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The Finite Union of Ideals in a Ring R.

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THE FINITE UNION OF IDEALS IN A RING R

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

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May, 1972
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<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>11</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER I. U-ideals and U-rings</td>
<td>7</td>
</tr>
<tr>
<td>CHAPTER II. C-rings and CF-rings</td>
<td>17</td>
</tr>
<tr>
<td>CHAPTER III. Ideal Union and Quasi-Equality</td>
<td>23</td>
</tr>
<tr>
<td>CHAPTER IV. Union of R-Modules, and Ú-rings</td>
<td>30</td>
</tr>
<tr>
<td>CHAPTER V. More on U-rings</td>
<td>41</td>
</tr>
<tr>
<td>CHAPTER VI. Examples concerning U-rings and Ú-rings</td>
<td>46</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>54</td>
</tr>
<tr>
<td>BIOGRAPHY</td>
<td>55</td>
</tr>
</tbody>
</table>

TABLE OF CONTENTS
Throughout this paper $R$ will denote a commutative ring with $1 \neq 0$ and total quotient ring $T$, and $D$ will be an integral domain with $1 \neq 0$ and quotient field $K$. An $R$-module will be understood to be unitary.

A classical theorem of ideal theory states that if $A$ is an ideal of $R$ and $A_1, \ldots, A_n$ are prime ideals of $R$, then $A \subseteq \bigcup_{1}^{n} A_i$ implies that $A \subseteq A_i$ for some $i$. One of the main problems we are concerned with in this paper is to determine conditions on a ring $R$ in order that $A \subseteq \bigcup_{1}^{n} A_i$ always implies $A \subseteq A_i$ for some $i$ (whether the $A_i$ are prime ideals or not). We say that an ideal $A$ of $R$ is a U-ideal if it has the following property: for any ideals $A_1, \ldots, A_n$ in $R$, $A \subseteq \bigcup_{1}^{n} A_i$ implies that $A \subseteq A_i$ for some $i$. If each ideal of $R$ is a U-ideal, then $R$ is a U-ring, or is said to have property $U$. It is easy to see that $A$ is a U-ideal if and only if $A$ has the property that $A = \bigcup_{1}^{n} A_i$ implies $A = A_i$ for some $i$; hence, in a U-ring a finite union of ideals is never an ideal, except trivially.

Some of the results we establish are: if every finitely generated ideal of $R$ is a U-ideal then each ideal of $R$ is a U-ideal, i.e. $R$ is a U-ring; an invertible ideal is a U-ideal; a Prüfer domain is a U-ring; if $R$ is a quasi-
local ring with maximal ideal $P$ then $R$ is a $U$-ring if and only if $R/P$ is infinite or finitely generated ideals of $R$ are principal; and if $D$ is a domain such that $D/P$ is finite for each maximal ideal $P$ of $D$ then $D$ is a $U$-ring if and only if $D$ is Prüfer.

We also consider the finite union of $R$-modules, and $\bar{U}$-rings, which we deal with in Chapter IV. A ring $R$ is a $\bar{U}$-ring if it has the following property: if $M_1, \ldots, M_n$ are submodules of an $R$-module $M$ such that $M = \bigcup_{i=1}^{n} M_i$, then $M = M_i$ for some $i$. Clearly a $\bar{U}$-ring is a $U$-ring. We show that: if there exists an infinite subset $S$ of $R$ such that $x - y$ is a unit of $R$ for all $x, y \in S$ with $x \neq y$ (for example $S$ could be an infinite subfield with the same $1$ as $R$), then $R$ is a $\bar{U}$-ring; if $R$ is a quasi-local ring with maximal ideal $P$, then $R$ is a $\bar{U}$-ring if and only if $R/P$ is infinite. We point out that a $U$-ring is not necessarily a $\bar{U}$-ring, and we give an example of a $U$-ring which does not contain an infinite field, and which is not Prüfer.
Throughout this paper R will denote a commutative ring with \( l \neq 0 \) and total quotient ring \( T \), and \( D \) will be an integral domain (referred to as domain) with \( l \neq 0 \) and quotient field \( K \). An R-module will be understood to be unitary unless otherwise stated. The letter \( Z \) will denote the ring of integers and \( Q \) the field of rational numbers. We will deal considerably with the finite set-theoretic union of ideals of a ring \( R \), and to simplify our notation \( UA_{i} \) will be used in place of \( U_{i}^{n}A_{i} \) (this notation will be used in the case \( i = 1, \ldots, n \); otherwise the limits of the union will be given). The remaining notation and terminology is that of Zariski and Samuel, Commutative Algebra, and in particular \( \subseteq \) will denote containment while \( < \) will denote proper containment.

While intersection of ideals is an important ideal theoretic operation, the union of a finite collection of ideals need not be an ideal, and very little seems to be known about the union of ideals. One important fact which is known is the following: if \( A_{1}, \ldots, A_{n} \) are prime ideals and \( A \) is an ideal of \( R \), then \( A \subseteq \bigcup A_{i} \) implies that \( A \subseteq A_{i} \) for some \( i \) [ZS₁, p. 215], [N, p. 6], [G, p. 40]. This property holds in general when \( n = 2 \) (i.e. whether \( A_{1} \) and
$A_2$ are prime ideals or not) as the following easy argument shows. Suppose $A \subseteq A_1U A_2$; if there are $x_1, x_2 \in A$ such that $x_1 \in A_1 \setminus A_2$ and $x_2 \in A_2 \setminus A_1$, then $x_1 + x_2 \notin A_1 U A_2$ which is a contradiction. It follows that $A \subseteq A_i$ for $i = 1$ or $2$, completing the argument. (Notice this argument holds for $A, A_1, A_2$ subgroups of a group $G$). It is also clear that $A \subseteq U A_i$ implies $A \subseteq A_i$ for some $i$, for any ideals $A_1, \ldots, A_n$ in case $A$ is a principal ideal. However, it is not true in general that $A \subseteq U A_i$ implies $A \subseteq A_i$ for some $i$; for example if $A = (2, x), A_1 = (4, x), A_2 = (2, x^2)$ and $A_3 = (2 + x, 2x)$ in the polynomial ring $\mathbb{Z}[x]$, then $A \subseteq U A_1$, but $A \not\subseteq A_i$ for $i = 1, 2, 3$. (for further details see Example 6.1).

One of the main problems we are concerned with in this paper is to determine conditions on the ring $R$ in order that $A \subseteq U A_i$ always implies $A \subseteq A_i$ for some $i$ (whether the $A_i$ are prime ideals or not). We say that an ideal $A$ of $R$ is a U-ideal if it has the following property: for any ideals $A_1, \ldots, A_n$ in $R$, $A \subseteq U A_i$ implies $A \subseteq A_i$ for some $i$. If each ideal of $R$ is a U-ideal, then $R$ is said to be a U-ring, or to have property U. It is easy to see that $A$ is a U-ideal if and only if $A$ has the property that $A = U A_i$ implies $A = A_i$ for some $i$; hence, in a U-ring a finite union of ideals is never an ideal except trivially.

The following are some of the results we establish
in Chapter I: if every finitely generated ideal of $R$ is a U-ideal, then $R$ is a U-ring; an invertible ideal is a U-ideal; a Prüfer ring is a U-ring; each homomorphic image of a U-ring is a U-ring; each quotient ring of a U-ring is a U-ring (where the quotient is with respect to some multiplicative system of $R$ [ZS1, p. 221]); and a (finite) direct sum of rings is a U-ring if and only if each summand is a U-ring.

In Chapter II we deal with property C and property CF on a ring $R$. We say that a ring $R$ has property C or is a C-ring if for $A$ and $B$ ideals of $R$, $B < A$ implies that there exists an ideal $C \not= R$ such that $B \subseteq AC$; $R$ has property CF or is a CF-ring if the above property holds for $A$ finitely generated. We show that a domain with property C is a Prüfer domain and hence a U-ring; in fact we prove that for a finitely generated regular ideal $A$ of a ring $R$, $A$ is invertible if and only if $A$ has the following property: for $B$ any ideal of $R$, $B < A$ implies the existence of an ideal $C \not= R$ in $R$ such that $B \subseteq AC$. We also show that a domain $D$ has property CF if and only if $D$ is Prüfer.

In Chapter III we investigate ideal union in conjunction with the equivalence relation quasi-equality: two ideals $A, B$ of a ring $R$ are quasi-equal (denoted $A \sim B$) if $A^* = B^*$ where $A^* = (A^{-1})^{-1}$ and $S^{-1} = [R:S]_T = \{x | x \epsilon T, xS \subseteq T\}$ with $S$ a subset of the total quotient ring $T$ of $R$; an
ideal $A$ of $R$ is quasi-invertible if $AA^{-1} \sim R$. The following are a few of the results we establish here: if $A_1, \ldots, A_n$ are ideals of $D$ and $A$ is a quasi-invertible ideal such that $A = U A_1$, then $A \sim A_i$ for some $i$; if $D$ is completely integrally closed and $A, A_1, \ldots, A_n$ are ideals of $D$, then $A = U A_1$ implies $A \sim A_i$ for some $i$; if $D$ is completely integrally closed and $A_1, \ldots, A_n$ are $v$-ideals (i.e. $A_1 = A_1^*$ for $i = 1, \ldots, n$) such that $A \subseteq U A_1$, then $A \subseteq A_i$ for some $i$.

Chapter IV deals with the finite union of unitary $R$-modules, $U$-rings, and more facts about $U$-rings. A ring $R$ is a $U$-ring if it has the following property: if $M_1, \ldots, M_n$ are submodules of an $R$-module $M$ such that $M = U M_1$, then $M = M_1$ for some $i$. Clearly a $U$-ring is a $U$-ring; however, the converse is false (see the remark following Theorem 4.15). Some of our results in this chapter are: if there exists an infinite subset $S$ of $R$ such that $x - y$ is a unit of $R$ for $x, y \in S$ with $x \neq y$, then $R$ is a $U$-ring; if $R$ contains an infinite subfield (where the subfield has the same 1 as $R$), then $R$ is a $U$-ring and hence a $U$-ring; if $R$ is a quasi-local ring with maximal ideal $P$, then $R$ is a $U$-ring if and only if $R/P$ is infinite; if $R$ is a quasi-local ring with maximal ideal $P$, then $R$ is a $U$-ring if and only if $R/P$ is infinite or finitely generated ideals of $R$ are principal; and if $R'$ is an overring of a $U$-ring $R$ such that $R$ and $R'$ have the same 1, then $R'$ is
a \( \mathbb{U} \)-ring. In addition to these results we give several conditions under which a ring \( R \) is not a \( \mathbb{U} \)-ring, and conditions under which \( R \) is not a \( \mathbb{U} \)-ring. We end Chapter IV with an example which shows that although containing an infinite subfield with \( 1 \) same as the \( 1 \) in \( R \), and being Prüfer are sufficient conditions for a domain to be a \( \mathbb{U} \)-ring, they are not necessary.

