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Complexity and heuristics in ruled-based algorithmic music composition

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COMPLEXITY AND HEURISTICS IN RULE-BASED ALGORITHMIC MUSIC COMPOSITION

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In Loving Memory
of
Mdm. Gwee Kwee Neo

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PREFACE

I first came to grips with automated composition species counterpoint when I was a graduate student in music. I had the privilege of attending Professor Jan Herlinger's class in medieval counterpoint, in which we wrote exercises in species counterpoint. We soon learnt that writing a technically correct melody was only the first, and relatively easy, step. Species counterpoint, for all its simplicity, provided not inconsiderable challenges to the students' musical creativity. Just for a lark, I began writing computer programs that could, as a start, churn out countermelodies to cantus firmi. I thought, we could insert the creativity later. Giving computers the creative touch turned out to be far more difficult that I had expected, and set me thinking for long after the final class of Professor Herlinger's counterpoint course, and for long after I presented my musicology dissertation. In this computer science dissertation, we investigate to what extend machine intelligence can approximate human intelligence in the realm of music composition.

In the first chapter, we introduce species counterpoint and justify its use as our compositional model: its rule-based approach facilitates computational expression. We hold that the key to successful algorithmic music composition lies in creating works that are not merely "correct," but sound like they were composed by a human being. In chapter 2, we show how, by adding "artistry" rules to the existing rule set, and by using what we term fuzzy rule application, we can give human-like guidance in choosing among computer-generated musical output.

The rest of the dissertation can be divided into two principal parts. In the first part, we lay theoretical foundations by discussing complexity issues in rule-based generation of melodies (chapters 3 and 4); these theoretical underpinnings justify our practical approach to the music problem in the subsequent part, where we use our concepts of artistry rules and fuzzy rule application to reflect human preferences, and develop algorithms to produce our desired human-like music (chapters 5 and 6).

More specifically, issues discussed in the first part (chapters 3 and 4) include descriptions of several formal problems. In chapter 3, we define restricted music composition problems that can be efficiently solved by the finite state machine; in chapter 4, we show why more general problems are

intractable, and prove that they belong in the NP-complete and other important complexity classes. In the second part (chapters 5 and 6), we discuss several algorithms adopted from machine learning to achieve certain goals. In chapter 5, we describe two methods to model human preferences: the "Jeppesen Experiment," whereby we calculate rule weights based on examples of an expert (here Knud Jeppesen); and training an artificial neural network to approximate an expert's ranking of melodies. We apply these, and related, techniques in chapter 6, where we develop two algorithms to produce music; these algorithms rest upon our acknowledgment of the intractability of the general music problem, which we established in chapter 4.

Throughout the dissertation, we have tried to provide musical examples with which most musicians can relate. Our attempt to link the computational complexity of music composition with problems in other fields involving computation will, we hope, indicate to both computer scientists and musicians that they have much in common. It was no accident that in Classical times, mathematics was nothing more nor less than arithmetic, geometry, astronomy, and music.

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ABSTRACT

Successful algorithmic music composition requires the efficient creation of works that reflect human preferences. In examining this key issue, we make two main contributions in this dissertation: analysis of the computational complexity of algorithmic music composition, and methods to produce music that approximates a commendable human effort. We use species counterpoint as our compositional model, wherein a set of stylistic and grammatical rules governs the search for suitable countermelodies to match a given melody.

Our analysis of the complexity of rule-based music composition considers four different types of computational problems: decision, enumeration, number, and optimization. For restricted versions of the decision problem, we devise a polynomial algorithm by constructing a non-deterministic finite state transducer. This transducer can also solve corresponding restricted versions of the enumeration and number problems. The general forms of the four types of problems, however, are respectively NP-complete, #P-complete, NP-complete in the strong sense, and NP-equivalent. We prove this by first reducing from the well known Three-Dimensional Matching problem to the music composition decision problem, and then by reducing among the music problems themselves.

In order to compose music both correct and human-like, we formulate new "artistry" rules to supplement traditional rules of musical style and grammar. We also propose the fuzzy application of these artistry rules, to complement the crisp application of the traditional rules. We then suggest two methods to model human preferences: (1) distinguish an expert's compositions from alternative compositions by determining rule weights; (2) train an artificial neural network to reflect an expert's musical preferences through the latter's evaluations of a set of compositions. We were able to approximate that elusive factor of human preference with better than 75% accuracy.

To solve the optimization problem, we adapt two different search algorithms: best-first search with branch-and-bound pruning (for $m \geq 1$ optimal solutions), and a genetic algorithm (for $m \geq 1$ near-optimal solutions). Through these algorithms, we test the techniques of rule weightings and of trained neural networks as evaluation functions. Our adaptation of the genetic algorithm produced optimal countermelodies in execution time favorably comparable to that taken by the best-first algorithm.

CHAPTER 1. INTRODUCTION

How well can a computer program compose music? How much can it learn from human example? Can its music be both technically correct and artistic? In attempting to answer such questions, we examine the process of music composition from the point of view of computational theory and machine learning. We use as our musical model the genre known as *species counterpoint*, which is a pedagogical method for composing music in the style of Palestrina, one of the foremost composers of the sixteenth century. We describe species counterpoint formally as rule-based music composition problems, analyze the computational complexity of these problems, and formulate practical algorithms to solve them. We also investigate how to improve upon the purely mechanical process of composition in order to produce music with some degree of artistry, i.e., music that shows human qualities. We hold that the key to successful algorithmic music composition lies in efficiently creating output that reflects human preferences.

We choose species counterpoint as the representative music problem for several reasons. This musical genre has long been highly regarded by composers and music educators. As a teaching tool, it is sufficiently abstract yet general enough that its methods suggest solutions to other more practical compositional styles. Its gradated mode of instruction facilitates systematic learning. The formulation of the five different *species* (see section 1.2) and the rules governing them are sufficiently precise as to lend themselves suitable for the rigorous approach we shall be undertaking.

In examining issues of automated music composition, we address various concerns of computer science as well as of music. By applying complexity analysis, we wish to lay a foundation upon which further examination rests: analysis from the perspective of machine intelligence shows us how closely we can model human thought processes; development of efficient music composition algorithms provides invaluable teaching aids to musicians.

The problem of musical composition is interesting in its own right, but here we also seek to link the problems in music composition with well-known non-trivial problems in other domains. We shall show that the species counterpoint problem (and its variants) belongs among the NP-complete group of problems, and shares with them the intriguing property of problems that are complete with respect to a complexity class: *An efficient solution (if it exists) to the species counterpoint problem will also give us an efficient solution to all NP problems*.

1.1. Previous Work on Algorithmic Music Composition

We have investigated the use of fuzzy set theory as heuristics for aesthetic evaluation of melodies, and incorporated these heuristics in efficient searches for valid and artistic melodies (Gwee 1998, 2002a, 2002b). We have also identified instances of the problem that are solvable efficiently by the finite state machine model, and those that are NP-complete (Gwee 2001).

Among other authors who have explored automated counterpoint composition, Schottstaedt (1989) introduced a heuristic based on "penalty points" to circumvent the combinatorial explosion inherent in an exhaustive tree search. Hörnel (1998) used tools from machine learning, e.g., neural networks, to provide computers human-like capabilities in learning and musical composition (Hörnel 1998).

One of the earliest attempts at a general algorithmic approach to composing music was that of Guido of Arezzo who in his *Micrologus* (1025–26 or 1028–32) devised a method to derive a melody from any given text (Guido 1955). Guido's method was based on what we would now call the "generate and test" model.

Among more contemporary researchers, Hiller and Isaacson (1959) proposed a non-deterministic method of composition, which they applied to various styles of music, including species counterpoint. They built a piece of music by applying a set of rules to candidate melodic elements that were determined by randomly generated numbers. In this way, they avoided an exhaustive search. Hiller and Isaacson, together with Schottstaedt, all considered a problem solved when they were able to obtain a single solution that best agreed with given rules.

Ames (1982) introduced the notion of preference ranking in the search for solutions, which he called "comparative search." He rejected Hiller's and Isaacson's non-deterministic method of random selection and based his decisions on stylistic criteria (Ames 1982, 1983).

Mathews (1965) proposed deterministic algorithms based on pitch quantization. Like Hiller's, Mathew's compositional method was rule-based. Ebcioglu (1988) also used a rule-based expert system to harmonize four-part chorales, and Temperley and Sleator (1999) used a preference rule system to analyze meter and harmony in common-practice music.

The majority of published studies on computer music composition emphasized musical methods and results, rather than theoretical computer science issues. The earlier researchers especially spent

much time in overcoming hardware problems. In this dissertation, we take a different perspective and present hardware-independent analysis based on computational complexity theory.

1.2. Species Counterpoint

Species counterpoint was developed by the eighteenth-century theorist Johann Joseph Fux and widely disseminated as a teaching method in his treatise entitled *Gradus ad parnassum* ("Steps to Parnassus," Fux 1725). The method is based on adding one or more melodic parts, called the "*countermelod*(*y*/*ies*)" to a given primary melody, called the *cantus firmus* [pl. *cantus firmi*]. The countermelodies and cantus firmi are strings of elements from a given set of musical notes called a *gamut*. Fux designated five gradated classes or species of writing styles: first species is the simplest, and fifth species is the most complex.

In a typical scenario, the teacher gives a student a cantus firmus, and requires the latter to compose a suitable countermelody. Throughout, the student is expected to adhere to a given set of rules of musical style and grammar. In first species counterpoint with one added part, the student provides a single note of the countermelody for each note of the cantus firmus. When the student becomes proficient in first species with one added part, he/she progresses to two added parts for the cantus firmus, then further to second species with one added part, and so on. In first species with one added part, each note of the added part sounds together with a single note of the cantus firmus, so too for first species with two added parts; in second species with one added part, two notes sound in succession in the added part for each note of the cantus firmus; in second species with two added parts, one added part moves note-by-note with the cantus firmus, and the other added part moves with two notes in succession with one note of the cantus firmus; in third species with one added part, four notes of the added part move in succession to one note of the cantus firmus; in third species with two added parts, the movements of the added parts are analogous to those of the added parts in the second species; in fourth species with one added part, the notes of the added part change in-between changes of notes of the cantus firmus; in the fourth species with two added parts, the movements are analogous once again to those of the other species already described. The culmination of this study is fifth species with one or more added parts, also known as *free counterpoint*. It is easy to understand the progression of difficulty as we advance among the species and the number of added parts. Naturally, different rules are necessary for each species, but certain musical principles hold consistently throughout. We shall thus confine ourselves at the moment to first species.

Example 1.1. The following cantus firmi and countermelodies in various species and various number of parts are from Jeppesen (1931):

First species, two parts:
countermelody: a aa g f e d c h d c# d
cantus firmus: D F E D G F a G f E D

(Jeppesen 1931, 112)

First species, three parts:
countermelody 1: d c e f g aa d c# d
countermelody 2: D F E D E F a A D
cantus firmus: D a G F E D F E D

(Jeppesen 1931, 177)

Second species, two parts:
countermelody: a h c d e d f F G a b F a h c# d
cantus firmus: D a G F E D F E D

(Jeppesen 1931, 117)

Second species, three parts:
countermelody 1: d a aa g e f e d e f d e d c b a d c# d
countermelody 2: a F G a b a a E F G a
cantus firmus: D F E D G F a G F E D

(Jeppesen 1931, 179)

Third species, two parts:
countermelody:
D E F G a h c d e d h c d c h a h c d e f g aa g f e c d e f g G a h c d e d h c# d
cantus firmus:
D F E D G F a G F E D

(Jeppesen 1931, 126–127)

Third species, three parts:
countermelody 1: D a G F G a h c d e d c d e f g G d c b a F G a E a G F#
countermelody 2: D D E A Bb D C# D
cantus firmus: D F G a G F E D

(Jeppesen 1931, 180–181)

Fourth species, two parts:
countermelody: a d c h e d c b a d c# d
cantus firmus: D F E D G F a G F E D

(Jeppesen 1931, 134)

Fourth species, three parts:
countermelody 1: d c h e d c b a d c# d
countermelody 2: D F G a G F D A D
cantus firmus: D a G F E D F E D

(Jeppesen 1931, 188) ■

From the computational point of view, the search for suitable countermelodies to a given cantus firmus is inherently exponential in nature, approximately proportional to $|\text{gamut}|^{|\text{cantus firmus}|}$ (i.e., exponential in the cantus firmus length, with the base the size of the gamut). Thus, extending the length of a cantus firmus by just one note results in considerably more than twice the number of computations. In our analysis, we clarify which variants of the general search problem are the most

"difficult" to solve, which are the "easiest," and which are the most "practical," whose solutions are efficiently computed yet musical. We identify various types of intractable music problems: Decision, Enumeration, Number, and Optimization. These correspond respectively to the NP-complete, #P-complete, "strong sense" NP-complete, and NP-equivalent complexity classes (Garey and Johnson 1975, 95).

From the artistic point of view, we investigate to what extent we may teach a machine to compose music that appears human-like to a listener. This is the musical equivalent of the Turing imitation game, where a computer program tries to fool a person that the latter is communicating with another human. Species counterpoint is an ideal model with which to study this issue because of its structure and well-established rule set. As the basis for composing music both correct and human-like, we propose modifying the traditionally crisp application of the rules of musical style and grammar.

CHAPTER 2. RULES OF MUSICAL STYLE AND GRAMMAR

What makes composing species counterpoint, and all other forms of music for that matter, particularly challenging is having to comply with strict rules at the same time one is giving free artistic expression. From the example of sixteenth-century composers, Fux and other scholars established rules that determine correctness of proposed countermelodies in a species counterpoint exercise. The rules can be divided into three categories, based on the musical aspects they control: melody, harmony and counterpoint. We introduce a new category, called "artistry," to encompass new rules that we shall use to model human musical preferences (see appendix A, section A.1 for a definition of all the rules we have used in the computation of countermelodies in first species, two parts).

In the following sections, we group the melody, harmony, and counterpoint rules under the single heading "correctness" rules, and our new rules under the heading "artistry" rules. The correctness rules are what historically have determined the stylistic and grammatical veracity of student solutions not only in species counterpoint exercises, but also in more "practical" styles of music.

2.1. Correctness Rules

Among the correctness rules, some dictate what notes may follow other notes in the countermelody (*melody* rules); other determine what notes of the countermelody may sound simultaneously with that of the given melody (*harmony* rules); yet others govern the combined harmonic and melodic aspects (*counterpoint* rules).

Example 2.1. Shown below in *if-then* form are examples each of harmony, melody, and counterpoint rules, followed by examples of violations of these rules.

Dissonant Melodic Intervals Rule (melody rule):
If $note_{\mathrm{n}}$ of the countermelody is p and $note_{\mathrm{n+1}}$ of the countermelody is q
Then the *interval type* (p, q) is *major* or the *interval type* (p, q) is *minor*

Preserve Pentachord Rule (harmony rule):
If $note_1$ of a given melody is p and $note'_1$ of the countermelody is q
Then the *interval type* (p, q) is *perfect* and the *interval size* (p, q) is *fifth*

Parallel Fifths [and Octaves] Rule (counterpoint rule):
If $note_{\mathrm{n}}$ of a given melody is p and $note'_{\mathrm{n}}$ of the countermelody is q
and $note_{\mathrm{n+1}}$ of a given melody is r and $note'_{\mathrm{n+1}}$ of the countermelody is s
and the *interval* (p, q) is *perfect fifth* [*perfect octave*]
Then the *interval* (r, s) is not *perfect fifth* [*perfect octave*]

The following countermelody violates the Dissonant Melodic Intervals Rule because notes E and b span the interval of a diminished fifth.

D **E** **b** a G F E D

The following two countermelodies (CM 1 and CM 2) to the same cantus firmus violate the Preserve Pentachord Rule. Notes D and F of cantus firmus CF and CM 1 are a minor third apart; correct is CM 2, which produces the perfect fifth.

CM 1: **F** a G F E D
CF: **D** F E D C# D

CM 2: **a** a G F E D
CF: **D** F E D C# D

The opening two harmonies of the following counterpoint violate the Parallel Fifths Rule. The interval D-a is a perfect fifth, hence the next interval must not be another perfect fifth, which it here is.

CM: **a** **h** a G F E D
CF: **D** **E** F E D C# D ■

2.1.1. Rule Redundancy and Indispensability

At this point, one may ask two related questions: "Are any rules redundant?" and "Are any rules indispensable?" For simplicity, we shall qualify this question by adding "… with respect to a given cantus firmus set." A rule is *redundant* if some other rule eliminates the same countermelodies it does, and at least one more; a rule is *indispensable* if, without it, at least one countermelody would otherwise be accepted.

Straightforward and efficient procedures are available for determining the existence of such rules. Knowing whether there are redundant or indispensable rules for a group of cantus firmi may or may not always be useful, however. Furthermore, the more exacting problem of finding the smallest necessary and sufficient rule set is intractable: we discuss this in chapter 4, sections 4.1.4 and 4.2.3.

We shall call sets of countermelodies accepted by a particular rule *acceptance sets*; similarly, we shall call sets of countermelodies rejected by a particular rule *rejection sets*. It should be clear that the acceptance and rejection sets of a particular rule are disjoint, and their union constitute the set of all valid countermelodies with respect to a given cantus firmus; therefore, the complement of the acceptance set is the rejection set, and vice versa. We formally define the simpler problem of rule redundancy as follows (SIMPLE RULE REDUNDANCY; see chapter 3, section 3.1 for an explanation of the format used here).

Problem 2.1. SIMPLE RULE REDUNDANCY
INSTANCE:
A cantus firmus set $CFSet = \{cf_1, cf_2, \ldots, cf_m\}$;
a rule set $RSet = \{r_1, r_2, \ldots, r_n\}$;
a collection of "acceptance sets" $Accept_{r, cf}$, $r \in RSet$,
which is the set of countermelodies for $cf \in CFSet$ that rule r accepts.
QUESTION:
Is there a proper subset $RSet' \subset RSet$
such that
$\forall cf \in CFSet, \forall r' \in RSet', \forall r \in RSet, Accept_{r', cf} \not\subset Accept_{r, cf}$?

We are concerned here with eliminating rules whose set of countermelodies it accepts is a proper superset of the acceptance set of some other rule. These rules are redundant since they are less restrictive than some other rule. The intersection of the acceptance sets of the remaining rules gives the same set of countermelodies as those accepted by the application of all rules. The answer to the question immediately follows: if some redundant rules were found, the answer is "yes," otherwise "no." The time complexity for achieving this is $O(n^2 (\max |Accept|)^2)$. Note that this procedure does not necessarily yield a minimal rule set, however. We shall show later (chapter 4, section 4.2.3) that this more ambitious problem of finding the minimal rule set is intractable.

Example 2.2. Consider the following instance of SIMPLE RULE REDUNDANCY:

$CFSet = \{cf_1\}$;
$RSet = \{r_1, r_2, r_3, r_4\}$;
$Accept_{r1,cf1} = \{1, 2, 3, 6\}$;
$Accept_{r2,cf1} = \{1, 3, 4, 5, 6\}$;
$Accept_{r3,cf1} = \{1, 3, 4, 6\}$;
$Accept_{r4,cf1} = \{1, 2, 3, 5, 6\}$.

The answer in this instance is "yes": r_2 and r_4 are redundant ($Accept_{r2,cf1} \supset Accept_{r3,cf1}$ and $Accept_{r4,cf1} \supset Accept_{r1,cf1}$), and so $RSet' = \{r_1, r_3\}$. The intersection of the acceptance sets of r_1 and r_3 is also the intersection of all four acceptance sets: {1, 3, 6}. In this instance, the above $RSet'$ happens also to be the minimal subset, with respect to the similarity of their intersecting acceptance sets with the intersection of the acceptance sets of all four rules. But notice, $\{r_1, r_2\}$ and $\{r_3, r_4\}$ are also minimal subsets. ■

In general, however, eliminating redundancy as described above does not necessarily result in minimal subsets, as example 2.2 shows.

Example 2.3. In the following collection of acceptance sets no one acceptance set properly contains elements of any other set, yet any two of them comprise a minimal subset, with respect to the similarity of their intersection ({1}) with the intersection of all the acceptance sets:

$CFSet = \{cf_1\}$;
$RSet = \{r_1, r_2, r_3\}$;
$Accept_{r1,cf1} = \{1, 2, 3\}$;
$Accept_{r2,cf1} = \{1, 4, 5\}$;
$Accept_{r3,cf1} = \{1, 6\}$. ■

Identifying redundant rules may not always be practical, since often very large acceptance sets are involved: this size is generally exponential in the length of the cantus firmus. The alternative is to examine the nature of the rules themselves to discover possibly redundant rules. For example, the rule that prohibits all repeated notes is obviously more restrictive than the rule that allows repeated but once in a countermelody; the Recovery Rule would make rules prohibiting a succession of leaps in the same direction redundant (Two Leaps in Succession and More Than Two Leaps in Succession Rules); a stronger form of the recovery rule, one that demands specifically a step in the opposite direction follow a leap, would make both the original Recovery and also Symmetric Leaps Rules redundant. Identifying other, less obvious, redundancies requires musical knowledge beyond that of the rules themselves: it might well be that Lewin's Rule makes cadential rules redundant, but this can be ascertained only by examining more closely the musical implications of both rules.

From the musical and aesthetic standpoint it may be desirable to remove redundant rules. From the computational standpoint, however, we might want to retain those redundant rules that nevertheless result in early pruning. Lewin's Rule, for example, might be found to make quite a number of other rules redundant, but we might still want to retain the latter if they manage to eliminate partial countermelodies higher in the search tree than does Lewin's Rule.

The related problem in finding indispensable rules, on the other hand, is amenable to computation without further musical knowledge. This problem can be defined as follows (SIMPLE RULE INDISPENSABILITY).

Problem 2.2. SIMPLE RULE INDISPENSABILITY
INSTANCE:
A cantus firmus set $CFSet = \{cf_1, cf_2, \ldots, cf_m\}$;
a rule set $RSet = \{r_1, r_2, \ldots, r_n\}$;
a collection of "acceptance sets" $Accept_{r,\ cf}$, $r \in RSet$,
which is the set of countermelodies for $cf \in CFSet$ that rule r accepts;.
QUESTION:
Is there a proper subset $RSet' \subseteq RSet$
such that

$$\exists cf \in CFSet, \forall r' \in RSet', \bigcap_{r \in (RSet - \{r'\})} Accept_{r,cf} - Accept_{r',cf} \neq \emptyset?$$

Example 2.4. In the following instance of SIMPLE RULE INDISPENSABILITY, $RSet' = \{r_1\}$ consists of the one indispensable rule, without which countermelody 7 could be accepted. On the other hand, in example 2.3, no indispensable rule as here defined can be found.

$CFSet = \{cf_1\}$;
$RSet = \{r_1, r_2, r_3, r_4\}$;
$Accept_{r1,cf1} = \{1, 2, 3, 4, 6\}$;
$Accept_{r2,cf1} = \{1, 3, 4, 6, 7\}$;
$Accept_{r3,cf1} = \{1, 2, 3, 6, 7\}$;
$Accept_{r4,cf1} = \{1, 3, 5, 6, 7\}$. ■

In a more practical setting, we can apply algorithm 2.1 to discover indispensable rules.

Algorithm FindIndispensableRules
Input: *CFSet*, *RSet*.
Output: $RSet' \subseteq RSet$.
1. $RSet' := \emptyset$
2. **for** each $cf \in CFSet$
Determine *numSolutions* with all rules applied
for each rule $r \in RSet$
Determine *numSolutions'* with all rules except rule *r* applied
if *numSolutions'* > *numSolutions*
Add rule *r* to *RSet'*.

Algorithm 2.1. Discovering indispensable rules, given a cantus firmus set (*CFSet*) and a rule set (*RFSet*)

Using this algorithm, we have determined that for cantus firmi <D, E, C#, D>, <a, D, F, E, D, C#, D>, and <D, F, E, D, G, F, a, G, F, E, D>, the following rules are indispensable: Ficta Notes, Lewin's, Dissonant Melodic Intervals, Other Inadmissible Intervals, Repeated Notes, Climax, Chromatic Resolution, Symmetric Leaps, Dissonant Harmonic Intervals, Unisons, Preserve Pentachord, Parallel Fifths and Octaves, Exposed Fifths and Octaves, Simultaneous Leaps, and Final Cadence. Note that the rule entitled Symmetric Leaps is one of our own: it prohibits leaps in one direction being followed immediately by a leap of the same size and type in the opposite direction. This was designed to prevent obvious melodic symmetry, a trait quite foreign to this style of melodic writing. For the cantus firmus <a, D, F, E, D, C#, D> this rule uniquely rejected two countermelodies: <D, F, D, G, F, E, D> and <D, a, D, G, F, E, D>. See appendix A, section A.2 for more details on these determinations.

For sheer number of unique rejections, two rules stand out: Lewin's Rule and Dissonant Harmonic Intervals Rule. For the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>, Lewin's Rule uniquely rejected 1752 countermelodies and as a result lessened by 45471 the total number of partial countermelodies that would otherwise have had to be analyzed; the Dissonant Harmonic Intervals Rule uniquely rejected 31027 countermelodies and lessened by 1255179 the total number of partial countermelodies analyzed. Giving these two rules the first opportunities at analyzing partial countermelodies would certainly reduce the number of applications of the other rules although this does not generally result in more efficient pruning of the search tree. For more efficiency, new correctness rules would have to be devised that uniquely eliminate short partial countermelodies.

2.2. Artistry Rules

Music students learn how to apply the correctness rules by writing exercises in species counterpoint. They soon discover, however, that mere correct application of these rules does not guarantee a musical result. Unfortunately too, these rules can be easily programmed in a machine to produce "correct" melodies in overwhelming quantities that would boggle the mind of even the most mechanically inclined student. Needless to say, only a minute proportion of these mass-produced melodies are artistic.

Example 2.5. Here are two instances of correct but inartistic melodies. Melody (a) is monotonous and has a limited range (a minor third in the space of seven notes); melody (b) begins too squarely and predictably (observe the ascending sequence of thirds):

(a) D E D E D C# D
(b) D F E G F a G F E D ■

With the aim of composing artistic music, we have added some of our own rules. These additions, which we call *artistry* rules, are rules that are mainly based on musical preferences of modern-day listeners. They are therefore heuristic in nature, and we do not claim that they are historically valid.

Example 2.6. The *Tonality Rule* is an artistry rule defined as follows:

Tonality Rule:
If (the *mode* of the given melody is *lydian*) or (the *mode* is *dorian*)
Then any *note* of the countermelody is not *h*

Under this rule, melody (a) is to be preferred over melody (b) because it has "b" as the second note, instead of "h," as in melody (b):

(a) D b a G F E D
(b) D h a G F E D ■

2.3. Rule Application: Crisp and Fuzzy

In evaluating a student's effort, the teacher traditionally applies the correctness rules *crisply*, i.e., if the student violates any of the rules at any point in the countermelody, the teacher fails the entire countermelody—just as would an English teacher an entire sentence should it break just one grammar rule. Crisp application of rules takes the form:

If *violation* of rule *r* by countermelody *m* is *yes [no]*
Then merit μ_r of countermelody *m* is *bad [good]*

In crisp rule application, a rule is either "violated" or "not violated." The countermelody is either "bad" or "good," with respect to the rule in question.

Example 2.7. The following melodies are "bad" and "good," with respect to the Dissonant Melodic Intervals Rule. Melody (a) is "bad" because G#–F is an augmented second; melody (b) is "good," because there are no augmented or diminished intervals (the note G# has been corrected to G). What a difference a single note makes!

(a) D F E G F a G# F E D
(b) D F E G F a G F E D ■

In most cases, however, such crisp rule application is too severe. What we have regarded as correctness rules resulted from historical observations, and were designed to capture "normal" practice. Clearly, composers of the period did not always abide by them—ironically several of their works would have failed many an elementary species counterpoint test.

