Louisiana State University
LSU Digital Commons

LSU Historical Dissertations and Theses | Graduate School

1972

Filters Based on Ultraspherical Rational Functions.

Mohsen David Kashefi
Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_disstheses

Recommended Citation

Kashefi, Mohsen David, "Filters Based on Ultraspherical Rational Functions." (1972). *LSU Historical Dissertations and Theses*. 2221.
https://digitalcommons.lsu.edu/gradschool_disstheses/2221

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.

INFORMATION TO USERS

This dissertation was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

University Microfilms

300 North Zeeb Road
Ann Arbor, Michigan 48106

A Xerox Education Company

72-28,356

KASHEFI, Mohsen David, 1930-
FILTERS BASED ON ULTRASPHERICAL RATIONAL FUNCTIONS.

The Louisiana State University and Agricultural and Mechanical College, Ph.D., 1972
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

FILTERS BASED ON ULTRASPHERICAL RATIONAL FUNCTIONS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Electrical Engineering

by
Mohsen David Kashefi
B.S., Louisiana State University, 1963
M.S., Louisiana State University, 1965
May, 1972

PLEASE NOTE:

Some pages may have

indistinct print.

Filmed as received.

University Microfilms, A Xerox Education Company

ACKNOWLEDGMENT

Grateful acknowledgment is expressed to Dr. David E. Johnson and Dr. Johnny R. Johnson for their patient and valuable guidance. Special gratitude is expressed to Dr. Leonard C. Adams for his encouragement and support. Also, appreciation is expressed to the members of the examining committee and the faculty of Electrical Engineering.

I wish to sincerely thank my wife, Linda, for her understanding and patience, and Miss Helen Chisholm who expertly typed this manuscript.

ABSTRACT

By varying the parameters a and b of the ultraspherical rational function, one may obtain the Chebyshev rational function for a given n and modulus k used in the magnitude function of the elliptic filter. In addition, these parameters may be varied to obtain a variety of other low-pass filters, band-pass filters, and filters with constant magnitude.

CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

The optimal approximation to the ideal low-pass filter that is realizable is the elliptic filter, which is described in Chapter II. The development of the elliptic filter to fit a set of prescribed tolerances requires the calculation of a modulus k and a certain amount of knowledge of the properties of the Jacobi elliptic functions, which are rather complex functions, to say the least.

In this development a generalization of the elliptic filter is considered which allows one to find directly the optimum filter for a given set of specifications and, moreover, the mathematics involved avoids the use of the elliptic functions altogether. The variation of the two parameters involved in the generalization allows one to obtain a wide variety of filters other than the elliptic filter. Some of these filters are entirely different in nature from the low-pass filter from which they come. For example, in addition to low-pass filters, the generalization includes high-pass, band-pass, and cases in which the amplitude function is constant.

CHAPTER II

OPTIMAL LOW-PASS FILTERS

A. Low-Pass Filters.

The ideal amplitude response $A(\omega)$ of a low-pass filter is defined by

$$A(\omega) = K \;, \; |\omega| < \omega_c$$
$$= 0 \;, \; |\omega| > \omega_c, \tag{2.1}$$

where K is a positive constant and ω_c is the cut-off frequency separating the pass-band $0 < \omega < \omega_c$ from the stop-band $\omega > \omega_c$. An approximation $A(\omega)$ to the ideal response satisfies the conditions

$$A_1 \leq A(\omega) \leq 1 \;, \; |\omega| \leq \omega_1 ,$$
$$A(\omega) \leq A_2 \;, \; |\omega| \geq \omega_2 , \tag{2.2}$$

where A_1, $A_2 < A_1$, ω_1, and $\omega_2 \geq \omega_1$ are specified, and $A(\omega)$ is a monotonically decreasing function in the transition interval, $\omega_1 \leq \omega \leq \omega_2$. The specification A_2 is generally much less than A_1, so that the cut-off point, defined for the nonideal case as that value ω_c at which $A(\omega)$ attains $1/\sqrt{2}$ times its maximum value, is also in the transition interval unless $A_1 < A_{max}/\sqrt{2}$.

An often used approximation to the ideal response is the amplitude function

$$A(\omega) = \frac{1}{\sqrt{1 + \epsilon^2 \; f(\omega^2)}} \;, \tag{2.3}$$

where ϵ is a real constant and $f(\omega^2)$ is relatively small in the pass-band and relatively large in the stop-band. From (2.3) it is clear that $A(\omega)$ is an even function, as is required of the amplitude. If $A(\omega)$ is the amplitude function of a finite filter it is necessary that $f(\omega^2)$ be either a polynomial or a rational function.

The <u>order of complexity</u> n of $A(\omega)$ is defined as half the degree 2n of $f(\omega^2)$, if $f(\omega^2)$ is a polynomial, and as half the degree 2n of the numerator or denominator, whichever is of higher degree, if $f(\omega^2)$ is a rational function. For a given order of complexity and a given set of specifications A_1, A_2, ω_1, ω_2, the function $A(\omega)$ is <u>optimal</u> if the length $\omega_2 - \omega_1$ of the transition interval is minimum.

B. <u>All-pole Filters</u>.

If $f(\omega^2)$ in (2.3) is restricted to be a polynomial the transfer function H(s) which ensues from $A(\omega)$, that is,

$$H(j\omega) = A(\omega)\ e^{j\phi(\omega)},$$

is a constant divided by a polynomial in s. Hence its zeros are all infinite, and finite critical frequencies thus are all poles. Such a filter is called an <u>all-pole filter</u>.

In this case ($f(\omega^2)$ a polynomial) it is known [1] that the optimal $A(\omega)$ occurs when

$$f(\omega^2) = C_n^2(\omega), \tag{2.4}$$

where $C_n(\omega)$ is the <u>Chebyshev polynomial</u> of degree n of the first kind, defined by

$$C_n(\omega) = \cos\ (n \cos^{-1}\ \omega). \tag{2.5}$$

The Chebyshev filter is characterized by an <u>equiripple</u> response in the interval $|\omega| < 1$, and a monotonically decreasing response for $|\omega| > 1$. The ripple width is $1 - 1/\sqrt{1 + \epsilon^2}$, since the maximum value of $A(\omega)$ given by (2.3) and (2.4) is 1, occurring at zeros of $C_n(\omega)$ (which are all on $-1 \le \omega \le 1$), and on $(-1, 1)$ the minimum value of $A(\omega)$ occurs when $C_n(\omega) = \pm 1$. Since $|C_n(\omega)| > 1$ for all $|\omega| > 1$, the response $A(\omega)$ is monotonic decreasing on $(1, \infty)$.

A generalization of the Chebyshev filter has been recently developed [2], [3], in which

$$f(\omega^2) = [F_n{}^a(\omega)]^2, \tag{2.6}$$

where $F_n{}^a(\omega)$ is related to the ultraspherical polynomial $P_n{}^{(a,a)}(\omega)$ of degree n, by

$$F_n{}^a(\omega) = \frac{n!}{(1 + a)_n} P_n{}^{(a,a)}(\omega);\ n = 0,1,2,\ldots;\ a > -1 \tag{2.7}$$

where $P_n{}^{(a,a)}(\omega)$ is the special case $b = a$ of the Jacobi polynomial,

$$P_n{}^{(a,b)}(\omega) = \frac{(1+a)_n}{n!} \sum_{k=o}^{n} \frac{(-n)_k(a+b+n+1)_k \left(\frac{1-\omega}{2}\right)^k}{k!\ (1+a)_k}, \tag{2.8}$$

and

$$\begin{aligned} (\alpha)_k &= \alpha(\alpha+1)\ldots\ldots(\alpha+k-1),\ k = 1,2,3,\ldots, \\ (\alpha)_o &= 1. \end{aligned} \tag{2.9}$$

The ultraspherical filter reduces to the Chebyshev when $a = -1/2$. It also includes, among others, the Butterworth filter $(a \to \infty)$ defined by

$$f(\omega^2) = \omega^{2n},\ n = 1,2,3,\ldots \tag{2.10}$$

Some properties of $F_n{}^a(\omega)$ are as follows, for $n = 0,1,2,\ldots,$ and $a > -1$:

$$F_n{}^a(1) = 1, \tag{2.11}$$

$$\frac{d^k}{d\omega^k} F_n^a(\omega) = \frac{(n-k+1)_k(2a+n+1)_k}{2^k(1+a)_k} F_{n-k}^{a+k}(\omega),$$

$$n \geq k,\ k = 0,1,2,\ldots, \qquad (2.12)$$

$$(2a+n)F_n^a(\omega) = (2a+2n-1)\,\omega\, F_{n-1}^a(\omega) - (n-1)\, F_{n-2}^a(\omega),$$

$$n \geq 2. \qquad (2.13)$$

Also, since the $F_n^a(\omega)$ are known to be orthogonal on $-1 < \omega \leq 1$, all their zeros must occur there.

C. <u>The Elliptic Filter</u>

If $f(\omega^2)$ in (2.3) is a rational function, then $A(\omega) = 1$ at the zeros of $f(\omega^2)$ and $A(\omega) = 0$ at the poles of $f(\omega^2)$. Hence, by selecting the zeros to be in the pass-band and the poles to be in the stop-band, a better approximation to the ideal response can be achieved than for the case $f(\omega^2)$ a polynomial.

If $f(\omega^2)$ is a rational function of complexity n, then it is known [4] that the optimal case is the <u>elliptic</u> <u>filter</u>, described by

$$f(\omega^2) = R_n^2(\omega) \qquad (2.14)$$

where $R_n(\omega)$ is the <u>Chebyshev</u> <u>rational</u> <u>function</u> defined by

$$R_n(\omega) = \prod_{i=1}^{[n/2]} \frac{(\omega_{2i-1}^2 - \omega^2)}{(1 - \omega_{2i-1}^2\,\omega^2)}, \quad n = 2,4,6,\ldots \qquad (2.15)$$

and

$$R_n(\omega) = \omega \prod_{i=1}^{[n/2]} \frac{(\omega_{2i}^2 - \omega^2)}{(1 - \omega_{2i}^2\,\omega^2)}, \quad n = 3,5,7,\ldots \qquad (2.16)$$

($[n/2]$ is the greatest integer $\leq n/2$.) In both cases the positive zeros of $R_n(\omega)$ are given by

$$\omega_j = \sqrt{k}\ \mathrm{sn}\ \frac{jK}{n}, \tag{2.17}$$

where $j = 1,3,5,\ldots,\ n-1$ if n is even; $j = 2,4,6,\ldots,\ n-1$, if n is odd, and sn u is the <u>elliptic sine</u> function,

$$\mathrm{sn}\ u = \sin\phi, \tag{2.18}$$

where u and ϕ are related by

$$u(k,\phi) = \int_0^{\phi} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}\ . \tag{2.19}$$

Finally, K is defined by

$$K = u(k,\pi/2), \tag{2.20}$$

where k is a real number, $0 < k < 1$, called the <u>modulus</u>. (For a discussion of the elliptic filter, see for example, [5], pp. 607-614.)

The Chebyshev rational function exhibits the equiripple property on the interval $0 \le \omega \le c$, where $c = \sqrt{k} < 1$. Its positive zeros are all on $(0,c)$ and its extremum values there are $\pm\delta$, given by

$$\delta = k^{n/2} \prod_{i=1}^{[n/2]} \left[\mathrm{sn}\ \frac{(2i-1)K}{n}\right]^2. \tag{2.21}$$

These occur at the points

$$p_j = \sqrt{k}\ \mathrm{sn}\ \frac{jK}{n}, \tag{2.22}$$

where $j = 2,4,6,\ldots,\ n-2$, for n even, and $j = 1,3,5,\ldots,\ n-2$, for n odd. Also, for n even, $p_0 = 0$ is a maxima. Therefore, on $0 \le \omega \le c$, we have

$$\frac{1}{\sqrt{1+\epsilon^2\delta^2}} \le A(\omega) \le 1 \tag{2.23}$$

From (2.15) and (2.16) it is evident that the poles and zeros of

$R_n(\omega)$ are reciprocals of each other, and that

$$R_n(1/\omega) = 1/R_n(\omega). \tag{2.24}$$

Hence $A(\omega)$ is equiripple also in the band $\omega \geq 1/c > 1$, assuming its minimum value of zero at the poles $1/\omega_j$ of $R_n(\omega)$ and its maximum value of $\frac{1}{\sqrt{1 + \epsilon^2/\delta^2}}$ at the points $1/p_j$; that is, for $\omega > 1/c$,

$$0 \leq A(\omega) \leq \frac{1}{\sqrt{1 + \epsilon^2/\delta^2}} . \tag{2.25}$$

For the elliptic filter the order of complexity is the subscript n on $R_n(\omega)$. Since it is the optimal filter, the transition interval length $1/c - c$ is a minimum for a given A_1, A_2, and n.

D. Procedure for Finding the Elliptic Filter Response

For a given n, A_1, and A_2, the procedure for finding $A(\omega)$ in the elliptic filter case is somewhat involved. From (2.23) and (2.24) we have

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{1 + \epsilon^2\delta^2}} , \\ A_2 &= \frac{1}{\sqrt{1 + \epsilon^2/\delta^2}} , \end{aligned} \tag{2.26}$$

from which we obtain

$$\begin{aligned} \epsilon^2 &= \frac{\sqrt{(1 - A_1^2)(1 - A_2^2)}}{A_1 A_2} , \\ \delta^2 &= \frac{A_2\sqrt{1 - A_1^2}}{A_1\sqrt{1 - A_2^2}} . \end{aligned} \tag{2.27}$$

Hence we select ϵ^2 as given by the first of (2.27) and seek a modulus k so that the second of (2.27) is satisfied. Since $A_2 < A_1 < 1$ it is

clear that δ may vary from 0 to 1. The range of k is also from 0 to 1. If $k = 0$, by (2.19) we have $u = \phi$ and by (2.20) we have $K = \pi/2$. From (2.21) these yield $\delta = 0$. From (2.18) and (2.19) we have

$$\mathrm{sn}\ K = \sin \pi/2 = 1,$$

and if $k \to 1$, we have from (2.19) and (2.20), $K \to \infty$. Therefore, for j and n positive integers,

$$\mathrm{sn}\ \frac{jK}{n} \sim \mathrm{sn}\ K = 1. \tag{2.28}$$

Therefore, by (2.21) we have if $k = 1$, then $\delta = 1$,

Since δ varies continuously from 0 to 1 as k ranges over (0,1), then given any δ, from (2.27) we may find a corresponding k. Graphs of δ versus $\sqrt{k} = c$ are shown in Figure 2-1 for $n = 2,3,\ldots\ldots,9$, and $\sqrt{k}$ ranging from .92 to 1. As is seen from the graphs for these values of n and for δ near 1, the choice of k may present arithmetical difficulties since k cannot actually attain the value 1. Once k is found we may determine K by (2.20) and, subsequently, the ω_j from (2.17). These then determine the Chebyshev rational function $R_n(\omega)$.

Examples of elliptic filter amplitude responses for $n = 4$ and $n = 6$, $\epsilon = 2.195$, and k as indicated, are shown in Figures 2-2 and 2-3.

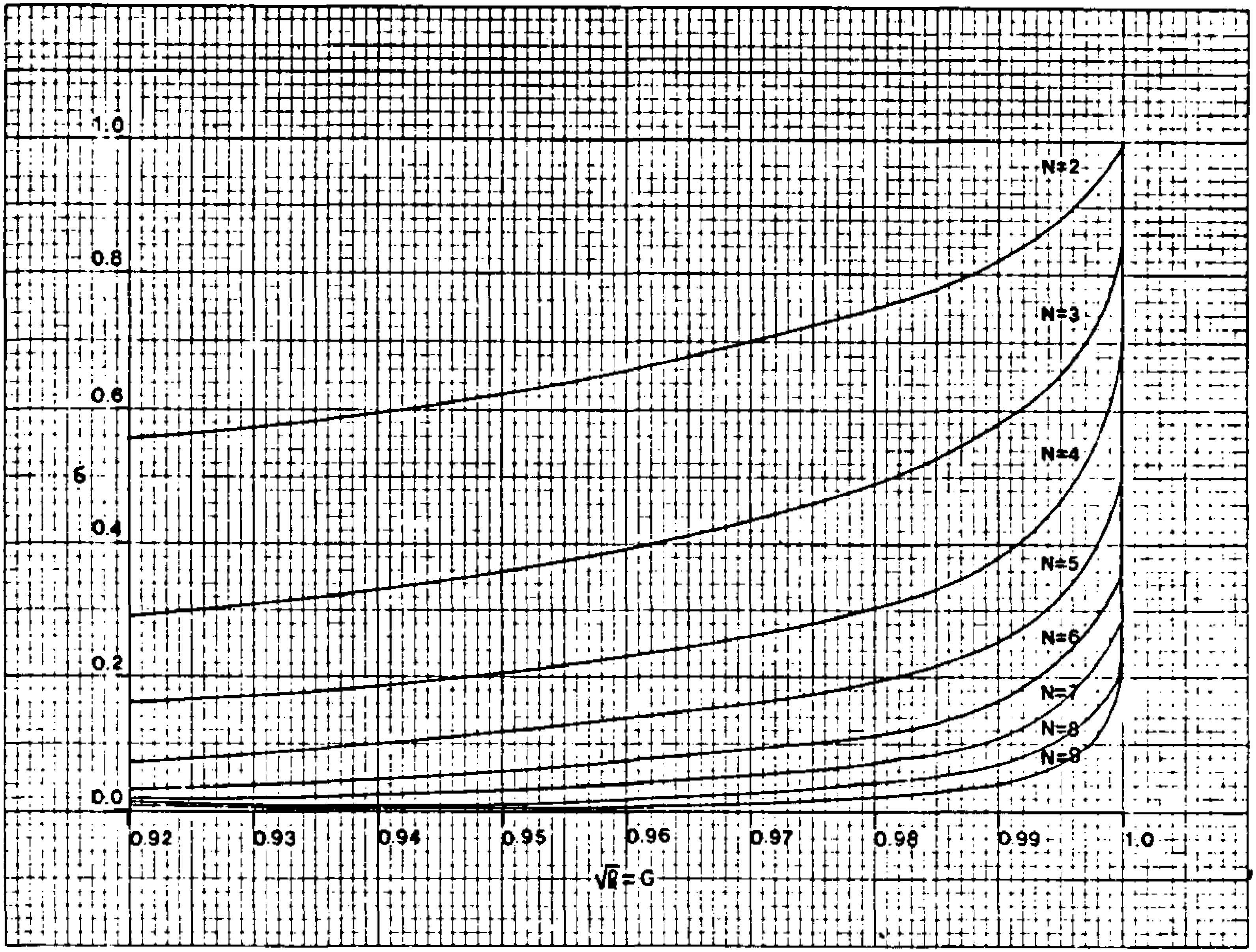

1.0
0.8
0.6
б
0.4
0.2
0.0
0.92
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1.0
√R = G
N=2
N=3
N=4
N=5
N=6
N=7
N=8
N=9

FIGURE 2-1

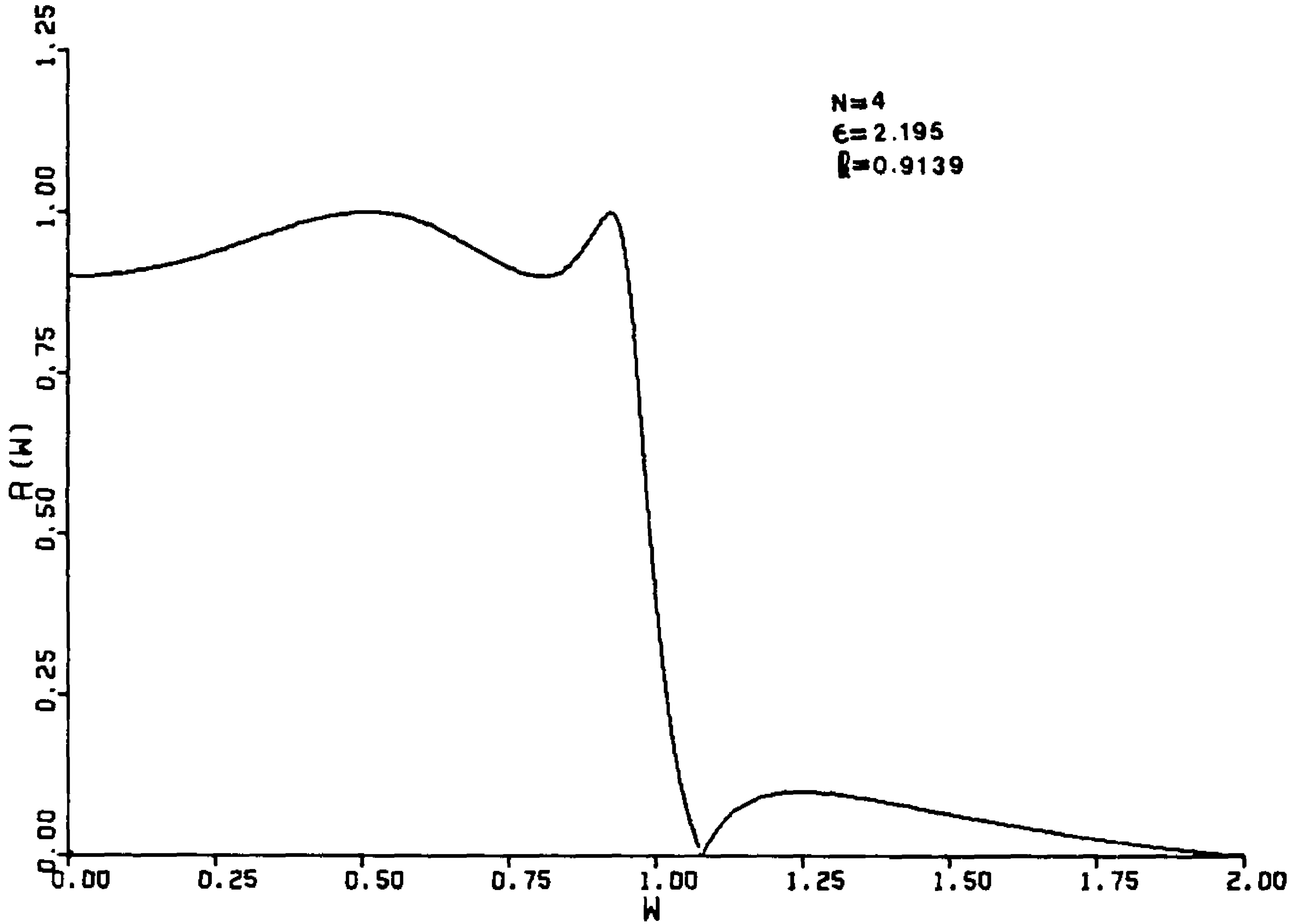


FIGURE 2-2

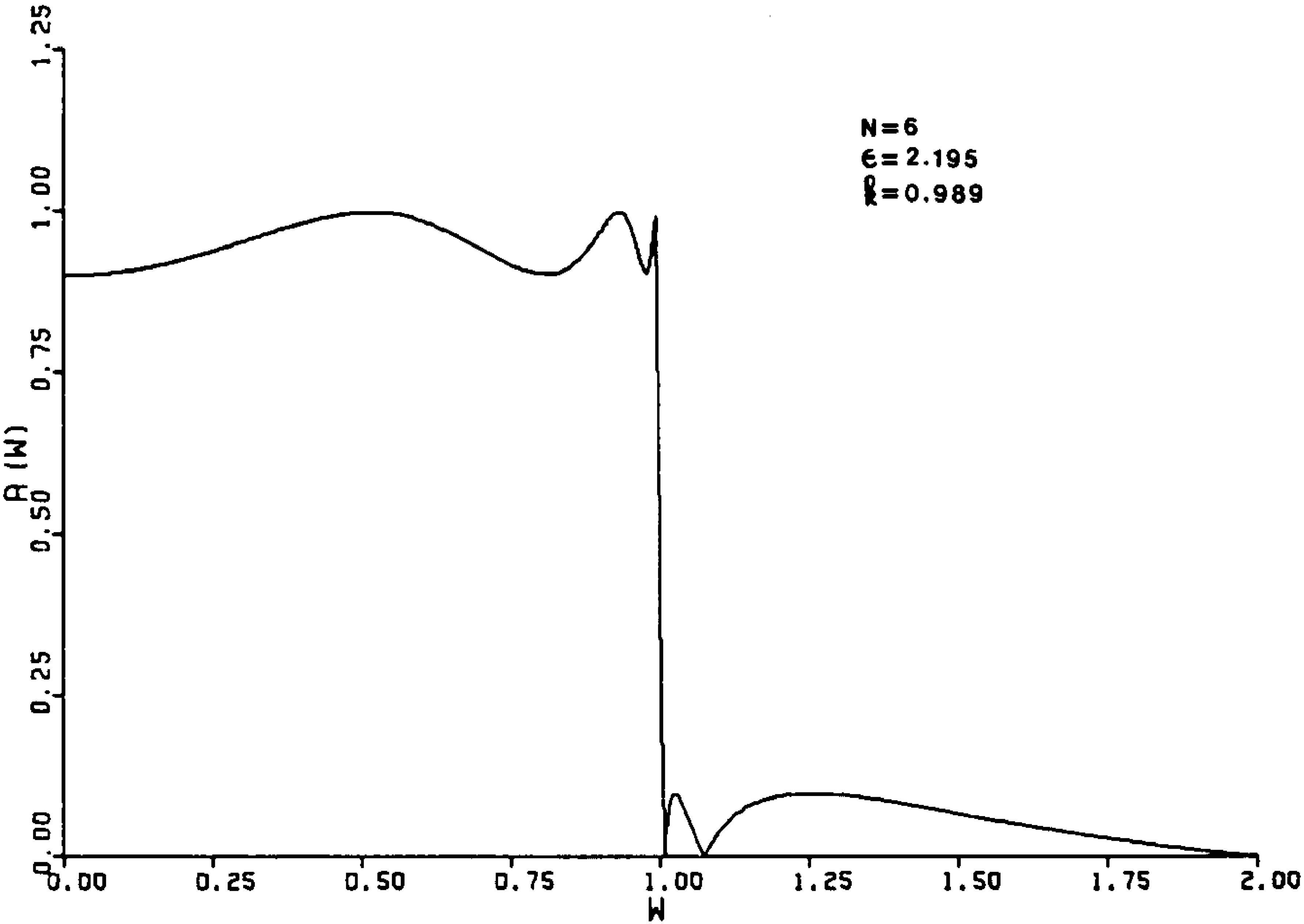


FIGURE 2-3

CHAPTER III

ULTRASPHERICAL RATIONAL FILTERS

A. The Ultraspherical Rational Function

The elliptic filter has the optimum amplitude response but as the previous chapter suggests the task of finding the proper modulus k and obtaining $A(\omega)$ may be a prodigious one. The work could be shortened and the theory made easier to comprehend if it could be shown that the Chebyshev rational function is a special case of a well-known set of functions having well-established characteristics. As was pointed out in section B of the previous chapter, this was accomplished in the case of the Chebyshev filter when it was shown that $C_n(\omega)$ was the special case $a = -\frac{1}{2}$ of the polynomial $F_n^a(\omega)$, based on the ultraspherical polynomials.