In Chapter V we study the relationship between an ideal \( A \) in \( R \) and the ideal \( AR_p \) of the ring \( R_p \) by means of the vector space \( A/\mathbb{AP} \) over the field \( R/P \) where \( P \) is a maximal ideal of \( R \). We use our results to give further facts about \( \mathbb{U} \)-rings. The important results of this chapter are: if \( P \) is a maximal ideal in a \( \mathbb{U} \)-ring \( R \) such that \( R/P \) is finite, then \( R_p \) is a Bezout ring; if \( D \) is a domain such that \( D/P \) is finite for each maximal ideal \( P \) of \( D \), then \( D \) is a \( \mathbb{U} \)-ring if and only if \( D \) is Prüfer; and if we let \( \bar{Z} \) be the integral closure of \( Z \) in a finite algebraic extension \( Q(\alpha) \) of \( Q \) and let \( D \) be a domain such that \( \bar{Z} \subseteq D \subseteq \bar{Z} \), then \( D \) is a \( \mathbb{U} \)-ring if and only if \( D \) is integrally closed.

In the last chapter we consider several examples which answer questions concerning the relationship between property \( \mathbb{U} \) and other important ring theoretic concepts such as the polynomial rings over a ring \( R \), the quotient ring of \( R \) with respect to a multiplicative system of \( R \), Noetherian
rings, and integrally closed rings. We prove the following theorem in Chapter VI, characterizing the U-rings which are polynomial rings over a field: if \( K \) is a field, \( X \) a set of indeterminates over \( K \) of cardinality \( \alpha \geq 1 \), and \( K[X] \) is the polynomial ring in the elements of \( X \) with coefficients in \( K \), then \( K[X] \) is a U-ring if and only if \( K \) is infinite or \( \alpha = 1 \).
CHAPTER I

In this chapter we will define and study U-ideals and U-rings (definitions 1.1 and 1.2). The union of ideals which we refer to in these definitions is finite set-theoretic union; \( UA_i \) will be used in place of \( \bigcup_{i=1}^{n} A_i \) (this notation will be used in the case \( i = 1, \ldots, n \); otherwise the limits of the union will be given). Some of our important results are: an invertible ideal is a U-ideal; a Prüfer domain is a U-ring; each homomorphic image of a U-ring is a U-ring; each quotient ring of a U-ring is a U-ring; and a (finite) direct sum of rings is a U-ring if and only if each summand is a U-ring.

**Definition 1.1** An ideal \( A \) of a ring \( R \) is a **U-ideal** if it has the following property: for \( A_1, \ldots, A_n \) ideals of \( R \), \( A \subseteq U A_i \) implies that \( A \subseteq A_i \) for some \( i \).

**Remark:** A principal ideal \( aR \) of \( R \) is clearly a U-ideal since \( aR \subseteq U A_i \) implies \( a \in A_i \) for some \( i \), and hence \( aR \subseteq A_i \).

**Definition 1.2** A ring \( R \) is a **U-ring** or is said to have property **U** if each ideal of \( R \) is a U-ideal.

**Remark:** A principal ideal ring is clearly a U-ring. We have
noted that \( \mathbb{Z}[x] \) is not a U-ring (see Example 6.1). In addition, it is easily seen that a non-principal ideal \( A \) containing only a finite number of elements can not be a U-ideal, since \( A = \bigcup_{a \in A} aR \); for example, if \( F[x,y] \) is a polynomial ring in two indeterminates \( x \) and \( y \) over a finite field \( F \), set \( R = F[x,y]/(x,y)^2 \) and \( A = \bar{x}R + \bar{y}R \) where \( \bar{x}, \bar{y} \) are the residue classes determined by \( x \) and \( y \) respectively.

**Theorem 1.3** If \( A, A_1, \ldots, A_n \) are arbitrary ideals of a ring \( R \), the following conditions are equivalent:

1. Each ideal of \( R \) is a U-ideal.
2. Each finitely generated ideal of \( R \) is a U-ideal.
3. If \( A \) is finitely generated and \( A = UA_1 \), then \( A = A_1 \) for some \( i \).
4. If \( A = UA_1 \), then \( A = A_1 \) for some \( i \).
5. If \( UA_1 = \sum_{1}^{n} A_1 \), then for some \( k \), \( A_1 \subseteq A_k \) for all \( i \).

**Proof:** It is clear that \( (1) \Rightarrow (2) \Rightarrow (3) \).

To show that \( (3) \Rightarrow (4) \), let \( A = UA_1 \) and suppose that \( A_1 \subset A \) for \( i = 1, \ldots, n \). There exists \( a_1 \in A \setminus A_1 \) for each \( i \); let \( A = (a_1, \ldots, a_n) \) and \( \bar{A}_1 = A_1 \cap A \). Notice that \( \bar{A} \subseteq A = UA_1 \), so it follows that \( \bar{A} = \bar{UA}_1 \). Since \( \bar{A} \) is finitely generated, we have, by (3), that \( \bar{A} = \bar{A}_1 \) for some \( i \), a contradiction since \( a_1 \notin A_1 \). It follows that there exists \( i \) such that \( A = A_i \).
To show that \((4) \Rightarrow (5)\), we assume that \(\cup A_1 = \sum_1^n A_1 = A\). Then by \((4)\), since \(\cup A_1 = A\), we have \(A = A_k\) for some \(i = k\) and \(A_1 \subseteq A = A_k\) for \(i = 1, \ldots, n\) and \((5)\) follows.

Consider \((5) \Rightarrow (1)\); assume \(A \subseteq \cup A_1\) and let \(A_1' = A \cap A_1\). It follows that \(A = \cup A_1' = \sum_1^n A_1'\). Hence by \((5)\) there exists \(k\) such that \(A_1' \subseteq A_k\) for \(i = 1, \ldots, n\) from which we get that \(A = \cup A_1' \subseteq A_k\). Since \(A \subseteq A_k \subseteq A_k\), \((1)\) follows.

Q.E.D.

Remark: From Theorem 1.3 we notice that a U-ring can be thought of as a ring in which a finitely generated ideal (or, an ideal) cannot be the union of a finite number of proper subideals. Also, a U-ring is a ring in which the union of a finite number of ideals is an ideal only in case that union is one of the finite number of ideals in question.

Definition 1.4 We say that \(R\) is a Bezout ring in case every finitely generated ideal of \(R\) is principal.

Corollary 1.5 A Bezout ring is a U-ring.

Remark: Two classical (nontrivial, i.e. not principal ideal rings) Bezout rings are the ring of algebraic integers and the ring of entire functions. For other examples of Bezout rings see [DB].

Definition 1.6 A valuation ring is an integral domain in which the ideals are linearly ordered under inclusion (see
Remark: It is easy to see that a valuation ring is a Bezout ring, and hence a U-ring.

Theorem 1.7 If \((0)\) is the annihilator of a finitely generated ideal \(A \neq (0)\) of \(R\) and \(G_1 \neq R\) is an ideal of \(R\) for \(i = 1, \ldots, n\), then \(A \not\subseteq U G_1\).

Proof: We will use induction on \(n\). For \(n = 1\), clearly \(A G_1 \subseteq A\) and if \(A G_1 = A\), then there exists \(x \in G_1\) such that \(a = ax\) for all \(a \in A\) [G,p. 58]. Since the annihilator of \(A\) is \((0)\), \(1 - x = 0\) which implies \(x = 1\), and this cannot happen since \(x \in G_1 \neq R\). Hence \(A G_1 \not\subseteq A\) and the theorem holds for \(n = 1\). We now show that if the theorem is valid for \(n-1 (\geq 1)\), then it is valid for \(n\). We consider two cases.

Case 1. Among \(G_1, \ldots, G_n\) there are two ideals, say \(G_1\) and \(G_2\), such that \(G_1 + G_2 \neq R\). Then the induction hypothesis implies that there exists \(a \in A\) such that \(a \not\in A(G_1 + G_2) \cup AG_3 \cup \ldots \cup AG_n\). Now since \(a \not\in A(G_1 + G_2) = AG_1 + AG_2\), it follows that \(a \not\in AG_1\) and \(a \not\in AG_2\). Hence the theorem holds in Case 1.

Case 2. If \(G_1 + G_j = R\) for all \(i \neq j\) then \(G_1 + (G_2 \ldots G_n) = R\) [ZS_1,p. 177]. Hence \(A G_1 + A(G_2 \ldots G_n) = A\).

If \(A(G_2 \ldots G_n) \subseteq AG_1\), it would follow that \(A = AG_1 + A(G_2 \ldots G_n) = AG_1\), which is impossible by case \(n = 1\).
So $A(G_2 \ldots G_n) \not\subseteq AG_1$; hence there exists $a_1 \in A(G_2 \ldots G_n) \setminus AG_1$. Likewise, there exists $a_i \in A(G_1 \ldots G_{i-1} \cdot G_{i+1} \ldots G_n) \setminus AG_1$ for each $i = 1, \ldots, n$.

Now letting $a = \Sigma^n_{i=1} a_i$, we see that $a \notin A$, and $a \not\subseteq AG_1$ for each $i = 1, \ldots, n$. The theorem follows by induction.

Q.E.D.

**Corollary 1.8** If $G_1, \ldots, G_n$ are ideals and $A \neq (0)$ is a finitely generated ideal of an integral domain $D$, then $A \not\subseteq UAG_1$.

**Definition 1.9** An $R$-submodule $F$ of $T$ is a fractional ideal of $R$ provided there exists a regular element (i.e. not a zero divisor) $r \in R$ such that $rF \subseteq R$. The inverse of $F$ is the fractional ideal $F^{-1} = [R:F]_T = \{t | t \in T, tF \subseteq R\}$. The fractional ideal $F$ is said to be invertible if $FF^{-1} = R$.

**Definition 1.10** A fractional ideal $F$ of $R$ is called a regular ideal if $F$ contains a regular element.

Note that the ideals of $R$ are simply the fractional ideals $F$ of $R$ such that $F \subseteq R$.

**Remark:** An invertible ideal is finitely generated and regular, [G,p. 65], [LM,p. 125].

**Theorem 1.11** If $A$ is an invertible ideal of a ring $R$, then $A$ is a $U$-ideal.

**Proof:** Assume that $A_1, \ldots, A_n$ are ideals of $R$ such that
A ⊆ UA_1 and let \( \bar{A}_1 = A \cap A_1 \) for \( i = 1, \ldots, n \); we have that \( A = \bigcup \bar{A}_i \) and \( \bar{A}_1 \subseteq A \) for each \( i \). Suppose \( \bar{A}_1 \subsetneq A \) for each \( i \). Then \( \bar{A}_1 = AG_1 \) for each \( i \), with \( G_1 = \bar{A}_1 A^{-1} \) an ideal of \( R \) (since \( \bar{A}_1 \subseteq A \)); furthermore, \( G_1 \not\subseteq R \) for each \( i \). Since \( A \) is invertible, \( A \) contains a regular element and therefore the annihilator of \( A \) is \((0)\); in addition, \( A \) is finitely generated. But Theorem 1.7 implies \( A \not\subseteq \bigcup \bar{A}_i \), which contradicts \( A = \bigcup \bar{A}_i \). Hence there exists \( k \) such that \( \bar{A}_k \not\subseteq A \), and since \( \bar{A}_1 \subsetneq A \) for \( i = 1, \ldots, n \) it follows that \( A = \bar{A}_k \subseteq A_k \).

Q.E.D.

**Definition 1.12**  
\( D \) is called a **Prüfer domain** provided every non-zero finitely generated ideal of \( D \) is invertible.

**Corollary 1.13**  
If \( D \) is a Prüfer domain, then \( D \) is a U-ring.

**Proof:** This follows directly from theorems 1.3 and 1.11.  
Q.E.D.

**Remark:** A **Dedekind domain** is an integral domain \( D \) such that every proper ideal of \( D \) is a product of prime ideals of \( D \). It is known that every proper ideal of a Dedekind domain is invertible. Consequently, a Dedekind domain is a Prüfer domain and therefore a U-ring.

