$ *and* $N_1 = M_1\backslash e, g/f$ *for some elements* f *and* g; *and* N_1 *has an element* x *such that* M_1 *has* $\{e, f, x\}$ *as a triangle and* $\{f, g, x\}$ *as a triad; moreover,* $M_1\backslash e$ *is* 3*-connected;*

(iv) $n = 3$ *and* M_1 *has a triad* $\{e, f, g\}$ *such that* $N_1 = M_1\backslash e, g/f = M_1\backslash e, f, g$; *moreover,* N_1 *has distinct elements* x *and* y *such that* $\{e, g, x\}$ *and* $\{e, f, y\}$ *are triangles of* M_1; *or*

(v) $n = 4$ *and* $N_1 = M_1\backslash e, g/f, h$ *for some elements* f, g, *and* h; *and* N_1 *has an element* x *such that* $\{e, f, x\}$ *and* $\{g, h, x\}$ *are triangles of* M_1, *and* $\{f, g, x\}$ *is a triad of* M_1; *moreover, each of* $M_1\backslash e$, $M_1\backslash e/f$, *and* $M_1\backslash h/g$ *is* 3*-connected.*

One direction of the equivalences in the main results of this chapter is an easy consequence of the well-known persistence of separations through minors of a matroid.

Proposition 2.7. *Let $k \geq 4$ be a fixed integer. Suppose M is a matroid having at least k elements, and $\mathcal{N}$ is a nonempty set of 3-connected matroids, each having at least k elements. If, for each k-element subset X of $E(M)$, there is an $\mathcal{N}$-minor of M using X, then M is 3-connected.*

Proof. Suppose M is not 3-connected. Let (A, B) be a j-separation of M for some $j < 3$. Choose a k-subset X of $E(M)$ such that $|X \cap A|, |X \cap B| \geq j$. Then X is in no 3-connected minor of M. □

Observe that the converse of this proposition also holds if, for instance, the set $\mathcal{N}$ is taken to be all 3-connected matroids having at least k elements. Certainly the characterization obtained in this manner is of questionable value. The results in this chapter concern the minimal lists needed to achieve these characterizations of 3-connectivity.

2.3 Matroid 3-connectivity in Terms of 4-element Sets

This section proves the main result of this chapter.

Proof of Theorem 2.3. Note that the list of matroids given in the statement of the theorem is closed under duality since each of its members is self-dual. Other than $M(\mathcal{W}_3)$, $\mathcal{W}^3$, and Q_6, the 3-connected matroids with at least four but not more than six elements consist of uniform matroids of rank and corank at least two, and P_6, the unique relaxation of Q_6. Each 4-element subset of each of these matroids is contained in a $\mathcal{W}^2$-minor of the matroid. Thus the theorem holds when $|E(M)| \leq 6$.

Now assume that M has at least seven elements and that there is some four-element subset $X = \{a, b, c, d\}$ of $E(M)$ such that no 3-connected proper minor of M uses X. First observe the following.

2.8. *Neither M nor M^* has a rank-2 flat containing more than three elements.*

If M has such a flat Y containing X, then $M|X \cong \mathcal{W}^2$; a contradiction. If $X \not\subseteq Y$ then, for any $y \in Y - X$, the matroid $M\backslash y$ is 3-connected; another contradiction. By duality, 2.8 holds. Note that 2.8 and Lemma 2.5 together restrict the possible structure of M enough to reduce the proof to a finite case check.

The following is an immediate consequence of 2.8.

2.9. $$3 \leq r(M) \leq |E(M)| - 3.$$

2.10. *If M has a triad, then $r(M) \geq 4$.*

Let Y be a triad of M. As $|E(M) - Y| \geq 4$, it follows by 2.8 that $r(E(M) - Y) \geq 3$, so $r(M) \geq 4$.

By Lemma 2.5, duality, and relabeling, we may assume the following.

2.11. *The set $\{e, a, b\}$ is a triangle of M for some $e \in E(M) - X$.*

Next, we show that:

2.12. *If, for some g in $E(M) - X - e$, there is a triad of M containing g and two elements of X, then M is isomorphic to $M(\mathcal{W}_4)$ or $\mathcal{W}^4$.*

By 2.11 and orthogonality, the triad must be $\{g, a, b\}$ or $\{g, c, d\}$. In each case, it follows by Lemma 2.5, orthogonality, and 2.8 that every f in $E(M) - X - e - g$ is in a triangle with $\{c, d\}$ or is in a triad with $\{a, b\}$ or $\{c, d\}$. But by 2.8, $E(M) - X - e$ has at most three elements, namely at most one element in a triad with $\{a, b\}$, at most one element in a triad with $\{c, d\}$, and at most one element in a triangle with $\{c, d\}$. By 2.10 and duality, $|E(M)| \geq 8$. Hence $|E(M)| = 8$ and each element of X is in both a triangle and a triad. Thus if $x \in X$, then neither $M\backslash x$ nor M/x is 3-connected. By assumption, if $x \in E(M) - X$, then neither

$M\backslash x$ nor M/x is 3-connected. Thus 2.12 holds by Theorem 1.2, Tutte's Wheels-and-Whirls Theorem.

We may asssume that each member of $E(M) - X$ is in a triangle with two elements of X. Let e_{xy} denote the unique element of $E(M) - X$ that is in a triangle with the pair $\{x, y\} \subseteq X$, if this element exists. Note that X spans M, and $M|X$ is a matroid on four elements not isomorphic to $U_{2,4}$ by 2.8, so $M|X$ is isomorphic to $U_{3,4}$, $U_{2,3} \oplus U_{1,1}$, or $U_{4,4}$.

As M has at least seven elements, we may assume by relabeling if necessary that M has triangles $T_1 = \{a, b, e_{ab}\}$ and $T_2 = \{a, c, e_{ac}\}$. A geometric representation for $M|(T_1 \cup T_2)$ is shown in Figure 2.1. Suppose $M|X$ is isomorphic to $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. Then d is in the plane of M spanned by $T_1 \cup T_2$. If d is on exactly one line spanned by two elements of $T_1 \cup T_2$, then M has a minor isomorphic to $\mathcal{W}^3$ using X. If d is on two such lines, then M has an $M(\mathcal{W}_3)$-minor using X. Otherwise, d lies on no such line, so M has a Q_6-minor that uses X.

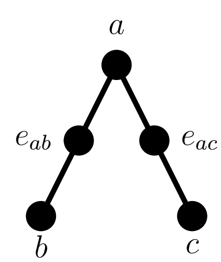


FIGURE 2.1: A geometric representation of $M|(T_1 \cup T_2)$ when X is not a basis of M.

Finally, suppose $M|X$ is isomorphic to $U_{4,4}$. Then d is not on the plane spanned by $T_1 \cup T_2$. As M has no 1-element or 2-element cocircuits, every element of X is in at least two triangles that each contain exactly one other element of X. Up to relabeling, assume that $\{e_{ab}, e_{ac}, e_{cd}, e_{bd}\} \subseteq E(M) - X$. Let $N = M|(X \cup \{e_{ab}, e_{ac}, e_{cd}, e_{bd}\})$. A geometric representation for N is shown in Figure 2.2 where possibly $\{e_{ab}, e_{ac}, e_{cd}, e_{bd}\}$ is a circuit. Thus N is isomorphic to one of $\mathcal{W}^4$ or $M(\mathcal{W}_4)$. $\square$

Binary and graphic corollaries to Theorem 2.3 are immediate.

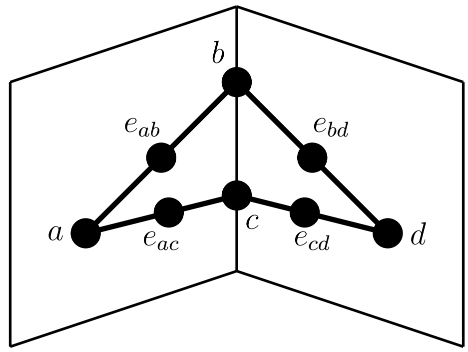


FIGURE 2.2: A representation for the matroid N when X is a basis of M.

Corollary 2.13. *A binary matroid M having at least four elements is 3-connected if and only if, for each 4-element subset X of $E(M)$, there is a minor M that uses X and is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$.*

Corollary 2.14. *A graph G having no isolated vertices and at least four edges is simple and 3-connected if and only if, for each 4-element subset X of $E(G)$, there is a minor H of G such that $X \subseteq E(H)$ and H is isomorphic to a 3- or 4-spoked wheel graph.*

2.4 Binary Matroid 3-connectivity in Terms of 5-element Sets

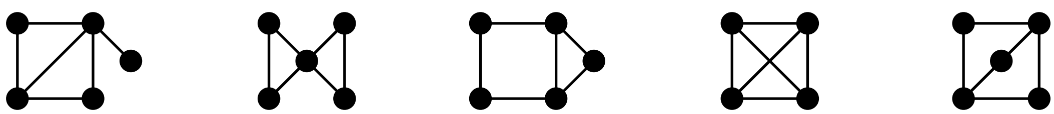
Observe that there is a 3-element version [10, Proposition 4.3.6] of Proposition 2.2.

Proposition 2.15. *A matroid M having at least three elements is 2-connected if and only if every 3-element subset of $E(M)$ is contained in a $U_{1,3}$- or $U_{2,3}$-minor of M.*

As noted in the introduction, there is an analogous characterization of 3-connectivity in terms of five-element sets. In this section, Theorem 2.4, the binary restriction of this characterization, is proved.

Proof of Theorem 2.4. Observe that the list of matroids in the statement of this theorem is closed under duality. Let M be a 3-connected binary matroid having at least five elements and at most eight elements. Suppose M is not isomorphic to a matroid in the list given in the theorem. Then M is isomorphic to F_7, F_7^*, or $AG(3,2)$. Every single-element deletion of F_7 is isomorphic to $M(\mathcal{W}_3)$, and every single-element deletion of $AG(3,2)$ is isomorphic to F_7^*. Thus, any set of five elements in M can be captured in an $M(\mathcal{W}_3)$-minor. Thus the theorem holds for matroids having at most eight elements.

Now suppose M is a simple binary matroid having exactly nine elements. Then M has rank 4 or 5. By duality, assume M is rank 4. View M as a restriction of $PG(3,2)$ and consider the complement of M in $PG(3,2)$; that is, consider $PG(3,2)\backslash E(M)$. This is a 6-element binary matroid and so is graphic. Thus the possibilities for M can be determined via consideration, up to 2-isomorphism, of all simple graphs on at most five vertices that have exactly six edges. These graphs are given in Figure 2.3.



(a) $G_{9,1}$ (b) $G_{9,2}$ (c) $G_{9,3}$ (d) $G_{9,4}$ (e) $G_{9,5}$

FIGURE 2.3: Graphs whose cycle matroids are the $PG(3,2)$-complements of 9-element, simple, rank-4 binary matroids.

The following argument shows that the theorem holds for M as either M is isomorphic to $M(K_5 - e)$, or M has at least six distinct elements f such that $M\backslash f$ is isomorphic to $M(\mathcal{W}_4)$ or S_8.

The $PG(3,2)$-complement of $M(G_{9,5})$ is not 3-connected. The complement of $M(G_{9,1})$ has four single-element deletions isomorphic to $M(\mathcal{W}_4)$ and two single-element deletions isomorphic to S_8. The complement of $M(G_{9,2})$ is isomorphic to $M^*(K_{3,3})$, so each of its single-element deletions is isomorphic to $M(\mathcal{W}_4)$. The complement of $M(G_{9,3})$ is isomorphic to $M(K_5 - e)$. The complement of $M(G_{9,4})$ is the tipped rank-4 binary spike, so the deletion of any element other than the tip is S_8.

The 10-element, rank-4, simple binary matroids are $PG(3,2)$-complements of the cycle matroids of the graphs in Figure 2.4. Moreover, each of the graphs in this figure is 2-isomorphic to a single-edge deletion of one of $G_{9,1}$, $G_{9,2}$, $G_{9,3}$, or $G_{9,4}$. Thus, each of the rank-4, simple, binary matroids having ten elements is 3-connected. Of the graphs in Figure 2.4, only $G_{10,1}$ has the property that an edge can be added to obtain a graph 2-isomorphic to $G_{9,5}$. It follows that the $PG(3,2)$-complement of $M(G_{10,1})$ is the only rank-4, simple, binary matroid with ten elements having a single-element deletion that is not 3-connected. Only three of its single-element deletions fail to be 3-connected, so the theorem holds for all rank-4 binary matroids having ten elements. Furthermore, the theorem holds for all rank-4 binary matroids having more than ten elements since it holds for every 10-element restriction of each of these matroids.

(a) $G_{10,1}$ (b) $G_{10,2}$ (c) $G_{10,3}$ (d) $G_{10,4}$

FIGURE 2.4: Graphs whose cycle matroids are the $PG(3,2)$-complements of 10-element, simple, rank-4 binary matroids.

Thus M is a 3-connected binary matroid with $r(M), r^*(M) \geq 5$, and there is some subset $X = \{a, b, c, d, e\}$ of $E(M)$ such that no 3-connected proper minor of M uses X. Since M is binary, no rank-2 flat of M or M^* contains more than three elements. By Lemma 2.5, every element of $E(M) - X$ is in a triangle or a triad with two elements of X. By orthogonality, $E(M) - X$ cannot contain a four-element subset $\{x_1, x_2, y_1, y_2\}$ such that x_1 and x_2 are each in triangles containing two members of X, and y_1 and y_2 are each in triads containing two members of X; otherwise M has at most nine elements. We may assume, by duality, that there is at most one $y \in E(M) - X$ such that y is in a triad with exactly two elements of X.

First consider the case when there is an element y such that, without loss of generality, $\{y, a, b\}$ is a triad. Each element of $E(M) - X - y$ is in a triangle with a pair of elements in $X = \{a, b, c, d, e\}$ having an even intersection with $\{a, b\}$. There are at most four such pairs. Since M has at least ten elements, all of these four possible triangles are present in M. Thus $|E(M)| = 10$, and $E(M)$ has a 6-element subset Z of rank 3. Evidently X

spans $E(M) - y$, so X spans M. Hence $r(M) \leq 5$, so $r(M) = 5$. Then $r^*(E(M) - Z) = |E(M) - Z| + r(Z) - r(M) = 2$. This is a contradiction since $|E(M) - Z| = 4$, and no coline of the binary matroid M has more than three elements.

Now assume that each element of $E(M) - X$ is in a triangle with exactly two elements of X. Note that $r(X) = r(M) = 5$. Let $A = [I_5|D]$, where D is the X-fundamental circuit incidence matrix of M. The matrix obtained by appending the row $[1\,1\,1\,1\,1\,0\,0 \cdots 0]$ to A yields a $GF(2)$-representation of M. View this representation as the vertex-edge incidence matrix of a 3-connected simple graph G having six vertices and at least ten edges. The elements of X label the five edges incident with a vertex v in G. If the 5-vertex graph $G - v$ has a Hamilton cycle, then M has a restriction using X that is isomorphic to $M(\mathcal{W}_5)$. Since G is 3-connected, the graph $G - v$ is 2-connected. The unique subgraph-minimal graph on five vertices having at least five edges that is 2-connected but not Hamiltonian is $K_{2,3}$. If $G - v$ has a subgraph isomorphic to $K_{2,3}$, then M has a restriction containing X that is isomorphic to $M(K_{1,2,3})$. □

The following result for graphs is an immediate consequence of Theorem 2.4.