To consider the possibility of generalizing the Chebyshev rational function $R_n(\omega)$ in a useful way, we observe, by (2.15), that for n even,

$$R_n(\omega) = \frac{(-1)^{n/2} \prod_{i=1}^{n/2} (\omega^2 - \omega_{2i-1}^2)}{\omega^n \prod_{i=1}^{n/2} (\omega^{-2} - \omega_{2i-1}^2)} . \qquad (3.1)$$

Since $F_n^a(\omega)$ for n even is an even polynomial of degree n, it may be written in the form

$$F_n^a(\omega) = a_n \prod_{i=1}^{n/2} (\omega^2 - \omega'^2_{2i-1}),$$

or

$$\prod_{i=1}^{n/2} (\omega^2 - \omega'^2_{2i-1}) = \frac{1}{a_n} F_n{}^a(\omega), \tag{3.2}$$

where a_n is the leading coefficient and ω'_{2i-1} are the zeros. If the ω'_{2i-1} were the same as the zeros ω_{2i-1} of $R_n(\omega)$ then evidently (3.1) could be written $R_n = G_n$, where

$$G_n(\omega) = \frac{(-1)^{n/2} F_n{}^a(\omega)}{\omega^n F_n{}^a(1/\omega)} . \tag{3.3}$$

An analysis for the odd case leads to the same result if $n/2$ is replaced by $[n/2]$.

In (3.3) we have one parameter, a, to vary in attempting to match the right and left members. We could match one coefficient on each side by selecting a, but this would not be sufficient in the general case to match the other coefficients, since one coefficient could be changed by scaling without changing the equiripple properties which characterize the Chebyshev rational function. In view of this, we are led to consider, as a possible generalization of $R_n(\omega)$, the function, which we define as the <u>ultraspherical rational function</u>,

$$f_n(a,b;\omega) = \frac{(-1)^{[n/2]} F_n{}^a(b\omega)}{\omega^n F_n{}^a(b/\omega)} , \tag{3.4}$$

where a and b are real, and n = 2,3,4,... . The cases n = o and n = 1 are trivial ($f_o = 1$, $f_1 = \omega$) and have no counterpart in the $R_n(\omega)$, as may be seen from (2.15) and (2.16). Evidently f_n has the same general form as R_n, its complexity is the same, its poles and zeros are reciprocals of each other; and, like R_n, it satisfies the relations

$$f_n(a,b;1) = (-1)^{[n/2]} , \tag{3.5}$$

$$f_n(a,b;1/\omega) = 1/f_n(a,b;\omega). \tag{3.6}$$

Alternately we may write (3.4) in the form

$$f_n(a,b;\omega) = \frac{(-1)^{[n/2]} P_n^{(a,a)}(b\omega)}{\omega^n P_n^{(a,a)}(b/\omega)} \tag{3.7}$$

where $P_n^{(a,a)}(\omega)$ is the ultraspherical polynomial defined by (2.8).

B. Determination of the Parameters

For purposes of comparing the Chebyshev rational function with the ultraspherical rational function we may write (2.15) and (2.16) in the form

$$R_n(\omega) = \frac{(-1)^{[n/2]}(\omega^n + \sigma_1\omega^{n-2} + \sigma_2\omega^{n-4} + \ldots\ldots)}{1 + \sigma_1\omega^2 + \sigma_2\omega^4 + \ldots\ldots}$$

$$= \frac{(-1)^{[n/2]} \sum_{i=o}^{[n/2]} \sigma_i\omega^{n-2i}}{\sum_{i=o}^{[n/2]} \sigma_i\omega^{2i}} \; ; \; n = 2,3,4,\ldots, \tag{3.8}$$

where $(-1)^i\sigma_i$ = sum of all the ω_j^2 taken i at a time, $i = 1,2,3,\ldots,$ $[n/2]$, and $\sigma_o = 1$.

The ultraspherical polynomials may be written in the form (see for example, [6], p. 277)

$$P_n^{(a,a)}(x) = \sum_{i=o}^{[n/2]} \frac{(-1)^i\, \Gamma(a+n+1)\, \Gamma(2a+2n-2i)\, x^{n-2i}}{i!(n-2i)!\, 2^{n-i}\, \Gamma(2a+n+1)\, \Gamma(a+n-i)}, \tag{3.9}$$

and hence by (3.7) we may write

$$f_n(a,b;\omega) = \frac{(-1)^{[n/2]} \sum_{i=o}^{[n/2]} A_{n-2i}\, \omega^{n-2i}}{\sum_{i=o}^{[n/2]} A_{n-2i}\, \omega^{2i}} \tag{3.10}$$

where

$$A_{n-2i} = \frac{(-1)^i\, n!\, \Gamma(a+n)\, \Gamma(2a+2n-2i+1)}{2(i!)(n-2i)!\, \Gamma(a+n-i+1)\, \Gamma(2a+2n)b^{2i}} \,. \tag{3.11}$$

We have normalized f_n by dividing the numerator and denominator by

$$\frac{\Gamma(a+n+1)\, \Gamma(2a+2n)b^n}{2^{n-1}\, n!\, \Gamma(2a+n+1)\, \Gamma(a+n)}$$

to make $A_n = 1$. The expression (3.11) may be simplified, if $\underline{a}$ is not a negative integer, to the form

$$A_{n-2i} = \frac{(-1)^i\, n!\, \Gamma(a+n)\, \Gamma(2a+2n-2i)}{i!\, (n-2i)!\, \Gamma(a+n-i)\, \Gamma(2a+2n)b^{2i}} \,. \tag{3.12}$$

For $R_n(\omega)$ and $f_n(a,b;\omega)$ to be identical it is necessary that

$$\sigma_1 = A_{n-2} \;,\; \sigma_2 = A_{n-4} \;,$$

or

$$\sigma_1 = -\frac{n!\, \Gamma(a+n)\, \Gamma(2a+2n-1)}{2(n-2)\,!\, \Gamma(a+n)\, \Gamma(2a+2n)b^2} \;,$$

$$\sigma_2 = \frac{n!\, \Gamma(a+n)\, \Gamma(2a+2n-3)}{4(n-4)!\, \Gamma(a+n-1)\, \Gamma(2a+2n)b^4} \,.$$

These reduce to

$$\begin{aligned} \sigma_1 &= -\frac{n(n-1)}{2(2a+2n-1)b^2} \;, \\ \sigma_2 &= \frac{n(n-1)(n-2)(n-3)}{8(2a+2n-1)(2a+2n-3)b^4} \;, \end{aligned} \tag{3.13}$$

from which we may solve for a and b, obtaining for n = 4,5,6,...,

$$a = \frac{(n-2)(n-3)(2n-1)\,\sigma_1^2 - 2n(n-1)(2n-3)\,\sigma_2}{2[2n(n-1)\,\sigma_2 - (n-2)(n-3)\,\sigma_1^2]} \tag{3.14}$$

and

$$b^2 = \frac{(n-2)(n-3)\,\sigma_1^2 - 2n(n-1)\,\sigma_2}{8\,\sigma_1\,\sigma_2} \tag{3.15}$$

The cases n=2 and n=3 are special cases since for these values of n, $\sigma_2 = 0$. (This is true also for σ_i, $i = 3,4,\ldots,[n/2]$.) Hence, by the first of (3.13), we have

$$\begin{aligned} (2a+3)b^2 &= -\frac{1}{\sigma_1}, \quad n \quad 2, \\ (2a+5)b^2 &= -\frac{3}{\sigma_1}, \quad n = 3. \end{aligned} \tag{3.16}$$

Thus, for these cases, a and b are not unique.

If we clear (3.10) of complex fractions by multiplying the numerator and denominator by the least common denominator of A_{n-2i}, the result is

$$f_n(a,b;\omega) = \frac{(-1)^{[n/2]}(B_n\omega^n + B_{n-2}\omega^{n-2} + B_{n-4}\omega^{n-4} + \ldots)}{B_n + B_{n-2}\omega^2 + B_{n-4}\omega^4 + \ldots} \tag{3.17}$$

The coefficients B_{n-2i} are tabulated in Table I for n = 2,3,...,9, and are given in general by

$$B_{n-2[n/2]} = \frac{(-1)^{[n/2]}\; n!\; b^{n-2[n/2]}}{([n/2])!\; 2^{[n/2]}},$$

$$B_{n-2i} = \frac{(-1)^i\; n!\; (2a+2n-2i-1)(2a+2n-2i-3)\ldots(2a+2n-2[n/2]+1)\,b^{n-2i}}{i!\,(n-2i)!\,2^i},$$

$$i = 0,1,\ldots,[n/2]-1.$$

TABLE I

COEFFICIENTS IN $f_n(a,b;\omega)$

n	B_n	B_{n-2}	B_{n-4}	B_{n-6}	B_{n-6}
2	$2\left(a+\frac{3}{2}\right)b^2$	-1			
3	$2\left(a+\frac{5}{2}\right)b^3$	$-3b$			
4	$4\left(a+\frac{5}{2}\right)\left(a+\frac{7}{2}\right)b^4$	$-12\left(a+\frac{5}{2}\right)b^2$	3		
5	$4\left(a+\frac{7}{2}\right)\left(a+\frac{9}{2}\right)b^5$	$-20\left(a+\frac{7}{2}\right)b^3$	$15b$		
6	$8\left(a+\frac{7}{2}\right)\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)b^6$	$-60\left(a+\frac{7}{2}\right)\left(a+\frac{9}{2}\right)b^4$	$90\left(a+\frac{7}{2}\right)b^2$	-15	
7	$8\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)b^7$	$-84\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)b^5$	$210\left(a+\frac{9}{2}\right)b^3$	$-150b$	
8	$16\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)\left(a+\frac{15}{2}\right)b^8$	$-224\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)b^6$	$840\left(a+\frac{9}{2}\right)\left(a+\frac{11}{2}\right)b^4$	$-840\left(a+\frac{9}{2}\right)b^2$	105
9	$16\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)\left(a+\frac{15}{2}\right)\left(a+\frac{17}{2}\right)b^9$	$-288\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)\left(a+\frac{15}{2}\right)b^7$	$1512\left(a+\frac{11}{2}\right)\left(a+\frac{13}{2}\right)b^5$	$-2520\left(a+\frac{11}{2}\right)b^3$	$945b$

Examples of ultraspherical rational filter amplitudes are shown in Figures 3-1 and 3-2 for values of a and b calculated from (3.14) and (3.15) for given elliptic filter amplitudes. In Figure 3-2 the effect on the ripple width of varying ϵ is illustrated.

C. Bounds on the Parameters

It is sufficient, of course, for R_n and f_n to be identical if, in addition to (3.13), we have

$$\sigma_i = A_{n-2i} \; ; \; i = 3,4,5,\ldots,[n/2] \tag{3.18}$$

for the values of a and b given by (3.14) and (3.15). For the cases n = 2,3,4, and 5, (3.18) is trivially satisfied since both its members are zero. For cases n = 6,7,8, and 9, (3.18) has been demonstrated to be true for various moduli k, and is here stated as a conjecture. Specifically, the conjecture is that $(-1)^i$ times the sum of all the ω_j^2, given by (2.17), taken i at a time, is equal to A_{n-2i}, given by (3.12). For example, for n = 6, we have

$$\sigma_3 = -k^3 \; sn^2 \frac{K}{6} \; sn^2 \frac{3K}{6} \quad sn^2 \frac{5K}{6} = \frac{-15}{(2a+11)(2a+9)(2a+7)b^6}$$

where

$$a = \frac{11\sigma_1^2 - 45\sigma_2}{10\sigma_2 - 2\sigma_1^2} ,$$

$$b^2 = \frac{3\sigma_1^3 - 15\sigma_2}{2\sigma_1\sigma_2} ,$$

$$\sigma_1 = -k\left(sn^2 \frac{K}{6} + sn^2 \frac{3K}{6} + sn^2 \frac{5K}{6}\right) ,$$

$$\sigma_2 = k^2\left(sn^2 \frac{K}{6} \; sn^2 \frac{3K}{6} + sn^2 \frac{K}{6} \; sn^2 \frac{5K}{6} + sn^2 \frac{3K}{6} \; sn^2 \frac{5K}{6}\right) .$$

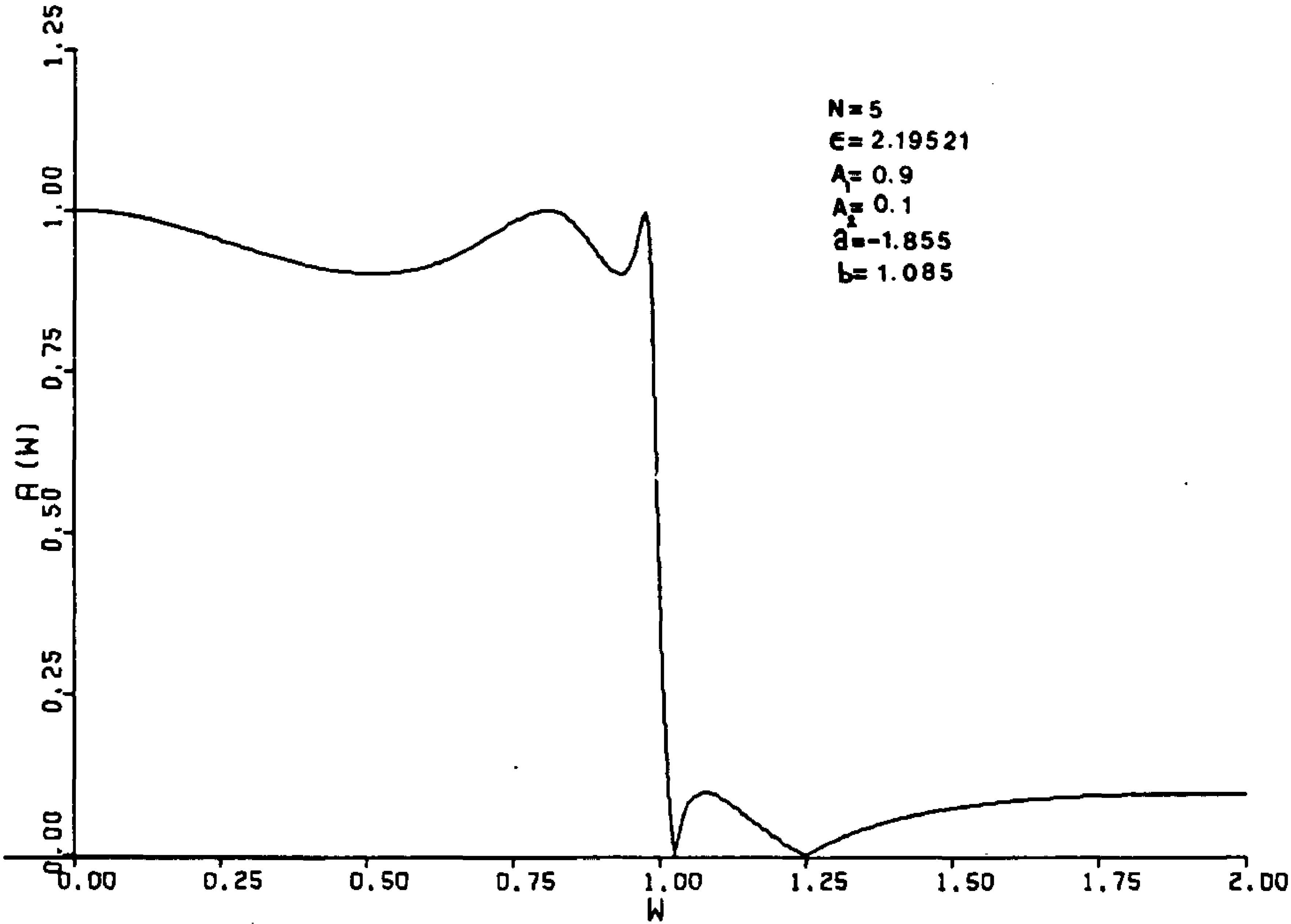
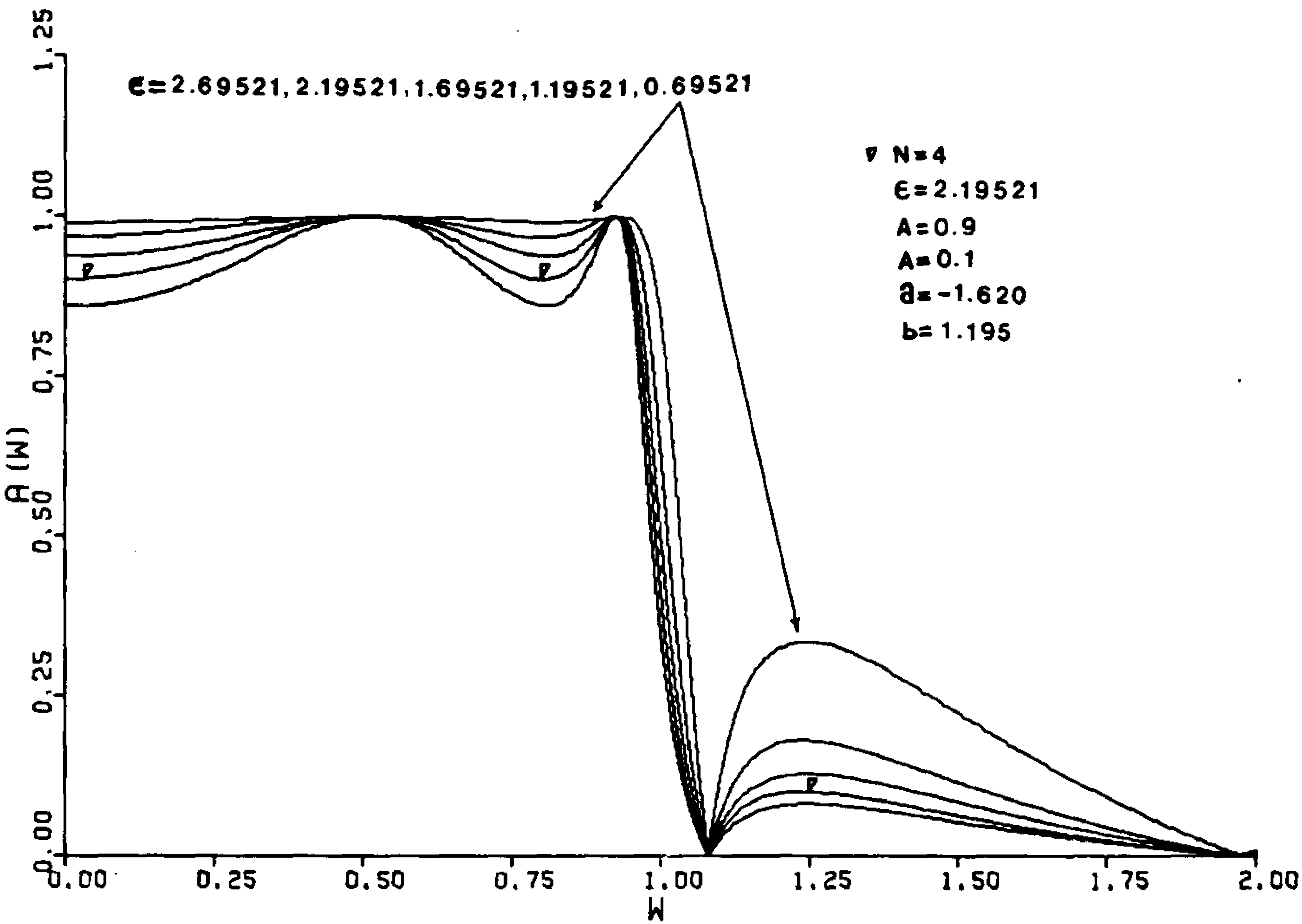

N = 5
ϵ = 2.19521
A1 = 0.9
A2 = 0.1
a = -1.855
b = 1.085
A (W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 3-1



ϵ=2.69521,2.19521,1.69521,1.19521,0.69521
N=4
ϵ=2.19521
A=0.9
A=0.1
a=-1.620
b=1.195
A (W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 3-2

It is not difficult to establish (3.18) for the limiting cases, k = 0 and k = 1, which also serve to establish the bounds on a and b. If k = 1, we have, by (2.28), that $\omega_j = 1$, and hence

$$\sigma_i = (-1)^i \binom{[n/2]}{i} ; \; i = 0,1,2,\ldots,[n/2]. \tag{3.19}$$

Also for this case, by (2.15) and (2.16) we have

$$R_n(\omega) = 1, \text{ n even}$$
$$= \omega, \text{ n odd.}$$

Let us now consider the function f_n for the case $a = -[n/2]$ and $b = 1$, for which we have by (3.12),

$$A_{n-2i} = \frac{(-1)^i \, n! \, \Gamma\,(n - [n/2])\, \Gamma\,(2n - 2[n/2] - 2i)}{i!\, (n - 2i)\, !\, \Gamma\,(n - [n/2] - i)\, \Gamma\,(2n - 2[n/2])} \, . \tag{3.20}$$

If n = 2m, then [n/2] = m, and (3.20) becomes

$$A_{n-2i} = \frac{(-1)^i (2m)\, !\, \Gamma\,(m)\, \Gamma\,(2m - 2i)}{i!\, (2m - 2i)\, !\, \Gamma\,(m - i)\, \Gamma\,(2m)}$$

$$= \frac{(-1)^i \,(2m)(m - 1)\, !}{i!\, (2m - 2i) \quad (m - i - 1)\, !}$$

$$= \frac{(-1)^i \, m\, !}{i!\, (m - i)\, !}$$

$$= (-1)^i \binom{m}{i}$$

$$= (-1)^i \binom{[n/2]}{i}$$

$$= \sigma_i \tag{3.21}$$

If n = 2m + 1, then [n/2] = m and for this case also (3.20) becomes (3.21).

Hence the limiting case k = 1 corresponds to the values $a = -[n/2]$, $b = 1$, and for this case (3.18) holds.