Example 2.8. In the following melody by Jeppesen, the opening leap of a perfect fifth should be followed by movement in the opposite direction, possibly a G. Jeppesen, however, follows the upward leap with a further upward motion to a b.

D a b a G D F E D ■

This suggests that a less rigid adherence to these rules would be truer to their spirit. As a first step towards our goal of human-like music composition, therefore, we propose the *fuzzy application* of a subset of these rules. Fuzzy application of rules takes the form:

If *violation of* rule *r* by countermelody *m* is *complete [somewhat/a little / slightly]*
Then merit μ_r of countermelody *m* is *bad [not so good/mediocre / not bad]*

In this case, a rule can be "completely violated," "somewhat violated," "a little violated," etc. Consequently, the countermelody can be "bad," "not so good,", "mediocre," etc. Thus, if the Recovery

Rule were so applied, Jeppesen's melody in example 2.8 would have violated the rule only "slightly," and thus his melody would still be considered "not bad," provided it violated no other rule.

Fuzzy rule application enables us to quantify the degree of rule satisfaction, which we shall normalize to stay within [0..1], as per fuzzy set theory convention. The Tonality Rule could be satisfied to, say, the degree 0.8 if, out of ten notes of a lydian or dorian melody, there occur two instances of the note h. Such a melody would then be considered partially good, instead of completely bad—"acceptable with reservation."

Some of our new artistry rules are more naturally expressed with a view to fuzzy application. For example, the two observations

> If for any note *n* of countermelody *m*, *degree of repetition* (*n*) is *not very high*
> Then the *note variety* of *m* is *quite high*

and

> If the *note variety* of countermelody *m* is *quite high*
> Then merit $\mu_{note\ variety}$ of countermelody *m* is *good*

result in the artistry rule we have called "Note Variety," e.g.:

> If for any note *n* of countermelody *m*, *degree of repetition* (*n*) is *not very high*
> Then merit $\mu_{note\ variety}$ of countermelody *m* is *good*

Under this rule, melody (a) of example 2.5 (<D, E, D, E, D, C#, D>) would fare poorly: the unimaginative repetition of the first two notes, especially of the D, results in the monotony described in the example above.

It is clear that musical rules play a central role in an algorithmic music composition system. In this dissertation, we explore this role in detail, viewing an algorithmic music composition system as an expert system. From this point of view, we require three specialists: the domain expert (the musician), the systems designer, and the programmer. The domain expert in species counterpoint expresses knowledge in the form of musical rules. The systems designer, in consultation with the domain expert, selects rules for fuzzy application, and formulates the determination of melodic merit by the application of these selected rules (see this and chapter 5); the designer also formulates the search engines and algorithms for the system (chapter 6). The programmer implements, tests, and maintains the system, initiating a cycle that loops back to the music expert and system designer (fig. 2.1). In this dissertation, we shall base our use of fuzzy rules on a simple formulation, described in the next section.

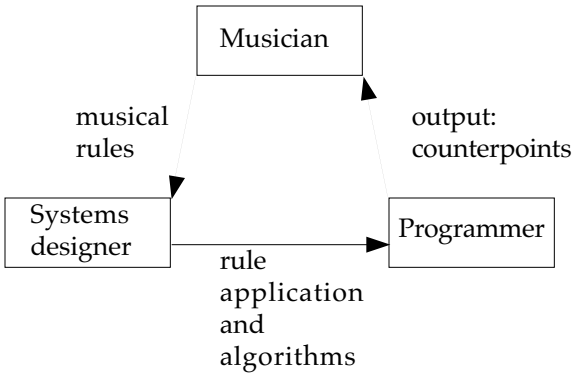


Fig. 2.1. An algorithmic music composition system

2.3.1. Implementation of Fuzzy Rule Application

In our implementation of fuzzy rule application, we calculate the merit of a countermelody as follows: (1) Derive the scores of individual artistry rules (and other rules selected for fuzzy application) on the basis of how many times the countermelody violates them. The larger the number of violations of a particular rule, the lower the score for that rule:

$$\mu_r(m) = \max\{1 - \frac{\rho_r\, t_r(m)}{T_r(m)}, 0\} \tag{2.1}$$

where
$\mu_r(m)$ is the merit of a countermelody m with respect to rule r, $0 \le \mu_r \le 1$,
$\rho_r \in \mathbb{R}+$ is the "severity rating" of rule r,
$t_r(m)$ is the number of times rule r is violated by m,
$T_r(m)$ is the number of times rule r is evaluated in m.

(2) Aggregate the individual rule scores to produce the overall merit of the countermelody. In the next section, we discuss rule weighting as a factor in the aggregation.

We apply the above formula to the first three of the artistry rules. For the fourth rule, Note Variety, we modify the formula to allow for one or more violations without penalty. To insist on strict adherence according to the formula would mean that only ascending or descending scales would be allowed, which would hardly be an artistic solution. Thus:

$$\mu_{\text{Note variety}}(M) = \max\left\{1 - \frac{\max\{\rho_{\text{Note variety}}(t(M) - e), 0\}}{T(M)}, 0\right\} \tag{2.2}$$

where
ρ, t, and T are as in formula (2.1),
$e \in \mathbb{N}_{T(M)}$ is the exemption factor.

Currently we have kept ρ = 1 a fixed quantity for all the four rules, and used e = 3. Thus, the score for melody (b) in example 2.6 (<D, h, a, G, F, E, D>) under the Tonality Rule is derived as follows: $\mu_{\text{Tonality}} = \max \{ 1 - 1/7, 0 \} = 0.86$. The score for melody (a) of example 2.5 (<D, E, D, E, D, C#, D>) is $\mu_{\text{Note variety}} = \max \{ 1 - \max \{ (4 - 3), 0 \} / 7, 0 \} = 0.86$. In this example, there were four violations: three for the repetitions of the note "D", and one for the repetition of the note "E".

We have used the above evaluation method (with modifications for certain rules) to guide heuristic searches for countermelodies. In the subsequent discussion, we refer to rules applied crisply as *crisp rules* and those applied as described in section 2.3.1 as *fuzzy rules*. See chapter 6 for a description of an exhaustive search algorithm (chapter 6, section 6.1), and for our use of the best-first branch-and-bound and genetic algorithms (chapter 6, sections 6.1, 2); all three applications use the notion of crisp and fuzzy rules for their respective evaluation functions.

2.3.2. Rule Weights

Humans compose music to a large extent by trial-and-error under the guidance of appropriate rules. Algorithmic music composition systematizes this process, but remains dependent upon the same rules. No rule set, however, can fully capture all the subtleties of a historical style, even for a simple model like species counterpoint. Moreover, as complexity analysis will indicate (chapter 4), a program armed with these rules can often produce a surfeit of countermelodies, albeit correct. A skilled composer has no trouble eliminating from consideration a large number of these, and even discriminate among many artistic possibilities. Clearly, the composer relies on more than just explicit rules in the decision-making process.

Example 2.9. Given the following four competing countermelodies (*CM* 1–4) for cantus firmus *CF*, most musicians would prefer countermelody *CM* 1 over the other three, for all that *CM* 3 is the most "correct" if we include Lewin's Rule in judging these countermelodies.

CM 4:	a	a	a	h	a	C#	D
CM 3:	D	E	D	E	D	C#	D
CM 2:	a	E	F	E	D	C#	D
CM 1:	a	a	F	E	D	C#	D
CF:	D	C#	D	G	F	E	D

■

To develop methods that more closely reflect expert preferences, we next consider the notion of *rule weights*, whereby we can distinguish among the relative importance of each rule. Not all rules are equal in importance, at least in the minds of composers and most listeners. Some are fundamental, and these are best applied crisply, to enforce the notion that no melody should ever violate them; others

are fairly significant, but their violation by a melody is not always fatal; yet others represent qualities that are nice to have but not absolutely essential. No masterpiece is perfect, but their very imperfection reflects their human artistry. Assigning weightings to rules that are less than fundamental allows us to reflect human artistry.

After assigning each of these rules a numerical weighting, the easiest way to calculate the overall merit of a countermelody would be to find the weighted average of the evaluations of all these rules. The formula for the overall merit of a countermelody m is thus:

$$\mu(m) = \frac{\sum_{r=1}^{n} w_r \mu_r(m)}{\sum_{r=1}^{n} w_r} \qquad (2.3)$$

where
$\mu(m)$ is the overall merit of countermelody m,
n is the total number of rules,
w_r is the weighting of rule r, $1 \le r \le n$, $w \ge 0$,
μ_r is the merit of m with respect to rule r, $1 \le r \le n$.

We use rule weights to calculate a countermelody's merit in a method to model human preferences (chapter 5, section 5.1).

2.4. Summary

The challenge in species counterpoint composition is in having to comply with melody, harmony, and counterpoint rules ("correctness" rules). These rules, unfortunately, while ensuring to some degree the correctness of a counterpoint solution, fall far short of ensuring artistic results. Our introduction of artistry rules, together with the concept of the fuzzy application of rules, addresses this difficulty.

We also considered the issue of rule redundancy and indispensability. We found that two rules in particular, Lewin's and the Dissonant Harmonic Intervals Rules, accounted for a significant number of rejections of potential solutions.

CHAPTER 3. SPECIES COUNTERPOINT AND THE FINITE STATE MODEL

Applying fuzzy rules is only a first step in the process of musical composition. This overall compositional process is quite computationally intensive, as we now show over this and the next chapter. We shall formally define several problems based on species counterpoint. We then prove that the more general among them belong to the NP-complete and other similarly complex categories, whereas more restricted, but still musically useful, problems can be solved efficiently using simple models like the finite state machine. In this chapter we discuss the restricted problems solvable on simple models; in the next chapter we discuss the more general problems.

In confining our analysis to first species counterpoint, we have not lost generality. To encompass the other species, we need not define entirely new problems, but simply alter the format of the cantus firmus; to obtain second species (two notes in the added melody to one in the cantus firmus) or fourth species (predominantly syncopated movement in the added part), we modify the given cantus firmus to repeat every note; to obtain third species (four notes in the added melody to one in the cantus firmus), we modify the cantus firmus to repeat every note thrice. For the fourth species, we specify the rules accordingly to obtain the predominantly syncopated movement; for the fifth species, we repeat the cantus firmus notes as many times as we wish to subdivide their beat.

Example 3.1. The following notations of counterpoints taken from Jeppesen (1931) show how to represent second and fourth species counterpoint by altering the format of the cantus firmus (compare the notation in example 1.1).

Second species, two parts:

countermelody: a h c d e d f F G a b F a h c#(c#) d
cantus firmus: D D a a G G F F E E D D F F E E D

(Jeppesen 1931, 117)

Fourth species, two parts:

countermelody: a a d d c c h h e e d e c c b a d d c# d
cantus firmus: D D F F E E D D G G F F a a G G F F E E D

(Jeppesen 1931, 134) ■

3.1. Restricted Problems Efficiently Solvable by the Finite State Model

The simplest problems are those dependent upon a history of a fixed number of notes in a single added part. We have defined one such version (HM2PAL RESTRICTED) in a standard format, which we shall use throughout this dissertation: we first describe the components of an instance of the problem, and then ask a question about an instance that requires a yes/no answer. For the more general form of this problem (HM2PAL), see chapter 4, section 4.1.1.

Problem 3.1. HM2PAL RESTRICTED
INSTANCE:
A finite set S;
$c \in \mathbb{N}$;
a sequence $CF \in S^c$;
H: $(S \times S) \rightarrow \{0, 1\}$;
M: $(S \times S) \rightarrow \{0, 1\}$;
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
$R\ (<cf_1, cf_2, \ldots, cf_c>, <m_1, m_2, \ldots, m_c>)$
$= 1$
if $H\ (cf_i, m_i) = 1,\ 1 \leq i \leq c$
and
$M\ (m_i, m_{i+1}) = 1,\ 1 \leq i \leq c - 1$
$= 0$ otherwise.
QUESTION:
Is there a $CP \in S^c$ such that $R\ (CF, CP) = 1$?

We shall show that this problem can be solved efficiently by using one of the simplest of computational models: the finite state machine. The input string is the cantus firmus, and arrival at the final state corresponds to a "yes" answer to this instance of the problem.

3.1.1. Non-Deterministic Finite State Machine to Solve HM2PAL RESTRICTED

From the rule specifications, we construct a non-deterministic finite state machine. The general method is as follows (algorithm 3.1).

(1) Create a start state (labeled "s").
(2) Create a distinct state for each permissible harmonic interval. For ease of subsequent modifications, we label the state "cm/cf", where "cm" is the note of the countermelody and "cf" that of the cantus firmus.
(3) Create transitions between source and destination states, with labels denoting cantus firmus notes (corresponding to the "cf" term in the destination state), wherever $\text{cm}_{\text{source}}$ to $\text{cm}_{\text{destination}}$ is a permissible melodic interval. From the start state, create transitions corresponding to the first note of the cantus firmus.
(4) Designate final states.

Algorithm 3.1. Building a non-deterministic finite state machine from rule specifications

Example 3.2. Consider the following concrete instance of HM2PAL RESTRICTED: Its gamut consists of the following notes: C#, D, E, F, G, a, and h. The only permissible harmonic intervals are major and minor thirds, perfect fourths and fifths, major and minor sixths, and unison only on D. The only permissible melodic intervals are seconds and thirds. We can even stipulate the following additional restrictions, not accommodated for in HM3PAL RESTRICTED's instance specification: No parallel perfect fifths are ever allowed, the first harmonic interval must be either a unison or a perfect fifth with the cantus firmus, which will always begin on D, and the final interval must be a unison on D. We now build the non-deterministic finite state machine for this instance, according to algorithm 3.1.

In step (2) of algorithm 3.1, the two possible beginning harmonies are "D/D" and "a/D". For this instance, it is necessary to distinguish these two states (that are used only for the initial note of the countermelody/cantus firmus) from two more states with similar intervals, but which indicate later notes of the counterpoint; this we have done so by labeling these latter states "D/D'" (final state) and "a/D'" respectively.

For step (3), there is a transition between state "a/D" to state "G/E" (labeled "E"), but there is no transition between state "a/D" to state "E/C#" because this would cause an inadmissible melodic interval of the fourth; neither is there a transition between state "a/D" to state "h/E" because this would cause parallel perfect fifths.

Using algorithm 3.1, we produce a finite state machine for the above instance to produce correct countermelodies (fig. 3.1). ■

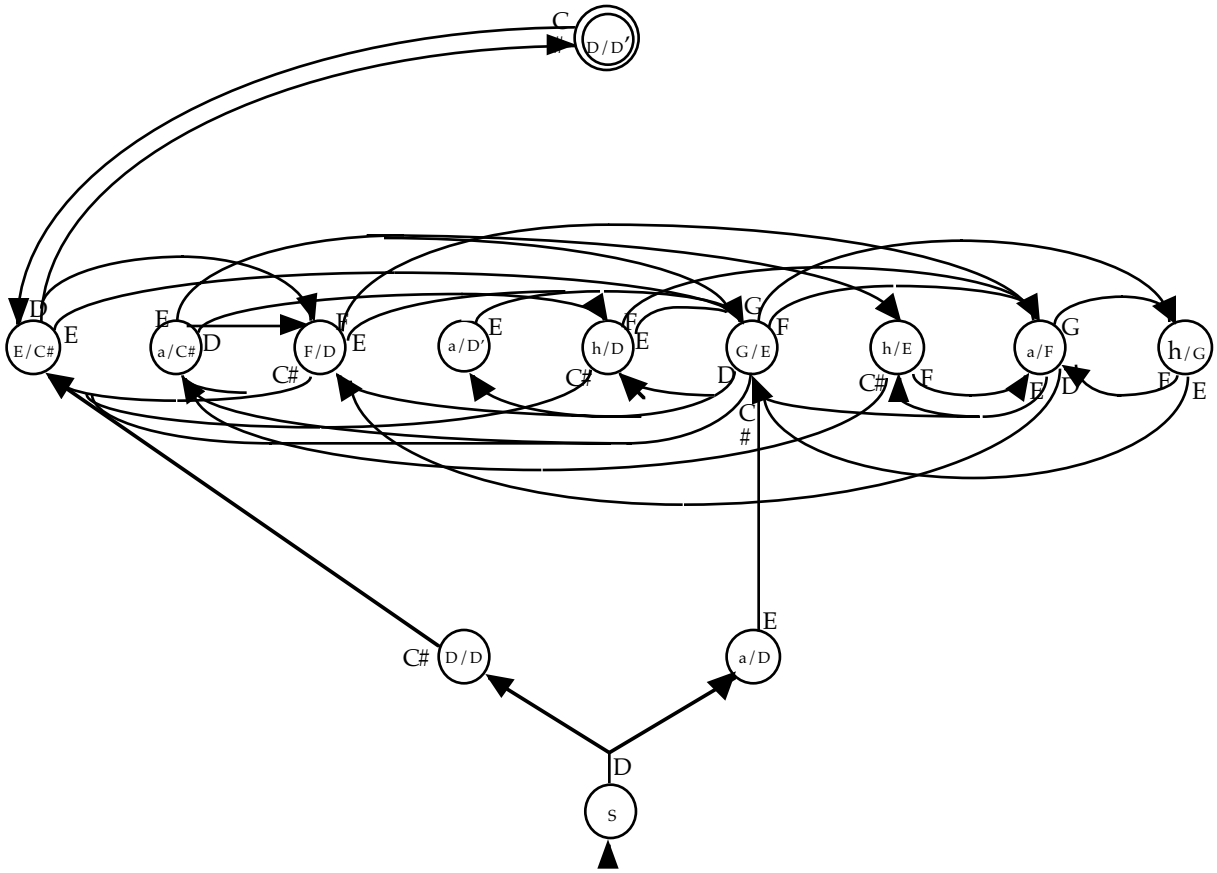


Fig. 3.1. Non-deterministic finite state machine for an instance of HM2PAL RESTRICTED

Computational theory tells us that both non-deterministic and the corresponding deterministic finite state machines are equally powerful. Furthermore, given a non-deterministic finite state machine as above, there exists an algorithm to verify whether there is a correct countermelody for a given cantus firmus without having to construct a deterministic version, and that algorithm is

polynomial in the number of states and in the length of the given cantus firmus. This is formalized in theorem 3.1 (Lewis and Papadimitriou 1998, 107).

Theorem 3.1. If $M = (Q, \Sigma, \Delta, q_0, F)$ is a non-deterministic finite state machine, then there is an algorithm to test "$w \in L(M)$" in time $O(|Q|^2 |w|)$.

We now implement this algorithm to test membership of a string (cantus firmus *CF*) in the language of the non-deterministic finite state machine (*M*). Our implementation applies the dynamic programming technique to compute sets of reachable states after having read a partial string (algorithm 3.2, after Lewis and Papadimitriou 1998, 106).

Algorithm TestCF
Input: $M = (Q, \Sigma, \Delta, q_0, F)$, $CF = \langle cf_1, cf_2, \ldots, cf_c \rangle$.
Output: *result* $\in$ {accept, reject}.
1. $S_0 := \{q_0\}$
2. **for** n from 1 to c
   $S_n := \cup \{q: p \in S_{n-1}, (p, cf_n, q) \in \Delta\}$
3. **if** $S_c \cap F \neq \varnothing$
   result := accept
else
   result := reject.

Algorithm 3.2. Testing acceptance of a cantus firmus (*CF*) by a non-deterministic finite state machine (*M*)

For a fixed gamut and fixed harmony and melody rules, we need construct a corresponding (non-deterministic) machine only once at the outset. Having constructed the finite state machine, we can store its specification for subsequent use.

Example 3.3. Given the finite state machine constructed in example 3.2, and the cantus firmus <D, E, D, C#, D>, the answer is "yes, there is/are countermelodies that satisfy/ies all the rules," but given the cantus firmus <D, C#, E, D>, the answer is "no, there are not." The set of states S_n processed by TestCF for <D, E, D, C#, D>, and for <D, C#, E, D>, and their respective paths, are shown in figure 3.2. ■

For restricted music decision problems solvable by the finite state machine, the following complexity analysis applies: If the length of the cantus firmus is fixed, or if the application of rules is limited to a fixed lookback distance, the number of states of the finite state machine and the required number of steps to solve an instance of the problem are both polynomially bounded, as shown below:

1) All cantus firmi ("CF") of fixed length LEN with LEN –1 history:

Number of states of an Non-Deterministic FSM: O ($|S|^{LEN}$).
Number of steps to search: O ($|S|^{2LEN}$).

2) All cantus firmi of any length with fixed length history k (k < |CF|):

Number of states of an Non-Deterministic FSM: O ($|S|^{k+1}$).
Number of steps to search: O ($|S|^{2k+2} . |CF|$).

(a)

n	cf_n	S_n	Path
0		S_0 = {s}	s
1	D	S_1 = {a/D, D/D}	a/D D/D
2	E	S_2 = {G/E}	G/E
3	D	S_3 = {F/D, h/D, a/D'}	F/D h/D a/D'
4	C#	S_4 = {E/C#, a/C#}	E/C# a/C#
5	D	S_5 = {D/D', F/D, h/D}	D/D' F/D h/D

Since D/D' ∈ F, result = accept.

(b)

n	cf_n	S_n	Path
0		S_0 = {s}	s
1	D	S_1 = {D/D, a/D}	D/D a/D
2	C#	S_2 = {E/C#}	E/C#
3	E	S_3 = {G/E}	G/E
4	D	S_4 = {F/D, a/D', h/D}	F/D a/D' h/D

Since neither F/D, a/D', nor h/D is a final state, result = reject.

Fig. 3.2. Processing of cantus firmi (a) <D, E, D, C#, D> and (b) <D, C#, E, D> by the non-deterministic finite state machine of figure 3.1

Although for every non-deterministic finite state machine, there is an equivalent deterministic finite state machine, and one that can process a cantus firmus in strictly linear time, there is a vast difference in the space requirements, as the number of states of the latter may be exponential (the power set of the number of states of the non-deterministic machine corresponding approximately to the number of acceptable harmonies, is exponentially larger). Also, the construction of the equivalent minimal deterministic finite state machine is super-polynomial in time complexity unless P = NP, as theorem 3.2 suggests (Lewis and Papadimitriou, 1998, 330, there expressed as a corollary).

Theorem 3.2. There is no algorithm polynomial in both input and output that, given a non-deterministic finite state machine, constructs an equivalent minimum-state deterministic finite state machine, unless P = NP.

The corresponding enumeration problem asks the question: "How many valid countermelodies are there?" We define it as follows (HM2PAL RESTRICTED ENUM).

Problem 3.2. HM2PAL RESTRICTED ENUM
INSTANCE:
A finite set S;
$c \in \mathbb{N}$;
a sequence $CF \in S^c$;
H: $(S \times S) \rightarrow \{0, 1\}$;
M: $(S \times S) \rightarrow \{0, 1\}$;
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
$R\ (<cf_1, cf_2, \ldots, cf_c>, <m_1, m_2, \ldots, m_c>)$
$= 1$
if $H\ (cf_i, m_i) = 1,\ 1 \leq i \leq c$
and
$M\ (m_i, m_{i+1}) = 1,\ 1 \leq i \leq c - 1$
$= 0$ otherwise.
QUESTION:
How many sequences $CP \in S^c$ are there such that $R\ (CF, CP) = 1$?

Using the non-deterministic finite state machine, the number of valid counterpoints can be computed by storing at each state arrived an integer value reflecting the number of partial paths leading to it, which is the sum of all partial paths to the states in its predecessor list (algorithm 3.3).

Example 3.4. Running the finite state machine shown in figure 3.1, the answer in the problem instance with cantus firmus <D, E, D, C#, D> is "one," because there is only one path leading to the goal state (corresponding to the transitions a/D–>G/E–>F/D–>E/C#–>D/D'); for cantus firmus <D, C#, E, D>, the answer is obviously "zero." Figure 3.3 illustrates the additional information needed to solve the enumeration problem. In this instance, the single final state D/D' shows only one partial path; had F/D been also a final state, the number of valid countermelodies would have been three. ■

Algorithm EnumerateCM

Input: $M = (Q, \Sigma, \Delta, q_0, F)$, $CF = <cf_1, cf_2, \ldots, cf_c>$.

Output: *rnumCountermelodies*.

1. $S_0 := \{q_0\}$
2. NumPathsTo $(q_0) := 1$
3. **for** n from 1 to c
   $S_n := \cup \{q: p \in S_{n-1}, (p, cf_n, q) \in \Delta\}$
   for each $s \in S_n$
   NumPathsTo $(s) := \Sigma_{p \in S_{n-1}}$ NumPathsTo $(p \in S_{n-1}, (p, cf_n, s) \in \Delta)$
4. $F' := S_c \cap F$
5. **if** $F' \neq \varnothing$
   numCountermelodies $:= \Sigma_{f \in F'}$ NumPathsTo (f)

else
   numCountermelodies $:= 0$.

Algorithm 3.3. Solving the enumeration problem by a non-deterministic finite state machine (M)

n	cf_n	S_n	Path
0		$S_0 = \{s\}$	s
1	D	$S_1 = \{a/D, D/D\}$	$a/D_{(1)}$ $D/D_{(1)}$
2	E	$S_2 = \{G/E\}$	$G/E_{(1)}$
3	D	$S_3 = \{F/D, h/D, a/D'\}$	$F/D_{(1)}$ $h/D_{(1)}$ $a/D'_{(1)}$
4	C#	$S_4 = \{E/C\#, a/C\#\}$	$E/C\#_{(1)}$ $a/C\#_{(2)}$
5	D	$S_5 = \{D/D', F/D, h/D\}$	$D/D'_{(1)}$ $F/D_{(3)}$ $h/D_{(2)}$

Fig. 3.3. Additional information stored to solve the enumeration problem (compare example 3.3). The notation, e.g., $a/C\#_{(2)}$ indicates that there are two paths leading to the state a/C# reached after the transition labeled "C#".

The number problem asks the question: "Are there at least K countermelodies?" (K HM2PAL RESTRICTED).

Problem 3.3. K HM2PAL RESTRICTED
INSTANCE:
A finite set S;
$c, K \in \mathbb{N}$;
a sequence $CF \in S^c$;

H: $(S \times S) \rightarrow \{0, 1\}$;
M: $(S \times S) \rightarrow \{0, 1\}$;
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
R ($<cf_1, cf_2, \ldots, cf_c>, <m_1, m_2, \ldots, m_c>$)
$= 1$
if H (cf_i, m_i) = 1, $1 \leq i \leq c$
and
M (m_i, m_{i+1}) = 1, $1 \leq i \leq c - 1$
$= 0$ otherwise.

QUESTION:
Are there at least K sequences $CP \in S^c$ such that R (CF, CP_i) = 1, $1 \leq i \leq K$?

The question can be answered by counting the number of distinct paths leading to the goal state.

Example 3.5. For cantus firmus <D, E, D, C#, D>, the answer is "yes" for $K = 1$, "no" for $K > 1$; for cantus firmus <D, C#, E, D>, the answer is "no" for any non-zero K. Using the same adaptation as for the enumeration problem, we can easily answer this question, by inspecting the sum of the number of partial paths to the final states in S_c. ■

3.1.2. Non-Deterministic Finite State Transducer for Output Related to HM2PAL RESTRICTED

By means of a simple modification to the finite state machine as constructed above, we obtain a finite state *transducer*, a machine that is capable of outputting a suitable countermelody to a given cantus firmus. We can adopt either the Moore (Moore 1956) or the Mealy (Mealy 1955) machine, since both models are equivalent. In the *Moore machine*, the states specify the output, whereas in the *Mealy machine*, the transitions specify the output. Using our convention of naming the states as "cm/cf", where "cm" is a note of the countermelody and "cf" is a note of the cantus firmus, the Moore machine would produce "cm" at each state, and the Mealy machine would produce at the transition the "cm" specified in the destination state. For the Moore machine, the start state produces a null character.