Remark: The converse to Corollary 1.13 is false, as can be seen by applying a later result (Corollary 4.5) which states
that any ring $R$ which contains an infinite field (with the same 1 as $R$) is a U-ring. Hence the polynomial ring $\mathbb{Q}[x,y]$ in two indeterminates $x,y$ over the rational field $\mathbb{Q}$ is a U-ring; however it is not a Prüfer domain as the following argument shows.

If $A = (x,y)$ is invertible, then $xA^{-1}$ is an ideal of $\mathbb{Q}[x,y]$ since $x \in A$, and $A(xA^{-1}) = (x)$. Since $(x)$ is a prime ideal and $A \not\subseteq (x)$, we have $xA^{-1} \subset (x)$ and $xA^{-1} = (x)$ since $(x) \subset xA^{-1}$. Hence $A(x) = (x)$, and $A = \mathbb{Q}[x,y]$, a contradiction.

Remark: The following theorem shows that a homomorphic image of a U-ring is a U-ring.

**Theorem 1.14** If $B$ is a proper ideal of a U-ring $R$, then $R/B$ is a U-ring.

**Proof:** Let $\bar{A}, \bar{A}_1, \ldots, \bar{A}_n$ be ideals of $R/B$ such that $\bar{A} \subseteq \bigcup \bar{A}_i$, and let $h: R \to R/B$ be the canonical homomorphism. It is easily checked that $h^{-1}(u\bar{A}_1) \subseteqUh^{-1}(\bar{A}_1), h^{-1}(\bar{A}) \subseteq h^{-1}(\bar{A}_1)$ for some $i$, and $\bar{A} \subseteq \bar{A}_i$ for some $i$, completing the proof. Q.E.D.

**Theorem 1.15** If $D/A$ is a U-ring for each proper ideal $A$ of the domain $D$, then $D$ is a U-ring.

**Proof:** Suppose that $A \subseteq UA_1$, where $A, A_1, \ldots, A_n$ are ideals of $D$. If $A = (0)$, then $A \subseteq A_i$ for each $i$. If
A \neq (0), consider B = A \cdot A_1 \ldots A_n. We can assume that A_i \neq (0) for i = 1, \ldots, n; B \neq (0), B \subseteq A, and B \subseteq A_i for i = 1, \ldots, n. Let \overline{A}, \overline{A}_i, denote the canonical image of A, A_i respectively in D/B, and note that \overline{A} \subseteq \bigcup \overline{A}_i. Hence \overline{A} \subseteq \overline{A}_k for some k and A \subseteq A_k.

Q.E.D.

Lemma 1.16 If R is a U-ring and S is a multiplicative system of R consisting of regular elements, then the quotient ring R_S is a U-ring.

Proof: Let A, A_1, \ldots, A_n be ideals of R_S such that A \subseteq UA_1 and set B = A \cap R, B_1 = A_i \cap R for i = 1, \ldots, n. Then B \subseteq UB_1. Suppose A \not\subseteq A_i and let a_i \in A \setminus A_i for i = 1, \ldots, n. There exists s \in S such that sa_i \in R, sa_i \in A, and sa_i \not\in A_i, so sa_i \in B \setminus B_1 for i = 1, \ldots, n; however, B \supseteq B_1 for some i since R is a U-ring, a contradiction. Hence A \subseteq A_i for some i.

Q.E.D.

Theorem 1.17 If R is a U-ring and S is any multiplicative system of R, then the quotient ring R_S is a U-ring.

Proof: The ring R_S in general is defined to be the ring \( (R/N)_{\overline{S}} \) where N = \{x \in R | xs = 0 \text{ for some } s \in S\}, and letting f be the canonical homomorphism f:R \rightarrow R/N, \overline{S} is the multiplicative system f(S) in R/N which consists of regular elements, [ZS_1, p. 221]. Hence the theorem follows.
immediately from Theorem 1.14 and Lemma 1.16.

Q.E.D.

Remark: The converse of Theorem 1.17 is false as can be seen by Example 6.2; in fact Example 6.3 shows that it can happen that every overring \( R' \) of \( R \), with \( R < R' < T \), is a U-ring and \( R \) is not a U-ring.

**Theorem 1.18** If \( R_1, \ldots, R_n \) are rings, then \( R = R_1 \oplus \cdots \oplus R_n \) is a U-ring if and only if \( R_1, \ldots, R_n \) are each U-rings.

**Proof:** If \( R = R_1 \oplus \cdots \oplus R_n \) is a U-ring, then \( R_1 \) is a U-ring for \( i = 1, \ldots, n \) by Theorem 1.14. Now consider the converse. It is sufficient to prove the theorem for \( n = 2 \) since the general result will then follow by induction.

Suppose \( A_1 \oplus A_2 = U(A_{1i} \oplus A_{2i}) \), where \( A_1 \) and \( A_{1i} \) are ideals in \( R_1 \) and \( A_2 \) and \( A_{2i} \) are ideals in \( R_2 \) for \( i = 1, \ldots, n \). Then \( A_1 = U A_{1i} \) and \( A_1 = A_{1k} \) for some \( k \) since \( R_1 \) is a U-ring. We consider two cases. First, suppose \( A_1 = A_{1i} \) for each \( i = 1, \ldots, n \); since \( A_2 = U A_{2i} \) and \( R_2 \) is a U-ring, then \( A_2 = A_{2j} \) for some \( j \) and it is clear that \( A_1 \oplus A_2 = A_{1j} \oplus A_{2j} \). Second, suppose \( A_1 \neq A_{1t} \) for some \( t \in \{1, \ldots, n\} \), say \( \{1, \ldots, n\} = MN \) where \( A_1 = A_{1i} \) for \( i \in M \) and \( A_1 \neq A_{1i} \) for \( i \in N \). Then \( A_1 \neq U A_{1i} \) since \( R_1 \) is a U-ring, hence there exists \( a_1 \in A_1 \) such that \( a_1 \notin A_{1i} \) for \( i \in N \). We claim that \( A_2 = U A_{2i} \), and hence \( A_2 = A_{2i} \) for some \( i \in M \) since \( R_2 \) is a U-ring, and \( A_1 \oplus A_2 = A_{1i} \oplus A_{2i} \) for some \( i \in M \). To settle the
claim, let \( x \in A_2 \) and consider \( a_{1} + x \); since \( a_{1} \not\in A_{11} \) for \( i \in N \) and \( a_{1} + x \in A_{1} \oplus A_{2} \), it follows that \( a_{1} + x \in A_{11} \oplus A_{21} \) for some \( i \in M \), and hence \( x \in A_{21} \) for some \( i \in M \), completing the argument.

Q.E.D.
In this chapter we define and study property C and property CF on a ring $R$, (Definitions 2.1 and 2.2). Some results obtained here are: a domain with property C is a Prüfer domain and hence a U-ring; a domain $D$ has property CF if and only if $D$ is Prüfer.

**Definition 2.1** A ring $R$ has property C or is called a C-ring if, for ideals $A, B$ of $R$, $B < A$ implies the existence of an ideal $C \subset R$ such that $B \subset AC$.

**Definition 2.2** A ring $R$ has property CF or is called a CF-ring if, for ideals $A, B$ of $R$ such that $A$ is finitely generated, $B < A$ implies there exists an ideal $C \subset R$ such that $B \subset AC$.

**Remark:** Property C implies property CF.

**Theorem 2.3** If $D$ is a domain with property CF, then $D$ is a U-ring.

**Proof:** Let $A$ be a finitely generated ideal and $A_1, \ldots, A_n$ ideals of $D$ such that $A = UA_1$. If $A \not\subset A_1$, then $A_1 < A$ and there exists $B_1 \not\subset D$ such that $A_1 \subset AB_1$ for $i = 1, \ldots, n$. By Corollary 1.8 there exists $a \in A$ such that $a \not\subset AB_1$ for
each \( i \), hence \( a \not \in A_i \) for each \( i \). But this contradicts 
\[ A = \bigcup A_i, \text{ implying } A = A_i \text{ for some } i \text{ and } D \text{ is a U-ring.} \]

**Q.E.D.**

**Note:** It is clear that if \( D \) is a \( C \)-ring, then \( D \) is a 
U-ring since property \( C \) implies property \( CF \).

**Remark:** The converse of Theorem 2.3 is false as the 
following example shows.

**Example 2.4** If \( D = \mathbb{Q}[x,y] \), the polynomial ring in two 
indeterminates over \( \mathbb{Q} \), then \( D \) is a U-ring by a later 
result (Corollary 4.5). Consider the ideals \((x,y^2) < (x,y)\) 
in \( D \), and suppose that an ideal \( A \neq D \) exists such that 
\((x,y^2) \subset (x,y)A\). Then there exists \( p,q \in A \) such that 
\[ px + qy = x. \]
Since \( D \) is a U.F.D. and \( x \mid qy \) in \( D \), \( q = dx \) 
for some \( d \in D \). Hence \( p + dy = 1 \) and \( py + dy^2 = y \), implying 
y \( \in A \) and \((x,y) \subset A\), which implies \( A = (x,y) \) since 
\((x,y)\) is maximal and \( A \neq D \). But \( x \not \in (x,y)^2 \), which is a 
contradiction. Hence \( D \) does not have property \( CF \).

**Q.E.D.**

**Theorem 2.5** If \( M \) is a maximal ideal in a domain \( D \) with 
property \( C \), then there are no ideals of \( D \) properly 
between \( M \) and \( M^2 \).

**Proof:** Suppose there is an ideal \( B \) of \( D \) such that 
\( M^2 \subset B < M \). Then there exists an ideal \( C \neq D \) such that 
\( B \subset MC \), and \( M^2 \subset B \subset C \). Now any proper prime ideal which
contains \( C \) must contain \( M^2 \), and therefore contains and hence equals \( M \). So we have \( C \subseteq M \). However \( C \subseteq M \) implies \( M^2 \subseteq B \subseteq MC \subseteq M^2 \). Therefore \( B = M^2 \), and there are no ideals properly between \( M \) and \( M^2 \).

Q.E.D.

**Theorem 2.6** If \( D \) is a Noetherian domain with property \( C \), then \( D \) is a Dedekind domain.

**Proof:** This follows immediately from Theorem 2.5 and the fact that a Noetherian domain \( D \) is Dedekind if and only if for each maximal ideal \( M \) of \( D \), there are no ideals properly between \( M \) and \( M^2 \) [G, Thm. 31.1, p. 443-44].

Q.E.D.

**Remark:** The following is an example of a \( C \)-domain (a \( C \)-ring which is a domain) which is not Dedekind, showing that Noetherian is a necessary assumption in the above theorem.

**Example 2.7** Consider the group \( G = Z \oplus Z \) ordered lexicographically and the mapping \( v: \mathbb{Q}[x,y] \rightarrow \mathbb{G} \cup \{\infty\} \) defined by \( v(\sum a_{i,j}x^i y^j) = \min\{(i,j)|a_{i,j} \neq 0\} \) and \( v(0) = \infty \). Now extend \( v \) to \( \mathbb{Q}(x,y) \) by setting \( v(r/s) = v(r) - v(s) \) for each \( r, s \in \mathbb{Q}(x,y), s \neq 0 \), [ZS_2, p. 37], hence producing a valuation on \( \mathbb{Q}(x,y) \), and consider the valuation ring \( D_v = D \).