For $k = 0$, we have $\mathrm{sn}K = \sin\frac{\pi}{2} = 1$ and since $\mathrm{sn}\,\frac{jK}{n} = \sin\frac{j\pi}{2n}$ is bounded, we have by (2.17) that $\omega_j = 0$. Therefore, for this case $\sigma_i = 0$, $i = 1,2,\ldots,[n/2]$. Hence by (3.8) we have

$$R_n^2(\omega) = \omega^{2n} \tag{3.22}$$

and thus for k = 0 the elliptic filter becomes the Butterworth filter.

Let us now consider what happens to a and b when $\sigma_i \to 0$. From (3.13) it is clear that for this case either $a \to \infty$, or $b \to \infty$, or both. It is known (see for example, [7], p. 573) that for k sufficiently near zero, we may approximate sn u by

$$\mathrm{sn}\,u \approx \sin u - \frac{k^2}{4}(u - \sin u \cos u)\cos u. \tag{3.23}$$

Since in this case, $K \approx \frac{\pi}{2}$, we may approximate σ_1 by

$$\sigma_1 = -\sum_j \omega_j^2 = -\sum_j k\,\mathrm{sn}^2\,\frac{jK}{n}$$

$$\approx -\alpha k, \tag{3.24}$$

where

$$\alpha = \sum_j \sin^2\frac{j\pi}{2n}\,. \tag{3.25}$$

The summation is on $j = 1,3,\ldots, n-1$ if n is even and on $j = 2,4,\ldots, n-1$ if n is odd. Also we may write

$$\sigma_2 \approx \beta k^2 \tag{3.26}$$

where β is the sum of all the products taken two at a time of the functions $\left\{\sin^2\frac{j\pi}{2n}\right\}$, where j is as described above. Hence we have

$$\alpha^2 = 2\beta + \sum_j \sin^4 \frac{j\pi}{2n} \tag{3.27}$$

where the summation is as in (3.25).

We may find the sums of the series defining α and β by the known formulas

$$\sum_{r=1}^{n} \cos r\theta = \frac{\sin \frac{1}{2} n\theta \cos \frac{1}{2}(n+1)\theta}{\sin \frac{1}{2}\theta}, \tag{3.28}$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta, \tag{3.29}$$

$$\sin^4 \theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta. \tag{3.30}$$

The results are, for both n even and n odd,

$$\alpha = \frac{n}{4}, \quad \beta = \frac{n^2 - 3n}{32}. \tag{3.31}$$

Therefore, for k sufficiently small, we have $\sigma_1 \approx -n\frac{k}{4}$ and $\sigma_2 \approx (n^2 - 3n)\frac{k^2}{32}$. Hence we have

$$\frac{\sigma_1^2}{\sigma_2} \approx \frac{2n}{n-3} \tag{3.32}$$

and hence, by (3.14), we have for the limiting case,

$$a = -\frac{1}{2} \tag{3.33}$$

Also, substituting (3.32) into (3.15), we have

$$b \approx \frac{1}{k} \tag{3.34}$$

and hence if $k \to 0$, then $b \to \infty$.

In summary, for the ultraspherical rational function to be the Chebyshev rational function, the parameters lie in the range

$$-[n/2] < a < -\frac{1}{2},$$

$$1 < b < \infty,$$

corresponding to the range of the modulus,

$$1 > k > 0.$$

In particular, $k = 1$ corresponds to $a = -[n/2]$ and $b = 1$, in which case

$$f_n = R_n = 1, \; n \text{ even}$$
$$= \omega, \; n \text{ odd}.$$

The case $k = 0$ corresponds to $a = -\frac{1}{2}$ and $b \to \infty$, in which case

$$f_n^2 = R_n^2 = \omega^{2n}.$$

It is interesting to note that the limiting case $a = -\frac{1}{2}$ is also the case for which the ultraspherical polynomial $F_n^a(\omega)$ becomes the Chebyshev polynomial $C_n(\omega)$.

D. <u>The Ripple Width</u>

In the case of the elliptic filter and the ultraspherical filter, when a and b are chosen by (3.14) and (3.15), the ripples are equal in the pass-band and in the stop-band. The ripple width on $0 \le \omega \le c = \sqrt{k}$ is given by

$$RW_p = 1 - A_1 = 1 - \frac{1}{\sqrt{1 + \varepsilon^2\delta^2}} \tag{3.35}$$

and on $\omega \ge \frac{1}{c} = \frac{1}{\sqrt{k}}$ by

$$RW_s = A_2 = \frac{1}{\sqrt{1 + \varepsilon^2/\delta^2}} \tag{3.36}$$

The values of A_1 and A_2 come from (2.26) and the deviation δ is given by (2.21).

In the case n even, we may determine the ripple width in terms of n, a, and b, since in this case we have

$$A_1 = A(0) = \frac{1}{\sqrt{1 + \epsilon^2 f_n^2(a,b;0)}}, \tag{3.37}$$

and

$$\delta = f_n(a,b;0). \tag{3.38}$$

For n even, by (3.10) and (3.12) we have

$$f_n(a,b;0) = \frac{(-1)^{n/2} A_0}{A_n}$$

where $A_n = 1$, and

$$A_0 = \frac{(-1)^{n/2} n! \, \Gamma(a+n) \, \Gamma(2a+n)}{(n/2)! \, \Gamma(a+n/2) \, \Gamma(2a+2n) b^n}.$$

Hence we have

$$\delta = \frac{n! \, \Gamma(a+n) \, \Gamma(2a+n)}{(n/2)! \, \Gamma(a+n/2) \, \Gamma(2a+2n) b^n}. \tag{3.39}$$

CHAPTER IV

OBTAINING THE ELLIPTIC FILTER

A. Graphs of the Parameters

As we have seen in Chapter II, the deviation δ varies continuously from 0 to 1 as the modulus k of the elliptic filter varies from 0 to 1. A graph of δ versus $c = \sqrt{k}$ was given in Figure 2-1.

It is also possible, for $0 < k < 1$, to calculate σ_1 and σ_2 for various values of n and use the results to obtain a and b from (3.14) and (3.15). Since each k corresponds to a δ, it is possible to plot graphs of δ versus b and δ versus a in this manner. Graphs of δ versus b, for $n = 4,5,\ldots,9$, and for $1 \le b \le 1.24$, are shown in Figure 4-1; and graphs of δ versus a for $n = 4,5,\ldots,9$ and various ranges of a, are shown in Figures 4-2, 4-3, and 4-4. This information may also be used to plot graphs of a versus b, which are shown for n = 4,5,6, and 7 in Figures 4-5 and 4.6.

Hence it is possible to select a given modulus k, $0 < k < 1$, and order n, and from the graphs obtain the corresponding a and b. From Figure 2-1, the values of k and n determine δ, which then may be used in Figure 4-1 to obtain b, and in Figures 4-2, 4-3, or 4-4 to obtain a. Alternately, the knowledge of b and n may be used in Figures 4-5 or 4-6 to obtain a. (Graphs for higher values of n could be plotted if necessary.)

For example, suppose it is given that $n = 4$ and $\sqrt{k} = .97$, or $k = .941$. From Figure 2-1 we find that $\delta = .26$. Then from Figure 4-1 we have $b = 1.182$ and from Figure 4-2 we have $a = -1.685$. As a check