We use this transducer to produce a valid countermelody to a cantus firmus, if one exists, by storing additional information in the form of a list of predecessor states (represented as $\text{Pred}_{s,n}$ in algorithm 3.4 below). We then use this list to reconstruct a valid countermelody, after ascertaining its existence by first running the transducer. The first part of the algorithm extends TestCF to produce the predecessor states list and the (possibly empty) set of final states reached after processing the input string (algorithm 3.4). For each state reached in the processing of the input string, the algorithm TestCF2 produces its predecessor list, which is a state of sets reached prior to the current state. We use

these sets of states to reconstruct a valid countermelody by starting with a final state that also belongs to the final set of states produced by TestCF2 (algorithm 3.5). Notice that this algorithm produces some valid countermelody from possibly several valid ones. Which countermelody it produces depends on non-deterministic choices of states from the set of final states F' and from the predecessor states list.

Algorithm TestCF2
Input: $M = (Q, \Sigma, \Delta, q_0, F)$, $CF = <cf_1, cf_2, \ldots, cf_c>$.
Output: $result \in$ {accept, reject},
$F' \subseteq F$,
Pred: $(Q \times \mathbb{N}) \rightarrow 2^Q$.
1. $S_0 := \{q_0\}$
2. $\text{Pred}_{q0,0} := \{\}$ // start state has no predecessor
3. **for** n from 1 to c
$S_n := \cup \{q: p \in S_{n-1}, (p, cf_n, q) \in \Delta\}$
for each $s \in S_n$
$\text{Pred}_{s,n} := \cup \{p \in S_{n-1}, (p, cf_n, s) \in \Delta\}$
4. $F' := S_c \cap F$
5. **if** $F' \neq \varnothing$
$result$:= accept
else
$result$:= reject.

Algorithm 3.4. Testing acceptance of a cantus firmus (*CF*) and outputting a state predecessor list (Pred) by a non-deterministic finite state machine (*M*)

Building the transducer, testing a cantus firmus, and deriving a valid countermelody still requires only polynomial computing time. The time complexity is $O(|CF||Q|^2) = O(|CF||S|^4)$. Should we desire a strictly linear computing time, we need to convert this non-deterministic machine into a deterministic one. In this case, as with the non-deterministic finite state machine (section 3.1.1), the number of states in the deterministic model may again become exponential in the size of the gamut, and the construction itself will again be super-polynomial, unless P = NP.

Example 3.6. Countermelodies produced by the finite state transducer for cantus firmi <D, C#, D, F, E, D, C#, D>, <D, E, D, C#, D>, and <D, F, E, D, C#, D> are shown below:

1. Countermelody:	D	E	F	a	G	F	E	D
Cantus firmus:	D	C#	D	F	E	D	C#	D
2. Countermelody:	a	G	F	E	D			
Cantus firmus:	D	E	D	C#	D			
3. Countermelody:	No countermelody possible							
Cantus firmus:	D	F	E	D	C#	D		

■

Algorithm ConstructCM
Input: $M = (Q, \Sigma, \Delta, q_0, F)$,
$CF = <cf_1, cf_2, \ldots, cf_c>$.
Output: $CM = <cm_1, \ldots, cm_c>$.
Assumption: TestCF2 accepts *CF*.
1. TestCF2 (M, CF, *result*, F', Pred)
2. f := some $s \in F'$
3. cm_c := Out (M, $f \in F'$)
4. $p := f$
5. **for** n from c–1 to 1
p' := some $s \in \text{Pred}_{p, n+1}$
cm_n := Out (M, p')
$p := p'$.

Subprogram Out
Input: $M = (Q, \Sigma, \Delta, q_0, F)$,
$q \in Q$, $q = <cm, cf>$, cm, $cf \in \Sigma$.
Output: *cm*.

Algorithm 3.5. Constructing a valid countermelody (*CM*) from output of an accepting non-deterministic finite state transducer (*M*)

With more information regarding the relative merits of the allowable intervals, both melodic and harmonic, it would also be possible to solve the corresponding optimization problem, which asks the question: "What is the best countermelody?" As it stands, the answer to all instances for which the answer to the decision problem is "yes" is the set of all the possible countermelodies, because they are all equally good.

The finite state transducer constructed for the instance specified in example 3.2 can be modified to enforce two extra rules that are commonly observed in much of the music composed during the sixteenth century and beyond: the Recovery Rule and the Leading-tone Rule (defined in appendix A as part of the Diatonic Resolution Rule).

• Recovery Rule (A leap must be followed by a step in the opposite direction):

If $note_n$ of the countermelody is p
and $note_{n+1}$ of the countermelody is q
and $note_{n+2}$ of the countermelody is r
Then if *ascending* (p, q) is *true*
then *descending* (q, r) is *true* and *interval size* (q, r) is *second*
and if *descending* (p, q) is *true*
then *ascending* (q, r) is *true* and *interval size* (q, r) is *second*.

• Leading-tone Rule (The leading tone (C#) must rise):

If $note_n$ of the countermelody is *C#*
Then $note_{n+1}$ of the countermelody is *D*.

Example 3.7. Consider the following four melodic fragments:

(a)
(i) D b c (ii) D b a

(b)
(i) … C# A … (ii) … C# D …

Melody (a) (i) violates the Recovery Rule because the ascending leap D–b is followed by further ascending motion; melody (a) (ii) is correct according to this rule: after the ascending leap, the melody falls (by a step) in the opposite direction. Melodies (b) (i) and (ii) both have a C#, which is the leading-tone in the dorian mode and has to rise stepwise to D, and not go anywhere else; thus, melody (b) (ii) is correct, but melody (b) (i) is not. Note that this definition of the Recovery Rule is the stronger form of the more common definition (for the latter definition, see appendix A; for a further illustration of this rule, see chapter 2, example 2.8). ■

The modified finite state transducer for the above instance of HM2PAL RESTRICTED would then require many more states to reflect direction and size of approach (fig. 3.4). The process is otherwise straightforward, if somewhat tedious. Although much more complicated than the original transducer, a correct countermelody can still be computed in polynomial time.

Example 3.8. Output of the modified finite state transducer (fig. 3.4) to cantus firmi <D, E, D, C#, D, F, E, D, C#, D, E, D, C#, D> and <D, C#, D, E, F, G, E, F, D, C#, D> are given below:

1. (2 possible solutions: countermelody 1 and 2):
countermelody 1: a G F E F a G F E F G F E D
countermelody 2: a G F E F a G h a F G F E D
cantus firmus: D E D C#D F E D C# D E D C# D

2. (2 attempted solutions):
countermelody 1: D E F G a h G a h a ? (no continuation is possible)
countermelody 2: D E F G a h G a F ? (no continuation is possible)
cantus firmus: D C# D E F G E F D C# D

3.
countermelody: D E F G a h G a F G F E D
cantus firmus: D C# D E F G E F D E D C#D ■

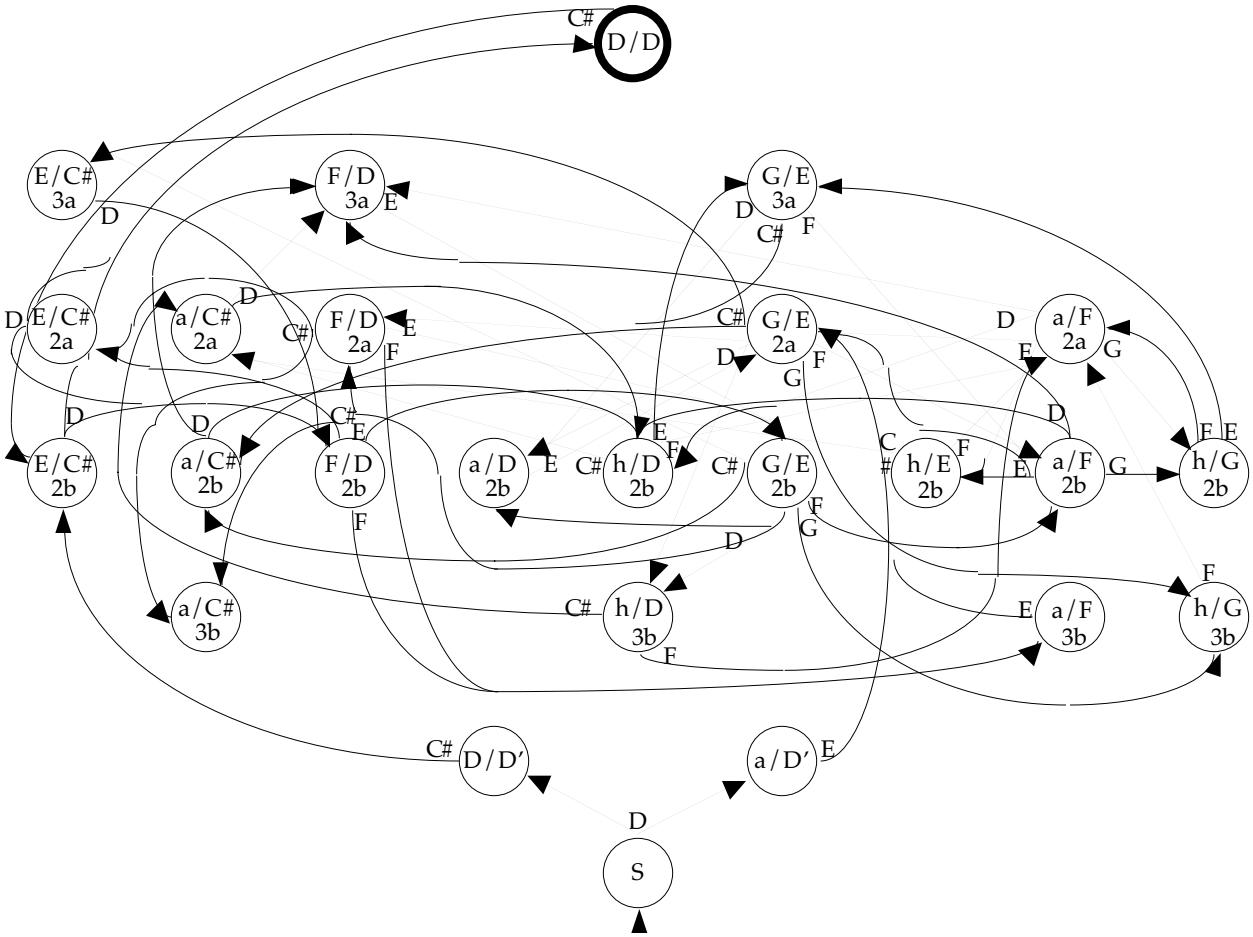


Fig. 3.4. Modified finite state machine to enforce two extra rules

3.1.3. Weighted Finite State Transducers to Solve the Optimization Problem

In the optimization problem (HM2PAL OPTIMIZATION), we would like to find the "best" countermelody, according to a criterion based on weights given to various harmonies and melodic progressions.

Problem 3.4. HM2PAL OPTIMIZATION
INSTANCE:
A finite set S;
$c \in \mathbb{N}$;
a sequence $CF \in S^c$;
H: $(S \times S) \rightarrow \{0, 1\}$;
M: $(S \times S) \rightarrow \{0, 1\}$;
$ValH$: $(S \times S) \rightarrow \mathbb{N}_0$;
$ValM$: $(S \times S) \rightarrow \mathbb{N}_0$;
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$

where R is defined as follows:

$R\ (<cf_1, cf_2, \ldots, cf_c>, <m_1, m_2, \ldots, m_c>)$

$= 1$

if $H\ (cf_i, m_i) = 1,\ 1 \le i \le c$

and

$M\ (m_i, m_{i+1}) = 1,\ 1 \le i \le c - 1$

$= 0$ otherwise.

QUESTION:

What is the optimal sequence $CP \in S^c$ that satisfies $R\ (CF, CP) = 1$ and produces $\alpha =$

$$\max_{CP} \left[\sum_{i=1}^{c} \left(ValH\ (cf_i, m_i) + ValM\ (cf_i, m_i) \right) \right] ?$$

We can extend the functionality, although not the language recognition power, of the finite state transducer to compute such "optimal" countermelodies. We establish the optimization criterion by assigning weights to each state and transition, corresponding to the weights of the harmonies and melodic progressions specified in the optimization problem; the optimal countermelody, then, is that countermelody accumulating the highest aggregate of these weight values in the processing of the cantus firmus. An extension to the dynamic programming method of determining string acceptance by a non-deterministic finite state transducer enables us to derive the optimal countermelody in polynomial time. As with Floyd's all-pairs shortest path algorithm (Floyd 1962), the method involves storing the optimal partial score for partial strings attained in the set of states reachable at that point. This exploits, as does Floyd's algorithm, the *principle of optimality*, which states that all subinstances of an optimal solution are also optimal. In the following algorithm (3.6), we assign weights to harmonies (corresponding to states); a simple extension would enable us to assign weights to melodic movements as well. We shall use the output of this algorithm to construct the optimal countermelody, hence the need to output $F' \subseteq F$ and Pred. As with Floyd's algorithm, we implement this algorithm with $|Q|$ by $|Q|$ matrices, but this time with two such matrices. Since any state may be arrived at more than once (unlike Floyd's algorithm on graphs, note repetition is possible), we need to store two matrices, and alternate them when calculating optimal partial paths.

Example 3.9. Figure 3.5 illustrates output of the finite state transducer (see figure 3.1) with cantus firmus <D, E, D, C#, D, C#, D>; suppose all harmonies have the weight of 1, except for the harmony F/D, which has a weight of 2. There are then three valid countermelodies: <a, G, F, E, F, E, D>, <a, G, F, a, F, E, D>, and <a, G, F, E, D, E, D>. Of these, the first two are optimal, both with a cumulative score of 9 (fig. 3.5). Note that at level $n = 5$, Pred $(F/D)5$ could be either {E/C#} or {a/C#};

we arbitrarily chose {E/C#}; had we chosen {a/C#} instead, we would have obtained <a, G, F, a, G, F, E> as an optimal countermelody. ■

Algorithm OptimalScore
Input: $M = (Q, \Sigma, \Delta, q_0, F)$,
$CF = <cf_1, cf_2, \ldots, cf_c>$.
Output: $result \in$ {accept, reject},
$optimalScore$,
$F' \subseteq F$,
Pred: $(Q \times \mathbb{N}) \rightarrow 2^Q$.
AccumulatedScore: $(Q \times \mathbb{N}) \rightarrow \mathbb{N}$
// accumulated score of a state during construction of partial melody
1. $S_0 := \{q_0\}$
2. AccumulatedScore $(q_0, 0) := 0$
3. $\text{Pred}_{q0,0} = \{\}$
4. **for** n from 1 to c
$S_n := \cup \{q: p \in S_{n-1}, (p, cf_n, q) \in \Delta\}$
for each $s \in S_n$
$maxP := p$: AccumulatedScore $(p, n-1) = \max_{p' \in S_{n-1}, (p', cf_n, s) \in \Delta}$ [AccumulatedScore $(p', n-1)$]
AccumulatedScore (s, n) := AccumulatedScore $(maxP, n-1)$ + Val (M, s)
$\text{Pred}_{s,n} := \{maxP\}$
5. $F' := S_c \cap F$
6. **if** $F' \neq \varnothing$
$result$:= accept
$optimalScore := \max_{p' \in F'}$ (AccumulatedScore (p', c))
else
$result$:= reject
$optimalScore$:= undefined.

Subprogram Val
Input: $M = (Q, \Sigma, \Delta, q_0, F)$,
$q \in Q$, $q = <cm, cf, v>$, $v \in \mathbb{N}$.
Output: v.

Algorithm 3.6. Outputting the score of the optimal countermelody to cantus firmus *CF* by a weighted non-deterministic finite state transducer *M*

The algorithm to construct the optimal countermelody is similar to ConstructCM (algorithm 3.5), except here Pred always yields a singleton set. Recall that in ConstructCM, we always chose one state among possibly several at each predecessor level; here, we simply take the single element in each predecessor set.

n	cf_n	S_n	Optimal predecessor states			Optimal CP
0		{s}	s			
1	D	{a/D, D/D}	a/D(1)	D/D(1)		a
2	E	{G/E}	G/E(2)			G
3	D	{F/D, a/D′, h/D}	F/D(4)	a/D′(3)	h/D(3)	F
4	C#	{E/C#, a/C#}	E/C#(5)	a/C#(5)		E
5	D	{D/D′, F/D, h/D}	D/D′(6)	F/D(7)	h/D(6)	F
6	C#	{E/C#, a/C#}	E/C#(8)	a/C#(8)		E
7	D	{D/D′, F/D, h/D}	D/D′(9)	F/D(10)	h/D(9)	D

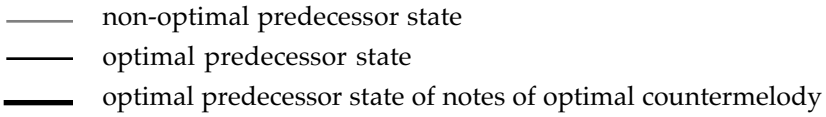


Fig. 3.5. Computing the optimal countermelody to cantus firmus <D, E, D, C#, D, C#, D>

We can also adapt algorithm 3.6 to output the m best countermelodies (algorithm 3.7, to solve problem HM2PAL m OPTIMIZATION).

Problem 3.5. HM2PAL m OPTIMIZATION
INSTANCE:
A finite set S; c, $m \in \mathbb{N}$;
a sequence $CF \in S^c$;
H: $(S \times S) \rightarrow \{0, 1\}$;
M: $(S \times S) \rightarrow \{0, 1\}$;
$ValH$: $(S \times S) \rightarrow \mathbb{N}_0$;
$ValM$: $(S \times S) \rightarrow \mathbb{N}_0$;
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
R ($<cf_1, cf_2, \ldots, cf_c>$, $<m_1, m_2, \ldots, m_c>$)
$= 1$
if H $(cf_i, m_i) = 1$, $1 \leq i \leq c$
and
M $(m_i, m_{i+1}) = 1$, $1 \leq i \leq c - 1$
$= 0$ otherwise.

QUESTION:

What are the m optimal sequences $CP_j \in S^c$ that satisfy $R(CF, CP_j) = 1, 1 \le j \le m$ and produce

$$\alpha_1 = \max_{CP} \left[\sum_{i=1}^{c} \left(ValH(cf_i, m_i) + ValM(cf_i, m_i) \right) \right]$$

$$\alpha_2 = \max_{CP-CP_1} \left[\sum_{i=1}^{c} \left(ValH(cf_i, m_i) + ValM(cf_i, m_i) \right) \right]$$

…

$$\alpha_m = \max_{CP-CP_1-\ldots-CP_{m-1}} \left[\sum_{i=1}^{c} \left(ValH(cf_i, m_i) + ValM(cf_i, m_i) \right) \right] ?$$

Here we store information not only for the best partial countermelody, but for the m best partial countermelodies. Using a max heap to compute the m best partially accumulated scores, the time complexity of this algorithm is $O(|CF| |Q| (m|Q| \log (m|Q|) + m \log m)) = O(|CF| |Q|^2 m (\log m + \log |Q|)) = O(|CF| |S|^4 m (\log m + \log |S|))$.

Example 3.10. Consider again the finite state transducer (fig. 3.1), with cantus firmus <D, E, D, C#, D, C#, D>, and assume again, as in example 3.9, all harmonies are equally weighted with the value of 1, except for F/D, which has the weight of 2. Storing accumulated scores enables us to derive the $m = 2$ best countermelodies <a, G, F, E, F, E, D> and <a, G, F, a, G, F, E> (fig. 3.6). ■

3.2. Restricted Problems that Cannot Be Solved by the Finite State Model

A finite state machine/transducer can implement only those rules that govern a fixed number of notes. With its fixed number of states (hence the adjective "finite"), this simple computational model cannot implement rules that govern notes spanning the entire length of the countermelody, or in fact all rules that govern a number of notes that is any non-constant function of the length of the countermelody. This length is variable, since it depends on the cantus firmus, whose length in turn is not fixed. For such rules, there does not seem to be available any less powerful model than the most general computing model itself, the Turing machine. Furthermore, these rules actually preclude the possibility of a polynomial solution (see PAL and related problems, defined in chapter 4). This may cause some surprise, since the validation of a countermelody by any one rule is never more than a linear function of its length.

Algorithm *m*Best

Input: $M = (Q, \Sigma, \Delta, q0, F)$, $CF = <cf1, cf2, \dots, cfc>$, $m \in \mathbb{N}$,

Val: $Q \rightarrow \mathbb{N}$,. // value of a state

Output: *result* ∈ {accept, reject},

$optimalScores = <opt1, opt2, \dots, opt_m>$, $opt_i \in \mathbb{N}$, $1 \le i \le m$, // best m optimal scores

$finalStates = <fin1, fin2, \dots, fin_m>$, $fin_i \in Q$, $1 \le i \le m$, // final states of the m optimal countermelodies

$predList_{q,i} = <q1, q2, \dots>$, $q, q_n \in Q$, $1 \le i \le c$. // predecessor states of state q at step i

AccumulatedScore: $(Q \times \mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{N}$. // accumulated best m scores of a state

1. $S1 := \cup \{q: (q0, cf1, q) \in \Delta\}$
2. **for** each $s \in S1$
   Insert $q0$ into $predList_{s,1}$
   AccumulatedScore $(s, 1, 1)$:= Val (M, s)
3. **for** n from 2 to c
   $Sn := \cup \{q: p \in S_{n-1}, (p, cf_n, q) \in \Delta\}$
   for each $s \in Sn$
   $predList_{s,n} := <>$
   for each $p \in S_{n-1}, (p, cf_n, s) \in \Delta$
   Index $(p) := 1$
   for i from 1 to m // find the m best among all predecessors to s
   $maxAccumulatedScore := 0$
   for each $p \in S_{n-1}, (p, cf_n, s) \in \Delta$, Index $(p) \le |predList_{p,n-1}|$
   if AccumulatedScore $(p, n\text{-}1,$ Index $(p))$ > $maxAccumulatedScore$
   $maxAccumulatedScore$:= AccumulatedScore $(p, n\text{-}1,$ Index $(p))$
   Index (p) := Index $(p) + 1$
   $maxP_i := p$
   if $maxP_i$ is defined
   Insert $maxP_i$ at end of $predList_{s,n}$
   AccumulatedScore $(s, n, i) := maxAccumulatedScore$ + Val (M, s)
4. $F' := S_C \cap F$
5. **if** $F' \neq \emptyset$
   result := accept
   for i from 1 to m
   for each $p \in F$
   Index $(p) := 1$
   $maxAccumulatedScore := 0$
   for each $p \in F'$
   if AccumulatedScore $(p, c,$ Index $(p))$ > $maxAccumulatedScore$
   $maxAccumulatedScore$:= AccumulatedScore $(p, c,$ Index $(p))$
   $optimalState := p$
   Index (p) := Index $(p) + 1$
   if $optimalState$ is defined
   $opt_i := maxAccumulatedScore$
   $fin_i := optimalState$

else
   result := reject
   $finalStates := \emptyset$.

Subprogram Val

Input: $M = (Q, \Sigma, \Delta, q0, F)$, $q \in Q$, $q = <cm, cf, v>$, $v \in \mathbb{N}$.

Output: v.

Algorithm 3.7. Outputting the m best countermelodies to cantus firmus CF by a weighted non-deterministic finite state transducer M

n	cf_n	S_n	Optimal predecessor states			$m = 2$ best CP's	
0		{s}	s				
1	D	{a/D, D/D}	a/D(1)	D/D(1)		a	a
2	E	{G/E}	G/E(2)			G	G
3	D	{F/D, a/D′, h/D}	F/D(4)	a/D′(3)	h/D(3)	F	F
4	C#	{E/C#, a/C#}	E/C#(5)	a/C#(5, 4)		E	a
5	D	{D/D′, F/D, h/D}	D/D′(6)	F/D(7, 7)	h/D(6, 5)	F	F
6	C#	{E/C#, a/C#}	E/C#(8, 8)	a/C#(8, 8)		E	E
7	D	{D/D′, F/D, h/D}	D/D′(9, 9)	F/D(10, 10)	h/D(9, 9)	D	D

non-optimal predecessor state
optimal predecessor state
optimal predecessor state of notes of optimal countermelody

Fig. 3.6. Computing $m = 2$ best countermelodies to the cantus firmus <D, E, D, C#, D, C#, D>

Two rules that encompass the entire length of the melody are Lewin's Rule (Lewin 1983) and the Stepwise Motion Rule:

- Lewin's Rule: For every non-final note m of a melody, there should exist a note n between m and the final note fi of the melody such that the musical interval m–n is a descending step if m is higher than fi, and an ascending step if m is lower than fi.

- Stepwise Motion Rule: Stepwise motion must predominate in a melody.

From the computational point of view, the complexity of validation of a countermelody by these two rules is O (| countermelody |).

Example 3.11. Consider the application of Lewin's and the Stepwise Motion Rule to the following counterpoints (countermelodies 1–3 are alternative solutions to the cantus firmus):

countermelody 3:	D	b	a	G	F	E	D
countermelody 2:	a	b	a	G	F	E	D
countermelody 1:	D	a	A	C#	D	E	D
cantus firmus:	a	D	F	E	D	C#	D

Countermelody 1 has an interval of a fourth between a and E: according to Lewin's Rule, this space must be filled up by G and F, as done by countermelodies 2 and 3, both of which do not violate the rule. Countermelody 1 also transgresses the Stepwise Motion Rule: of the six melodic intervals, three are leaps (D–a, a–A, A–C#), far too high a proportion. Countermelody 2 consists entirely of stepwise motion, and thus does not violate this rule, but runs afoul of the Note Variety Rule, should the latter be applied. Countermelody 3 has the best balance of leap and step (one leap and five steps). The aim of both Lewin's Rule and the Stepwise Motion Rule is to maintain smoothness; a countermelody that violates one rule often violates the other, as here. ■

3.3. Summary

The non-deterministic finite state machine solves the decision problem HM2PAL RESTRICTED; the non-deterministic finite state transducer can additionally output countermelodies. Adjustments to this model result in the solution to the corresponding enumeration and number problem. A weighted finite state transducer can be used to output the optimal countermelody, and also the *m* best countermelodies.

These machine models also solve their respective problems efficiently. Table 3.1 shows the results of time complexity analysis presented in this chapter.

Table 3.1. Complexity of various properties of and operations with finite state machines

	Property, activity	Complexity
1.	Testing membership in a non-deterministic finite state automaton	$O(\|Q\|^2\|w\|)$
2.	Number of states of an NDFSM with fixed-length CF, $\|CF\|-1$ history	$O(\|S\|^{\|CF\|})$
3.	Number of states of an NDFSM with fixed-length history $k < \|CF\|$	$O(\|S\|^{k+1})$
4.	Deriving a valid countermelody with a finite state transducer	$O(\|CF\|\|S\|^4)$
5.	Deriving *m* best countermelodies with a finite state transducer	$O(\|CF\|\|S\|^4 m(\log m + \log \|S\|))$

Finite state machines can solve music composition problems that require the retention of a history of no more than a fixed number of notes. When the cantus firmus length is variable, this model is inadequate. When the computational complexity of rule validation is non-constant, the music composition problem becomes intractable, as we show in the next chapter.

CHAPTER 4. INTRACTABLE PROBLEMS IN SPECIES COUNTERPOINT

The species counterpoint problem in general is computationally extremely complex. We have seen that the addition of rules that govern notes spanning entire lengths of countermelodies causes the problem to be unsolvable by finite state machines. Indeed, it may prevent the problem from being solvable efficiently at all. In this chapter, we shall identify and define several variations of these problems. The form of these definitions enables us to prove their membership in specific complexity categories by linking them to known problems with similar complexity properties.