If one studies the value group \( G \) and its segments [G, p. 184-86] it becomes clear that (1) if \( A \) and \( B \) are ideals of \( D \) such that \( B \subset A \), then \( B \subseteq MA \) where \( M \) is the maximal ideal.
of $D$ (hence $D$ is a $C$-ring), and (2) $D$ is not Noetherian and hence not Dedekind.

Q.E.D.

**Definition 2.8** A ring $R$ is said to be **quasi-local** if it has a unique maximal ideal.

**Theorem 2.9** Let $A$ be a finitely generated ideal in a quasi-local domain $D$. If $A$ has the property that for any ideal $B$ of $D$, $B < A$ implies there exists an ideal $C \not= D$ of $D$ such that $B \subseteq AC$ - then $A$ is principal.

**Proof:** Let $A = (a_1, \ldots, a_n)$ and suppose $(a_i) < A$ for $i = 1, \ldots, n$. Then there exists an ideal $C_1 \not= D$ in $D$ such that $(a_i) \subseteq AC_1$ for $i = 1, \ldots, n$ and $\Sigma_1^n C_1 \subseteq M$, the maximal ideal of $D$. Now $AC_1 \subseteq A$ for each $i$, so $A\Sigma_1^n C_1 = \Sigma_1^n AC_1 \subseteq A$ and $A \not= A\Sigma_1^n C_1$ by Nakayama's lemma, [L,p. 242]. However, $A = \Sigma_1^n (a_i) \subseteq \Sigma_1^n AC_1 < A$ implies $A < A$, a contradiction. Hence $A = (a_i)$ for some $i$.

Q.E.D.

**Theorem 2.10** If $A$ is an ideal in a domain $D$, then the following are equivalent.

(a) $A$ is invertible.

(b) $A$ is finitely generated and $B < A$

$(B$ an ideal of $D)$ implies the existence of an ideal $C \not= D$ of $D$ such that $B \subseteq AC$.

**Proof:** We may assume $A \not= D$; suppose $A$ satisfies (b) and
let $M$ be a maximal ideal of $D$ such that $A \subset M$. In the quasi-local ring $D_M$ the ideal $AD_M = A^e$ is finitely generated. Suppose $B < A^e$, an ideal of $D_M$, and note that $A \cap B^c < A$ (where $B^c = B \cap D$) and $(A \cap B^c)^e = A^e \cap B^e = A^e \cap B$ by [G,Thm. 3.4,p. 34]. There is an ideal $C \neq D$ in $D$ such that $A \cap B^c \subset AC$, and consequently $B = (A \cap B^c)^e \subset A^e C^e = (AC)^e$. It follows from Theorem 2.9 that $A^e$ is a principal ideal in $D_M$ for each maximal ideal $M$ such that $A \subset M$, and it is clear that $D_M = AD_M = A^e$ for each maximal ideal $M$ such that $A \not\subset M$. Since $A$ is finitely generated and $AD_M$ is principal for every maximal ideal $M$ of $D$, it follows that $A$ is invertible in $D$ from [G,Cor. 6.5,p. 72]. Conversely, suppose $A$ is invertible and $B < A$, ($B$ an ideal of $D$). Then $C = BA^{-1}$ is an ideal of $D$, $AC = B$, and $A$ is finitely generated [G,p. 65], [LM,p. 125].

Q.E.D.

Remark: The proofs given in Theorems 2.9 and 2.10 are valid for a regular ideal $A$ of a ring $R$.

Corollary 2.11 A domain $D$ has property CF if and only if $D$ is Prüfer.

Proof: Since finitely generated ideals are invertible in a Prüfer domain (Definition 1.12), it follows from Theorem 2.10 that a Prüfer domain has property CF. The converse is clear from Theorem 2.10, since finitely generated non-zero ideals
in a CF-domain are invertible.

Q.E.D.

Corollary 2.12 The domain D is a Noetherian CF-domain if and only if D is a Dedekind domain.

Proof: Since every non-zero ideal is invertible in a Noetherian CF-domain by Theorem 2.10, and since D is Dedekind if and only if each non-zero ideal of D is invertible [LM,p. 137], the theorem follows.

Q.E.D.
CHAPTER III

In this chapter we study the finite union of ideals in conjunction with the equivalence relation quasi-equality, denoted \( \sim \) (see Definition 3.1). In particular we consider ideals \( A \) such that \( A = \bigcup A_i \) implies that \( A \sim A_i \) for some \( i \) and we consider rings in which this is true for all ideals \( A \). Among our results are the following: if \( A_1, \ldots, A_n \) are ideals and \( A \) is a quasi-invertible ideal of a domain \( D \) such that \( A = \bigcup A_i \) then \( A \sim A_i \) for some \( i \); if \( D \) is a completely integrally closed domain and \( A, A_1, \ldots, A_n \) are ideals of \( D \), then \( A = \bigcup A_i \) implies \( A \sim A_i \) for some \( i \); and if \( D \) is completely integrally closed and \( A_1, \ldots, A_n \) are \( v \)-ideals such that \( A \subseteq \bigcup A_i \), then \( A \subseteq A_i \) for some \( i \).

Definition 3.1 Two ideals \( A, B \) of \( R \) are said to be \textit{quasi-equal}, denoted \( A \sim B \), if \( A^* = B^* \) where \( A^* = (A^{-1})^{-1} \), (see \([G, \S\ 26\ and\ \S\ 28]\)).

Remark: \( A^* = B^* \) if and only if \( A^{-1} = B^{-1} \), and hence the definition of quasi-equality given here is the same as that of van der Waerden and Artin.

Definition 3.2 An ideal \( A \) of \( R \) is called a \textit{v-ideal} if \( A = A^* \).
Definition 3.3  An ideal $A$ of $R$ is **quasi-invertible** if $AA^{-1} \sim R$.

Remark: Next we will state without proof some of the elementary properties of "\(\sim\)" , "inverse", and "\(*\)". Most of these are found in [G, §26 and §28] for domains, however they are true for rings and their proofs follow easily from those for domains.

Theorem 3.4. Let $A, B, C$ be ideals of $R$. Each of the following holds in $R$.

1. The relation "\(\sim\)" is an equivalence relation on the set of ideals of $R$.
2. If $A \subseteq B$, then $B^{-1} \subseteq A^{-1}$, $A* \subseteq B*$, and $AB^{-1} \subseteq R$.
3. $A \subseteq A*$; $A \sim A*$; $(A*)* = A*$; $(A*)^{-1} = A^{-1}$.
4. If $A \sim B$, then $AC \sim BC$ and $A + C \sim B + C$; if $AC \sim BC$ and $C$ is quasi-invertible, then $A \sim B$.

Lemma 3.5  If $A, A_1, \ldots, A_n$ are ideals of $R$ and $(A + A_1) \sim R$ for $i = 1, \ldots, n$, then $[A + (A_1 \ldots A_n)] \sim R$.

Proof: We will use induction on $n$. The lemma is clear for $n = 1$. Assuming its truth for $n = k - 1$, we have
\[ A + (A_1 \ldots A_{k-1}) \sim R \] which implies \[ AA_k + (A_1 \ldots A_k) \sim A_k, \]
and this implies \[ A + [AA_k + (A_1 \ldots A_k)] \sim A + A_k \sim R. \]
But \[ A + AA_k = A \] so that \[ A + (A_1 \ldots A_k) \sim R, \]
and the lemma follows.

Q.E.D.

**Theorem 3.6** If \( A \) is a quasi-invertible ideal of a domain \( D \) and \( G_1, \ldots, G_n \) are ideals of \( D \) such that \( G_1 \not\subset D \) for each \( i \), then \( A \not\subset U(AG_1)^* \).

**Proof:** We use induction on \( n \). We note that \( AG_1 \subset A \) and hence \( (AG_1)^* \subset A^* \). Since \( A \) is quasi-invertible and \( G_1 \not\subset D \), it follows that \( A \not\subset AG_1 \). If \( A \subset (AG_1)^* \), then \( A^* \subset (AG_1)^* \) which implies \( A^* = (AG_1)^* \), \( A \sim AG_1 \), \( G_1 \sim D \) a contradiction. Therefore \( A \not\subset (AG_1)^* \) and the theorem is true for \( n = 1 \). Now suppose the theorem is true for \( n = k - 1 \) (\( 1 < k \)) and consider \( n = k \). There are two cases.

**Case 1.** There exist two of \( G_1, \ldots, G_n \), say \( G_1 \) and \( G_2 \), such that \( (G_1 + G_2) \not\subset D \). Then by the induction hypothesis there exists \( a \in A \) such that

\[ a \not\in (A(G_1 + G_2))^* \cup (AG_3)^* \cup \ldots \cup (AG_k)^* \cdot \]

But

\[ (A(G_1 + G_2))^* = (AG_1 + AG_2)^* = ((AG_1)^* + (AG_2)^*)^* \],

which contains \( (AG_1)^* \) and \( (AG_2)^* \). Hence \( a \not\in \cup_1^k (AG_i)^* \), completing the proof in this case.
Case 2. $G_i + G_j \sim D$ for all $i \neq j$ in $\{1, \ldots, k\}$. Lemma 3.5 implies that $G_i + \prod_{j \neq i} G_j \sim D$ for $i = 1, \ldots, k$. Hence

(1) $(A \cdot \prod_{j \neq i} G_j)^* \not\subseteq (AG_i)^*$ for $i = 1, \ldots, k$ ;

for, if containment holds in (1) for some $i$, then

$$(AG_i)^* = (A \cdot \prod_{j \neq i} G_j)^* + (AG_i)^* = ((A \cdot \prod_{j \neq i} G_j)^* + (AG_i)^*)^* = (A \cdot \prod_{j \neq i} G_j + AG_i)^* = A^*$$

which implies $AG_i \sim A$ and $G_i \sim D$, a contradiction. Now let $a_i \in (A \cdot \prod_{j \neq i} G_j)^* \setminus (AG_i)^*$ for $i = 1, \ldots, k$ and set

$$a = \sum_{i=1}^{k} a_i \ .$$

Now $a \not\in (AG_i)^*$ for $i = 1, \ldots, k$ since

$$a_t \in (A \cdot \prod_{j \neq t} G_j)^* \subset (AG_i)^*$$

for each $t \neq i$. This completes the argument in this case and the theorem follows.

Q.E.D.

Theorem 3.7 If $A_1, \ldots, A_n$ are ideals and $A$ is a quasi-invertible ideal of $D$ such that $A \sim UA_1$, then $A \sim A_1$ for some $i$.

Proof: If $A = UA_1$, then $A \subset UA_1^*$ since $A_1 \subset A_1^*$.

Suppose $A \not\sim A_1$ for each $i$. Then $A_1 A_1^{-1}$ is an ideal of $D$ (note we have $A_1 \subset A$), and $A_1 A_1^{-1} \not\subset D$. Hence by Theorem 3.6, $A \not\subseteq U(A(A_1^{-1}A_1))^*$, a contradiction since $(A(A_1^{-1}A_1))^* = A_1^*$ and $A \subset UA_1^*$.

Q.E.D.

Definition 3.8 An element $t$ in the total quotient ring $T$ of $R$ is said to be almost integral over $R$ if there exists
some regular element \( r \in R \) such that \( t^n r \in R \) for all positive integers \( n \). If \( M \) is the set of all elements of \( T \) almost integral over \( R \) and \( M \subseteq R \), then \( R \) is said to be **completely integrally closed**.