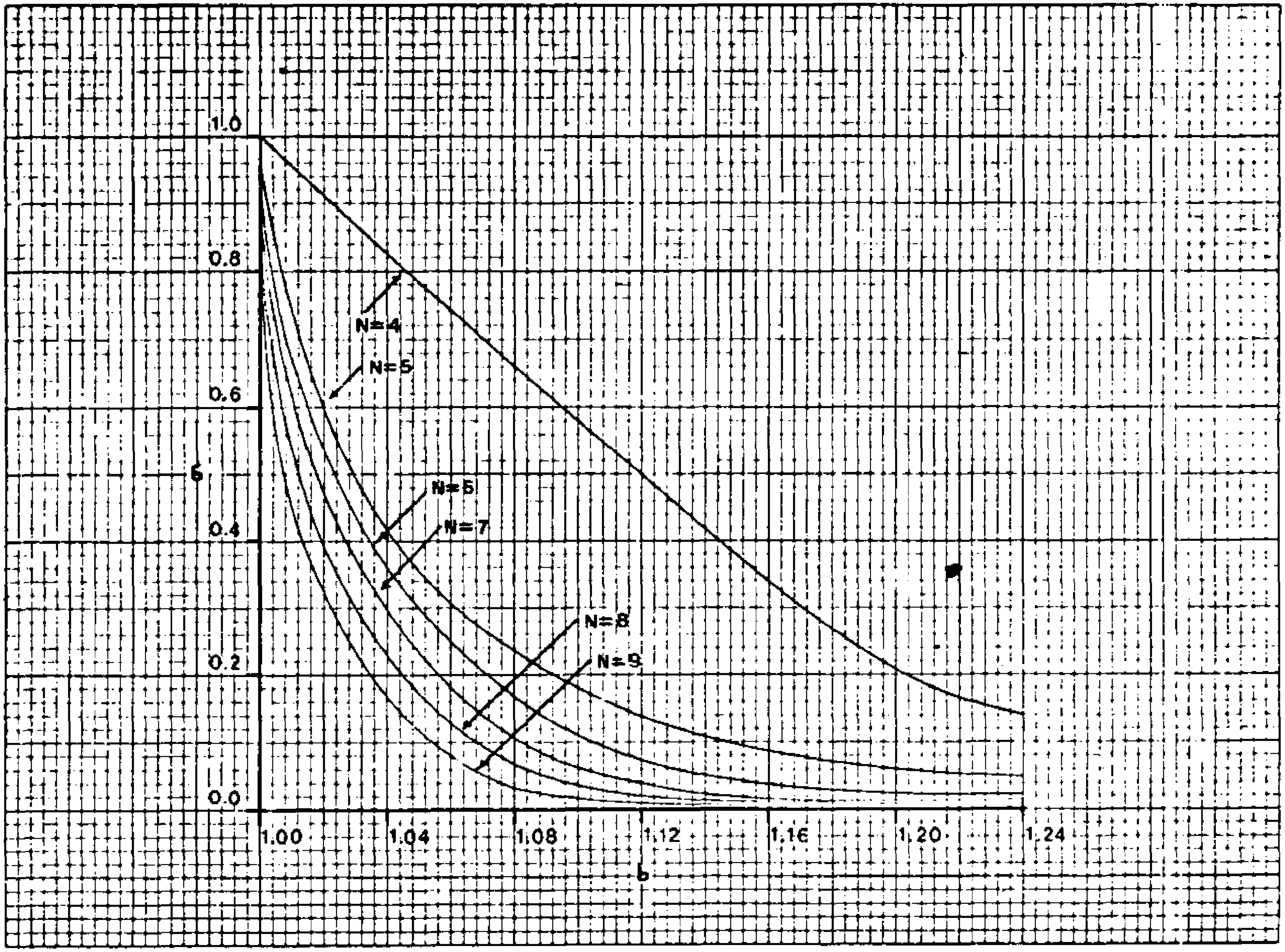


FIGURE 4-1

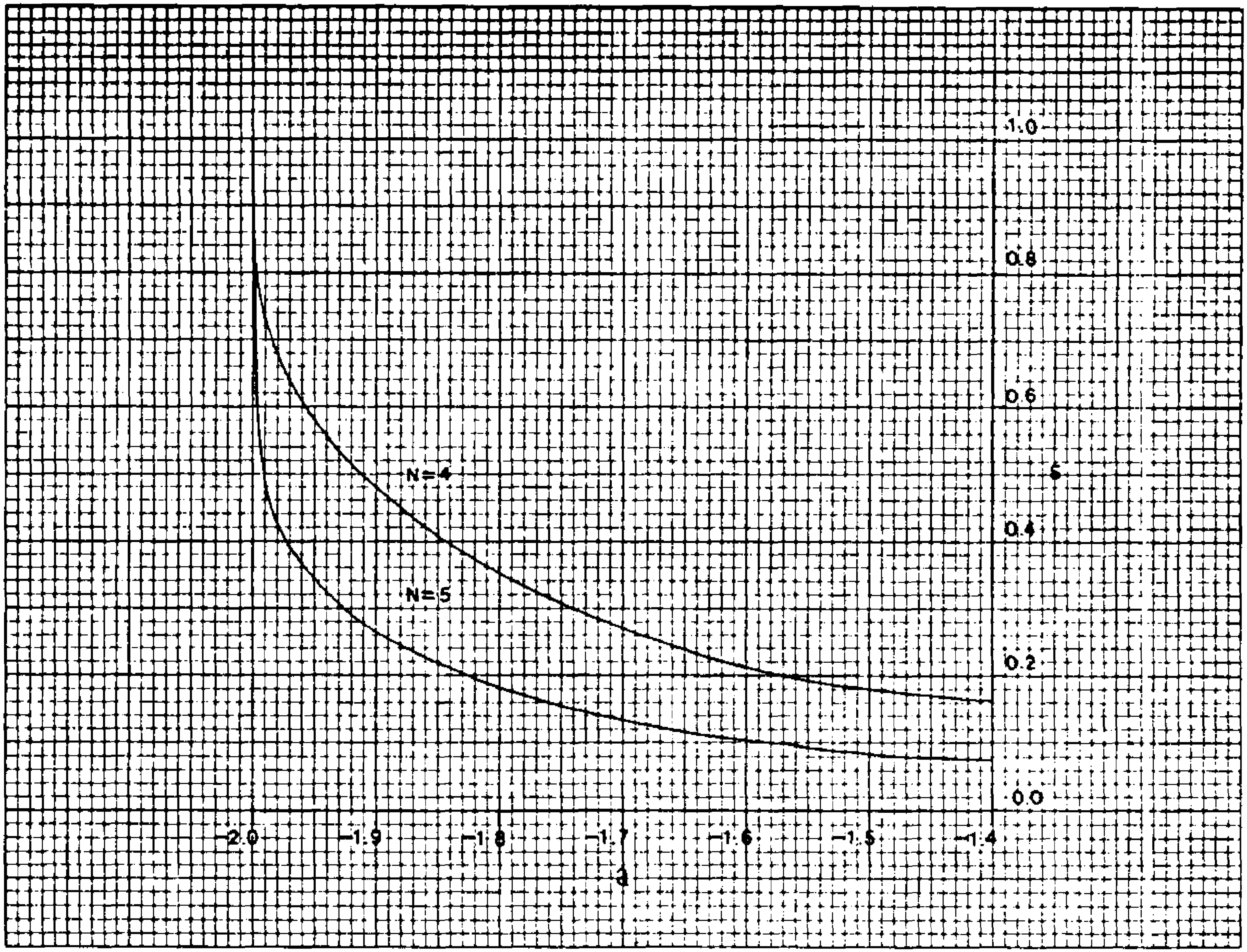


FIGURE 4-2

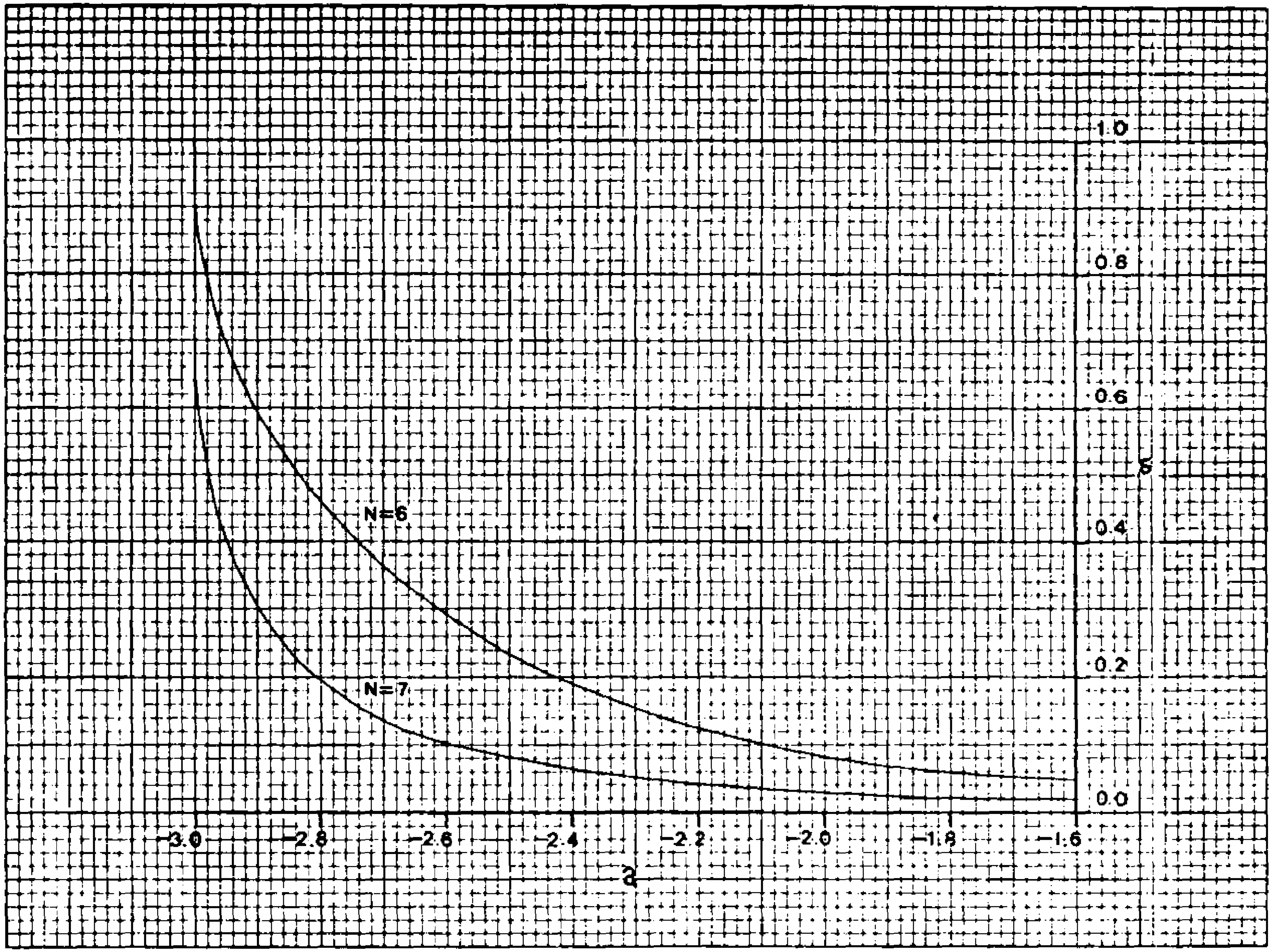


FIGURE 4-3

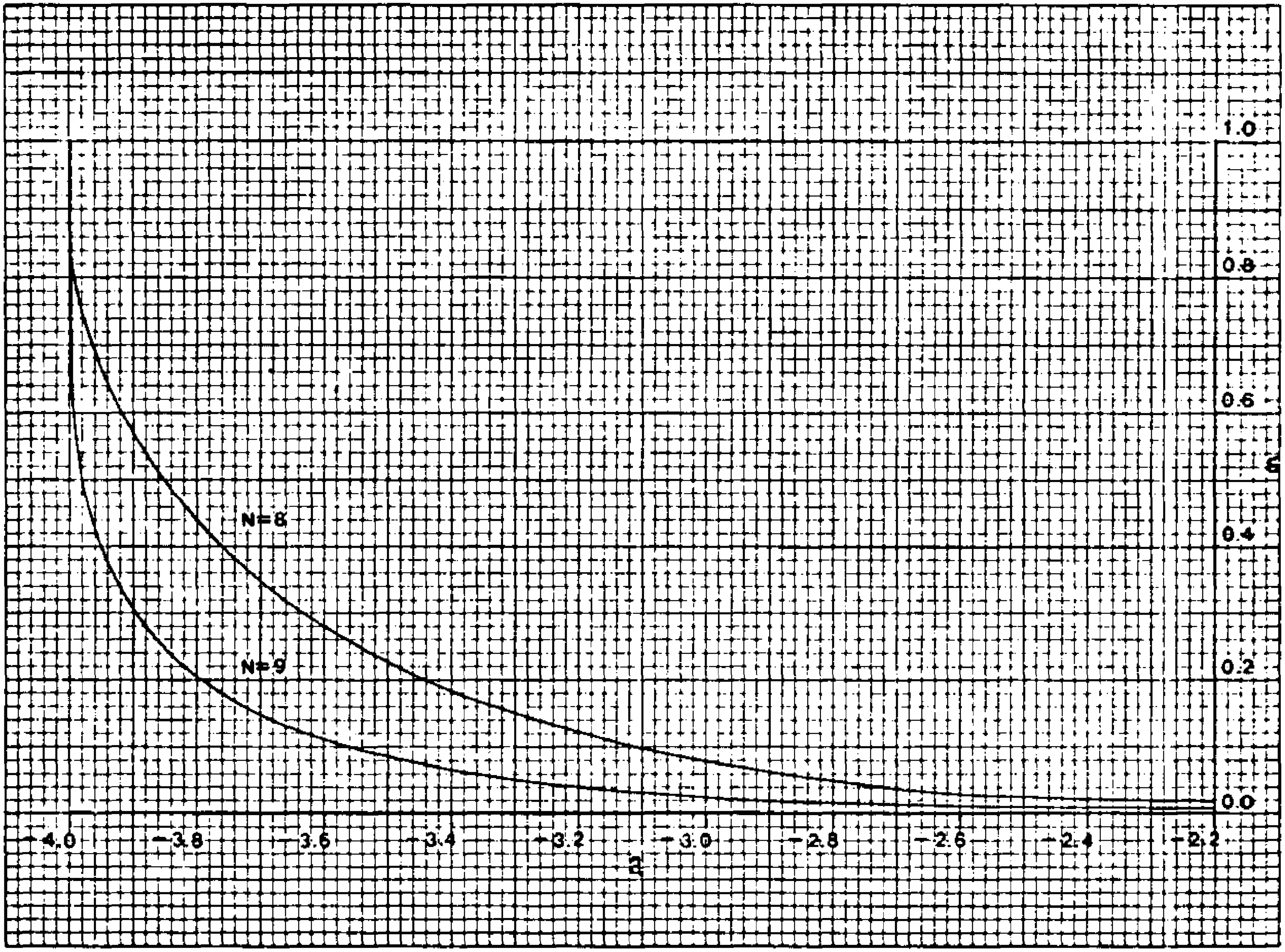


FIGURE 4-4

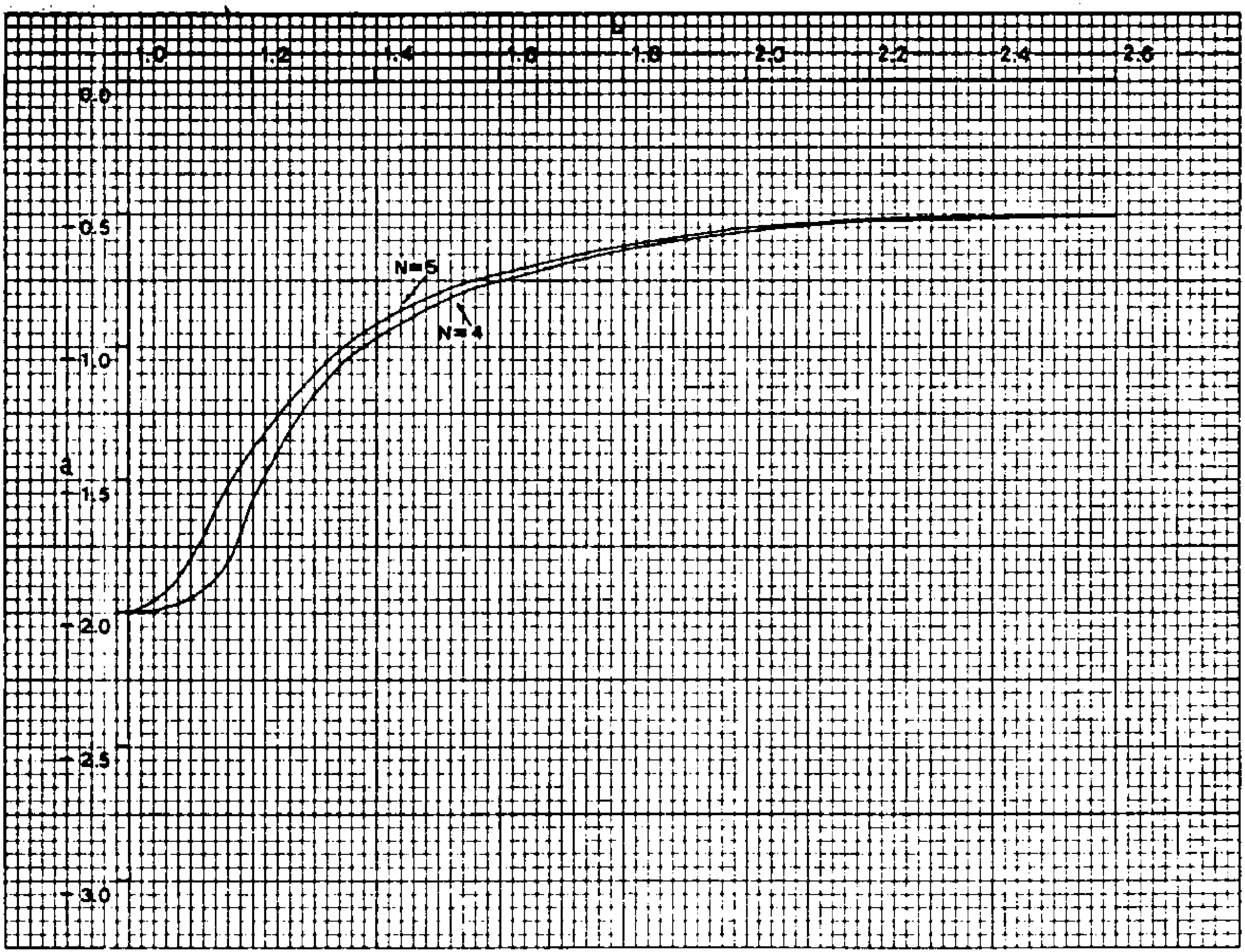


FIGURE 4-5

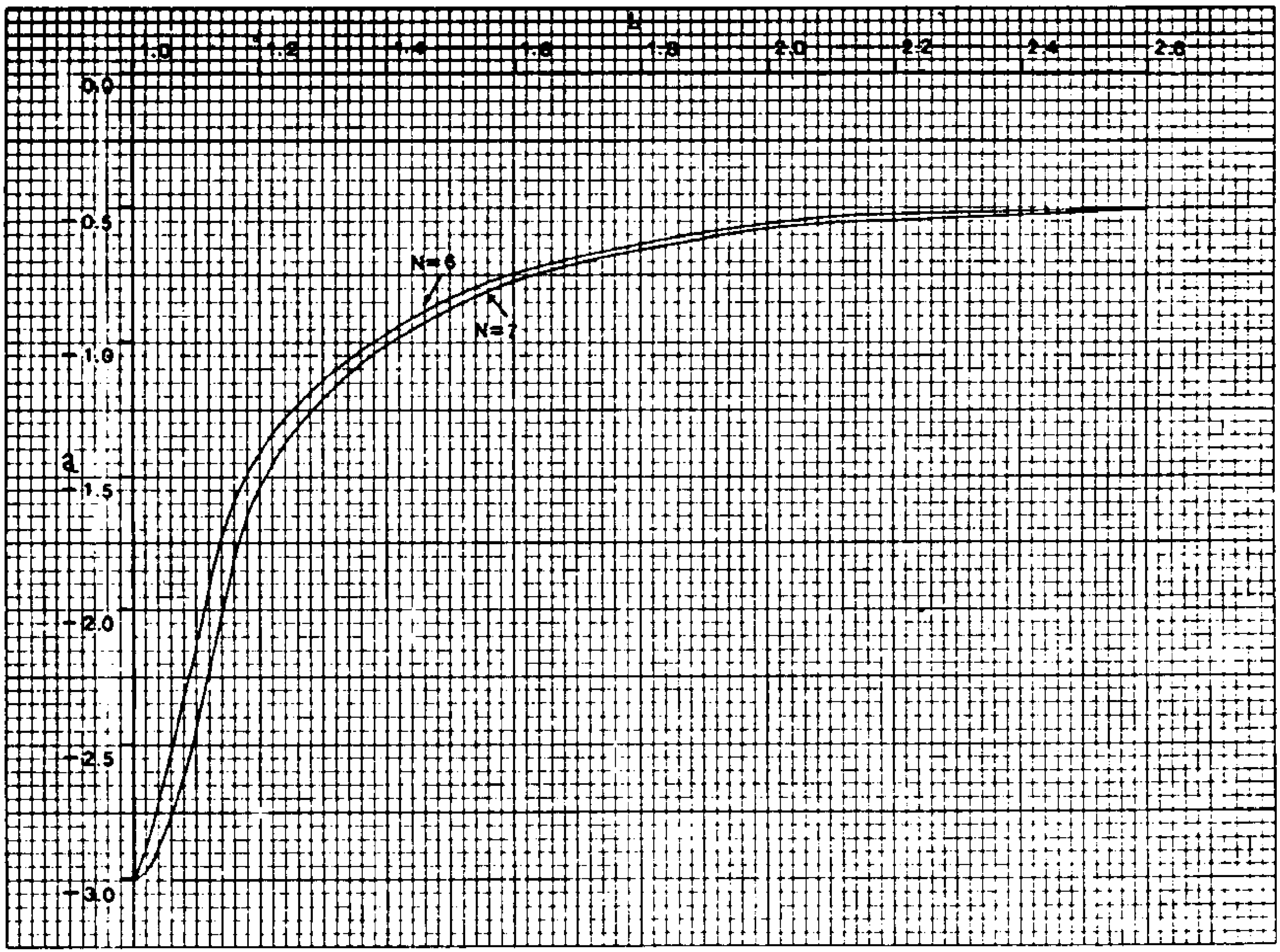


FIGURE 4-6

we see from Figure 4-5 that if $a = -1.685$, then $b = 1.180$, which checks to two decimal places.

B. Procedure for Finding the Elliptic Filter

In this section we shall consider a direct method of obtaining the optimal (elliptic) filter from a given set of specifications. Suppose that it is desired to obtain a magnitude function of minimum complexity whose graph lies in the shaded area shown in Figure 4-7, with transition interval specified by c. Since we know that such an optimal filter is the elliptic filter, the problem is then that of finding an appropriate a and b with which to construct the ultraspherical rational function $f_n(a,b;\omega)$ for a minimum n.

From the given values of A_1 and A_2 we calculate ϵ^2 and δ^2 given by (2.27), and from the calculated value of δ and the known value of c, we may select from Figure 2-1 the minimum n which either achieves or betters the given c. Using the known values of δ and n, we obtain the appropriate value of b from Figure 4-1, and the appropriate value of a from one of Figures 4-2, 4-3, or 4-4. Alternately we may obtain a from the knowledge of n and b by using one of Figures 4-5 or 4-6. (A more complete set of graphs could be drawn by considering values of n beyond $n = 9$. It should be noted that an elliptic filter having $n = 9$ is comparable, in fact superior, to a Chebyshev filter of order 18.)

C. An Example

To illustrate the method of obtaining the parameters a and b which yield the elliptic filter for a given set of specifications, as shown in Figure 4-7, let us consider the following example. Suppose it is desired to obtain a magnitude function of an optimal filter having

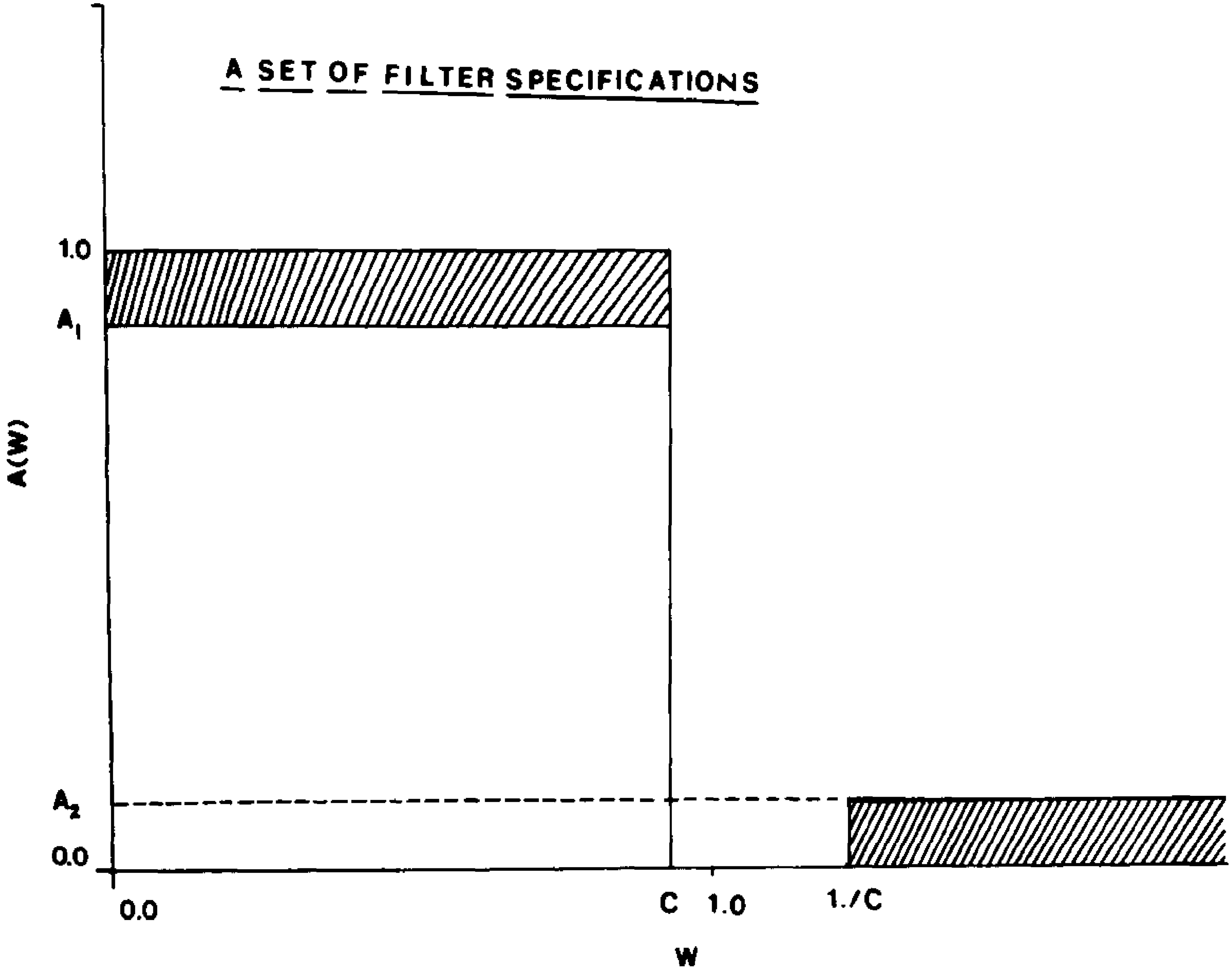


FIGURE 4-7

$$A_1 = .9,$$
$$A_2 = .1, \qquad (4.1)$$
$$.99 \le c < 1.$$

By (2.27) we have

$$\varepsilon^2 = \frac{\sqrt{(1 - A_1^2)(1 - A_2^2)}}{A_1 A_2} = 4.819,$$

and

$$\delta^2 = \frac{A_2 \sqrt{1 - A_1^2}}{A_1 \sqrt{1 - A_2^2}} = .0487,$$

or $\delta = .221$. From Figure 2-1 it is seen that the minimum n for this value of δ and the given c is $n = 6$. (This value actually corresponds to $c = .9945$, whereas $n = 5$ yields only $c = .986$.) From Figure 4-1 we have $b = 1.066$ and from Figure 4-3 we have $a = -2.47$.

Hence the elliptic filter satisfying the specifications of (4.1) has magnitude function given by

$$A(\omega) = \frac{1}{\sqrt{1 + 4.819\ f_6^2(-2.47, 1.066;\ \omega)}} \qquad (4.2)$$

A plot of this function is shown in Figure 4-8, where it may be seen that the equiripple property is present and that (4.1) is satisfied.

D. <u>The PATERN Search Strategy</u>

Another method of obtaining the elliptic filter by varying the parameters a and b is to use the computer to exploit the fact that the elliptic filter is the optimal filter. This may be done by using a computer program which determines one or more parameters which minimize some pre-selected <u>cost</u> function. This is done by computing the cost for a wide range of parameters and selecting the ones which produce the minimum cost.

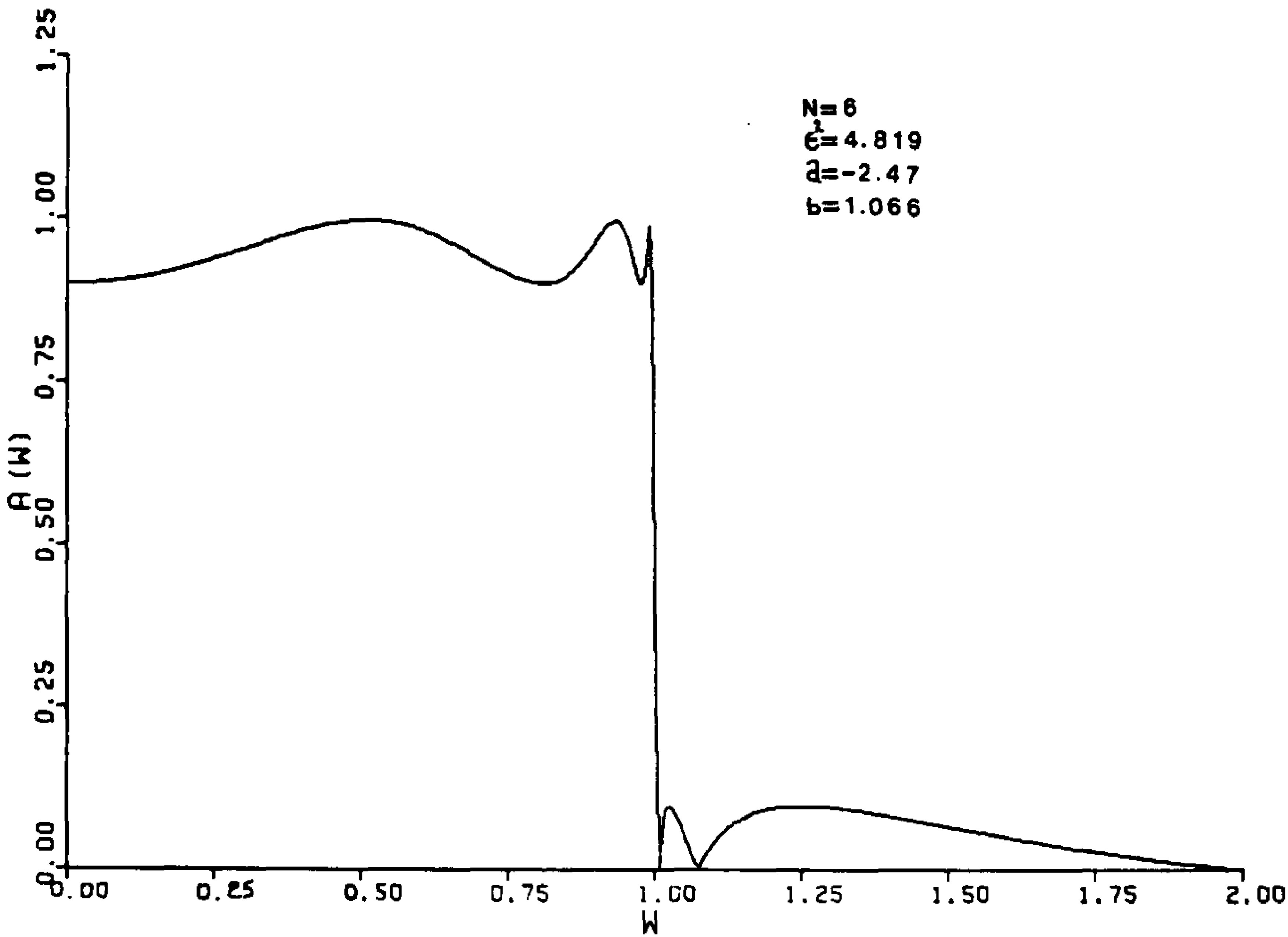


FIGURE 4-8

Such a program is the PATERN search strategy [8], which is available at the College of Engineering, Louisiana State University. In order to use Patern in a digital search, three programs are required besides subroutine Patern as shown in the Macro flow chart, which follows.

MAIN PROGRAM

Contains the overall logic peculiar to the particular study being made. It also defines all the initial search Parameters and calls Subroutine Patern.

PATERN

Optimization Subroutine which conducts the search specified in the main program and RETURNs only after the optimum answer has been determined.

PROC.

Subroutine called by Patern. It contains the particular function to be searched. Given a value for a set of search Parameters PROC calculates a corresponding value of cost = f(P).

BOUNDS

Subroutine called by Patern to determine if a particular set of search parameters have violated any constraint.

The subroutine Patern contains all the necessary logic required to perform a particular optimization. It is essentially self-contained and is in ready-to-use form. However, PROC and BOUNDS are both subroutines which are written particularly for the problem under study.

A set of filter specifications is given in Figure 4-7. One is asked to determine $A(\omega)$ such that

$$A_{max} \geq A(\omega) \geq A_1 \quad , \quad |\omega| \leq c \ ,$$
$$A(\omega) \leq A_2 \quad , \quad |\omega| \geq \frac{1}{c} \ , \tag{4.3}$$

for the smallest possible value of $\frac{1}{c}$. (Since minimizing $\frac{1}{c}$ results in an optimal filter.)

$A(\omega)$ is written in the form

$$A(\omega) = \frac{1}{\sqrt{1 + \epsilon^2 [f_n(a,b;\omega)]^2}} \tag{4.4}$$

where

$$f_n(a,b;\omega) = \frac{(-1)^{[n/2]} \ F_n{}^a(b\omega)}{\omega^n \ F_n{}^a \ (b/\omega)} \quad , \tag{4.5}$$

as previously defined, is the ultraspherical rational function with a and b real, and n = 2,3,4,... .

We define cost as

$$\text{Cost} = \frac{1}{c} + |Z_1 - A_1| + |Z_2 - A_2| \ , \tag{4.6}$$

where for any particular value of a and b,

Z_1 = minimum value of $A(\omega)$ in $0 < \omega < c$

Z_2 = maximum value of $A(\omega)$ in $\omega > \frac{1}{c}$.

Since PATERN calls for PROC and Proc calls for Cost, then subroutine Patern conducts the search to find the best a and b, such that the cost becomes minimum. Minimum cost results in $|Z_1 - A_1|$ and $|Z_2 - A_2|$

virtually going to zero, which guarantees (4.3) to be satisfied; also $\frac{1}{c}$ has the smallest value which results in the optimal filter.

The cost function may be weighted, equally or otherwise, in order to improve the output results. The constraints for subroutine BOUNDS are

$$a < -\frac{1}{2}\ , \quad b > 1\ ,$$

$$1 \geq Z_1 \geq A_1\ , \quad 0 < \omega < c\ ,$$

$$0 \leq Z_2 \leq A_2\ , \quad \omega > \frac{1}{c}\ .$$

E. An Example

To illustrate the method of obtaining the parameters a and b which yield the optimal filter for a given set of specifications, let us use $A_1 = .98$ and $A_2 = .075$ for $N = 5$; as an input data. The output after 281 functional evaluations with cpu-time equal to one minute and nine seconds is as follows:

$$\begin{aligned}
\text{Cost} &= 1.04802 \\
a &= -1.67656 \\
b &= 1.12964 \\
c &= \sqrt{k} = .955344 \\
Z_1 &= A_1 = .978727 \\
Z_2 &= A_2 = .075 \\
\epsilon &= 1.64311
\end{aligned}$$

A plot of this specification is shown in Figure 4-9.

By (2.27), for $A_1 = .98$ and $A_2 = .075$, we have $\delta = .1225$ and $\epsilon = 1.64311$. From Figure 4-1, for $n = 5$, we have $b = 1.128$, and from Figure 4-2 we have $a = -1.68$ which shows a very small difference. A

plot with the above specifications is shown in Figure 4-10, and is virtually identical to that of Figure 4-9.

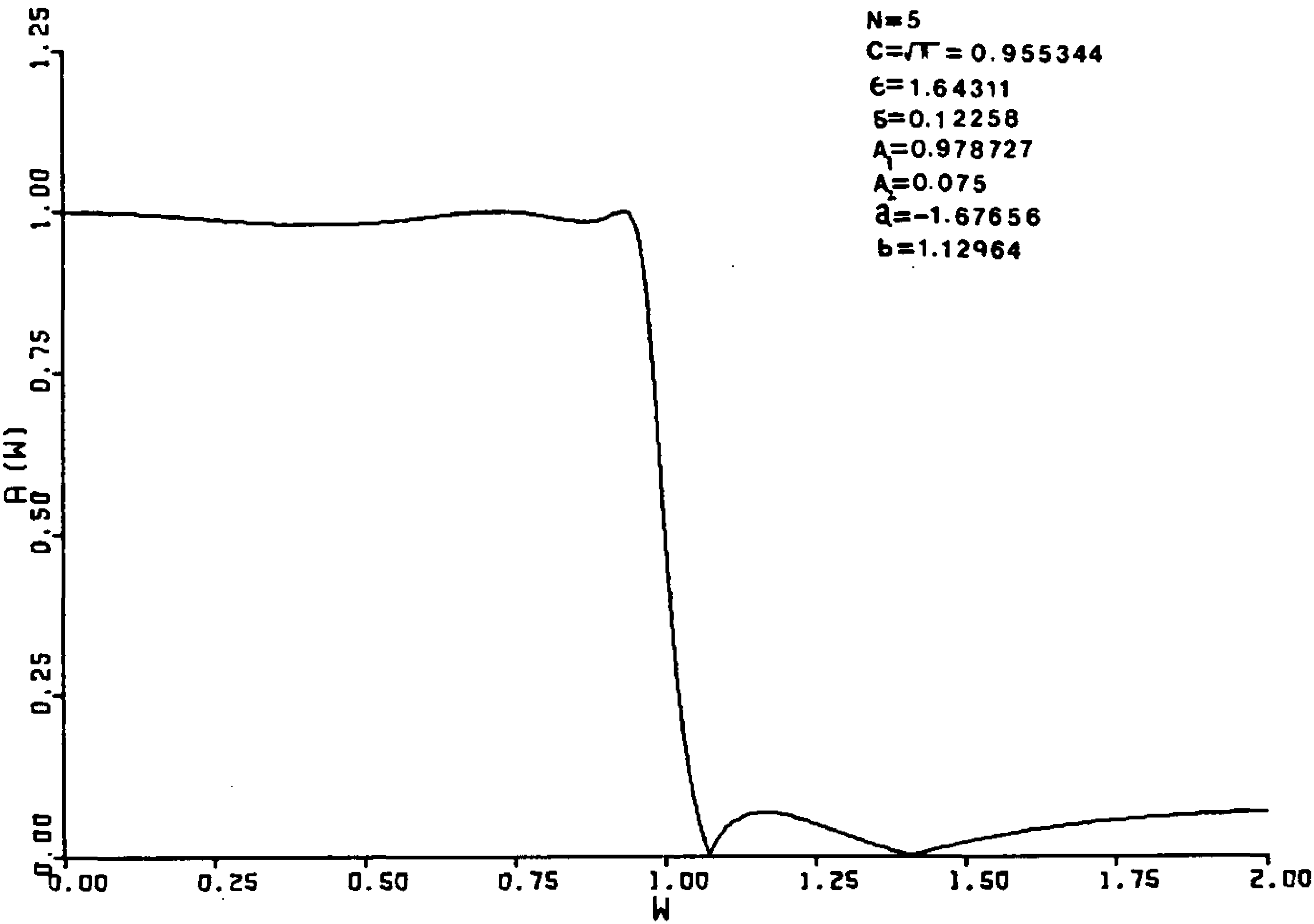

N=5
C=√K = 0.955344
ϵ=1.64311
δ=0.12258
A1=0.978727
A2=0.075
a=-1.67656
b=1.12964
A (W)
0.00
0.25
0.50
0.75
1.00
1.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
W

