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Beyond the Tails of the Colored Jones Polynomial

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BEYOND THE TAILS OF THE COLORED JONES POLYNOMIAL

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Table of Contents

List of Figures

Abstract

In [2] Armond showed that the heads and tails of the colored Jones polynomial exist for adequate links. This was also shown independently by Garoufalidis and Le for alternating links in [8]. Here we study coefficients of the "difference quotient" of the colored Jones polynomial.

We begin with the fundamentals of knot theory. A brief introduction to skein theory is also included to illustrate those necessary tools. In Chapter 3 we give an explicit expression for the first coefficient of the relative difference. In Chapter 4 we develop a formula of t_2, the number of regions with exactly 2 crossings in the diagram of a link, for a specific class of alternating links, and then improve with this result the upper bound of the volume for a hyperbolic alternating link which Dasbach and Tsvietkova gave in the coefficients of the colored Jones polynomial in [7].

Chapter 1
Introduction

In knot theory a link L is a finite disjoint union of S^1 in R^3 or S^3. A link with just one component is called a knot. A major portion of knot theory is the study of link invariants, e.g. polynomial invariants such as the colored Jones polynomial.

In [5] Dasbach and Lin proved that the first two and the last two coefficients of the un-normalized colored Jones polynomial $\widetilde{J}(K,n)$ for an alternating knot K are stable, i.e. they are independent of the color n. They then investigate the third term and show that it is stable when $n \geqslant 3$. They conjectured that the first k coefficients of $\widetilde{J}(K,n)$ for an alternating knot should be independent of the color n when $n \geqslant k+1$. Armond later proved in [2] the existence of the heads and tails for the unreduced colored Jones polynomial of an adequate link, which confirmed that conjecture. Independently, Garoufalidis and Le also obtained the same result in [8] for alternating links, and they extended it to the higher order stability, which Katherine Walsh checked and gave an expression for the second stable sequence of the colored Jones polynomial for a certain class of knots in [15].

With these known results, it is then natural to ask whether the coefficients right after (or before) the heads (or tails) preserve some relations. We start with an introduction of some fundamentals of knot theory. Tools form skein theory is then discussed, and we give a brief review of the colored Jones polynomial.

In Chapter 3, we first recall some useful results from Cody Armond's paper [2] and prepare key lemmas needed for the proof of the first main result. Then we consider the "relative difference". Let $\widetilde{J}(n,K)$ and $\widetilde{J}(n+1,K)$ to be the n-th and $n+1$-th unreduced colored Jones polynomial of an alternating link K. By multiplying with suitable powers $\pm A^{\epsilon_1}$ and $\pm A^{\epsilon_2}$ we can adjust them to get two polynomials $\widetilde{AJ}(n,K)$ and $\widetilde{AJ}(n+1,K)$ such that both of them now have leading coefficients 1 (called adjusted unreduced colored Jones polynomial). Denote the coefficient in front of l-th power of $\widetilde{AJ}(n, K)$ by a_n^l. Since the heads and tails for alternating links exist, the difference polynomial $\widetilde{AJ}(n+1, K) - \widetilde{AJ}(n, K)$ can be written as $A^{4n+4}\widetilde{RJ}(n, K)$ where $\widetilde{RJ}(n, K)$ is a polynomial. We call this $\widetilde{RJ}(n, K)$ the n-th relative difference of K. Denote $r_n^l(K)$ the coefficient in front of power A^{l-4n-4}. Therefore $r_n^l(K) = a_{n+1}^l(K) - a_n^l(K)$ by definition.

The first main result is about the first non-trivial coefficient in the relative difference $\widetilde{RJ}(n, K)$.

Theorem 3.11 (Main Theorem 1)**.** *Let G_B to be the B-graph of an alternating link K, E the number of edges in G_B, and V the number of vertices in G_B. Then*

$$r_n^{4n+4}(K) = V - 2E + t_1.$$

To give an example we discuss the result for the knot 6_3 in Rolfsen's table.

In Chapter 4, we focus on r_1^{12} for a specific class of alternating links. A *chain* in the B-graph G_B of a knot K is a non-loop path P_n in G with $n \geqslant 3$, where the two endpoints of it have valency strictly greater than 2 and all other vertices on it have valency 2. The class $\mathcal{C}$ of alternating links that we consider can be defined recursively: all links with a B-graph G_B that is a polygon with at least 5 edges are in $\mathcal{C}$. If removing a chain from a B-graph G_B of K yields an element of $\mathcal{C}$ then K is in $\mathcal{C}$.

For links in $\mathcal{C}$ we can give an expression for r_1^{12}, which is stated in the second main theorem:

Theorem 4.13 (Main Theorem 2)**.** *Suppose an alternating link K is in $\mathcal{C}$. Then*

$$r_1^{12} = (E - t_1)(1 + F) + F(F - 1) - t_2.$$

We discuss the result on examples. A useful application is to refine the bound of the volume for an hyperbolic alternating link if it involves the coefficients of the colored Jones polynomial. Let L be a link in the three sphere S^3. We say L is hyperbolic if its link complement has a hyperbolic structure with a finite volume. Kashaev introduced a complex number valued link invariant by using quantum dilogarithm in [9]. He also observed that this invariant is a quantum generalization of the hyperbolic volume. Murakami and Murakami pointed out later in [13] that the invariant in Kashaev's paper turns out to be an evaluation of the colored Jones polynomial. Their result links the study of the colored Jones polynomial and the hyperbolic volume of link complements.

Since the accurate volume function in general is hard to find for an arbitrary hyperbolic link, bounds for the hyperbolic volume are studied. In [7] Oliver Dasbach and Anastasiia Tsvietkova gave the following result

Theorem 1.1 (Theorem 2.3 in [7])**.** *Given a diagram D of a hyperbolic alternating link K; denote the number of twists that have exactly i crossings by $t_i(D)$, and the number of twists that have at least i crossings by $g_i(D)$. Let v_3 be the volume of a regular ideal hyperbolic tetrahedron. Then $Vol(S^3 - K) \leqslant (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - a)v_3$, where $a = 10$ if g_4 is non-zero, $a = 7$ if t_3 is non-zero, and $a = 6$ otherwise.*

The authors then bound the volume in terms of coefficients of the colored Jones polynomial:

Theorem 1.2 (Theorem 3.3 in [7])**.** *Let K be an alternating, prime, non-torus link, and let*

$$J_K(n) = \pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \cdots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$$

be the colored Jones polynomial of K, where a_n and α_n are positive. Then

$$Vol(S^3 - K) \leqslant (6((c_2 + \gamma_2) - (c_3 + \gamma_3)) - 2(b_2 + \beta_2) - a)v_3 \leqslant 10(b_2 + \beta_2 - 1)v_3,$$

where $a = 10$ *if* $b_2 + \beta_2 \neq (c_2 - c_3) + (\gamma_2 - \gamma_3)$ *and* $a = 4$ *otherwise.*

Using Main Theorem 2, we can replace t_2 in this result with coefficients in $J(2, K)$ and $J(3, K)$ if K is in $\mathcal{C}$, and hence get a finer bound for the result using coefficients of the colored Jones polynomial as follows

Theorem 4.23. *Let* K *be an alternating, prime, non-torus link in* $\mathcal{C}$*, and let*

$$J_K(n) = \pm(a_n q^{k_n} - b_n q^{k_n - 1} + c_n q^{k_n - 2}) + \cdots \pm (\gamma_n q^{k_n - r_n + 2} - \beta_n q^{k_n - r_n + 1} + \alpha_n q^{k_n - r_n})$$

be the colored Jones polynomial of K*, where* a_n *and* α_n *are positive. Then*

$$\begin{aligned} Vol(S^3 - K) \leqslant & \{(2 - 4\beta_2)[(c_2 + \gamma_2) - (c_3 + \gamma_3)] + (2 + 4\beta_2)(b_2 + \beta_2) \\ & + \beta_2(\beta_2 - 1) - [(-b_3 + c_3 - d_3) - (c_2 - d_2)] - a\} v_3, \end{aligned}$$

where $a = 10$ *if* $b_2 + \beta_2 \neq (c_2 - c_3) + (\gamma_2 - \gamma_3)$ *and* $a = 4$ *otherwise.*

Chapter 2
Background and Preliminaries

2.1 Knot Theory

Knot theory is a branch of topology which studies knots and links. A knot K is a subset of R^3 (or S^3) which is homeomorphic to S^1. A link L is a finite disjoint union of knots K_1, K_2, $\cdots$, K_l, where each K_i is also called a component of L. l is called the multiplicity of L, and is denoted by $\mu(L)$. Hence a knot K is a link with $\mu(K) = 1$.

We can define an equivalence relation on the set of all links as follows: let h be a homotopy of a space $X \subset \mathbb{R}^3$. If h_t is injective for every $t \in [0,1]$, then h is called *isotopy*. Two links L_1 and L_2 are *ambient isotopic* if there is an isotopy $h : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $h(L_1, 0) = h_0(L_1) = L_1$ and $h(L_1, 1) = h_1(L_1) = L_2$.

We can also define another equivalence relation on links. Let $L \subset \mathbb{R}^3$ be a link and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ a projection map. A point $x \in \pi(L)$ is *regular* if $\pi^{-1}(x)$ is a single point, and is *singular* otherwise. If $|\pi^{-1}(x)| = 2$ then x is called a *double point*. If $\pi(L)$ has a finite number of singular points and they are all transverse double points, the projection is said to be *regular*.

A *diagram* is a regular projection of a link that has relative height information added to it at each of the double points. The convention is to make breaks in the line corresponding to the strand that passes underneath. The double points in the projection become *crossings* in the diagram.

Two link diagrams represent the same link if and only if one can be achieved from the other one via a finite sequence of the three types Reidemeister moves.

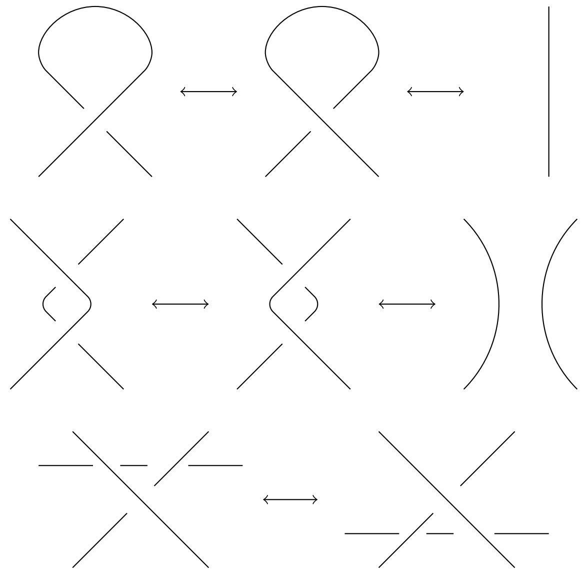


Figure 2.1: From the top to the bottom: the Reidemeister moves of Types I, II and III.

(a) $+1$ (b) -1

Figure 2.2: Assign a sign to a crossing.

We can assign an orientation to each component of a link, and hence get an oriented link. For a given diagram D of an oriented link L, we also define the sign for each crossing by Figure 2.2.

The writhe of D, which is denoted by $w(D)$, can then be defined as the sum of the signs of all crossings. No matter which orientation is assigned, the writhe for a given knot diagram is unchanged.

A *framed link* is a link together with a smooth section of the normal bundle over the link which is called a *framing*. If all the vectors are perpendicular to the plane where the link diagram sits, this framing is called the *blackboard framing*. Two link diagrams both equipped with the blackboard framing represent the same framed link if and only if one can be derived from the other by a finite sequence of Reidemeister moves of types II and

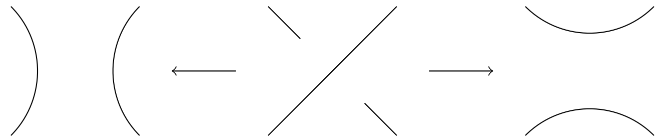


Figure 2.3: Two ways to smooth a crossing. The left one gives the A-smoothing, and the right one gives the B-smoothing.

III only. This sequence is called a *regular isotopy.* In this paper we mainly deal with framed links.

A link diagram is called *alternating* if the position of the strand alternates between over and under as we go through any component of the link. A link is called alternating if it has an alternating diagram.

For a crossing of a given link diagram, we have two ways to smooth it: A-smoothing and B-smoothing, as in Figure 2.3. After smoothing all the crossings of a link diagram by either the A-smoothing or the B-smoothing, we obtain a new diagram S without any crossing which is called a *Kauffman state.* There are two special Kauffman states. If we make as the choice of the smoothing to all the crossings the A-smoothing, the corresponding Kauffman state S_A is called the *all-A state.* We can give the all-B state S_B similarly. In a Kauffman state S, if we replace any circle by a vertex, and add an edge connecting two such vertices whenever these circles are originally related by a crossing, we construct a graph, which is called the *state graph* and is denoted by G_s. With this notation, we can also denote the all-A state graph by G_A and the all-B state graph G_B, and call them A-graph and B-graph for short.

We say a link diagram *A-adequate* (respectively *B-adequate*) if there is no loop in G_A (respectively G_B). A link diagram is called *adequate* if it is both A-adequate and B-adequate. A link is called *adequate* if it has an adequate diagram. It is known that a reduced alternating link diagram is adequate; here "reduced" means there is no *nugatory* or *removable* crossing as Figure 2.4 shows in the link diagram. See [11] for a proof.

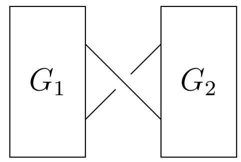


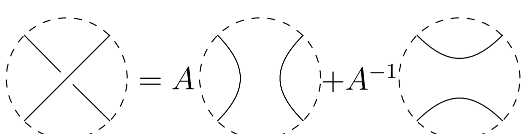
Figure 2.4: A nugatory or removable crossing in a link diagram. G_1 and G_2 are both part of this link diagram.

A major portion of knot theory is the study of link invariants. A *link invariant* is a function from the set of links to some other set whose value depends only on the equivalence class of the link. Any representative from the class can be chosen to calculate the invariant. In the following section, we will review an important link invariant: the colored Jones polynomial.

2.2 The Skein Theory and the Colored Jones Polynomial

We briefly introduce the Kauffman bracket skein module, the Temperley-Lieb algebra (especially the Jones Wenzl idempotent), the colored Jones polynomial and the related properties we need to use in this paper. The material covered here can also be found in [11], [14] and [12].

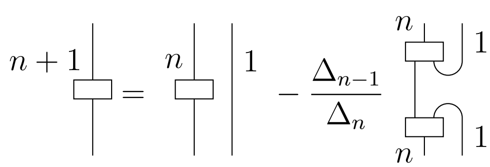
Let M to be a 3-manifold and R a commutative ring with identity and a fixed invertible element A. The *Kauffman bracket skein module* which is denoted by $S(M; R, A)$ is then generated by isoptopy classes of framed links in M (including the empty link) module the submodule generated by the *Kauffman relations*:



and $L \cup O = (-A^2 - A^{-2})L$,

where L represents any framed link and O the trivial framed knot.

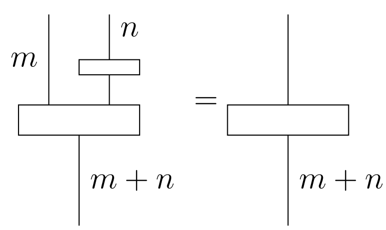
In this paper we always use $R = \mathbb{Q}(A)$, which is the field generated by A over the rational numbers. If M is chosen to be D^3 with $2n$ boundary points, $S(D^3; R, A)$ is also called the *Temperley-Lieb algebra* and is denoted by TL_n. An special element in TL_n plays an important role in the colored Jones polynomial, called the *Jones-Wenzl idempotent* and denoted by $f^{(n)}$, is represented diagrammatically by a little box with n parallel strands in and n parallel strands out. We make the convention that writing an n next to a strand means replacing it with n parallel strands and often say this strand is colored by n, where n is a non-negative integer. There is a recursion formula due to Wenzl [16], which can be then described graphically as follows:



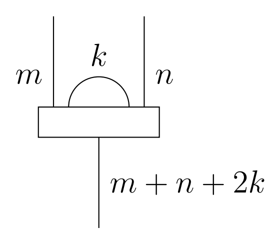
, (2.1)

where $\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$.

The Jones-Wenzl idempotent satisfies the following properties:

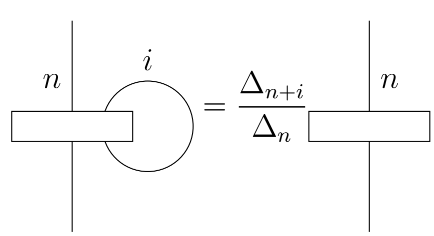


, (2.2)



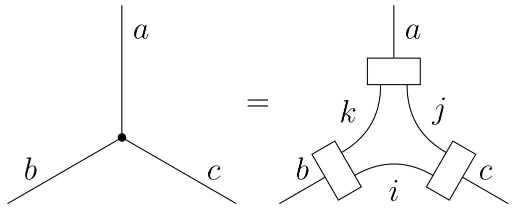
$= 0$ (2.3)

and



. (2.4)

A triple of colors (a, b, c) is called *admissible* if $a + b + c$ is even and $|a - b| \leqslant c \leqslant a + b$. We can define a 3-*valent vertex* by



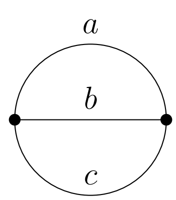
,

if a triple of colors (a,b,c) is given. Here $i=\frac{b+c-a}{2}$, $j=\frac{a+c-b}{2}$ and $k=\frac{a+b-c}{2}$ are called the *inner colors* of this 3-valent vertex.