**Corollary 3.9** If \( D \) is completely integrally closed and \( A, A_1, \ldots, A_n \) are ideals of \( D \), then \( A = U A_1 \) implies \( A \sim A_1 \) for some \( i \).

**Proof:** Each ideal of a completely integrally closed domain is quasi-invertible [G,p. 395], so this result follows immediately from Theorem 3.8.

**Q.E.D.**

**Remark:** The converse of Corollary 3.9 is false, as can be seen by letting \( D \) be a valuation ring of dimension greater than \( 1 \). \( D \) is not completely integrally closed [G, Thm. 14.5.3,p. 178], but \( A = U A_1 \) implies \( A = A_1 \) (and hence \( A \sim A_1 \)) for some \( i \).

**Remark:** We observe that a Krull domain is completely integrally closed, so Corollary 3.9 applies to Krull domains. (see [G,§ 35]).

In the following corollaries, \( A, A_1, \ldots, A_n \) are ideals in the domain \( D \).

**Corollary 3.10** If \( A \) is quasi-invertible in \( D \) and \( A \subseteq U A_1 \), then \( A^* \subseteq A_1^* \) for some \( i \).
Proof: $A \subseteq UA_1$ implies that there exist $B_1 = A \cap A_1$ for $i = 1, \ldots, n$ such that $A = UB_1$. By Theorem 3.7 $A \sim B_1$ for some $i$ so $A^* = B_1^* \subseteq A_1^*$. Q.E.D.

Corollary 3.11 If $D$ is completely integrally closed, then $A \subseteq UA_1$ implies $A^* \subseteq A_1^*$ for some $i$.

Corollary 3.12 If $A$ is quasi-invertible and $A_1, \ldots, A_n$ are $v$-ideals such that $A \subseteq UA_1$, then $A \subseteq A_i$ for some $i$.

Proof: By Corollary 3.10 $A \subseteq A^* \subseteq A_1^* = A_1$ for some $i$. Q.E.D.

Corollary 3.13 If $D$ is completely integrally closed and $A_1, \ldots, A_n$ are $v$-ideals such that $A \subseteq UA_1$, then $A \subseteq A_1$ for some $i$.

Proof: $A$ is quasi-invertible since $D$ is completely integrally closed. Q.E.D.

Remark: We point out the parallel between Corollary 3.13 for completely integrally closed domains and $v$-ideals, and the classical theorem for a ring $R$ and prime ideals.

Theorem 3.14 If $A, A_1, \ldots, A_n$ are ideals of a ring $R$ such that $A = UA_1$, then $A^{-1} = \cap_1^n A_1^{-1}$.

Proof: Since $A_1 \subseteq A$, it is clear that $A^{-1} \subseteq A_1^{-1}$ for $i = 1, \ldots, n$ and $A^{-1} \subseteq \cap_1^n A_1^{-1}$. If $y \in \cap_1^n A_1^{-1}$, then
yA_i \subset R$ for $i = 1, \ldots, n$ and $y(UA_i) = yA \subset R$, and
$y \in A^{-1}$.

Q.E.D.

Remark: The converse of Theorem 3.14 is false in general. To see this let $R = \mathbb{Z}/(30)$, $A = \overline{2R}$, $A_1 = \overline{3R}$, $A_2 = \overline{5R}$. Now $A^{-1} = A_1^{-1} = A_2^{-1} = R$, hence $A_1^{-1} \cap A_2^{-1} = A^{-1}$, but $A \neq A_1 \cup A_2$.

Theorem 3.15 If $D$ has the property that for any finitely generated ideal $A$, $A = UA_i$ implies $A \sim A_i$ for some $i$, then the same property holds for all ideals of $D$, finitely generated or not.

Proof: Let $B$ be an ideal of $D$ such that $B = \bigcup B_i$ for ideals $B_i \subset D$, and suppose $B \not\subset B_i$ for each $i$. Then since $B_i \subset B \subset B^*$ we must have $B_i^* \subset B^*$ and $B^* \not\subset B_i^*$ which implies $B \not\subset B_i^*$ for each $i$. Now let $A = (a_1, \ldots, a_n)$ where $a_i \in \mathbb{B} \setminus B_i^*$ for each $i$. We have $A \subset B = \bigcup B_i$, which implies $A = \bigcup B_i'$ where $B_i' = A \cap B_i$, and note $A$ is finitely generated. Hence $A \sim B_j'$ for some $j$ (since the property holds for finitely generated ideals). But this implies $A \subset (B_j')^* \subset B_j^*$ for some $j$ which is a contradiction since $a_j \in A$ and $a_j \not\in B_j^*$. It follows that $B \sim B_i$ for some $i$.

Q.E.D.
CHAPTER IV

This chapter deals with the finite union of unitary R-modules, Û-rings (Definition 4.2), and more facts about U-rings. It is obvious that a Û-ring is a U-ring, however the converse is false (see the remark following Theorem 4.15). Some of our results in this chapter are: if there exists an infinite subset S of R such that x - y is a unit of R for x, y ∈ S with x ≠ y, then R is a Û-ring; if R contains an infinite subfield (where the subfield has the same 1 as R), then R is a Û-ring and hence a U-ring; if R is a quasi-local ring R with maximal ideal P, then R is a Û-ring if and only if R/P is infinite; if R is a quasi-local ring with maximal ideal P, then R is a U-ring if and only if R/P is infinite or finitely generated ideals of R are principal; and if R' is an overring of a Û-ring R such that R and R' have the same 1, then R' is a Û-ring. We close the chapter with an example of a domain R which is a U-ring but it does not contain an infinite field and it is not a Prüfer domain.

Lemma 4.1 If M is an arbitrary R-module and M₁, ..., Mₙ are arbitrary R-submodules of M, the following conditions are equivalent.
(a) \( M = \bigcup M_i = M = M_i \) for some \( i \).

(b) \( M \) is finitely generated and \( M = \bigcup M_i = M = M_i \)
for some \( i \).

Proof: Clearly (a) \( \Rightarrow \) (b). Suppose (b) holds and consider (a). Suppose \( M_i < M \) for each \( i \) and let \( m_i \in M\backslash M_i \) for \( i = 1, \ldots, n \). Set \( M' = \bigoplus M_i \) and \( M_i = M' \cap M_i \) for \( i = 1, \ldots, n \). Then \( M' = \bigcup M_i \) and \( M' \) is finitely generated so that \( M' = M_i' \) for some \( i \), and \( m_i \in M_i' \subseteq M_i \), a contradiction.

Q.E.D.

Definition 4.2 R is called a \( \bar{U} \)-ring if the following holds: if \( M_1, \ldots, M_n \) are submodules of an \( R \)-module \( M \) such that \( M = \bigcup M_i \), then \( M = M_i \) for some \( i \).

Remark: If \( R \) is a \( \bar{U} \)-ring then \( R \) is a \( U \)-ring, but the converse is false. The quotient ring \( \mathbb{Z}(2) \) of \( \mathbb{Z} \) is an example of a \( U \)-ring which is not a \( \bar{U} \)-ring (see Remark following Theorem 4.15).

Remark: If \( M \) is a principal \( R \)-module and if \( M = \bigcup M_i \), then \( M = M_i \) for some \( i \), where \( M_1, \ldots, M_n \) are submodules of \( M \).

Theorem 4.3 If there exists an infinite subset \( S \) of \( R \) such that \( x - y \) is a unit of \( R \) for all \( x, y \in S \) with \( x \neq y \), then \( R \) is a \( \bar{U} \)-ring.

Proof: Let \( M_1, \ldots, M_n \) be submodules of a finitely generated
R-module $M$ such that $M = \bigcup M_1$, and suppose $M = \Sigma^r_1 Rm_1$ with $r$ the minimal number of generators of $M$. We may suppose $1 < r$ by the "Remark" above. Consider the set

$E = \{e_x = \Sigma^r_1 x^{i-1}m_1 | x \in S, x^0 = 1\}$ where $S$ is as in the hypothesis, and notice that $x \neq y$ in $S$ implies that $e_x \neq e_y$ in $E$, since $x - y$ is a unit in $R$ and $r$ is minimal. Since $E \subseteq M = \bigcup M_1$ and $E$ is infinite, we see that some $M_t$ contains at least $r$ elements of $E$, say $e_j = e_{x_j} = \Sigma^r_1 x_j^{i-1}m_1$ for $j = 1, \ldots, r$. Now by Cramer's rule [MB, p. 304-5] we solve the system $\Sigma^r_1 x_j^{i-1}m_1 = e_j$, for $j = 1, \ldots, r$ and obtain

(1) $d_m^r = \Sigma^r_1 d_{i,j}^r e_1$ for $j = 1, \ldots, r$

where $d$ is the determinant of the coefficient matrix

$(x_j^{i-1})$ and $d_{i,j} = (-1)^{i+j} \det(A_{1,j})$ where $A_{i,j}$ is the matrix obtained from the coefficient matrix by striking out row $i$ and column $j$. Now $d$ is a Vandermonde determinant, so

$d = \Pi_{j>k}(x_j - x_k)$ is a unit in $R$ and (1) implies that

$m_j \in \Sigma^r_1 R e_1 \subseteq M_t$ for $j = 1, \ldots, r$ and $M = M_t$.

Q.E.D.

**Corollary 4.4** If there exists an infinite subset $S$ of $R$ such that $x - y$ is a unit of $R$ for $x, y \in S$ with $x \neq y$, then $R$ is a U-ring.

**Proof**: If $R$ is a U-ring, then $R$ is a U-ring.

Q.E.D.
Corollary 4.5 If \( R \) contains an infinite subfield \( K \) (where \( R \) and \( K \) have the same \( 1 \)), then \( R \) is a \( \tilde{U} \)-ring and hence a \( U \)-ring. In particular, if \( V \) is a vector space over an infinite field \( K \), then \( V \) is not the union of a finite number of proper subspaces.

Remark: We note that \( \mathbb{Q}[x,y] \) is a \( U \)-ring and \( \mathbb{Z}[x] \) is not (see Example 6.1), while both are polynomial rings in one indeterminate over a Euclidian ring.

Corollary 4.6 Using the notation of Theorem 4.3, if there are at least \( n(r-1)+1 \) elements in \( S \) then \( M \) is not the union of \( n \) or fewer proper submodules.

Proof: This follows from the proof given for Theorem 4.3.

Q.E.D.

Definition 4.7 A ring \( R \) is said to be quasi-local if it has a unique maximal ideal.

Theorem 4.8 If \( R \) is a quasi-local ring with maximal ideal \( P \) and \( R/P \) is infinite, then \( R \) is a \( \tilde{U} \)-ring (in fact, there exists an infinite set \( S \) of the type described in Theorem 4.3).

Proof: Letting \( S \) be a complete set of representatives of the collection of all non-zero cosets of \( P \) in \( R \), the result follows immediately from Theorem 4.3 since \( x,y \in S \) (with \( x \neq y \)) implies \( x-y \in R\setminus P \) and \( x-y \) is a unit in \( R \).

Q.E.D.
Corollary 4.9 Under the hypothesis of Theorem 4.8, R is a U-ring.

Remark: Several of the following theorems deal with the field \( R/P \) - where \( R \) is a ring and \( P \) is a maximal ideal of \( R \) - and the vector space \( M/PM \) over \( R/P \), where \( M \) is an \( R \)-module. The operation between \( R/P \) and \( M/PM \) is the "natural" or "usual" one defined as \((r+P)\cdot(m+PM) = rm+PM\) where \( r+P \in R/P \) and \( m+PM \in M/PM \) (see [ZS, p. 139-40 and p. 145]).