Corollary 2.16. *A graph G having no isolated vertices and at least five edges is simple and 3-connected if and only if, for each 5-element subset X of $E(G)$, there is a minor H of G such that $X \subseteq E(H)$ and H is isomorphic to a 3-, 4-, or 5-spoked wheel graph; or $K_5 - e$, or its planar dual, the 3-prism; or $K_{1,2,3}$.*

2.5 Largest Matroids Characterizing 3-connectivity in Terms of k-element Sets

For each $k \geq 4$, let $\mathcal{N}_k$ be the set of 3-connected matroids M having a k-element subset X such that no 3-connected proper minor of M uses X. The following result is a straightforward extension of Proposition 2.7

Proposition 2.17. *For each $k \geq 4$, a matroid M with at least k elements is 3-connected if and only if every k-element subset of $E(M)$ is contained in an $\mathcal{N}_k$-minor of M.*

The definition of $\mathcal{N}_k$ means that the last result fails if $\mathcal{N}_k$ is replaced by any proper subset. It is now not difficult to see that $\mathcal{N}_k$ is the unique minimal set of matroids characterizing 3-connectivity in terms of k-element sets. By the next result, members of $\mathcal{N}_k$ have at most $3k-4$ elements. Hence, $\mathcal{N}_k$ is certainly finite.

Proposition 2.18. *For each $k \geq 4$, each member of $\mathcal{N}_k$ has at most $3k-4$ elements.*

Proof. The proposition holds when $k = 4$ by Theorem 2.3. Suppose that $k > 4$ and that the proposition holds for $\mathcal{N}_{k-1}$. Let M be a member of $\mathcal{N}_k$. Let X be a k-element subset of $E(M)$ that is used by no 3-connected proper minor of M. Let $e \in X$. Then there is an $\mathcal{N}_{k-1}$-minor N of M using $X - e$. By Theorem 2.6, $|E(M) - E(N)| \leq 4$. However, since the matroid $M_1 \backslash h/g$ is 3-connected in case (v) of Theorem 2.6, we tighten that bound to $|E(M) - E(N)| \leq 3$. Since $|E(N)| \leq 3(k-1) - 4$, it follows that $|E(M)| \leq 3k - 4$. □

The family of matroids described next shows that, indeed, the largest members of $\mathcal{N}_k$ have size exactly equal to $3k-4$. For a fixed value of $k \geq 4$, a *codex with $k-2$ pages* is any matroid isomorphic to a 3-connected rank-k matroid on the $(3k-4)$-element ground set $\{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i \in [k-2]}$ where, for each $i \in [k-2]$, the set $\{a_i, b_i, c_i\}$ is a triad, and the sets $\{s_1, a_i, b_i\}$ and $\{s_2, b_i, c_i\}$ are triangles.

One might visualize a codex as $k-2$ *page* planes, each spanned by a triad $\{a_i, b_i, c_i\}$ and joined together at a common *binding* containing s_1 and s_2, subject to the dependence conditions specified. Figure 2.5 gives a geometric representation of a codex.

Note that the codices with two pages are the rank-4 wheel and whirl. In these rank-4 cases, there is a particular ambiguity regarding the labeling of elements. The permutation of labels given by $(s_1\ b_1)(s_2\ b_2)(a_2\ c_1)$ yields another labeling satisfying the conditions in the

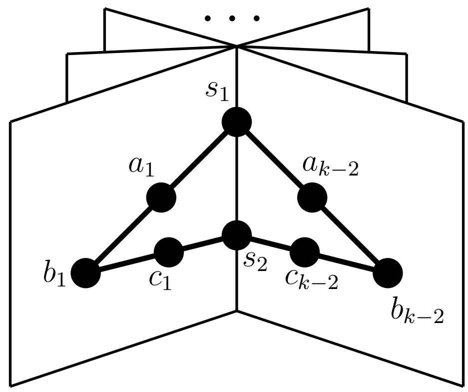


FIGURE 2.5: A visualization of a rank-k codex with $k-2$ pages.

definition. In a codex with more than two pages, there is no ambiguity concerning which pair of elements is in the binding since s_1 and s_2 are the only elements that are in no triads.

Suppose M is a codex with $k-2$ pages. Take $X = \{s_1, s_2\} \cup \{b_i\}_{i \in [k-2]}$. It is evident that no 3-connected proper minor of M contains X in its ground set. Therefore, M and its dual are isomorphic to largest members of $\mathcal{N}_k$. Moreover, by the next lemma, the set X is the only k-element subset of $E(M)$ not contained in a 3-connected proper minor of M unless M is a smallest codex.

Lemma 2.19. *Suppose M is a codex with $k-2$ pages for some $k \geq 4$, and $E(M) = \{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i \in [k-2]}$. Then a k-element subset X of $E(M)$ is not contained in the ground set of a 3-connected proper minor of M if and only if one of the following holds.*

(i) M is a rank-4 wheel or whirl, and X is the set of spokes or the rim; or

(ii) $X = \{s_1, s_2\} \cup \{b_i\}_{i \in [k-2]}$

Proof. From the remarks above, it suffices to show that if M has a k-element subset X that is not contained in the ground set of a 3-connected proper minor of M, then (i) or (ii) holds. First let $k = 4$. Then M is a wheel or a whirl. Write the ground set of M in a fan ordering. Consider the ordering cyclically so that the first and last elements are taken to be consecutive. If two consecutive elements in the ordering are not in X, then some combination of the deletion and contraction of those two elements yields a rank-3 wheel or whirl containing X in its ground set; a contradiction. The fan ordering therefore alternates between X and $E(M) - X$, so X is either the rim of M or the set of spokes of M.

Now suppose that $k > 4$. Note that $\mathrm{si}(M/s_1)$ is not 3-connected as it contains a 2-cocircuit. It follows, by Bixby's Lemma, that $\mathrm{co}(M\backslash s_1)$ is 3-connected. Since, by orthogonality, s_1 is in no triad, this matroid is just $M\backslash s_1$. Hence s_1 is in X. By symmetry, s_2 is in X. If X misses a triad $\{a_i, b_i, c_i\}$ altogether for some $i \in [k-2]$, then the deletion of this triad preserves X and is 3-connected. Therefore, X meets every such triad. Suppose X misses b_i for some $i \in [k-2]$. Then X contains either a_i or c_i but not both. Without loss of generality, suppose X contains a_i. Certainly $\mathrm{co}(M\backslash c_i)$ is not 3-connected since it contains a 2-circuit. Therefore, $\mathrm{si}(M/c_i) \cong M/c_i\backslash b_i$ is 3-connected. This is a contradiction, so $X = \{s_1, s_2\} \cup \{b_i\}_{i\in[k-2]}$. $\square$

Note that if X is the rim of the rank-4 wheel or whirl, then $X = \{a_1, a_2, c_1, c_2\}$. If X is the set of spokes, then $X = \{s_1, s_2, b_1, b_2\}$, the rank-4 case of (ii) in the statement of Lemma 2.19. Furthermore, the limitation on X given by the previous lemma is crucial to the proof of the next result, which shows inductively that the codices and their duals are the only largest members of $\mathcal{N}_k$.

Theorem 2.20. *Fix $k \geq 4$. Suppose M is a largest member of $\mathcal{N}_k$. Then one of M or M^* is a codex with $k-2$ pages.*

Proof. Theorem 2.3 proves the result when $k = 4$. Suppose $k > 4$ and that the statement holds for $\mathcal{N}_{k-1}$. Now $E(M)$ has a k-subset X that is contained in no 3-connected proper minor of M, and $|E(M)| = 3k - 4$.

Choose e in X. Then M has a minor-minimal 3-connected minor N that uses $X - e$. Thus $N \in \mathcal{N}_{k-1}$, so

$$|E(N)| \leq 3(k-1) - 4 = |E(M)| - 3 \tag{2.1}$$

Now M is a minor-minimal 3-connected matroid that uses e and has N as a minor. Thus, by Theorem 2.6, $|E(M)| \leq |E(N)| + 4$. But, when $|E(M)| = |E(N)| + 4$, which arises in (v) of that theorem, $M \backslash h/g$ is 3-connected and uses X; a contradiction. Thus

$$|E(M)| \leq |E(N)| + 3 \tag{2.2}$$

Then combining 2.1 and 2.2 shows that equality holds throughout each, so N is a largest member of $\mathcal{N}_{k-1}$. By the induction assumption, N or N^* is a codex with $k-3$ pages. Moreover, since $|E(M) - E(N)| = 3$, either (iii) or (iv) of Theorem 2.6 must hold. Assume $M_1 = M$ in that theorem by duality.

Suppose first that (iii) holds. If $x \notin X$, then $M/f \backslash x, g$ is 3-connected and uses X; a contradiction. Thus $x \in X$. As $\mathrm{co}(M \backslash g)$ is not 3-connected, $\mathrm{si}(M/g)$ is. If there is no element y of $E(M)$ such that $\{g, x, y\}$ is a triangle, then M/g or $M/g \backslash f$ is 3-connected; a contradiction. If there is such a y but $y \notin X$ then $M/g \backslash y$ or $M/g \backslash f, y$ is 3-connected; a contradiction. Thus $y \in X$. Then interchanging the labels on e and x gives case (iv) of Theorem 2.6. Therefore, it suffices to treat that case. By an argument similar to the above, both x and y are in X. Moreover, M has a 5-element fan (x, g, e, f, y) where $\{g, e, f\}$ is a triad whose deletion from M gives N. The case analysis that follows is structured around the possible identities of x and y in N.

First, consider the case that N is a codex with $k-3$ pages, taking $E(N) = \{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i \in [k-3]}$. Suppose $k = 5$. Then N is a rank-4 wheel or whirl and, by the previous lemma, $X - e$ is the rim or the set of spokes of N. Suppose first that $X - e$ is the rim $\{a_1, a_2, c_1, c_2\}$ of N. Assume then that $x = a_1$ and $y \in \{a_2, c_2\}$. The unique triad of N containing s_1 is $\{a_1, s_1, a_2\}$. Thus if M has a triad containing s_1, it must be $\{a_1, s_1, a_2\}$. But M has a triangle that meets $E(N)$ in $\{a_1\}$. Hence, by orthogonality, M has no triad containing s_1. Since M also has no triangle that contains s_1 and two elements of X, this is a contradiction to Lemma 2.5.

The case when $k = 5$ and $X - e$ is the set of spokes of N is included in the general argument to follow. Suppose $k > 4$. First observe that if both x and y are in the binding of N, then M is a codex with $k - 2$ pages.

Now suppose that at least one of x and y is not in the binding. Assume then that $x = b_1$ and that $y \in \{s_1, b_2\}$. The following argument shows that $M\backslash a_1$ is 3-connected. Assume the contrary, letting (A, B) be a 2-separation of $M\backslash a_1$. First, observe that the only triad of N containing a_1 is $\{a_1, b_1, c_1\}$. This triad is not a triad of M since M has a triangle that meets $E(N)$ in $\{b_1\}$. Hence M has no triad containing a_1, so $M\backslash a_1$ has no minimal 2-separations. Without loss of generality, assume that $|A \cap \{e, f, g\}| \geq 2$. Thus $(A \cup \{e, f, g\}, B - \{e, f, g\})$ is a 2-separation of $M\backslash a_1$, so we may assume that $\{e, f, g\} \subseteq A$. Then A spans $\{x, y\}$, so we may assume that A contains $\{e, f, g, x, y\}$. If $y = s_1$, then $(A \cup a_1, B)$ is a 2-separation of M; a contradiction. Thus $y = b_2$. Note that $(A - \{e, f, g\}, B)$ is a 2-separation of $N\backslash a_1$. But $N\backslash a_1$ is the parallel connection of the triangle $\{b_1, c_1, s_2\}$ and a 3-connected matroid. Hence the only 2-separations of $M\backslash a_1$ have one side equal to $\{b_1, c_1\}$ or $\{b_1, c_1, s_2\}$. Since neither $A - \{e, f, g\}$ nor B is equal to $\{b_1, c_1\}$ or $\{b_1, c_1, s_2\}$, this is a contradiction. Therefore $M\backslash a_1$ is 3-connected, which contradicts the minimality of M.

It remains to consider the case that N is the dual of a codex with $k-3$ pages. Assume that $k > 5$ since codices with two pages are self-dual. Recall that $M\backslash\{e, f, g\} = N$. Without loss of generality, assume that $x \in \{b_1, s_1\}$. Clearly a_1 is in no triangle of N containing two members of X. Moreover, the only triad of N containing a_1 is $\{a_1, b_1, s_1\}$ and, by orthogonality, it is not a triad of M. Therefore M has no triad containing a_1. This contradiction to Lemma 2.5 completes the proof of the theorem. $\square$

2.6 Conclusion

The characterizations given by Theorems 2.3 and 2.4 are admittedly not very useful in a computational sense for testing matroid 3-connectivity. A weakening of the equivalent condition to 3-connectivity still gives a characterization and improves the computational expense. Recall that $\mathcal{N}_k$ is the unique minimal set of matroids characterizing 3-connectivity with respect to k-subsets.

Proposition 2.21. *Fix $k \geq 4$, and let M be a matroid on at least k elements containing a fixed $(k-2)$-subset Y in its ground set. Then M is 3-connected if and only if, for each pair $\{e, f\} \subseteq E(M) - Y$, there is an $\mathcal{N}_k$-minor of M using $Y \cup \{e, f\}$.*

The proof of this proposition is not difficult using the techniques presented in this chapter. The proof is omitted.

It is natural to ask whether there are analogs of the main results of this chapter for higher connectivity. While, for example, there must be some minimal set of matroids that characterizes 4-connectivity with respect to k-element subsets for $k \geq 6$, there are currently no inductive tools in the style of Bixby's Lemma for 4-connectivity or higher. Therefore the methods used in the proofs of these results cannot be extended to find the appropriate lists.

Chapter 3

Matroids That Are Extremal with Respect to Bixby's Lemma

3.1 Introduction

In a 2-connected matroid, at least one of the deletion and the contraction of any element is 2-connected. One analog of this result for 3-connectivity is Tutte's Wheels-and-Whirls Theorem [14], which states that there is an element whose deletion or contraction maintains 3-connectivity except in the cases of wheels and whirls.

Unlike Tutte's Wheels-and-Whirls Theorem, Bixby's Lemma (Lemma 1.3) lets us choose the element we wish to remove with the caveat that the resulting matroid might have parallel or series pairs.

The work in this chapter is motivated by a natural extremal problem related to Bixby's Lemma. In particular, we consider the binary matroids for which each element can only be removed in exactly one of the ways given by Bixby's Lemma.

A 3-connected matroid M with $T \subseteq E(M) \neq \emptyset$ is called *Bixby-extremal on T* if, for each $e \in T$, one of $\mathrm{si}(M/e)$ and $\mathrm{co}(M\backslash e)$ is not 3-connected. The matroid M is *Bixby-extremal* if it is Bixby-extremal on $E(M) \neq \emptyset$.