Some coefficients and formulas needed in the calculation oh this paper are also listed here:

trihedron coefficient:

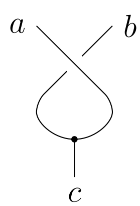
$$\theta(a,b,c) :=$$



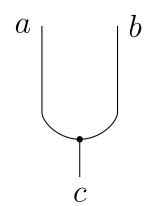
$$= \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{x+y-1}!\Delta_{x+z-1}!} \quad , \tag{2.5}$$

where x, y and z are the inner colors of the given triple of colors (a,b,c), and $\Delta_n! := \Delta_n\Delta_{n-1}\cdots\Delta_1$.

half-twist coefficient:

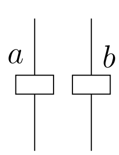


$$= (-1)^{\frac{a+b-c}{2}} A^{a+b-c+\frac{a^2+b^2-c^2}{2}}$$

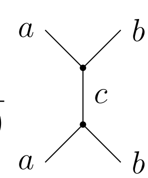


$$, \tag{2.6}$$

and *fusion formula:*



$$= \sum_c \frac{\Delta_c}{\theta(a,b,c)}$$



$$, \tag{2.7}$$

where the triple of colors (a,b,c) is admissible.

The *un-normalized colored Jones polynomial* (or *unreduced colored Jones polynomial*) $\widetilde{J}_{n,L}(A)$ of a framed link L is then defined to be the value of the link with each component decorated by the n-th Jones-Wenzl idempotent $f^{(n)}$ viewed as an element in $S(\mathbb{R}^3; \mathbb{Q}(A), A)$. It is known that $\widetilde{J}_{n,L}(A)$ actually lies in $\mathbb{Z}[A, A^{-1}]$.

Another version of the colored Jones polynomial is the *normalized colored Jones polynomial*, in which we consider links without a framing. It can be derived from the un-normalized colored Jones polynomial by

$$J_{n+1,L}(q) := \frac{\widetilde{J}_{n,L}(A)}{\Delta_n}\Big|_{A=q^{-\frac{1}{4}}}.$$

In this paper, we use $\widetilde{J}(n, L)$ for $\widetilde{J}_{n,L}(A)$.

Chapter 3
The Coefficient After the Tails

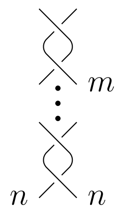
In [5] Dasbach and Lin proved that the first two and last two coefficients of the unreduced colored Jones polynomial $\tilde{J}(K,n)$ for an alternating knot K are stable, i.e. they are independent of the color n. They then investigated the third term and showed that it also becomes stable when $n \geqslant 3$. This result gave rise to the conjecture that the first k coefficients of $\tilde{J}(K,n)$ for an alternating knots should be independent of the color n when $n \geqslant k$. Armond [2] proved this conjecture by showing the existence of the heads and tails for the unreduced colored Jones polynomial of an adequate link. Independently, Garoufalidis and Le also showed the result in [8] for alternating links and extended it to the higher order stability, which Katherine Walsh studied. She gave an expression for the second stable sequence of the colored Jones polynomial for a certain class of knots in [15].

With these known results, it is then natural to ask whether the coefficients right after (or before) the heads (or tails) preserve some relations. In this chapter, we discuss the relation between $a_n^{4n+4}(K)$ and $a_{n+1}^{4n+4}(K)$, the immediate coefficient after the tails of the unreduced colored Jones polynomial of an alternating link K. By elaborating on the B-adequate diagram of K, we obtain Theorem 3.11, which gives the first main result. At the end, we give as example the colored Jones polynomial of the knot 6_3 in Rolfsen's table.

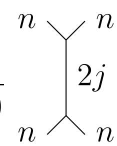
Unless otherwise stated, we will assume the link K discussed in the remaining paper is an alternating (hence adequate) link, and that all diagrams are reduced.

3.1 Proof of Main Theorem 1

Let D be the diagram for a given adequate link K. we assign an idempotent to each component of K. By using the fusion formula 2.7 on a maximal negative twist region, we obtain



$$= \sum_{j=0}^{n} (\gamma(n,n;2j))^m \frac{\Delta_{2j}}{\theta(n,n,2j)}$$



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where m represents the number of negative crossings in this twist region. Note that $\gamma(a,b;c) = (-1)^{a+b-c} A^{a+b-c+\frac{a^2+b^2-c^2}{2}}$.

Applying this result to all maximal negative twist regions, we obtain trivalent graphs $\Gamma_{n;j_1,j_2,\cdots,j_k}$ from D where k is the number of maximal negative twist regions and $2j_i$ the color for the i-th twist region in the fusion operation, with all other edges colored n. Note that $0 \leqslant j_i \leqslant n$. We eventually get the unreduced colored Jones polynomial for K as

$$\widetilde{J}(n,K) = \sum_{j_1,j_2,\cdots,j_k=0}^{n} \prod_{i=1}^{k} (\gamma(n,n;2j_i))^{m_i} \frac{\Delta_{2j_i}}{\theta_{n,n,2j_i}} \Gamma_{n;j_1,j_2,\cdots,j_k},$$

where m_i is the number of twist crossings in the i-th maximal negative twist region.

Here are some notations we will need to state the following lemmas.

Definition 3.1. *Let $f \in \mathbb{Q}(A)$. We use $d(f)$ to denote the minimal degree of f when it is written as a Laurent series in A.*

Definition 3.2. *Let $S \in (S^3; R, A)$. Denote $\bar{S}$ its crossing-less diagram obtained by replacing all idempotents in S with the identities of TL_n. Then we define $D(S) := d(\bar{S})$.*

The following three lemmas are due to Cody Armond. Proofs can be found in [2].

Lemma 3.3 (Armond)**.**

$$d(\gamma(n,n;2n)) = d(\gamma(n,n;2(n-1))) - 4n$$

$$d(\gamma(n,n;2j)) \leqslant d(\gamma(n,n;2(j-1)))$$

Lemma 3.4 (Armond)**.**

$$d(\frac{\Delta_{2n}}{\theta(n,n,2j)}) = d(\frac{\Delta_{2(j-1)}}{\theta(n,n,2(j-1))}) - 2$$

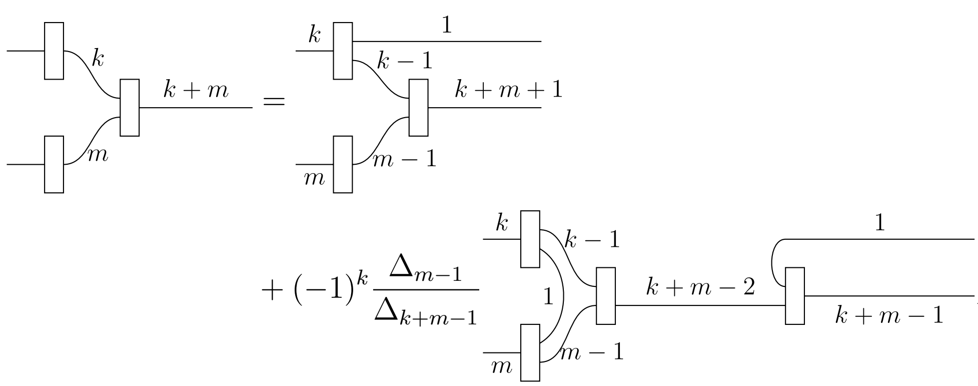
Lemma 3.5 (Armond)**.** *If Γ is the graph coming from a B-adequate diagram, then*

$$D(\Gamma_{n,(j_1,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_k)}) = D(\Gamma_{n,(j_1,\cdots,j_{i-1},j_i-1,j_{i+1},\cdots,j_k)}) \pm 2$$

$$d(\Gamma_{n,(n,\cdots,n,\cdots,n)}) = D(\Gamma_{n,(n,\cdots,n-1,\cdots,n)}) - 2$$

We also need another lemma which was first proven by Armond in [2].

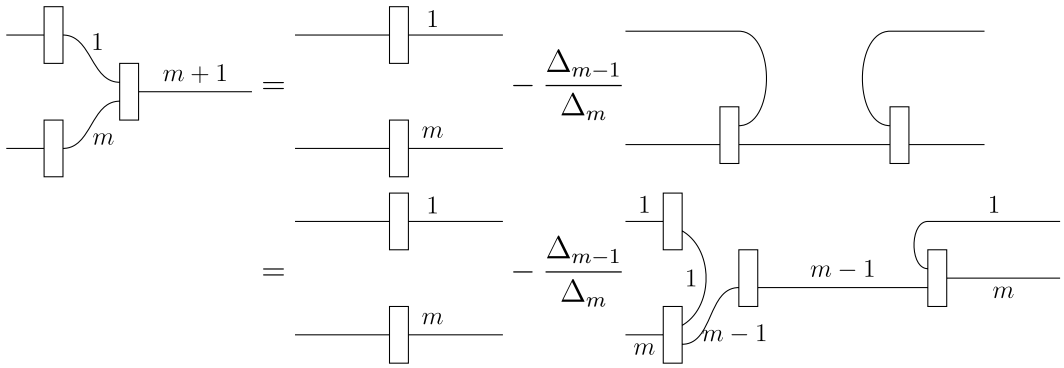
Lemma 3.6.



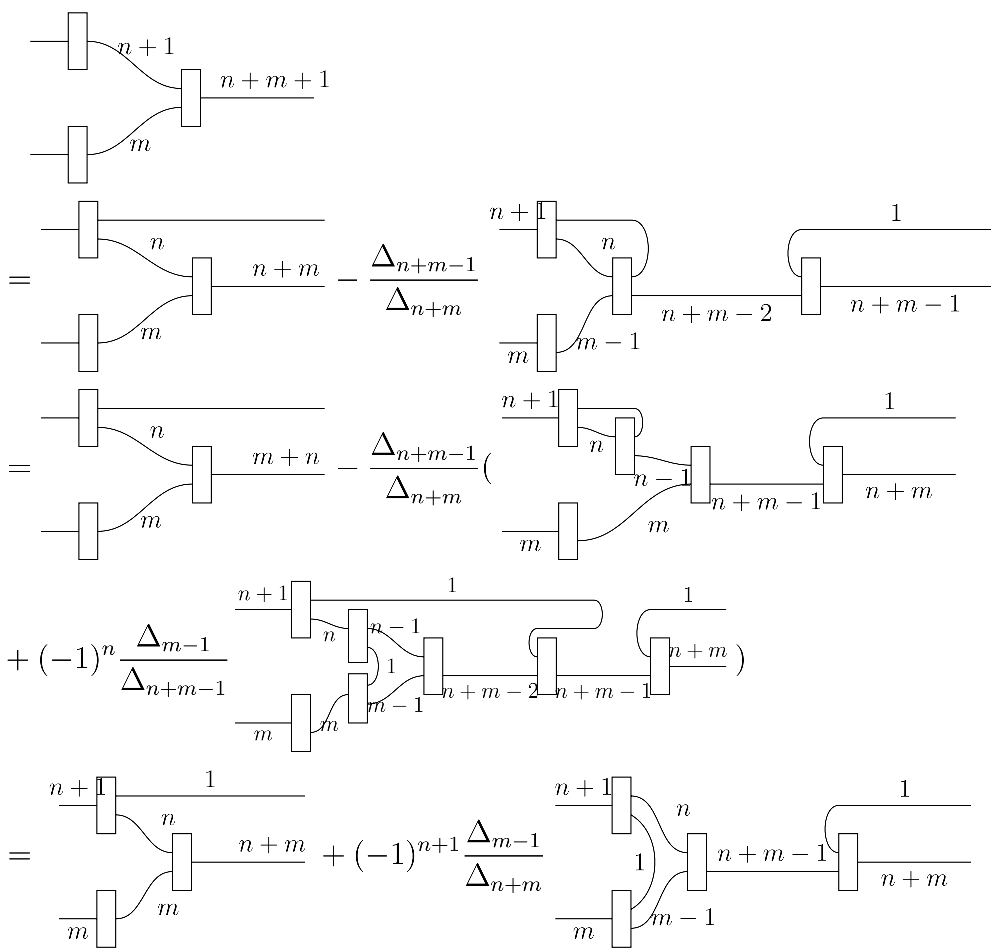
where colors $k \geqslant 1$ and $m \geqslant 1$.

Proof. We will prove this lemma by using mathematical induction on k.

When $k = 1$, applying the recursive relation 2.1, we have



Now assume the statement holds for $k = n$. Let $k = n + 1$. Combining the recursive relation 2.1 and the assumption, we get

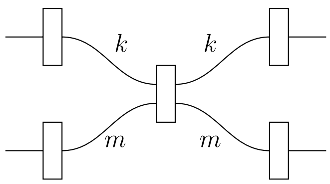


Therefore, by induction we obtain the result.

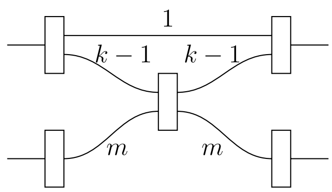
□

Adding a "mirror image" to the left hand diagram and applying Lemma 3.6, we obtain the following corollary which is useful in the computation of the tails of the unreduced colored Jones polynomial for K.

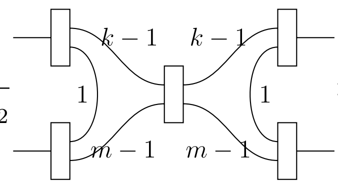
Corollary 3.7.



$=$



$$- \frac{\Delta_{m-1}^2}{\Delta_{k+m-1}\Delta_{k+m-2}}$$



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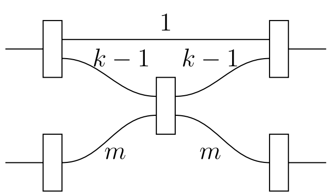

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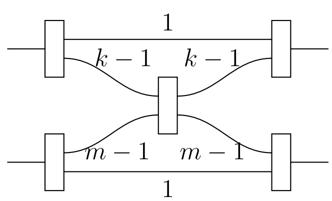
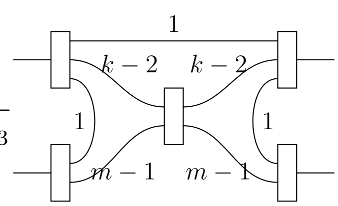
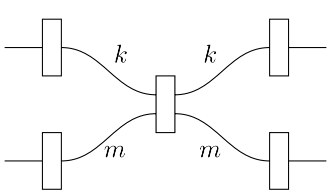


$$- \frac{\Delta_{k-2}^2}{\Delta_{k+m-2}\Delta_{k+m-3}}$$

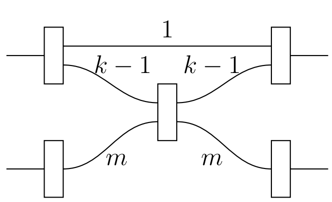


.

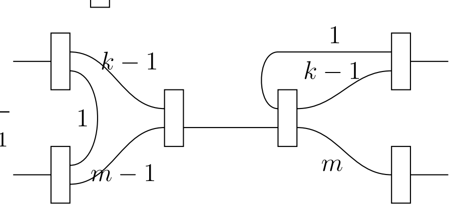
Proof. Applying Lemma 3.6, we have

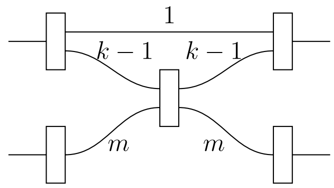


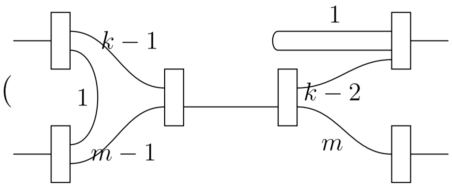
$=$

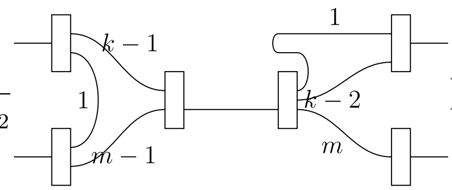


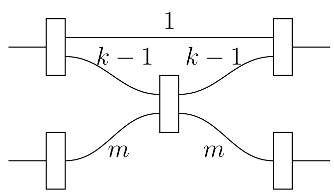
$$+ (-1)^k \frac{\Delta_{m-1}}{\Delta_{k+m-1}}$$

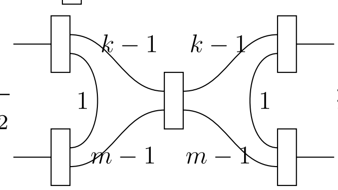


$=$ 

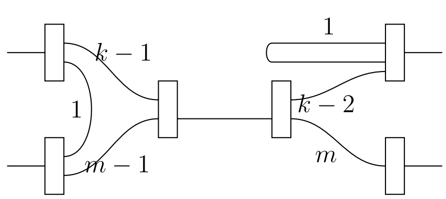
$+ (-1)^k \frac{\Delta_{m-1}}{\Delta_{k+m-1}}($ 

$+ (-1)^{k-1} \frac{\Delta_{m-1}}{\Delta_{k+m-2}}$  $)$

$=$ 

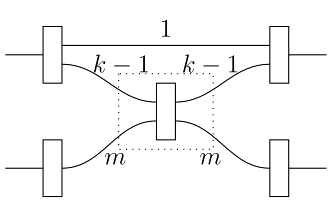
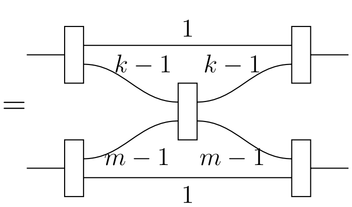
$- \frac{\Delta_{m-1}\Delta_{m-1}}{\Delta_{k+m-1}\Delta_{k+m-2}}$  ;

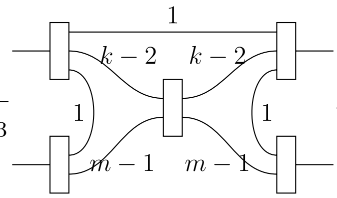
here

 $= 0$

by the second property of the Jones-Wenzl idempotent.