Theorem 4.10 If there exists a maximal ideal \( P \) of \( R \) and an \( R \)-module \( M \) such that \( R/P \) is finite (with \( q = n-1 \) elements) and the vector space \( M/PM \) is not one-dimensional over the field \( R/P \), then there exist submodules \( M_i < M \) for \( i = 1, \ldots, n \) such that \( M = \bigcup M_i \) and hence \( R \) is not a U-ring.

Proof: Assume that the dimension of \( \bar{M} = M/PM \) as a vector space over \( \bar{R} = R/P \) is greater than 1, and let \( B \) be a basis for \( \bar{M} \) over \( \bar{R} \). Let \( b_1 \) and \( b_2 \) be distinct elements of \( B \) and consider the following subsets of \( \bar{M} \),

\[
E_1 = B - \{b_1\}, E_2 = B - \{b_2\}, E_{2+1} = [B - \{b_1, b_2\}] \cup \{b_1 + x_1 b_2\}
\]

where \( x_1, \ldots, x_{q-1} \) are the non-zero elements of \( \bar{R} \) (\( \bar{R} \) of order \( q \)). Let \( A_i \) be the submodule of \( M \) generated by \( E_i \) for \( i = 1, \ldots, q+1 \). We claim that \( \bar{M} = \bigcup^q A_i \) and \( A_i < \bar{M} \) for each \( i \). It is clear that \( A_i < \bar{M} \) for \( i = 1, q \) and we will show that \( b_1 \notin A_i \) for \( q + 1 < i \). If \( b_1 \notin A_i \) for \( 2 < i \), then
(1) \[ b_1 = r(b_1 + x_1 b_2) + \sum_{j=1}^{n} r_j b_j' \]

where \( r \in \mathbb{R} \), \( r_j \in \mathbb{R} \), \( b_j' \in B \) and \( b_j' \neq b_1, b_2 \) for \( j = 1, \ldots, n \).

From (1) we have

(2) \[ (1-r)b_1 = r x_1 b_2 + \sum_{j=1}^{n} r_j b_j' , \]

which implies that \( r = 1 \) and \( x_1 = 0 \) since \( B \) is a basis, a contradiction since \( x_1 \neq 0 \) in \( \mathbb{R} \).

It is clear that \( U^{q+1}_1 A_1 \subset \bar{M} \), so choose \( x \in \bar{M} \). Then

(3) \[ x = \sum_{i=1}^{n} r_i b_i, \quad r_i \in \mathbb{R}, \quad b_i \in B, \quad i = 1, \ldots, n . \]

If either \( r_1 \) or \( r_2 = 0 \), then \( x \) belongs to \( A_1 \) or \( A_2 \).

Suppose \( r_1 r_2 \neq 0 \) and let \( x_k = r_2 r_1^{-1} \). It follows from (3) that \( r_1^{-1} x \in A_k \) and \( x \in A_k \). Let \( f: M \to \bar{M} \) be the canonical homomorphism, and set \( M_i = f^{-1}(A_i) \) for \( i = 1, \ldots, q+1 \). Then \( M = U^{q+1}_1 M_1 \) and \( M_1 < M \) for each \( i \); the result follows with \( n = q+1 \).

Q.E.D.

Remark: The following is clear by Corollary 4.5 and Theorem 4.10: If \( V \) is a vector space over a field \( K \) then \( V \) is the union of a finite number of proper subspaces if and only if the dimension of \( V \) is greater than one and \( K \) is finite. In fact, if \( V \) is such a vector space and \( K \) is of order \( q \) then by Theorem 4.10 we have that \( V \) is the union of \( q+1 \) proper subspaces.

Theorem 4.11 If there exist a maximal ideal \( P \) of \( R \) and an ideal \( A \) of \( R \) such that \( R/P \) is finite and \( A/\text{AP} \) is
not one dimensional over \( R/P \), then \( R \) is not a \( U \)-ring.

**Proof:** The proof of Theorem 4.10 is valid here if the \( R \)-modules involved are replaced by ideals of \( R \).

Q.E.D.

**Theorem 4.12** If \( R/P \) is finite, where \( P \) is the maximal ideal of the quasi-local ring \( R \), and there exists a finitely generated \( R \)-module \( M \) which is not principal (i.e. not generated by one element), then \( R \) is not a \( U \)-ring.

**Proof:** If \( M \) is finitely generated and not principal, then so is \( M/MP \) as a vector space over \( R/P \) \([N,p. 13]\) and the conclusion follows from Theorem 4.10.

Q.E.D.

**Theorem 4.13** If \( R/P \) is finite, where \( P \) is the maximal ideal of the quasi-local ring \( R \), and there exists a finitely generated ideal of \( R \) which is not principal, then \( R \) is not a \( U \)-ring.

**Proof:** This follows from Theorem 4.11 as Theorem 4.12 follows from Theorem 4.10.

Q.E.D.

**Theorem 4.14** If \( R \) is a quasi-local ring with maximal ideal \( P \), then \( R \) is a \( U \)-ring if and only if \( R/P \) is infinite.

**Proof:** Suppose \( R/P \) is finite and consider the polynomial ring \( R[x,y] \) in two indeterminates \( x,y \) over \( R \). The \( R \)-module \( Rx + Ry \) is finitely generated and not principal, so \( R \) is
not a $\bar{U}$-ring by Theorem 4.12. The converse is clear from Theorem 4.8.

Q.E.D.

Theorem 4.15 If $R$ is a quasi-local ring with maximal ideal $P$, then $R$ is a $U$-ring if and only if $R/P$ is infinite or finitely generated ideals of $R$ are principal.

Proof: If $R$ is a $U$-ring and $R/P$ is finite, then finitely generated ideals of $R$ are principal by Theorem 4.13. Conversely, if $R/P$ is infinite, then $R$ is a $U$-ring by Corollary 4.9, and if $R$ is a Bezout ring then $R$ is a $U$-ring by Corollary 1.5.

Q.E.D.

Remark: Every $\bar{U}$-ring is a $U$-ring, but the converse is false - e.g. a quasi-local Bezout ring in which $R/P$ is finite (see Theorems 4.14 and 4.15) such as the quotient ring $\mathbb{Z}((2))$ of $\mathbb{Z}$ with respect to $(2)$.

Theorem 4.16 If $R'$ is any overring of a $\bar{U}$-ring $R$ such that $R$ and $R'$ have the same $I$, then $R'$ is a $\bar{U}$-ring.

Proof: If $M, M_1, \ldots, M_n$ are $R'$-modules, then they are $R$-modules hence $UM_1 = M$ implies $M = M_1$ for some $i$, and $R'$ is a $\bar{U}$-ring.

Q.E.D.

Remark: It is clear that if $R$ contains an infinite field
K such that $K$ and $R$ have the same 1, then $R$ has a subset $S$ as described in Theorem 4.3, and $R$ is a U-ring. It is also clear that if $R$ is a Prüfer domain, then $R$ is a U-ring. The question as to whether containing an infinite field, or being Prüfer are necessary conditions in order that a domain $R$ be a U-ring is answered by the following example.

Example 4.17 We construct a quasi-local domain $R$ which:

(a) Contains a set $S$ as in Theorem 4.3 and hence is a U-ring.
(b) Does not contain an infinite field, and
(c) Is not Prüfer.

Let $D$ be a non-maximal order in an algebraic number field $\mathbb{Q}(a)$; that is, $D$ is integral over $\mathbb{Z}$, $D$ is not integrally closed, and the quotient field of $D$ is $\mathbb{Q}(a)$. For example, take $D = \mathbb{Z} [\sqrt{5}]$ in $\mathbb{Q}[\sqrt{5}]$. Note that $D$ is a Noetherian domain in which proper prime ideals are maximal, and consequently every proper ideal of $D$ is a product of non-factorable primary ideals ($N$ is a non-factorable ideal provided $N$ is a proper ideal and $N = AB$, with $A, B$ ideals of $D$, implies one of $A$ and $B$ is $D$). Since $D$ is not a Dedekind domain, there exist non-factorable primary ideals of $D$ which are not prime ideals. Let $P$ be a proper prime ideal of $D$ such that $P$ contains a non-factorable primary ideal $H$ which is not prime and set
\( J = \mathbb{D}_p \), the quotient ring of \( D \) with respect to \( P \). Let \( R = J(x) = J[x]_S \), where \( S \) is the multiplicative system in the polynomial ring \( J[x] \) consisting of all polynomials \( a_0 + a_1 x + \ldots + a_n x^n \) with \((a_0, a_1, \ldots, a_n)J = J\), \([G, p. 379],[N, p. 17]\). We observe that \( J \) is not integrally closed (if it is, then \( J \) is a rank one discrete valuation ring and \( HJ \) is a power of \( PJ \), contradicting the assumptions on \( H \) and \( P \)), and hence \( J \) is not a Prüfer domain. However, \( J \) is a local ring and thus \( R \) is a local ring \([N, p. 18]\). We claim that \( R \cap Q(\alpha) = J \) (recall that \( Q(\alpha) \) is the quotient field of \( J \)). Let \( (p(x)/q(x)) = k \in Q(\alpha) \) where \( 0 \neq p(x) \in J[x] \) and \( q(x) \in S \); equating coefficients we get equations \( p_i = kq_i \) where \( p_i, q_i \in J \) for \( i = 1, \ldots, n \) and hence \( k \in J \) since \((q_1, \ldots, q_n)J = J\), completing the proof of the claim. It now follows that \( R \) is not integrally closed, since \( J \) is not integrally closed, and therefore \( R \) is not a Prüfer domain. Now \( R/PR \cong (J/PJ)(x) \) \([N, p. 18]\), \((J/PJ)(x)\) is a finite field of prime characteristic \( p \), \( (J/PJ)(x) \) is an infinite field, and \( R/PR \) is an infinite field. Hence a subset of \( R \) of the desired type exists by Theorem 4.8. If \( R \) contains an infinite subfield \( F \), then \( F \) must be of characteristic zero since \( R \supseteq \mathbb{Z} \), implying \( Q \subset F \subset R \), a contradiction since \( Q \not\subset R \).

Q.E.D.

Theorem 4.18 A homomorphic image \( \bar{R} \) of a \( \bar{U} \)-ring \( R \) is a
U-ring, and any unitary overring of $\bar{R}$ is a U-Ring. In particular, a quotient ring $R_S$ of $R$ with respect to a multiplicative system $S$ is a U-ring.

Proof: If $f:R \rightarrow \bar{R}$ is a ring homomorphism and $M$ is an $\bar{R}$-module, then $M$ is also an $R$-module where the module action $*$ is defined by $r*m = f(r)m$ for $r \in R$ and $m \in M$. The proof is completed by applying Theorem 4.16 and observing that $R_S$ is a unitary overring of a homomorphic image of $R$.

Q.E.D.

Theorem 4.19 If $R$ is a U-ring, then $R/A$ is infinite for every ideal $A \neq R$ of $R$.

Proof. For each proper prime ideal $P$ of $R$ the quotient ring $R_P$ is a U-ring, and hence $F = R_P/PR_P$ is infinite by Theorem 4.14. Since $F$ is isomorphic to the quotient field of $R/P$ [ZS1,p. 227], it follows that $R/P$ is infinite. Let $A \neq R$ be an ideal and let $P$ be a prime ideal containing $A$. Since $(R/A)/(P/A) \cong R/P$, it follows that $R/A$ is infinite.