The main result of this chapter gives a method of decomposing binary Bixby-extremal matroids into pieces that are either covered by large fans and certain six-element 3-separating sets containing many four-element fans, or are sequential binary Bixby-extremal matroids.

Given a 3-separation of a 3-connected matroid M, Beavers [1, Theorem 2.1.1] proved the existence of a natural extension of M formed by adding points on the boundary line between the sides of that 3-separation. Theorem 3.1 is a specialization of Beavers's result to binary matroids, although, for such matroids, the result is well known. This chapter makes frequent use of the extension of M described in the next result.

Theorem 3.1. *Let M be a 3-connected binary matroid and let (A, B) be a 3-separation of M. Then there is a matroid M' that is either M or an extension of M by a set L' with the following properties:*

(i) *M' has a triangle L such that $L' \subseteq L$.*

(ii) *M' has an exact 3-separation $(A \cup L', B)$, and $L = \mathrm{cl}_{M'}(A) \cap \mathrm{cl}_{M'}(B)$.*

(iii) *M', $M'|(A \cup L)$, and $M'|(B \cup L)$ are 3-connected.*

(iv) *M' is binary, and $M' = P_L(M'|(A \cup L), M'|(B \cup L))$.*

It is not difficult to see, by first embedding M in a binary projective geometry, that M' in the above theorem is unique up to isomorphism. We shall refer to the matroid constructed in the manner of M' as *the (A, B)-completion of M by L'*. Note that L' is a set of size $3 - |\mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)|$ that is disjoint from $E(M)$.

The following is the main result of this chapter. It shows that every binary Bixby-extremal matroid that does not fall into one of two natural classes can be decomposed into two smaller binary Bixby-extremal matroids.

Theorem 3.2. *Let M be a binary Bixby-extremal matroid. Then*

(i) *M is a sequential matroid; or*

(ii) *each element of M is an end of a four-element fan; or*

(iii) *M has size exceeding ten, and there is a 3-separation (X, Y) of M for which there are matroids M_X^1, M_X^2, M_Y^1, and M_Y^2 satisfying the following:*

Let M' be the (X, Y)-completion of M by L'. Let $L = \mathrm{cl}_{M'}(X) \cap \mathrm{cl}_{M'}(Y)$. For each Z in $\{X, Y\}$, the matroid M_Z^1 is the generalized parallel connection of $M'|(Z \cup L)$ and a copy of $M(K_4)$ across L. For each Z in $\{X, Y\}$, the matroid M_Z^2 is the deletion of

L' from the generalized parallel connection of $M'|(Z \cup L)$ and a copy of $M(\mathcal{W}_4)$ across L. Furthermore, at least one matroid in each of $\{M_X^1, M_X^2\}$ and $\{M_Y^1, M_Y^2\}$ is a binary Bixby-extremal matroid that is smaller than M.

Based on (i) of the previous result, it is natural to ask for a precise description of the binary sequential matroids that are Bixby-extremal. Such a description is given in Corollary 3.16.

Note that the decomposition in Theorem 3.2 gives two possible structures for gluing onto each side of the 3-separation. We prefer to use the structure involving the rank-4 wheel whenever possible. The $M(K_4)$-structure is only used in certain cases where the line separating the sides of the 3-separation already contains exactly three elements, and in certain well-understood cases where using the 4-wheel structure does not yield a 3-connected matroid.

Evidently, Theorem 3.2 lends itself to iteration. In Section 3.4, we obtain a corollary to Theorem 3.2 that says in short that a binary Bixby-extremal matroid can be constructed from a finite set of matroids, each of which is sequential or is a matroid in which every element is an end of a four-element fan. The operations used to piece these building blocks together are extension along the line dividing the sides of a 3-separation, generalized parallel connection, deletion of certain wheel-like 3-separating sets, and dualization.

In Section 3.2, we compile tools for working with 3-separating sets in binary Bixby-extremal matroids. In Section 3.3, we consider some methods of constructing Bixby-extremal matroids. Section 3.4 proves the main decomposition results of this chapter. Section 3.5 gives a catalog of small binary Bixby-extremal matroids.

3.2 Preliminaries

Suppose M is a 3-connected matroid, and (X, Y) is a 3-separation of M for which there is some $e \in E(M)$ such that both $(X \cup e, Y - e)$ and $(X - e, Y \cup e)$ are 3-separations. When this occurs, e is in exactly one of $\mathrm{cl}_M(X-e) \cap \mathrm{cl}_M(Y-e)$ and $\mathrm{cl}^*_M(X-e) \cap \mathrm{cl}^*_M(Y-e)$. In the former case, we say that e is in the *guts* of (X, Y). In the latter case, we say that e is in the *coguts* of (X, Y). Note that these definitions of guts and coguts are slightly different from what is standard in the literature. This is because in this chapter, we only ever consider an element e in the guts or coguts of a 3-separation (X, Y) when each of $X \cup e$ and $Y \cup e$ has at least four elements. Rather than repeatedly adding this requirement on such 3-separations, we have chosen to use these alternative definitions.

By submodularity of the rank function, for a partition (X, Y) of $E(M)$, we have $\lambda_M(X) = r_M(\mathrm{cl}_M(X)) + r_M(\mathrm{cl}_M(Y)) - r(M) \geq r_M(\mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y))$. Hence, for an exact 3-separation (X, Y), the respective closures of X and Y meet in a flat of rank at most two. For this reason, when M is binary, we often think of the guts elements of a 3-separation as being part of a *guts line*, which we may consider to exist by Theorem 3.1 even when there are not two guts elements for a given 3-separation.

Suppose M is a 3-connected matroid and A and B are disjoint subsets of $E(M)$, each of size at least two, for which $(e_1, e_2, \ldots, e_n)$ is an ordering of the elements of $E(M) - A - B$ such that $A \cup \{e_1, e_2, \ldots, e_i\}$ and $\{e_{i+1}, e_{i+2}, \ldots, e_n\} \cup B$ are exactly 3-separating for each $0 \leq i \leq n$. Then $(A, e_1, e_2, \ldots, e_n, B)$ is called a 3-*sequence*. Note that this means $E(M) - A - B \subseteq \mathrm{fcl}_M(A) \cap \mathrm{fcl}_M(B)$. Moreover, if $|A|, |B| \geq 3$, then each e_i is on the guts or coguts of a 3-separation in M, namely, $(A \cup \{e_1, \ldots, e_i\}, \{e_{i+1}, \ldots, e_n\} \cup B)$. Thus, if a 3-connected matroid M is Bixby-extremal on a set T, then T is a subset of the set T' of all $x \in E(M)$ such that there is a 3-sequence (A_x, x, B_x) for some partition (A_x, B_x) of $E(M) - x$ with $|A_x|, |B_x| \geq 3$. Furthermore, M is Bixby-extremal on T'.

An exactly 3-separating set X in a matroid is *sequential* if X can be given an ordering $(x_1, x_2, \ldots, x_n)$ such that $\{x_1, x_2, \ldots, x_i\}$ is 3-separating for all i such that $1 \leq i \leq n$. We call this ordering a *sequential ordering*. A 3-separation (X, Y) is *sequential* if X or Y is sequential. If M has a 3-sequence $(A, e_1, e_2, \ldots, e_n, B)$ with $|A| = |B| = 2$, then we say that

M is *sequential*. Beavers [1, Corollary 4.3.10] proved a characterization of the sequential 3-connected binary matroids.

The next result is well known. We include a proof for completeness.

Proposition 3.3. *Suppose M is a 3-connected matroid. There is an e in $E(M)$ such that either $\mathrm{si}(M/e)$ or $\mathrm{co}(M\backslash e)$ is not 3-connected if and only if e is on the guts or coguts, respectively, of a 3-separation of M.*

Proof. Suppose $\mathrm{si}(M/e)$ is not 3-connected. Then M/e has a non-minimal 2-separation (X, Y). We have then that $|X|, |Y| \geq 3$ and

$$\begin{aligned} 2 \leq \lambda_M(X \cup e) &= r_M(X \cup e) + r_M(Y) - r(M) \\ &= r_{M/e}(X) + r_M(e) + r_M(Y) - (r(M/e) + 1) \\ &= r_{M/e}(X) + r_M(Y) - r(M/e) \\ &= \lambda_{M/e}(X) + r_M(Y) - r_{M/e}(Y) \\ &\leq \lambda_{M/e}(X) + 1 = 2. \end{aligned}$$

Since equality must hold throughout the above, we deduce that $r_{M/e}(Y) = r_M(Y) - 1$. Hence $e \in \mathrm{cl}_M(Y)$. By symmetry, we have also that $e \in \mathrm{cl}_M(X)$, and that both $(X \cup e, Y - e)$ and $(X - e, Y \cup e)$ are 3-separations in M. Therefore, e is in the guts of a 3-separation in M. The converse of the above follows similarly from the rank function of M/e. The proposition holds by duality. $\square$

Each element of the ground set of a Bixby-extremal matroid M is on the guts or coguts of a 3-separation of M. This suggests that a typical Bixby-extremal matroid should have many 3-separations, so the notion of *uncrossings* is important for studying such matroids. Uncrossings are applications of the next lemma, the proof of which follows from the submodularity of the connectivity function. The reader may find more on uncrossings and their applications in [11].

Lemma 3.4. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.

(ii) If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.

We now give some results on fans in Bixby-extremal matroids. If F is a fan with at least four elements and $(e_1, e_2, \ldots, e_n)$ is a fan ordering of F, then every fan ordering of F has its first and last elements in $\{e_1, e_n\}$. We call e_1 and e_n the *ends* of the fan. Note that some authors only define the ends of a fan when the fan is maximal.

Proposition 3.5. *If M is a 3-connected matroid on at least seven elements, and $e \in E(M)$ is the end of a fan of length at least four, then e is on the guts or coguts of a 3-separation in M.*

Proof. Let (a, b, c, e) be a fan ordering of a four-element fan in M. Then

$$(\{a, b, c\}, e, E(M) - \{a, b, c, e\})$$

is a 3-sequence. □

In a fan having at least six elements, every element is an end of a four-element fan. Therefore, we have the following immediate consequences of Proposition 3.5.

Corollary 3.6. *If M is a 3-connected matroid, and e is an element of M that is contained in a fan of length at least six, then e is on the guts or coguts of a 3-separation in M.*

Restating the above corollary, we have the following:

Corollary 3.7. *If M is a 3-connected matroid, and F is a fan in M of length at least six, then M is Bixby-extremal on F.*

This might suggest that fans are somehow natural structures to consider when studying binary Bixby-extremal matroids. Indeed, fans will appear often in this chapter. Note that a fan of odd length $2k+1$ for some $k \geq 1$ either has triangles at both ends or has triads at both ends. We shall refer to these as $(2k+1)$-*fans* and $(2k+1)$-*cofans*, respectively. Small exactly 3-separating sets in binary matroids contain fans or four-element circuit-cocircuits. The next proposition of Oxley, Semple, and Whittle [12, Lemma 2.9], concerns the second of these two structures. We give the proof here for completeness.

Proposition 3.8. *Suppose M is a 3-connected matroid, and $X \subseteq E(M)$ such that X is a circuit-cocircuit of size four. If $e \in X$, then both $\mathrm{si}(M/e)$ and $\mathrm{co}(M\backslash e)$ are 3-connected.*

Proof. Suppose $e \in X$ and (A, B) is a partition of $E(M\backslash e)$ such that each of $|A|$ and $|B|$ is at least three. Up to exchanging the labels on A and B, we have $|A \cap (X - e)| \geq 2$. Since $X - e$ is a triad of $M\backslash e$ and $e \in \mathrm{cl}(X - e)$, we have that

$$\begin{aligned}
\lambda_{M\backslash e}(A) &= \lambda_{M\backslash e}(A \cup (X - e)) \\
&= r_M(A \cup (X - e)) + r_M(B - X) - r(M) \\
&= r_M(A \cup X) + r_M(B - X) - r(M) \\
&= \lambda_M(A \cup X) \geq 2.
\end{aligned}$$

Thus, $M\backslash e$ has no non-minimal 2-separations. By duality, M/e has no non-minimal 2-separations. □

We call a circuit-cocircuit of size four a *quad*. Since a Bixby-extremal matroid has no quads, the possibilities for small 3-separating sets in binary Bixby-extremal matroids are quite limited. The next proposition identifies a particular 3-separating set that will be important in what follows.

Proposition 3.9. *Suppose M is a 3-connected binary matroid on at least nine elements where $X \subseteq E(M)$ with $M|X \cong M(K_4)$ and $M^*|X \cong U_{2,3} \oplus U_{3,3}$. Then X is 3-separating in M, and each element of X is an end of a four-element fan in M.*

Proof. The set X is 3-separating because $r(X) + r^*(X) - |X| = 3 + 5 - 6 = 2$. The guts line of $(X, E - X)$ does not meet the triad in X by orthogonality since each guts element is in the closure of $E - X$, so this triad is disjoint from one of the triangles in X. This implies that the proposition holds. □

We refer to a 3-separating set X as a K_4-*separator* of a 3-connected binary matroid M if $|E(M)| \geq 9$ and, for some N in $\{M, M^*\}$, we have that $N|X \cong M(K_4)$ and $N^*|X \cong U_{2,3} \oplus U_{3,3}$. Two examples of K_4-separators in graphic matroids are illustrated in Figure 3.1.

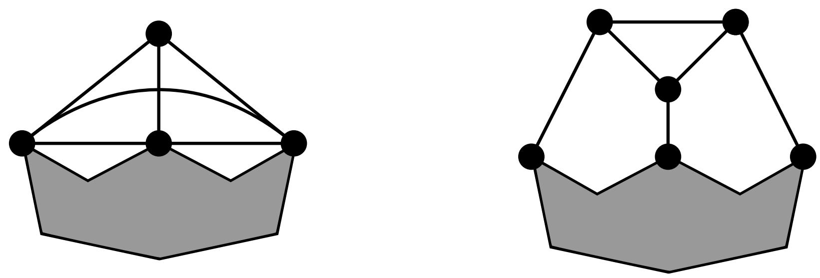


FIGURE 3.1: Graphic examples of K_4-separators. In each drawing, the shaded shape represents the part of the simple, 3-connected graph other than the edges of the K_4-separator and its incident vertices.

Proposition 3.10. *Suppose M is a binary Bixby-extremal matroid, and $X \subseteq E(M)$ is a 3-separating set of size four, five, or six.*

(i) If $|X| = 4$, then X is a 4-fan.

(ii) If $|X| = 5$, then X is a 5-fan or 5-cofan.