The most extensively researched of the complexity categories is that known as "NP-complete." We begin by discussing intractable music composition problems that belong to this category. We then define intractable music problems that belong to related complexity categories. These are the enumeration, number, and optimization problems, which we shall show to belong to the respective #P-complete, "strong sense" NP-complete, and NP-equivalent complexity classes.

To prove the membership of these problems to the respective intractability classes, we use reduction techniques, some first described by Garey and Johnson (1979). We shall then be in a position to outline a hierarchy of proofs of NP-completeness and other complexity properties with respect to reductions from previously known problems with similar complexity properties.

4.1. NP-Complete Music Problems

NP-complete problems are the most well-known among the intractable problems. They are called *NP* (for "non-deterministic polynomial") because for each of them, if a solution were proposed for a particular instance, that solution could be verified in a length of time of no more than a polynomial factor of the input length of the problem itself. They are called *NP-complete* because they are in some sense the most complex of these non-deterministic polynomial problems: if the NP-complete problems were somehow shown to be solvable in polynomial time, then all problems in NP are also solvable in polynomial time.

Many problems in algorithmic music composition can be shown to be NP-complete, and in this chapter we examine several such problems. We begin by defining the most general problem, which we call PAL, then we derive two specialized versions, called HM3PAL and HM2PAL respectively. In chapter 3, section 3.1, we defined a restricted form of HM2PAL for which we presented a polynomial solution based on the finite state machine model.

We cast the three problems named above (PAL, HM3PAL, and HM2PAL) as *decision problems*, viz., those problems with output of either "yes" or "no." This allows us to incorporate the problems into the context of NP-completeness theory, which is limited to decision problems. The general problem that we call "PAL" is one that, given a gamut of notes, a cantus firmus constructed from notes of this gamut, the number of parts to be added, and a set of rules providing the necessary and sufficient conditions for a solution consisting of the previously specified number of added parts, asks the question whether it is possible to create the specified number of added parts that satisfy the given rules.

The more restricted HM3PAL problem specializes the general problem to two added parts, and to two musical rules only: one pertaining to a harmonic feature and the other to a melody feature. The other specialized HM2PAL problem also specifies two rules, as for HM3PAL, but restricts the problem to a single added part. Both these problems are prefixed with "HM-" to indicate that they are based on two rule types: harmony and melody. We define each of these problems in the following section (4.1.1).

4.1.1. Formal Definitions of PAL-Related Decision Problems

The definitions below (PAL, HM3PAL, HM2PAL, etc.) use the format as used in chapter 3, section 3.1. Since these are decision problems, the question in each case requires a yes/no answer.

Problem 4.1. PAL
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
a sequence $CF \in S^c$;
a function R: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, whose computation is bounded by a polynomial in n^*p.
QUESTION:
Is there a p-tuple of sequences $CP \in (S^c)^p$ such that $R\ (CF, CP) = 1$?

Problem 4.2. HM3PAL
INSTANCE:
A finite set S;
positive integers $c, d, c > 1, 1 \leq d < c$; ($d$ is used to indicate the desired extent of melodic measurement between notes in the same part [see definition of R])
a sequence $CF \in S^c$;
a function H: $S \times S \times S \rightarrow \{0, 1\}$; (signifying "allowable" harmonic intervals);
a function M: $S \times S \rightarrow \{0, 1\}$; (signifying "allowable" melodic intervals);
a function R: $S^n \times (S^n)^2 \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
$R\ (<cf_1, cf_2, \ldots, cf_c>, <<m_{11}, m_{12}, \ldots, m_{1c}>, <m_{21}, m_{22}, \ldots, m_{2c}>>)$
$= 1$

if $H\ (cf_i, m1_i, m2_i) = 1,\ 1 \le i \le c$
and
$M\ (m1_i, m1_j) = 1,\ 1 \le i < j \le c,\ j - i \le d$
and
$M\ (m2_i, m2_j) = 1,\ 1 \le i < j \le c,\ j - i \le d$
$= 0$ otherwise.

QUESTION:
Is there a $CP \in (S^c)^2$ such that $R\ (CF, CP) = 1$?

Problem 4.3. HM2PAL
INSTANCE:
A finite set S;
positive integers $c, d, c > 1, 1 \le d < c$; (d is used to indicate the desired extent of melodic measurement between notes in the same part [see definition of R])
a sequence $CF \in S^c$;
a function H: $S \times S \rightarrow \{0, 1\}$; (signifying "allowable" harmonic intervals);
a function M: $S \times S \rightarrow \{0, 1\}$; (signifying "allowable" melodic intervals);
a function R: $S^n \times S^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$
where R is defined as follows:
$R\ (<cf_1, cf_2, \ldots, cf_c>, <m_1, m_2, \ldots, m_c>)$
$= 1$
if $H\ (cf_i, m_i) = 1,\ 1 \le i \le c$
and
$M\ (m_i, m_j) = 1,\ 1 \le i < j \le c,\ j - i \le d$
$= 0$ otherwise.

QUESTION:
Is there a $CP \in S^c$ such that $R\ (CF, CP) = 1$?

An even more formal definition of these problems uses the concept of languages, and the notion of acceptance or rejection by Turing machines. Thus, L_{PAL} defines the language corresponding to PAL.

Problem 4.4. L_{PAL}
$= \{<S, CF, CP, R>:$
S is a finite set,
CF is a sequence $<cf_1, cf_2, \ldots, cf_c>$ over S,
R: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n, p \in \mathbb{N}$,
is a function whose computation is bounded by a polynomial in n^*p,
and
there exists a CP′ for which R (CF, CP′) = 1}.

A Turing machine determines membership of a proposed instance; an NP problem is one in which a succinct certificate exists with which a non-deterministic Turing machine can verify, in a number of steps no greater than a polynomial factor of the input which encodes that instance, whether that input belongs in the language. For ease of explanation, we adopt the less formal definitions presented here.

Example 4.1. An instance of PAL is the following:

S = {FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
$c = 10$
$p = 1$
CF = <d, c, h, c, h, a, h, a, F#, G>
R is effected by the melody, harmony, and counterpoint rules (see appendix A, section A.1).

The answer to this problem is "yes." ■

PAL and HM3PAL are NP-complete. We could prove this in the following order: first show that HM3PAL is NP-complete by reduction from a known NP-complete problem called "Three-Dimensional Matching" (abbreviated to "3DM," proven NP-complete by Karp [1972]). Then we may use this result to show that PAL is NP-complete. We may also use this result to show that HM2PAL is NP-complete. This latter result is rather unusual, because of the better-known NP-complete problems—3SAT, 3DM, and 3-MATROID INTERSECTION—their two-dimensional counterparts, 2SAT, 2DM, and 2-MATROID INTERSECTION, are all tractable.

The above decision problems used crisp rules. It is straightforward to create their fuzzy rule counterparts. For example, FUZZY PAL DECISION is the fuzzy rule equivalent of PAL.

Problem 4.5. FUZZY PAL DECISION
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
$\alpha \in \mathbb{N}_0$;
a sequence $CF \in S^c$;
a set of functions ("crisp rules") $R = \{ru_1, ru_2, \ldots, ru_r\}$, ru_i: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, $1 \leq i \leq r$,
a set of functions ("fuzzy rules") $A = \{a_1, a_2, \ldots, a_q\}$, a_i: $S^n \times (S^n)^p \rightarrow \mathbb{N}_{0,n^*p}$, $n \in \mathbb{N}$, $1 \leq i \leq q$
where the computation of each element of R and A is bounded polynomially in n^*p.
QUESTION:
Is there a $CP \in (S^c)^p$ such that
$ru_i\,(CF, CP) = 1$, $1 \leq i \leq r$
and
$\sum_{i=1}^{q} a_i\,(CF, CP) \geq \alpha$?

Example 4.2. The fuzzy version of the instance of PAL in example 4.1 is:

S, c, p, CF, R as for example 4.1, A is effected by the four artistry rules (see appendix A, section A.1.4), $\alpha = 34$.

The answer to this instance is "yes," but "no" if $\alpha = 36$. ■

Naturally, several variants of these decision problems come to mind: they include one that asks about possible solutions that are similar to a given solution, correct or otherwise (PAL MIN CHANGES), and another that asks about complete solutions given a partial one (FUZZY PAL EXTENSION DECISION). The PAL MIN CHANGES problem relates to problems of melodic similarity and clustering; the FUZZY PAL EXTENSION DECISION problem relates to a practical application to favor an expert's choice among alternatives (see the "Jeppesen Experiment", chapter 5, section 5.1).

Problem 4.6. PAL MIN CHANGES
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
a sequence $CF \in S^c$;
a p-tuple of sequences $CP \in (S^c)^p$;
a function R: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, whose computation is bounded by a polynomial in n^*p;
a function $Dist$: $(S^n)^p \times (S^n)^p \rightarrow \mathbb{N}_0$, $n \in \mathbb{N}$, whose computation is bounded polynomially in n^*p;
$k \in \mathbb{N}_0$.
QUESTION:
Is there a p-tuple of sequences $CP' \in (S^c)^p$
such that
$R\ (CF, CP') = 1$
and
Dist (CP, CP') ≤ k?

Problem 4.7. FUZZY PAL EXTENSION DECISION
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
$\alpha \in \mathbb{N}_0$;
a sequence $CF \in S^c$;
a set of functions ("crisp rules") $R = \{ru_1, ru_2, \ldots, ru_r\}$, ru_i: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, $1 \leq i \leq r$,
a set of functions ("fuzzy rules") $A = \{a_1, a_2, \ldots, a_q\}$, a_i: $S^n \times (S^n)^p \rightarrow \mathbb{N}_{0,n^*p}$, $1 \leq i \leq q$
where the computation of each element of R and A is bounded polynomially in n^*p;
a p-tuple of sequences $PM = <<mel_{11}, mel_{12}, \ldots, mel_{1c}>, <mel_{21}, mel_{22}, \ldots, mel_{2c}>, \ldots, <mel_{p1}, mel_{p2}, \ldots, mel_{pc}>>$, where each sequence in PM is over $S \cup \{\lambda\}$.
QUESTION:
Is there a p-tuple of sequences $CP' = <<mel'_{11}, mel'_{12}, \ldots, mel'_{1c}>, <mel'_{21}, mel'_{22}, \ldots, mel'_{2c}>, \ldots, mel'_{p1}, mel'_{p2}, \ldots, mel'_{pc}>$, where each sequence in CP' is over S and $mel_{xy} \neq \lambda \rightarrow mel_{xy} = mel'_{xy}$, $1 \leq x \leq p$, $1 \leq y \leq c$,
such that
$ru_i\ (CF, CP') = 1$, $1 \leq i \leq r$

and
$\sum_{i=1}^{q} a_i(CF, CP') \geq \alpha$?

We now define the following two problems (GEN PAL and GEN FUZZY PAL DECISION), which are the most general forms of PAL and PAL FUZZY. They enable us to describe the problems in all the other species, which require more than one note in the countermelody for each note of the cantus firmus. Since these other species do not result in more complex problems, we do not deal with these two problems further.

Problem 4.8. GEN PAL
INSTANCE:
A finite set S;
$c, p, sp \in \mathbb{N}$;
a sequence $CF \in S^c$;
a function $R: S^n \times ((S^{sp})^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$,
whose computation is bounded by a polynomial in sp^*n^*p.
QUESTION:
Is there a $CP \in ((S^{sp})^n)^p$ such that $R(CF, CP) = 1$?

Problem 4.9. GEN FUZZY PAL DECISION
INSTANCE:
A finite set S;
$c, p, sp \in \mathbb{N}$;
$\alpha \in \mathbb{N}_0$;
a sequence $CF \in S^c$;
a set of functions ("crisp rules") $R = \{ru_1, ru_2, \ldots, ru_r\}$, $ru_i: S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, $1 \leq i \leq r$,
a set of functions ("fuzzy rules") $A = \{a_1, a_2, \ldots, a_q\}$, $a_i: S^n \times ((S^{sp})^n)^p \rightarrow \mathbb{N}_0$, sp^*n^*p,
$n \in \mathbb{N}$, $1 \leq i \leq q$,
where the computation of each element of R and A is bounded polynomially in sp^*n^*p.
QUESTION:
Is there a $CP \in ((S^{sp})^c)^p$ such that
$ru_i(CF, CP) = 1$, $1 \leq i \leq r$
and
$\sum_{i=1}^{q} a_i(CF, CP) \geq \alpha$?

4.1.2. NP-Completeness Proofs

3DM extends the classical *marriage problem* (actually an informal specification of 2DM): Given n single men and women, and a list of agreed pairings of one man with one woman, the marriage problem asks whether it is possible to arrange n monogamous marriages that satisfy every man and woman in

the problem (see example 4.3 for instances of the marriage problem). With 3DM we may imagine three different sexes (of n elements each) and a list of three-way matches. We then ask the question whether it is possible to arrange n groupings that are "monogamous" and mutually acceptable. The marriage problem can be solved in polynomial time, and is therefore tractable. 3DM, on the other hand, is NP-complete.

Example 4.3. Two instances of the marriage problem are as follows:

(a)
men = {a, b, c, d}
women = {e, f, g, h}
pairings = {<a, g>, <b, e>, <c, f>, <d, e>}

(b)
men = {a, b, c, d}
women = {e, f, g, h}
pairings = {<a, g>, <b, e>, <b, h>, <c, f>, <d, e>}

The answer is "no" to the first instance, "yes" to the second (matching = {<a, g>, <b, h>, <c, f>, <d, e>}). ■

Garey and Johnson (1978, 46) define 3DM as follows:

Problem 4.10. 3DM
INSTANCE:
A set $Ma \supseteq W \times X \times Y$, where W, X, and Y are disjoint sets, each having q elements.
QUESTION:
Does Ma contain a matching, that is, a subset $Ma' \subseteq Ma$ such that $|Ma'| = q$ and no two elements of Ma' are repeated in any coordinate?

To prove that HM3PAL is NP-complete, we first argue that it is an NP problem, i.e., it is solvable by a non-deterministic polynomial algorithm. This is relatively easy to do, as is the case of many NP-completeness proofs. We then show that 3DM polynomially reduces to HM3PAL, i.e., that for any instance of 3DM, we can convert it to an instance of HM3PAL in polynomial time such that a "yes" answer for that instance of 3DM results in a "yes" answer for the corresponding HM3PAL instance, and such that a "no" answer results in a "no" answer for the corresponding HM3PAL instance.

Theorem 4.1. HM3PAL is NP-complete.

Proof: HM3PAL $\in$ NP since a non-deterministic algorithm need only guess at a CP and evaluate $R(CF, CP)$. The entire process can be accomplished in time polynomially bounded in c.

To show that HM3PAL is NP-hard, we transform an instance of 3DM to one of HM3PAL .

Let W, X, and Y be disjoint sets having the same number q of elements, and $Ma \subseteq W \times X \times Y$ be an instance of 3DM. We shall construct an instance of HM3PAL: a finite set S, a sequence $CF = <cf_1, cf_2, \ldots, cf_c>$ on S, and functions H and M

such that

there is a sequence $CP = \langle mel_{11}, mel_{12}, \ldots, mel_{1c}, mel_{21}, mel_{22}, \ldots, mel_{2c} \rangle$ on S

such that $R(CF, CP) = 1$ if and only if Ma contains a matching $Ma' \subseteq Ma$.

Let i = $W \cup X \cup Y$.

Let $c = q$.

Let $d = q - 1$. (together with M's definition, ensures no notes are repeated in each melody)

Let $CF = \langle cf_1, cf_2, \ldots, cf_c \rangle$, $cf_i \in W$ and $cf_i \neq cf_j$ whenever $i \neq j$, $1 \leq i \leq c$, $1 \leq j \leq c$.

I.e., the elements of CF are a permutation of all the elements of W.

Define H as follows:

$H(z_1, z_2, z_3)$ =1 if $\langle z_1, z_2, z_3 \rangle \in Ma$

= 0 otherwise.

Define M as follows:

$M(z_1, z_2)$ = 1 if $z_1 \neq z_2$

= 0 otherwise.

It can be verified that this is a polynomial transformation, and that there is a CP such that $R(CF,$ i) = 1 if and only if there is a matching $Ma' \subseteq Ma$. ■

The following example (4.4) illustrates a transformation from 3DM to HM3PAL.

Example 4.4. Consider the following two instances of 3DM:

(a)
W = {a, b, c, d}
X = {e, f, g, h}
Y = {i, j, k, l}
Ma = {<a, g, j>, <b, e, i>, <b, h, j>, <b, h, k>, <c, f, l>, <d, e, k>}

(b)
W = {a, b, c, d}
X = {e, f, g, h}
Y = {i, j, k, l}
Ma = {<a, g, j>, <b, e, i>, <b, h, i>, <b, h, k>, <c, f, l>, <d, e, k>}

The answers to the two instances are "no" and "yes," respectively. The following is a matching for the second instance:

Ma' = {<a, g, j>, <b, h, i>, <c, f, l>, <d, e, k>}

Transforming the second instance gives the following instance of HM3PAL:

S = {A, B, C#, D, E, F, G, a, b, c#, d, e}
c = 4
d = 3
H = {<c#, E, A>, <c#, a, A>, <b, G, B>, <e, E, C#>, <c#, a, C#>, <d, F, D>}
M = {<z_1, z_2>: $z_1 \neq z_2$}
CF = <A, B, C#, D> ■

Once we have shown that HM3PAL is NP-complete, we can readily prove that the more general problem PAL is NP-complete. We do so by reducing from the now "known" NP-complete problem HM3PAL since the latter is a special case of the former.

Theorem 4.2. PAL is NP-complete.

Proof: Clearly, PAL is in NP. Restrict to HM3PAL by allowing only instances in which $c > 1$, $p = 2$, and R combines the functions H and M as specified. ■

We next prove that HM2PAL is NP-complete, also by reducing from HM3PAL.

Theorem 4.3. HM2PAL is NP-complete.

Proof: HM2PAL $\in$ NP since a non-deterministic algorithm need only guess at a CP and evaluate $R(CF, CP)$ for $1 \leq i \leq c$. The entire process can be accomplished in time polynomially bounded in c.
To show that HM2PAL is NP-hard, we transform HM3PAL to HM2PAL as follows:
Let (S = S, c = c, d = d, CF = CF, H = H, M = M) be an instance of HM3PAL. We shall construct an instance (S′, c′, d′, CF′, H′, M′) of HM2PAL
such that
there is a sequence CP′ = <mel_1, mel_2, … , mel_c> on S′ such that R_{HM2PAL} (CF, CP) = 1 if and only if there is a sequence CP = <mel_{11}, mel_{12}, … , mel_{1c}, mel_{21}, mel_{22}, … , mel_{2c}> on S such that R_{HM3PAL} (CF, CP) = 1, where R_{HM2PAL} and R_{HM3PAL} refer to the R defined in HM2PAL and HM3PAL respectively.
Let S′ = {a, b, ab, ba: a, b $\in$ S}.
Let c′ = c.
Let d′ = d.
Let CF′ = CF.
Let H′ = {<a, bc, x>: <a, b, c, x> $\in$ H}.
Let M′ = {<ac, bd, z>: <a, b, x>, <c, d, y> $\in$ M, z = 1 if x = y = 1, z = 0 otherwise}.
This is clearly a polynomial transformation of complexity O ($|H| + |S|^2 + |M|^2$). ■

Example 4.5. Consider the following instance of HM3PAL :

S = {a, b, c, d}
d = 1
H = {<a, b, c, 1>, <b, c, d, 1>, <_,_,_, 0>}
M = {<a, b, 1>, <b, c, 1>, <_, _, 0>}
CF = <a, b, c>.

A transformation gives the following instance of HM2PAL:

S′ = {a, b, c, d, aa, bb, cc, dd, ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc}
d = 1
H′ = {<a, bc, 1>, <b, cd, 1>, <–,–, 0>}
M′ = {<ab, bc, 1>, <ba, cb, 1>, <aa, bb, 1>, <bb, cc, 1>, <_,_, 0>}
CF′ = <a, b, c>.

An example of a counterpoint in the above instance of HM3PAL is:

CM2: c a b
CM1: b c a
CF: a b c

whose equivalent in HM2PAL is:

CM′: bc ca ab
CF′: a b c

Both these counterpoints will not be accepted in both problems. In fact, for the above instances of HM3PAL and HM2PAL, no valid counterpoint is possible. Thus, the answer to the above instances of HM3PAL and HM2PAL is "no". ■

Example 4.6. Consider, however, the following instance of HM3PAL:

S = {a, b, c, d}
d = 1
H = {<a, b, c, 1>, <b, c, d, 1>, <_,_,_, 0>}
M = {<a, b, 1>, <b, c, 1>, <c, d, 1>, <_, _, 0>}
CF = <a, b>.

whose transformation into HM2PAL yields:

S′ = {a, b, c, d, aa, bb, cc, dd, ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc}
d = 1
H′ = {<a, bc, 1>, <b, cd, 1>, <–,–, 0>}
M′ = {<ab, bc, 1>, <ba, cb, 1>, <aa, bb, 1>, <bb, cc, 1>,
<ac, bd, 1>, <ca, db, 1>, <bc, cd, 1>, <cb, dc, 1>,
CF′ = <a, b>.

The answer to this instance is "yes". There is one solution (for each of these instances of HM3PAL and HM2PAL):

HM3PAL:
CM2: c d
CM1: b c
CF: a b

HM2PAL:
CM′: bc cd
CF′: a b ■

We next prove that FUZZY PAL DECISION and PAL MIN CHANGES are NP-complete. To show that the problems are in NP is trivial. We describe only the reductions necessary to complete the proof.

Theorem 4.4. FUZZY PAL DECISION is NP-complete.

Proof: Restrict to PAL by allowing only instances having $\alpha = 0$ and $A = \varnothing$. ■

Theorem 4.5. PAL MIN CHANGES is NP-complete.

Proof: Restrict to PAL by allowing only instances having $Dist\ (CP, CP') \in \{0\}$ and $k = 0$, with arbitrary CP. I.e., we ignore the distance component of the problem. ■

We next show that FUZZY PAL EXTENSION DECISION is an NP problem. The reason we included this problem is to provide a lemma to prove the theorem that the PAL-related optimization problem—we shall call FUZZY PAL OPTIMIZATION—is NP-equivalent (theorem 4.11, section 4.2.3).

Lemma 4.1. FUZZY PAL EXTENSION DECISION is in NP.

Proof: A non-deterministic algorithm need only guess at a CP to complete PM, check that it extends PM in linear time, and verify it by applying the polynomially bounded rules. The entire process can be accomplished in time polynomially bounded in $c^*p^*r^*q$. ■

Note that FUZZY PAL DECISION is really a special case of FUZZY PAL EXTENSION DECISION, with PM as the empty sequence; thus, FUZZY PAL EXTENSION DECISION can be easily proven to be NP-complete as well. But its membership in NP is sufficient to prove NP-equivalence of FUZZY PAL OPTIMIZATION (section 4.2.3).

4.1.3. An Alternative Proof of the NP-Completeness of PAL

Another NP-complete problem that can be polynomially transformed to PAL is the so-called BOUNDED TILING problem. In this problem, we are asked whether it is possible to cover an s by s square with various unit-length tiles given the tiling of the first row, and restrictions on which tiles may be placed adjacent to each other both horizontally and vertically. It is defined as follows (problem 4.11).

Problem 4.11. BOUNDED TILING
INSTANCE:
A tiling system $\mathbb{D} = <D, H, V>$ where
D is a finite set of "tiles";
H: $D \times D \rightarrow \{0, 1\}$;
V: $D \times D \rightarrow \{0, 1\}$;
$s > 0$;
a function f_0: $\{0, \ldots, s-1\} \rightarrow D$.
QUESTION:
Is there a function f: $\{0, 1, \ldots, s-1\} \times \{0, 1, \ldots, s-1\} \rightarrow D$ such that
$f(m, 0) = f_0(m)$, $0 \leq m < s$;
$<f(m, n), f(m+1, n)> \in H$, $0 \leq m < s-1$, $0 \leq n < s$;
$<f(m, n), f(m, n+1)> \in V$, $0 \leq m < s$, $0 \leq n < s-1$?

It is easy to see that D corresponds to S; H and V correspond to two rules in R (one governing melodic progressions and the other governing harmonies); $s = |CF| = p + 1$; and f_0 corresponds to CF.

Example 4.7. Consider the following instance of BOUNDED TILING:

D = {C#, D, E, F, G, a}
H = {<C#, C#, 1>, <C#, D, 1>, <C#, E, 0>, <C#, F, 0>, <C#, G, 0>, <C#, a, 0>,
<D, C#, 1>, <D, D, 1>, <D, E, 1>, <D, F, 0>, <D, G, 1>, <D, a, 0>,
<E, C#, 0>, <E, D, 1>, <E, E, 1>, <E, F, 1>, <E, G, 0>, <E, a, 1>,
<F, C#, 0>, <F, D, 0>, <F, E, 1>, <F, F, 1>, <F, G, 1>, <F, a, 0>,
<G, C#, 0>, <G, D, 0>, <G, E, 0>, <G, F, 1>, <G, G, 1>, <G, a, 1>,
<a, C#, 0>, <a, D, 0>, <a, E, 0>, <a, F, 0>, <a, G, 1>, <a, a, 1>}
V = {<C#, C#, 0>, <C#, D, 0>, <C#, E, 1>, <C#, F, 0>, <C#, G, 0>, <C#, a, 0>,
<D, C#, 0>, <D, D, 0>, <D, E, 0>, <D, F, 1>, <D, G, 0>, <D, a, 0>,
<E, C#, 0>, <E, D, 0>, <E, E, 0>, <E, F, 0>, <E, G, 1>, <E, a, 1>,
<F, C#, 0>, <F, D, 0>, <F, E, 0>, <F, F, 0>, <F, G, 0>, <F, a, 1>,
<G, C#, 0>, <G, D, 1>, <G, E, 0>, <G, F, 0>, <G, G, 0>, <G, a, 0>,
<a, C#, 0>, <a, D, 0>, <a, E, 1>, <a, F, 0>, <a, G, 0>, <a, a, 0>}
$s = 3$
f_0 = <D, C#, D>.

Transforming this gives the following instance of PAL:

S = {C#, D, E, F, G, a}
$c = 3$
$p = 2$
CF = <D, C#, D>
CP = <CM_1, CM_2>
R (CF, CP) is such that only melodic movement of stepwise or unison motion is allowed, and only harmonies of consonant thirds and fourths are allowed.

The function R is described informally using musical terminology, but corresponds exactly to the combined effect of functions H and V of the BOUNDED TILING instance.

The answer to the BOUNDED TILING instance is "yes": the following two tilings are valid:

(a)			(b)		
a	a	a	a	G	a
F	E	F	F	E	F
D	C#	D	D	C#	D

These are also exactly the valid counterpoints for the corresponding PAL instance. ■

In BOUNDED TILING, the size of the quadrant we are asked to tile is bounded by s; in the unbounded version of BOUNDED TILING, we are asked to tile an infinite quadrant. It is interesting to note that the unbounded version of this problem is undecidable.

The reader might also wish to consider the similarity of PAL with the following NP-complete problem (FINITE STATE AUTOMATA INTERSECTION; Garey and Johnson 1979, 266; Kozen 1977), which also involves the acceptance of strings by an external function, in this case a deterministic finite state machine:

Problem 4.12. FINITE STATE AUTOMATA INTERSECTION
INSTANCE:
A sequence $A_1, A_2, \ldots, A_n$ of deterministic finite state machines, all with the input alphabet Σ.
QUESTION:
Is there a string $x \in \Sigma^*$ accepted by A_i, $1 \leq i \leq n$?

In PAL, the function $R(CF, CP)$ could be viewed as a deterministic Turing machine, and $R(CF, CP) = 1$ means that the machine accepts CP (with respect to CF) and $R(CF, CP) = 0$ means that the machine rejects CF.