Q.E.D.

Remark: We observe that a general overring of a U-ring need not be a U-ring (see Theorem 4.16); e.g. if $R_1$ is a U-ring and $R_2$ is not, then $R_1 \oplus R_2$ is not a U-ring by Theorem 4.18.
CHAPTER V

In this chapter we study the relationship between an ideal $A$ in $R$ and the ideal $AR_p$ of the ring $R_p$ by means of the vector space $A/\mathbb{P}$ over the field $R/P$, where $P$ is a maximal ideal of $R$. We use the results to give further facts about U-rings. The important results of this chapter are: if $P$ is a maximal ideal in a U-ring $R$ such that $R/P$ is finite, then $R_p$ is a Bezout ring; if $D$ is a domain such that $D/P$ is finite for each maximal ideal $P$ of $D$, then $D$ is a U-ring if and only if $D$ is Prüfer; and if we let $\bar{Z}$ be the integral closure of $Z$ in a finite algebraic extension $Q(\alpha)$ of $Q$ and let $D$ be a domain such that $Z \subseteq D \subseteq \bar{Z}$, then $D$ is a U-ring if and only if $D$ is integrally closed.

Lemma 5.1 Let $R$ be a ring, $P$ a maximal ideal of $R$, $A$ a finitely generated non-principal ideal of $R$ such that $A/\mathbb{P}$ is a one-dimensional vector space over $R/P$ (in the natural way). Then if $A$ is generated by $n$ elements ($1 < n$), $AR_p$ is generated by $n-1$ elements.

Proof: Let $A = (a_1, \ldots, a_n)$. Since $A/\mathbb{P}$ is a one-dimensional vector space over $R/P$, there exists $\alpha \in A$ such that
(1) \[ A + AP = (a) + AP \]

Let \( a = \sum_{i=1}^{n} x_i a_i \) with \( x_i \in R \). Since \( a_j \in A \) for \( j = 1, \ldots, n \), from (1) we have that there exist \( r_1 \in R \) and \( p_{ij} \in P \) such that

(2) \[ a_j = r_j \sum_{i=1}^{n} x_i a_i + \sum_{j=1}^{n} p_{ij} a_i, \quad j = 1, \ldots, n. \]

Suppose that one of the \( x_i \) belongs to \( P \), say \( x_k \in P \). Take \( j=k \) in (2) and note that the coefficient of \( a_k \) in (2) is \( c_k = 1 - r_k x_k - p_{kk} \); now notice \( c_k \notin P \), so \( c_k \) becomes a unit in \( R_P \). This implies that \( a_k \) can be expressed in terms of the other \( a_i \) (i.e., \( i \neq k \)) in \( R_P \), and \( A_R \) is generated by the images of the \( \{a_i\}_{i \neq k} \) in \( R_P \) under the canonical map \( R \to R_P \). Now, suppose that \( x_i \notin P \) for \( i = 1, \ldots, n \). Take \( j=1 \) in (2) and consider the coefficient \( c_2 = r_1 x_2 + p_{21} \) of \( a_2 \) which occurs in (2). If \( r_1 \notin P \), then \( c_2 \notin P \) and we can express \( a_2 \) in terms of the other \( a_1 \) in \( R_P \) as above. If \( r_1 \in P \), then the coefficient \( c_1 = 1 - r_1 x_1 - p_{11} \) of \( a_1 \) in (2) is not in \( P \) and \( a_1 \) can be expressed in terms of the other \( a_i \).

Q.E.D.

Remark: Recall from Definition 1.4 that a Bezout ring is a ring \( R \) in which finitely generated ideals are principal.

Lemma 5.2 Let \( P \) be a maximal ideal of a ring \( R \). If \( A/AP \) is a one dimensional vector space over \( R/P \) for each finitely generated ideal \( A \) of \( R \), then \( R_P \) is a Bezout ring.
Proof: Let \( h:R \to R_P \) be the canonical homomorphism \([ZS_1,p.\ 221-22]\), and let \( B = \sum_{i=1}^{n}(h(b_i)/h(s_i))R_P \) be a finitely generated ideal of \( R_P \) with the \( n \) generators \( h(b_i)/h(s_i) \) where the \( b_i \in R \), the \( s_i \in R\setminus P \), and \( n \) is minimal. It is clear that \( B = \sum_{i=1}^{n}h(b_i)R_P \), since \( h(s_i) \) is a unit. Suppose \( l < n \) and consider the ideal \( A = (b_1, \ldots, b_l) \) in \( R \). By Lemma 5.1 \( AR_P = B \) is generated by \( n-1 \) of the \( h(b_i) \), a contradiction since \( n \) is minimal.

Q.E.D.

Theorem 5.3 If \( P \) is a maximal ideal in a U-ring \( R \) such that \( R/P \) is finite, then \( R_P \) is a Bezout ring.

Proof: If \( R \) is a U-ring and \( R/P \) is finite, then by Theorem 4.11 \( A/AP \) is one dimensional over \( R/P \) for every ideal \( A \) of \( R \). In particular \( A/AP \) is one dimensional for each finitely generated ideal \( A \), and \( R_P \) is a Bezout ring by Lemma 5.2.

Q.E.D.

Corollary 5.4 If \( D \) is a domain such that \( D/P \) is finite for each maximal ideal \( P \) of \( D \), then \( D \) is a U-ring if and only if \( D \) is Prüfer.

Proof: If \( D \) is a U-ring, then by Theorem 5.3 we know that \( D_P \) is Bezout, hence Prüfer, hence a valuation ring. But this implies that \( D \) is a Prüfer domain \([G,p.\ 254]\).

Conversely, if \( D \) is a Prüfer domain then \( D \) is a U-ring
by Corollary 1.13.

Q.E.D.

**Theorem 5.6** Let \( \bar{Z} \) be the integral closure of \( Z \) in a finite extension \( Q(\alpha) \) of \( Q \) and let \( D \) be a domain such that \( Z \subset D \subset \bar{Z} \). Then \( D \) is a U-ring if and only if \( D \) is integrally closed.

**Proof:** If \( D \) is integrally closed, then \( D \) is a Dedekind domain (hence Prüfer) and a U-ring by Corollary 1.13. Conversely, suppose \( D \) is not integrally closed. By the argument in Example 4.17, there exists a prime ideal \( P \) in \( D \) such that \( D_P \) is not integrally closed. Setting \( P^e = PD_P \), we claim that \( D_P/P^e \) is finite. Now \( D_P/P^e \cong D/P \) [ZS, p. 226-27]. Since \( \bar{Z} \) is a finite \( Z \)-module, \( D \) is also a finite \( Z \)-module and \( D/P \) is a finite \( Z/(P^e) \) module. This means that \( D/P \) is a finitely generated module over a finite field, hence finite. Since \( D_P \) is not integrally closed, it is not Prüfer and there is a finitely generated ideal which is not invertible (hence not principal). By Theorem 4.15 it follows that \( D_P \) is not a U-ring, and by Theorem 1.17 we have that \( D \) is not a U-ring.

Q.E.D.

**Theorem 5.7** Let \( D \) be a Noetherian domain in which proper prime ideals are maximal. If \( P \) is a prime ideal of \( D \) such that \( D/P \) is finite and \( D_P \) is not integrally closed, then \( D \) is not a U-ring.
Proof: Since $D_p$ is not integrally closed it is not Prüfer, so there is some finitely generated ideal which is not invertible and hence not principal. This, together with $D_p/P^e \cong D/P$, implies that $D_p$ is not a U-ring and hence neither is $D$ by Theorem 1.17.

Q.E.D.
CHAPTER VI

This chapter is devoted to considering several examples of rings in the interest of studying the relationship between property \( U \) and other important ring-theoretic considerations such as the polynomial rings over a \( U \)-ring \( R \), the quotient rings of \( R \) with respect to a multiplicative system, the property "Noetherian", and the property "integrally closed". One important result we notice while considering these examples is that a domain \( D \) of polynomials in \( X \) over a field \( K \) (where \( X \) is a set of indeterminates over \( K \) of cardinality \( a > 1 \)) is a \( U \)-ring if and only if \( K \) is infinite or \( a = 1 \).

The following is an example which shows that a polynomial ring over a \( U \)-ring is not necessarily a \( U \)-ring.

**Example 6.1** Let \( D = \mathbb{Z}[x] \) where \( x \) is an indeterminate over the integers \( \mathbb{Z} \). Consider in \( D \) the ideals \( A = (2, x) \), \( A_1 = (4, x) \), \( A_2 = (2, x^2) \), \( A_3 = (2 + x, 2x) \). We will show that \( A \subseteq \bigcup_{1}^{3} A_1 \) and \( A \not\subseteq A_1 \) for each \( 1 \). To see that \( A \subseteq \bigcup_{1}^{3} A_1 \), let \( r \in A \). Then \( r \) can be written as \( p(x) \cdot x^2 + a_1 x + 2a_0 \) where \( p(x) \in \mathbb{Z}[x] \) and \( a_1, a_0 \in \mathbb{Z} \). Now if \( 2 | a_0 \), then \( r \in A_1 \); if \( 2 | a_1 \), then \( r \in A_2 \); if \( 2 \nmid a_0 \) and \( 2 \nmid a_1 \), then
\[ r = p(x) \cdot x^2 + (2m+1)x + 2(2n+1) = p(x) \cdot x^2 + 2mx + 4n + x + 2 \in A_3 \]
since \( x^2 \in A_3 \) and \( 4 \in A_3 \). So \( A \subseteq \bigcup_{i=1}^{3} A_i \) (since \( A_i \subseteq A \) for each \( i \), we see in fact that \( A = \bigcup_{i=1}^{3} A_i \)). Now to see that \( A \subseteq A_i \) for each \( i \), we note that \( x \not\in (2,x^2) \), \( 2 \not\in (4,x) \), and we show that \( 2 \not\in (2+x,2x) \). If \( 2 \in (2+x,2x) \) then \( 2 = a(2+x) + b2x \), for some \( a, b \in D \), and \( 2(1-a-bx) = ax \). This implies that \( a \in 2D \) and \( 1-a-bx \in xD \). But \( a \in 2D \) implies that \( a+bx \in (2,x) \) which implies that \( 1-a-bx \not\in (2,x) \) which implies that \( 1-a-bx \not\in xD \). This contradiction implies that \( 2 \not\in (2+x,2x) \) and hence \( A \not\subseteq A_i \) for each \( i \).

Q.E.D.

The next example gives a domain \( D \) which is not a U-ring, but \( D_S \) is a U-ring for each multiplicative system \( S \) of \( D \) such that \( D < D_S \). This shows that the converse of Corollary 1.17 is false. It also shows that although valuation rings are quasi-local, and valuation rings are U-rings, not all quasi-local rings are U-rings.