(iii) If $|X| = 6$ and there is an element e in X that is on the guts of $(X, E - X)$, then

(a) X is a 6-fan; or

(b) $X - e$ *is a* 5*-cofan, and* e *is in a plane spanned by three elements of the cofan so that* e *is in the guts of* $(X, E - X)$*; or*

(c) X *is a* K_4*-separator.*

Proof. If $4 \leq |X| \leq 5$, the result is elementary since M contains no quads. Suppose that $|X| = 6$ and that there is some guts element $e \in X$. Then $X - e$ is a five-element 3-separating set. Therefore, $X - e$ is a 5-fan or 5-cofan. If $X - e$ is a 5-fan, then X is a K_4-separator. Suppose then that $X - e$ is a 5-cofan. If e is spanned by one of the triads contained in $X - e$, then X is a 6-fan. Otherwise, e is contained in a circuit C that meets both of the cocircuits contained in $X - e$. Let $(a_1, a_2, a_3, a_4, a_5)$ be a 5-cofan ordering of $X - e$. We have that $\{a_1, a_2, a_3\} \triangle \{a_3, a_4, a_5\} = (X - e) - a_3$ is a 4-cocircuit C^* of M. Since M has no quads and $C \cap C^*$ is nonempty having an even number of elements, we see that $|C \cap C^*| = 2$. Therefore, $C = \{a_1, a_3, a_5, e\}$. □

Corollary 3.11. *If* M *is a binary Bixby-extremal matroid, and there is some* e *in* $E(M)$ *that is in no fan of* M*, then* $|E(M)| \geq 11$.

If an element is an end of a 4-fan, we can determine whether its deletion or contraction has a non-minimal 2-separation. The next two propositions examine whether the deletion or contraction of the middle element of a fan of length five has a non-minimal 2-separation.

Proposition 3.12. *Suppose* $(1, 2, 3, 4, 5)$ *is a fan ordering of a* 5*-fan* X *in a* 3*-connected binary matroid* M *such that* X *is in no* K_4*-separator. Then* $\mathrm{co}(M \backslash 3)$ *is* 3*-connected.*

Proof. The matroid $\mathrm{co}(M \backslash 3)$ is isomorphic to the matroid that is obtained from $M \backslash \{2, 3, 4\}$ by adding the third point z on the line spanned by $\{1, 5\}$. This matroid is 3-connected by Theorem 3.1. □

Proposition 3.13. *Suppose* M *is a* 3*-connected binary matroid on at least nine elements, and* $(1, 2, 3, 4, 5)$ *is a fan ordering of a* 5*-cofan* X *in* M *such that* X *is in no* K_4*-separator.*

Suppose further that $(X, E-X)$ has a single guts element, and this guts element does not extend X to a larger fan. Then both $\mathrm{co}(M\backslash 3)$ *and* $\mathrm{si}(M/3)$ *are* 3*-connected.*

Proof. By Proposition 3.12 and duality, we have that $\mathrm{si}(M/3)$ is 3-connected. Note that $M\backslash 3$ has two series pairs, namely $\{1,2\}$ and $\{4,5\}$. Neither of these meets a triangle, so $\mathrm{co}(M\backslash 3)$ is isomorphic to $M\backslash 3/\{1,4\}$, which is isomorphic to the deletion of X from the $(X, E-X)$-completion of M by L'. By Theorem 3.1, this matroid is 3-connected. □

We have then that a binary Bixby-extremal matroid cannot contain the configuration in Proposition 3.13, so we obtain the following corollaries.

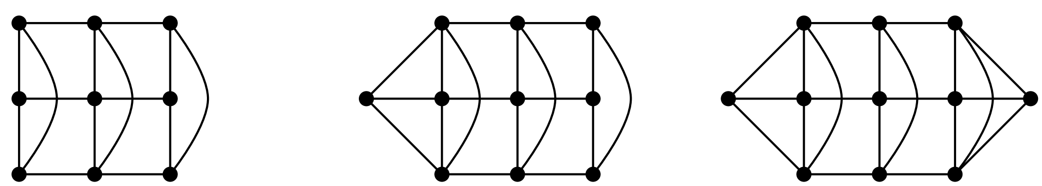
Corollary 3.14. *Suppose M is a binary Bixby-extremal matroid on at least nine elements, and $(1,2,3,4,5)$ is a fan ordering of a 5-cofan X in M spanning an element 6 that does not extend X to a larger fan. Then X is in a K_4-separator, or there is some element 7 of $E(M)-X-6$ for which $X \cup 7$ is a six-element fan.*

Corollary 3.15. *Suppose M is a binary Bixby-extremal matroid on at least nine elements. If A is a sequential* 3*-separating set in M with $|A| \geq 6$, then there is some $e \in E(M)$ such that $A \cup e$ is a sequential* 3*-separating set with a sequential ordering in which the first six elements of the ordering comprise a six-element fan or a K_4-separator.*

Proof. Obtain a five-element sequential 3-separating subset B of A by truncating a sequential ordering of A. By Proposition 3.10, B is a 5-fan or a 5-cofan. Up to duality, assume B is a 5-cofan. There is an some e in $A-B$ that is in the closure or coclosure of B. If e is in the coclosure of B, then $B \cup e$ is a K_4-separator. Suppose then that e is in the closure of B. If $B \cup e$ is not a 6-fan, then B is in a K_4-separator or a 6-fan by Corollary 3.14.

Let f be the element of $E(M)-B$ such that $B \cup f$ is a 6-fan or a K_4-separator. Evidently, there is a sequential ordering of $A \cup f$ such that the first six elements are $B \cup f$. □

Beavers [1, Corollary 4.3.10] characterized the binary 3-connected matroids that are sequential. From this characterization, we can describe the sequential binary 3-connected matroids that are Bixby-extremal. First, we define a family of graphs that are used in Beavers's characterization. For $n \geq 0$, let P_1, P_2, and P_3 be pairwise vertex-disjoint paths, the vertices of which can be ordered $(x_0, x_1, \ldots, x_n)$, $(y_0, y_1, \ldots, y_n)$, and $(z_0, z_1, \ldots, z_n)$, respectively, such that, for each i in $\{0, 1, \ldots, n-1\}$, there are edges $x_i x_{i+1}$, $y_i y_{i+1}$, and $z_i z_{i+1}$. Let P^n_Δ be the graph obtained by adding edges $x_i y_i$, $x_i z_i$, and $y_i z_i$ to $P_1 \cup P_2 \cup P_3$, for each i in $\{0, 1, \ldots, n\}$. We say that P^n_Δ is a *sequential* 3*-path with triangle ends*. We say that the graph P^n_Y is a *sequential* 3*-path with triad ends* if P^n_Y is obtained from P^{n-2}_Δ by adding vertices p_0 and p_{n-2} such that p_0 is adjacent only to each vertex in $\{x_0, y_0, z_0\}$ and p_{n-2} is adjacent only to each vertex in $\{x_{n-2}, y_{n-2}, z_{n-2}\}$. We say that the graph P^n_M is a *sequential* 3*-path with mixed ends* if P^n_M is obtained from P^{n-1}_Δ by adding a vertex p_0 that is adjacent only to each vertex in $\{x_0, y_0, z_0\}$. A graph that is of any of the three types given is called a *sequential* 3*-path*. For each sequential 3-path P^n_Δ, P^n_Y, and P^n_M, we say that the *length* of the sequential 3-path is n. Each sequential 3-path is 3-connected. Figure 3.2 shows some examples of small sequential 3-paths.



(a) P^3_Δ (b) P^4_M (c) P^5_Y

FIGURE 3.2: Three examples of sequential 3-paths.

Note that the edges of a sequential 3-path with triangle ends P^n_Δ can be partitioned $(A^0_\Delta, A^0_Y, A^1_\Delta, A^1_Y, \ldots, A^{n-1}_\Delta, A^{n-1}_Y, A^n_\Delta)$ such that A^i_Δ is the triangle $\{x_i, y_i, z_i\}$ for each i in $\{0, 1, \ldots, n\}$, and A^j_Y is the triad $\{x_j x_{j+1}, y_j y_{j+1}, z_j z_{j+1}\}$ for each j in $\{0, 1, \ldots, n-1\}$. We may partition the edges of a sequential 3-path with triad ends P^n_Y by obtaining such a partition on $E(P_{\Delta}^{n-2})$ before prepending $\{p_0x_0, p_0y_0, p_0z_0\}$ and appending $\{p_{n-2}x_{n-2}, p_{n-2}y_{n-2}, p_{n-2}z_{n-2}\}$. Similarly, the edges of a sequential 3-path with mixed ends P_M^n can be partitioned by prepending $\{p_0x_0, p_0y_0, p_0z_0\}$ to such a partition of the edges of P_{Δ}^{n-1}. For any sequential 3-path P, let $\Pi_Y^{\Delta}(P)$ be a partition of the edges of P obtained in one of three ways above. Note that $\Pi_Y^{\Delta}(P)$ alternates between triangles and triads in $M(P)$. We say that the edges in an end of the ordered partition $\Pi_Y^{\Delta}(P)$ comprise an *end* of the matroid $M(P)$. When N is a minor of $M(P)$ that uses an end of $M(P)$, we shall say that this set of edges is also an *end* of N. An element of the partition $\Pi_Y^{\Delta}(P)$ that is not an end is called an *internal triangle* or *internal triad* if it is a triangle or triad, respectively, of $M(P)$. We now specialize Beavers's characterization of binary sequential 3-connected matroids to those that are Bixby-extremal.

Corollary 3.16. *Let M be a sequential 3-connected binary Bixby-extremal matroid. Then M is isomorphic to a matroid N that can be obtained from the cycle matroid of a sequential 3-path by*

(i) contracting members of internal triads,

(ii) deleting members of internal triangles, and then

(iii) deleting all but one element from each parallel class and contracting all but one element from each series class,

such that each end of N is in a six-element fan or a K_4-separator in N. Moreover, any binary matroid M that is obtained in such a way from the cycle matroid of a sequential 3-path is a sequential 3-connected binary Bixby-extremal matroid.

Proof. This is a consequence of [1, Corollary 4.3.10] and Corollary 3.15. □

The elementary proof of the following proposition is omitted.

Proposition 3.17. *Suppose F is a maximal fan in a 3-connected matroid and $|F| = 5$. If (a, b, c, d, e) is a fan ordering of F such that c is an end of a four-element fan, then there is an element $f \notin F$ such that (c, b, d, f) is a fan, and $F \cup f$ is a K_4-separator.*

Lemma 3.18. *Suppose X is a four- or five-element fan in a binary Bixby-extremal matroid M. Then*

(i) X is not a maximal fan; or

(ii) X is a subset of a K_4-separator; or

(iii) there is a 3-sequence $(A, x_1, x_2, \ldots, x_n, B)$ in M with $|A|, |B| \geq 3$, where $(x_1, x_2, \ldots, x_n)$ is a fan ordering of X.

Proof. Suppose X is a fan that is not in a K_4-separator. The set X has $(1, 2, 3, 4)$ or $(1, 2, 3, 4, 5)$ as a fan ordering. If $|X| = 5$, we have that, for some N in $\{M, M^*\}$, the set $\{1, 2, 3\}$ is a triangle. By Proposition 3.12, $\mathrm{co}(N\backslash 3)$ is 3-connected. If $|X| = 4$, then $\mathrm{co}(M\backslash 3)$ is 3-connected in one of M and M^*; furthermore, we may reverse the fan ordering of X or swap the order of 2 and 3 to obtain another fan ordering of X. Hence, we assume up to duality and by possibly relabeling the elements of X that $\mathrm{co}(M\backslash 3)$ is 3-connected, and that $\{1, 2, 3\}$ is a triangle of M.

We have that $M/3$ has a non-minimal 2-separation (A, B) such that $1 \in A$. By closure, we may assume that $2 \in A$. Then $4 \notin A$ because $(A \cup 3, B)$ is not a 2-separation of M. By closure, we may assume that $5 \notin A$. We have that $(A \cup 3, B)$ and $(A, B \cup 3)$ are 3-separations in M. Moreover, the element 3 is in the guts of those 3-separations. If $|A| \leq 4$, then $A \cup 3$ contains a four-element fan of M containing the triangle $\{1, 2, 3\}$. This fan extends X to a larger fan. Suppose now that $|A| \geq 5$. Then $(A - \{1, 2\}, 1, 2, 3, B)$ is a 3-sequence in M. We have by orthogonality with the cocircuit $\{2, 3, 4\}$ that B is not a triangle. Therefore, $|B| \geq 4$. If $|X| = 4$, we have that $(A - \{1, 2\}, 1, 2, 3, 4, B - 4)$ is a 3-sequence, and the lemma holds.

Suppose then that $|X| = 5$. We apply the above argument by the symmetry of the 5-fan. Either $|B| \leq 4$, and X is not a maximal fan; or $|B| \geq 5$, and $(A-\{1,2\}, 1, 2, 3, 4, 5, B-\{4,5\})$ is a 3-sequence in M. $\square$

We now consider a certain structure involving triads that meet both sides of a 3-separation. This will be important to understand when proving the main decomposition results in this chapter.

Proposition 3.19. *Suppose M is a 3-connected binary matroid having a 3-separation (X, Y) with guts S. If T is a triad of M satisfying $|(X - S) \cap T| = 2$, then $|S| \leq 1$.*

Proof. Suppose M has such a triad T. By orthogonality between T and each of the circuits meeting S that are spanned by Y, we have that $T \cap S = \emptyset$. Note that $M|(X \cup S)$ contains a series pair $X \cap T$, so $M|(X \cup S)$ is not 3-connected. Let M' be the (X, Y)-completion of M by L'. Let $L = S \cup L'$. Since $M'|(X \cup L)$ is 3-connected and has no cocircuits of size less than three, there is a nonempty subset R of L' for which $R \cup T$ is a cocircuit of M'. By orthogonality with the guts line of $(X \cup L', Y)$ in M', we have that $|R| = 2$. Thus $|S| \leq 1$. $\square$

The next lemma and the two propositions that follow it are not employed in the rest of the chapter, but they are included here because the inductive-type results they give are in line with this chapter's aim.

Lemma 3.20. *Suppose M is a Bixby-extremal matroid, and $e \in E(M)$ such that $M\backslash e$ is 3-connected and e is in no 3-separating set of M of size four. Then $M\backslash e$ is Bixby-extremal.*

Proof. Let $a \in E(M) - e$. Then a is on the guts or coguts of a 3-separation (A, B) of M with $e \in A$. Since $(A - a, B \cup a)$ is a 3-separation and $e \in A - a$, we have that $|A - a| \geq 5$. Hence $|A - a - e| \geq 4$. Furthermore, by [10, Lemma 8.2.4], we have that $\lambda_{M\backslash e}(B - a) \leq \lambda_M(B - a) = 2$ and $\lambda_{M\backslash e}(B \cup a) \leq \lambda_M(B \cup a) = 2$, so a is on the guts or coguts of the 3-separation $(A - e, B)$ of $M\backslash e$. $\square$

The following propositions follow easily from Lemma 3.20.

Proposition 3.21. *Suppose M is a binary Bixby-extremal matroid, and $e \in E(M)$ with e in no triad or triangle of M. Then the unique 3-connected member of $\{M\backslash e,\ M/e\}$ is also Bixby-extremal.*

Proof. The 3-separating sets of size four in a binary Bixby-extremal matroid are covered by triangles and triads. As e is in no triad or triangle, e is in no 3-separating set of size four. Exactly one of $\mathrm{co}(M\backslash e) = M\backslash e$ and $\mathrm{si}(M/e) = M/e$ is 3-connected. By Lemma 3.20 and duality, the proposition holds. □

Proposition 3.22. *Suppose M is a binary Bixby-extremal matroid, and $e \in E(M)$ with e contained in no fan of M having at least four elements. If $M\backslash e$ is 3-connected, then $M\backslash e$ is Bixby-extremal. Dually, if M/e is 3-connected, then M/e is Bixby-extremal.*

Proof. Since e is in no fan of length at least four, we have that e is in no 3-separating set of size four. The proposition follows from Lemma 3.20 and duality. □

3.3 Constructions

The results in this section are used to describe how the generalized parallel connection affects the Bixby-extremal property. In particular, we have that gluing two binary matroids together along a triangle does not destroy the property that an element is on the guts or coguts of a 3-separation.