FIGURE 4-9

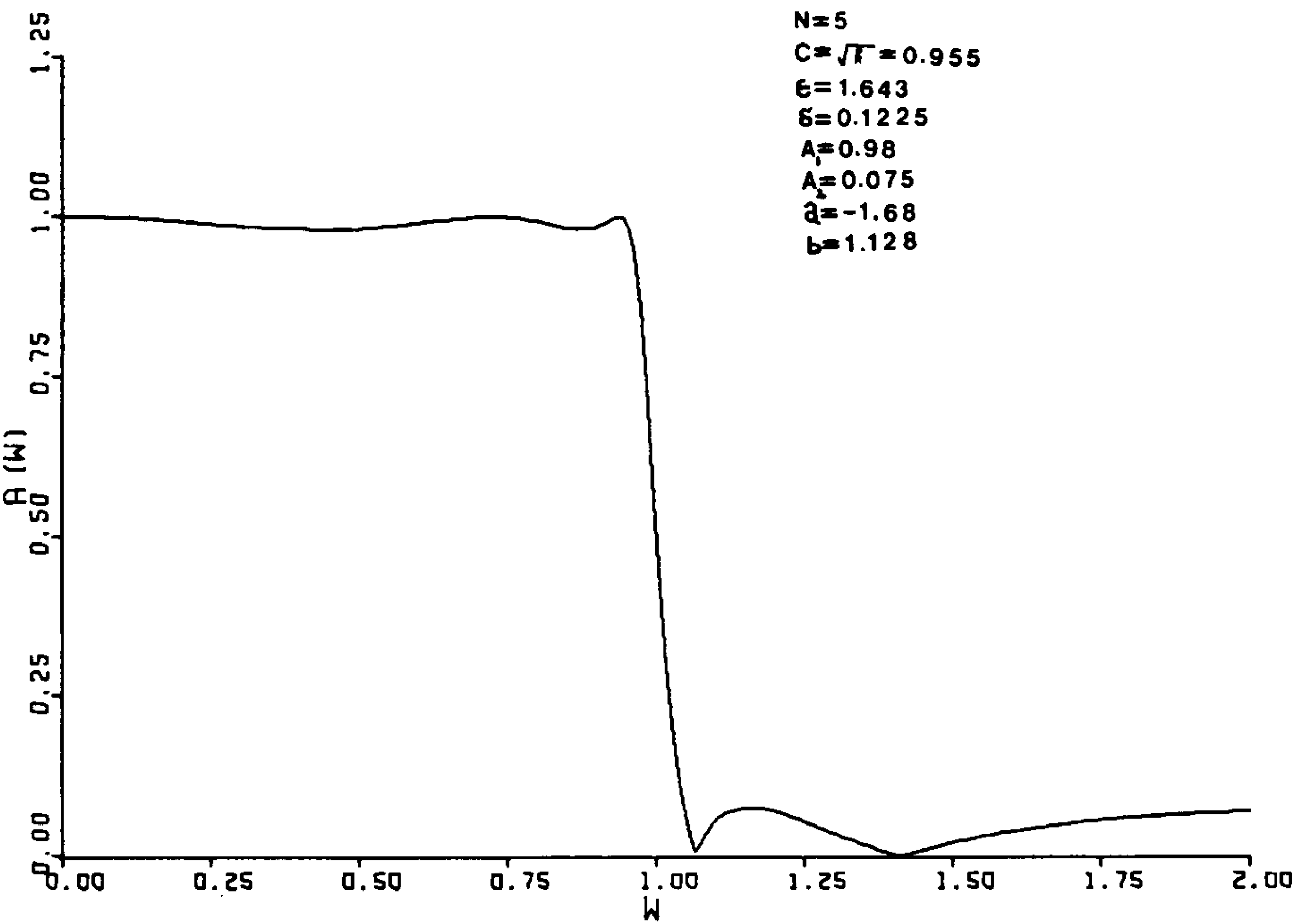

N=5
C=√Γ=0.955
ε=1.643
δ=0.1225
A₁=0.98
A₂=0.075
a=-1.68
b=1.128
A(W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 4-10

CHAPTER V

VARIATIONS OF THE PARAMETERS

A. Varying the Parameter a

We have observed in the previous chapters that in order for the ultraspherical rational function to be the Chebyshev rational function, the parameters a and b must be given by (3.14) and (3.15), and the ranges on these parameters must be given by

$$\begin{aligned} -[n/2] \le a \le -\frac{1}{2}, \\ 1 \le b < \infty . \end{aligned} \tag{5.1}$$

These correspond to the range on the modulus k given by

$$1 \ge k \ge 0 . \tag{5.2}$$

By (3.4) we see that the zeros of $f_n(a,b;\omega)$ are the zeros of $F_n{}^a(b\omega)$, and that their nature (that is, real, imaginary, etc.) is determined by the parameter a. The parameter b is merely a scale factor as far as the zeros are concerned. This may be seen by examining the case $b = 1$, for which the zeros are those of the ultraspherical polynomial $F_n{}^a(\omega)$. If ω_1 is a zero of the ultraspherical polynomial, then ω_1/b is a zero of the ultraspherical rational function $f_n(a,b;\omega)$. Hence, if $b > 1$, the real zeros are shifted toward the origin and, if $0 < b < 1$, they are shifted toward infinity. In this section we consider the

effect on the zeros of varying $\underline{a}$ for a fixed $\underline{b}$ and in section C we consider variations in b.

Let us consider first the case for the extreme values $b = 1$ and $b \to \infty$, which b may assume in order that $R_n = f_n$. If $b \to \infty$, no matter what value $\underline{a}$ has, the ultraspherical rational filter becomes the Butterworth filter, as was pointed out in Chapter III. For the case $b = 1$, if $a > -1$ we know that the zeros of f_n are all real and on $(-1,1)$, since this is true of the ultraspherical polynomial zeros. Hence $A(\omega)$ for this case will resemble the elliptic filter magnitude except that the ripples will be unequal.

If we define the ultraspherical polynomial for $a = -1$ by

$$P_n^{(-1,-1)}(\omega) = \lim_{a \to -1} P_n^{(a,a)}(\omega)$$

then by (2.8) we have

$$P_n^{(-1,-1)}(\omega) = \sum_{k=1}^{n} \frac{(-n)_k (n-1)_k \, \Gamma(n) \left(\frac{1-\omega}{2}\right)^k}{n! \; k! \; \Gamma(k)}, \tag{5.3}$$

and hence

$$P_n^{(-1,-1)}(1) = 0. \tag{5.4}$$

Therefore by (3.7) we see that if $b = 1$, $a = -1$, then $\omega = 1$ is a zero (and a pole) of f_n. This agrees with the known property of the ultraspherical polynomials that for $a > -1$, the zeros migrate toward -1 as a decreases. (This may be developed by using oscillation theory. See for example, [9], Chapter 10.) Figure 5-1 shows the case $a = -1$, $b \neq 1$, where b is chosen small enough to shift the zeros into the range $\omega > 1$.

Information concerning the nature of the zeros as $\underline{a}$ changes may be obtained by using Descartes' rule of signs, which states that the

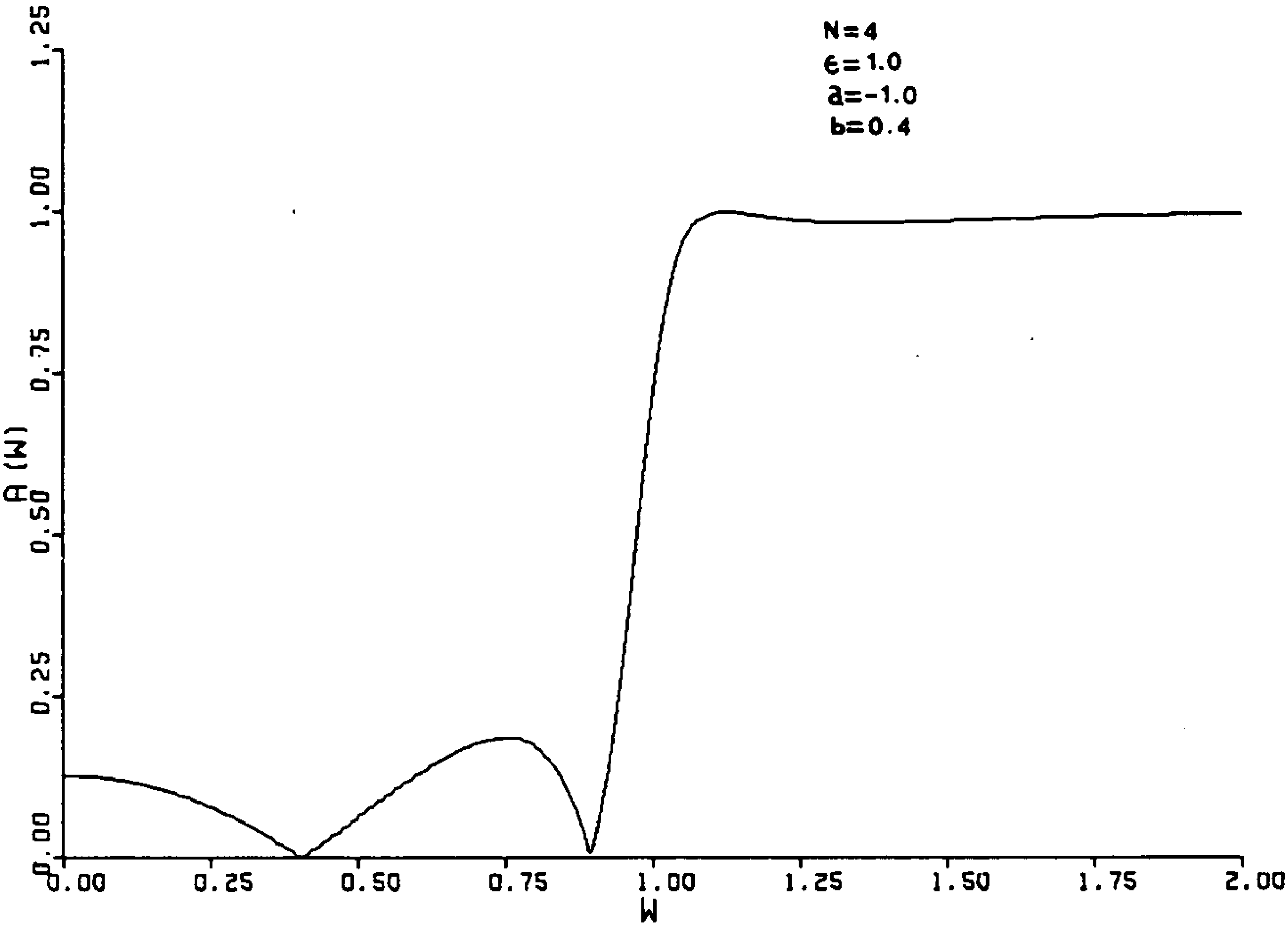

N=4
ϵ=1.0
a=-1.0
b=0.4
A (W)
0.00
0.25
0.50
0.75
1.00
1.25
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 5-1

number of positive zeros of a polynomial equals the number of variations in sign of its coefficients minus a nonnegative even number.

For $b = 1$, the coefficients in the numerator of f_n as given in (3.12) may be written $A_n = 1$ and

$$A_{n-2i} = \frac{(-1)^i \, n!}{i! \; (n-2i)!(2a+2n-1)(2a+2n-3)\ldots(2a+2n-2i+1)2^i} \tag{5.5}$$

for $i = 1,2,\ldots,[n/2]$. Let us define the quantity

$$a_j = -n + j + \frac{1}{2} \; ; \; j = 0,1,2,\ldots,[n/2] - 1. \tag{5.6}$$

Then we see that $A_n = 1$, A_{n-2} contains $(a - a_0)(-1)^1$, A_{n-4} contains $(a - a_0)(a - a_1)(-1)^2$, A_{n-6} contains $(a - a_0)(a - a_1)(a - a_2)(-1)^3$, and in general A_{n-2i} contains i factors, $(a - a_0)$, $(a - a_1)$, $\ldots$, $(a - a_{i-1})$ as well as the factor $(-1)^i$. Hence if $a < a_0$, all A_{n-2i} are positive and have no variations in sign. Thus if $a < a_0$, then f_n has no positive zeros. Since f_n is even it follows that it has no negative zeros either, and except for the zero, $\omega = 0$ in the odd case, none of the zeros of f_n are real.

If $a_0 < a < a_1$, then $A_n = 1$, but all the A_{n-2i} for $i \geq 1$ change signs. Thus there is for this case one variation in sign and hence f_n has one positive zero. Continuing in this fashion we have that

$$a < a_0 \tag{5.7}$$

implies no variation in sign,

$$a_{j-1} < a < a_j \; ; \; j = 1,2,\ldots[n/2] - 1 \tag{5.8}$$

implies j variations in sign, and

$$a > a_{[n/2]-1} \tag{5.9}$$

implies $[n/2]$ variations in sign.

Since the amplitude function $A(\omega)$ reaches its maximum value of 1 at the zeros of f_n and its minimum value of 0 at the poles (the reciprocals of the zeros) of f_n, it is evident that the parameter $\underline{a}$ may be varied to completely change the nature of the filter. For example, if $n = 4$, $b = 1$, we have

$$a < a_0 = -\frac{7}{2} \quad \text{(no variations)},$$

$$a_0 < a < a_1 = -\frac{5}{2} \quad \text{(1 variation)},$$

$$a_1 < a \qquad \text{(2 variations)}.$$

Hence if $a < -\frac{7}{2}$, f_4 has no real zeros; if $-\frac{7}{2} < a < -\frac{5}{2}$, f_4 has one positive zero; and if $-\frac{5}{2} < a$, f_4 has either two or no positive zeros.

In this simple case ($n = 4$), we may show that for $b = 1$, and

$$-\frac{5}{2} < a < -2$$

there are no positive zeros. They are all complex. Also if

$$-2 < a < -1$$

there are two positive zeros, one on $\omega > 1$ and one on $0 < \omega < 1$. Finally, as is generally the case, if $a > -1$, all four zeros are real and on $-1 \le \omega \le 1$. The positive zero for the case $-\frac{7}{2} < a < -\frac{5}{2}$ is on $\omega > 1$.

The cases where the zeros of f_n are on the interval $\omega > 1$ we have $A(\omega)$ achieving its maximum value of 1 on $\omega > 1$. Also since in

this case the poles of f_n are on $(-1,1)$ and for these values of ω we have $A(\omega) = 0$. Thus the nature of the filter is completely changed from the low-pass characteristic to a band-pass characteristic.

These different cases of a for $n = 4$, $b = 1$, and $\epsilon = 1$ are shown in Figures 5-2 through 5-7. In Figure 5-2 we have two cases of $a > -1$ which resemble the elliptic filter magnitude but are not equiripple. The zeros of f_4 are all on $(-1,1)$. In Figure 5-3, we have $-2 < a < -1$ in which case one zero is on $\omega > 1$ and one is on $(0,1)$. One pole is thus on $(0,1)$ and one on $\omega > 1$. The filter in this case passes two bands. Figure 5-4 illustrates the case $-\frac{5}{2} < a < -2$. All the zeros are complex so that f_4 does not attain its maxima of 1 or its minima of 0 at real values of ω. In this case $a = -2.4$ is near the break-point $a = -2.5$, at which f_4 degenerates to $\frac{1}{\omega^4}$, an inverse Butterworth function. Hence $A(\omega)$ resembles the high-pass inverse Butterworth order of 4.

Figure 5-5 shows the case $-\frac{7}{2} < a < -\frac{5}{2}$ for which the only real positive zero of f_4 is on $\omega > 1$. Hence the maxima $A(\omega) = 1$ is achieved on $\omega > 1$ and the minima $A(\omega) = 0$ is achieved on $(0,1)$. Finally Figures 5-6 and 5-7 show the case $a < -\frac{7}{2}$. All the zeros of f_4 are imaginary so that $A(\omega) = 0$ and $A(\omega) = 1$ are not achieved for real ω. The values of a in these figures are -3.8 and -4.2, which are on opposite sides of $a = -4$. The filters are different in nature also. The value $a = -4$ is a special case resulting in

$$f_4(-4,1;\omega) = \frac{\omega^4 + 6\omega^2 + 1}{1 + 6\omega^2 + \omega^4} = 1 \tag{5.10}$$

The zeros are reciprocals as may be seen by the symmetry of the coefficients. Since the poles are reciprocals of the zeros, the function