4.1.4. An Intractable Rule Redundancy Decision Problem

We return briefly to problems we introduced in chapter 2, section 2.1.1, viz., that of determining redundancy and indispensability in rule sets. Here we discuss a more demanding version of these two problems and prove that it is NP-complete. Recall that the simpler problems involved determining the existence of a proper subset of the rules that did not duplicate the selectivity of any other rule in the subset, and determining the existence of a set of rules that were uniquely responsible for the elimination of some candidate countermelodies. We found there that the practical methods used did not always yield optimal results: a great deal depended on the acceptance sets themselves. This would lead us to suspect that the general problem might be intractable. We now define the generalized decision problem (PAL RULE SUBSET) as that of determining the existence of a subset of specified size that succeeds in eliminating the same countermelodies as would all the rules combined. First we define a related NP-complete problem, VERTEX COVER (Garey and Johnson 1978, 46).

Problem 4.13. VERTEX COVER
INSTANCE:
A graph $G = (V, E)$ and a positive integer $K \leq |V|$.
QUESTION:
Is there a vertex cover of size K or less for G, i.e., a subset $V' \subseteq V$ such that $|V'| \leq K$ and, for each edge $\{u, v\} \in E$, at least one of u and v belongs to V'?

Our definition of PAL RULE SUBSET follows (problem 4.14).

Problem 4.14. PAL RULE SUBSET
INSTANCE:
A cantus firmus set $CFSet = \{cf_1, cf_2, \ldots, cf_m\}$;
a rule set $RSet = \{r_1, r_2, \ldots, r_n\}$;
a collection of "Acceptance sets" $Accept_{r,c}$, $1 \leq r \leq n$, $1 \leq c \leq m$, which is the set of countermelodies for $cf_c \in CFSet$ that rule r_r accepts;
a positive integer $K \leq |RSet|$.
QUESTION:
Is there a subset $RSet' \subseteq RSet$, $|RSet'| \leq K$

such that

$$\bigcap_{1 \le r \le |RSet'|} Accept_{r,\,c} = \bigcap_{1 \le s \le |RSet|} Accept_{s,\,c} \quad , 1 \le c \le m \,?$$

Theorem 4.6. PAL RULE SUBSET is NP-complete.

Proof: PAL RULE SUBSET is in NP, because given *RSet'*, checking the condition in the question can be accomplished in polynomial time. To prove PAL RULE SUBSET is NP-hard, restrict to VERTEX COVER. Note that for each $Accept_{r,\,c}$ we have $Reject_{r,\,c}$ = complement of $Accept_{r,\,c}$, where $Reject_{r,\,c}$ is the set of countermelodies rejected by rule r_r. Then, the intersection of all the acceptance sets = complement of the union of all the rejection sets (this follows from the definitions of acceptance and rejection sets made in chapter 2, section 2.1.1). Given an instance of VERTEX COVER, we create an instance of PAL RULE SUBSET as follows: let $m = 1$, let $RSet = V$, $Reject_{r,\,1}$ = set of all edges incident on the corresponding vertex; the integer K is identical for both problems. ■

Example 4.8. Consider the following instance of VERTEX COVER (fig. 4.1):

$G = (V, E)$; $V = \{a, b, c, d\}$; $E = \{ \{a, c\}, \{a, d\}, \{b, c\} \}$.

The transformation gives us the following instance of PAL RULE SUBSET:

$m = 1$; $RSet = \{r_1, r_2, r_3, r_4\}$;
$Reject_{1,1} = \{ ac, ad \}$;
$Reject_{2,1} = \{ bc \}$;
$Reject_{3,1} = \{ ac, bc \}$;
$Reject_{4,1} = \{ ad \}$.

The answer for both problems is "yes" for $K \ge 2$, "no" otherwise. There are three minimal vertex covers: {a, b}, {a, c}, are {c, d}; the three minimal rule sets are, of course, $\{r_1, r_2\}$, $\{r_1, r_3\}$, and $\{r_3, r_4\}$ (fig. 4.1).

Observe that edges in G incident upon a particular vertex correspond to countermelodies rejected by the corresponding rule, while edges in G not incident upon a vertex correspond to countermelodies accepted by the corresponding rule, but rejected by some other rule/s. Also, all possible edges in G not actually in G correspond to countermelodies accepted by all the rules, or equivalently, the intersection of all the acceptance sets (fig. 4.2). ■

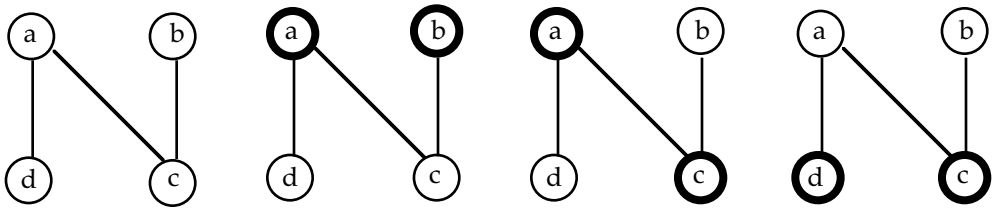
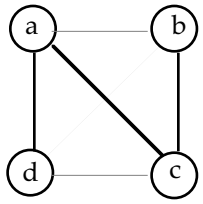


Fig. 4.1. Three distinct vertex covers ({a, b}, {a, c}, and {c, d}) for a graph G (represented in 4.1a); (b)–(d) illustrate the effect of the three vertex covers.



Rejection sets	Acceptance sets
$Reject_{1,1}$ = {ac, ad}	$Accept_{1,1}$ = {ab, bc, 2–4, cd}
$Reject_{2,1}$ = {bc}	$Accept_{2,1}$ = {ab, ac, ad, bd, cd}
$Reject_{3,1}$ = {ac, bc}	$Accept_{3,1}$ = {ab, ad, bd, cd}
$Reject_{4,1}$ = {ad}	$Accept_{4,1}$ = {ab, ac, bc, bd, cd}

Fig. 4.2. Correspondences between edges in (and not in) G and rejection and acceptance sets. The edges in graph G correspond to the countermelodies rejected by the musical rules; edges indicated by dotted lines (remaining edges to make a complete graph) correspond to the countermelodies accepted by all the musical rules. Unbroken lines incident upon a vertex correspond to countermelodies rejected by the corresponding rule; unbroken lines not incident upon a vertex correspond to countermelodies accepted by the corresponding rule, but rejected by some other rule/s.

Note the similarity between PAL RULE SUBSET and MINIMUM TEST SET (Garey and Johnson 1978, 222), which is also NP-complete.

Problem 4.15. MINIMUM TEST SET
INSTANCE:
Collection C of subsets of a finite set S, positive integer $K \leq |C|$.
QUESTION:
Is there a subcollection $C' \subseteq C$ with $|C'| \leq K$ such that
$\forall\, u, v \in S, u \neq v, \exists\, c \in C'$ that contains exactly one of u and v?

The optimization problem corresponding to the decision problem PAL RULE SUBSET is NP-equivalent (see section 4.2.3).

4.2. Other Intractable Music Problems

We discuss music problems related to the above NP-complete problems, some of which which belong to even more complex computational classes. They are the enumeration, number, and optimization problems, and we show that these problems are "#P-complete," "NP-complete in the strong sense," and "NP-equivalent." We describe only problems related to the general PAL and FUZZY PAL DECISION problems.

4.2.1. Enumeration Music Problems

Enumeration problems are search problems that ask how many solutions there are in a given instance. In these problems, we do not have to list all the solutions: all we have to do is answer "how many." Therefore, they may not necessarily be intractable even though the number of solutions could be exponential. It is easy to see, however, that enumeration problems related to NP-complete problems are at least NP-hard (if the number of solutions is non-zero, then the answer to the decision problem is

"yes", otherwise "no"). They may be intractable even if P = NP, so they belong to an interesting complexity category. Valiant (1979, 1979b) introduced the complexity class #*P*, which contains enumeration problems for which a non-deterministic polynomial algorithm exists that can guess exactly the number of distinct solutions. The most difficult of the #P problems are the *#P-complete problems*: they are those to which all other #P problems are Turing-reducible. To prove that a problem is #P-complete, we may perform a special kind of Turing reduction known as *parsimonious transformation* (first used by Simon [1975]): one in which the number of solutions of the transformed instance is the same as that the original instance.

The enumeration music composition problems related to the PAL and FUZZY PAL DECISION are PAL ENUM and FUZZY PAL ENUM. In the enumeration problems, we would like to know how many solutions there are for a given cantus firmus. Some cantus firmi are more fruitful than others, and if we could quickly determine their potential yield, we could adapt our search strategies accordingly; a teacher of species counterpoint would know better the variety of answers to expect from students.

Problem 4.16. PAL ENUM
INSTANCE:
As for PAL.
QUESTION:
How many sequences *CP* are there such that
$R\ (CF, i) = 1$?

Problem 4.17. FUZZY PAL ENUM
INSTANCE:
As for FUZZY PAL DECISION.
QUESTION:
How many sequences *CP* are there
such that
$ru_i\ (CF, CP) = 1,\ 1 \le i \le r$
and
$\sum_{i=1}^{q} a_i\ (CF, CP) \ge \alpha$?

Example 4.9. Consider the following instance of PAL ENUM (compare example 4.1):

S = {FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
$c = 10$
$p = 1$
CF = <d, c, h, c, h, a, h, a, F#, G>
R combines the effect of the melody, harmony, and counterpoint rules (see appendix A, section A.1).

The answer for this instance of PAL ENUM is "55" (for the output of these 55 countermelodies, see appendix E, section E.3.2). ■

Example 4.10. The corresponding instance of FUZZY PAL ENUM is:

S = {FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
$c = 10$
$p = 1$
CF = <d, c, h, c, h, a, h, a, F#, G>
R combines the effect of the melody, harmony, and counterpoint rules (see appendix A, section A.1)
$\alpha = 34$.

The answer for this instance is "17". ■

Theorem 4.7. PAL ENUM is #P-complete.

To prove this we need to define the enumeration problem associated with 3DM (let us call it 3DM ENUM).

Problem 4.18. 3DM ENUM
INSTANCE:
As for 3DM.
QUESTION:
How many matchings does M contain?

Example 4.11. An instance of 3DM ENUM that corresponds to the second instance of 3DM in example 4.4 would have the answer "1". ■

Proof (theorem 4.7): 3DM ENUM is #P-complete (Garey and Johnson 1979, Moret 1997). The transformation from 3DM ENUM to the enumeration version of HM3PAL (call it HM3PAL ENUM) is parsimonious, since clearly only those matchings in 3DM ENUM correspond to acceptable solutions in HM3PAL ENUM, and vice versa. The NP-completeness proof of PAL was a "restriction" proof by reduction from HM3PAL; consequently the transformation from HM3PAL ENUM to PAL ENUM is (strictly) parsimonious. ■

Example 4.12. Using the instance of HM3PAL transformed above from the second instance of 3DM of example 4.4, the corresponding HM3PAL ENUM instance would have the answer "1" as well. ■

Theorem 4.8. FUZZY PAL ENUM is #P-complete.

Proof: Restrict to PAL ENUM by allowing only instances having $\alpha = 0$ and $A = \emptyset$. This is a strictly parsimonious transformation, as are all "restriction" proofs (Moret 1997). ■

4.2.2. Number Music Problems

A *number problem* is one in which the description of an instance contains one or more numbers whose magnitude cannot be bounded by any polynomial in the size of the instance, using what is commonly agreed as a reasonable encoding scheme of the instance. By "reasonable" we include almost all encoding schemes except the unary encoding scheme; the most common reasonable scheme is binary because of its ideal use in computers. A dynamic programming approach applied to these problems would still be

inefficient, all appearances to the contrary. For example, the well-known dynamic programming solution to the KNAPSACK problem has a time complexity of $\Theta(nW)$, where n is the number of items that can be stolen and W is the maximum capacity of the knapsack. Although it may appear that the dynamic programming solution has a polynomial complexity (and a very good linear one at that), unfortunately W is a number, or magnitude, and using a reasonable encoding scheme for it would render the complexity of the solution exponential, not polynomial. Such solutions that are polynomial with respect to the magnitude of some aspect of the input, but not necessarily to the size, are referred to as *pseudo-polynomial*.

The number music problems related to the problems we have already defined are *NP-complete in the strong sense* (terminology introduced by Garey and Johnson [1978]). These are decision problems that cannot be solved even by a pseudo-polynomial algorithm, unless P = NP. Note, however, that the definition of strong-sense NP-completeness implies that all NP-complete problems that are not number problems are also NP-complete in the strong sense. Thus, the NP-complete problems defined in section 4.1 are all also NP-complete in the strong sense.

The related number music problems are K PAL and K FUZZY PAL. As defined below, these number problems are similar in usefulness to the enumeration problems. Solutions to these problems enable us to predict the fruitfulness of various cantus firmi.

Problem 4.19. K PAL
INSTANCE:
A finite set S;
$c, p, K \in \mathbb{N}$;
a sequence $CF \in S^c$;
a function R: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, whose computation is bounded by a polynomial in n^*p.
QUESTION:
Are there at least K p-tuple of sequences $CP \in (S^c)^p$
such that
$R(CF, CP_i) = 1$, $1 \leq i \leq K$?

Problem 4.20. K FUZZY PAL
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
$K \in \mathbb{N}_0$;
$\alpha \in \mathbb{N}_0$;
a sequence $CF \in S^c$;

a set of functions ("crisp rules") $R = \{ru_1, ru_2, \ldots, ru_r\}$, ru_i: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, $1 \leq i \leq r$,

a set of functions ("fuzzy rules") $A = \{a_1, a_2, \ldots, a_q\}$, a_i: $S^n \times (S^n)^p \rightarrow \mathbb{N}_{0,n^*p}$, $n \in \mathbb{N}$, $1 \leq i \leq q$

where the computation of each element of R and A is bounded polynomially in n^*p.

QUESTION:

Are there at least K p-tuple of sequences $CP \in (S^c)^p$
such that
$ru_i\,(CF, CP_j) = 1$, $1 \leq i \leq r$, $1 \leq j \leq K$
and
$\sum_{i=1}^{q} a_i\,(CF, CP_j) \geq \alpha$, $1 \leq j \leq K$?

Example 4.13. The instance of K FUZZY PAL corresponding to the instance of FUZZY PAL DECISION of example 4.2 has the answer "yes" if $K = 17$, but "no" for $K > 17$. ■

In the following proofs, we again only show the reductions.

Lemma 4.2. K PAL is NP-complete.

Proof: Restrict to PAL by allowing only instances having $K = 1$. ■

Theorem 4.9. K PAL is NP-complete in the strong sense.

Proof: K PAL is a number problem, since $Max\,[I] = K$ and there exists no polynomial pol such that $K \leq pol\,(Length\,[I])$ for all $I \in D_{\text{K-PAL}}$, where $Max\,[I]$ is the magnitude of the largest number in problem instance I and $Length\,[I]$ is the number of symbols used to describe problem instance I using a reasonable encoding scheme (definition by Garey and Johnson [1979, 92]). By Lemma 4.1, K PAL is NP-complete, proved by using a polynomial transformation from PAL by restricting K PAL to the subproblem with $K = 1$. With this restriction, $Max\,[I] \leq Length\,[I]$, and the subproblem of K PAL made up of all those instances satisfying this inequality is itself NP-complete. It follows that K PAL is NP-complete in the strong sense. ■

Theorem 4.10. K FUZZY PAL is NP-complete in the strong sense.

Proof: Restrict to K PAL by allowing only instances having $\alpha = 0$ and $A = \varnothing$. ■

We show below a partial reduction tree (fig. 4.3), which includes the polynomial transformations we used in our proofs of the NP-complete music problems. The root problem is SAT (the first problem proven to be NP-complete [Cook 1971]), and our tree shows the branch containing 3SAT (proved NP-complete by Cook [1971]) and leading to 3DM, which we used to prove the NP-completeness of HM3PAL. An analogous tree could be constructed for the reductions used in the enumerated versions of these problems.

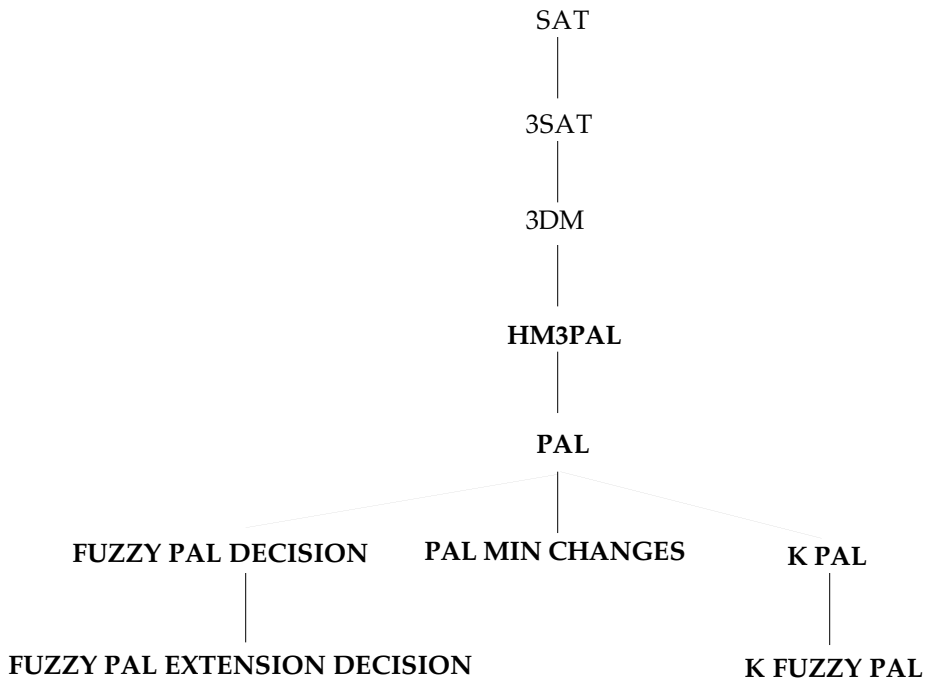


Fig. 4.3. NP-completeness proof transformation hierarchy for PAL-related problems

4.2.3. Optimization Music Problems

An *optimization* (sometimes referred to as *combinatorial optimization*) *problem* is one in which we seek a solution that satisfies either a maximal or minimal condition. For example, an optimization music problem requires a countermelody that has the highest possible merit. The complexity category to which related optimization music problems belong will be shown to be *NP-equivalent*: these problems are defined to be both NP-hard and NP-easy. An *NP-hard problem* is one to which every NP problem is Turing reducible (equivalently, to which any NP-complete problem is Turing reducible); an *NP-easy problem* is one which is Turing reducible to any NP problem. The terms "NP-equivalent" and "NP-easy" were introduced by Garey and Johnson (1979) to encompass problems that are as difficult and as easy as the NP-complete problems, within a polynomial factor; NP-equivalent problems are not necessarily NP-complete themselves, because they are not necessarily decision problems.

The optimization problems are the decision problems modified to contain both fuzzy and crisp rules; this combination of fuzzy and crisp rules is the basis of algorithms to reflect human preferences. Our subsequent discussions on creating "human-like" music rest upon the problem we call FUZZY PAL OPTIMIZATION.

Problem 4.21. FUZZY PAL OPTIMIZATION
INSTANCE:
A finite set S;
$c, p \in \mathbb{N}$;
a sequence $CF \in S^c$;
a set of functions ("crisp rules") $R = \{ru_1, ru_2, \ldots, ru_r\}$, ru_i: $S^n \times (S^n)^p \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, $1 \leq i \leq r$,
a set of functions ("fuzzy rules") $A = \{a_1, a_2, \ldots, a_q\}$, a_i: $S^n \times (S^n)^p \rightarrow \mathbb{N}_{0,n^*p}$, $n \in \mathbb{N}$, $1 \leq i \leq q$
where the computation of each element of R and A is bounded polynomially in n^*p.
QUESTION:
What is the optimal p-tuple of sequences $CP \in (S^c)^p$ that satisfies $ru_i\,(CF, CP) = 1$, $1 \leq i \leq r$
and produces $M =$
$\max_{CP} [\sum_{i=1}^{q} a_i\,(CF, CP)]$?

Lemma 4.3. FUZZY PAL OPTIMIZATION is NP-hard.

Proof: The Turing reduction from FUZZY PAL DECISION to FUZZY PAL OPTIMIZATION is straightforward. The solution to the optimization problem tells us the maximum possible value of α (call it α_{opt}) in the decision problem, and the answer to the decision problem depends on the actual value of α in a given instance (call it α'): "yes" if $\alpha' \leq \alpha_{opt}$, "no" otherwise. ■

Lemma 4.4. FUZZY PAL OPTIMIZATION is NP-easy.

Proof: We present a constructive proof by showing a Turing reduction to FUZZY PAL EXTENSION DECISION. We use the solution to FUZZY PAL EXTENSION DECISION as an oracle to compute the optimal score M and then to derive the optimal sequence CP. To begin, notice that M ranges from 0 to c^*p^*r, and can be discovered for any instance of the problem with a polynomial number of calls to the FUZZY PAL EXTENSION DECISION oracle, using binary search: we start the search by querying the oracle for $\alpha = \lfloor c^*p^*r/2 \rfloor$, and proceed by successively modifying α according to the oracle's answers, always giving PM as the empty sequence. The complexity of deriving the optimal score in this way is no more than a polynomial in $\log (c^*p^*r) = (\log c + \log p + \log r)$, since FUZZY PAL EXTENSION DECISION is in NP; let this optimal score be opt. If $opt = 0$, we are done, for any CP will do; otherwise, continue as follows: Build CP by querying the oracle, from now on always giving $\alpha = opt$. We derive CP by building upon PM, each time querying the oracle for the validity of our choice of an element from S. The maximum number of queries is $c^*p^*|S|$; thus, the complexity of deriving CP is no more than a polynomial in $c^*p^*|S|$. The entire procedure queries the oracle a polynomial number of times, and therefore the entire reduction from FUZZY PAL OPTIMIZATION to FUZZY PAL EXTENSION DECISION runs in polynomial time. By definition, then, FUZZY PAL OPTIMIZATION is NP-easy. ■

We now present the algorithm that is used in the above constructive proof of this theorem and call it PALOptimization; it calls two subfunctions, FindOptimalCountermelodyGrade and FindOptimalCountermelody. Both these subfunctions rely on an *oracle*, which is the putative solution for the decision problem FUZZY PAL EXTENSION DECISION.

Algorithm PALOptimization
Input: *S*, *CF*, *c*, *p*, *R*, *A*.
Output: *optimalCountermelody*.
1. α := FindOptimalCountermelodyGrade (*S*, *CF*, *c*, *p*, *R*, *A*)
2. *optimalCountermelody* := FindOptimalCountermelody (*S*, *CF*, *c*, *p*, *R*, *A*, α).

Subprogram FindOptimalCountermelodyGrade
Input: *S*, *CF*, *c*, *p*, *R*, *A*.
Output: α.
1. **for** *i* from 1 to *p*
    for *j* from 1 to *c*
        CP [*i*][*j*] := λ
2. *low* := 0
3. *high* := *c***p** | *S* |
4. **while** *low* ≤ *high*
    mid := ⌊ (*low* + *high*)/2 ⌋
    if ORACLE (*S*, *CF*, *R*, *A*, *CP*, *mid*) = "yes"
        low := *mid* + 1
    else
        high := *mid* - 1
5. α := *high*.

Subprogram FindOptimalCountermelody
Input: *S*, *CF*, *c*, *p*, *R*, *A*, α.
Output: *optimalCountermelody*.
1. **for** *i* from 1 to *p*
    for *j* from 1 to *c*
        optimalCountermelody [*i*][*j*] := empty
2. **for** *i* from 1 to *p*
    for *j* from 1 to *c*
        for each *s* ∈ *S*
            optimalCountermelody [*i*][*j*] := *s*
            if ORACLE (*S*, *CF*, *R*, *A*, *optimalCountermelody*, α) = "yes"
                break.

Algorithm 4.1. Solving FUZZY PAL OPTIMIZATION using the solution to FUZZY PAL EXTENSION DECISION

Example 4.14. Consider the following instance of FUZZY PAL OPTIMIZATION:

S= {C, C#, D, E, F}
c = 4, *p* = 1, *r* = number of "crisp rules" (see appendix A, sections A.1.1–3), q = 3
CF = <D, E, C#, D>
R is equivalent to the set of "crisp" rules (see appendix A, sections A.1.1–3)
A is equivalent to the set consisting of the Tonality, Final in Leaps of Fourths and Fifths, and False Relations artistry rules (see appendix A, section A.1.4).

As outlined in the above proof, querying the FUZZY PAL EXTENSION DECISION oracle occurs in two stages:

Stage 1 (finding α):

QUESTION ANSWER

Is there a p-tuple of sequences $CP' = <<mel'_{11}, mel'_{12}, \ldots, mel'_{1c}>, <mel'_{21}, mel'_{22}, \ldots, mel'_{2c}>, \ldots, mel'_{p1}, mel'_{p2}, \ldots, mel'_{pc}>$, where each sequence in CP' is over S and $mel_{xy} \neq \lambda \rightarrow mel_{xy} = mel'_{xy}$, $1 \leq x \leq p$, $1 \leq y \leq c$,
such that
$ru_i(CF, CP') = 1$, $1 \leq i \leq r$
and
$\sum_{i=1}^{q} a_i(CF, CP') \geq \alpha$?

Note: $0 \leq \alpha \leq 12$.

α =	
6	yes
9	yes
11	yes
12	no

CONCLUSION: $\alpha = 11$.

Stage 2 (finding CP):

QUESTION ANSWER

Is there a p-tuple of sequences $CP' = <<mel'_{11}, mel'_{12}, \ldots, mel'_{1c}>, <mel'_{21}, mel'_{22}, \ldots, mel'_{2c}>, \ldots, mel'_{p1}, mel'_{p2}, \ldots, mel'_{pc}>$, where each sequence in CP' is over S and $mel_{xy} \neq \lambda \rightarrow mel_{xy} = mel'_{xy}$, $1 \leq x \leq p$, $1 \leq y \leq c$,
such that
$ru_i(CF, CP') = 1$, $1 \leq i \leq r$
and
$\sum_{i=1}^{q} a_i(CF, CP') \geq 11$?

PM =	
<C, λ, λ, λ>	no
<C#, λ, λ, λ>	no
<D, λ, λ, λ>	yes
<D, C, λ, λ>	yes
<D, C, C, λ>	no
<D, C, C#, λ>	no
<D, C, D, λ>	no
<D, C, E, λ>	yes
<D, C, E, C>	no
<D, C, E, C#>	no
<D, C, E, D>	yes

CONCLUSION: The optimal CP is <D, C, E, D>. ■

Theorem 4.11. FUZZY PAL OPTIMIZATION is NP-equivalent.

Proof: Since FUZZY PAL OPTIMIZATION is both NP-hard and NP-easy, by definition it is NP-equivalent. ■

Finally, we consider the optimization problem related to the decision problem PAL RULE SUBSET. Recall that in this problem we wanted to know if there was a rule subset of a given size that eliminated the same countermelodies to a given cantus firmus as that by the entire rule set. In the optimization version of this problem, we wish to know the smallest such subset that would eliminate the same countermelodies. We now define PAL RULE SUBSET and prove that it is NP-equivalent.

Problem 4.22. PAL MINIMAL RULE SUBSET
INSTANCE:
A cantus firmus set $CFSet = \{cf_1, cf_2, \ldots, cf_m\}$;
a rule set $RSet = \{r_1, r_2, \ldots, r_n\}$;
a collection of "Acceptance sets" $Accept_{r,\,c}$, $1 \le r \le n$, $1 \le c \le m$,
which is the set of countermelodies for $cf_c \in CFSet$ that rule r_r accepts.
QUESTION:
What is the minimal subset $RSet' \subseteq RSet$
such that
$\bigcap_{1 \le r \le |\,RSet'\,|} Accept_{r,\,c} = \bigcap_{1 \le s \le |\,RSet\,|} Accept_{s,\,c}$, $1 \le c \le m$?