Now by applying this result to the dotted region, we have

 $=$ 

$$-\frac{\Delta_{k-1}^2}{\Delta_{k+m-2}\Delta_{k+m-3}}$$  .

□

Remark 3.8. *The four idempotents colored k and m in the X-shape "tangle" on the left hand side of the equation play an important role as the proof shows. This feature is essential in the proof of Theorem 3.11.*

3.1.1 The Relative Difference of the Colored Jones polynomial

Now we consider the so-called "relative difference".

Let $\widetilde{J}(n,K)$ and $\widetilde{J}(n+1,K)$ to be the n-th and $n+1$-th unreduced colored Jones polynomial of an alternating link K. By multiplying with suitable powers $\pm A^{\epsilon_1}$ and $\pm A^{\epsilon_2}$, we can normalize them to get two polynomials $\widetilde{AJ}(n,K)$ and $\widetilde{AJ}(n+1,K)$ such that both of them now have leading coefficients 1 (called adjusted unreduced colored Jones polynomial). Denote the coefficient in front of l-th power of $\widetilde{AJ}(n,K)$ by a_n^l. Since the heads and tails for alternating links exist, the difference polynomial $\widetilde{AJ}(n+1,K) - \widetilde{AJ}(n,K)$ can be written as $A^{4n+4}\widetilde{RJ}(n,K)$ where $\widetilde{RJ}(n,K)$ is a polynomial. We call this $\widetilde{RJ}(n,K)$ the n-th relative difference of K. Denote $r_n^l(K)$ the coefficient in front of power A^{l-4n-4}. Therefore $r_n^l(K) = a_{n+1}^l(K) - a_n^l(K)$ by definition.

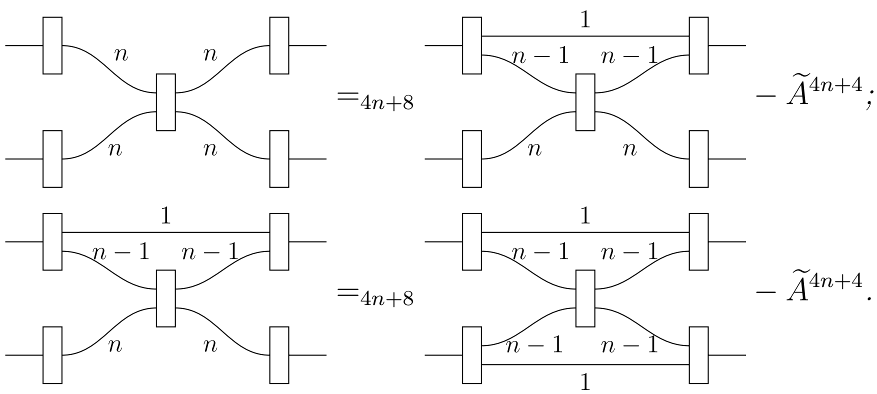
Also, for a Laurent polynomial P in A having minimal degree $d(P)$, we call $A^{n+d(P)}$ the relative n-th power of P, and denote it by $\widetilde{A}^n$.

Definition 3.9. *Let $P_1(A)$ and $P_2(A)$ be two Laurent series in A, and denote $\{Co_i(j)\}_{j=1}^n$ the corresponding coefficient sequence containing the first n coefficients of P_i (i = 1,2). Then $P_1(A) =_n P_2(A)$ if either $\forall j \in \{1,\cdots,n\}$ $Co_1(j) = Co_2(j)$ or $\forall j \in \{1,\cdots,n\}$ $Co_1(j) = -Co_2(j)$, and we say $P_1(A)$ is n-equivalent to $P_2(A)$.*

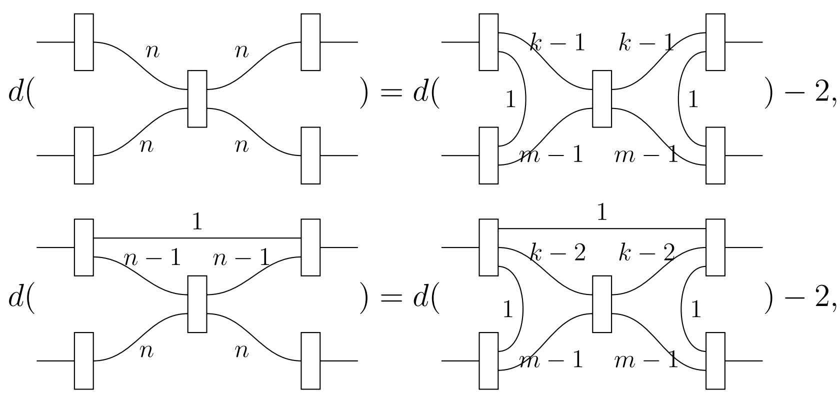
$-1 + A^4 - A^5 =_5 A^{-2} - A^2 - 3A^4$ gives a quick example.

When k and m are specifically assigned the same color n, we can get a further result regarding $(4n+8)$-equivalence by comparing the minimal degrees among all terms in this relation.

Corollary 3.10.



Proof. According to Lemma 3.4 and Lemma 3.5, we have



$$d\left(\frac{\Delta_{n-1}^2}{\Delta_{2n-1}\Delta_{2n-2}}\right) = 4n+2$$

and

$$d\left(\frac{\Delta_{n-2}^2}{\Delta_{2n-2}\Delta_{2n-3}}\right) = 4n+2.$$

The conclusion is now straightforward. □

We can state the first main result now.

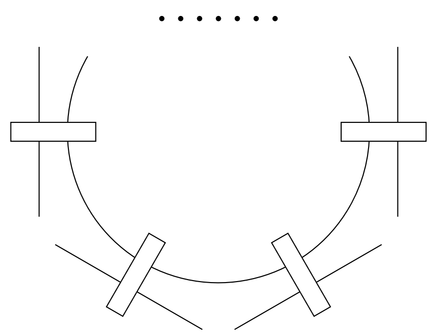


Figure 3.1: Local picture around one B-cycle. All strands are colored by $n+1$.

Theorem 3.11 (Main Theorem 1)**.** *Let G_B to be the B-graph of an alternating link K, E the number of edges in G_B, and V the number of vertices in G_B. Then*

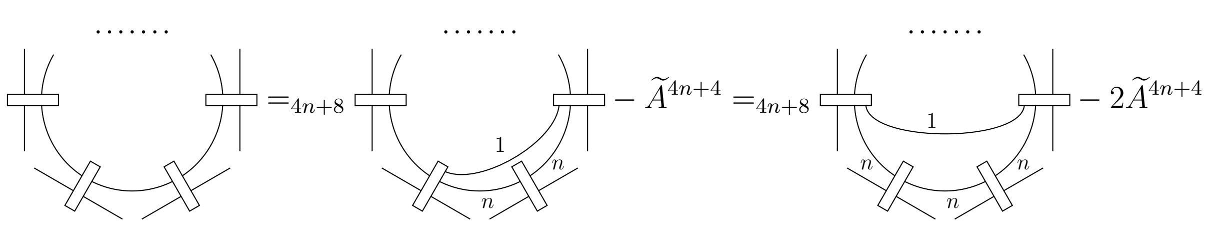
$$r_n^{4n+4}(K) = V - 2E + t_1.$$

Proof. To prove this result, we need to investigate both $\widetilde{J}(n, K)$ and $\widetilde{J}(n+1, K)$.

When all idempotents with color $2n+2$ in $\widetilde{J}(n+1, K)$ are ignored, the graph correspond to the all-B state contains only groups of "nested" circles which correspond to B-disks in the all-B state. For $\Gamma_{n;n,n,\cdots,n}$, we investigate one group such circles with associated idempotents added back. The local picture is as Figure 3.1 shows.

Start with any X-shape "tangle". Applying the operation in Corollary 3.7 to it, we can spin one color off the original idempotents on both sides. Focusing on the side where the target circle locates, we keep operating on the following X-shape "tangle" next to it, say, in clockwise fashion. Repeat this process until there is only one $2n+2$ idempotent left.

Assume we have in total c many $2n+2$ idempotents involved on the side of this B-disk. Applying Corollary 3.7 and Corollary 3.10, we now have



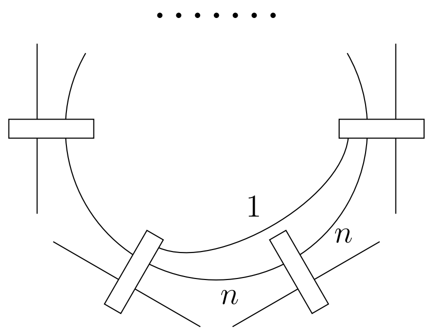
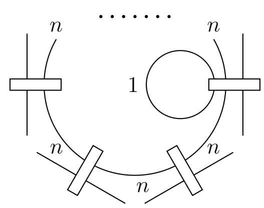


Figure 3.2: Spin one color off one $(2n+2)$-idempotent. All other unlabelled strands are colored by $n+1$.

$$=_{4n+8} \cdots =_{4n+8}$$



$$- (c-1)\widetilde{A}^{4n+4} \quad ,$$

where all other unlabelled strands are colored by $n+1$.

Repeat this process to all B-disks. Note that for some of them, we may need the second result of Corollary 3.7 and Corollary 3.10; although it still gives us the same result in the case of $(4n+8)$-equivalence. Since the all-B state diagram has E edges and V vertices in total, and each edge should be counted twice if without the different behaviour of the last circle associated to each B-disk, we have

$$\Gamma_{n+1;n+1,\cdots,n+1} =_{4n+8} \widetilde{\Gamma}_{n+1;n+1,\cdots,n+1} - (2E-V)\widetilde{A}^{4n+4},$$

where $\widetilde{\Gamma}_{n;n,\cdots,n}$ denotes the diagram with one circle not spun off for each B-disk.

For the inner circle associated to the remaining $(2n+2)$-idempotent in each B-disk, however, we cannot apply Corollary 3.7 to it. Instead, we use the third property of the Jones-Wenzl idempotent to spin off the 1-color circle. When all those 1-color circles are spun off, the remaining one is exactly $\Gamma_{n;n,\cdots,n}$. Note that

$$\frac{\Delta_{2n+2}}{\Delta_{2n+1}} =_{4n+8} \frac{\Delta_{2n+1}}{\Delta_{2n}} =_{4n+8} 1;$$

we now have

$$\Gamma_{n+1;n+1,\cdots,n+1} =_{4n+8} \widetilde{\Gamma}_{n;n,\cdots,n} - (2E-V)\widetilde{A}^{4n+4} =_{4n+8} \Gamma_{n;n,\cdots,n} - (2E-V)\widetilde{A}^{4n+4}.$$

Combine this result with the existence of the tails, we then get

$$\widetilde{J}(n+1,K) =_{4n+8} \Gamma_{n+1;n+1,\cdots,n+1} =_{4n+8} \Gamma_{n;n,\cdots,n} - (2E-V)\widetilde{A}^{4n+4}.$$

Now let us investigate $\widetilde{J}(n,K)$.

Recall that the unreduced colored Jones polynomial for K is

$$\widetilde{J}(n,K) = \sum_{j_1,j_2,\cdots,j_k=0}^{n} \prod_{i=1}^{k} (\gamma(n,n;2j_i))^{m_i} \frac{\Delta_{2j_i}}{\theta_{n,n,2j_i}} \Gamma_{n;j_1,j_2,\cdots,j_k},$$

where m_i is the number of twist crossings in the i-th maximal negative twist region. The general term in this sum is a product $\prod_{i=0}^{k}(\gamma(n,n;2j_i))^{m_i} \frac{\Delta_{2j_i}}{\theta_{n,n,2j_i}} \Gamma_{n;j_1,j_2,\cdots,j_k}$. According to Lemma 3.4 and Lemma 3.5, we get

$$d(\frac{\Delta_{2j_i}}{\theta_{n,n,2j_i}} \Gamma_{n;j_1,j_2,\cdots,j_i,\cdots,j_k}) \leqslant d(\frac{\Delta_{2(j_i-1)}}{\theta_{n,n,2(j_i-1)}} \Gamma_{n;j_1,j_2,\cdots,j_i-1,\cdots,j_k}).$$

Also, by Lemma 3.3 we have $d(\gamma(n,n;2j)) \leqslant d(\gamma(n,n;2(j-1)))$. In fact, we can get a more precise result by definition

$$\begin{aligned} & d(\gamma(n,n;2j)) - d(\gamma(n,n;2(j-1))) \\ = & (n+n-2j+\frac{n^2+n^2-(2j)^2}{2}) - (n+n-2(j-1)+\frac{n^2+n^2-(2j-2)^2}{2} = 4j. \end{aligned}$$

Note that when $m_i \geqslant 2$, $d(\gamma^{m_i}(n,n;2(n-1))) = 4nm_i \geqslant 4n+8$ for $n \geqslant 2$.

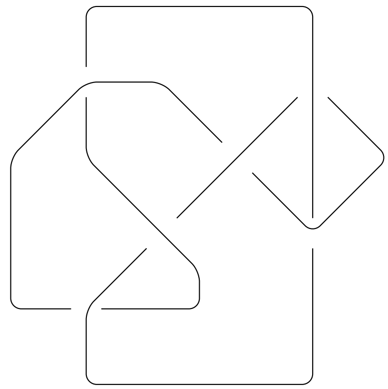


Figure 3.3: Knot 6_3 from Rolfsen's table

Hence

$$\begin{aligned}\widetilde{J}(n,K) &=_{4n+8} \Gamma_{n;n,\cdots,n} \\ &\quad - \sum_{j\in J}(\gamma(n,n;2n-2)\frac{\Delta_{2n-2}}{\theta_{n,n,2n-2}})(\prod_{i\neq j}^{k}(\gamma(n,n;2n))^{m_i}\frac{\Delta_{2n}}{\theta_{n,n,2n}})\Gamma_{n;n,n,\cdots,j=n-1,\cdots,n} \\ &=_{4n+8} \Gamma_{n;n,\cdots,n} - t_1\widetilde{A}^{4n+4},\end{aligned}$$

where J contains all those j such that the j-th edge corresponds to a maximal negative twist region which has just 1 crossing.

With the results on $\widetilde{J}(n+1,K)$ and $\widetilde{J}(n,K)$ together, now we obtain

$$\begin{aligned}A^{4n+4}\widetilde{RJ}(n,K) &=_{4n+8} (\Gamma_{n;n,\cdots,n} - (2E-V)\widetilde{A}^{4n+4}) - (\Gamma_{n;n,\cdots,n} - t_1\widetilde{A}^{4n+4}) \\ &=_{4n+8} (V-2E+t_1)A^{4n+4}.\end{aligned}$$

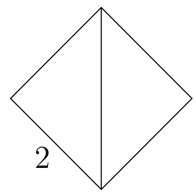
That completes the proof.

□

3.2 Example: Knot 6_3

As an example we consider the knot 6_3 in Rolfsen's table, which is shown in Figure 3.3.

The B-graph of this diagram is



,

where the number 2 on that edge means it is a twist region with 2 negative crossings. From the B-graph, we have $E = 5$,$V = 4$ and $t_1 = 4$.

Using Mathematica, we have the first 4 adjusted unreduced colored Jones polynomials of 6_3 are as follows:

$\widetilde{J}(1, 6_3) = 1 - A^4 - A^{12} - A^{16} - A^{24} + A^{28}$,

$\widetilde{J}(2, 6_3) = 1 - A^4 - 2A^8 + 2A^{12} - 2A^{20} + 2A^{24} + A^{28} + \cdots$,

$\widetilde{J}(3, 6_3) = 1 - A^4 - 2A^8 + 3A^{16} + 2A^{20} - 4A^{24} - 2A^{28} + \cdots$,

$\widetilde{J}(4, 6_3) = 1 - A^4 - 2A^8 + A^{16} + 5A^{20} - 4A^{28} + \cdots$.

That gives $r_1^8(6_3) = r_2^{12}(6_3) = r_3^{16}(6_3) = -2$.

Indeed, Theorem 3.11 states that $r_n^{4n+4}(6_3) = V - 2E + t_1 = -2$.

Chapter 4

r_1^{12} for Specific Class of Links

We have discussed r_n^{4n+4} in general. When it comes to r_n^{4n+8}, however, the same method doesn't work efficiently since the calculation of the corresponding coefficients (i.e. a_n^{4n+8} and a_{n+1}^{4n+8}) involves too many terms even for very simple alternating knots. Therefore, in this chapter we focus mainly on a_1^{12} and a_2^{12}.