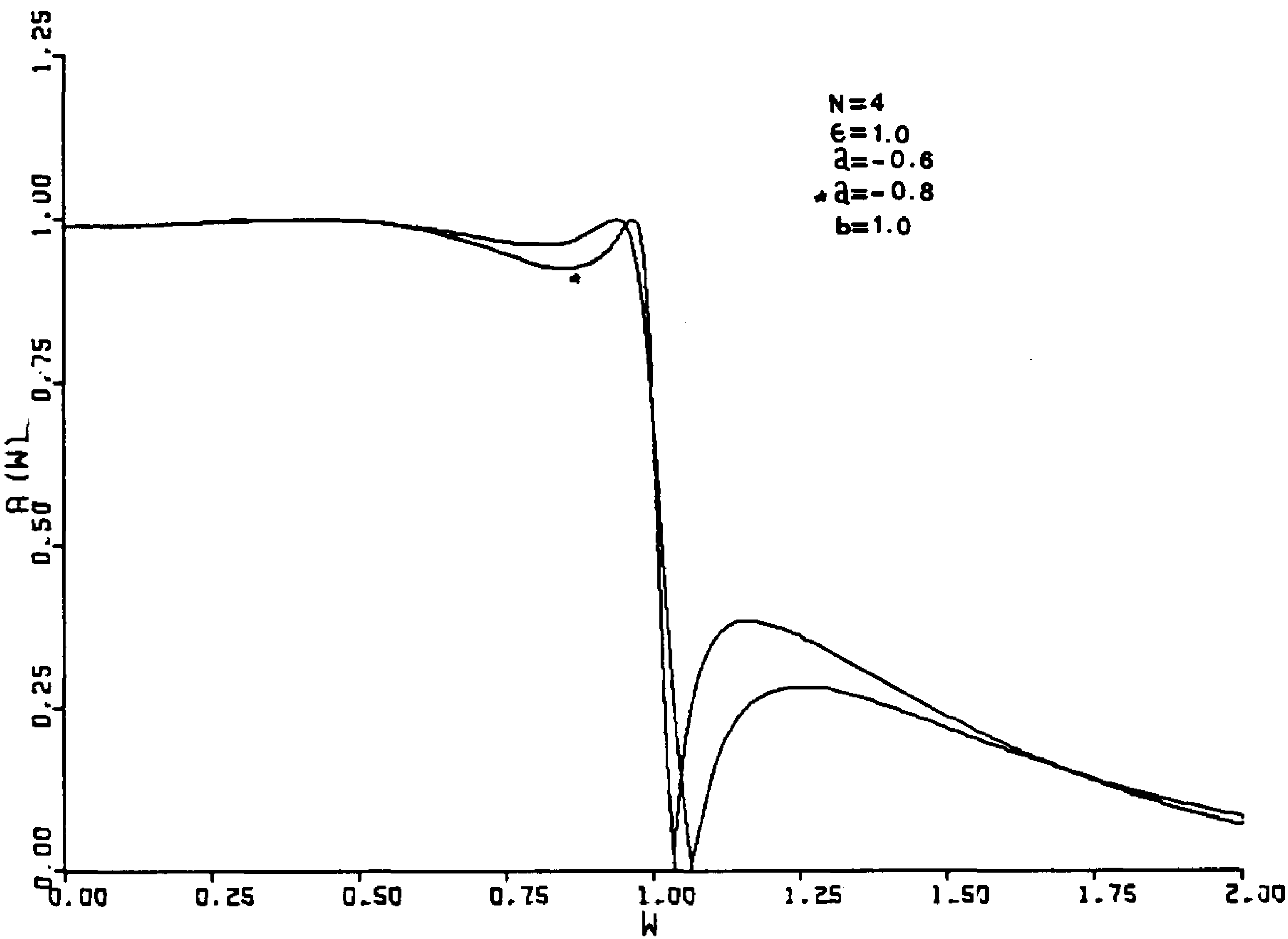


FIGURE 5-2

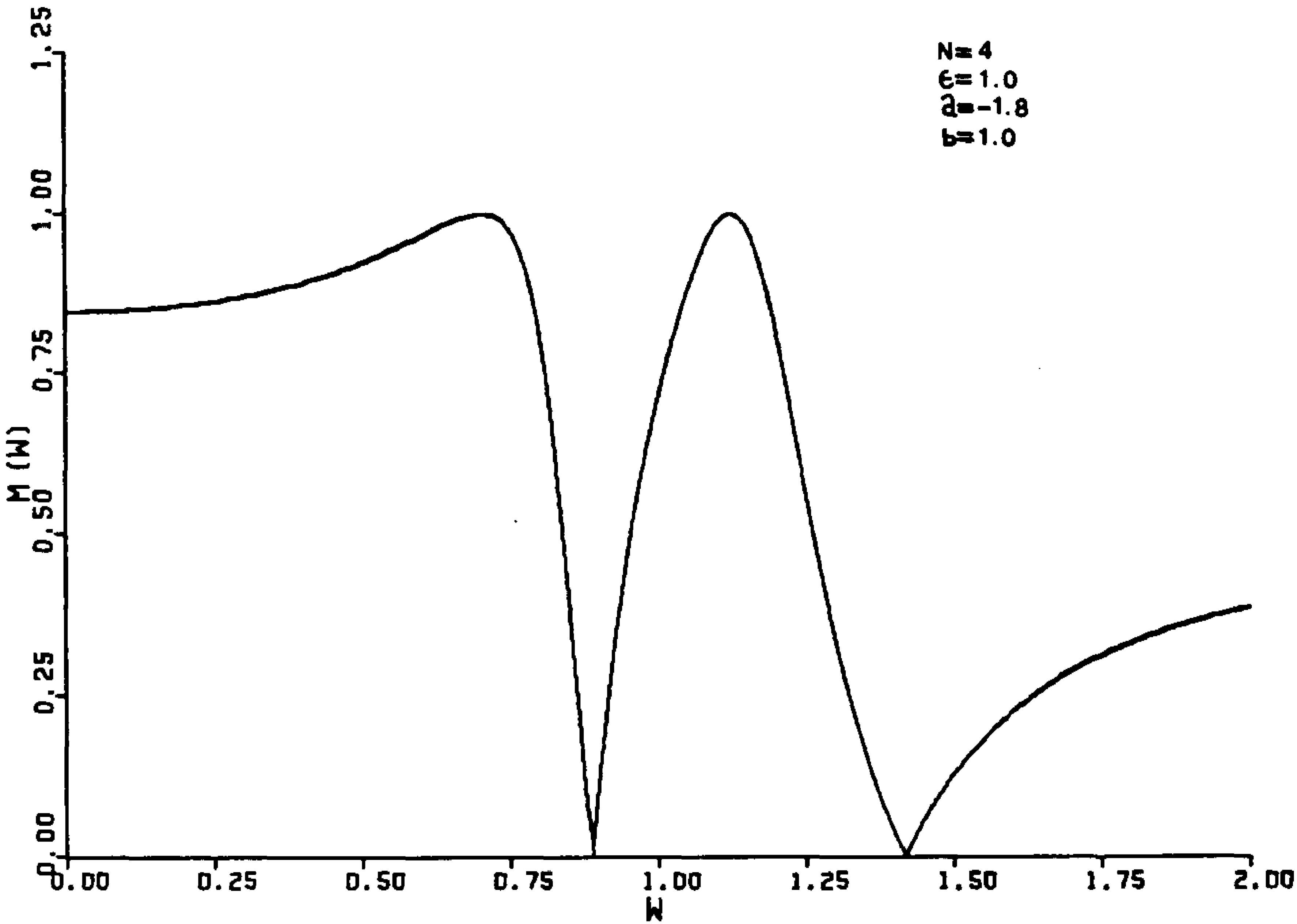


FIGURE 5-3

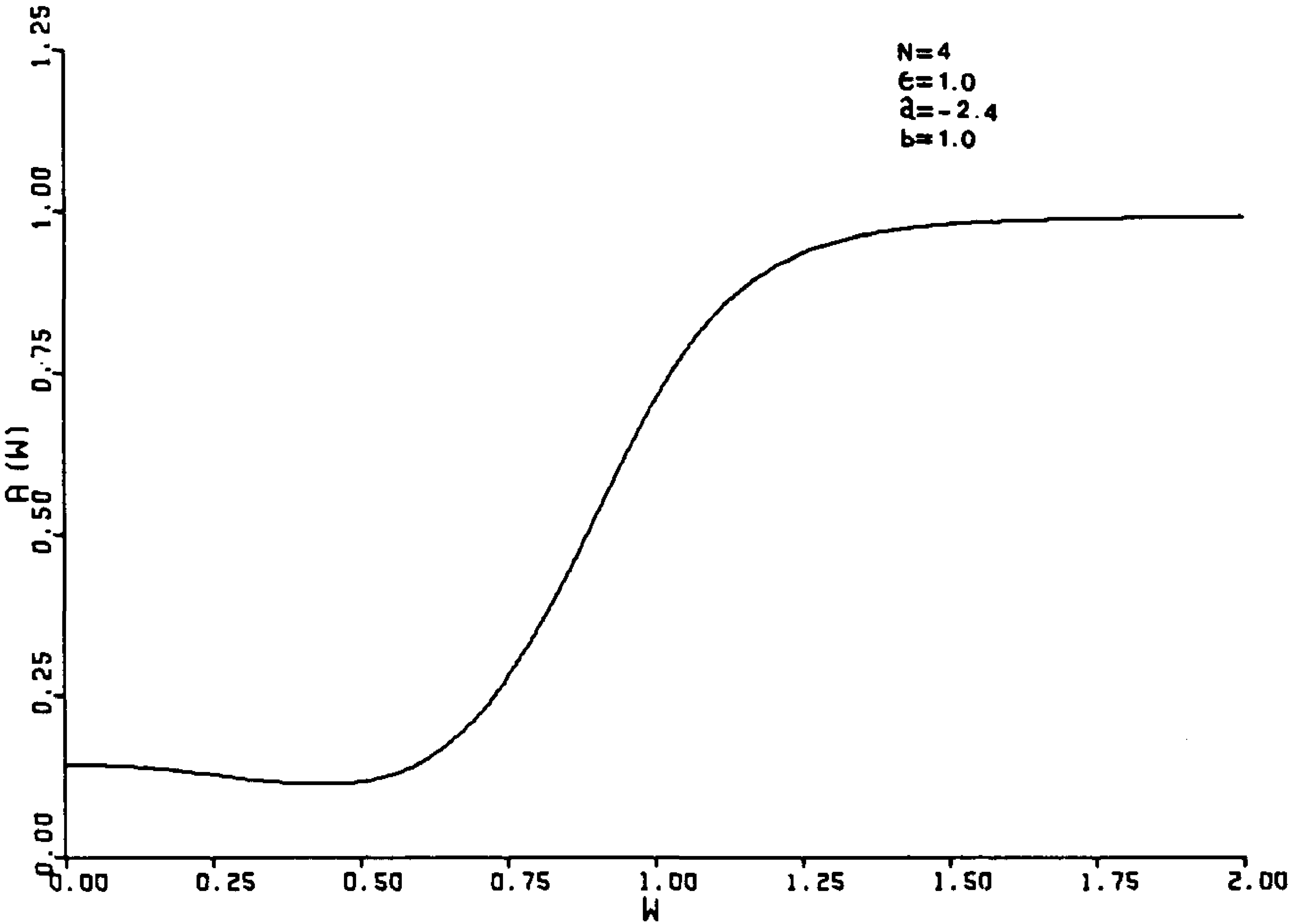


FIGURE 5-4

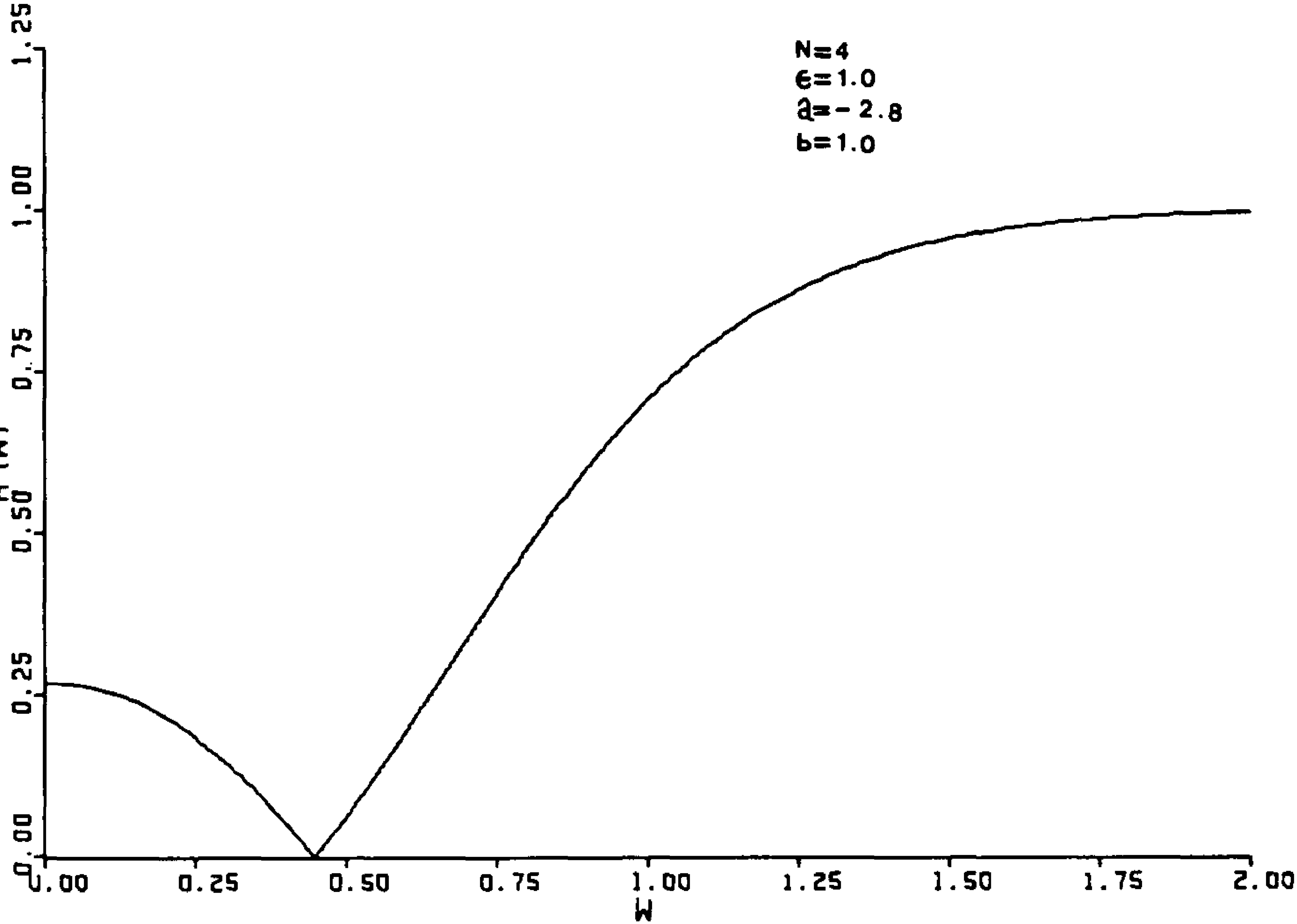


FIGURE 5-5

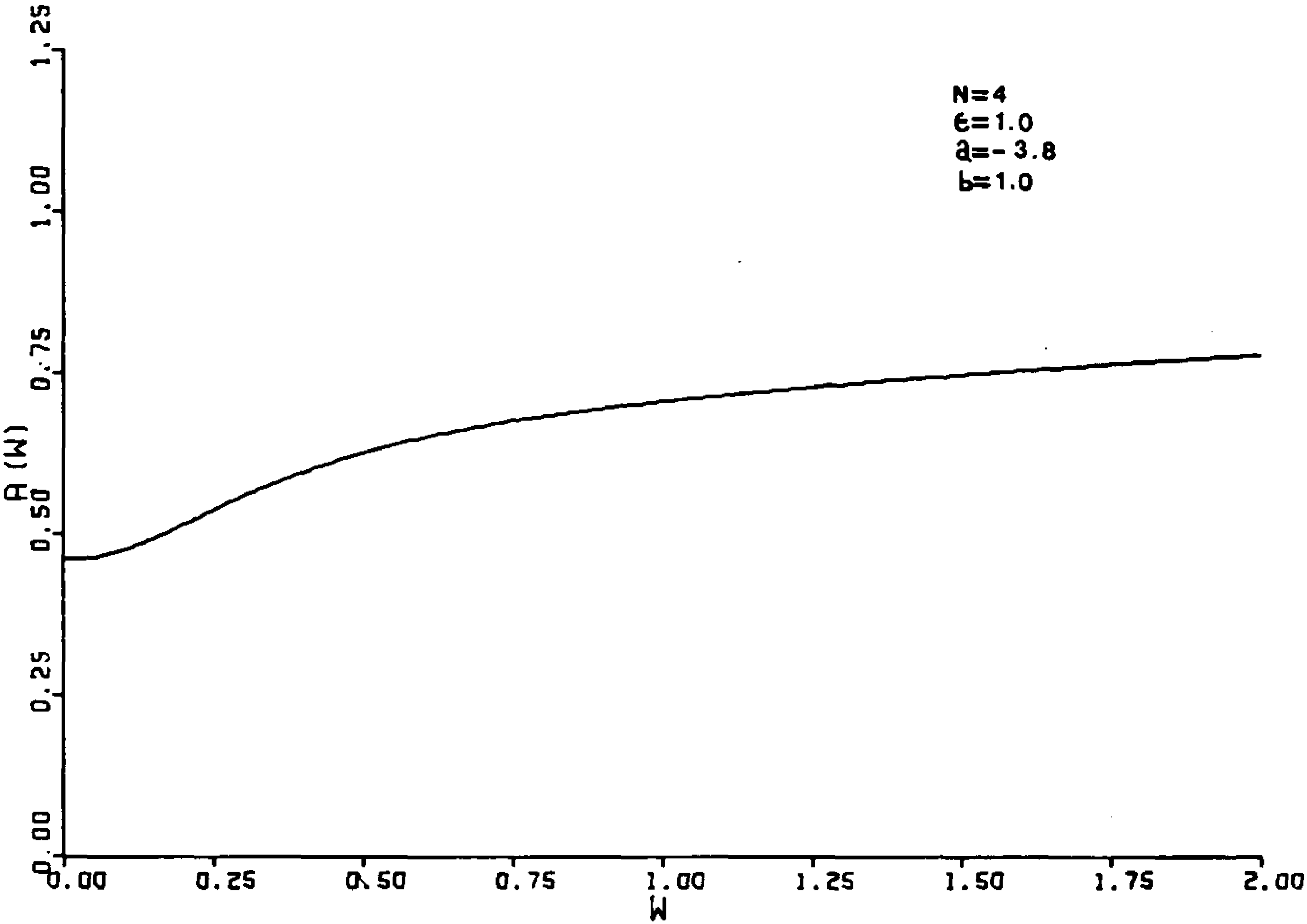


FIGURE 5-6

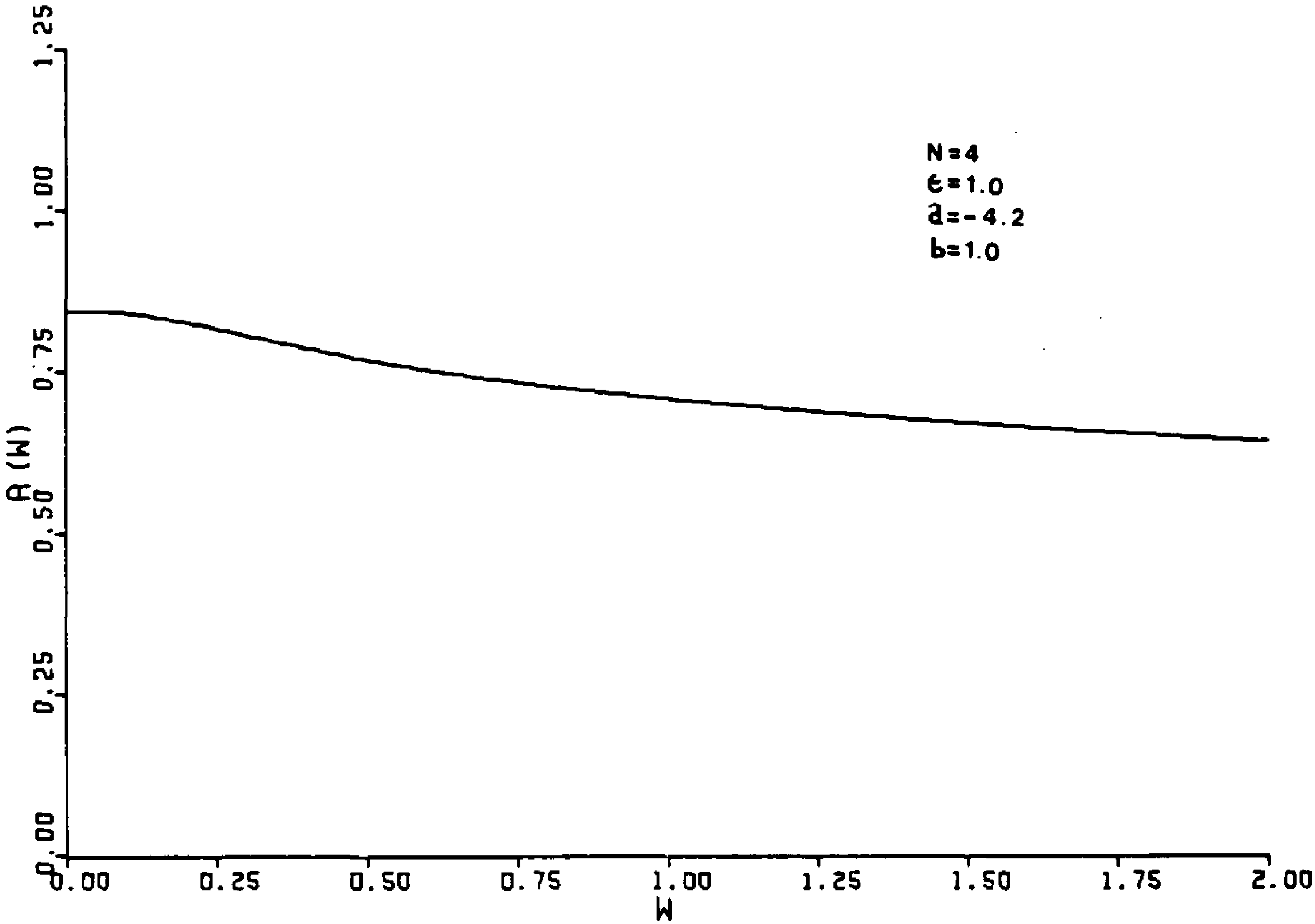


FIGURE 5-7

degenerates to $f_4 = 1$, as indicated. Hence the cases in Figures 5-6 and 5-7 represent approaches to $A = \frac{1}{\sqrt{2}}$ from two different directions.

Finally in this section we consider the case $a = -\frac{1}{2}$, for which as has been previously observed, the ultraspherical polynomial becomes the Chebyshev polynomial C_n of first kind. That is,

$$f_n(-\tfrac{1}{2},b;\omega) = \frac{(-1)^{[n/2]}\ C_n(b\omega)}{\omega^n\ C_n(b/\omega)}\ . \tag{5.11}$$

Hence if $b = 1$, since $a > -1$, the zeros of f_n are on $(-1,1)$ so that the filter resembles the elliptic filter except that the ripples are unequal. Since $a = -\frac{1}{2}$ corresponds to $b \to \infty$ in the elliptic case, we should expect the ripples to tend toward equality as b increases. Also as b increases, the amplitude approaches the Butterworth amplitude, as has been previously noted. A graph of case $a = -\frac{1}{2}$, $b = 1$, $\epsilon = 1$ is shown in Figure 5-8 and for $b = 1.8$ in Figure 5-9.

In this case of the Chebyshev polynomials, the zeros are known to be

$$\omega = \frac{1}{b}\cos\frac{(2j-1)\pi}{2n}\ ;\ j = 1,2,\ldots,n\ ,$$

(see for example [10], page 175) and the values of f_n at the origin are given by

$$\begin{aligned} f_n(-\tfrac{1}{2},b;0) &= 0, && n \text{ odd} \\ &= \frac{(-1)^{n/2}}{2^{n-1}\ b^n}\ , && n \text{ even.} \end{aligned}$$

Thus by varying b we may adjust the zeros and for n even control the starting point of the amplitude function.

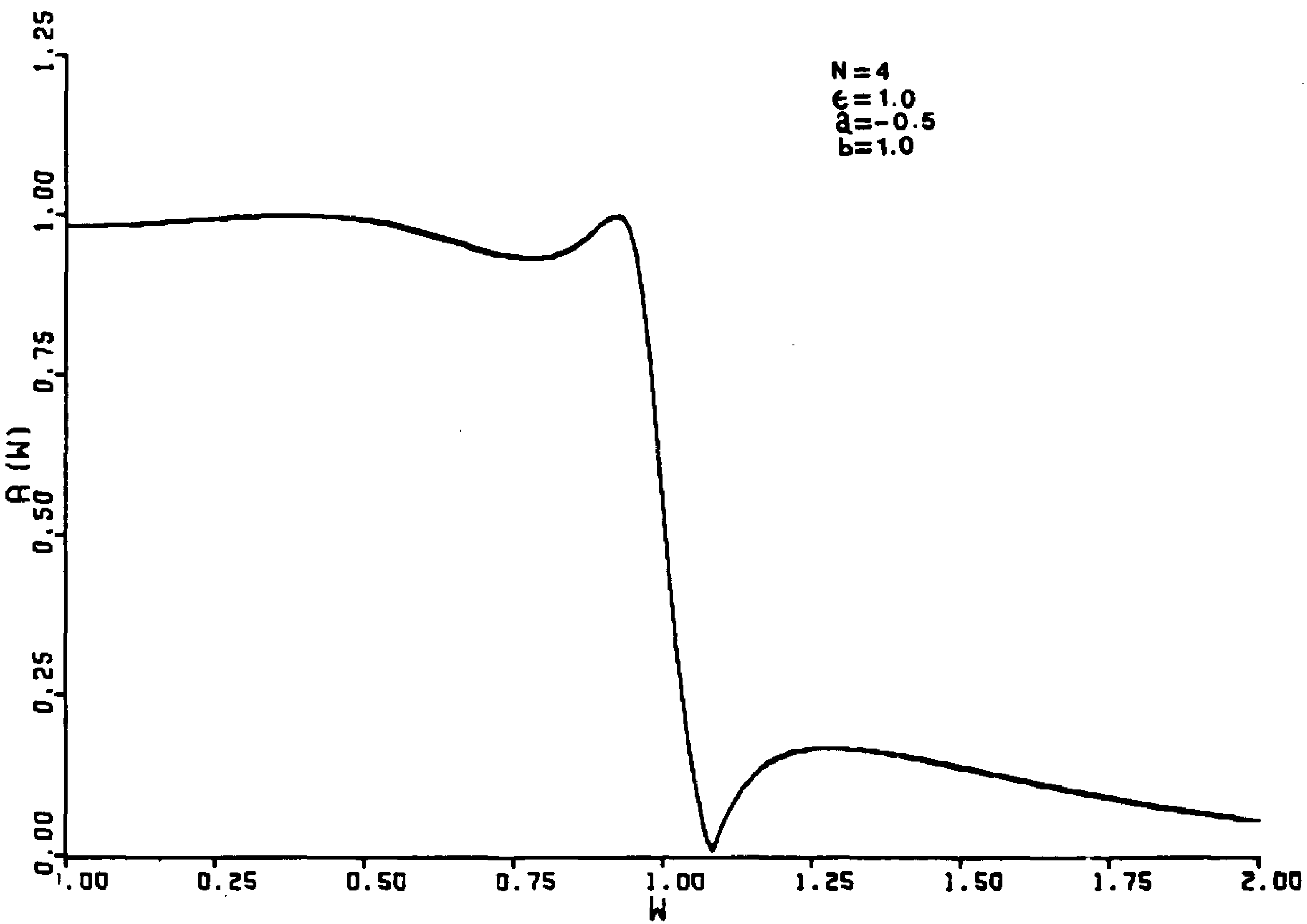

A (W)
0.00
0.25
0.50
0.75
1.00
1.25
W
.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
N = 4
ϵ = 1.0
a = -0.5
b = 1.0

FIGURE 5-8

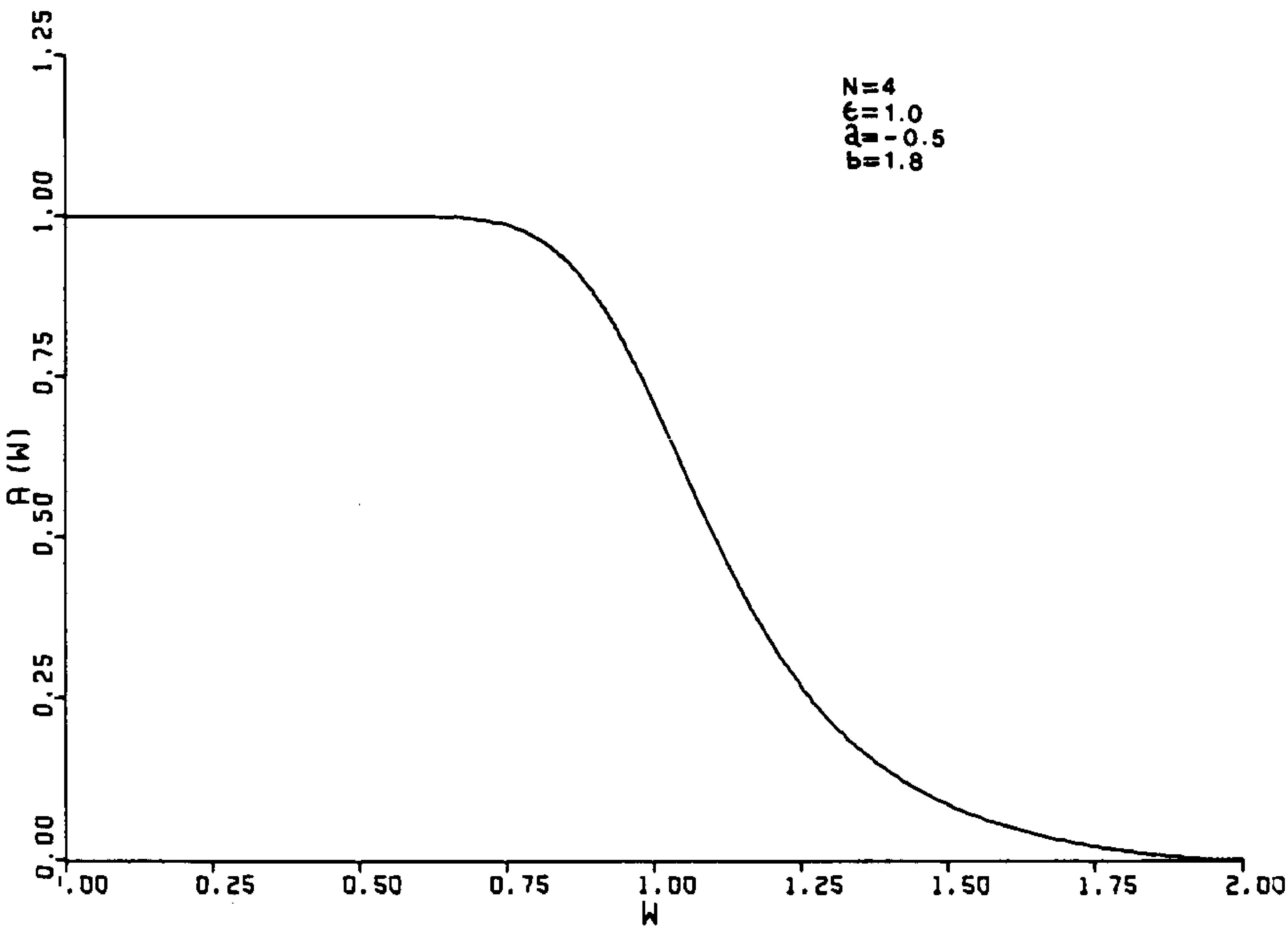


FIGURE 5-9

B. The Break-Points

As indicated in the previous section there are a number of break-points a_j, given by

$$a_j = -n + j + \frac{1}{2} \; ; \; j = 0,1,2,\ldots,[n/2] - 1, \tag{5.12}$$

which divide the a-axis into intervals on each of which the nature of the zeros of f_n is different. We consider in this section what the nature of the filter is when a assumes the values of the break-points.

If $a = a_{[n/2]-1} = [n/2] - n - \frac{1}{2}$, then as may be seen from the form of f_n given by (3.17), all the coefficients B_{n-2i} are zero except $B_{n-2[n/2]}$. Hence in this case we have

$$\begin{aligned} f_n(a,b;\omega) &= \frac{1}{\omega^n} \quad , \qquad n \text{ even} \\ &= \frac{1}{\omega^{n-2}} \; , \qquad n \text{ odd} \end{aligned} \tag{5.13}$$

This results in the high-pass inverse Butterworth as in the case $a = -2.5$ discussed in previous section for $n = 4$. Other examples of this case are

$$a = -\frac{7}{2} \; ; \quad n = 5,6,$$

$$a = -\frac{9}{2} \; ; \quad n = 7,8,$$

$$a = -\frac{11}{2} \; ; \quad n = 9,10.$$

Most of these may be readily checked from Table I of Chapter III.