Lemma 4.5. PAL MINIMAL RULE SUBSET is NP-hard.

Proof: Use a Turing reduction from PAL RULE SUBSET to PAL MINIMAL RULE SUBSET. ■

Lemma 4.6. PAL MINIMAL RULE SUBSET is NP-easy.

Proof: We use the solution to the decision problem PAL RULE SUBSET as an oracle first to find the size of the minimal subset, and then to build this minimal subset, using a technique similar to that used in the NP-easiness proof of FUZZY PAL OPTIMIZATION. The first part of the proof requires no more than $\log_2 |\,RSet\,|$ calls to the oracle. Suppose the size of the minimal rule set was found to be K'. We then proceed to build this minimal rule set, starting with the empty set. To determine which rules belong in this minimal rule set, we try adding each rule in turn to the minimal rule set. For each candidate rule considered for inclusion, we create modified acceptance sets of the rules that have not yet been added to the minimal rule set by removing those countermelodies implicitly rejected by the candidate rule and by rules that have already been determined to belong to the minimal rule subset. We present to the oracle the modified rule set which consists of $RSet$ less the candidate rule and rules that have been determined to belong to the minimal rule set, the modified acceptance sets of rules not in the minimal rule set, and as the integer K the value of K' (found in the first part of the proof) less the current size of the minimal rule set, and less one. If the oracle answers "yes", we accept the candidate rule for inclusion into the minimal rule subset, otherwise we reject it. We proceed thus until the integer K becomes 0, in which case we cease asking the oracle (since the problem is not defined for $K = 0$), and instead select the final rule to be added to the minimal rule set by observing which one results in the remaining modified acceptance sets all becoming equal to the intersection of the original acceptance sets. In building the minimal rule subset in this way, the maximum number of calls to the oracle is $|\,RSet\,|$. ■

We now present the algorithm that is used in the above constructive proof of this theorem, which solves the optimization problem along similar lines as did the algorithm to solve FUZZY PAL OPTIMIZATION. We call this present algorithm PALMinSubset, which calls two subfunctions, FindMinSubsetSize and FindMinSubset. Both these subfunctions rely on an oracle, which is the putative solution for the decision problem PAL RULE SUBSET.

Algorithm PALMinSubset
Input: $CFSet$, $RSet$, $Accept_{r,c}$, $1 \le r \le |RSet|$, $1 \le c \le |CFSet|$.
Output: $RSet'$.
1. K := FindMinSubsetSize ($CFSet$, $RSet$, $Accept$)
2. $RSet'$:= FindMinSubset ($CFSet$, $RSet$, $Accept$, K).

Subprogram FindMinSubsetSize
Input: $CFSet$, $RSet$, $Accept_{r,c}$, $1 \le r \le |RSet|$, $1 \le c \le |CFSet|$.
Output: K.
1. $low := 1$
2. $high := |RSet|$
3. **while** $low \le high$
   $mid := \lceil (low + high)/2 \rceil$
   if ORACLE ($CFSet$, $RSet$, $Accept$, mid) = "yes"
      $high := mid$ - 1
   else
      $low := mid + 1$
4. $K := low$.

Subprogram FindMinSubset
Input: $CFSet$, $RSet$, $Accept_{r,c}$, $1 \le r \le |RSet|$, $1 \le c \le |CFSet|$, K.
Output: $RSet'$.
1. $RSet' := \{\}$
2. $K' := K$ - 1
3. **for** i from 1 to $|RSet|$
   $tempRSet := RSet' \cup \{r_i\}$
   for j from i to $|RSet|$
      for k from i to $|CFSet|$
         $Accept'_{j,k} := Accept_{j,k} \cap Accept_{r,k}$, $1 \le r \le tempRSet$
   if $K' > 0$
      if ORACLE ($CFSet$, $RSet$ - $tempRSet$, $Accept'$, K') = "yes"
         $K' := K'$ - 1
         $RSet' := tempRSet$
   else // $K' = 0$
      if $Accept'_{j,k} = \cap_{1 \le s \le |RSet|} Accept_{s,k}$, $1 \le k \le |CFSet|$
         $RSet' := tempRSet$
         break.

Algorithm 4.2. Solving PAL MINIMAL RULE SUBSET using the solution to PAL RULE SUBSET

Example 4.15. Consider the following instance of PAL MINIMAL RULE SUBSET:

A single element *CFSet*, $m = 1$, *RSet* = {a, b, c, d, e}, Acceptance sets as follows:

Rule	Acceptance set						
a	{1,		3,	4,	5,	6,	7,	8}
b	{		3,	4,	5,	6,	7	}
c	{	2,	3,	4,	5,	6,	7,	8}
d	{		3,	4,	5,		7,	8}
e	{		3,	4,	5,	6,		8}

Queries to the oracle occur in two stages:

Stage 1 (finding the size of the minimal subset);

QUESTION ANSWER

Is there a subset $RSet' \subseteq RSet$, $|RSet'| \leq K$
such that
$\bigcap_{1 \leq r \leq |RSet'|} Accept_{r,\, c} = \bigcap_{1 \leq s \leq |RSet|} Accept_{s,\, c}$, $1 \leq c \leq m$?

Note: The possible values of K are in [1, 5].

$K' = 3$? yes

$K' = 2$? no

CONCLUSION: $K' = 3$.

Stage 2 (building the minimal subset of size 3):

QUESTION
Is there a subset $RSet' \subseteq RSet$, $|RSet'| \leq 3$
such that
$\bigcap_{1 \leq r \leq |RSet'|} Accept_{r,\, c} = \bigcap_{1 \leq s \leq |RSet|} Accept_{s,\, c}$, $1 \leq c \leq m$?

given

RSet = {b′, c′, d′, e′} // (Add rule a to CURRENT MINIMAL SUBSET = {}?)

Rule	(Modified) Acceptance set					
b′	{3,	4,	5,	6,	7	}
c′	{3,	4,	5,	6,	7,	8}
d′	{3,	4,	5,		7,	8}
e′	{3,	4,	5,	6,		8}

K = 2

ANSWER: no

RSet = {a′, c′, d′, e′} // (Add rule b to CURRENT MINIMAL SUBSET = {}?)

Rule	(Modified) Acceptance set				
a′	{3,	4,	5,	6,	7}
c′	{3,	4,	5,	6,	7}
d′	{3,	4,	5,		7}
e′	{3,	4,	5,	6	}

$K = 2$

ANSWER: yes

RSet = {a′, d′, e′} // (Add rule c to CURRENT MINIMAL SUBSET = {b}?)

Rule	(Modified) Acceptance set				
a′	{3,	4,	5,	6,	7}
d′	{3,	4,	5,		7}
e′	{3,	4,	5,	6	}

$K = 1$

ANSWER: no

RSet = {a′, c′, e′} // (Add rule d to CURRENT MINIMAL SUBSET = {d}?)

Rule	(Modified) Acceptance set			
a′	{3,	4,	5,	7}
c′	{3,	4,	5,	7}
e′	{3,	4,	5	}

$K = 1$

ANSWER: yes

(Add rule e to CURRENT MINIMAL SUBSET = {b, d}?)

a′	{3	4	5}
c′	{3	4	5}

$a' = c' = \{3, 4, 5\} = \underset{1 \le s \le |\,RSet\,|}{Accept_s}, c \quad , 1 \le c \le m$

Accept rule e

CONCLUSION: Minimal rule subset = {b, d, e}. ■

Note that in the second stage in building the minimal rule subset, once a rule has been considered for inclusion in the minimal rule set but rejected because of a negative reply from the oracle, that rule need not ever again be considered. Thus, in the instance in example 4.15, rules “a” and “b”, having drawn negative responses from the oracle when considered initially for inclusion into the current empty rule subset, need not be considered in subsequent attempts. Eliminating from consideration such rules, however, does not improve the efficiency of the procedure by any order of magnitude.

Theorem 4.12. PAL MINIMAL RULE SUBSET is NP-equivalent.

Proof: Since MINIMAL RULE SUBSET is both NP-hard and NP-easy (lemmata 4.5, 4.6), by definition it is NP-equivalent. ■

4.3. Summary

Rules that govern notes spanning entire countermelodies render the counterpoint composition problem intractable. Among intractable music composition problems, we identified and classified such problems as decision, enumeration, number, and optimization problems. Initially proving NP-completeness of one decision music composition problem, HM3PAL, by reduction from 3DM (and alternatively, from BOUNDED TILING), we established a hierarchy of proofs that show all related compositional problems to be intractable as well. Table 4.1 summarizes the complexity classes to which the music problems discussed in this chapter belong:

Table 4.1. Complexity classes of PAL-related problems

	Problem	**Type**	**Complexity class**
1.	PAL	Decision	NP-complete
2.	HM2PAL	Decision	NP-complete
3.	HM3PAL	Decision	NP-complete
4.	FUZZY PAL DECISION	Decision	NP-complete
5.	PAL MIN CHANGES	Decision	NP-complete
6.	PAL ENUM	Enumeration	#P-complete
7.	FUZZY PAL ENUM	Enumeration	#P-complete
8.	K PAL	Number	Strong sense NP-complete
9.	K FUZZY PAL	Number	Strong sense NP-complete
10.	FUZZY PAL OPTIMIZATION	Optimization	NP-equivalent

The general problems dealing with rule redundancy and indispensability, PAL RULE SUBSET and PAL MINIMAL RULE SUBSET, are also intractable. We proved this initially by reduction from VERTEX COVER.

CHAPTER 5. MODELING HUMAN PREFERENCES IN MUSIC COMPOSITION

In the classic *Turing imitation game*, a computer program tries to fool a person that its discourse springs from another human being. We now try to write a program that can fool a person that its music, too, springs from the creative imagination of another human being. For this to happen, that music needs to reflect human preferences. We present two methods that use fuzzy rules to model such preferences: in the first method (the "Jeppesen Experiment"), we collect a set of melodies composed by an expert, and determine the relative importance the expert placed upon the various rules used in the compositional process by calculating their weights (for our use of rule weights, see chapter 2, section 2.3.2); in the second method, we train an artificial neural network to predict how a particular listener might evaluate test melodies. We shall then be able to imbue heuristic methods (described in chapter 6) with the ability to make choices among alternatives that reflect what a human expert might possibly select.

5.1. The Jeppesen Experiment

In this method, we infer rule preferences directly from examples of a human expert. We first collect countermelodies composed by an expert in species counterpoint; then we generate alternative countermelodies at various points of the original compositions, and apply the fuzzy evaluation of a predetermined set of rules. We assign weights to these rules in such a way that they favor the choice the expert actually made over the alternatives. The goal is to create a weighted rule set that reflects the expert's value system. An authority on this style of music who has written a considerable corpus is the Danish musicologist Knud Jeppesen (1931).

We carried out an experiment according to the procedure described in algorithm 5.1.

1. Select fuzzy rules.
2. Fuzzify any remaining crisp rules violated by Jeppesen's solutions.
3. *valid weight vector* := <1, 1, …>.
4. *valid data set* := Jeppesen's original solutions.
5. *num blanks* := 1.
6. **do**
   Prepare (*data set*, *num blanks*)
   if Adjust (*weight vector*, *data set*) is successful
   valid weight vector := *weight vector*
   valid data set := *data set*
   increment *num blanks*
   while *num blanks* ≤ |CF| **and** Adjust was successful.
7. **output** *valid weight vector* and *valid data set*.

Algorithm 5.1. Procedure for the Jeppesen Experiment

In the experiment, we wanted to find a fuzzy rule weight vector that would favor one of Jeppesen's solutions to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>. The four fuzzy rules we used were the artistry rules, and for Jeppesen's solution in this instance (<a, aa, g, f, e, d, c, h, d, c#, d>) there was no need to fuzzify any crisp rule. The initial *valid weight vector* was therefore <1, 1, 1, 1> to weight each of the four fuzzy rules equally. The initial *valid data set* comprised Jeppesen's solution and the cantus firmus (listing 5.1). Listing 5.2 shows the *data set* with a single blank (characterized in algorithm 5.1 as the output by subroutine Prepare), where "*" indicates blanks, or the opportunity for alternatives. With the initially equally weighted fuzzy rules, the countermelody <a, aa, g, f, e, d, *, h, d, c#, d> yielded two alternatives (Jeppesen's solution is the second among these) that tied for the highest score (listing 5.3). By adjusting the weight of the NoteVariety Rule by doubling it (i.e., *weight vector* = <1, 1, 1, 2>), Jeppesen's solution became the best choice among all alternatives (shown in listing 5.4 is Jeppesen's solution followed by the formerly tied-for-first, now relegated to second-best, alternative). Since we were able to adjust the weights successfully with one blank, we proceeded to create a new data set with two blanks. Listing 5.5 shows the *data set* with two blanks. For this data set, no further weight adjustments were necessary to preserve Jeppesen's solution as the best. In fact, the same weight vector was valid for all subsequent data sets with three and more blanks. The final data set comprised a single countermelody with eleven blanks (corresponding to the eleven notes of the cantus firmus), which essentially served to ask for all possible solutions to the cantus firmus. For this final data set, there were 76 solutions. With the weight vector thus calculated, Jeppesen's solution still prevailed (shown in listing 5.6 is Jeppesen's solution, followed by four alternatives that tied for second place).

The need to increase the weightage of the NoteVariety rule in this instance indicates to some extent that, for this cantus firmus at least, Jeppesen laid greater emphasis on note variety than he did on, say, "false relations." See appendix E, section E.1 for output of the genetic algorithm (chapter 6, section 6.2) based on rules weighted according to the outcome of this instance of the Jeppesen experiment: the genetic algorithm succeeded in finding the optimal countermelody (Jeppesen's) very quickly.

Listing 5.1. Initial valid data set

```
a aa g f e d c h d c# d
D F  E D G F a G F E  D
```

Listing 5.2. Data set with one blank

```
* aa g f e d c h d c# d
D F  E D G F a G F E  D
a *  g f e d c h d c# d
D F  E D G F a G F E  D
a aa * f e d c h d c# d
D F  E D G F a G F E  D
a aa g * e d c h d c# d
D F  E D G F a G F E  D
a aa g f * d c h d c# d
D F  E D G F a G F E  D
a aa g f e * c h d c# d
D F  E D G F a G F E  D
a aa g f e d * h d c# d
D F  E D G F a G F E  D
a aa g f e d c * d c# d
D F  E D G F a G F E  D
a aa g f e d c h * c# d
D F  E D G F a G F E  D
a aa g f e d c h d *  d
D F  E D G F a G F E  D
a aa g f e d c h d c# *
D F  E D G F a G F E  D
```

Listing 5.3. Solutions for data set with one blank

Countermelody: a aa g f e d * h d c# d
Cantus firmus: D F E D G F a G F E D
Alpha cut: 0.0

Rule	ρ	**w**
Tonality	1.0	1.0
Final in Leaps of 4ths and 5ths	1.0	1.0
False Relations	1.0	1.0
Note Variety	1.0	1.0

*** 1.
Countermelody: a aa g f e d e h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	00001100100	0.727273

	Overall rating:	0.909091

*** 2.
Countermelody: a aa g f e d c h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000010000	0.909091

(listing 5.3 continued)

Note Variety	00000100100	0.818182

	Overall rating:	0.909091

Number of solutions: 2.

Listing 5.4. Solutions for data set with one blank, and adjusted rule weights

Countermelody: a aa g f e d * h d c# d
Cantus firmus: D F E D G F a G F E D
Alpha cut: 0.0

Rule	ρ	**w**
Tonality	1.0	1.0
Final in Leaps of 4ths and 5ths	1.0	1.0
False Relations	1.0	1.0
Note Variety	1.0	2.0

*** 1.
Countermelody: a aa g f e d c h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000010000	0.909091
Note Variety	00000100100	0.818182

	Overall rating:	0.890909

*** 2.
Countermelody: a aa g f e d e h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	00001100100	0.727273

	Overall rating:	0.872727

Number of solutions: 2.

Listing 5.5. Data set with two blanks

```
* *  g f e d c h d c# d
D F  E D G F a G F E  D
a *  * f e d c h d c# d
D F  E D G F a G F E  D
a aa * * e d c h d c# d
D F  E D G F a G F E  D
a aa g * * d c h d c# d
D F  E D G F a G F E  D
a aa g f * * c h d c# d
D F  E D G F a G F E  D
a aa g f e * * h d c# d
```

(listing 5.5 continued)

```
D F  E D G F a G F E  D
a aa g f e d * * d c# d
D F  E D G F a G F E  D
a aa g f e d c * * c# d
D F  E D G F a G F E  D
a aa g f e d c h * *  d
D F  E D G F a G F E  D
a aa g f e d c h d *  *
 D F  E D G F a G F E  D
```

Listing 5.6. Solutions for data set with eleven blanks, and adjusted rule weights

Countermelody: * * * * * * * * * * *
Cantus firmus: D F E D G F a G F E D
Alpha cut: 0.0

Rule	ρ	w
Tonality	1.0	1.0
Final in Leaps of 4ths and 5ths	1.0	1.0
False Relations	1.0	1.0
Note Variety	1.0	2.0

*** 1.
Countermelody: a aa g f e d c h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	Violations	μ
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000010000	0.909091
Note Variety	00000100100	0.818182

	Overall rating:	0.890909

*** 2.
Countermelody: a a G F E D A B D C# D
Cantus firmus: D F E D G F a G F E D

Rule	Violations	μ
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	10000100100	0.727273

	Overall rating:	0.872727

*** 3.
Countermelody: a D G F E D A B D C# D
Cantus firmus: D F E D G F a G F E D

Rule	Violations	μ
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	01000100100	0.727273

(listing 5.6 continued)

Overall rating: 0.872727

*** 4.

Countermelody: a aa g f e f e h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	00011000100	0.727273

Overall rating: 0.872727

*** 5.

Countermelody: a aa g f e d e h d c# d
Cantus firmus: D F E D G F a G F E D

Rule	**Violations**	***μ***
Tonality	00000001000	0.909091
Final in Leaps of 4ths and 5ths	00000000000	1.000000
False Relations	00000000000	1.000000
Note Variety	00001100100	0.727273

Overall rating: 0.872727

Number of solutions: 5.

Notice that in proceeding with an increasing number of blanks in our data set, we produce sets of countermelodies that are successively supersets of those produced with fewer blanks. Unlike in the above experiment, it may happen that no valid weight vector can be found that makes Jeppesen's solution the best among new alternatives occasioned by a larger number of blanks, in which case we will have to be satisfied with a sub-optimal solution related to a data set with a smaller number of blanks than we would like. The optimal solution for a single cantus firmus, as above, is to have a weight vector that makes Jeppesen's solution prevail among all alternatives (*num blanks* = cantus firmus length). The ultimate optimal solution is a weight vector valid for all cantus firmi attempted by Jeppesen.

We implemented this experiment by using the best-first search algorithm (described in chapter 6, section 6.1; for the implementation of the "fill in the blanks" algorithm, see appendix D, section D.2.3). In each iteration of stage (6) of our procedure, we might have used this algorithm to generate all alternatives for all the instances with *num blanks* filled. The algorithm automatically outputs the alternatives in non-increasing order of merit, and it was not necessary for it to generate all possibilities, only alternatives that included Jeppesen's choice for us to know where the latter stands in each instance. In the above experiment, Jeppesen's choice always appeared as the best countermelody (either

tied or alone), both before and after weight adjustment, and we needed only to ask the best-first algorithm to produce the best countermelody. This was sufficient for us to ensure that Jeppesen's choice was indeed favored above all alternatives after a suitable rule weight adjustment. In a less fortuitous instance, we might have to ask for the two, or three, etc., best countermelodies, and at the worst, for all alternatives. In appendix B, we show all the alternatives that could be generated at each iteration of stage (6) of our procedure.

5.1.1. Complexity Analysis of the Jeppesen Experiment

In general, adjusting the rule weights in order to make Jeppesen's solutions rate higher than the alternatives can be formalized as problems that can be shown to relate closely to various forms of the INTEGER PROGRAMMING problem, which in turn have been shown to be NP-complete or NP-hard by several authors (Karp [1972], Sahni [1974], Borosh and Treybig [1976], Kannan and Monma [1978], Papadimitriou [1981]). The simplest problem we have identified is what we have named FAVOR JEPPESEN 0-1. This corresponds to the problem of whether or not to include a particular rule into a set of equally weighted rules.

Problem 5.1. FAVOR JEPPESEN 0-1
INSTANCE:
A collection of finite string sets $\boldsymbol{JM}$ =
$\{ \{jepp_1, m_{11}, m_{12}, \ldots, m_{1m1} \}, \{jepp_2, m_{21}, m_{22}, \ldots, m_{2m2}\}, \ldots, \{jepp_n, m_{n1}, m_{n2}, \ldots, m_{nmn}\} \}$,
$jepp_i, m_{i1}, m_{i2}, \ldots \in$ String, $1 \le i \le n$;
a set of rules $\boldsymbol{R} = \{R_1, R_2, \ldots, R_r\}$, R_i: String$\rightarrow\mathbb{Z}$, $1 \le i \le r$.
QUESTION:
Is there a function w: $\boldsymbol{R}\rightarrow\{0, 1\}$ such that

$$\sum_{R\in\boldsymbol{R}} (w(R)\, R(jepp_k)) \ge \max_{km_k} \left\{ \sum_{R\in\boldsymbol{R}} (w(R)\, R(m_{km_k})) \right\}, 1 \le k \le |\boldsymbol{JM}|, \sum_{R\in\boldsymbol{R}} w(R) > 0 \,?$$

Theorem 5.1. FAVOR JEPPESEN 0-1 is NP-complete.

The proof of the above theorem restricts the problem to an instance of KNAPSACK (proven NP-complete by Karp [1972]).

Problem 5.2. KNAPSACK
INSTANCE:
Finite set U, for each $u \in U$ a size $s(u) \in \mathbb{N}$ and a value $v(u) \in \mathbb{N}$,
and positive integers B and K.
QUESTION:
Is there a subset $U' \subseteq U$
such that

$$\sum_{u \in U'} s(u) \leq B$$

and such that

$$\sum_{u \in U'} v(u) \geq K \ ?$$

Proof (theorem 5.1): FAVOR JEPPESEN 0-1 is clearly in NP, a "guess" can easily be verified in polynomial time. To show that it is NP-hard, restrict it to KNAPSACK by allowing only instances in which

$\boldsymbol{JM} = \{ \{jepp_1, m_{11}\}, \{jepp_2, m_{21}\} \}$;

$\boldsymbol{R} = \{R_1, R_2, \ldots, R_{|U|}, R_{|U|+1}\}$;

$R_i(jepp_1) = v(u_i)$, and $R_i(m_{11}) = 0$, $1 \leq i \leq |U|$;

$R_{|U|+1}(jepp_1) = 0$, and $R_{|U|+1}(m_{11}) = K$;

$R_i(jepp_2) = 0$, and $r_i(m_{21}) = s(u_i)$, $1 \leq i \leq |U|$;

$R_{|U|+1}(jepp_2) = B$, and $R_{|U|+1}(m_{21}) = 0$; ■

In this reduction, $\boldsymbol{JM}$ consists of two pairs of melodies. $\boldsymbol{R}$ contains $|U| + 1$ rules: with respect to the set containing $jepp_1$, the evaluations of the first $|U|$ rules correspond to $v(u_i)$, while the evaluation of the last rule corresponds to K; with respect to the set containing $jepp_2$, the evaluations of the first $|U|$ rules correspond to $s(u_i)$, while the evaluation of the last rule corresponds to B.

In the definition of FAVOR JEPPESEN 0-1, note that the range space of the rules (represented as functions) differs from that in the definition of PAL and related problems. If desired, the range space of all rules could be restricted to $[0..\max_i\{|jepp_i|\}]$, and in the proof specify further that $\max[|jepp_1|, |jepp_2|] = \max_i[v(u_i)]$. This will not alter the complexity of FAVOR JEPPESEN 0-1.

A more general problem is the following, where the chosen rules are allowed to be unequally weighted (FAVOR JEPPESEN INTEGER). The proof of lemma 5.1 transforms a version of INTEGER PROGRAMMING, proved NP-hard by Sahni (1974).

Problem 5.3. FAVOR JEPPESEN INTEGER

INSTANCE:

A collection of finite string sets $\boldsymbol{JM} =$

$\{ \{jepp_1, m_{11}, m_{12}, \ldots, m_{1m1} \}, \{jepp_2, m_{21}, m_{22}, \ldots, m_{2m2}\}, \ldots, \{jepp_n, m_{n1}, m_{n2}, \ldots, m_{nmn}\} \}$,

$jepp_i, m_{i1}, m_{i2}, \ldots \in$ String, $1 \leq i \leq n$;

a set of rules $\boldsymbol{R} = \{R_1, R_2, \ldots, R_r\}$, R_i: String$\rightarrow\mathbb{Z}$, $1 \leq i \leq r$.

QUESTION:

Is there a function w: $\boldsymbol{R}\rightarrow\mathbb{Z}$ such that

$$\sum_{R \in \boldsymbol{R}} (w(R)\, R(jepp_k)) \geq \max_{km_k} \left\{ \sum_{R \in \boldsymbol{R}} (w(R)\, R(m_{km_k})) \right\}, 1 \leq k \leq |\boldsymbol{JM}|, \sum_{R \in \boldsymbol{R}} w(R) > 0 \ ?$$

Lemma 5.1. FAVOR JEPPESEN INTEGER is NP-hard.

Problem 5.4. INTEGER PROGRAMMING
INSTANCE:
Finite set $\boldsymbol{X}$ of m-tuples $\boldsymbol{x}$ of integers.
QUESTION:
Is there an m-tuple $\boldsymbol{y}$ of integers such that $\boldsymbol{x}.\boldsymbol{y} \geq 0$ for all $\boldsymbol{x}$ in $\boldsymbol{X}$?

Proof (lemma 5.1): Restrict to INTEGER PROGRAMMING by allowing only instances in which $|\boldsymbol{JM}| = |\boldsymbol{X}|$;
Each element in $\boldsymbol{JM}$ is $\{jepp_i, m_i1\}$, $1 \leq i \leq |\boldsymbol{JM}|$;
$|\boldsymbol{R}| = m$;
$R_j\,(jepp_i) - R_j\,(m_i1) = j$th element of ith m-tuple in $\boldsymbol{X}$. ■

In this reduction, we restrict each element in $\boldsymbol{JM}$ to have only a single alternative countermelody.

The integer programming problem appears in the literature in various forms, most of which have been proven to be NP-complete. One variation is the following (we call it INTEGER PROGRAMMING B; defined in Garey and Johnson [1979]), which we shall use below in a convenient transformation to prove that FAVOR JEPPESEN INTEGER is in NP.

Problem 5.5. INTEGER PROGRAMMING B
INSTANCE:
Finite set $\boldsymbol{X}$ of pairs $<\boldsymbol{x}, b>$, where $\boldsymbol{x}$ is an m-tuple of integers and b is an integer, an m-tuple $\boldsymbol{c}$ of integers, and an integer B.
QUESTION:
Is there an m-tuple $\boldsymbol{y}$ of integers such that $\boldsymbol{x}.\boldsymbol{y} \leq b$ for all $(\boldsymbol{x}, b)$ in $\boldsymbol{X}$ and such that $\boldsymbol{c}.\boldsymbol{y} \geq B$?