Lemma 3.23. *Suppose M_1 and M_2 are 3-connected binary matroids having a common restriction $N \cong U_{2,3}$ with $E(M_1) \cap E(M_2) = E(N)$. Let $M = P_N(M_1, M_2)$, and suppose $a \in E(M_1) - E(M_2)$. If a is on the guts of a 3-separation of M_1, then a is on the guts of a 3-separation of M; and if a is on the coguts of a 3-separation of M_1, then a is on the coguts of a 3-separation of M.*

Proof. We have that a is on the guts or coguts of a 3-separation (A, B) in M_1 such that $B - a$ contains at least two elements of $E(N)$. Then

$$\begin{aligned}
\lambda_M(A-a) &= r_M(A-a) + r_M(E(M) - (A-a)) - r(M) \\
&\leq r_{M_1}(A-a) + (r_{M_1}(B \cup a) + r(M_2) \\
&\quad - 2) - (r(M_1) + r(M_2) - 2) \\
&= r_{M_1}(A-a) + r_{M_1}(B \cup a) - r(M_1) \\
&= \lambda_{M_1}(A-a) \\
&= 2.
\end{aligned}$$

□

Proposition 3.24. *Suppose M_1 and M_2 are binary 3-connected matroids having a common restriction $N \cong U_{2,3}$ such that $E(M_1) \cap E(M_2) = E(N)$. If each M in $\{M_1, M_2\}$ is Bixby-extremal on $E(M) - E(N)$, then $P_N(M_1, M_2)$ is a binary Bixby-extremal matroid.*

Proof. It is clear that N is the guts line of a 3-separation in M. Suppose $a \in E(M) - E(N)$. Then a is on the guts or coguts of a 3-separation of M_1 or M_2. By Lemma 3.23, the element a is on the guts or coguts of a 3-separation of $P_N(M_1, M_2)$. □

3.4 Decompositions

In this section, we prove the main decomposition results of the chapter. Before that, we prove a number of lemmas and propositions needed to lay some groundwork for the proofs. The first lemma helps us to keep track of cocircuits as we perform decomposition operations.

Lemma 3.25. *Let M be a 3-connected binary matroid with a non-minimal 3-separation (X, Y) having guts L_0, and let M' be the (X, Y)-completion of M by L'. If T is a cocircuit of M and $T \subseteq X \cup L_0$, then T is a cocircuit of M'.*

Proof. The lemma follows from orthogonality since $E(M') - E(M) \subseteq \mathrm{cl}_{M'}(Y - (L_0 \cup L'))$. □

An attractive property of Theorem 3.1 is noted in the following proposition.

Proposition 3.26. *Suppose (X, Y) is a 3-separation of a 3-connected binary matroid M with $L_0 = \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Let M' be the (X, Y)-completion of M by L', and let $N = M'|(X \cup L_0 \cup L')$. If X is a sequential 3-separating set in M, then N is a sequential matroid. A sequential ordering in M of the elements of X is a sequential ordering of the elements of X in N.*

Proof. Let $(x_1, x_2, \ldots, x_n)$ be a sequential ordering of X in M. Each circuit of M contained in X is still a circuit in N. By Lemma 3.25, if $Z \subseteq X$ and $x \in X \cap \mathrm{cl}^*_M(Z)$, then $x \in X \cap \mathrm{cl}^*_N(Z)$. It follows that $(x_1, x_2, \ldots, x_n)$ is a sequential ordering of X in N. As any elements of N not in X are in the closure of X in N, we have that N is sequential. □

The next lemma shows that if X is a Bixby-extremal set of elements in a matroid M, and N is a 3-connected minor of M that uses X, then the set X is Bixby-extremal in N.

Lemma 3.27. *Suppose a is on the guts or coguts of a 3-separation (A, B) of M. Then, for each 3-connected minor N of M with $A \cup a \subseteq E(N)$ and $|E(N) - (A \cup a)| \geq 3$, the element a is on the guts or coguts of the 3-separation $(A, E(N) - A)$ of N.*

Proof. Suppose $N = M \backslash D / C$. By Lemma 8.2.4 of [10], we have $2 = \lambda_M(A - a) \geq \lambda_N(A - a)$ and $2 = \lambda_M(A \cup a) \geq \lambda_N(A \cup a)$. Since N is 3-connected, equality holds throughout. □

The following lemma and its corollaries tell us how decomposing across a line of separation as in Theorem 3.1 can affect the set of Bixby-extremal elements in a matroid.

Lemma 3.28. *Suppose M is a 3-connected binary matroid having a 3-separation (X, Y). Let L_0 be the guts of (X, Y), and suppose that $|X \cap L_0| \leq 1$, that $|X| \geq 6$, and that $|Y| \geq 4$. Let M' be the (X, Y)-completion of M by L'. Suppose $a \in X - L_0$ is on the guts or coguts of a 3-separation (A, B) of M, where neither $A - a$ nor $B - a$ is a triad having exactly two elements in $X - L_0$. Let $L = L_0 \cup L'$. Then a is on the guts or coguts of a 3-separation in $M'|(X \cup L)$.*

Proof. Suppose $A \subseteq Y$ or $B \subseteq Y$. Then a is in the closure or coclosure of Y. Since $a \notin L_0$, we have that a is in the coclosure of Y. Therefore, a is in the coguts of (X, Y). Thus, there is a cocircuit of M containing a in $Y \cup a$, so there is a cocircuit of M' containing a in $Y \cup a \cup L$. We have then that a is in a cocircuit in $M'|(X \cup L)$ that is contained in $L \cup a$. Since $M'|(X \cup L)$ is binary and 3-connected, this cocircuit is a triad, so a is an end of a four-element fan in $M'|(X \cup L)$, and the lemma holds by Proposition 3.5.

Suppose then that neither A nor B is a subset of Y. Since $|X| \geq 6$, we have that one of $|(A \cap X) \cup a|$ and $|(B \cap X) \cup a|$ is at least four. Up to relabeling, assume the former. Furthermore, we may assume $a \in A$ by closure or coclosure. We have that $|B \cap X| \geq 1$ since $B \not\subseteq Y$.

Suppose $|B \cap Y| \geq 2$. Then $|A \cap X|$ is 3-separating by uncrossing. Since L' is in the closure of Y in M', we have that $r_M(E(M) - (A \cap X)) = r_{M'}((E(M') - (A \cap X)) \cup L)$ and $r_M((E(M) - (A \cap X)) \cup a) = r_{M'}((E(M') - (A \cap X)) \cup L \cup a)$. Therefore, $\lambda_{M'}(A \cap X) = \lambda_M(A \cap X) = \lambda_M((A \cap X) - a) = \lambda_{M'}((A \cap X) - a) = 2$. Since $|B \cap X| \geq 1$ and there are at least two members of L not in X by assumption, the lemma holds by Lemma 3.27.

Suppose now that $|B \cap Y| < 2$. If $B \cap Y = \emptyset$, then $|(B \cap X) \cup a|$ is at least four and $|(A - a) \cap Y| \geq 2$, and we have the lemma by exchanging the labels on A and B and applying the above argument. Suppose then that $|B \cap Y| = 1$. If $|B \cap X| > 2$, then $|A \cap Y| \geq 2$, and the lemma holds by relabeling. Assume then that $|B \cap X| = 2$. Thus, B is a triangle or a triad. If B is a triangle, then the element in $B \cap Y$ is a member of L_0. In this case, $B \cup a$ is a four-element fan that is also a fan of $M'|(X \cup L)$ by Lemma 3.25, so the lemma holds. Suppose instead that B is a triad. By assumption, an element of B must also be a member of L_0. The element a is in a triangle with two of the elements in B. It must be that a is in a triangle with the two elements in $B \cap X$, as otherwise the element in $B \cap Y$ would be in both the closure and coclosure of X. Since $M'|(X \cup L)$ is 3-connected, we have that $B \cap X$

is contained in a cocircuit of $M'|(X \cup L)$ that meets L. Furthermore, this cocircuit meets L in exactly two elements by orthogonality. It follows that a is again an end of a four-element fan in $M'|(X \cup L)$, so the lemma holds by Proposition 3.5. □

Corollary 3.29. *Suppose M is a binary Bixby-extremal matroid having a 3-separation (X, Y) with guts $L_0 \neq \emptyset$. Suppose also that $|X \cap L_0| \leq 1$, that $|X| \geq 6$, and that $|Y| \geq 4$. Let M' be the (X, Y)-completion of M by L'. Let $L = L_0 \cup L'$. Let $\mathcal{A}$ be the set of all elements $a \in X - L_0$ for which a is an end of some four-element fan F satisfying the following:*

(i) $F - a$ *is a triad,*

(ii) $|(F - a) \cap (X - L_0)| = 2$, *and*

(iii) $|(F - a) \cap (Y - L - 0)| = 1$.

Then $M'|(X \cup L)$ is Bixby-extremal on $X - (L \cup \mathcal{A})$.

Proof. Suppose e is an element of $X - \mathcal{A} - L_0$. Then e is on the guts or coguts of a 3-separation (A, B) of M where neither $A - e$ nor $B - e$ is a triad T for which $|T \cap (X - L_0)| = 2$ and $|T \cap (Y - L_0)| = 1$. By Lemma 3.28, e is on the guts or coguts of a 3-separation in $M'|(X \cup L)$. □

Recall that Proposition 3.19 says that if there are at least two elements on the guts of the 3-separation (X, Y) in Corollary 3.29, then the set $\mathcal{A}$ in that corollary is empty. As a consequence, we have the next result.

Corollary 3.30. *Suppose M is a binary Bixby-extremal matroid having a 3-separation (X, Y) with guts $L_0 \neq \emptyset$. Suppose also that $|X \cap L_0| \leq 1$, that $|X| \geq 6$, and that $|Y \cup L_0| \geq 4$. Let M' be the (X, Y)-completion of M by L', and let $L = L_0 \cup L'$. If $|L_0| \geq 2$, then $M'|(X \cup L)$ is Bixby-extremal on $X - L$.*

Our decomposition involves replacing one side of a 3-separation with a specially chosen gadget matroid, namely, a wheel of rank three or four. The rank-three wheel is not itself a Bixby-extremal matroid, but the next proposition tells us that the elements of that wheel become Bixby-extremal elements after gluing.

Proposition 3.31. *Suppose M is a 3-connected binary matroid that is Bixby-extremal on $E(M)-S$, and N is a matroid isomorphic to $M(K_4)$ such that $E(M)\cap E(N)=T$, where T is a triangle of each of M and N. Then $P_T(M,N)$ is Bixby-extremal on $(E(M)\cup E(N))-(S-T)$.*

Proof. The proposition follows by Lemma 3.23 and Proposition 3.9. □

The statements of the following two results are somewhat lengthy and technical, so we now give an outline of what these results accomplish. Lemma 3.32 enables our iterative decomposition strategy. This lemma says that, given a 3-separation satisfying certain conditions in a binary Bixby-extremal matroid, we can replace one side of the 3-separation with one of our two wheel-like gadgets to obtain another binary Bixby-extremal matroid. Theorem 3.33 essentially provides all of the substance of the main result of this chapter. That theorem shows that a binary Bixby-extremal matroid that is not small, sequential, or covered by many large fans or K_4-separators has a 3-separation satisfying the conditions in Lemma 3.32. Furthermore, the theorem tells us that we can apply the lemma to both sides of the 3-separation and obtain two smaller matroids after decomposing.

Lemma 3.32. *Let M be a binary Bixby-extremal matroid. Suppose that $e\in E(M)$, and that e is on the guts L_0 of a 3-separation (X,Y) of M. Suppose further that $|X\cap L_0|\leq 1$, that $|X|\geq 6$, and that $|Y|\geq 4$. Let M' be the (X,Y)-completion of M by L'. Let $L=L_0\cup L'$. Let f label an element of $L-e$ such that if $|L_0|\geq 2$, then $f\in L_0-e$. Take copies of $M(K_4)$ and $M(\mathcal{W}_4)$ for which the following hold:*

(i) the ground set of each of $M(K_4)$ and $M(\mathcal{W}_4)$ meets $E(M')$ in L;

(ii) L is a triangle of each of $M(K_4)$ and $M(\mathcal{W}_4)$; and

(iii) e and f are spokes of $M(\mathcal{W}_4)$.

Then each of the following holds:

3.32.1. *If $\mathrm{fcl}_M(Y) \cap X \subseteq L_0$, then $P_L(M'|(X \cup L), M(\mathcal{W}_4))\backslash L'$ is 3-connected.*

3.32.2. *If $P_L(M'|(X \cup L), M(\mathcal{W}_4))\backslash L'$ is 3-connected, then it is a binary Bixby-extremal matroid.*

3.32.3. *If $\mathrm{fcl}_M(X) = X \cup L_0$, then $P_L(M'|(X \cup L), M(K_4))$ is a binary Bixby-extremal matroid.*

3.32.4. *If $P_L(M'|(X \cup L), M(K_4))$ is not a Bixby-extremal matroid, then $|L_0| = 1$, and $P_L(M'|(X \cup L), M(\mathcal{W}_4))\backslash L'$ is a binary Bixby-extremal matroid.*

3.32.5. *If X is a sequential 3-separating set, then*

(a) $P_L(M'|(X \cup L), M(K_4))$ is a sequential matroid; and

(b) if $P_L(M'|(X \cup L), M(\mathcal{W}_4))\backslash L'$ is 3-connected, then it is a sequential matroid.

For each of the proofs of statements 3.32.1–3.32.5 from Lemma 3.32, let $\mathcal{A}$ be the set of all elements $c \in X - L_0$ such that c is an end of a four-element fan containing a triad T_c for which $|T_c \cap (X - L_0)| = 2$ and $|T_c \cap (Y - L_0)| = 1$.