**Example 6.2** Let \( D = \mathbb{Z}[x]_{(2,x)} \). It follows from the discussion in Example 6.1 that

\[
(2,x)D = (4,x)D \cup (2,x^2)D \cup (2+x,2x)D \cup (2,x)D \text{ properly contains each of these ideals, and hence } D \text{ is not a U-ring.}
\]

Now let \( S \) be a multiplicative system of \( D \) such that \( D < D_S \). It follows that \( (2,x)D \cap S \neq \emptyset \), and \( [(2,x)D]D_S = D_S \). Now since \( D \) is of dimension two (this follows from the fact that \( \mathbb{Z}[x] \) is two dimensional and \( (x,2)D \) is maximal; see

\[
(2,x)D = (4,x)D \cup (2,x^2)D \cup (2+x,2x)D \cup (2,x)D \text{ properly contains each of these ideals, and hence } D \text{ is not a U-ring.}
\]

Now let \( S \) be a multiplicative system of \( D \) such that \( D < D_S \). It follows that \( (2,x)D \cap S \neq \emptyset \), and \( [(2,x)D]D_S = D_S \). Now since \( D \) is of dimension two (this follows from the fact that \( \mathbb{Z}[x] \) is two dimensional and \( (x,2)D \) is maximal; see
[G, Thm. 25.5, p. 343]), prime ideals of $D_S$ are extensions of prime ideals of $D$, and $[(2, x)D]D_S = D_S$, it follows that the dimension of $D_S$ is less than or equal to one. Hence, since $D_S$ is a Krull domain [G, Thm. 35.6, p. 513] of dimension less than or equal to one it follows that $D$ is a Dedekind domain [G, Thm. 35.6, p. 525], hence a U-ring.

Q.E.D.

The following is also an example which shows that the converse of Theorem 1.17 is false; in fact it is a domain $R$ such that $R$ is not a U-ring, but each overring of $R$ (contained in $F$ the quotient field of $R$) is a U-ring.

Example 6.3 Let $J = \{a + b\sqrt{5} | a, b \in \mathbb{Z}\}; S = \{\text{odd integers}\}; R = \{a/s | a \in J, s \in S\} = J_S$. Let also $M = (2, 1 + \sqrt{5})$, $B = (4, 1 + \sqrt{5}) = (1 + \sqrt{5})$, $\bar{B} = (4, 1 - \sqrt{5}) = (1 - \sqrt{5}$, $C = (2)$ each of which is an ideal of $R$. We will show that $M \subset B \cup \bar{B} \cup C$, (in fact, $M \subset B \cup \bar{B} \cup C$), and since $M$ is not principal and $B, \bar{B}, C$ are principal and contained in $M$ it will follow that since $M \not\subset B$, $M \not\subset \bar{B}$, and $M \not\subset C$, we have $R$ is not a U-ring.

To show $M \subset B \cup \bar{B} \cup C$ let $x \in M$, and consider $x = (a + b\sqrt{5})/s$ where $a, b \in \mathbb{Z}$, and $s \in S$. Since $s$ is a unit, $x \in A$ if and only if $sx \in A$ for $A$ any ideal of $R$; hence we may assume $s = 1$. Now, it is easy to check that $x \in M$ implies that $a \equiv b \pmod{2}$ so that $a$ and $b$ are of
the same parity.

Part I Let \( a, b \) be odd.

**Case 1.1** Let \( a, b \equiv 1 \pmod{4} \). Then \( a = 4n + 1, \)
\( b = 4m + 1 \) for \( m, n \in \mathbb{Z} \). Now
\( x = (4n + 1) + (4m + 1)\sqrt{5} = 4(n + m\sqrt{5}) + (1 + \sqrt{5}) \in \mathbb{B}, \) so \( x \in \mathbb{B}. \)

**Case 1.2** Let \( a, b \equiv 3 \pmod{4} \). As in Case 1.1, it follows that \( x \in \mathbb{B}. \)

**Case 1.3** Let \( a \equiv 1 \pmod{4}, b \equiv 3 \pmod{4} \). An easy argument shows \( x \in \overline{\mathbb{B}}. \)

**Case 1.4** Let \( a \equiv 3 \pmod{4}, \) and \( b \equiv 1 \pmod{4} \).
Again \( x \) is in \( \overline{\mathbb{B}}. \)

Part II Let \( a \) and \( b \) both be even. Then
\( x = 2n + 2m\sqrt{5} = 2(n + m\sqrt{5}) \in (2) = \mathbb{C}. \)

Now [JG, Ex. 40, p. 37-39] establishes that the integral closure \( \overline{R} \) of \( R \) in its quotient field \( F \) is a valuation ring (hence a U-ring) and is the only proper overring of \( R \) in \( F \). We now have that \( R \) is not a U-ring, but each proper overring of \( R \) in \( F \) is a U-ring.

Q.E.D.

Remark: The polynomial ring \( D[x] \) over a Euclidean domain \( D \) may be a U-ring (e.g. \( D = \mathbb{Q} \)) or not (e.g. \( D = \mathbb{Z} \)). Since both \( \mathbb{Z}[x] \) and \( \mathbb{Q}[x] \) are Noetherian rings, a Noetherian ring may or may not be a U-ring. We observe that \( \mathbb{Z}[x] \) is
a two dimensional, Noetherian U.F.D. which is not a U-ring; hence two dimensional Noetherian Krull rings need not be U-rings. The polynomial ring \( \mathbb{Q}[x_1,x_2,\ldots,x_n,\ldots] \) in infinitely many indeterminates over the rational field \( \mathbb{Q} \) is a non-Noetherian U.F.D. which is a U-ring. The domain in the following example is a non-Noetherian U.F.D. which is not a U-ring.

Example 6.4 Let \( D = \mathbb{Z}[x_1,\ldots,x_n,\ldots] \) (where \( x_i \) is an indeterminate over \( \mathbb{Z} \) for all \( i \)). It is clear that \( D \) is not Noetherian, and if we consider the ideals
\[
A = (2,x_1,x_2,\ldots), \quad A_1 = (2,x_1^2,x_2,\ldots), \quad A_2 = (4,x_1,x_2,\ldots), \quad A_3 = (2+x_1,2x_1,x_2,\ldots)
\]
with \( D \) it follows by an argument similar to that in Example 6.1 that \( A = \bigcup_1^3 A_i \) and \( A \neq A_i \) for \( i = 1,2,3 \); hence \( D \) is not a U-ring.

Q.E.D.

The following theorem characterizes those polynomial rings over a field which are U-rings.

Theorem 6.5 Let \( K \) be a field, \( X \) a set of indeterminates over \( K \) of cardinality \( \alpha \), where \( \alpha \geq 1 \), and \( K[X] \) the polynomial ring in the elements of \( X \) with coefficients in \( K \). Then \( K[X] \) is a U-ring if and only if \( K \) is infinite or \( \alpha = 1 \).

Proof: If \( K \) is infinite, then \( K[X] \) is a U-ring (Corollary 4.5); and if \( \alpha = 1 \), then \( K[X] \) is a principal ideal domain,
hence a U-ring. Now suppose $K$ is finite and $a > 1$, and consider $D = K[X]$. Consider also the set $S = X - \{x_1, x_2\}$ for $x_1, x_2$ fixed elements of $X$, and the following ideals of $D$: $A$, generated by the elements of $X$, $A_1$, generated by the elements of $SU\{x_1, x_2\}$, $A_2$, generated by the elements of $SU\{x_1^2, x_2\}$, and $A_{2+i}$, generated by the elements of $SU\{x_1 + a_1 x_2, x_1 x_2\}$ where $K = \{0, a_1, \ldots, a_n\}$.

Since $D$ is a unique factorization domain we have $x_1 \notin A_2$, $x_2 \notin A_1$, and $x_i \notin A_{2+i}$ for $i = 1, \ldots, n$, and it follows that $A$ properly contains each of the $A_t$ for $t = 1, 2$ or $2+i$ for each $i$. We now claim that $A \subseteq U_2^{2+n}A_t$ (and hence $A = U_1^{2+n}A_t$). Let $p(X) \in D$. If $p(X) \in A$ then $p(X) = a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1^2 + a_5 x_2^2 + f(X)$ where each term of $f(X)$ either has some $x \in X, x \notin \{x_1, x_2\}$ as a factor or has degree larger than two, and $a_1, \ldots, a_5 \in K$. Now if $a_2 = 0$, then $p(X) \in A_1$; if $a_1 = 0$ then $p(X) \in A_2$; and if $a_1, a_2 \neq 0$, then $a_1^{-1}p(X) \in A_{2+i}$ and hence $p(X) \in A_{2+i}$ where $i$ is such that $a_i = a_1^{-1}a_2$. Hence $D$ is not a U-ring.

Q.E.D.

Remark: Concerning the relationship between property U and the property integrally closed, we note that we have already seen examples of integrally closed rings which are U-rings e.g. $Q[x]$, and integrally closed rings which are not U-rings
e.g. $K[x,y]$, $K$ a finite field. We will now look at two examples of rings which are not integrally closed; one of them is a U-ring (Example 6.6) and the other is not a U-ring (Example 6.7).

**Example 6.6** Let $D = \{ p(x) = a_0 + a_1 x + \ldots + a_n x^n | p(x) \in Q[x] \}$ and $a_1 = 0$. $D$ is clearly a U-ring since $D$ contains $Q$, and since $x \not\in D$ and $x$ satisfies the equation $y^2 = x^2 = 0$ (monic polynomial equation in $y$ with coefficients in $D$) it follows that $D$ is not integrally closed.

**Q.E.D.**

**Example 6.7** Let $D = \{ p(x) = a_0 + a_1 x + \ldots + a_n x^n | p(x) \in K[x], K = Z/(2), \text{ and } a_1 = 0 \}$, and let $A = (x^2, x^3)D$, $A_1 = (x^2)D$, $A_2 = (x^3, x^4)D$, $A_3 = (x^2 + x^3, x^4)D$. Now $x^3 \not\in A_1$, $x^2 \not\in A_2$, and $x^2 \not\in A_3$, so it follows that $A_1 < A$ for $i = 1, 2, 3$. We will now show that $A \subset \bigcup_{1}^{3} A_{1}$ (and hence $A = \bigcup_{1}^{3} A_{1}$). Let $q \in A$. Then $q = a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$, where $a_i \in K = Z/(2)$ for $i = 2, \ldots, n$. Now if $a_3 = 0$, then $q \in A_1$; if $a_2 = 0$, then $q \in A_2$; if $a_1, a_2 \neq 0$ then $q \in A_3$. It now follows that $D$ is not a U-ring, and by considering the monic polynomial equation $y^2 - x^2 = 0$ in $y$ with coefficients in $D$, we see that $x$ is a root and $x \not\in D$ so $D$ is not integrally closed.

**Q.E.D.**

Remark: One other question which arose early in our
consideration of U-rings, when we realized that valuation rings are U-rings, and in fact all Prüfer domains are U-rings, and that question is: are all quasi-local U-rings valuation rings? Corollary 4.5 sheds some light on this situation however, and it is easily seen that $D = \mathbb{Q}[x,y]/(x,y)$ is quasi-local, is not a valuation ring ($xD \not\subset yD$ and $yD \not\subset xD$), and is a U-ring.
BIBLIOGRAPHY


BIOGRAPHY

Philip Quartararo, Jr. was born on March 26, 1942 in New Orleans, Louisiana. He attended public schools there, and in May of 1964 received a Bachelor of Arts degree in Mathematics from Louisiana State University in New Orleans. In July of 1964 he married Dorothy Marie Morvant, now the mother of his four children. In September of 1964 he entered Louisiana State University in Baton Rouge, and there in August of 1966 received a Master of Science degree in Mathematics. In September of 1967 he accepted the position of Instructor of Mathematics at Southern University in Baton Rouge. He is presently Assistant Professor of Mathematics at Southern University and a candidate for the Doctor of Philosophy degree in the Mathematics Department of Louisiana State University.
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Title of Thesis:  THE FINITE UNION OF IDEALS IN A RING R.

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