We have shown that FAVOR JEPPESEN INTEGER is NP-hard. In order to show FAVOR JEPPESEN INTEGER is NP-complete, we need also to show that it belongs in NP, i.e., that any proposed solution contains numbers that are polynomially bounded. If this were not the case, a non-deterministic Turing machine would not be able to verify the proposed solution in a polynomial number of steps. Unlike most NP-complete problems, those associated with INTEGER PROGRAMMING are not very easy to prove to be in NP. For NP proofs of versions of INTEGER PROGRAMMING, see Borosh and Treybig (1976), Kannan and Monma (1978), Papadimitriou (1981).

We show that FAVOR JEPPESEN INTEGER is also in NP by transforming in polynomial time an instance of the problem into an instance of INTEGER PROGRAMING B. Then, by the closure property of the complexity class NP under polynomial-time transformation (Papadimitriou 1995, 166), we infer that FAVOR JEPPESEN INTEGER is also in NP. We can thus conclude that FAVOR JEPPESEN INTEGER is NP-complete.

Theorem 5.2. FAVOR JEPPESEN INTEGER is NP-complete.

Proof: We first show a polynomial-time transformation of FAVOR JEPPESEN INTEGER to INTEGER PROGRAMMING B. Restrict to INTEGER PROGRAMMING B by allowing only instances in which $m = |\boldsymbol{R}|$, $\boldsymbol{x} = \boldsymbol{0}$ and $b = 0$ for all $(\boldsymbol{x}, b)$ in $\boldsymbol{X}$; $c_i = \sum_k R_i\,(jepp_k) - \sum_k \max_{kmk}\,[\mathrm{R_i}\,(m_{kmk})]$, for each c_i in $\boldsymbol{c}$, $1 \le i \le m$; $B = 0$. Then, since INTEGER PROGRAMMING B is in NP, FAVOR JEPPESEN INTEGER is also in NP, by the closure property under polynomial-time transformation of NP. This, together with lemma 5.1, implies that FAVOR JEPPESEN INTEGER is NP-complete. ■

In INTEGER PROGRAMMING, we require integer solutions, and this requirement makes the problem NP-complete. On the other hand, LINEAR PROGRAMMING, wherein we accept any real number solution, is a polynomial-time problem (Dobkin et al. [1976]), for which efficient algorithms have been devised (see Dantzig [1963], Khachiyan [1979], and Karmarkar [1984]). Since FAVOR JEPPESEN INTEGER is really INTEGER PROGRAMMING in disguise, if we accept any real number solution, we could easily adapt algorithms for LINEAR PROGRAMMING to solve a tractable version of FAVOR JEPPESEN INTEGER, which we shall call FAVOR JEPPESEN.

Problem 5.6. FAVOR JEPPESEN
INSTANCE:
A collection of finite string sets $\boldsymbol{JM}$ =
$\{ \{jepp_1, m_{11}, m_{12}, \ldots, m_{1m1} \}, \{jepp_2, m_{21}, m_{22}, \ldots, m_{2m2}\}, \ldots, \{jepp_n, m_{n1}, m_{n2}, \ldots, m_{nmn}\} \}$,
$jepp_i, m_{i1}, m_{i2}, \ldots \in$ String, $1 \le i \le n$;
a set of rules $\boldsymbol{R} = \{R_1, R_2, \ldots, R_r\}$, R_i: String$\to\mathbb{Z}$, $1 \le i \le r$.
QUESTION:
Is there a function w: $\boldsymbol{R}\to\mathbb{R}$ such that

$$\sum_{R\in\boldsymbol{R}} (w\,(R)\,R\,(jepp_k)) \ge \max_{km_k} \left\{ \sum_{R\in\boldsymbol{R}} (w\,(R)\,R\,(m_{km_k})) \right\},\ 1 \le k \le |\boldsymbol{JM}|,\ \sum_{R\in\boldsymbol{R}} w\,(R) > 0\ ?$$

5.1.2. Related Jeppesen Problems

In this section, we consider related problems, and indicate why they are also intractable if we require integer solutions. In the decision problems for the Jeppesen Experiment, we asked about the existence of a function that would evaluate Jeppesen's compositions favorably against all alternatives. What we really would like to know in this experiment is, of course, what that function is, if it exists. Better yet, if possible, we would like to know the function whose evaluation maximizes the scores of Jeppesen's compositions. To define the optimization problem that asks these questions, we proceed in the same way that led to the optimization problem related to PAL. As a preliminary to the optimization problem formulation, we define the following number problem (FAVOR JEPPESEN K).

Problem 5.7. FAVOR JEPPESEN K
INSTANCE:
A collection of finite string sets $\boldsymbol{JM}$ =
$\{ \{jepp_1, m_{11}, m_{12}, \ldots, m_{1m1} \}, \{jepp_2, m_{21}, m_{22}, \ldots, m_{2m2}\}, \ldots, \{jepp_n, m_{n1}, m_{n2}, \ldots, m_{nmn}\} \}$,
$jepp_i, m_{i1}, m_{i2}, \ldots \in$ String, $1 \le i \le n$;
a set of rules $\boldsymbol{R} = \{R_1, R_2, \ldots, R_r\}$, R_i: String$\rightarrow\mathbb{Z}$, $1 \le i \le r$;
a non-negative integer K.
QUESTION:
Is there a function w: $\boldsymbol{R}\rightarrow\mathbb{Z}$
such that

$$\sum_{R\in\boldsymbol{R}} (w(R)\, R(jepp_k)) \ge K + \max_{k\,m_k} \left\{ \sum_{R\in\boldsymbol{R}} (w(R)\, R(m_{k\,m_k})) \right\}, 1 \le k \le |\boldsymbol{JM}|, \sum_{R\in\boldsymbol{R}} w(R) > 0 \,?$$

The related optimization problem is:

Problem 5.8. FAVOR JEPPESEN OPTIMIZATION
INSTANCE:
A collection of finite string sets $\boldsymbol{JM}$ =
$\{ \{jepp_1, m_{11}, m_{12}, \ldots, m_{1m1} \}, \{jepp_2, m_{21}, m_{22}, \ldots, m_{2m2}\}, \ldots, \{jepp_n, m_{n1}, m_{n2}, \ldots, m_{nmn}\} \}$,
$jepp_i, m_{i1}, m_{i2}, \ldots \in$ String, $1 \le i \le n$;
a set of rules $\boldsymbol{R} = \{R_1, R_2, \ldots, R_r\}$, R_i: String$\rightarrow\mathbb{Z}$, $1 \le i \le r$.
QUESTION:
What is the function w: $\boldsymbol{R}\rightarrow\mathbb{Z}$ that maximizes

$$\sum_{R\in\boldsymbol{R}} (w(R)\, R(jepp_k)) - \max_{k\,m_k} \left\{ \sum_{R\in\boldsymbol{R}} (w(R)\, R(m_{k\,m_k})) \right\}, 1 \le k \le |\boldsymbol{JM}|, \sum_{R\in\boldsymbol{R}} w(R) > 0 \,?$$

Expectedly, both the number problem and the optimization problem are NP-hard, i.e., at least as hard as NP-complete problems. Additionally, if FAVOR JEPPESEN K were in NP, it would be NP-complete in the strong sense.

Theorem 5.3. FAVOR JEPPESEN K is NP-hard.

Proof: Restrict to FAVOR JEPPESEN INTEGER by allowing only instances in which $K = 0$. ■

Theorem 5.4. FAVOR JEPPESEN OPTIMIZATION is NP-hard.

Proof: The Turing reduction from the decision problem FAVOR JEPPESEN K to the optimization problem FAVOR JEPPESEN OPTIMIZATION is straightforward. The solution to the optimization problem tells us the maximum possible value of K (call it K_{opt}) in the decision problem, and the answer to the decision problem depends on the actual value of K in a given instance (call it K'): "yes" if $K' \le K_{opt}$, "no" otherwise. ■

In the foregoing problem descriptions, we have made the simplifying assumption that Jeppesen's preferences can be expressed accurately enough as linear inequalities. That this assumption may not hold is the motivation behind the second method, described next.

5.2. Training an Artificial Neural Network to Predict Human Evaluation of Melodies

In this second method, we train a backpropagation neural network to predict human evaluations of counterpoints. We present a listener with a fairly comprehensive set of melody-countermelody combinations (henceforth referred to as *counterpoint/s*) and ask him/her to evaluate them as consistently as possible. The listener's evaluations become part of the training data for an artificial neural network. The goal is to be able to use the output of the trained network to rank, as the listener might, other test melodies.

To gather network data and to train the network, we used the procedure described in algorithm 5.2.

1. A human expert ("Expert 1") composes a set of *m* cantus firmi.
2. **for** each of these *m* cantus firmi
   Generate a set of "correct" countermelodies
   ("Palestrina 1", using the generate-and-test method: see chapter 6, section 6.1)
   and their respective violations of each artistry rule ("Palestrina 2").
3. Randomly select *n* examples from the countermelodies generated.
   Randomize the order of these *n* examples.
   for each of the *n* examples
   Another human expert ("Expert 2") evaluates by giving a numerical grade ∈ [0..1].
   Classify the numerical grades under various letter-grading schemes:
   e.g., melodies with scores between 0.9 and 1.0 receive an "A," 0.8 and 0.89 a "B," etc.
4. Randomly partition the resulting set of counterpoints, rule violations, and Expert 2's grades into disjoint training and validation sets.
   Train a backpropagation neural network application with this data;
   the network learns the function *f*: (counterpoints x rule violations)→grades.
   Prevent overfitting of the training data by means of a common cross-validation technique.

Algorithm 5.2. Procedure to train an artificial neural network to reflect human preferences

The above procedure is illustrated in figure 5.1. See appendix C, section C.1 for the training and validation data used in the above experiment.

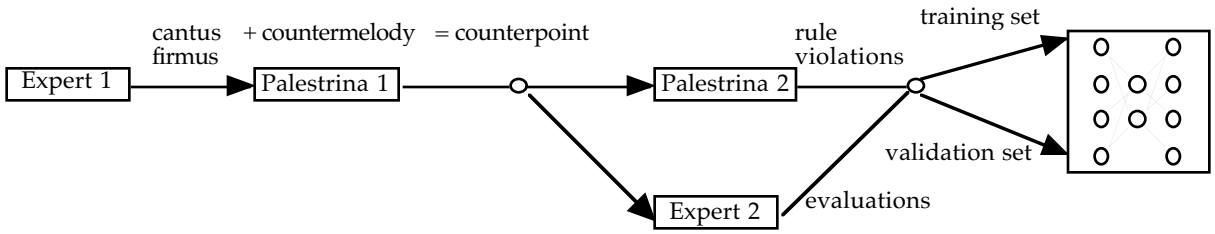


Fig. 5.1. System to produce counterpoint and to train an artificial neural network to predict an expert's evaluation of new counterpoints

In our experiment we began with $m = 10$ cantus firmi, two for each of five *modes*, or scale patterns that are the basis of the musical style we are trying to emulate. In step (3) we randomly chose $n = 100$ examples, 10 per cantus firmus. We then partitioned these 100 examples into 80 training and 20 validation examples with various letter-grading schemes.

The experiment yielded the following results with respect to the validation set (table 5.1). In (a) we trained the artificial neural network separately for each cantus firmus, and input the diagnoses of four artistry rules; in (b) we trained the network once for each grading scheme, and provided as input additional information on the cantus firmus and on the countermelody. The first grading scheme (0, 0.5, 1.0) corresponds to a "pass-fail" grading; grading schemes 2, 3, and 5 used arbitrary partitions into five classes (the cutoff points are shown); grading schemes 4, 6, and 7 used partitions into five classes produced by Kohonen mapping (Kohonen 1982, 1984), fuzzy c-means (Bezdek 1981), and histogram equalizer algorithms, respectively; finally, grading scheme 8 used partitions into ten classes (cutoff points 0, 0.1, 0.2, … , 0.9). Each unit difference between the expert's rating and the artificial neural network output was counted as a unit error, and the total error was the sum of these errors over the 20 exemplars; the average error was the total error divided by the number of validation exemplars (20); the percentage error was the total error divided by the maximum possible error over the 20 exemplars. Note that a grading scheme may have a lower total (and average) error than another scheme, but have a higher percentage error, as shown in the first two schemes. It will be observed that the separately trained networks, in pooling their results, as shown in (a), fared slightly better on the whole than networks trained with input information on the cantus firmus and countermelodies, but trained once for all cantus firmi, as shown in (b).

With separately trained networks, grading schemes 1, 2, 3, and 8 resulted in better than 75% prediction accuracy (using the percentage error reckoning). In all cases, grading schemes 5–7 resulted in fairly equal distribution of grades, and apparently higher error. Nevertheless, results are as yet inconclusive, and we have yet to determine which grading scheme, if any, would yield reliable predictive performance by the artificial neural network. See appendix C, section C.3 for detailed comparisons between the expert's and the neural network's outputs within these various grading schemes. The architectures of the artificial neural network are illustrated in figures 5.2 and 5.3.

Table 5.1. Artificial neural network prediction of expert ratings based on various grading schemes

Grading scheme	(a) Total	(a) Error Avg	(a) %	(b) Total	(b) Error Avg	(b) %
1. 0, 0.50, 1.0	4	0.20	0.20	6	0.30	0.30
2. 0, 0.20, 0.40, 0.60, 0.80, 1.0	13	0.65	0.24	16	0.80	0.30
3. 0, 0.50, 0.70, 0.80, 0.90, 1.0	15	0.75	0.24	17	0.85	0.27
4. 0, 0.40, 0.55, 0.67, 0.82, 1.0	22	1.10	0.40	22	1.10	0.40
5. 0, 0.40, 0.50, 0.60, 0.70, 1.0	25	1.25	0.41	25	1.25	0.41
6. 0, 0.38, 0.50, 0.60, 0.75, 1.0	25	1.25	0.41	23	1.15	0.38
7. 0, 0.42, 0.50, 0.57, 0.69, 1.0	26	1.30	0.40	22	1.10	0.34
8. 0, 0.1, 0.2, 0.3, … , 0.9, 1.0	25	1.25	0.21	30	1.50	0.25
Average	19.38	0.97	0.32	20.13	1.01	0.32

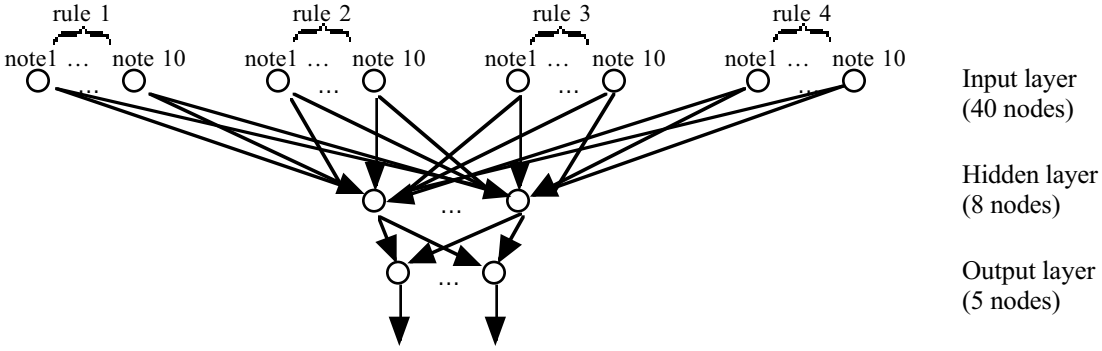


Fig. 5.2. Artificial neural network architecture to learn human musical preferences. The input layer consists of 40 nodes, 10 per rule; each node represents the error detection by the rule of a single note of the ten-note countermelody. The output layer here shows a 1-of-5 encoding of the expert's grade.

In combining all cantus firmi into a single training and validation data set, we additionally provided the neural network input information on the respective cantus firmus and countermelody, again using 1-of-*n* encoding. This resulted in an additional 20 input nodes (10 for the cantus firmus, and 10 for the countermelody, see figure 5.3). The notes are indicated as a fraction dependent upon its position in a 58-note gamut. Thus, e.g., the 29th note of the gamut is indicated by the value of 29/58 = 0.5. The use of 1-of-10 grade partitioning had the effect of truncating the expert's score; thus, the expert's score of 0.85 was considered to be 0.8, and a score of 0.78 was considered to be 0.7, etc.

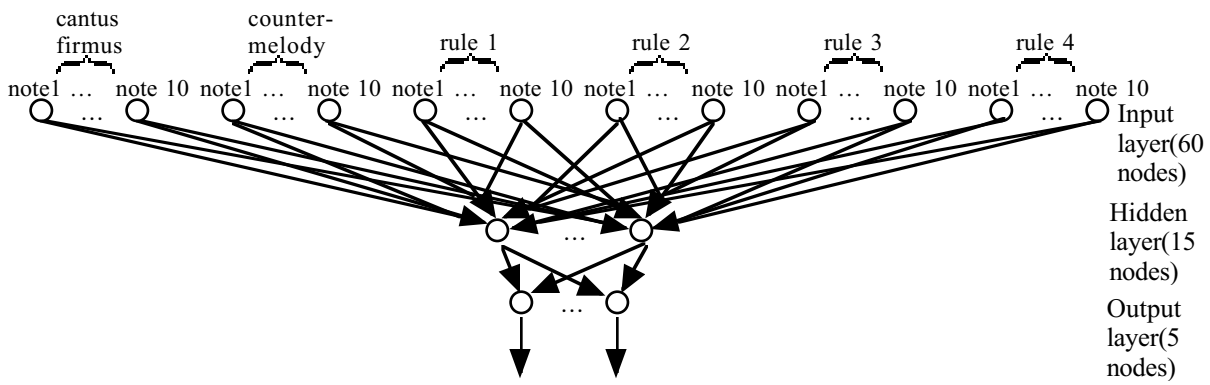


Fig. 5.3. Artificial neural network with input information on cantus firmus and countermelody. The output layer here shows a 1-of-5 encoding of the expert's grade.

Example 5.1. The following shows (a) a training exemplar represented as (b) a string of values in [0, 1] (here spaced across several lines for clarity) for neural net input and output nodes. In (b), the first row is the error detection by the Tonality, Final in Leaps of Fourths and Fifths, False Relations, and Note Variety rules, represented as 10-bit vectors; e.g., the first two rules detected no errors, but the False Relations rule detected an error in the sixth and eighth notes of the countermelody. The next two rows show the cantus firmus and the countermelody , and the final row shows the expert's score in 1-of-10 representation.

(a)
countermelody / cantus firmus expert's score
D E D E G F E F a G 0.65
d c h c h a h a F# G

(b)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1 1 0 1 1 1 0 0 0 0
0.534483 0.482759 0.465517 0.482759 0.465517 0.413793 0.465517 0.413793 0.344828 0.362069
0.241379 0.293103 0.241379 0.293103 0.362069 0.310345 0.293103 0.310345 0.413793 0.362069
0 0 0 0 0 0 1 0 0 0

■

We have also trained the neural network using partially filled countermelodies (more on this latter topic in section 5.2.1). See appendix C, section C.2 for the input data formatted using partially filled countermelodies.

It should be noted that the function computed by the artificial neural network is implicit: there is no way to derive the actual weights (i.e., w_R in formula [2]), or even to determine the nature, linear or otherwise, of the computation of the overall merit of a countermelody. It is also not possible to analyze precisely the complexity of this method.

All told, training an artificial neural network is more an art than a science. We can suggest several training variations: train the network on a more restricted rule set by asking the expert to

evaluate countermelodies based only on given criteria (note variety, smoothness of melody, etc.); restrict training sets to one or two different cantus firmi only; train using non-experts whose discriminatory powers may not be as acute as those of an expert's but may show decided preferences nevertheless. Regarding the latter, experiments on non-experts suggest, for example, that certain individuals evaluate exemplars based solely on *tessitura* (average pitch of a melody): one subject manifestly preferred low-lying melodies, while in contrast another individual preferred just the opposite.

Despite their unique difficulties, artificial neural networks have a singular advantage in that they are robust mechanisms in the face of noisy or contradictory data—quite appropriate for use in our situation, as music experts are notoriously inconsistent. For example, music experts often rate two very similar melodies very differently, and this poses a problem if the available rules are not able to distinguish the two melodies; to differentiate exemplars as much as possible, we have suggested inputting not only the rule evaluations, but also information on the cantus firmus and countermelody; this results in a more comprehensive feature set.

5.2.1. Non-Increasing Scores of Partial Countermelodies

We intend to use the evaluations of a trained network in algorithms to produce countermelodies, using the best-first and genetic algorithms (chapter 6). If using the neural network for countermelody evaluations, both algorithms naturally expect the network to evaluate complete countermelodies; the best-first algorithm, in addition, requires the network to evaluate partial countermelodies (see algorithm 6.2). Since the original training data was based on the expert's evaluations of complete countermelodies, we need to extend the training data set to give the network the ability to evaluate partial countermelodies as well.

Here is the method we used to prepare partial countermelodies and their appropriate scores: for each exemplar, based on a single n-note cantus firmus, create n-1 exemplars comprising the same expert rating together with the partial melodies consisting of the [nth], [(n-1)th, n th], [(n-2)th, (n-1)th, and nth], … , [2nd, 3rd, … , and nth] notes of the original melody, respectively. We use this enhanced training set to train the network to output the same grade for any partial countermelody as for the complete countermelody. See appendix C, section C.2 for an example of a data set configured this way (using only training and validation examples in the mixolydian mode).

Such a trained network can then be used in the best-first branch-and-bound algorithm without contradiction of one of the algorithm's requirements: that the predicted score should be non-increasing as we descend a branch on the search tree. This is because partial countermelodies tend to have fewer rule violations, and hence a higher score, than would a longer countermelody lower down in the same branch of the search tree. As appendix E, section E.3 shows, a network trained with partial countermelodies clearly produces more consistent evaluations with respect to partial countermelodies than does one trained only on complete countermelodies.

5.3. Summary

In order to be convincingly human-like, music produced algorithmically must reflect human preferences. We describe two methods to achieve this. Performing the Jeppesen Experiment, we weight a set of selected fuzzy rules to reflect an expert's value system; adopting a machine learning approach, we train an artificial neural network by an expert's grading of a set of countermelodies, using the diagnosis of artistry rules. Certain aspects of the Jeppesen Experiment involves yet another set of intractable problems, shown to be such by reduction from INTEGER PROGRAMMING and KNAPSACK.

In artificial neural network training, partitioning of the grades according to various grading schemes affects prediction accuracy. Non-increasing grades of partial countermelodies, with respect to the length of these countermelodies, is also a vital issue if the network's output is to be used in the best-first branch-and-bound algorithm, described in the next chapter.

CHAPTER 6. HEURISTIC ALGORITHMS BASED ON FUZZY RULE APPLICATION

When faced with difficult problems like those described in chapter 4, we have four options. The first option is simply to avoid such problems entirely. The second is to solve easier versions of these problems: this we have done by identifying those versions that are solvable efficiently by finite state transducers (chapter 3). The third is to attempt only small instances of the intractable problem: solve for short cantus firmi, for which even exhaustive search does not take intolerably long a time to complete. The fourth is to find approximate algorithms and settle for near-optimal countermelodies. We describe here two heuristic methods for the third and fourth options.

The first of our two heuristic methods applies a best-first search with branch-and-bound pruning. It improves upon a straightforward "generate-and-test" exhaustive tree search with pruning, which is the "brute-force" approach that serves as a benchmark for proposed improvements in all search algorithms. The second method applies a genetic algorithm to obtain approximately optimal results. This method adapts an algorithm based on biological evolutionary principles that has been used successfully in many optimization problems. Although we originally designed our methods to solve the optimization problem of composing counterpoint, they can easily be adapted to solve the other three problems we have identified, viz., decision, enumeration, and number.

Both heuristic methods use fuzzy rule application (chapter 2, section 2.3). We can choose among using either the weighted average of all fuzzy rules (applying formula[2.3], with all rules equally weighted, with weights individually adjusted, or determined by means of the method described in the Jeppesen Experiment), or the output of a trained artificial neural network. For the implementation of these methods, see appendix D, sections D.2–3.

6.1. Best-First Search with Branch-and-Bound Pruning Algorithm

The best-first search with branch-and-bound pruning algorithm is an improvement over an exhaustive search algorithm. Exhaustive algorithms are rarely practical for all but small instances. Yet, they serve usefully as a benchmark against which other, possibly more efficient, algorithms can be measured. We therefore first describe an exhaustive algorithm for the species counterpoint problem.

Our version of the exhaustive algorithm performs a tree search method with pruning (the *generate-and-test* method) and incorporates the notion of crisp and fuzzy rule application. We apply the set of crisp rules to every note that is considered in the construction of a countermelody. At any point in the construction of this countermelody, as soon as a crisp rule is broken, we abandon that (partial)

countermelody, and backtrack to consider another countermelody. In this way, we avoid exploring many futile branches. If, however, a fuzzy rule is broken, we keep a tally on the frequency of violation of that rule. A countermelody that does not break any crisp rules emerges with a score depending on the degree it satisfies all the fuzzy rules. We may thus rank all "correct" countermelodies. We can also express the merit of a countermelody by an interval of values representing a "fuzzy" score; in this case we can adopt the *preference ranking* method based on a fuzzy measure of "leftness" relations (Kundu 1997). For the derivation of such values, and the application of preference ranking, see Gwee (1998).

Algorithm 6.1 shows a recursive implementation of the generate-and-test algorithm. See appendix D, section D.2.1 for a more efficient iterative implementation of this algorithm.

Algorithm GenerateAndTest
Input: finite set of notes *S*, cantus firmus *CF*.
Output: list of melodies *L*.
1. **for** each note $n \in S$
2. Extend m by adding n as a prefix
3. **if** m does not satisfy all the rules
Remove n from m
else
if m is a complete melody
Add a copy of m to L
Remove n from m
else
GenerateAndTest (*S*, *CF*).

Algorithm 6.1. Generate-and-test algorithm to produce valid countermelodies (*L*) to cantus firmus *CF*

Example 6.1. Figure 6.1 shows the search tree for the generation of countermelodies to cantus firmus <D, E, C#, D>. Numbers above each node indicate the depth-first order in which the node was created. The algorithm generates countermelodies in reverse and paths starting from leaves at the lowest level to the topmost level nodes represent "correct" countermelodies. In this example, the algorithm found five correct countermelodies: <D, C, E, D>, <a, E, E, D>, <a, G, E, D>, <D, E, e, d>, and <d, c, e, d>. Numbers in leaf nodes of correct countermelodies indicate the score of that countermelody. Thus, the optimal countermelodies are <D, C, E, D> and <d, c, e, d>, with a score of 0.97. ■

Pruning in the manner illustrated in example 6.1 enables us to abandon fruitless searches on extensions of any faulty substrings. Besides pruning, we have found other means of improving upon the straightforward exhaustive search. We may order the rules to apply the more selective rules before the less selective ones: since once any rule is broken in the construction of a countermelody, no further application of other rules or further exploration of the countermelody is necessary. For example, it may be expedient to allow such a (computationally) simple rule as the Dissonant Harmonic Intervals Rule to

evaluate a proposed countermelody before most other rules, owing to its high selectivity (see chapter 2, section 2.1.1). Such ordering of the rules, while cutting down on extraneous rule application, will not improve the efficiency of the search algorithm by any order of magnitude. For this to happen, we still need to be able to prune the search tree more extensively and preferably at higher levels.