Proof of 3.32.1. If $L' = \emptyset$, the statement holds. Suppose then that $g \in L'$. Evidently, L is the guts line of a 3-separation in the 3-connected matroid $N = P_L(M'|(X \cup L), M(\mathcal{W}_4))$. We have that $\mathrm{co}(N\backslash g)$ is 3-connected. It is clear by orthogonality that g is in no triad of N, so $N\backslash g$ is 3-connected. Now suppose that f is also in L'. If f is in a triad of $N\backslash g$, then f is in a four-element cocircuit containing g in N. However, such a cocircuit must meet $X - L_0$ in some element z. This implies that $\{f, g, z\}$ is a triad of $M'|(X \cup L)$, which in turn implies that there are cocircuits in M' and M that meet $X - L_0$ in z. This contradicts the fact that $\mathrm{fcl}_M(Y) \cap X \subseteq L_0$, so the statement holds. □

Proof of 3.32.2. Let $N = P_L(M'|(X \cup L), M(\mathcal{W}_4))$. Every element of $M(\mathcal{W}_4)$ is an end of a four-element fan. We have by Corollary 3.29 and Lemma 3.23 that N is Bixby-extremal on $E(N) - \mathcal{A}$. Suppose $z \in X - L - \mathcal{A}$. Then z is on the guts or coguts of a 3-separation (Z_1, Z_2) of $M'|(X \cup L)$ by Lemma 3.28. Up to exchanging the labels of Z_1 and Z_2, we have $|Z_2 \cap L| = 2$. If $Z_1 - z$ meets L, then $Z_1 - z$ is not a minimal 3-separating set as otherwise $Z_1 \cup z$ would be a four-element fan, a violation of orthogonality. Therefore, we may assume that $Z_1 \cap L = \emptyset$. Then z is on the guts or coguts of a 3-separation $(Z_1, Z_2 \cup E(M(\mathcal{W}_4)))$ in N. By Lemma 3.27, we have that z is on the guts or coguts of a 3-separation in $N \backslash L'$.

Now suppose that $z \in \mathcal{A}$. We have that $L' = \{f, g\}$, and there is a triad $\{a, b, c\}$ in M with $\{a, b\} \subseteq X - L_0$, and $c \in Y - L_0$, where $\{a, b, z\}$ is a triangle. We have that $\{a, b, f, g\}$ is a four-element cocircuit of the 3-connected matroid $M'|(X \cup L)$. Therefore, $\{a, b, f, g, h\}$ is a five-element cocircuit of N for some $h \in E(N) - E(M')$. Thus, $\{a, b, h\}$ is a triad of $N \backslash L'$, so z is an end of a four-element fan in $N \backslash L'$.

We have shown that each $z \in X - L$ is on the guts or coguts of a 3-separation in $N \backslash L'$, so the statement holds. □

Proof of 3.32.3. Let $N = P_L(M'|(X \cup L), M(K_4))$. We have by Corollary 3.29, Lemma 3.23, and Proposition 3.31 that N is Bixby-extremal on $E(N) - \mathcal{A}$. By assumption, $\mathrm{fcl}_M(X) = X \cup L_0$, so $\mathcal{A} = \emptyset$. □

Proof of 3.32.4. We have that $P_L(M'|(X \cup L), M(K_4))$ is Bixby-extremal on $(X \cup E(M(K_4))) - \mathcal{A}$. If $\mathcal{A} \neq \emptyset$, then $|L_0| = 1$. Suppose $z \in \mathcal{A}$. If there is some $y \in X - L$ that is in the coclosure of (X, Y), then $y \cup L'$ is a triad of $M'|(X \cup L)$. There is a triad $\{a, b, c\}$ of M such that z is spanned by $\{a, b\}$, and $\{a, b\} \subseteq X - L_0$. We have that $\{a, b\} \cup L'$ is a cocircuit of

$M'|(X\cup L)$, so $\{a,b,y\}$, the symmetric difference of the cocircuits $\{a,b\}\cup L'$ and $L'\cup y$, is a triad of $M'|(X\cup L)$. Hence, z is on the guts of a 3-separation $(\{a,b,y\},(X\cup L)-\{a,b,y\})$ in $M'|(X\cup L)$. Therefore, if $P_L(M'|(X\cup L),M(K_4))$ is not Bixby-extremal, we have that $\mathrm{fcl}_M(Y)\cap X = L_0$, so $P_L(M'|(X\cup L),M(\mathcal{W}_4))\backslash L'$ is Bixby-extremal by 3.32.1 and 3.32.2. □

Proof of 3.32.5. This follows from Proposition 3.26 since a sequential ordering of X in M persists as a sequential ordering in either of the matroids given, and the complement of X is clearly sequential in each case. □

Now we apply the above lemma by finding appropriate 3-separations in matroids that are not building blocks in our decompositions.

Theorem 3.33. *Let M be a non-sequential binary Bixby-extremal matroid on at least ten elements. Suppose there is some element of M that is not in a six-element fan or a K_4-separator. Up to duality, there is a 3-separation (X,Y) of M having guts $L_0\neq\emptyset$ satisfying the following:*

Let M' be the (X,Y)-completion of M by L'. Let $L=L_0\cup L'$. Let e be an element of L_0. Let f label an element of $L-e$ such that if $|L_0|\geq 2$, then $f\in L_0-e$. Take copies of $M(K_4)$ and $M(\mathcal{W}_4)$ such that each of the following holds:

(i) the ground set of each of $M(K_4)$ and $M(\mathcal{W}_4)$ meets $E(M')$ in L;

(ii) L is a triangle of each of $M(K_4)$ and $M(\mathcal{W}_4)$; and

(iii) e and f are spokes of $M(\mathcal{W}_4)$.

Then each of the following holds:

3.33.1. *If $|L_0|<3$, set $N_X=P_L(M'|(X\cup L),M(\mathcal{W}_4))\backslash L'$. Otherwise, set $N_X=P_L(M'|(X\cup L),M(K_4))$. The matroid N_X is a binary Bixby-extremal matroid such that $|E(N_X)|<|E(M)|$.*

3.33.2. *If* $\mathrm{fcl}_M(X)\cap Y \subseteq L_0$ *and* $|L_0| < 3$*, set* $N_Y = P_L(M'|(Y\cup L), M(\mathcal{W}_4))\backslash L'$*. Otherwise, set* $N_Y = P_L(M'|(Y\cup L), M(K_4))$*. The matroid* N_Y *is a binary Bixby-extremal matroid with* $|E(N_Y)| < |E(M)|$*.*

Proof. There is some $e' \in E(M)$ that is not in a six-element fan or a K_4-separator. Since M is non-sequential, we have that e' is in a 3-sequence (X_0, e', Y_0) such that X_0 is non-sequential. We have that $|X_0| \geq 6$ because smaller 3-separating sets are sequential by Proposition 3.10. Moreover, $|Y_0| \geq 5$ by Lemma 3.18. Suppose $|Y_0| = 5$. Then Y_0 is a 5-fan or a 5-cofan. Since e' is not in a 6-fan or K_4-separator we have, up to duality, that $Y_0 \cup e'$ is a 5-cofan for which e' is a guts element that does not extend Y_0 into a 6-fan. By Corollary 3.14, there is some $d \in E(M)$ such that $(X_0 - d, e', d, Y_0)$ is a 3-sequence. Since $X_0 - d$ is not sequential, we have that $|X_0 - d| \geq 6$ and $|Y_0 \cup d| \geq 6$. Hence, we may assume that $|X_0|, |Y_0| \geq 6$ by relabeling if necessary.

We have that $(X_0 - \mathrm{fcl}_M(Y_0), \mathrm{fcl}_M(Y_0))$ is a 3-separation (X_1, Y_1) of M in which $|X_1| \geq 6$. The full closure operation induces a 3-sequence $(X_1, e_1, e_2, \ldots, e_n, e', Y_0)$ in M. Dualize if necessary so that e_1 is a guts element, and let $(X, Y) = (X_1 \cup e_1, E(M) - (X_1 \cup e_1))$. As X and X_0 differ by a subsequence of a 3-sequence, we have that X is non-sequential. Therefore, $|X| \geq 6$. Also note that $|Y| \geq |Y_0| \geq 6$.

By construction, $\emptyset \neq \mathrm{fcl}_M(Y) \cap X \subseteq L_0$. The statements in 3.33.1 and 3.33.2 follow easily from Lemma 3.32. □

We now have the following iterated decomposition using Theorem 3.33. A tree T is a *rooted tree* if exactly one vertex v_R of T has be designated as the *root.* If $v \in V(T) - v_R$ and u is the vertex adjacent to v on a path in T from v to v_R, then we say that v is a *child* of u and that u is the *parent* of v. A *rooted binary tree* is a rooted tree in which every vertex has at most two children. We say that the binary Bixby-extremal matroid M *decomposes into* M_1

and M_2 if M_1 and M_2 are matroids that can be formed, as in Theorem 3.33, in the manner of N_X and N_Y for some 3-separation (X, Y) in M.

Corollary 3.34. *Let M be a binary Bixby-extremal matroid. There is a binary rooted tree T whose root is labeled by M such that the following hold:*

(i) *Each vertex of T is labeled by a binary Bixby-extremal matroid whose ground set has strictly fewer elements than the matroid labeling its parent.*

(ii) *Each leaf vertex of T is labeled by a sequential matroid; or a matroid in which each element is in a K_4-separator or a fan of length at least six.*

(iii) *Suppose $v \in V(T)$ has a child. Then it has exactly two children. Furthermore, the matroid M_v labeling v decomposes into M_1 and M_2. One child of v is labeled by M_1, and the other child of v is labeled by M_2.*

3.5 The Smallest Binary Bixby-Extremal Matroids

This section identifies all of the 3-connected binary Bixby-extremal matroids with at most ten elements.

Proposition 3.35. *Suppose M is a 3-connected binary Bixby-extremal matroid on at most ten elements. Then M or M^* is isomorphic to one of the following:*

(i) $M(\mathcal{W}_4)$,

(ii) $M(\mathcal{W}_5)$, *or*

(iii) *the cycle matroid of one of the graphs in Figure 3.3.*

Proof. The 3-connected binary matroids on at most seven elements are well known to be $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, $M(K_4)$, F_7, and F_7^*. It is routine to check that none of these is Bixby-extremal, so $|E(M)| \geq 8$.

(a) $K_5 - e$ (b)

FIGURE 3.3: Two non-wheel graphs whose cycle matroids are small binary Bixby-extremal matroids.

Up to duality, there is some e in $E(M)$ such that (X, e, Y) is a 3-sequence, where e is on the guts of $(X \cup e, Y)$, and $|X| \geq |Y|$.

Suppose $|E(M)| = 8$. Then $|X| = 4$ and $|Y| = 3$, so $X \cup e$ is a 5-fan and $Y \cup e$ is a 4-fan by Proposition 3.10. Thus, M is isomorphic to a wheel matroid of rank four.

Suppose next that $|E(M)| = 9$. Then either $|X| = 5$ and $|Y| = 3$, or $|X| = |Y| = 4$. If $|X| = |Y| = 4$, then $X \cup e$ and $Y \cup e$ are 5-fans. Thus, M is isomorphic to the generalized parallel connection of two copies of $M(K_4)$ across a triangle. This is the cycle matroid of $K_5 - e$. Figure 3.3 gives an illustration of $K_5 - e$.

If $|X| = 5$, then $Y \cup e$ is a 4-fan, and X is either a 5-fan or a 5-cofan. If X is a 5-fan, then $M \cong M(K_5 - e)$. Suppose then that X is a 5-cofan. In this case, M is isomorphic to the matroid obtained by forming the generalized parallel connection of $M(K_4)$ and $M(\mathcal{W}_4)$ across a triangle T that is common to each, and then deleting two elements of T. Up to isomorphism, there are two possible matroids constructed in this way. The matroid formed by deleting the two elements of T that are spokes of the 4-wheel is $M^*(K_5 - e)$. The other possible matroid contains a series pair and so is not 3-connected.

Suppose now that $|E(M)| = 10$. Note that M contains no K_4-separator K because K would contain a guts triangle or coguts triad of the 3-separation $(K, E(M)-K)$, and $E(M)-K$ would be a four-element fan containing a guts element and a coguts element of $(K, E(M)-K)$. Such a structure cannot occur in a binary matroid.

We have that $3 \leq |Y| \leq 4$. Thus, one of $Y \cup e$ and Y is a four-element fan D. By Lemma 3.18, either M has a six-element fan containing D, or there is a fan ordering (d_1, d_2, d_3, d_4) of D such that M has a 3-sequence $(A, d_1, d_2, d_3, d_4, B)$. In the latter case, $|A| = |B| = 3$, so M is a sequential matroid. By Corollary 3.16, M has a 6-fan.

We now know that M has a six-element fan. The complement of this fan in M is a four-element fan. Thus, M is isomorphic to the matroid obtained from the generalized parallel connection of $M(K_4)$ and $M(\mathcal{W}_4)$ across a common triangle T by deleting an element of T. Therefore, M is isomorphic to either $M(\mathcal{W}_5)$ or the cycle matroid of the six-vertex graph in Figure 3.3. $\square$

Note that each of these smallest matroids is sequential. Upon combining Proposition 3.35 with Theorem 3.33, we obtain Theorem 3.2.

3.6 Conclusion

The decomposition in this chapter is motivated by a desire for an inductive tool for dealing with Bixby-extremal matroids. Since we only describe a decomposition for binary Bixby-extremal matroids, it is natural to ask about other classes of Bixby-extremal matroids. For a prime power $q > 2$, some of the techniques in this chapter using generalized parallel connection might extend to $GF(q)$-representable matroids by gluing across $(q+1)$-element lines rather than triangles. The work in this chapter relied upon exploiting the limited number of possible 3-separating structures in binary matroids. Moving to $GF(q)$-representable matroids would require consideration of a greater number of possible structures. This problem is compounded when we look outside the class of matroids representable over a finite field. Moreover, considering nonrepresentable matroids may lead to issues in developing a well-defined method of decomposition similar to Theorem 3.33.

Chapter 4

Unavoidable Minors of Large 4-connected Bicircular Matroids[1]

4.1 Introduction

The following result of Ding, Oporowski, Oxley, and Vertigan [6] shows that each sufficiently large 3-connected matroid is guaranteed to contain a large minor isomorphic to one of a few types of 3-connected matroids.

Theorem 4.1. *For every integer n exceeding two, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to one of $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, $M(\mathcal{W}_n)$, $\mathcal{W}^n$, or an n-spike.*

Evidently, corollaries for various minor-closed classes of matroids follow by filtering out the members of the list in Theorem 4.1 that are not in the class of interest. For instance, we may choose to restrict to graphic matroids.

Corollary 4.2. *For every integer n exceeding two, there is an integer $N(n)$ such that every simple, 3-connected graph having at least $N(n)$ edges has a minor isomorphic to one of $K_{3,n}$ or $\mathcal{W}_n$.*

The following result of Oporowski, Oxley, and Thomas [9] is a stronger version of Corollary 4.2. For $k \geq 3$, let $V = \{v_0, v_1, x_1, y_1, x_2, y_2, \ldots, x_{k-2}, y_{k-2}\}$ be a set of k isolated vertices. Add edges v_0v_1, v_0x_1, v_0y_1, v_1x_{k-2}, and v_1y_{k-2}. Add edge x_iy_i for each i in $\{1, 2, \ldots, k-2\}$. If $k \geq 4$, then for each i in $\{1, 2, \ldots, k-3\}$, add the edges x_ix_{i+1} and y_iy_{i+1}. The resulting graph is V_k, which is illustrated in Figure 4.1.

[1]Reprinted from [5] with permission from Springer as given in Appendix: Copyright Agreement.