We will give a formula for $r_1^{12}(K)$ of the colored Jones polynomials for K in a specific class of links. First we discuss the general formula for $\widetilde{AJ}(1,K)$ up to A^{12} for the unreduced colored Jones polynomial of any link K. Then we study $\widetilde{AJ}(2,K)$ case by case, for a specific class of links, and obtain the second main theorem. Finally, we use this formula to improve a result from [7].

4.1 General Formula for $\widetilde{AJ}(1,K)$ up to A^{12}

To give the second main theorem, we first focus on $\widetilde{J}(1,K)$ for an alternating link K. Assume all strands in this section are colored by 1.

Let G_B to be the B-graph of K. Denote V the number of vertices of G_B, E the number of edges of G_B, F the number of faces (without counting the outer face) of G_B, and R the number of 4-faces in G_B. Also, we use T_i to denote the category of all edges in G_B who correspond to maximal twist regions that have exactly i negative twist crossings.

With all these notations we state the general result for $\widetilde{AJ}(1,K)$ up to A^{12}.

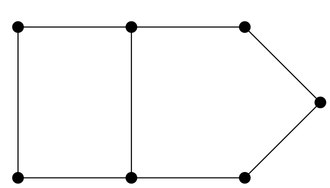


Figure 4.1: A B-graph for some link K. Note that $E = 8$, $V = 7$, $F = 2$ and $R = 1$.

Theorem 4.1. *For an alternating link K,*

$\widetilde{A}\widetilde{J}(1,K) =_{16} \Gamma_{1;(1,\cdots,1)} + [t_{1,N_3}(E-V+1) + t_{1,Y_3}(E-V) + t_{1,D_3}(E-V-1) + t_2]\widetilde{A}^{12}$,

where t_{1,N_3} is the number of edges in T_1 not in a triangle face, t_{1,Y_3} the number of edges in T_1 that are in exactly one triangle face, and t_{1,D_3} the number of edges in T_1 that are shared by two triangle faces.

Remark 4.2. *By definition we have: $t_1 = t_{1,N_3} + t_{1,Y_3} + t_{1,D_3}$.*

Proof. First we consider the special case that the B-graph G_B of a link K is C_4, where C_n is a circle with n edges. Assume $m_i = 1$ for all i, where m_i is the number of twist crossings in the i-th maximal negative twist region.

We already know the unreduced colored Jones polynomial for K is

$$\widetilde{J}(n,K) = \sum_{j_1,j_2,\cdots,j_k=0}^{n} \prod_{i=1}^{k} (\gamma(n,n;2j_i))^{m_i} \frac{\Delta_{2j_i}}{\theta_{n,n,2j_i}} \Gamma_{n;j_1,j_2,\cdots,j_k}.$$

Now we have

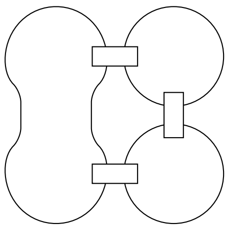
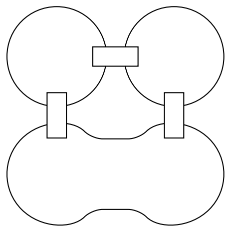
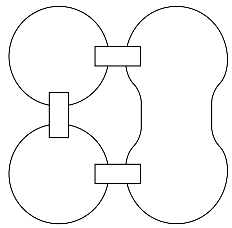
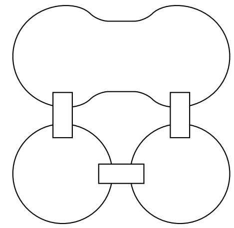
$$\begin{aligned}\widetilde{J}(1,C_4) = \Gamma_{1;(1,\cdots,1)} &+ \sum_{i=1}^{4} \gamma(1,1;0)^{m_i} \frac{\Delta_0}{\theta_{1,1,0}} \Gamma_{1;(1,\cdots,j_i=0,\cdots,1)} \\ &+ \sum_{i,k=1,i<k}^{4} \gamma(1,1;0)^{m_i}\gamma(1,1;0)^{m_k} (\frac{\Delta_0}{\theta_{1,1,0}})^2 \Gamma_{1;(1,\cdots,j_i=0,\cdots,j_k=0,\cdots,1)} + \cdots \\ &=_{16} \Gamma_{1;(1,\cdots,1)} + \sum_{i=1}^{4} \gamma(1,1;0) \frac{\Delta_0}{\theta_{1,1,0}} \Gamma_{1;(1,\cdots,j_i=0,\cdots,1)},\end{aligned}$$

by combining Lemma 3.3, Lemma 3.4 and Lemma 3.5 with the fact that C_2 in G_B is in fact P_1.

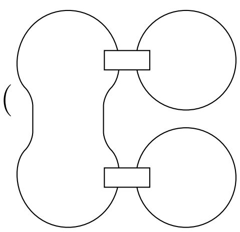
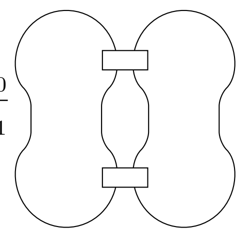
Moreover, detailed computation shows that

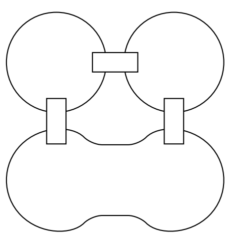
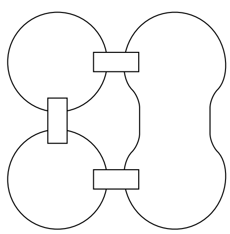
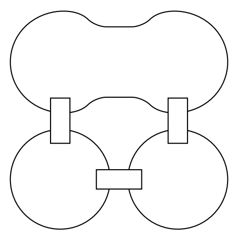
$$\widetilde{J}(1,C_4) =_{16} \Gamma_{1;(1,\cdots,1)} + \sum_{i=1}^{4}\gamma(1,1;0)\frac{\Delta_0}{\theta_{1,1,0}}\Gamma_{1;(1,\cdots,j_i=0,\cdots,1)}$$

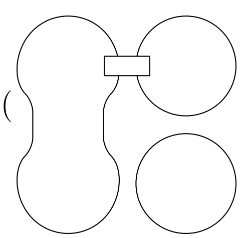
$$=_{16} \Gamma_{1;(1,\cdots,1)} + \gamma(1,1;0)\frac{\Delta_0}{\theta_{1,1,0}}$$

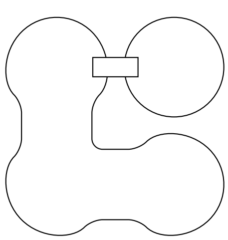
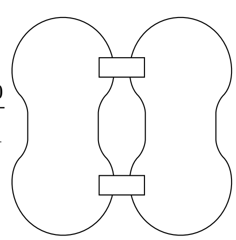
( +  +  + ).

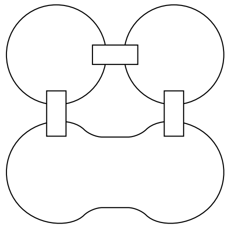
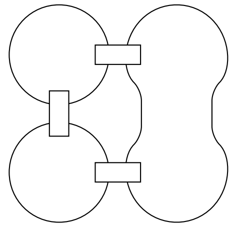
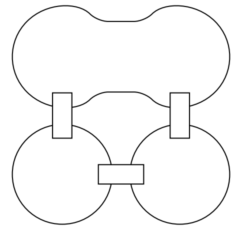
We focus on the first C_3 in the decomposition, and have

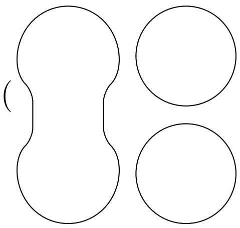
$$\widetilde{J}(1,C_4) =_{16} \Gamma_{1;(1,\cdots,1)} + \gamma(1,1;0)\frac{\Delta_0}{\theta_{1,1,0}}($$  $$-\frac{\Delta_0}{\Delta_1}$$ 

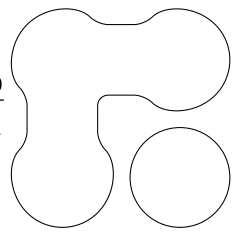
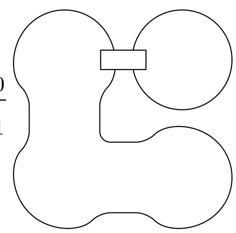
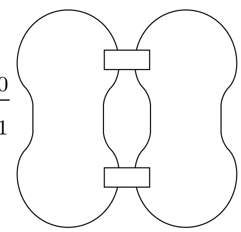
\+  +  + )

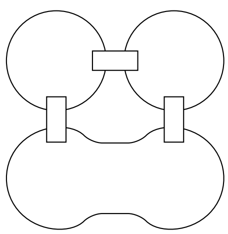
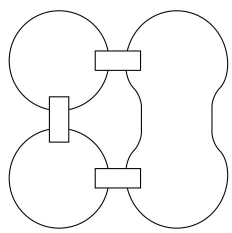
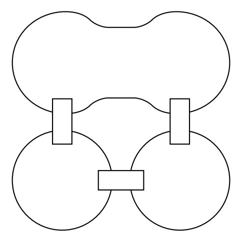
$$=_{16} \Gamma_{1;(1,\cdots,1)} + \gamma(1,1;0)\frac{\Delta_0}{\theta_{1,1,0}}($$ 

$$-\frac{\Delta_0}{\Delta_1}$$  $$-\frac{\Delta_0}{\Delta_1}$$ 

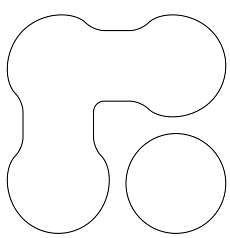
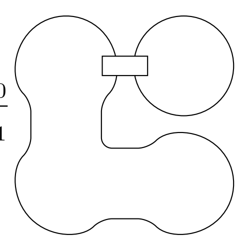
$+$  $+$  $+$  $)$

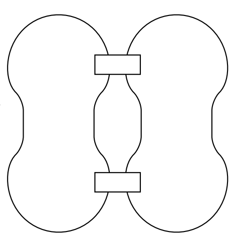
$=_{16} \Gamma_{1;(1,\cdots,1)} + \gamma(1,1;0)\dfrac{\Delta_0}{\theta_{1,1,0}}($ 

$-\dfrac{\Delta_0}{\Delta_1}$  $-\dfrac{\Delta_0}{\Delta_1}$  $-\dfrac{\Delta_0}{\Delta_1}$ 

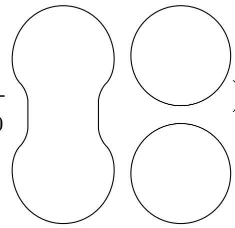
$+$  $+$  $+$  $).$

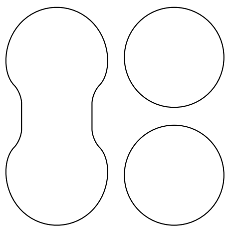
Note that

$d(\gamma(1,1;0)\dfrac{\Delta_0}{\theta_{1,1,0}}\dfrac{\Delta_0}{\Delta_1}$  $) = d(\gamma(1,1;0)\dfrac{\Delta_0}{\theta_{1,1,0}}\dfrac{\Delta_0}{\Delta_1}$  $)$

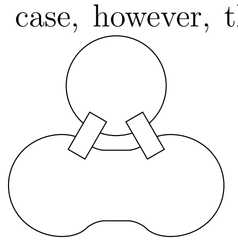
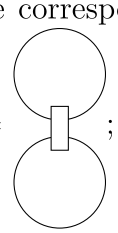
$= d(\gamma(1,1;0)\dfrac{\Delta_0}{\theta_{1,1,0}}\dfrac{\Delta_0}{\Delta_1}$  $) = 12 + d(\Gamma_{1;(1,\cdots,1)}).$

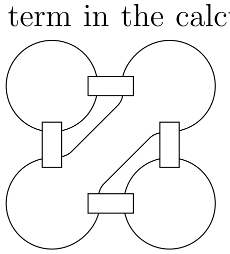
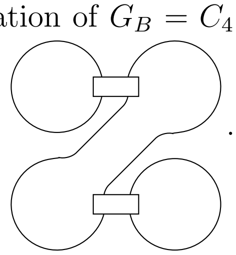
Each of them then contributes 1 to $\widetilde{A}^{12}$, whereas

$$d(\gamma(1,1;0)\frac{\Delta_0}{\theta_{1,1,0}}$$  $$) = 8 + d(\Gamma_{1;(1,\cdots,1)})$$

and  $= \Delta_1^{4-1}$ so together with the coefficient in front of it, the contribution of this term is $-[(4-1)-1]$ to $\widetilde{A}^{12}$.

As for the other three C_3 terms, similar calculation can be done as well. Therefore except for $\Gamma_{1;(1,\cdots,1)}$, from each edge in C_4, we have $-[(4-1)-1]+(4-1)$ contributed to $\widetilde{A}^{12}$. If in general we replace C_4 with C_n where $n \geqslant 4$, each edge will contribute $-[(E-1)-1]+(V-1) = E-V+1$ to $\widetilde{A}^{12}$ provided $\Gamma_{1;(1,\cdots,1)}$ is excluded.

When it comes to C_3 case, however, the corresponding C_3 term in the calculation of $G_B = C_4$ is changed to  $=$ ; therefore we should change E in the previous formula of C_n ($n \geqslant 4$) to $E-1$ in C_3 case, and hence each edge in exactly one C_3 contributes $E-V$ to $\widetilde{A}^{12}$.

As for the last case, where one edge is shared by two C_3's, we have the corresponding C_3 term in the calculation of $G_B = C_4$ which is related to the shared edge here changed to  $=$ . Therefore we should substitute E in the formula of C_n ($n \geqslant 4$) case by $E-2$, which leads to the contribution to $\widetilde{A}^{12}$ from this shared edge $E-V-1$.

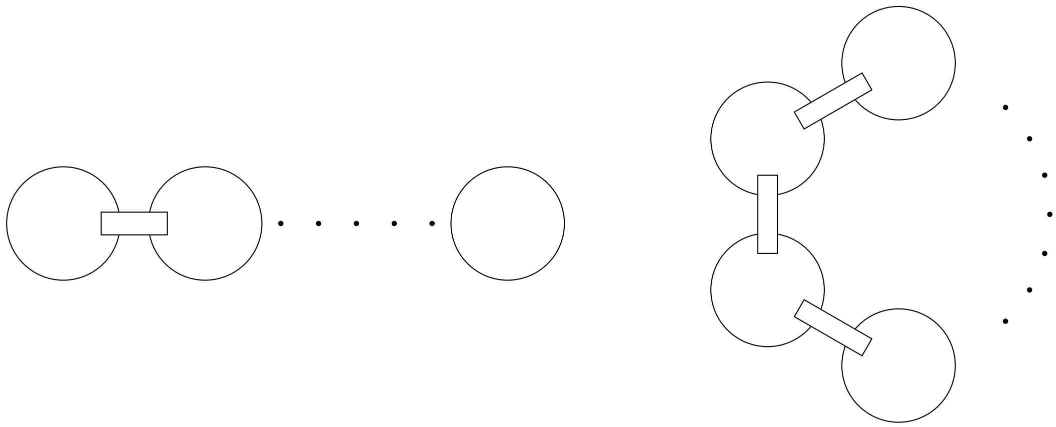


Figure 4.2: $P_n^{ⓘ}$ and $C_n^{ⓘ}$; all "arcs" are colored by i.

Note that if $m_i > 2$ for some i, those terms will contribute more degrees than needed, and hence can be dropped as well. If $m_j = 2$ for some j, these terms will contribute 1 to $\widetilde{A}^{12}$. That completes the proof.

□

4.2 Proof of Main Theorem 2

In this section we show the second main result. Based on Theorem 4.1, to get $r_1^{12}(K)$ for an alternating link K, we just need to compute the coefficient in front of $\widetilde{A}^{12}$ term of $\Gamma_{1;(1,\cdots,1)}$, since all other terms in the bracket of that expression can be easily read off from the B-graph G_B. Let's start with a simple case: G_B of K is C_n. In this section, we use $C_n^{ⓘ}$ to denote the $\Gamma_{i;(i,\cdots,i)}$ obtained from $\widetilde{J}(i,K)$. Similarly we use $P_n^{ⓘ}$ to denote the $\Gamma_{i;(i,\cdots,i)}$ obtained from $\widetilde{J}(i,K)$ if G_B is P_n, which is a path with n edges. Unless otherwise stated, we will assume all unlabelled strands in this section are colored by 2.