For the case

$$a = a_{[n/2]-2} = [n/2] - n - \frac{3}{2}$$

all the B_{n-2i} except the last two ($i = [n/2], [n/2] - 1$) are zero, so that

$$f_n = \frac{(-1)^{[n/2]}\,(B_2\omega^2 + B_0)}{B_2\omega^{n-2} + B_0\omega^n} \quad ; \quad n \text{ even},$$

$$= \frac{(-1)^{[n/2]}\,(B_3\omega^3 + B_1\omega)}{B_3\omega^{n-3} + B_1\omega^{n-1}} \quad ; \quad n \text{ odd}. \tag{5.14}$$

Hence f_n has zeros for n even at

$$\omega = \pm\sqrt{\frac{-B_0}{B_2}} = \pm\ j/b\,\sqrt{n}$$

and poles at

$$\omega = \pm\ j\ b\,\sqrt{n}\ .$$

The other poles are at $\omega = 0$ and the other zeros are at infinity. For n odd, f_n has a zero at zero and zeros at

$$\omega = \pm\sqrt{\frac{-B_1}{B_3}} = \pm\ j\,\sqrt{3}/b\,\sqrt{n-1}\ .$$

The finite poles are at

$$\omega = \pm\ j\ b\,\sqrt{n-1}/\sqrt{3}\ .$$

The other poles are at $\omega = 0$ and the other zeros are at infinity.

Hence the nature of the filter is radically changed at these two break-points. A similar situation exists at the other break-points where one or more of the B_{n-2i} will be zero.

As an example, let us consider the case n = 6 for which the break-points are $a_0 = -\frac{11}{2}$, $a_1 = -\frac{9}{2}$, and $a_2 = -\frac{7}{2}$. By Table I of Chapter III we see that f_n for these cases are given by

$$f_6(-\frac{11}{2},b;\omega) = \frac{8b^4\omega^4 - 12b^2\omega^2 + 1}{8b^4\omega^2 - 12b^2\omega^4 + \omega^6},$$

$$f_6(-\frac{9}{2},b;\omega) = \frac{6b^2\omega^2 + 1}{6b^2\omega^4 + \omega^6},$$

$$f_6(-\frac{7}{2},b;\omega) = \frac{1}{\omega^6}.$$

Thus in each case f_6 has degenerated from a sixth to a lower degree numerator.

C. Varying the Parameter b

As we noted in Section A, the parameter b is a scale factor as far as the zeros of the ultraspherical rational function is concerned. The zeros of $f_n(a,b;\omega)$ are those of the ultraspherical polynomial multiplied by $\frac{1}{b}$. Hence if b is increased from the value that yields the Chebyshev rational function for a fixed value of a, the ripples become unequal and the zeros shift toward the origin. The poles, of course, shift toward infinity. If b decreases, the zeros shift toward infinity and the poles shift toward the origin.

Hence if a is such that all the zeros of f_n are real and on $(-1,1)$, it is possible to decrease b to shift any given number of the zeros into the interval $\omega > 1$. Then $A(\omega)$ attains its maxima $A(\omega) = 1$ at one or more points in $\omega > 1$ and attains its minima $A(\omega) = 0$ at one or more points in $(-1,1)$, and thus the nature of the filter is entirely changed. For example, in Figure 5-10 the case $n = 6$ is shown for $\epsilon = 2.25916$, $a = -2.4761$, and $b = 1.066$, which is the elliptic case. Also in Figure 5-10 the parameter b is changed to .366, which shifts all the zeros into $\omega > 1$ and changes the filter to a high-pass filter.

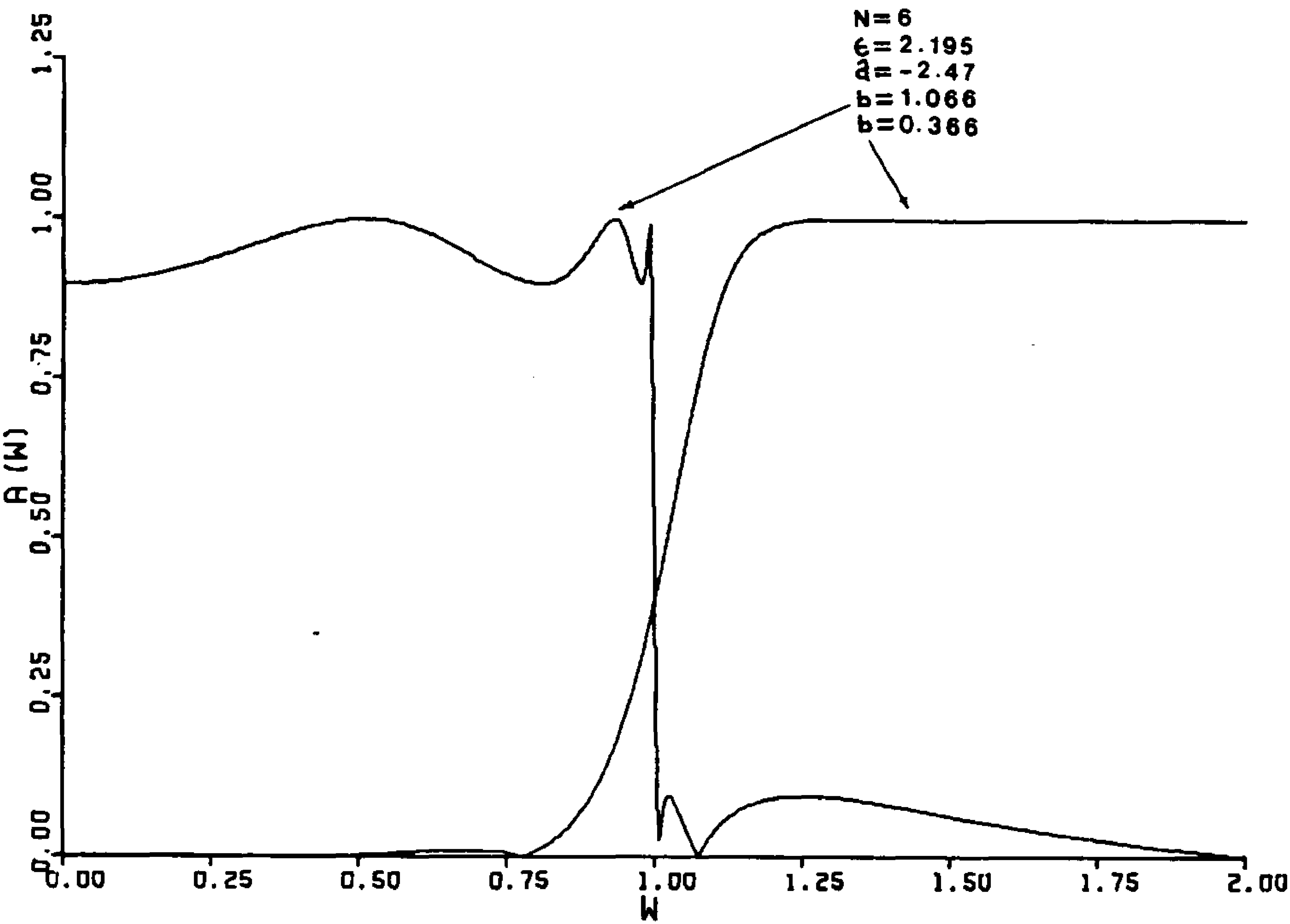

N=6
ε=2.195
a=-2.47
b=1.066
b=0.366
A (W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 5-10

A case where b is sufficiently small to shift only one zero into $\omega > 1$, resulting in a band-pass filter is shown in Figure 5-11. Other interesting cases are shown in Figure 5-12 where b is selected to shift poles very near zeros, and in Figure 5-13, where a variety of responses is obtained. In Figure 5-14 we see the progress toward the Butterworth filter from the elliptic filter as b is increased. Finally in Figure 5-15 we see the case where b is sufficiently small to shift all the zeros for $n = 4$ to $\omega > 1$.

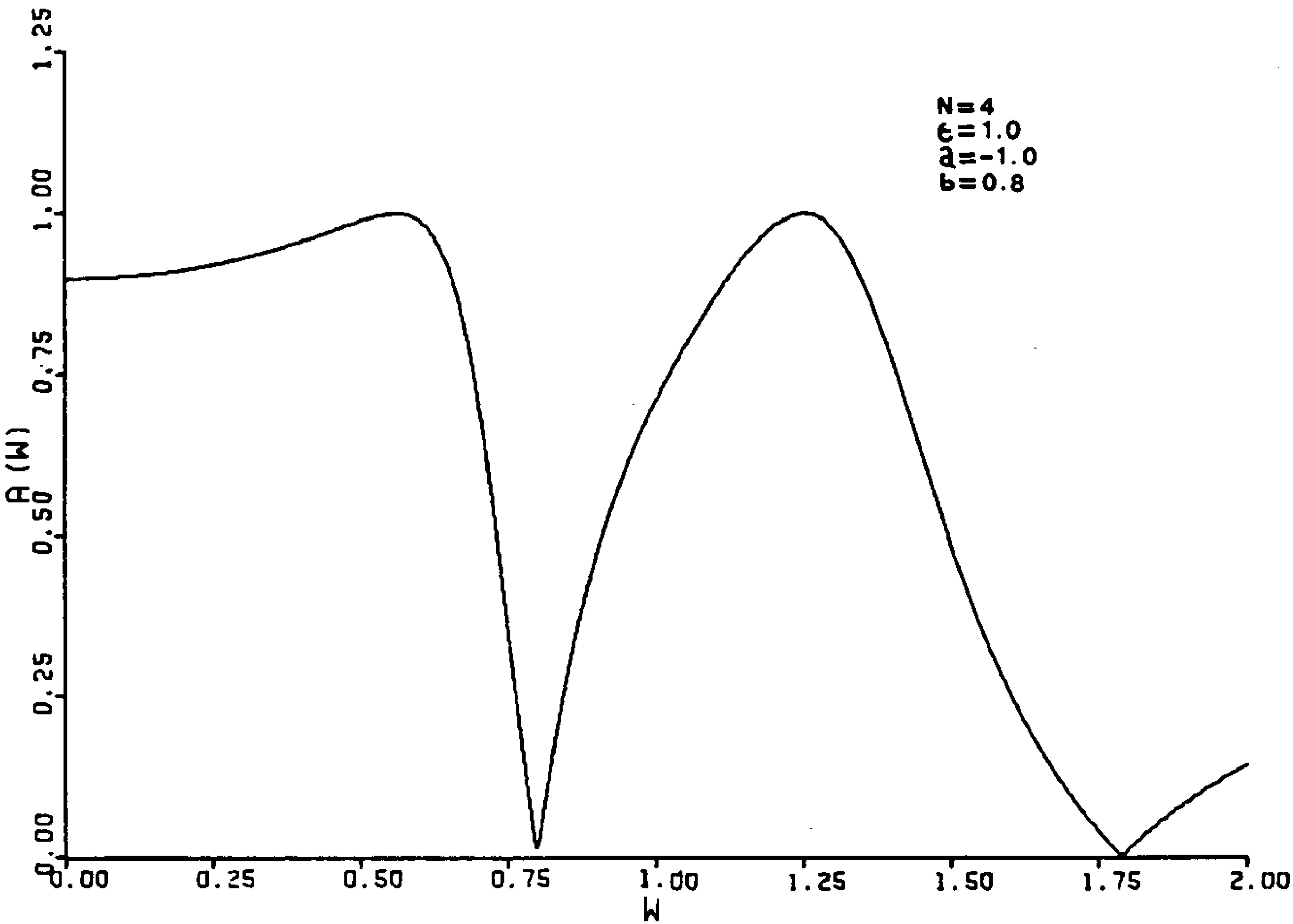
Let us now consider the extreme value $a = -[n/2]$ on the a-b curve. For this value of a the corresponding b is 1, resulting, as we saw in Section C of Chapter III, in $f_n = 1$, for n even and $f_n = \omega$, for n odd. If $b \neq 1$; we have, by (3.21), for the general case when $a = -[n/2]$,

$$A_{n-2i} = (-1)^i \binom{[n/2]}{i} \quad b^{-2i} \quad . \tag{5.15}$$

Hence we have for this case,

$$\begin{aligned} f_n(-[n/2],b;\omega) &= \left(\frac{b^2\omega^2 - 1}{\omega^2 - b^2} \right)^{[n/2]} , \quad n \text{ even} \\ &= \omega \left(\frac{b^2\omega^2 - 1}{\omega^2 - b^2} \right)^{[n/2]} , \quad n \text{ odd.} \end{aligned} \tag{5.16}$$

Therefore for this case, all the zeros (except $\omega = 0$ in the case n odd) occur at $\omega = \pm \frac{1}{b}$, and all the poles (except $\omega = \infty$ in the case n odd) occur at $\omega = \pm b$. Hence we have a phenomenon somewhat like that of the Butterworth filter in which all the zeros are concentrated at a single point. If $b > 1$, then $A(\omega)$ attains its maxima $A = 1$ at a point on $(0,1)$ and attains its minima $A = 0$ at a point on $\omega > 1$. Moreover if n is large, A tends to be flat in the vicinity of its maxima and



1.25
1.00
0.75
A (W)
0.50
0.25
0.00
N=4
ϵ=1.0
a=-1.0
b=0.8
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
W

FIGURE 5-11

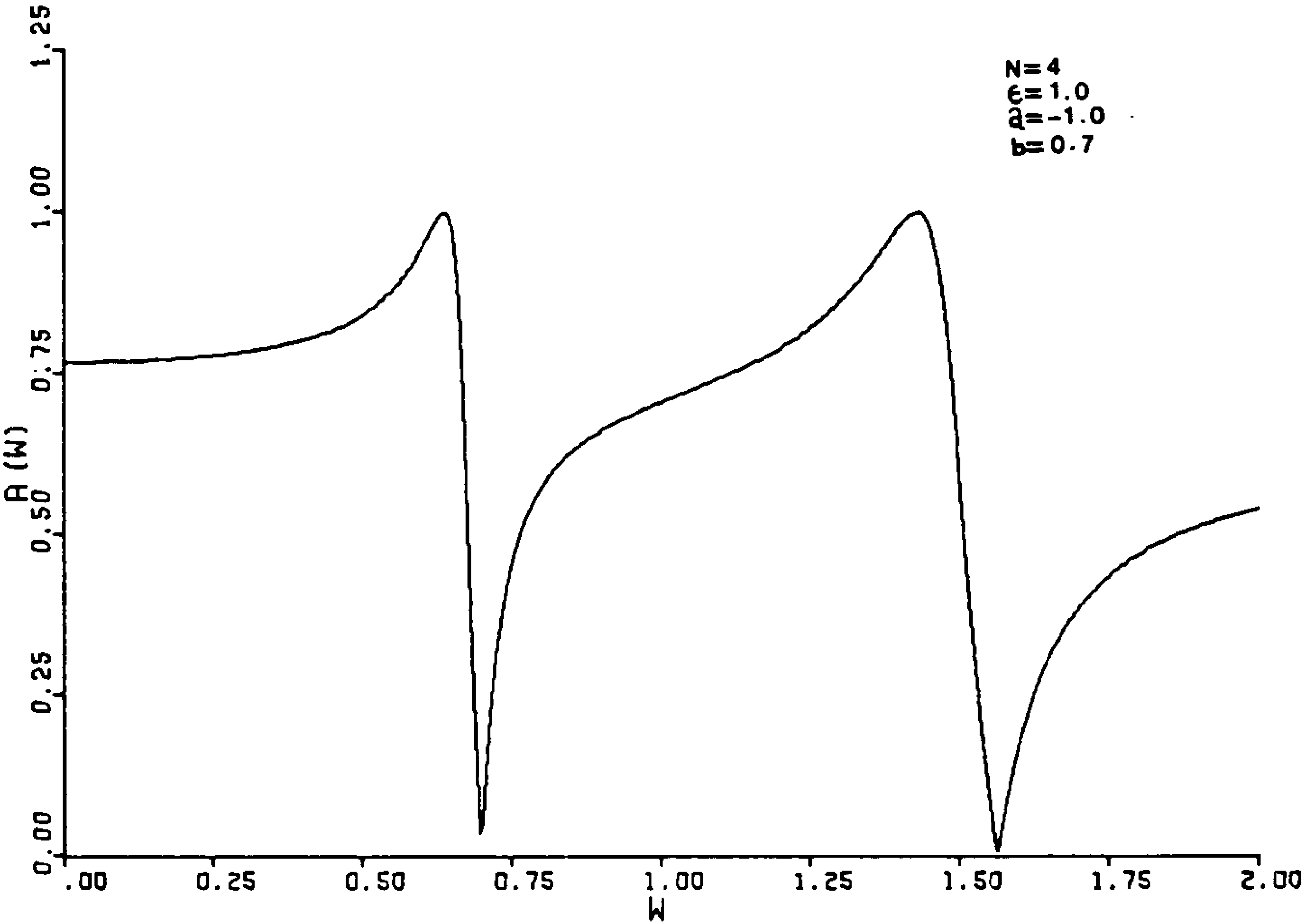

N=4
ϵ=1.0
a=-1.0
b=0.7
A (W)
0.00
0.25
0.50
0.75
1.00
1.25
.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
W

FIGURE 5-12

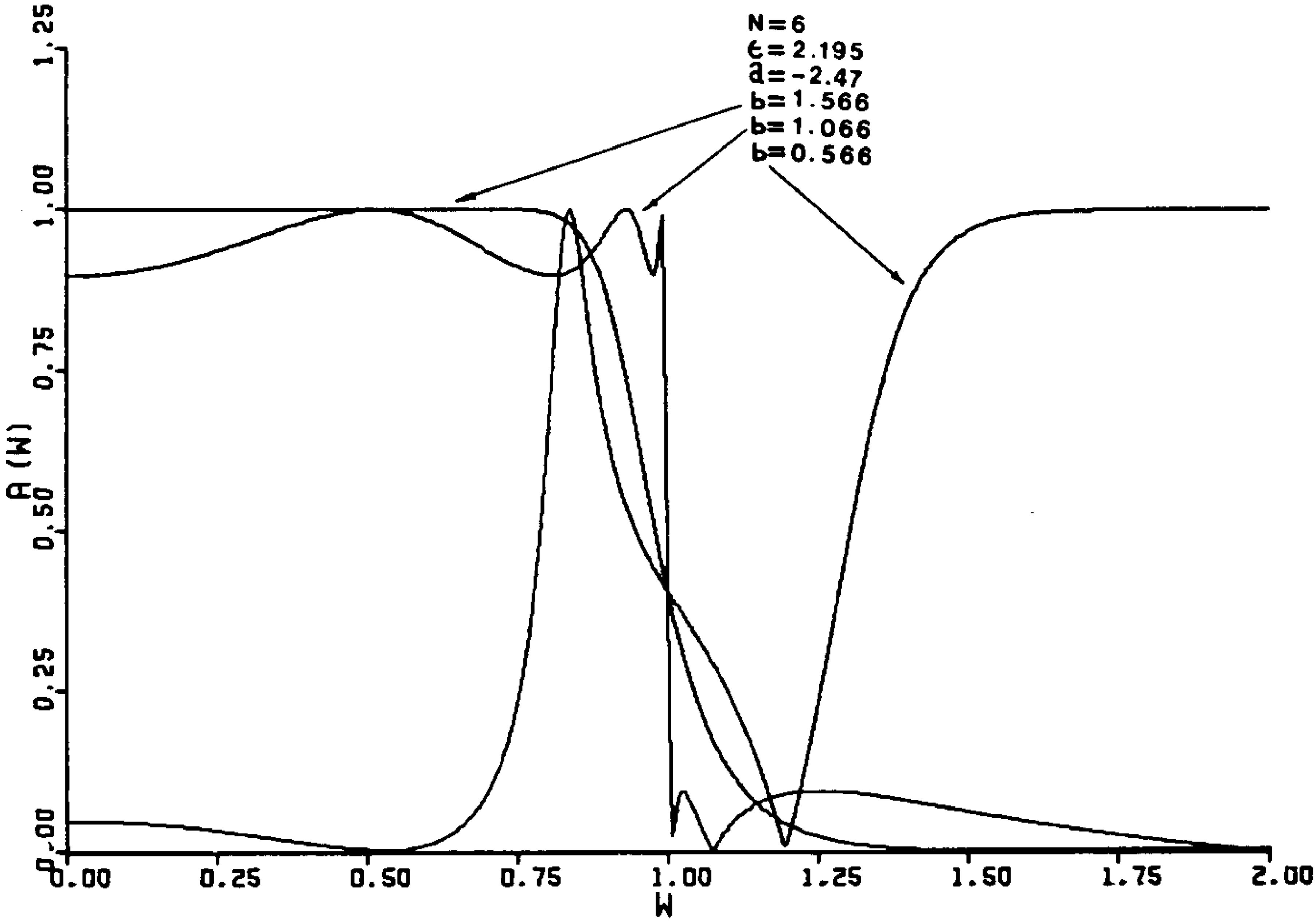

N=6
ϵ=2.195
a=-2.47
b=1.566
b=1.066
b=0.566
A (W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 5-13

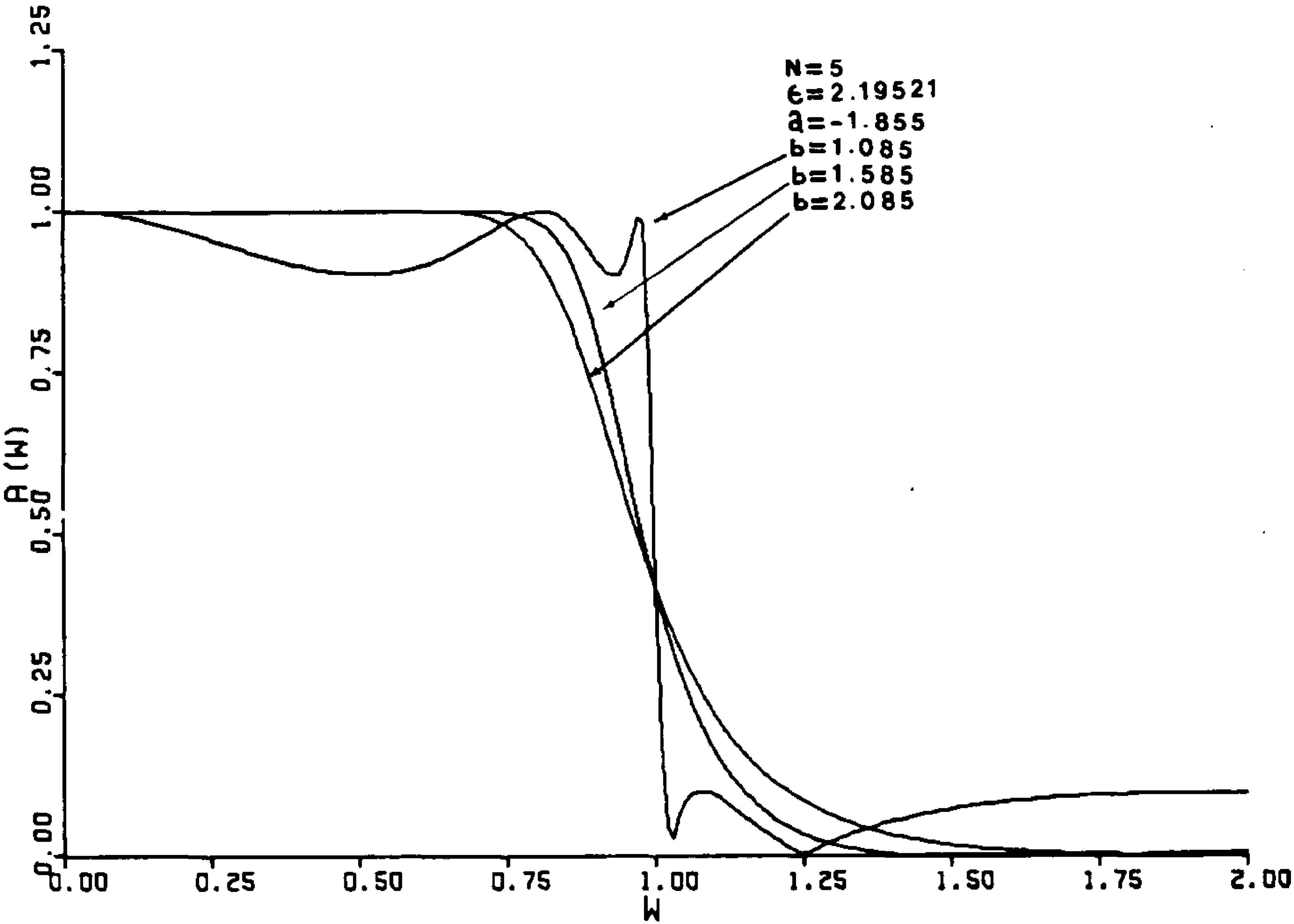

N=5
ϵ=2.19521
a=-1.855
b=1.085
b=1.585
b=2.085
A (W)
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 5-14