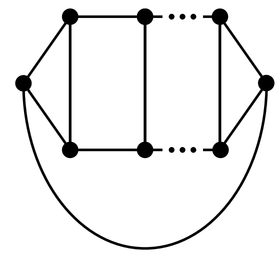
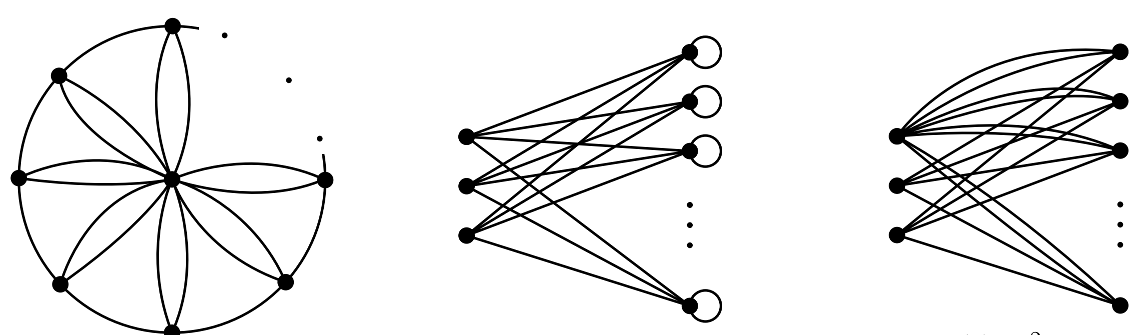


FIGURE 4.1: Illustration of V_k.

Theorem 4.3. *For every integer $k \geq 3$, there is an integer N such that every 3-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of $\mathcal{W}_k$, V_k, and $K_{3,k}$.*

The focus of this chapter is an unavoidable-minor result for bicircular matroids. As noted above, a result of this type for 3-connected bicircular matroids is merely a corollary of Theorem 4.1. However, a 4-connected analog of Theorem 4.1 is not known. The following theorem is the main result of this chapter. Let $n \geq 3$. The graph $\mathcal{W}_n^2$ can be constructed from the n-spoked wheel by adding an edge in parallel to each spoke. The graph $K_{3,n}^+$ is formed by adding a loop at each of the n degree-3 vertices of $K_{3,n}$. Finally, $K_{3,n}^2$ is constructed from $K_{3,n}$ by adding an edge in parallel to each of the edges incident with a single degree-n vertex.



(a) $\mathcal{W}_n^2$ (b) $K_{3,n}^+$ (c) $K_{3,n}^2$

FIGURE 4.2: Unavoidable minors for 4-biconnectivity.

Theorem 4.4. *For every integer n exceeding four, there is an integer $N(n)$ such that every 4-connected bicircular matroid with at least $N(n)$ elements has a minor isomorphic to one of $B(\mathcal{W}_n^2)$, $B(K_{3,n}^+)$, or $B(K_{3,n}^2)$.*

The proof of this result makes use of a type of graph connectivity called *biconnectivity*. Section 4.2 provides an equivalent characterization of n-biconnectivity that is used in Section 4.4 to prove Theorem 4.4.

In Section 4.3, we analyze the graphic structure of size-n cocircuits in n-connected bicircular matroids. This is used in Section 4.5 to prove the following internally 4-connected analog of Theorem 4.4.

Theorem 4.5. *For every integer n exceeding four, there is an integer $N'(n)$ such that every internally 4-connected bicircular matroid with at least $N'(n)$ elements has a minor isomorphic to $B(\mathcal{W}_n)$ or $B(K_{3,n})$.*

Finally, we prove a vertically 4-connected version of the main result in Section 4.6. Recall that, by definition, a vertically 4-connected need not be 3-connected. For simplicity, we assume in the next result the matroids under consideration are 3-connected.

Theorem 4.6. *For each integer n exceeding four, there is an integer $N''(n)$ such that every vertically 4-connected and 3-connected bicircular matroid on at least $N''(n)$ elements has a restriction isomorphic to $U_{2,n}$, or a minor isomorphic to one of $B(\mathcal{W}_n^2)$, $B(K_{3,n}^+)$, or $B(K_{3,n}^2)$.*

4.2 Preliminaries

Wagner defines n-biconnectivity in [15] with respect to k-biseparations as follows. Let (E_1, E_2) partition the edge set E of a connected graph $G = (V, E)$. For $i \in \{1, 2\}$, let G_i denote the subgraph of G induced by E_i. We say (E_1, E_2) is a *k-biseparation* of G, for $k \geq 1$, if each of

$|E_1|$ and $|E_2|$ is at least k, and

$$|V(G_1) \cap V(G_2)| = \begin{cases} k-1 & \text{if neither } G_1 \text{ nor } G_2 \text{ is acyclic} \\ k & \text{if an odd number of } G_1, G_2, \text{ and } G \text{ are acyclic} \\ k+1 & \text{if both } G_1 \text{ and } G_2 \text{ are acyclic, but } G \text{ is not acyclic} \end{cases}$$

For n a positive integer, a graph is *n-biconnected* if it has no k-biseparation for $k < n$.

The next theorem of Wagner [15] shows that biconnectivity is the version of graphic connectivity corresponding to matroid connectivity in bicircular matroids.

Theorem 4.7. *Let G be a connected graph. Then $B(G)$ is n-connected if and only if G is n-biconnected.*

Here we give an equivalent characterization for n-biconnectivity. The size of a bicycle is the number of edges in it. For a graph G, we denote the minimum degree of a vertex in G by $\delta(G)$.

Lemma 4.8. *For $n \geq 3$, a graph G with at least n vertices and at least $2n-2$ edges is n-biconnected if and only if each of the following holds:*

(i) G has no vertex cut of size at most $n-2$,

(ii) $\delta(G) \geq n$,

(iii) and G has no bicycle of size at most $n-1$.

Proof. Wagnger [15] proved this result when $n = 3$. Suppose that $G = (V, E)$ is n-biconnected for a fixed $n > 3$ and that the lemma holds for all smaller values of n exceeding two. Since G is $(n-1)$-biconnected, $\delta(G) \geq n-1$, and G has no vertex cut of size less than $n-2$. Suppose G has a vertex cut of size $n-2$ separating edges E_1 from edges E_2; that is, let E_1 and E_2 partition E such that the subgraphs G_1 and G_2 induced by E_1 and E_2, respectively, meet only in vertices of the size-$(n-2)$ vertex cut. As $\delta(G) \geq n-1$, we have that $|E_1|, |E_2| \geq n-1$ and

$|V(G_1) \cap V(G_2)| = n-2$. Since G is n-biconnected, G has no $(n-1)$-biseparation. Therefore, either G_1 or G_2 is acyclic – assume G_1. Then a leaf vertex of $V(G_1) - V(G_2)$ is adjacent to all $n-2$ vertices of the vertex cut. Hence, there is only one vertex in $V(G_1) - V(G_2)$. However, this contradicts the fact that G_1 is acyclic since $\delta(G) \geq n-1$.

Suppose G has a vertex v of degree $n-1$. Since G has no vertex cut of size at most $n-2$, the subgraph induced by the edges incident with v is acyclic. Thus $G - v$ is acyclic since G has no $(n-1)$-biseparation. Let $N(v)$ denote the set of vertices that are adjacent to v. Each leaf vertex of $G - v - N(v)$ is adjacent to at least $\delta(G) - 1 \geq n-2$ members of $N(v)$. Since $G - v - N(v)$ is acyclic, each connected component of $G - v - N(v)$ consists of exactly one vertex. Since $\delta(G) \geq n-1$, every such vertex must be adjacent to all vertices of $N(v)$. Therefore, $G - v - N(v)$ consists of exactly one vertex of degree $n-1$, so G is isomorphic to $K_{2,n-1}$, contradicting the fact that $\delta(G) > 2$.

By the induction assumption, G has no bicycle of size less than $n-1$. Suppose G has a bicycle of size $n-1$ with edge set E_1. Let $E_2 = E - E_1$. Then $|E_2| \geq 2n-2-(n-1) = n-1$, and $|V(G_1) \cap V(G_2)| = |V(G_1)| = n-2$. Since G has no $(n-1)$-biseparation, G_2 must be acyclic. However, G_2 has at least $n-(n-2) = 2$ vertices and therefore has at least two leaf vertices; every such leaf vertex is adjacent to all members of $V(G_1)$. This contradicts the fact that G_2 is acyclic.

Now suppose $G = (V, E)$ is a graph satisfying the three conditions in the statement of the lemma for some $n > 3$ and that the equivalence holds for all smaller values of n exceeding two. By assumption, G has no k-biseparation for $k < n-1$. Suppose (E_1, E_2) is an $(n-1)$-biseparation of G, and let G_1 and G_2 be the subgraphs induced by E_1 and E_2, respectively. First, suppose that $|V(G_1) \cap V(G_2)| = n-2$. Since G has no size-$(n-2)$ vertex cut, at least one of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ is empty – assume the former. Then $|E_1| \geq n-1$

and $|V_1| = n-2$, so G_1 contains a bicycle of size at most $n-1$, a contradiction. Hence $|V(G_1) \cap V(G_2)| \geq n-1$.

Next suppose that $|V(G_1) \cap V(G_2)| = n-1$. The graph G is not acyclic by assumption, so we may assume G_1 is acyclic. Since $|E_1| \geq n-1$ and $|V(G_1) \cap V(G_2)| = n-1$, it follows that $V(G_1) - V(G_2) \neq \emptyset$. Since $\delta(G) \geq n$, a leaf vertex of $V(G_1) - V(G_2)$ is adjacent to all vertices of $V(G_1) \cap V(G_2)$. As G_1 is acyclic, there is only one such vertex. This contradicts the fact that $\delta(G) \geq n$.

We may now assume that $|V(G_1) \cap V(G_2)| = n$, so both G_1 and G_2 are acyclic. First we show that one of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ is empty. Suppose that neither $V(G_1) - V(G_2)$ nor $V(G_2) - V(G_1)$ is empty. Since each of G_1 and G_2 is acyclic and $\delta(G) \geq n$, each of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ must have only one vertex by the pigeonhole principle. So G is isomorphic to $K_{2,n}$, a contradiction.

We may now assume that $V(G_1) - V(G_2) = \emptyset$. Then $|E_1| = n-1$. Thus $|V(G_2) - V(G_1)| \in \{0, 1\}$. If $V(G_2) - V(G_1) \neq \emptyset$, then a leaf of G_1 has degree 2 in G, a contradiction. Therefore $V(G_2) - V(G_1) = \emptyset$. Hence G is a graph on $2n-2$ edges and n vertices. The sum of the degrees of vertices of G is at least $n\delta(G) \geq 4n$. However, $2|E| = 4n-4$, a contradiction. Thus, G has no $(n-1)$-biseparation, so G is n-biconnected. □

4.3 The Graphic Structure of Small Cocircuits in n-connected Bicircular Matroids

The following is Matthews's [7] description of a hyperplane of $B(G)$.

Lemma 4.9. *Let G be a connected graph that contains a bicycle. A hyperplane H is a collection of edges of G such that the subgraph with vertex set $V(G)$ and edge set H consists of*

(i) exactly one acyclic component H_0, which may be an isolated vertex; and

(ii) a collection of other components, none of which is acyclic;

such that all edges of $E(G)\backslash H$ have at least one endpoint in H_0.

Evidently, a cocircuit of $B(G)$ is a minimal set of edges X such that $G - X$ has exactly one acyclic component. In general, the edges of a cocircuit need not form a bond in G as they would in the case of graphic matroids. The results below describe small cocircuits in the underlying graphs of n-connected bicircular matroids. Before exploring this graphic structure, we consider the following corollary of the minimum-degree condition in Lemma 4.8 that will be used frequently in our description of these small cocircuits.

Corollary 4.10. *Let G be a connected graph. Suppose $B(G)$ is n-connected, for some $n \geq 3$. Let X be a cocircuit of $B(G)$. Let H_0 denote the unique acyclic component of $G - X$. Then*

$$2|X| \geq \sum_{\substack{v \in V(H_0); \\ d_{G-X}(v) < n}} (n - d_{G-X}(v))$$

Recall that a triangle is a 3-element circuit and a triad is a 3-element cocircuit. We now consider triads in 3-connected bicircular matroids.

Lemma 4.11. *Let G be a connected graph with at least seven edges. Suppose $B(G)$ is 3-connected. Let X be a triad of $B(G)$. Either*

(i) the edges of X are all incident with a common vertex;

(ii) or $G|X$ is a path on four vertices, and the set of edges in G incident to either of the two internal vertices of this path consists of the edges of X along with a single edge in parallel to the middle edge of the path.

Proof. We have by Lemma 4.9 that $G - X$ contains exactly one acyclic component H_0. Evidently $G - X$ has at most one non-acyclic component H_1 since G is 2-connected by Lemma 4.8. If H_0 has exactly one vertex, we are done. Assume H_0 is a tree containing at least two vertices. Thus, H_0 has at least two leaf vertices. By Corollary 4.10, H_0 has at most three leaf vertices.

If all edges of X have both ends in H_0, then H_0 is a tree and $|E(H_0)| = |E(G)| - 3 \geq 7 - 3 = 4$. Since H_0 has at most three leaves, it is easy to see that either H_0 is a path of length at least 4, or H_0 has exactly three leaves and at least one degree-2 vertex. However, each of these contradicts Corollary 4.10.

We may now assume that some edge of X has one end in H_0 and one end in a non-acyclic component H_1 of $G - X$. Since G is 2-connected, there is at least one other edge of X with one end in H_0 and the other in H_1. Therefore, H_0 has exactly two leaf vertices, say u and v, and these are the only vertices in H_0. Each is incident with an H_0-H_1 edge of X. Since $\delta(G) \geq 3$, the third edge of X must be incident to both u and v.

□

A similar proof technique establishes the graphic structure of n-cocircuits in n-connected bicircular matroids for $n \geq 4$.

Lemma 4.12. *Suppose G is a connected graph having at least seven edges, and $B(G)$ is n-connected for some $n \geq 4$. If $X \subseteq E(G)$ is a size-n cocircuit of $B(G)$, then the edges of X are all incident with a common vertex.*

Proof. Let H_0 be the acyclic component of $G - X$ as specified by Lemma 4.9. As in the proof of Lemma 4.11, we may assume that H_0 has at least two vertices. Since $2n < 3(n-1)$, we have that H_0 has exactly two leaf vertices by Corollary 4.10, so H_0 is a path. Furthermore, $2n < 2(n-1) + 2(n-2)$, so H_0 is a path on two or three vertices.

First suppose that all edges of X have both ends in H_0. So $|V(G)| = 2$ or 3, and $|E(G)| \geq 7$. It is easy to see that G must contain a bicycle of size at most 3, contradicting the n-biconnectivity.

We now know that there is an edge in X that has an end in a non-acyclic component H_1 of $G - X$. By the $(n-1)$-connectivity of G, there are least two such edges. Then there are at most $2n-2$ ends of the edges of X in H_0. Thus H_0 is P_2. Since bicycles of G must have at least four edges, at most one edge of X has both ends in H_0. Then there are at most $n-1+2 = n+1$ ends of the edges of X in H_0. Since $n+1 < 2n-2$, this is a contradiction. $\square$

4.4 Unavoidable Minors of 4-connected Bicircular Matroids

Before proving the main result of this chapter, we recall that if a graph H is a minor of a graph G, then the bicircular matroid $B(H)$ is a minor of $B(G)$ [16]. The next result can be found in Biedl [2]; one may prove it by a simple counting argument.