Lemma 4.3. *If G_B is P_n, then*

$$P_n^{①} = \frac{\Delta_2^n}{\Delta_1^{n-1}}.$$

Proof. Using the third property of the Jones-Wenzl idempotent, we can absorb those "circles" one by one; we then have

$$P_n^{\textcircled{1}} = (\frac{\Delta_2}{\Delta_1})^n \Delta_1 = \frac{\Delta_2^n}{\Delta_1^{n-1}}.$$

$\square$

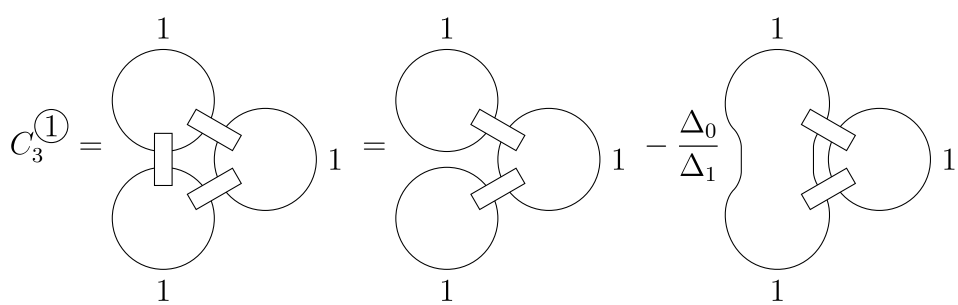
Lemma 4.4. *If G_B is C_n, then*

$$C_n^{\textcircled{1}} = \begin{cases} \frac{\Delta_2^2 - \Delta_2}{\Delta_1} & \text{if } n = 3 \\ \frac{\Delta_2^3 - \Delta_2^2 + \Delta_2}{\Delta_1^2} & \text{if } n = 4 \\ \frac{\Delta_2^{n-1} - \Delta_2^{n-2} + \Delta_2^{n-3} - \Delta_2^{n-4} + \cdots}{\Delta_1^{n-2}} & \text{if } n \geqslant 5, \end{cases}$$

and hence

$$C_n^{\textcircled{1}} =_{16} \begin{cases} 1 + 2A^8 - A^{12} & \text{if } n = 3 \\ 1 + 4A^8 - 3A^{12} & \text{if } n = 4 \\ 1 + nA^8 - nA^{12} & \text{if } n \geqslant 5. \end{cases}$$

Proof. Let us start with the case $n = 3$.



$$= P_2^{\textcircled{1}} - \frac{\Delta_0}{\Delta_1} C_2^{\textcircled{1}} = P_2^{\textcircled{1}} - \frac{\Delta_0}{\Delta_1} P_1^{\textcircled{1}} = \frac{\Delta_2^2}{\Delta_1} - \frac{\Delta_0 \Delta_2}{\Delta_1} =_{16} 1 + 2A^8 - A^{12}.$$

When $n = 4$, we have

$$C_4^{①} = P_3^{①} - \frac{\Delta_0}{\Delta_1}C_3^{①} = \frac{\Delta_2^3}{\Delta_1^2} - \frac{\Delta_0}{\Delta_1}(\frac{\Delta_2^2}{\Delta_1} - \frac{\Delta_0\Delta_2}{\Delta_1}) = \frac{\Delta_2^3 - \Delta_2^2 + \Delta_2}{\Delta_1^2} =_{16} 1 + 4A^8 - 3A^{12}.$$

Similarly, when $n \geqslant 5$, we have

$$\begin{aligned}
C_n^{①} &= P_{n-1}^{①} - \frac{\Delta_0}{\Delta_1}C_{n-1}^{①} = \frac{\Delta_2^{n-1}}{\Delta_1^{n-2}} - \frac{\Delta_0}{\Delta_1}C_{n-1}^{①} = \frac{\Delta_2^{n-1}}{\Delta_1^{n-2}} - \frac{\Delta_0}{\Delta_1}(\frac{\Delta_2^{n-2}}{\Delta_1^{n-3}} - \frac{\Delta_0}{\Delta_1}C_{n-2}^{①}) \\
&= \frac{\Delta_2^{n-1}}{\Delta_1^{n-2}} - \frac{\Delta_2^{n-2}}{\Delta_1^{n-2}} + \frac{1}{\Delta_1^2}C_{n-2}^{①} = \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} - \Delta_2^{n-2}) + \frac{1}{\Delta_1^2}(\frac{\Delta_2^{n-3}}{\Delta_1^{n-4}} - \frac{1}{\Delta_1}C_{n-3}^{①}) \\
&= \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} + (-1)\Delta_2^{n-2} + (-1)^2\Delta_2^{n-3}) + (-1)^3\frac{1}{\Delta_1^3}C_{n-3}^{①} \\
&= \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} - \Delta_2^{n-2} + \Delta_2^{n-3} - \cdots + (-1)^{n-4}\Delta_2^3) + (-1)^{n-3}\frac{1}{\Delta_1^{n-3}}C_3^{①} \\
&= \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} - \Delta_2^{n-2} + \Delta_2^{n-3} - \cdots + (-1)^{n-3}\Delta_2^2) + (-1)^{n-2}\frac{1}{\Delta_1^{n-2}}C_2^{①} \\
&= \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} - \Delta_2^{n-2} + \Delta_2^{n-3} - \cdots + (-1)^{n-2}\Delta_2).
\end{aligned}$$

Note that $d(\frac{\Delta_2^{n-5}}{\Delta_2^{n-1}}) = 16$. We then get

$$C_n^{①} =_{16} \frac{1}{\Delta_1^{n-2}}(\Delta_2^{n-1} - \Delta_2^{n-2} + \Delta_2^{n-3} - \Delta_2^{n-4}) =_{16} 1 + nA^8 - nA^{12}.$$

□

Remark 4.5. *The case $n = 2$ is straight forward, which gives the result $C_2^{①} = \Delta_2$, hence $C_2^{①} =_{16} 1 + A^4 + A^8$. However, when we construct the Γ graph, we have maximized all negative crossings, so there will be no such case in the corresponding B-graph. Actually in the case of C_2 we have P_1 instead as its B-graph. See Figure 4.3.*

Before discussing the result for $C_n^{②}$, we need the following fundamental formula first.

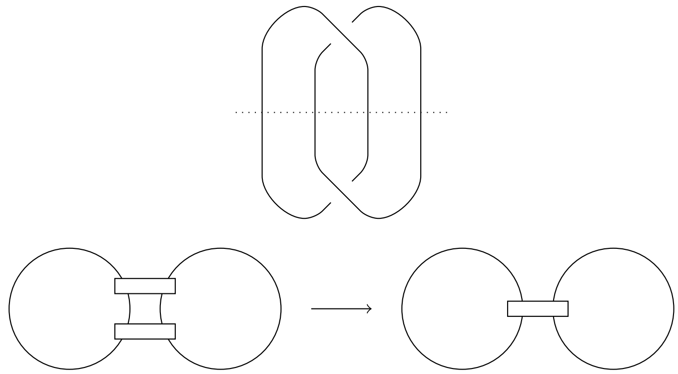
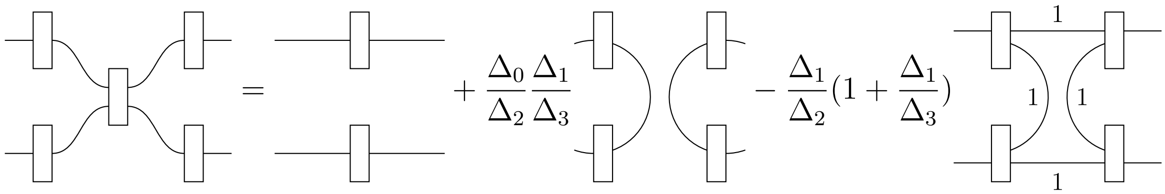
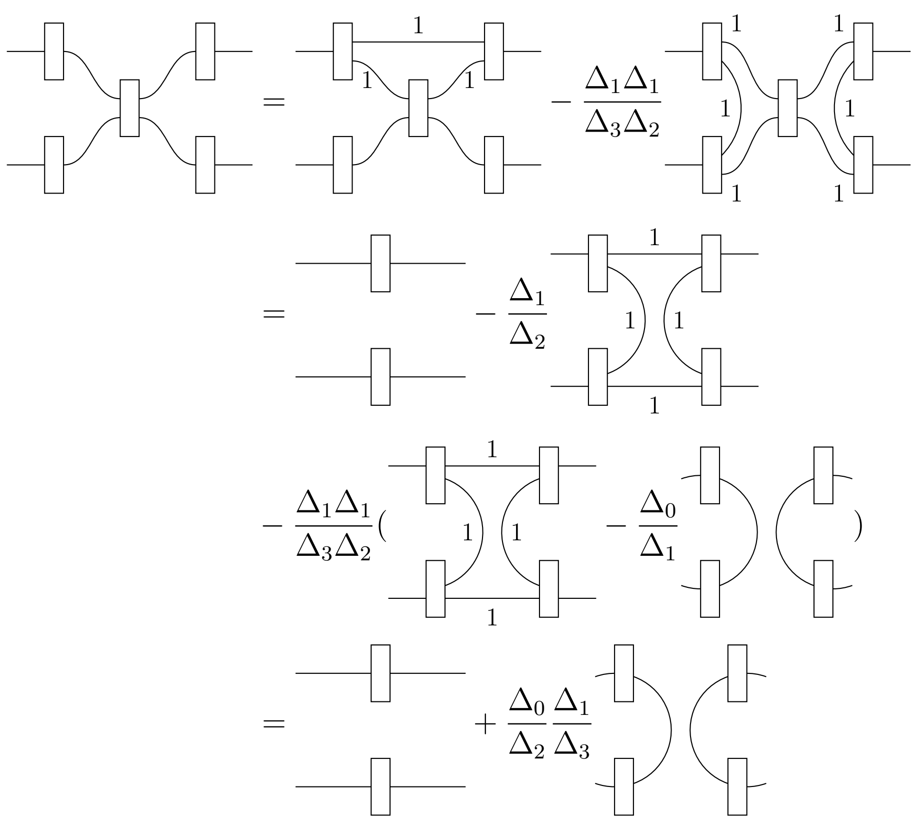


Figure 4.3: A possible link diagram corresponding to C_2 which is actually a P_1.

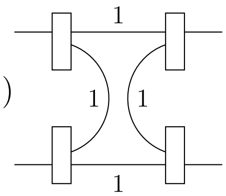
Lemma 4.6.



Proof. We prove this result by using Lemma 3.6 and the recursive relation 2.1.



$$-\frac{\Delta_1}{\Delta_2}\left(1+\frac{\Delta_1}{\Delta_3}\right)$$



$\square$

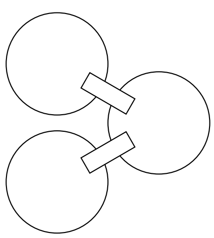
Lemma 4.7. *Assume G_B of K is given by C_n . Then*

$$C_n^{\textcircled{2}} =_{16} \begin{cases} 1 - A^8 + 2A^{12} & \text{if } n = 3 \\ 1 + 5A^{12} & \text{if } n = 4 \\ 1 + nA^{12} & \text{if } n \geqslant 5 \end{cases}$$

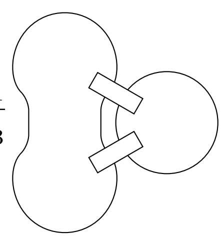
Remark 4.8. *Note that when $n = 2$, we actually get P_1 instead of C_2. Hence $C_2^{\textcircled{2}} = \Delta_4 =_{16} 1 + A^4 + A^8 + A^{12}$.*

Proof. Let us start with $n = 3$.

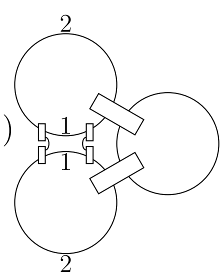
$$C_3^{\textcircled{2}} =$$



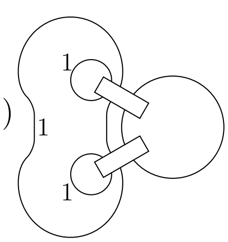
$$+\frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}$$



$$-\frac{\Delta_1}{\Delta_2}\left(1+\frac{\Delta_1}{\Delta_3}\right)$$

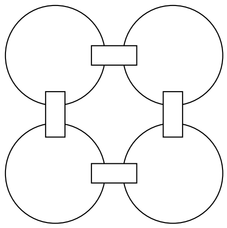
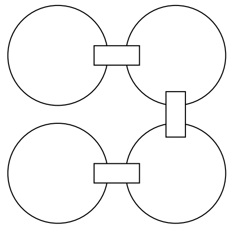
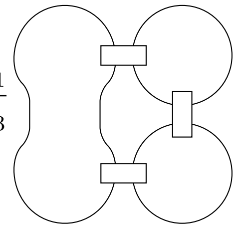


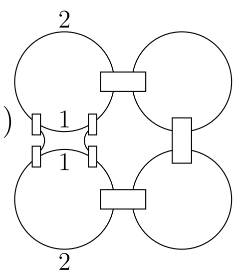
$$= P_2^{\textcircled{2}} + \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}P_1^{\textcircled{2}} - \frac{\Delta_1}{\Delta_2}\left(1+\frac{\Delta_1}{\Delta_3}\right)$$

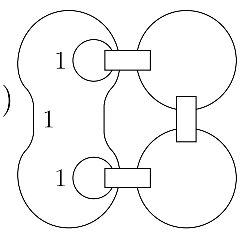


$$= \frac{\Delta_4^2}{\Delta_2} + \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}\Delta_4 - \frac{\Delta_1}{\Delta_2}\left(1+\frac{\Delta_1}{\Delta_3}\right)\frac{\Delta_4}{\Delta_3}\Delta_4 =_{16} 1 - A^8 + 2A^{12}.$$

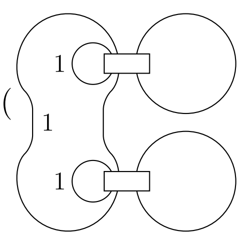
When $n = 4$, we have

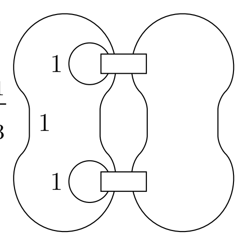
$$C_4^{\textcircled{2}} =$$

$$=$$

$$+ \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}$$


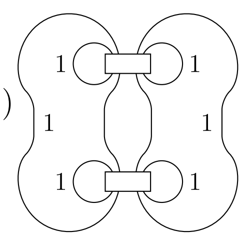
$$- \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})$$


$$= P_3^{\textcircled{2}} - \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}C_3^{\textcircled{2}} - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})$$


$$= \frac{\Delta_4^3}{\Delta_2^2} + \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}C_3^{\textcircled{2}} - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})\cdot$$

$$($$


$$+ \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}$$


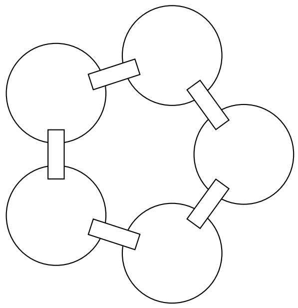
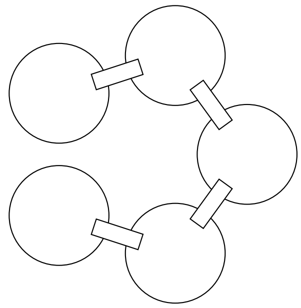
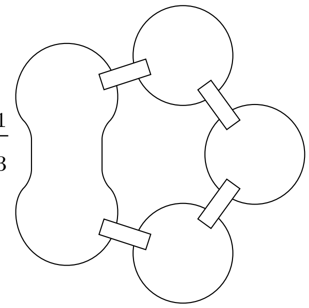
$$- \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})$$


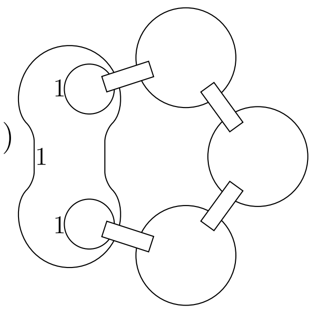
$$)$$

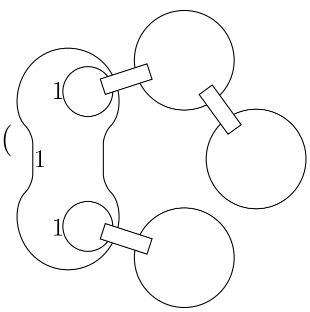
$$=_{16} \frac{\Delta_4^3}{\Delta_2^2} + \frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}C_3^{\textcircled{2}} - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})[\frac{\Delta_4}{\Delta_1}\Delta_4 - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_2}\Delta_4]$$

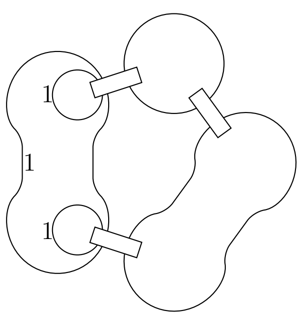
$$=_{16} 1 + 5\widetilde{A}^{12}.$$

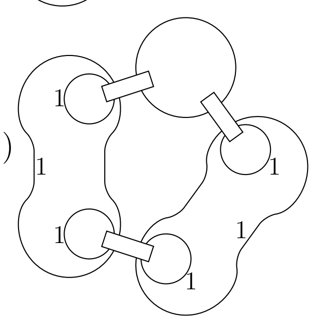
When $n = 5$, we have

$C_5^{②} =$  $=$  $+ \dfrac{\Delta_0}{\Delta_2}\dfrac{\Delta_1}{\Delta_3}$ 

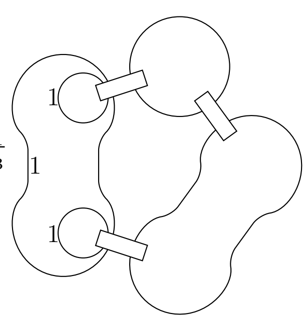
$- \dfrac{\Delta_1}{\Delta_2}(1 + \dfrac{\Delta_1}{\Delta_3})$ 

$= P_4^{②} + \dfrac{\Delta_0}{\Delta_2}\dfrac{\Delta_1}{\Delta_3} C_4^{②} - \dfrac{\Delta_1}{\Delta_2}(1 + \dfrac{\Delta_1}{\Delta_3})($ 