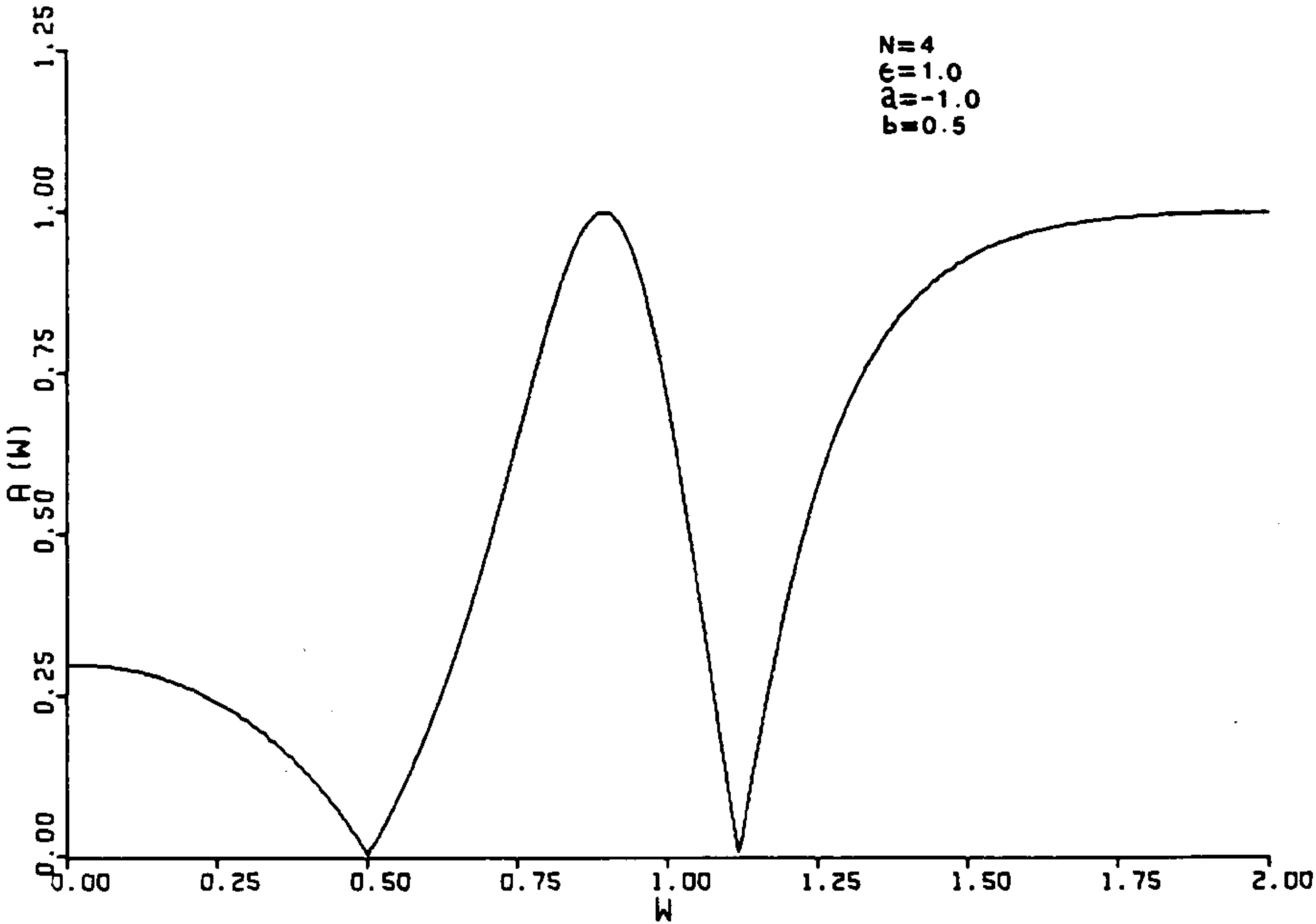

N=4
ϵ=1.0
a=-1.0
b=0.5
A (W)
0.00
0.25
0.50
0.75
1.00
1.25
W
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

FIGURE 5-15

minima. If $b < 1$, the positions of the maxima and minima are reversed. The maxima occurs at a point on $\omega > 1$ and the minima at a point on $(0,1)$.

By (5.16) we may calculate $A(0)$ for $a = -[n/2]$, obtaining

$$A(0) = \frac{1}{\sqrt{1 + \epsilon^2 b^{-2n}}} \quad , \quad n \text{ even}$$
$$= 1 \quad , \quad n \text{ odd} \tag{5.17}$$

Hence if b is small (near 0), in the even case, $A(0)$ is small and the maxima occurs on $\omega > 1$. The minima occurs on $(0,1)$. Also for $\omega \to \infty$, we have $f_n \to b^n$, which is small, so that $A(\omega)$ tends toward a number near 1. This is a high-pass filter, an example of which is shown in Figure 5-16, for $n = 4$, $b = .5$. If n is odd and b small, the same type $A(\omega)$ arises except that $A(0) = 1$ and $A(\infty) \to 0$. An example of this case is shown in Figure 5-17, for $n = 7$.

The case $n = 4$, $b = 1.5$ is shown in Figure 5-18, where it may be seen to resemble the Butterworth filter. It attains $A = 0$ at a finite point, $\omega = 1.5$ and attains $A = 1$ at $\omega = \frac{2}{3}$. For ω large, $A \to 0$ for this case.

A case where $b < 1$, but very near 1 is shown in Figure 5-19, which illustrates what happens when the poles and zeros are chosen close together (b near 1).

The other extreme, $a = -\frac{1}{2}$, on the a-b curve is of course the case when the ultraspherical polynomial becomes the Chebyshev polynomial. In this case all the zeros of f_n are real and b may be varied to shift them either toward or away from the origin. This case was discussed in Section A of this Chapter.

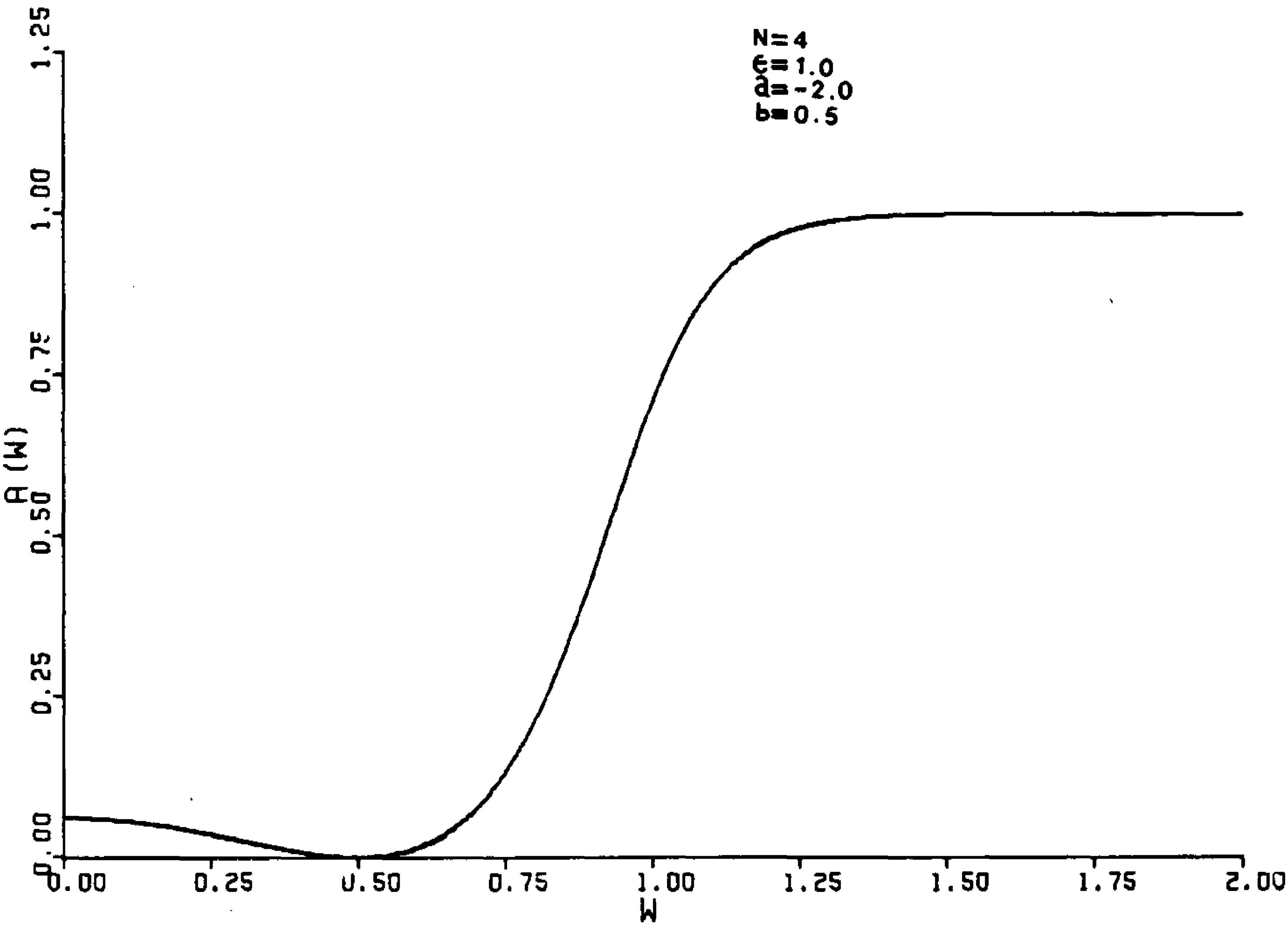


FIGURE 5-16

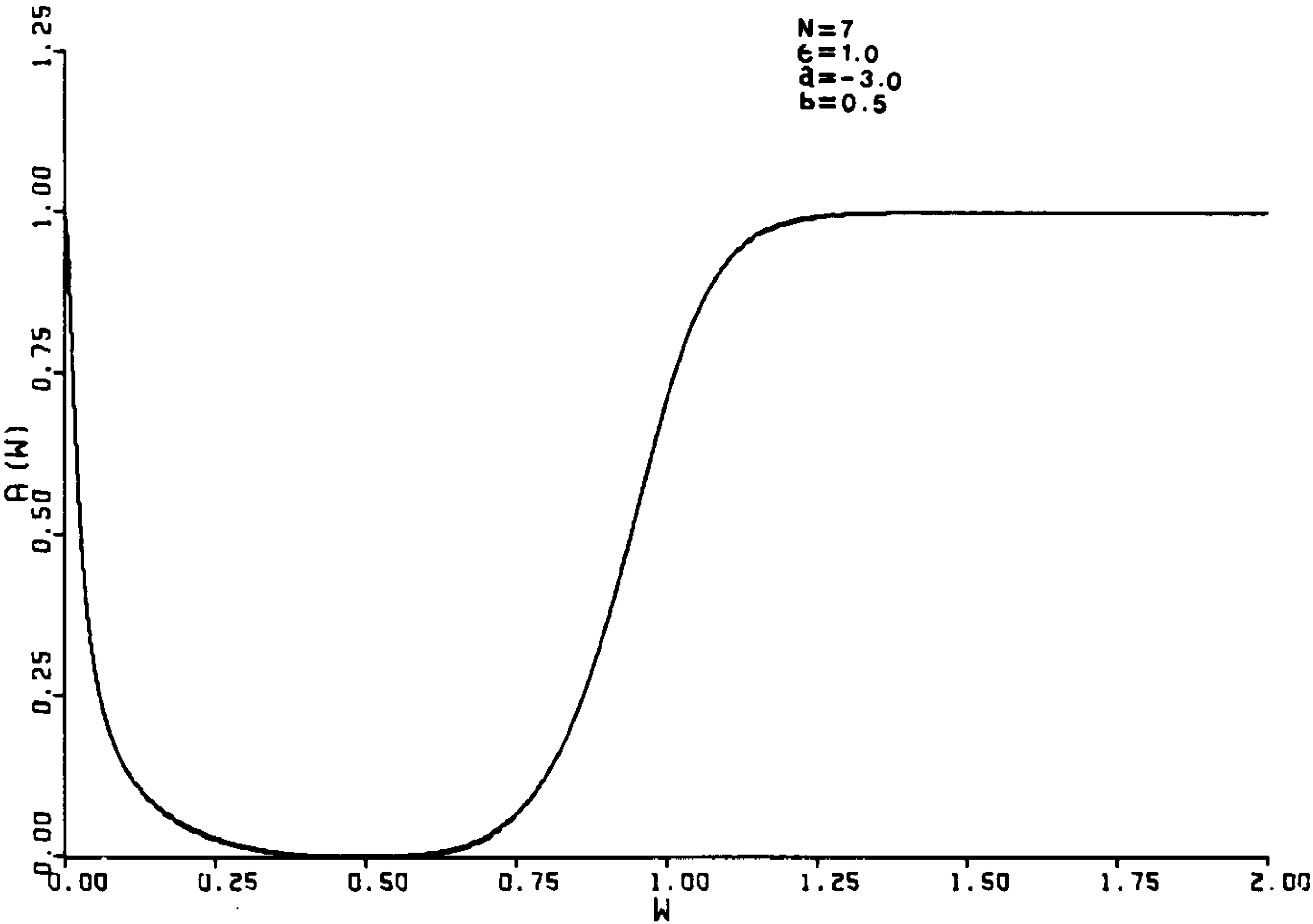


FIGURE 5-17

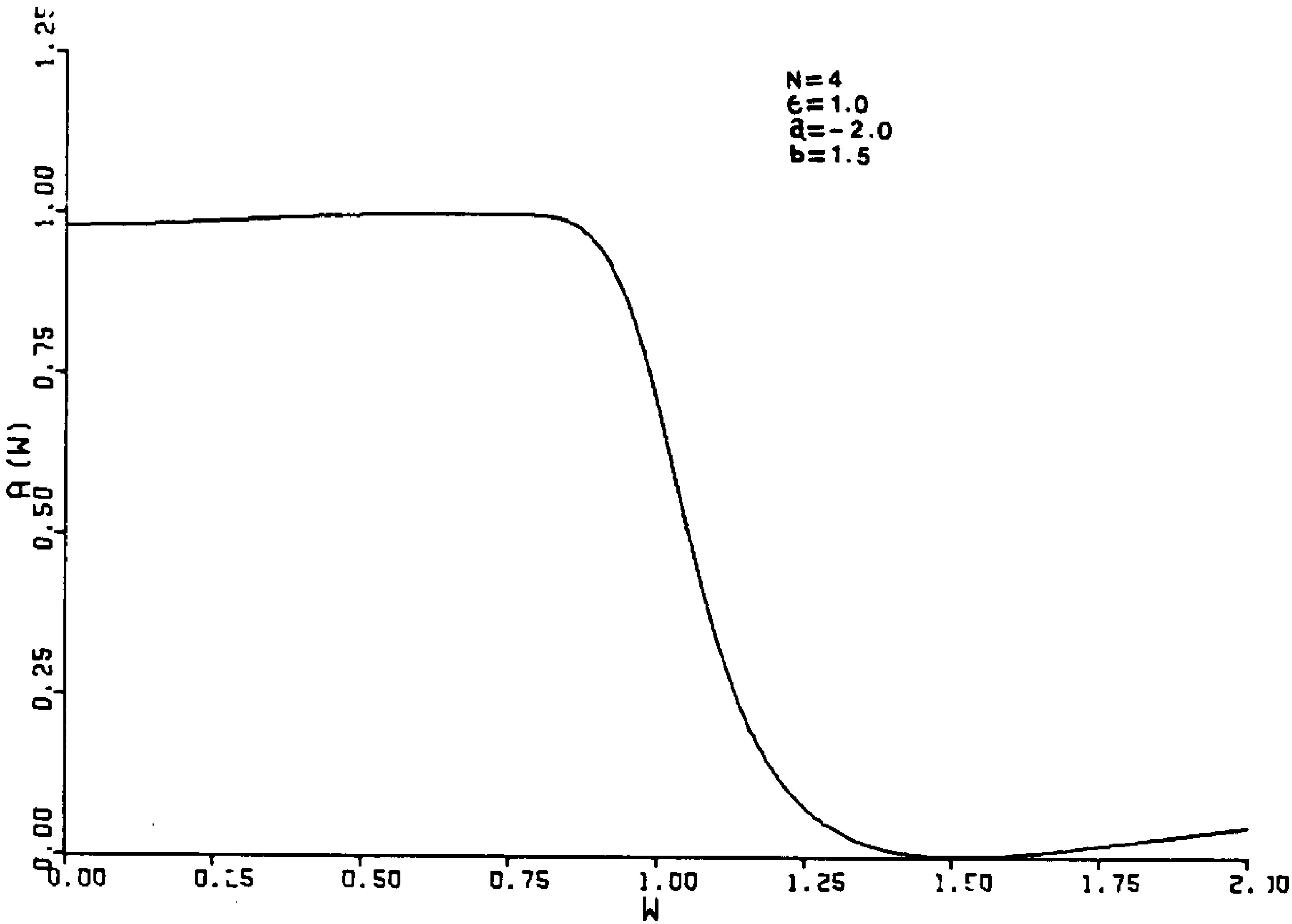


FIGURE 5-18

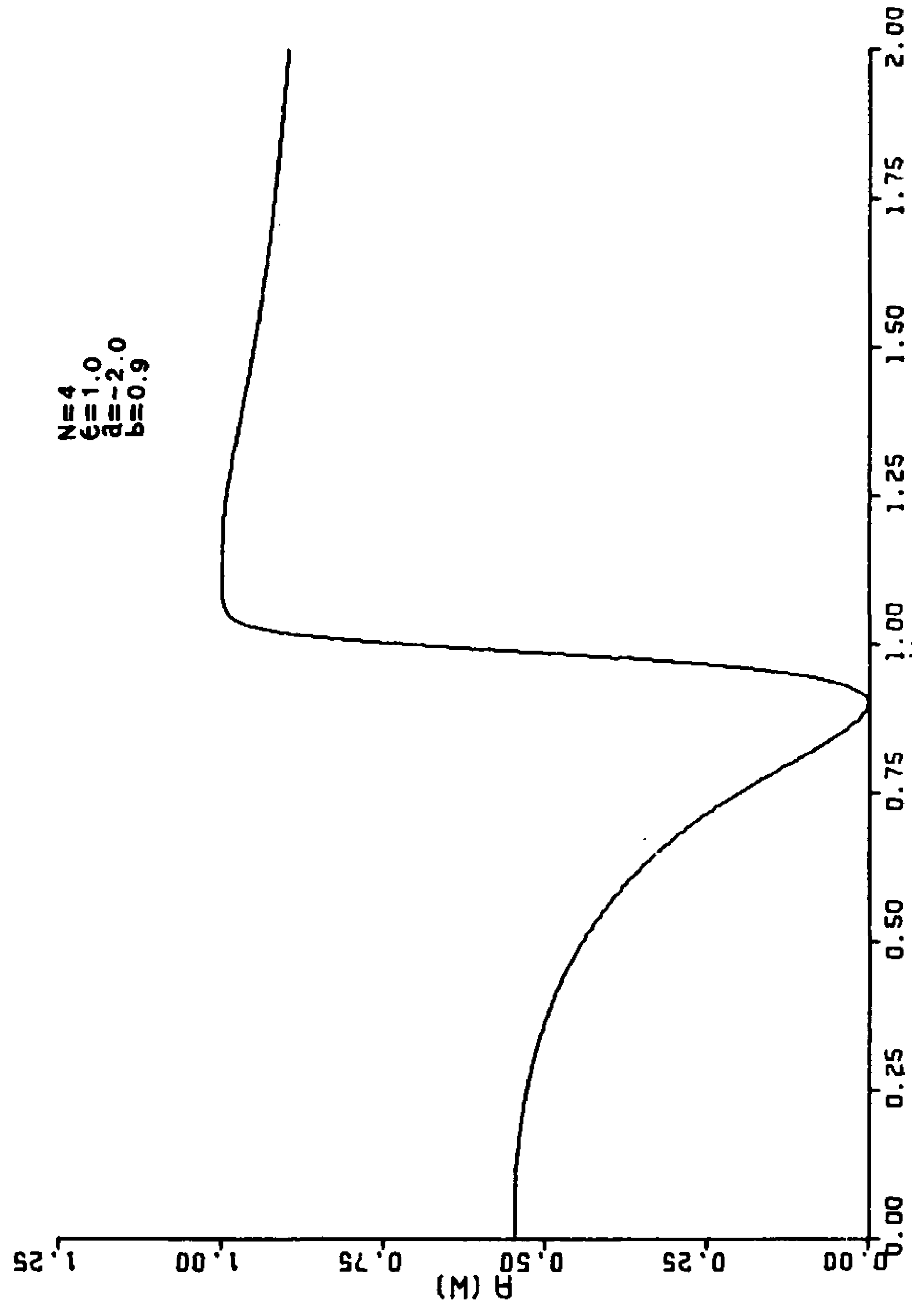


FIGURE 5-19

CHAPTER VI

CONCLUSIONS

The ultraspherical rational function includes the Chebyshev rational function used in obtaining the elliptic filter magnitude response for appropriate values of the parameters a and b. By using the graphs relating a and b to the modulus k, the deviation δ, and to each other, one may obtain directly the elliptic filter for a given set of tolerance. No trial and error procedures are required in the process.

The ultraspherical rational filter is more versatile than the elliptic filter. By varying the parameters a and b from their values which yield the elliptic filter, one may obtain other low-pass filters, such as Butterworth, for example. In addition, other filters may be obtained, such as high-pass, band-pass, and filters having constant magnitude responses.

BIBLIOGRAPHY

[1] Richards, P., "Universal optimum-response curve for arbitrarily selected coupled resonators," Proc. I.R.E., Vol. 34, pp. 624-629, September 1946.

[2] Kashefi, M., "Ultraspherical filter design," (unpublished thesis), Louisiana State University, 1965.

[3] Johnson, D. E., and J. R. Johnson, "Low-pass filters using ultraspherical polynomials," IEEE Trans. Circuit Theory, Vol CT-13, pp. 364-369, December 1966.

[4] Papoulas, A., "On the approximation problem in filter design," IRE Convention Record, Vol. 5, Pt. 2, pp. 175-185, 1957.

[5] Guillemin, E. A., Synthesis of Passive Networks. New York: John Wiley and Sons, Inc., 1957.

[6] Rainville, E. D., Special Functions. New York: McGraw-Hill Book Co., Inc., 1960.

[7] Handbook of Mathematical Functions. Washington, D.C.: NBS, Appl. Math. Ser. 55, June 1964.

[8] Moore, C. F., C. L. Smith, and P. W. Murrill, "Multidimensional optimization using PATERN search," College of Engineering, Louisiana State University, Rept. T-TR-23, 1968.

[9] Ince, E. L., Ordinary Differential Equations. New York: Dover Publications, Inc., 1956.

[10] Johnson, D. E. and J. R. Johnson, Mathematical Methods in Engineering and Physics: Special Functions and Boundary Value Problems. New York: The Ronald Press Co., 1965.

VITA

Mohsen D. Kashefi was born on May 10, 1930, in Theran, Iran. He was graduated from Air-Force Academy with a B.S. degree in 1953. He received his B.S. and M.S. in Electrical Engineering from Louisiana State University in 1963 and 1965. He then spent one year teaching in Delgado College before returning to Louisiana State University as an instructor.

EXAMINATION AND THESIS REPORT

Candidate: Mohsen David Kashefi

Major Field: Electrical Engineering

Title of Thesis: Filters Based on Ultraspherical Rational Functions

Approved:

David E. Johnson
Major Professor and Chairman

Max Goodrich
Dean of the Graduate School

EXAMINING COMMITTEE:

Leonard C. Adams

Robert C. McIlhenny

Marion E. Council

Owen T. Fox

Johnny R. Johnson

A. H. Fezonni

Date of Examination:

May 10, 1972