Lemma 4.13. *A maximal matching in an m-edge graph with maximum degree k has size at least $\frac{m}{2k-1}$.*

The next lemma is the main result of this section.

Lemma 4.14. *For each n, there is an $R(n)$ such that every 3-connected graph on at least $R(n)$ vertices having minimum degree at least four has a minor isomorphic to one of W_n^2, $K_{3,n}^+$, or $K_{3,n}^2$.*

Proof. Let $k = 4n^2 - 2n - 4$. By Theorem 4.3, there is an R such that each 3-connected graph on at least R vertices has a subgraph isomorphic to a subdivision of W_k, $K_{3,k}$, or V_k. Suppose G is a 3-connected graph on at least R vertices. Since $k = 4n^2 - 2n - 4 > 4n$, if G has a W_k- or V_k-subdivision as a subgraph, then G has a W_n^2-minor, and we are done. Assume then that G has a $K_{3,k}$-subdivision as a subgraph; that is, G has vertices u_1, u_2, u_3, v_1, v_2, $\ldots$, v_k such that, for each $i \in \{1, 2, \ldots, k\}$, there are paths $P_{i,1}$, $P_{i,2}$, and $P_{i,3}$ from v_i to u_1, u_2, and u_3, respectively, such that P_{i_1,j_1} and P_{i_2,j_2} are internally vertex-disjoint whenever $(i_1, j_1) \neq (i_2, j_2)$.

Let $e \in E(G)$. Note that if e satisfies either of the following conditions, then G/e contains a $K_{3,k}$-subdivision having small and large sides $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, \dots, v_k\}$, respectively, such that $d_{G/e}(v_j) \geq 4$ for each $j \in \{1, 2, \dots, k\}$.

(i) For some $a \in \{1, 2, 3\}$ and $b \in \{1, 2, \dots, k\}$, e is an edge on the path $P_{a,b}$ that is incident with u_a but has its other end in $V(G) - \{v_1, v_2, \dots, v_k\}$.

(ii) Each path $P_{i,j}$ has length one, and e is an edge of G with one end in $\{v_1, v_2, \dots, v_k\}$ and the other end in $V(G) - \{u_1, u_2, u_3, v_1, v_2, \dots, v_k\}$.

Obtain a minor H of G by first consecutively contracting edges of the types given above until no such edges remain, and then deleting all edges not incident with some $\{v_1, v_2, \dots, v_k\}$. Now, H consists of a $K_{3,k}$-subgraph with some extra edges added incident with the vertices on the large side of the bipartition. By construction, no step of the algorithm above decreases the degree of a vertex in $\{v_1, v_2, \dots, v_k\}$. Hence, each of the $k = 4n^2 - 2n - 4$ vertices is incident with at least one such extra edge. If at least n of these vertices have adjacent loops, then H has a $K_{3,n}^+$-minor. If at least $3n - 2$ of these vertices are adjacent to a vertex in $\{u_1, u_2, u_3\}$ by an edge not in the $K_{3,k}$-graph, then at least n are adjacent to the same vertex by the pigeonhole principle, so H has a $K_{3,n}^2$-minor. Assume neither of these last two situations occurs. Let E_1 be the set of non-loop edges of H that have both ends in $\{v_1, v_2, \cdots, v_k\}$, let H_1 be the subgraph of H induced by E_1, and let $Z = V(H_1)$. Then $|Z| \geq (4n^2 - 2n - 4) - (n - 1) - (3n - 3) = 4n^2 - 6n$ and every vertex in Z is adjacent to some other vertex in Z. We have that H_1 has at least $\frac{|Z|}{2} \geq 2n^2 - 3n$ edges. If some vertex $v_i \in Z$ has degree greater than $n-1$ in H_1, then H has $K_{3,n}^2$-minor by contraction of the edge $v_i u_1$. Assume then that the maximum degree in H_1 is at most $n - 1$. Then by Lemma 4.13, H_1 has a matching of size at least $\frac{2n^2-3n}{2n-3} = n$. Thus H has a $K_{3,n}^2$-minor by contraction of each edge in this matching. $\square$

Corollary 4.15. *For each n, there is an $N(n)$ such that every 4-biconnected graph on at least $N(n)$ edges has a minor isomorphic to one of W_n^2, $K_{3,n}^+$, or $K_{3,n}^2$.*

Proof. Note that a 4-biconnected graph G contains at most one loop at each vertex, and each parallel class of edges has size at most two. Therefore, $|E(G)| \leq |V(G)| + 2\binom{|V(G)|}{2} = |V(G)|^2$. Hence $|V(G)| \geq \sqrt{|E(G)|}$. Fix n. Let $R(n)$ be given as in Lemma 4.14. If $|E(G)| \geq R(n)^2$ then G is a 3-connected graph with $\delta(G) \geq 4$ on at least $R(n)$ vertices, so G has one of the required minors. □

It is straightforward to prove Theorem 4.4 from the above corollary.

Proof of Theorem 4.4. The theorem follows from Corollary 4.15 since a sufficiently large 4-connected bicircular matroid can be represented by a large 4-biconnected graph, which in turn must have one of the given large minors. □

4.5 Unavoidable Minors of Internally 4-connected Bicircular Matroids

Recall that a matroid M is internally 4-connected if M is 3-connected and for every 3-separation (X, Y) of M, either $|X| = 3$ or $|Y| = 3$. It is clear that a triangle in a bicircular matroid $B(G)$ consists of one of the following in G:

(i) a set of three parallel edges,

(ii) a set of two parallel edges and a loop at one end,

(iii) or two loops at two distinct vertices and an edge between.

Lemma 4.11 describes triads in the graphs underlying 3-connected bicircular matroids. Note that the exceptional case in Lemma 4.11 gives rise to a 3-separating set of size 4, and so does not occur in an internally 4-connected bicircular matroid $B(G)$ when $|E(G)| \geq 8$. Therefore, every triad in an internally 4-connected bicircular matroid corresponds to either a degree-3 vertex, or a degree-4 vertex incident to exactly one loop in the underlying graph.

By Lemma 4.8, the graph underlying an internally 4-connected bicircular matroid is 2-connected and has minimum degree at least three. However, using Wagner's original definition of biconnectivity, we see that the 2-separations in such a graph are highly restricted.

Lemma 4.16. *Let G be a connected graph having at least six edges. If $B(G)$ is internally 4-connected and G has a 2-vertex cut, then one side of the separation consists of a single vertex having exactly three incident edges.*

Proof. Since $\delta(G) \geq 3$, neither side of the 2-separation is acyclic. Therefore, the 2-vertex cut in G induces a 3-biseparation (E_1, E_2) in G. Assume $|E_1| = 3$ since G is internally 4-connected. Thus $|V(G_1) - V(G_2)| = 1$. □

By the last result, each 2-separation in the graph underlying an internally 4-connected bicircular matroid must have one of the configurations given in Figure 4.3.

FIGURE 4.3: In a graph underlying an internally 4-connected bicircular matroid, a 2-separation must have one of the configurations above. One side of the 2-separation must be one of the 3-edge subgraphs shown. The shaded shape in each drawing represents the other side of the separation.

Now it is easy to see that we have the following graphic characterization for a bicircular matroid to be internally 4-connected.

Lemma 4.17. *Let G be a connected graph having at least eight edges. Then $B(G)$ is internally 4-connected if and only if each of the following holds.*

(i) G is 2-connected.

(ii) There is at most one loop at each vertex.

(iii) $\delta(G) \geq 3$

(iv) *Every vertex cut of size two must have one of the forms shown in Figure 4.3; moreover, there are no edges between and no loops at these vertices.*

(v) *Every parallel class of edges has size at most three.*

(vi) *For each parallel class of size three, there is no loop at either end.*

(vii) *For each parallel class of size two, there is at most one loop at the two ends.*

We now prove our result on the unavoidable minors of large internally 4-connected bicircular matroids.

Proof of Theorem 4.5. First note that the matroids $B(\mathcal{W}_n)$ and $B(K_{3,n})$ are internally 4-connected by Lemma 4.17.

Suppose G is a connected graph for which $B(G)$ is internally 4-connected. A parallel class of edges in G has size at most three, and there is at most one loop at each vertex. Therefore, $|E(G)| \leq |V(G)| + 3\binom{|V(G)|}{2} \leq \frac{3}{2}|V(G)|^2$. Thus $|V(G)| \geq \sqrt{\frac{2}{3}|E(G)|}$.

By Corollary 4.2, there is an integer R such that any 3-connected graph on at least R vertices has a minor isomorphic to $\mathcal{W}_n$ or $K_{3,n}$. Now suppose G is a connected graph underlying an internally 4-connected bicircular matroid $B(G)$ having at least $\frac{3}{2}R^4$ elements in its ground set.

If G has a 2-separation, we have by Lemma 4.16 that one side of the separation consists of a single degree-3 vertex that is adjacent to exactly two vertices, namely the two cut vertices. Call such a degree-3 vertex a *tick*. A vertex that is not a tick is a *non-tick*. There is a natural injection between the set of ticks and the set of pairs of non-ticks given by matching a tick with its associated pair of 2-separating non-tick vertices. Let τ denote the number of ticks in G, and let η denote the number of non-tick vertices. We have that $\tau \leq \binom{\eta}{2}$ and

$\eta+\tau=|V(G)|$. Since $\eta \geq 1$, we have $\frac{\eta-1}{2}+1 \leq \eta$, so

$$\eta^2 \geq \eta\left(\frac{\eta-1}{2}+1\right) = \binom{\eta}{2}+\eta \geq \tau+\eta = |V(G)|$$

Note that the graph resulting from the contraction of a non-loop edge incident with a tick is still 2-connected. Furthermore, any 2-separations of the resultant graph are also 2-separations of G up to identification of vertices via contraction. Thus, we can consecutively contract non-loop edges incident with ticks to obtain a 3-connected graph H having $\eta \geq \sqrt{|V(G)|}$ vertices.

Recall that G has at least $\frac{3}{2}R^4$ edges, so G has at least R^2 vertices. Hence, G has a 3-connected minor having at least R vertices. Thus, G has a minor isomorphic to one of $\mathcal{W}_n$ or $K_{3,n}$, so $B(G)$ has a minor isomorphic to one of $B(\mathcal{W}_n)$ or $B(K_{3,n})$. $\square$

4.6 Bicircular Matroids That Are Vertically 4-connected and 3-connected

In this section, we study bicircular matroids that are both vertically 4-connected and 3-connected. Since a rank-2 flat in a 3-connected bicircular matroid is a class of parallel non-loop edges plus the set of loops at the two end vertices, the next result follows easily from Lemma 4.8.

Lemma 4.18. *If G is a connected graph on at least four vertices such that $B(G)$ is vertically 4-connected and 3-connected, then G is 3-connected and $\delta(G) \geq 4$.*

We are now ready to prove the unavoidable minors of large vertically 4-connected bicircular matroids.

Proof of Theorem 4.6. Suppose G is a connected graph such that $B(G)$ is 3-connected and vertically 4-connected and $|E(G)| \geq N'' = \frac{n-1}{2}R(n)^2$, where $R(n)$ is given as in Lemma 4.14.

If G has a parallel class of edges of size at least n, then $B(G)$ has a $U_{2,n}$-restriction. So we may assume that each parallel class of edges has size at most $n-1$. Since $B(G)$ is 3-connected, G has at most one loop at each vertex. Therefore we have $|E(G)| \leq |V(G)| + (n-1)\binom{|V(G)|}{2} = \frac{n-1}{2}|V(G)|^2 - \frac{n-3}{2}|V(G)|$. Since $n \geq 4$, $|E(G)| \leq \frac{n-1}{2}|V(G)|^2$. Therefore,

$|V(G)| \geq \sqrt{\frac{2}{n-1}|E(G)|} \geq \sqrt{\frac{2}{n-1} \cdot \frac{n-1}{2} R^2(n)} = R(n)$. By Lemma 4.18, G is a 3-connected graph having minimum degree at least four. By Lemma 4.14, G has a minor isomorphic to one of $\mathcal{W}_n^2$, $K_{3,n}^+$, or $K_{3,n}^2$. Thus, G has one of these minors, so $B(G)$ has a minor isomorphic to the bicircular matroid of one of these graphs.

□

4.7 Conclusion

The class of 4-connected bicircular matroids is admittedly restrictive. However, the list of unavoidable minors given in this chapter is a partial list of unavoidable minors of large 4-connected matroids in general, a full list of which is not currently known. The techniques in this chapter center around the biconnectivity property and do not readily extend to more general classes of bias matroids. Slilaty and Qin [13] offer a version of Wagner's biconnectivity that is generalized to bias matroids. Evidently, the extra attention that must be paid to balanced cycles is the inherent complication in obtaining an analog of Lemma 4.8, which we have relied upon in the proof of our unavoidable-minors result. An extension to 4-connected signed graphic matroids might be much more easily obtained and would still have the benefit of providing the list of unavoidable minors of large 4-connected graphs.

References

[1] B. Beavers, Doctoral dissertation, Louisiana State University, 2006.

[2] T. Biedl, E.D. Demaine, C.A. Duncan, R. Fleischer, S.G. Kobourov, Tight bounds on maximal and maximum matchings, Discrete Math. 285 (2004) 7–15.

[3] R.E. Bixby, A simple theorem on 3-connectivity, Linear Algebra Appl. 45 (1982) 123–126.

[4] R.E. Bixby, C.R. Coullard, Finding a smallest 3-connected minor maintaining a fixed minor and a fixed element, Combinatorica 7 (1987) 231–242.

[5] D. Chun, T. Moss, D. Slilaty, X. Zhou, Unavoidable minors of large 4-connected bicircular matroids, Ann. Comb. (to appear).

[6] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected matroids, J. Combin. Theory Ser. B 71 (1997) 244–293.

[7] L.R. Matthews, Bicircular matroids, Quart. J. Math. Oxford Ser. (2) 28 (1977) 213–228.

[8] T. Moss, A minor-based characterization of matroid 3-connectivity, Adv. in Appl. Math. 50 (2013), 132–141.

[9] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, J. Combin. Theory Ser. B 57 (1993) 239–257.

[10] J. Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.

[11] J. Oxley, C. Semple, G. Whittle, The structure of the 3-separations of 3-connected matroids, J. Combin. Theory. Ser. B 92 (2004) 257–293.

[12] J. Oxley, C. Semple, G. Whittle, Wild triangles in 3-connected matroids, J. Combin. Theory Ser. B 98 (2008) 291–323.

[13] D. Slilaty, H. Qin, Connectivity in frame matroids, Discrete Math. 308 (2008) 1994–2001.

[14] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.

[15] D.K. Wagner, Connectivity in bicircular matroids, J. Combin. Theory Ser. B 39 (1985) 308–324.

[16] T. Zaslavsky, Biased graphs. II. the three matroids, J. Combin. Theory Ser. B 51 (1991) 46–72.

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Vita

Tyler Moss was born in 1985, in Laurel, Mississippi. He finished his undergraduate studies at Millsaps College in May 2008. He earned a Master of Science degree in mathematics from Louisiana State University in May 2010. In August 2008, he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2014.