$+ \dfrac{\Delta_0}{\Delta_2}\dfrac{\Delta_1}{\Delta_3}$ 

$- \dfrac{\Delta_1}{\Delta_2}(1 + \dfrac{\Delta_1}{\Delta_3})$ 

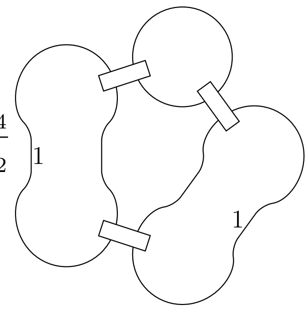
$)$

Since the degree of $\frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_0}{\Delta_2}\frac{\Delta_1}{\Delta_3}$ 

is too high, we can just throw it away. Hence we have

$$C_5^{②} =_{16} \frac{\Delta_4^4}{\Delta_2^3} + \widetilde{A}^{12}$$

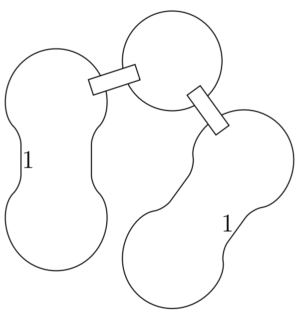
$$- \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})[\frac{\Delta_4}{\Delta_1}\frac{\Delta_4}{\Delta_2}\Delta_4 - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})(\frac{\Delta_4}{\Delta_3})^2\frac{\Delta_4}{\Delta_2}$$



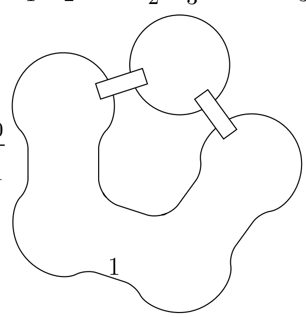
$$]$$

$$=_{16} \frac{\Delta_4^4}{\Delta_2^3} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})[\frac{\Delta_4^3}{\Delta_1\Delta_2} - \frac{\Delta_1\Delta_4^3}{\Delta_2^2\Delta_3^2}(1 + \frac{\Delta_1}{\Delta_3})\cdot$$

$($



$$- \frac{\Delta_0}{\Delta_1}$$



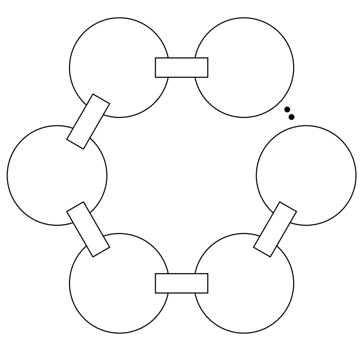
$)]$

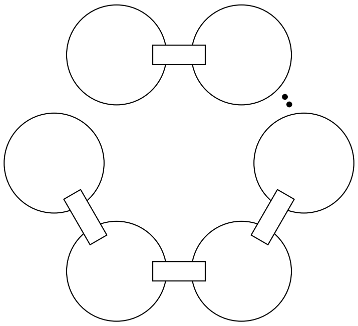
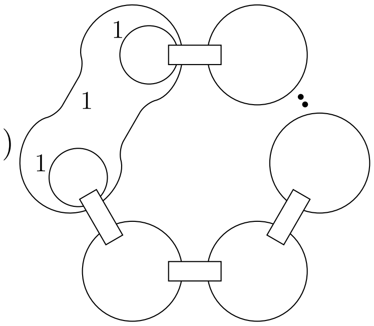
$$=_{16} \frac{\Delta_4^4}{\Delta_2^3} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1 + \frac{\Delta_1}{\Delta_3})[\frac{\Delta_4^3}{\Delta_1\Delta_2} - \frac{\Delta_4^3\Delta_1}{\Delta_2^2\Delta_3^2}(1 + \frac{\Delta_1}{\Delta_3})(\frac{\Delta_3}{\Delta_2}\Delta_3 - \frac{\Delta_0}{\Delta_1}\Delta_3)]$$

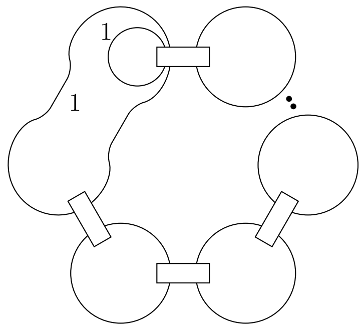
$$=_{16} 1 + 5\widetilde{A}^{12}.$$

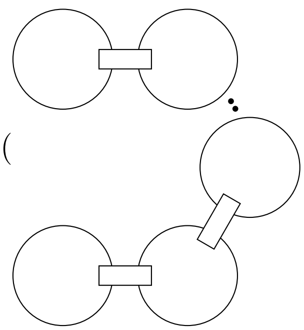
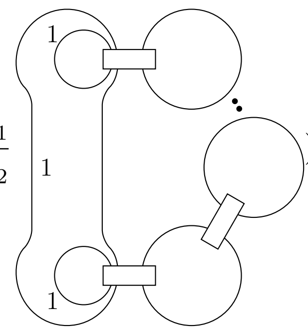
Similarly, we have for $n \geqslant 6$ that

$$C_n^{②} =$$

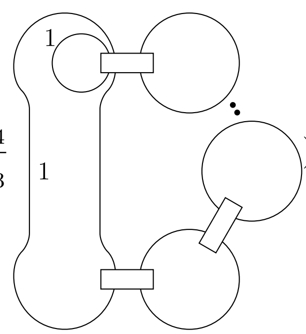


$=_{16}$  $+\widetilde{A}^{12}+\frac{\Delta_1}{\Delta_2}(1-\frac{\Delta_1}{\Delta_3})$ 

$= P^{\textcircled{2}}_{n-1}+\widetilde{A}^{12}-\frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}$ 

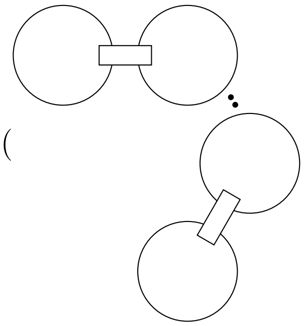
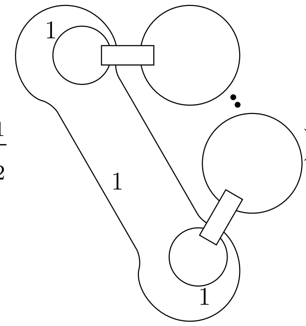
$= P^{\textcircled{2}}_{n-1}+\widetilde{A}^{12}-\frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}($  $-\frac{\Delta_1}{\Delta_2}$ 

$)$

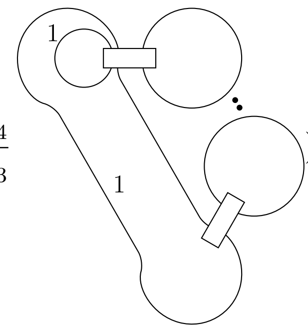
$= P^{\textcircled{2}}_{n-1}+\widetilde{A}^{12}-\frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}(P^{\textcircled{2}}_{n-2}-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$ 

$)$

$= P^{\textcircled{2}}_{n-1}+\widetilde{A}^{12}-\frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}$

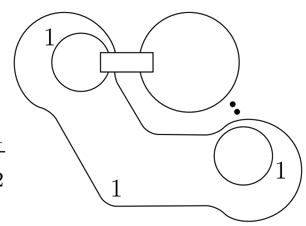
$[P^{②}_{n-2} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}($  $- \frac{\Delta_1}{\Delta_2}$ 

$)]$

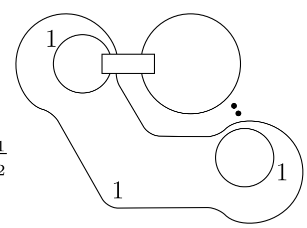
$= P^{②}_{n-1} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}[P^{②}_{n-2} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}(P^{②}_{n-3} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$ 

$)]$

$= P^{②}_{n-1} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}$

$[P^{②}_{n-2} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}(P^{②}_{n-3} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}(P^{②}_{n-4} - \frac{\Delta_1}{\Delta_2}$ 

$))];$

for the same reason, the degree of $\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}\frac{\Delta_1}{\Delta_2}$ 

is too high, hence we have

$$\begin{aligned}
C^{②}_n &=_{16} P^{②}_{n-1} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}[P^{②}_{n-2} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}(P^{②}_{n-3} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}P_{n-4})] \\
&= \frac{\Delta_4^{n-1}}{\Delta_2^{n-2}} + \widetilde{A}^{12} - \frac{\Delta_1}{\Delta_2}(1+\frac{\Delta_1}{\Delta_3})\frac{\Delta_4}{\Delta_3}[\frac{\Delta_4^{n-2}}{\Delta_2^{n-3}} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}(\frac{\Delta_4^{n-3}}{\Delta_3^{n-4}} - \frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}\frac{\Delta_4^{n-4}}{\Delta_3^{n-5}})] \\
&=_{16} 1 + n\widetilde{A}^{12}.
\end{aligned}$$

□

Therefore we have

Proposition 4.9. *If G_B of K is C_n defined above, we have*

$$r_1^{12} = \begin{cases} 3-(t_1+t_2) & \text{if } n=3 \\ 2n-(2t_1+t_2) & \text{if } n \geqslant 4 \end{cases}$$

where t_i denotes the number of edges in the B-graph which correspond to the negative twisted regions with exactly i crossings.

Proof. When $n=3$, $E=V=3$, $t_{1,N_3}=t_{1,D_3}=0$, $t_{1,Y_3}=t_1$, and we have that

$$\begin{aligned}
\widetilde{AJ}(1,K) &=_{16} \Gamma_{1;(1,\cdots,1)} + [t_{1,N_3}(E-V+1)+t_{1,Y_3}(E-V)+t_{1,D_3}(E-V-1)+t_2]\widetilde{A}^{12} \\
&=_{16} C_3^{\textcircled{1}} + [t_{1,N_3}(E-V+1)+t_{1,Y_3}(E-V)+t_{1,D_3}(E-V-1)+t_2]\widetilde{A}^{12} \\
&=_{16} 1+2A^8-A^{12}+t_2A^{12} \\
&=_{16} 1+2A^8+(t_2-1)A^{12}.
\end{aligned}$$

Also we know from [2] that

$$\widetilde{AJ}(2,K) =_{16} \Gamma_{2;(2,\cdots,2)} - t_1\widetilde{A}^{12} = C_3^{\textcircled{2}} - t_1\widetilde{A}^{12} =_{16} 1-A^8+(2-t_1)A^{12}.$$

Hence we have

$$r_1^{12} = (2-t_1)-(t_2-1) = 3-(t_1+t_2).$$

When $n = 4$, $E = V = 4$, $t_{1,Y_3} = t_{1,D_3} = 0$, and $t_{1,N_3} = t_1$. Hence we have

$$\begin{aligned}\widetilde{AJ}(1,K) &=_{16} \Gamma_{1;(1,\cdots,1)} + [t_{1,N_3}(E-V+1) + t_{1,Y_3}(E-V) + t_{1,D_3}(E-V-1) + t_2]\widetilde{A}^{12} \\ &=_{16} C_4^{\textcircled{2}} + [t_{1,N_3}(E-V+1) + t_{1,Y_3}(E-V) + t_{1,D_3}(E-V-1) + t_2]\widetilde{A}^{12} \\ &=_{16} 1 + 4A^8 - 3A^{12} + (t_1+t_2)A^{12} \\ &=_{16} 1 + 4A^8 + (t_1+t_2-3)A^{12}\end{aligned}$$

and

$$\widetilde{AJ}(2,K) =_{16} \Gamma_{2;(2,\cdots,2)} - t_1\widetilde{A}^{12} = C_4^{\textcircled{2}} - t_1\widetilde{A}^{12} =_{16} 1 + (5-t_1)A^{12},$$

therefore

$$r_1^{12} = (5-t_1) - (t_1+t_2-3) = 8 - (2t_1+t_2).$$

As for $n \geqslant 5$, $E = V = n$, $t_{1,Y_3} = t_{1,D_3} = 0$, and $t_{1,N_3} = t_1$. Similar calculation shows that

$$\begin{aligned}\widetilde{AJ}(1,K) &=_{16} \Gamma_{1;(1,\cdots,1)} + [t_{1,N_3}(E-V+1) + t_{1,Y_3}(E-V) + t_{1,D_3}(E-V-1) + t_2]\widetilde{A}^{12} \\ &=_{16} C_n^{\textcircled{2}} + [t_{1,N_3}(E-V+1) + t_{1,Y_3}(E-V) + t_{1,D_3}(E-V-1) + t_2]\widetilde{A}^{12} \\ &=_{16} 1 + nA^8 - nA^{12} + (t_1+t_2)A^{12} \\ &=_{16} 1 + nA^8 + (t_1+t_2-n)A^{12}\end{aligned}$$

and

$$\widetilde{AJ}(2,K) =_{16} \Gamma_{2;(2,\cdots,2)} - t_1\widetilde{A}^{12} = C_n^{\textcircled{2}} - t_1\widetilde{A}^{12} =_{16} 1 + (n-t_1)A^{12},$$

hence

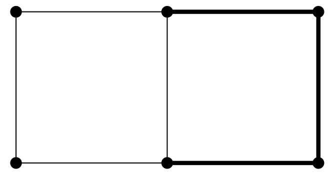


Figure 4.4: A chain in G_B for some link K. The thickened line represents a chain.

$$r_1^{12} = (n - t_1) - (t_1 + t_2 - n) = 2n - (2t_1 + t_2).$$

□

With all these preparations, now we can discuss a more general case, which gives the main result in this section. To state the result, first we need to define what a "chain" is.

Definition 4.10. *A* chain *in a graph G is a non-loop path P_n in it with $n \geqslant 3$, where the two endpoints of it have valency strictly greater than 2 and all other vertices on it have valency 2.*

Remark 4.11. *The subgraph obtained from detaching the first few chains will always be at least 2-connected, i.e. we can not change this subgraph into 2 graphs which are not connected to each other by removing just 1 vertex. This property is guaranteed by the definition of the chain.*

Definition 4.12. *The class $\mathcal{C}$ of alternating links that we consider can be defined recursively: all links with a B-graph G_B that is a polygon with at least 5 edges are in $\mathcal{C}$. If removing a chain from a B-graph G_B of K yields an element of $\mathcal{C}$ then K is in $\mathcal{C}$.*

Here comes

Theorem 4.13 (Main Theorem 2)**.** *Suppose K is in $\mathcal{C}$. Then*

$$r_1^{12} = (E - t_1)(1 + F) + F(F - 1) - t_2.$$

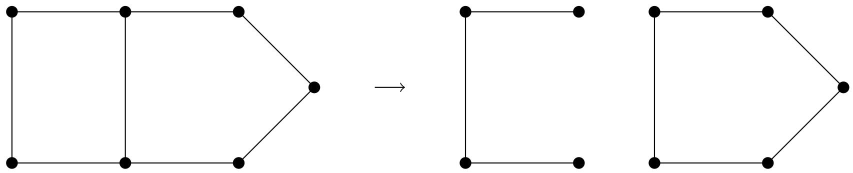


Figure 4.5: A possible decomposition of a given G_B for some K in $\mathcal{C}$. Note that if G_B is decomposed into P_4 and C_5 then by definition this K is not in $\mathcal{C}$.

To prove Theorem 4.13, we need some lemmas and corollaries. Also in this section, we abuse the notation G_s to represent both the state graph and the corresponding $\Gamma_{n;(n,\cdots,n)}$ if it won't cause any confusion.

Lemma 4.14. *Assume K is in $\mathcal{C}$; then*

$$\Gamma_{1;(1,\cdots,1)} =_{16} 1-(F-1)A^4+[E+\frac{1}{2}F(F-1)]A^8 + [R-EF-\frac{F}{6}(F^2-1)]A^{12}.$$

Proof. Assume all strands in this proof are colored by 1, and the polygon C obtained in the decomposition of G_B has length n ($n>4$) and all faces in G_B have at least 5 edges. Denote E_1,E_2,$\cdots$,E_m the chains in the decomposition of G_B where E_{i+1} is removed after E_i, the length of E_i e_i, and G_{K_i} the corresponding graph after removing E_1,$\cdots$,E_i. Therefore $C = G_{K_m}$ by this notation. We first show that in this case the formula is given by

$$\Gamma_{1;(1,\cdots,1)} =_{16} 1-(F-1)A^4+[E+\frac{1}{2}F(F-1)]A^8 + [-EF-\frac{F}{6}(F^2-1)]A^{12}.$$

Let us focus on one general chain E_i in $G_{K_{i-1}}$. We start with the case $e_i=3$.

Using the recursive relation 2.1 repeatedly, we have

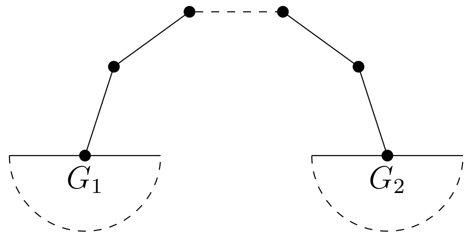
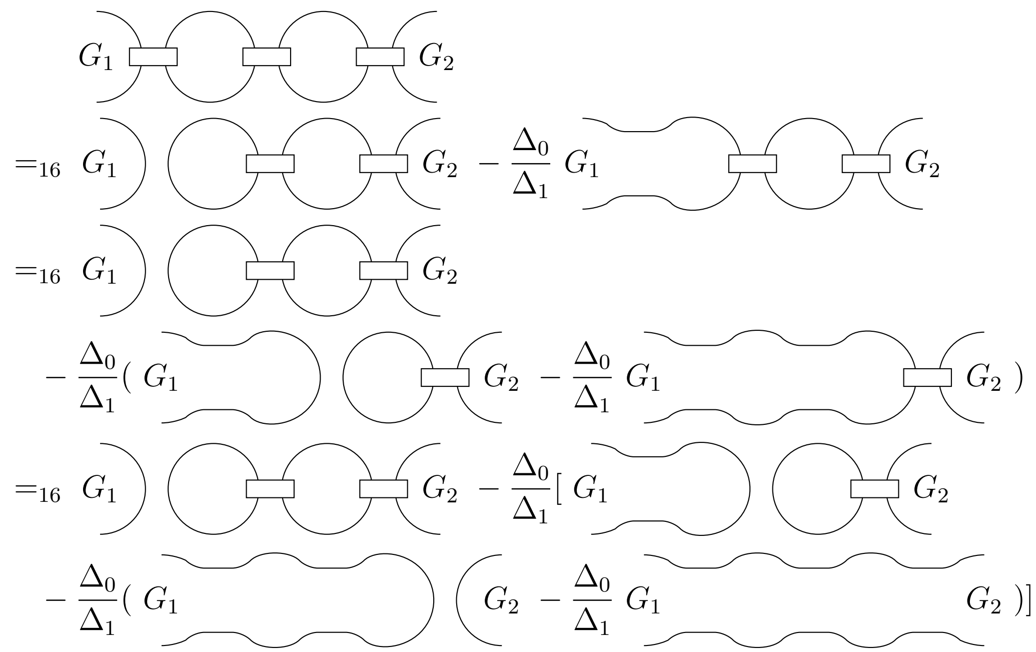
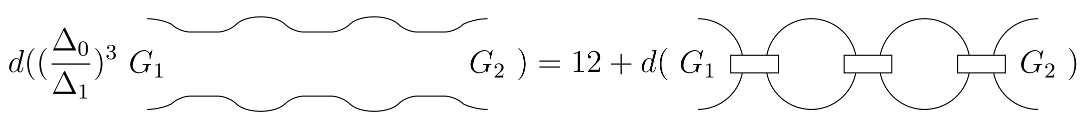


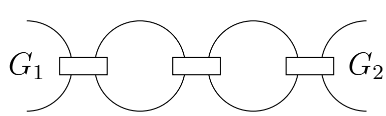
Figure 4.6: How a chain in the decomposition of G_B appears in G_B. The valency for both endpoints needs to be higher than 3. Note that G_1 and G_2 are actually connected to each other.



Note that



we then have



$$=_{16} (\frac{\Delta_2}{\Delta_1})^2 G_{K_i} - \frac{\Delta_0}{\Delta_1}\frac{\Delta_2}{\Delta_1} G_{K_i} + (\frac{\Delta_0}{\Delta_1})^2 G_{K_i} - \widetilde{A}^{12}$$

$$=_{16} (\frac{\Delta_2}{\Delta_1})^2 [1 - \frac{\Delta_0}{\Delta_2} + (\frac{\Delta_0}{\Delta_2})^2] G_{K_i} - \widetilde{A}^{12}.$$

As for the case $e_i > 3$, similar calculation for that $\Gamma_{1;(1,\cdots,1)}$ still applies and gives the same result, as those extra edges in the chain will not contribute to the coefficient of $\widetilde{A}^{12}$ due to the fact that the degree of those extra terms are too high. We now have

$$G_{K_{i-1}} =_{16} (\frac{\Delta_2}{\Delta_1})^{e_i-1}[1-\frac{\Delta_0}{\Delta_2}+(\frac{\Delta_0}{\Delta_2})^2]G_{K_i}-\widetilde{A}^{12}.$$

We can now prove the formula by induction on the number of faces F, which is also $m+1$.

If $F=1$, $G_B=C$. Lemma 4.4 says that $C_n^{②} =_{16} 1+nA^8-nA^{12}$ if $n \geqslant 5$ which satisfies the conclusion.

Now assume it is true for $F=k$. When $F=k+1$, we have from the result above that

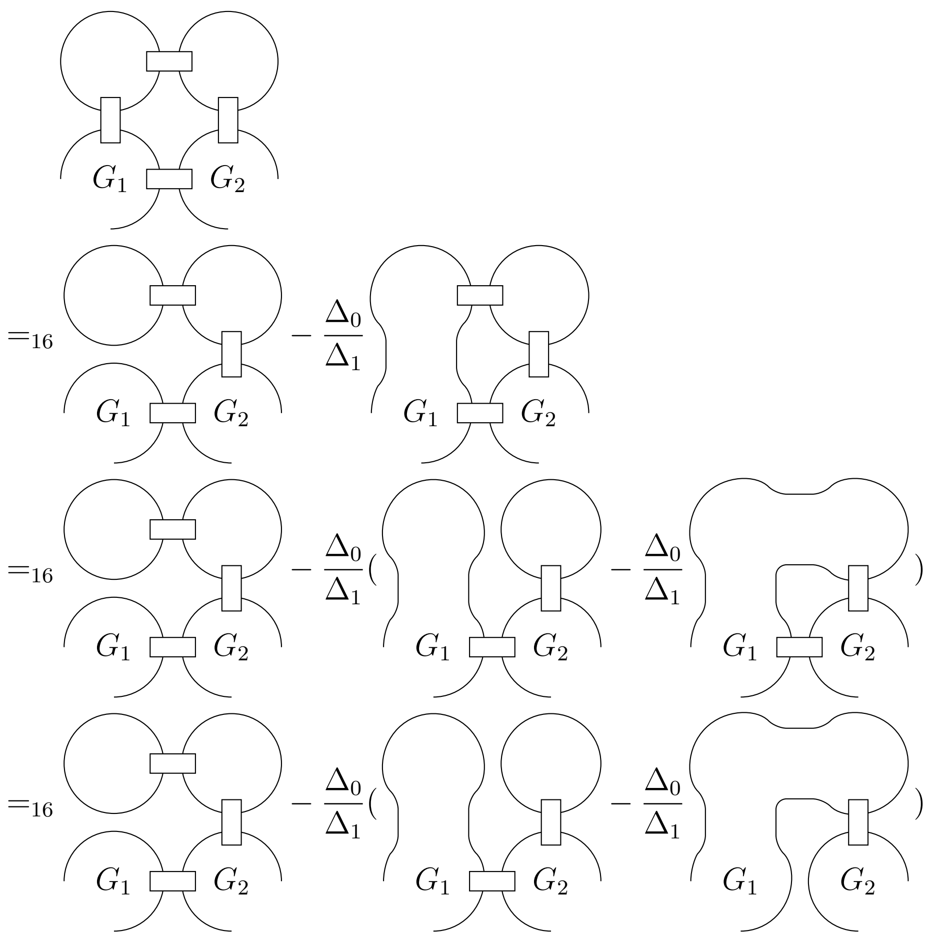
$$G_K =_{16} (\frac{\Delta_2}{\Delta_1})^{e_1-1}[1-\frac{\Delta_0}{\Delta_2}+(\frac{\Delta_0}{\Delta_2})^2]G_{K_1}-\widetilde{A}^{12}.$$

Note that G_{K_1} has only k faces, and $d(G_{K_1}) = d(G_K)+2(e_1-1)$. By assumption, we have

$$\begin{aligned}
G_K &=_{16} (\frac{\Delta_2}{\Delta_1})^{e_1-1}[1-\frac{\Delta_0}{\Delta_2}+(\frac{\Delta_0}{\Delta_2})^2]G_{K_1}-\widetilde{A}^{12} \\
&=_{16} (\frac{\Delta_2}{\Delta_1})^{e_1-1}[1-\frac{\Delta_0}{\Delta_2}+(\frac{\Delta_0}{\Delta_2})^2] \\
&\quad A^{-2e_1+2}(\widetilde{A}^0-(k-1)\widetilde{A}^4+[E+\frac{1}{2}k(k-1)]\widetilde{A}^8+[-kE-\frac{k}{6}(k^2-1)]\widetilde{A}^{12})-\widetilde{A}^{12} \\
&=_{16} 1-(k)A^4+[E+\frac{1}{2}k(k+1)]A^8+[-(k+1)E-\frac{k+1}{6}((k+1)^2-1)]A^{12} \\
&=_{16} 1-(F-1)A^4+[E+\frac{1}{2}F(F-1)]A^8+[-EF-\frac{F}{6}(F^2-1)]A^{12},
\end{aligned}$$

which completes the induction.

Now let us consider the general case that there are faces with only 4 edges in G_B. Since G_B should satisfy the needed condition, locally it has to be surrounded by a chain with 3 edges and a single edge. Without loss of generality, consider the one partially bounded by E_1. Note that $e_1 = 3$. Still use the recursive relation 2.1 and we have



$$=_{16} (\frac{\Delta_2}{\Delta_1})^2 G_{K_1} - \frac{\Delta_2}{\Delta_1}\frac{\Delta_0}{\Delta_1} G_{K_1} + (\frac{\Delta_0}{\Delta_1})^2 G_{K_1}$$

$$=_{16} (\frac{\Delta_2}{\Delta_1})^2 G_{K_1}[1 - \frac{\Delta_0}{\Delta_2} + (\frac{\Delta_0}{\Delta_2})^2],$$

which indicates that each 4-face will introduce an extra $\widetilde{A}^{12}$; therefore we have in general that

$$\Gamma_{1;(1,\cdots,1)} =_{16} 1 - (F-1)A^4 + [E + \frac{1}{2}F(F-1)]A^8$$
$$+ [R - EF - \frac{F}{6}(F^2 - 1)]A^{12}.$$

□

Corollary 4.15. *Assume K is in $\mathcal{C}$; then*

$$\widetilde{AJ}(1,K) =_{16} 1-(F-1)A^4+[E+\frac{1}{2}F(F-1)]A^8 + [R-EF-\frac{F}{6}(F^2-1)+t_1F+t_2]A^{12}.$$

Proof. Recall from Theorem 4.1,

$$\widetilde{AJ}(1,K) =_{16} \Gamma_{1;(1,\cdots,1)} + [t_{1,N_3}(E-V+1)+t_{1,Y_3}(E-V)+t_{1,D_3}(E-V-1)+t_2]\widetilde{A}^{12}.$$

Lemma 4.14 tells us that

$$\Gamma_{1;(1,\cdots,1)} =_{16} 1-(F-1)A^4+[E+\frac{1}{2}F(F-1)]A^8 + [R-EF-\frac{F}{6}(F^2-1)]A^{12}.$$

Also we note that all faces in G_B have at least 4 edges and hence $t_{1,Y_3}=t_{1,D_3}=0$ and $t_{1,N_3}=t_1$. Besides, $E-V+1=F$ under our notation. Combining all these information, we achieve the conclusion.

□

Lemma 4.16. *Assume K is in $\mathcal{C}$; then*

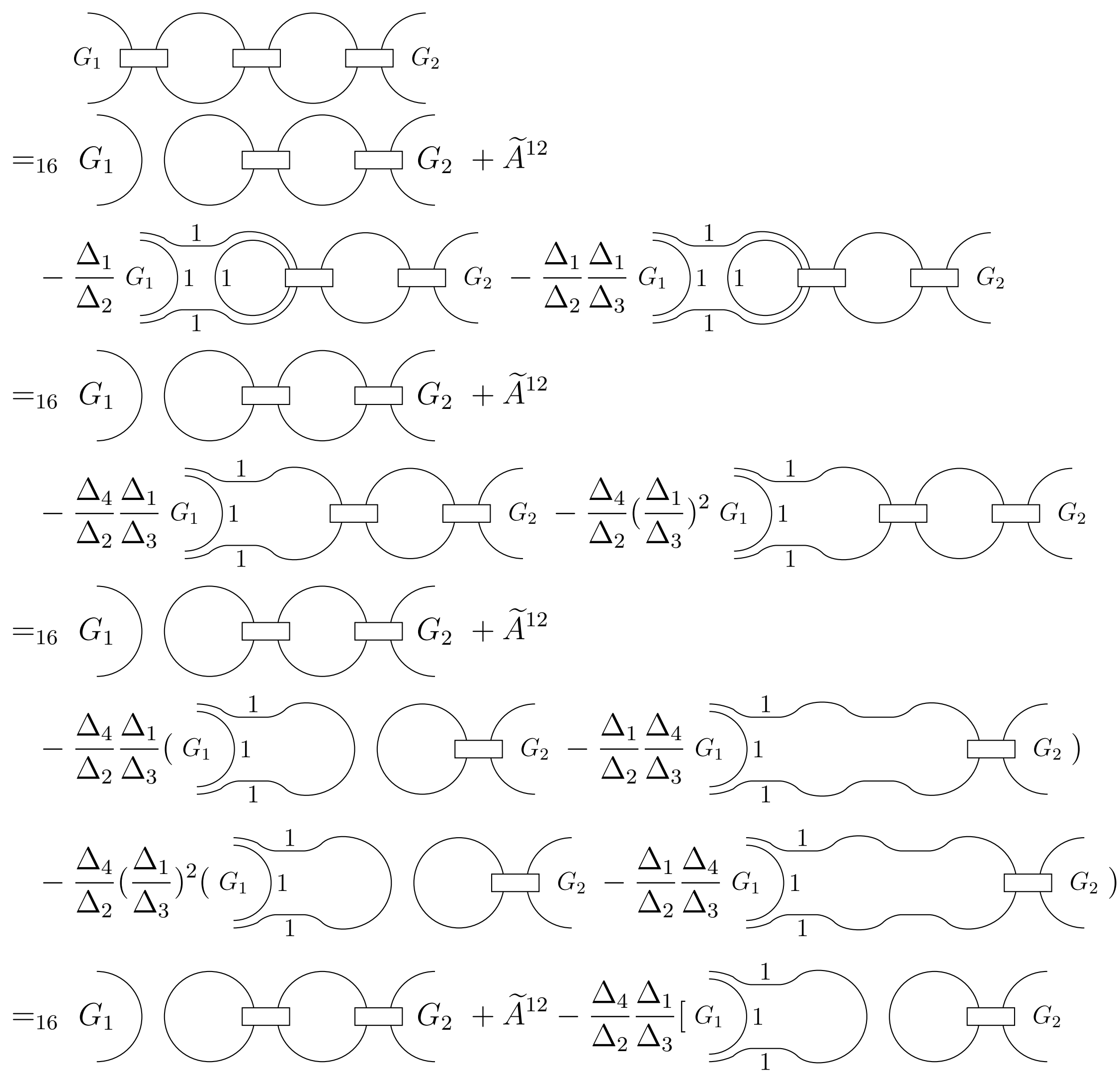
$$\Gamma_{2;(2,\cdots,2)} =_{16} 1-(F-1)A^4+\frac{1}{2}(F-2)(F-1)A^8 + [R+E-\frac{F}{6}(F-1)(F-5)]A^{12}.$$

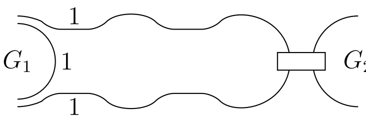
Proof. The proof of this lemma is very similar to the one of Lemma 4.14. Here we assume all unlabelled strands are colored by 2. Same notations used in Lemma 4.14 will be used here as well. Still, let us first assume the polygon C obtained in the decomposition of G_B

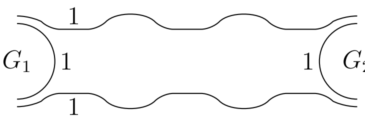
has length n ($n > 4$) and all faces in G_B have at least 5 edges. In this case the formula is given by

$$\begin{aligned}\Gamma_{2;(2,\cdots,2)} &=_{16} 1 - (F-1)A^4 + \frac{1}{2}(F-2)(F-1)A^8 \\ &+ [E - \frac{F}{6}(F-1)(F-5)]A^{12}.\end{aligned}$$

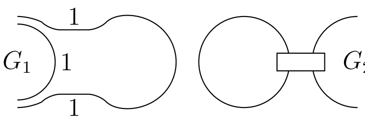
We start with the case $e_i = 3$ for one general chain E_i in $G_{K_{i-1}}$. Using both the recursive relation 2.1 and Lemma 4.6 leads to

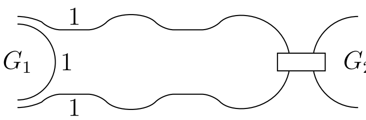


$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}($$ 

$$-\frac{\Delta_1}{\Delta_2}$$ 

$$)]$$

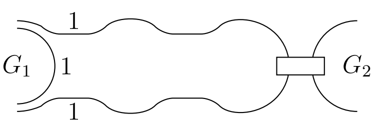
$$-\frac{\Delta_4}{\Delta_2}(\frac{\Delta_1}{\Delta_3})^2($$ 

$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$$ 

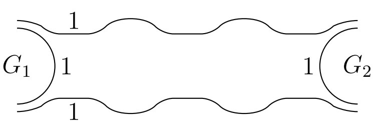
$$)$$

$$=_{16} (\frac{\Delta_4}{\Delta_2})^2 G_{K_i}[1-\frac{\Delta_1}{\Delta_3}+(\frac{\Delta_1}{\Delta_3})^2-(\frac{\Delta_1}{\Delta_3})^2+\widetilde{A}^{12}-\widetilde{A}^{12}+\widetilde{A}^{12}$$
$$=_{16} (\frac{\Delta_4}{\Delta_2})^2 G_{K_i}(1-\frac{\Delta_1}{\Delta_3})+\widetilde{A}^{12};$$

here in the calculation, both

$$\frac{\Delta_4}{\Delta_2}(\frac{\Delta_1}{\Delta_3})^2\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$$ 

and

$$(\frac{\Delta_4}{\Delta_2})^2(\frac{\Delta_1}{\Delta_3})^2\frac{\Delta_1}{\Delta_2}$$ 

contribute just 1 $\widetilde{A}^{12}$.

As for the case $e_i > 3$, a similar calculation as for $\Gamma_{2;(2,\cdots,2)}$ still applies and gives the same result, since those extra edges in the chain will not contribute to the coefficient of $\widetilde{A}^{12}$ due to the fact that the degree of those extra terms are too high. We now have

$$G_{K_{i-1}} =_{16} (\frac{\Delta_4}{\Delta_2})^{e_i-1} G_{K_i}(1-\frac{\Delta_1}{\Delta_3})+\widetilde{A}^{12}.$$

We again apply induction on the number of faces F, which is also $m+1$.

If $F=1$, $G_B = C$. Lemma 4.7 says that $C_n^{②} =_{16} 1+nA^{12}$ if $n \geqslant 5$ which satisfies the conclusion.

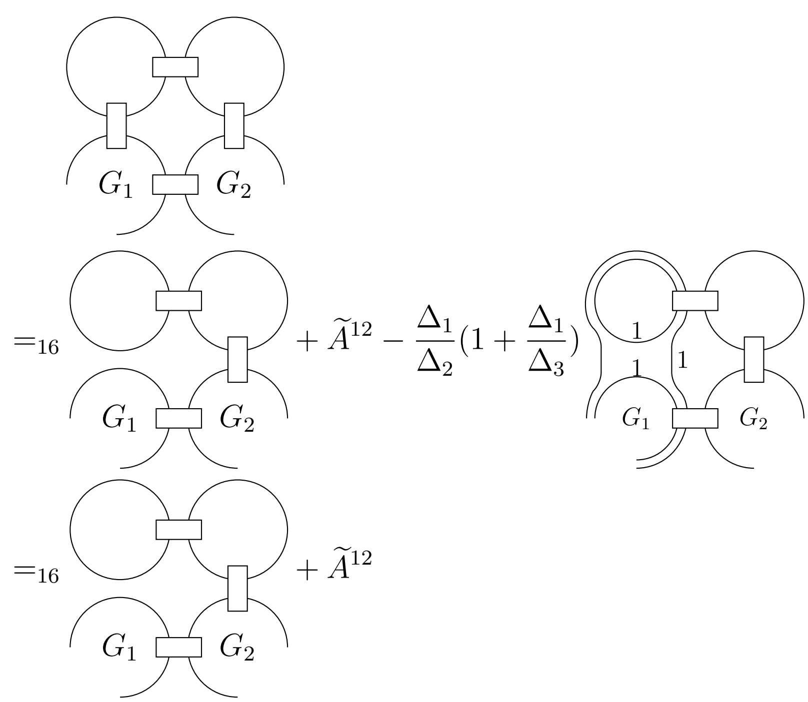
Now assume it is true for $F=k$. When $F=k+1$, we have from the result above that

$$G_K =_{16} (\frac{\Delta_4}{\Delta_2})^{e_i-1} G_{K_1}(1-\frac{\Delta_1}{\Delta_3})+\widetilde{A}^{12}.$$

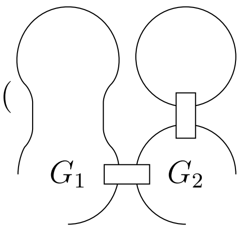
Note that G_{K_1} has only k faces, and $d(G_{K_1}) = d(G_K) + 4(e_1 - 1)$. By assumption, we have

$$
\begin{aligned}
G_K &=_{16} (\frac{\Delta_4}{\Delta_2})^{e_i-1} G_{K_1}(1 - \frac{\Delta_1}{\Delta_3}) + \widetilde{A}^{12} \\
&=_{16} (\frac{\Delta_4}{\Delta_2})^{e_i-1} \\
&\quad A^{-4e_1+4}(1 - (k-1)A^4 + \frac{1}{2}(k-2)(k-1)A^8 \\
&\quad + [E - \frac{k}{6}(k-1)(F-5)]A^{12})(1 - \frac{\Delta_1}{\Delta_3}) + \widetilde{A}^{12} \\
&=_{16} 1 - (F-1)A^4 + \frac{1}{2}(F-2)(F-1)A^8 + [E - \frac{F}{6}(F-1)(F-5)]A^{12}.
\end{aligned}
$$

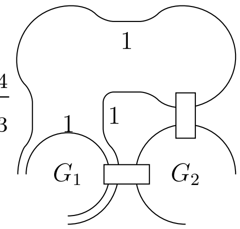
As the general case that there are faces with only 4 edges in G_B, without loss of generality, consider the one partially bounded by E_1. Note that $e_1 = 3$. We have



$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}($$

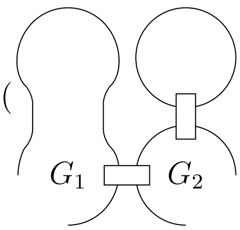


$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$$

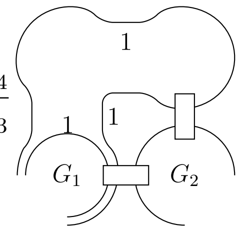


$$)$$

$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_1}{\Delta_3}\frac{\Delta_4}{\Delta_3}($$



$$-\frac{\Delta_1}{\Delta_2}\frac{\Delta_4}{\Delta_3}$$



$$)$$

$$
\begin{aligned}
&=_{16} (\frac{\Delta_4}{\Delta_2})^2 G_{K_1} + \widetilde{A}^{12} - \frac{\Delta_4}{\Delta_2}\frac{\Delta_1}{\Delta_3}\frac{\Delta_4}{\Delta_2} G_{K_1} \\
&\quad + (\frac{\Delta_4}{\Delta_2})^2 (\frac{\Delta_1}{\Delta_3})^2 G_{K_1} - (\frac{\Delta_4}{\Delta_2})^2 (\frac{\Delta_1}{\Delta_3})^2 G_{K_1} + (\frac{\Delta_4}{\Delta_2})^2 (\frac{\Delta_1}{\Delta_3})^3 G_{K_1} \\
&=_{16} (\frac{\Delta_4}{\Delta_2})^2 G_{K_1} (1 - \frac{\Delta_1}{\Delta_3}) + \widetilde{A}^{12} + \widetilde{A}^{12} \\
&=_{16} (\frac{\Delta_4}{\Delta_2})^2 G_{K_1} (1 - \frac{\Delta_1}{\Delta_3}) + 2\widetilde{A}^{12},
\end{aligned}
$$

which indicates that each 4-face will introduce an extra $\widetilde{A}^{12}$; therefore we have in general that

$$
\begin{aligned}
\Gamma_{2;(2,\cdots,2)} =_{16} \; & 1 - (F-1)A^4 + \frac{1}{2}(F-2)(F-1)A^8 \\
& + [R + E - \frac{F}{6}(F-1)(F-5)]A^{12}.
\end{aligned}
$$

□

Corollary 4.17. *Assume K is in $\mathcal{C}$; then*

$$
\begin{aligned}
\widetilde{AJ}(2,K) =_{16} \; & 1 - (F-1)A^4 + \frac{1}{2}(F-2)(F-1)A^8 \\
& + [R + E - \frac{F}{6}(F-1)(F-5) - t_1]A^{12}.
\end{aligned}
$$

Proof of Theorem 4.13. With Corollary 4.15 and Corollary 4.17, Theorem 4.13 follows immediately.

□

4.3 Examples

In this section we give examples to illustrate Theorem 4.13.

Example 4.18. *Here the knots are presented by a Gauss code.*

1. *For the Gauss code given by*

$$[1,-2,3,-4,5,-6,7,-8,9,-3,4,-10,8,-7,6,-11,-1,2,-9,10,-5,11]$$

with $t_1 = E$*;* $E = 11$*,* $F = 3$ *and* $V = 9$*.*

Bar-Natan's Mathematica package KnotTheory [3] computes:

$$\begin{aligned} \widetilde{AJ}(1,K) &= \sqrt{\frac{1}{A^4}}(1-2A^4+3A^8-3A^{12}+3A^{16}+\cdots) \\ \widetilde{AJ}(2,K) &= \quad 1-2A^4+A^8+3A^{12}+\cdots, \end{aligned}$$

and hence $r_1^{12} = 3-(-3) = 6$*.*

Theorem 4.13 verifies that $r_1^{12} = F(F-1) = 3(3-1) = 6$*.*

2. *For the Gauss code given by* $[1,-2,3,-4,5,-6,7,-3,8,-1,2,-7,6,-5,4,-8]$ *with* $t_1 = E$*;* $E = 8$*,* $F = 2$ *and* $V = 7$ *the KnotTheory package computes:*

$$\begin{aligned} \widetilde{AJ}(1,K) &= 1-A^4+A^8-A^{24}+\cdots \\ \widetilde{AJ}(2,K) &= 1-A^4+2A^{12}+\cdots, \end{aligned}$$

and hence $r_1^{12} = 2-0 = 2$*.*

Theorem 4.13 verifies that $r_1^{12} = F(F-1) = 2(2-1) = 2$*.*

3. Suppose the Gauss code is given by

$$[1,-2,3,-4,5,-6,7,-8,4,-3,9,-1,2,-5,6,-7,8,-9]$$

where the crossings 3 *and* 4 *create a negative twisted region with* 2 *crossings. Hence* $E=9,\ F=2,\ V=7,\ t_2=1$ *and* $t_1=7$

then

$$\widetilde{AJ}(1,K) = 1 - A^4 + 2A^8 - A^{12} + \cdots$$
$$\widetilde{AJ}(2,K) = 1 - A^4 + 3A^{12} + \cdots,$$

and hence $r_1^{12} = 3-(-1) = 4$.

Theorem 4.13 verifies this: $r_1^{12} = (E-t_1)(1+F)+F(F-1)-t_2 = (8-7)(1+2)+2(2-1)-1 = 4$.

4.4 Application to Hyperbolic Volume

Let L be a link in the three sphere S^3. We say L is hyperbolic if its link complement has a hyperbolic structure with a finite volume. Kashaev introduced a complex number valued link invariant by using quantum dilogarithm in [9]. He also observed that this invariant is a quantum generalization of the hyperbolic volume. Murakami and Murakami pointed out later in [13] that the invariant in Kashaev's paper turns out to be an evaluation of the colored Jones polynomial. Their result links the study of the colored Jones polynomial and the hyperbolic volume of link complements.

Since the accurate volume function in general is hard to find for an arbitrary hyperbolic link, the bounds for the hyperbolic volume are usually studied. In [7] Oliver Dasbach and Anastasiia Tsvietkova gave the following result

Theorem 4.19 (Theorem 2.3 in [7]). *Given a diagram D of a hyperbolic alternating link K; denote the number of twists that have exactly i crossings by $t_i(D)$, and the number of twists that have at least i crossings by $g_i(D)$. Let v_3 be the volume of a regular ideal hyperbolic tetrahedron. Then $Vol(S^3 - K) \leqslant (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - a)v_3$, where $a = 10$ if g_4 is non-zero, $a = 7$ if t_3 is non-zero, and $a = 6$ otherwise.*

They then bound the volume in terms of coefficients of the colored Jones polynomial and give

Theorem 4.20 (Theorem 3.3 in [7]). *Let K be an alternating, prime, non-torus link, and let*

$$J_K(n) = \pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \cdots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n}) \quad (4.1)$$

be the colored Jones polynomial of K, where a_n and α_n are positive. Then

$$Vol(S^3 - K) \leqslant (6((c_2 + \gamma_2) - (c_3 + \gamma_3)) - 2(b_2 + \beta_2) - a)v_3 \leqslant 10(b_2 + \beta_2 - 1)v_3,$$

where $a = 10$ if $b_2 + \beta_2 \neq (c_2 - c_3) + (\gamma_2 - \gamma_3)$ and $a = 4$ otherwise.

Note that in Theorem 4.13, the formula involves t_2. Hence we can represent t_2 as

$$t_2 = (E - t_1)(1 + F) + F(F - 1) - r_1^{12}.$$

Recall that Dasbach and Lin proved

Theorem 4.21 (Theorem 4.4 in [5]). *Let K be an alternating knot. Write*

$$J'_K(n) = \pm\,(a_n A^{k_n} - b_n A^{k_n-4} + c_n A^{k_n-8}) \pm \cdots$$
$$\pm\,(\gamma_n A^{k_n-4r_n+8} - \beta_n A^{k_n-4r_n+4} + \alpha_n A^{k_n-4r_n})$$

with positive a_n and α_n.

Let $A(D)$ and $B(D)$ be the A- and B-graphs of a reduced alternating diagram D of K with crossing number c. The reduced graphs $A(D)'$ and $B(D)'$ have e_A and e_B edges and

v_A and v_B vertices. Note, that $v_A + v_B = c + 2$. Furthermore, there are τ_A and τ_B triangles in $A(D)'$ and $B(D)'$.

Then $\beta_n = e_B - v_B + 1$.

Also recall the immediate proposition following from the results in [5]

Proposition 4.22 (Proposition 3.1 in [7]). *Let the colored Jones polynomial be given in the form as in Equation 4.1. Then*

$$\begin{aligned} b_2 + \beta_2 &= t(D) \\ &= t_1(D) + g_2(D) \\ (c_2 + \gamma_2) - (c_3 + \gamma_3) &= t(D) + g_2(D) \\ &= t_1(D) + 2g_2(D) \end{aligned}$$

Note that F defined in this thesis is actually $F = e_B - v_B + 1 = \beta_n = \beta_2$ and hence $t_2 = (E - t_1)(1 + \beta_2) + \beta_2(\beta_2 - 1) - r_1^{12}$. Also $\widetilde{J}(n, K) = \Delta_n J_K(n + 1)$ implies that $r_1^{12} = (-b_3 + c_3 - d_3) - (c_2 - d_2)$. Combining these results with Theorem 4.19 and Proposition 4.22, we have

Theorem 4.23. *Let K be an alternating, prime, non-torus link in $\mathcal{C}$, and let*

$$J_K(n) = \pm(a_n q^{k_n} - b_n q^{k_n - 1} + c_n q^{k_n - 2}) + \cdots \pm (\gamma_n q^{k_n - r_n + 2} - \beta_n q^{k_n - r_n + 1} + \alpha_n q^{k_n - r_n})$$

be the colored Jones polynomial of K, where a_n and α_n are positive. Then

$$\begin{aligned} Vol(S^3 - K) \leqslant \{&(2 - 4\beta_2)[(c_2 + \gamma_2) - (c_3 + \gamma_3)] + (2 + 4\beta_2)(b_2 + \beta_2) \\ &+ \beta_2(\beta_2 - 1) - [(-b_3 + c_3 - d_3) - (c_2 - d_2)] - a\}v_3, \end{aligned}$$

where $a = 10$ if $b_2 + \beta_2 \neq (c_2 - c_3) + (\gamma_2 - \gamma_3)$ and $a = 4$ otherwise.

References

[1] C. C. Adams, *The Knot Book: an elementary introduction to the mathematical theory of knots*, W. H. Freeman and Company, New York, 1994.

[2] C. Armond, *The head and tail conjecture for alternating knots*, arXiv:1112.3995v1.

[3] D. Bar-Natan, and S. Morrison, and et al., *The Knot Atlas*, http://katlas.org.

[4] P. R. Cromwell, *Knots and Links*, Cambridge University Press, 2004.

[5] O. Dasbach, and X.-S. Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math., 142 (2006), No. 5, 1332-1342.

[6] O. Dasbach, and X.-S. Lin, *A Volum-ish Theorem for the Jones polynomial of alternating knots*, Pacific J. Math., 231 (2007), No. 2, 279-291.

[7] O. Dasbach, and A. Tsvietkova, *A refined upper bound for the hyperbolic volume of alternating links and the colored Jones polynomial*, Math Res. Lett., 22 (2015), No. 4, 1047-1060.

[8] S. Garoufalidis, and T. T. Q. Le, *Nahm sums, stability and the colored Jones polynomial*, arxiv1112.3905v1.

[9] R. M. Kashaev, *The Hyperbolic Volume of Knots from the Quantum Dilogarithm*, Letters in Mathematical Physics, 39 (1997), 269-275.

[10] L. H. Kauffman, *On Knots*, Princeton University Press, Princeton, New Jersey, 1987.

[11] W. B. R. Lickorish, *An Introduction to Knot Theory*, Springer, 1997.

[12] G. Masbaum, and P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J. Math., (164) 1994, No. 2, 361-381.

[13] H. Murakami, and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Mathematica, 186 (2001), 85-104.

[14] J. H. Przytycki, *Fundamentals of Kauffman bracket skein module*, Kobe J. Math. 16 (1999), No. 1, 45-66.

[15] K. Walsh, *Higher order stability in the coefficients of the colored Jones polynomial*, arxiv:1603.06957v1.

[16] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci., Canada, IX (1987), 5-9.

Vita

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