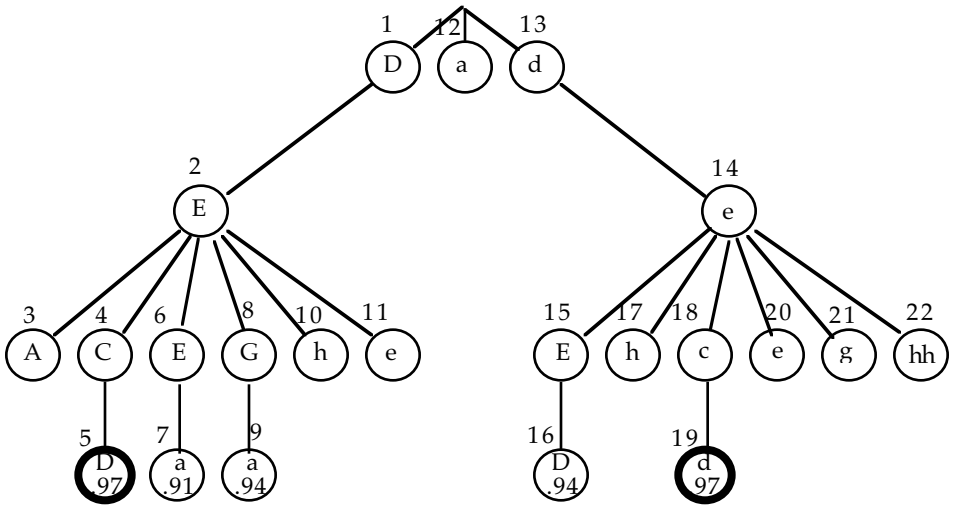


Fig. 6.1. Search tree for the generation of countermelodies to the cantus firmus <D, E, C#, D>. Paths starting from leaves at the lowest level to the topmost level nodes represent "correct" countermelodies. There are five correct countermelodies, and two optimal countermelodies: <D, C, E, D> and <d, c, e, d>. The two optimal countermelodies have a score of 0.97. Nodes are labeled 1, 2 … 19 in the depth-first order.

We can facilitate such earlier pruning by constructing countermelodies in reverse: the cadence rules (Final Cadence Rule and Approach to Cadence Rule) are among the most selective of the rules and can eliminate many branches very early on; Lewin's Rule (Lewin 1983), another relatively selective rule, can be most effective if applied to a reversed countermelody (example 6.2).

Example 6.2. Consider the following (grammatically incorrect) melody:

a F G a b a G **F** **D**

The error lies in the last two notes of the melody: the cadence rules and Lewin's Rule require that the penultimate note be a step away from the final note (here, only E or C# would be valid). If this melody were built in the normal order (i.e., successively a, F, G, etc.), very many alternatives will be considered correct, until the penultimate note is reached. In this case, all alternatives with F and D as the final pair of notes would ultimately be rejected. On the other hand, if this melody were built in reverse (i.e., successively D, F, etc.), no branch extending beyond the sequence D–F would be admitted. Here in fact only branches extending beyond D–E and D–C# would be considered. ■

In sum, if a melody were built front to back (as illustrated in example 6.2), very many alternatives present themselves at the beginning, many of which will ultimately be rejected; if the melody were

built back to front, very few alternatives are correct at the outset, and fewer fruitless branches will be explored.

The difference in the number of nodes visited during the search between a forward and a reverse scanning order is drastically shown in example 6.3, which illustrates the benefits in search efficiency of early pruning of the search tree brought about by searching a reverse melody with application of rules that are extremely selective on the ending portion of the melody.

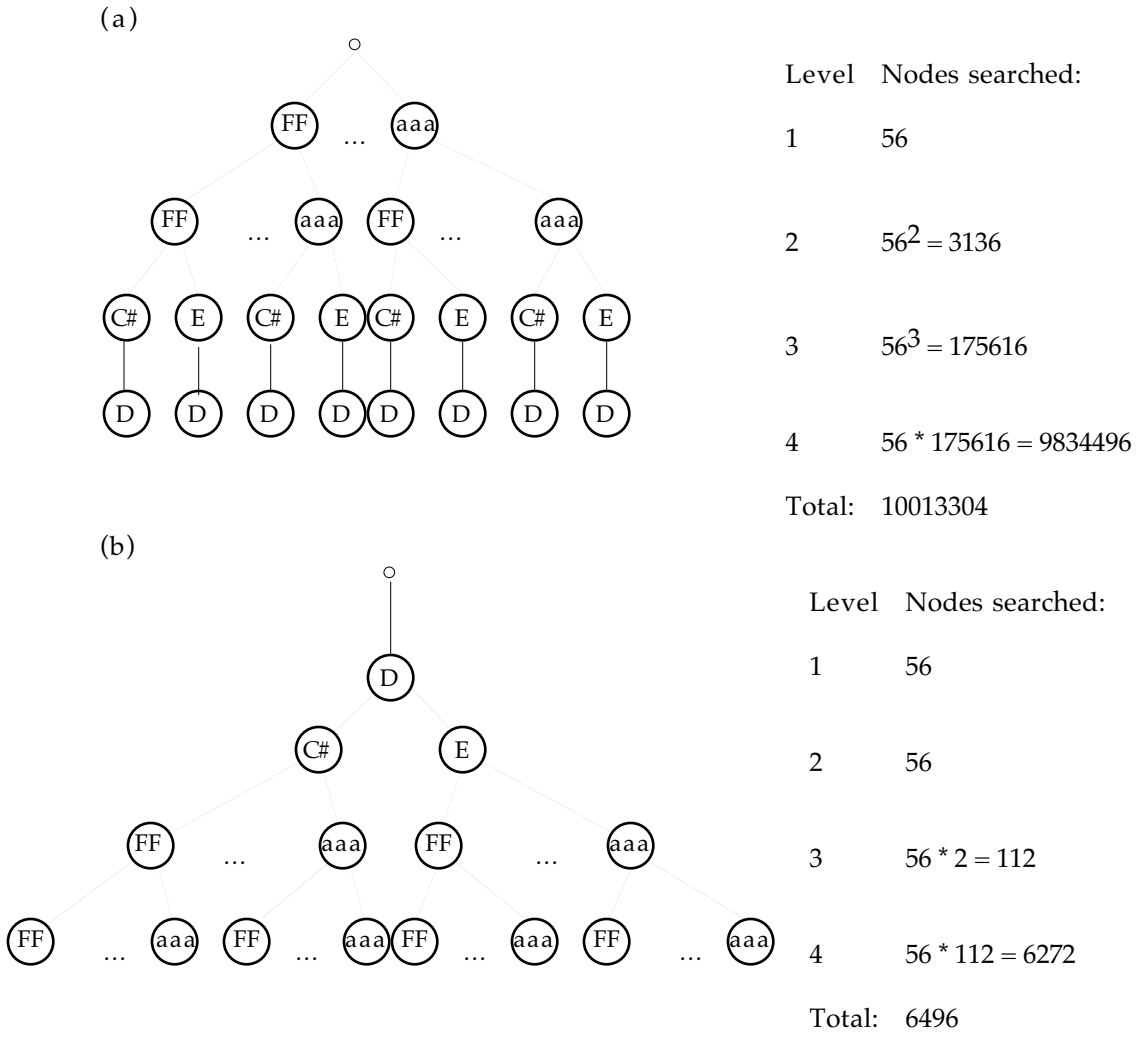


Fig. 6.2. Comparison of the number of nodes visited during the search of a four-note countermelody with a gamut of 56 notes, given the application of only the cadence rules in (a) forward order; (b) reverse order

Example 6.3. Figure 6.2 illustrates the number of nodes searched in forward and reverse order, respectively. A forward order search examined 10013304 nodes, while a reverse order search examined only 6496. ■

The best-first search algorithm improves upon the exhaustive search by selectively exploring branches of the search tree (algorithm 6.2). Unlike the exhaustive search, which systematically, but blindly, searches branches in a set order (e.g., leftmost- and depth-first), the best-first search tries its best to avoid fruitless explorations by applying fuzzy rule heuristics. Use of these heuristics also enables the user to abbreviate the search by specifying a threshold fuzzy rating, or to ask for only the *m* best melodies.

```
Algorithm BestFirst
    Input: finite set of notes S , cantus firmus CF.
    Output: optimal_melody.
    1. for each note n ∈ S
        Create partial melody m by making n its first note
        if bound (m, CF) > 0                          // bound computes the merit of m w.r.t. CF,
            node := <m , bound (m, CF) >              // and returns 0 if m is ungrammatical
            Enqueue node into priority queue PQ       // priority based on bound (m, CF ) and | m |
    2. found := false
    3. optimal_melody := blank
    4. while not found and PQ is not empty
        Dequeue node from PQ // priority queue ensures the best is chosen first
        if | node .m | < | CF |
            for each note n in S
                Create partial melody m' from node . m extended by addition of n
                if bound (m', CF) > 0
                    node2 := < m', bound (m', CF) >
                    Enqueue node2 into PQ
        else
            optimal_melody := node .m
            found := true.
```

Algorithm 6.2. Best-first algorithm to produce an optimal countermelody to cantus firmus *CF*

The algorithm performs a best-first search and places a bound on each node by calculating the fuzzy score of the partial countermelody (see figure 6.3). The bounds computed are such that they are non-increasing with the level of the nodes, and are therefore upper bounds. In case of a tie in the bounds between nodes in the same level, the lexicographical ordering prevails, and between nodes at different levels, the node at the lower level is chosen for expansion. The rationale for the latter criterion is that melodies nearer completion stand a better chance of obtaining the best ultimate score.

Example 6.4. The pruned search tree produced for the cantus firmus <D, E, C#, D> is shown in figure 6.3 (compare the search tree traversed by the exhaustive algorithm for the same cantus firmus, figure 6.1). Stored in each node are the musical note and the bound, with only nodes of non-zero bounds shown; as with figure 6.1, the numbers above each node indicate the order in which the nodes are created. This algorithm in general explores fewer nodes than does the exhaustive algorithm: in this instance, nodes in lighter shade were not explored. The optimal countermelodies are: <D, C, E, D> and

<d, c, e, d>. For output of the best-first and genetic algorithms to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>, using rules weighted to favor Jeppesen's preferences, see appendix B (output for *num blanks* = 11) and appendix E, section E.1. ■

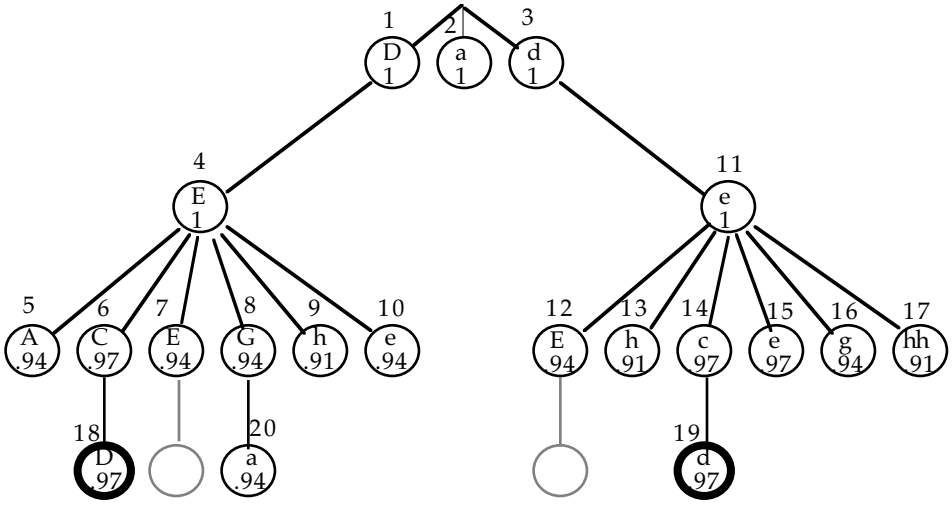


Fig. 6.3. Pruned search tree produced by the branch-and-bound algorithm for cantus firmus <D, E, C#, D>. Paths starting from leaves at the lowest level to the topmost level nodes represent "correct" melodies. Optimal countermelodies here are <D, C, E, D> and <d, c, e, d>.

To evaluate complete countermelodies and to calculate the bounds of partial countermelodies, we may use the output of an artificial neural network trained to reflect an expert's preferences. As we pointed out in chapter 5, section 5.2.1, if we do so, we need to preserve the integrity of the best-first search algorithm: it is crucial that the artificial neural network produce non-increasing scores with respect to any partial countermelody when descending a particular branch of the search tree. We have attempted to train networks to produce consistent scores in this respect (see appendices C and E, sections C.2 and E.3). Despite our best intentions in training the neural network with partial melodies, however, it turned out that its evaluations were not always non-increasing. In our implementation of the best-first algorithm, we had to allow for this, and so directed the algorithm to keep open the possibility that a more recent complete melody may actually have a higher score than an earlier one; if evaluation scores were absolutely non-increasing, the algorithm could assume that no more recently completed melody could ever be better than previous ones. See appendix E, section E.3.1 for an instance where a neural network not producing non-increasing output misleads the best-first algorithm into outputting suboptimal countermelodies ahead of some optimal ones, subverting the mechanism behind its main advantage over an exhaustive search. On the other hand, a neural network trained on data

prepared as described in chapter 5, section 5.2.1 (almost) succeeds in producing non-increasing output with respect to the partial countermelodies; the best-first algorithm that uses such a neural network runs (almost) as expected, with (most) optimal countermelodies output before suboptimal ones (appendix E, section E.3.2).

Example 6.5. The best-first search algorithm produced the following optimal countermelodies to the cantus firmus <d, c, h, c, h, a, h, a, F#, G>. The evaluation function used a neural network trained on the same cantus firmus. The grading scale was based on the 0–0.2–0.4–0.6–0.8–1.0 partitioning, which gave about the best predictive ability (see chapter 5, section 5.2). The third countermelody (<D, E, G, F, G, F, E, F, a, G>) actually also belonged to the validation set used in the training of the network; using the current grading scale, the expert gave the same grade to this countermelody. The absence of an "A"-grade countermelody might be due to the fact that the highest grade obtained for all training data corresponding to this cantus firmus was only a "B".

	Countermelody										**Grade**
1.	D	E	G	E	G	F	E	F	a	G	"B"
2.	D	F	G	E	G	F	E	F	a	G	"B"
3.	D	E	G	F	G	F	E	F	a	G	"B"
4.	D	F	G	F	G	F	E	F	a	G	"B"
5.	D	E	G	E	G	F	G	F	a	G	"B"
6.	G	E	G	E	G	F	G	F	a	G	"B"
7.	D	F	G	E	G	F	G	F	a	G	"B"
8.	d	e	g	e	g	f	g	f	aa	g	"B"

■

6.2. Genetic Algorithm

Both the exhaustive and branch-and-bound algorithms conduct thorough and deterministic searches. The common enemy of both these algorithms is the combinatorial explosion of the search space size; their dedication to thoroughness condemns both algorithms to long and possibly fruitless searches. Using a completely different search method, genetic algorithms model evolutionary processes to search equally large spaces in less thorough but more expedient ways. In part randomized, these algorithms use historical information to set up favorable conditions for promising solutions to manifest themselves.

In its most common implementation, a genetic algorithm maintains a set of strings each encoding a candidate solution to the problem (in genetic algorithm terminology, a *population* of *chromosomes*). Each candidate solution is assessed according to some fitness criterion. Through an iterative process, solutions that are "more fit" stand a better chance of preservation than those that are "less fit"; fitter solutions also stand a better chance of participating in the creation of new candidate solutions (in biological systems, these correspond to *survival* and *reproduction*). It is hoped that in the course of many iterations (*epochs*), a steadily improving set of solutions emerges, ultimately resulting in an optimal or near-optimal solution.

We have developed a system using the genetic algorithm to produce approximately optimal counterpoints to given cantus firmi. The chromosomes represent the candidate melodies and the genes are the various notes themselves: thus, the four-gene chromosome "D C E D" represents the melody <D, C, E, D>. In our implementation, we have used the technique of fitness scaling (Goldberg 1989, 76) to avoid premature convergence (see appendix D, algorithm D.5).

The basic genetic algorithm randomly generates an initial population. If all possible encodings are valid, then all chromosomes produced are also valid. In the case of species counterpoint, however, valid chromosomes (corresponding to countermelodies) are not necessarily musically valid: in the majority of cases, the countermelody would violate one or more crisp rules. By our system of melodic evaluation, such randomly produced chromosomes would form initial populations with very low, probably zero, average fitness. Subsequent epochs would result in no better than random string generation. In order to give the algorithm a more promising start, we introduce domain specific modifications not just to ensure valid strings, but also to create highly diversified initial populations with relatively high average fitness (as far as reasonably possible). We do so by allowing several initialization attempts for each chromosome, with the hope of producing at least some chromosomes with non-zero fitness. With initial non-zero fitness populations, we then have the option of retaining optimal melodies in future generations using the elitist model, or we may store such optimal melodies so that, even if they do not survive, we ultimately will have all optimal melodies that have been created during the execution of the algorithm. The sample outputs of the genetic algorithm in appendix E, sections E.1–2 all show non-zero fitness initial populations.

As we had for the best-first algorithm we have here several options for evaluation functions. Burton and Vladimirova (1999) classify music composition genetic algorithm systems by the evaluation functions they use: deterministic mathematical functions (Horner and Goldberg 1991; Ralley 1995); user-determined (Biles 1994; Horowitz 1994); formalistic, i.e., rule-based (McIntyre 1994; Horner and Ayers 1995; Thywissen 1996); and neural network-based (Gibson and Byrne 1991).

We have used evaluation functions characterizing the latter two classifications listed above, and our functions are based on methods described in chapter 5. The rule-based evaluation functions use fuzzy rules weighted to favor Jeppesen's choice for the cantus firmus <D, F, E, D, G, F, a, G, F, E, D> (see chapter 5, section 5.1). Two runs of the genetic algorithm succeeded in producing the optimal

countermelody: <a, aa, g, f, e, d, c, h, d, c#, d> (this was also Jeppesen's choice). The first run took seven iterations, while the second took only three (appendix E, section E.1).

Example 6.6. We illustrate below how crossover and selection operations produced the optimal countermelody <a, aa, g, f, e, d, c, h, d, c#, d> in the second run of the genetic algorithm to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D> (only the ancestors to the optimal countermelody are shown here):

Generation	Countermelody	Fitness	Operation	Ancestry
Initial				
0.	a aa g f e f e h d c# d	0.87	INITIAL	
4.	aa d g f e f e h d c# d	0.84	INITIAL	
11.	d a e f e d c h d c# d	0.80	INITIAL	
14.	d c e f e d c h d c# d	0.78	INITIAL	
18.	a a h d e d c h d c# d	0.76	INITIAL	
19.	d d c d e d c h d c# d	0.75	INITIAL	
1.				
0.	a aa g f e f e h d c# d	0.87	CROSSOVER	0 4
2.	aa d g f e f e h d c# d	0.84	SELECTION	4
11.	d a e f e d c h d c# d	0.80	CROSSOVER	19 11
14.	d c e f e d c h d c# d	0.78	CROSSOVER	14 18
2.				
1.	a aa g f e f e h d c# d	0.87	CROSSOVER	2 0
17.	d c e f e d c h d c# d	0.78	CROSSOVER	14 11
3.				
0.	a aa g f e d c h d c# d	0.89	CROSSOVER	17 1

■

We may also make arbitrary weight assignments. In this latter case, we could give equal weights to all but perhaps one of the fuzzy rules to see if by giving emphasis to that one rule, whether the set of optimal melodies could change. Not unexpectedly, this was found to be so (example 6.7).

Example 6.7. We compare the outputs of the best-first search and genetic algorithms to cantus firmus <D, F, E, D, G, F, a, G, F, E, D> (Jeppesen 1931, 112). The best-first algorithm produced the same optimal countermelodies to satisfy (a) rules equally weighted, and (b) rules weighted to give preference to the Stepwise Motion Rule, but a different optimal countermelody when (c) the Note Variety Rule was given preference. In comparison, the genetic algorithm quite quickly produced the same optimal countermelodies with equally weighted rules (further illustrated in figure 6.4). The grade for each countermelody is the weighted average of the individual rule scores.

Best-first algorithm

(a) Equally weighted fuzzy rules

	Countermelody	Grade
1.	aa aa g f e d c e d c# d	0.889091
2.	a a G F E D C E D C# D	0.889091
3.	a a G F E D C B D C# D	0.889091
4.	aa aa g f e d c h d c# d	0.889091

(b) Preference for Stepwise Motion Rule

```
1. aa aa g  f  e  d  c  e  d  c# d  0.893182
2. a  a  G  F  E  D  C  E  D  C# D  0.893182
3. a  a  G  F  E  D  C  B  D  C# D  0.893182
4. aa aa g  f  e  d  c  h  d  c# d  0.893182
```

(c) Preference for Note Variety Rule

```
1. a  aa g  f  e  d  c  h  d  c# d  0.861364
```

Genetic algorithm

```
crossover rate: 0.6
mutation rate: 0.1
max num initialization attempts: 20
max num crossover attempts: 20
max num mutation attempts: 20
fitness multiple: 1.2
population size: 100
max num iterations: 50
```

Countermelody	Grade	Iteration
1. a aa G F E D C E D C# D	0.89	Initial population
2. a a G F E D C B D C# D	0.89	Initial population
3. aa aa g f e d c h d c# d	0.89	2
4. aa aa g f e d c e d c# d	0.89	3

■

It is not surprising, however, that the genetic algorithm does not always produce optimal solutions to instances of the species counterpoint problem, or all the possible optimal solutions. Perhaps more typical are the results shown in example 6.8.

Example 6.8. The following is the output of the genetic algorithm to cantus firmus <D, F, E, D, G, F, a, G, F, E, D> (same cantus firmus as for example 6.7), evaluated by (a) equally weighted rules, rules weighted with preference for (b) Stepwise Motion, and for (c) Note Variety, respectively:

(a) Equally weighted fuzzy rules

Countermelody	Grade	Iteration
1. a aa g f e d c h d c# d	0.89	19

(b) Preference for Stepwise Motion Rule

1. d d e f e d c h d c# d	0.87	26

(c) Preference for Note Variety Rule

1. aa aa g f e a c h d c# d	0.85	8

■

The other evaluation function we used is based on a trained neural network that reflects the choice of one particular expert, and exploits the diagnostic feedback of fuzzy rules. In order to generate good rhythms, Gibson and Byrne (1991) rely on the proximity of candidate rhythms to "good" ones to guide their evaluations; but in order to generate good countermelodies, similarly relying on the

proximity of candidate countermelodies to "good" ones as an indication of fitness is extremely risky. See example 2.7 for an illustration of the difference a single semitone can make to a countermelody.

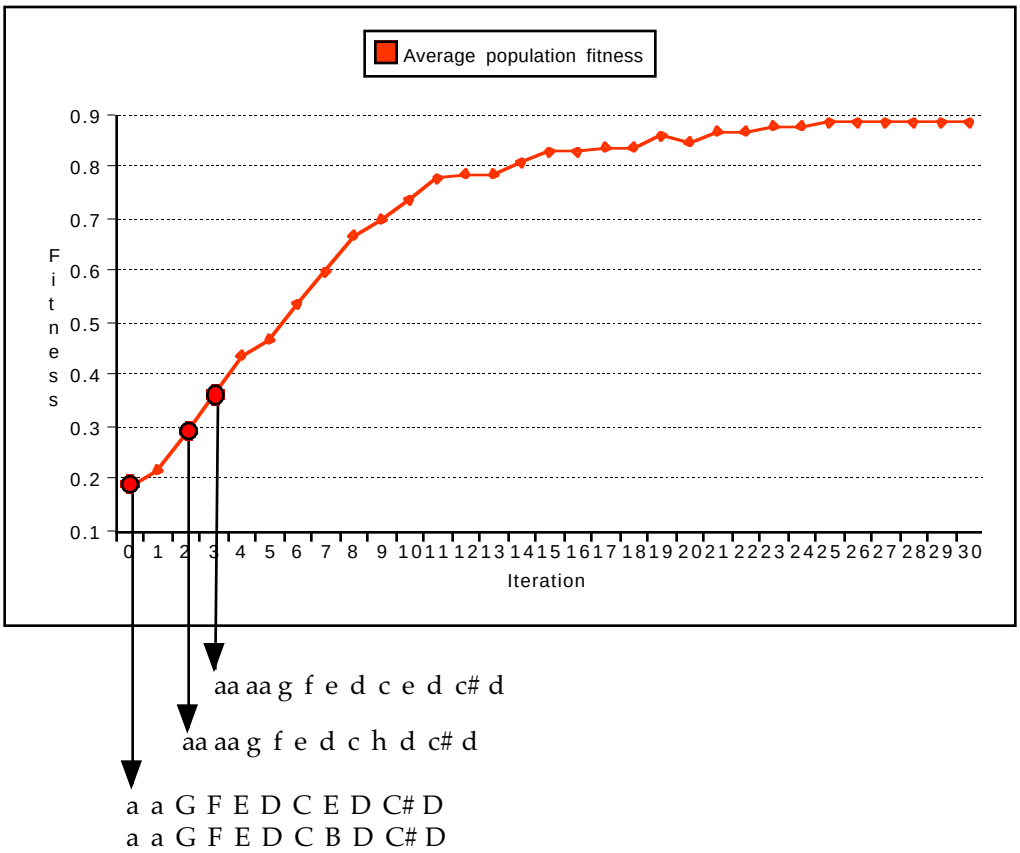


Fig. 6.4. Generation of optimal countermelodies by the genetic algorithm to cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

Example 6.9. The genetic algorithm produced the following optimal countermelodies to the cantus firmus <d, c, h, c, h, a, h, a, F#, G>, the same one for which the best-first algorithm produced the output shown in example 6.5. The evaluation function was carried out by the same trained neural network as for the best-first algorithm. For the initial population, each chromosome was allowed a maximum of 50 initialization attempts to produce a countermelody of non-zero fitness. With relatively short cantus firmi, this often results in several very fit chromosomes in the initial population: notice the presence of several optimal countermelodies in the initial population. Notice also the speed with which the set of eight optimal countermelodies was produced (fig. 6.5). This set is the same set as that produced by the best-first algorithm.

```
crossover rate: 0.6
mutation rate: 0.1
max num initialization attempts: 50
max num crossover attempts: 20
max num mutation attempts: 20
fitness multiple: 1.2
population size: 50
```

```
max num iterations: 50
elitist model, max num elites: 50
```

Initial population

chromosome	fitness
D E G E G F E F a G	0.75
d e g e g f g f aa g	0.75
D F G E G F G F a G	0.75
D F G F G F E F a G	0.75
D F G E G F G F a G	0.75
d e d e g f g f aa g	0.5
d e g e d e g f aa g	0.5
d e g e d e g f aa g	0.5
G C D E D F E F a G	0.5
d g d e d e g f aa g	0.5
D E G E D F E F a G	0.5
d e d e g f g f aa g	0.5
d e d e g f g f aa g	0.5
d e d e g f g f aa g	0.5
d g d e d e g f aa g	0.5
G E D E G F E F a G	0.5
G C D E G F G F a G	0.5
G C E C D F E F a G	0.5
d e g e d e g f aa g	0.5
d e d e d e g f aa g	0.5
D F D E D F E F a G	0.5
d g d e g f g f aa g	0.5
D F G E D F E F a G	0.5
d e g e d e g f aa g	0.5
d e g e d e g f aa g	0.5
d e d e d e g f aa g	0.5
d g d e d e g f aa g	0.5
G E G F G F E F a G	0.5
G E G F G F G F a G	0.25
G E D E G F G F a G	0.25
GG A B A GG A GG FF A GG	0.25
D E G F G F G F a G	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
D F D E G F G F a G	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG A B A GG A GG FF A GG	0.25
GG A B A GG A GG FF A GG	0.25
G E D E G F G F a G	0.25
GG A B A GG A GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
D E G F G F G F a G	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG A B A GG A GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25
GG A B A GG A GG FF A GG	0.25
GG FF GG FF GG FF GG FF A GG	0.25

Countermelody	Grade	Iteration
1. D E G E G F E F a G	"B"	Initial population
2. d e g e g f g f aa g	"B"	Initial population
3. D F G E G F G F a G	"B"	Initial population
4. D F G F G F E F a G	"B"	Initial population
5. D E G F G F E F a G	"B"	2
6. G E G E G F G F a G	"B"	2
7. D F G E G F E F a G	"B"	2
8. D E G E G F G F a G	"B"	7

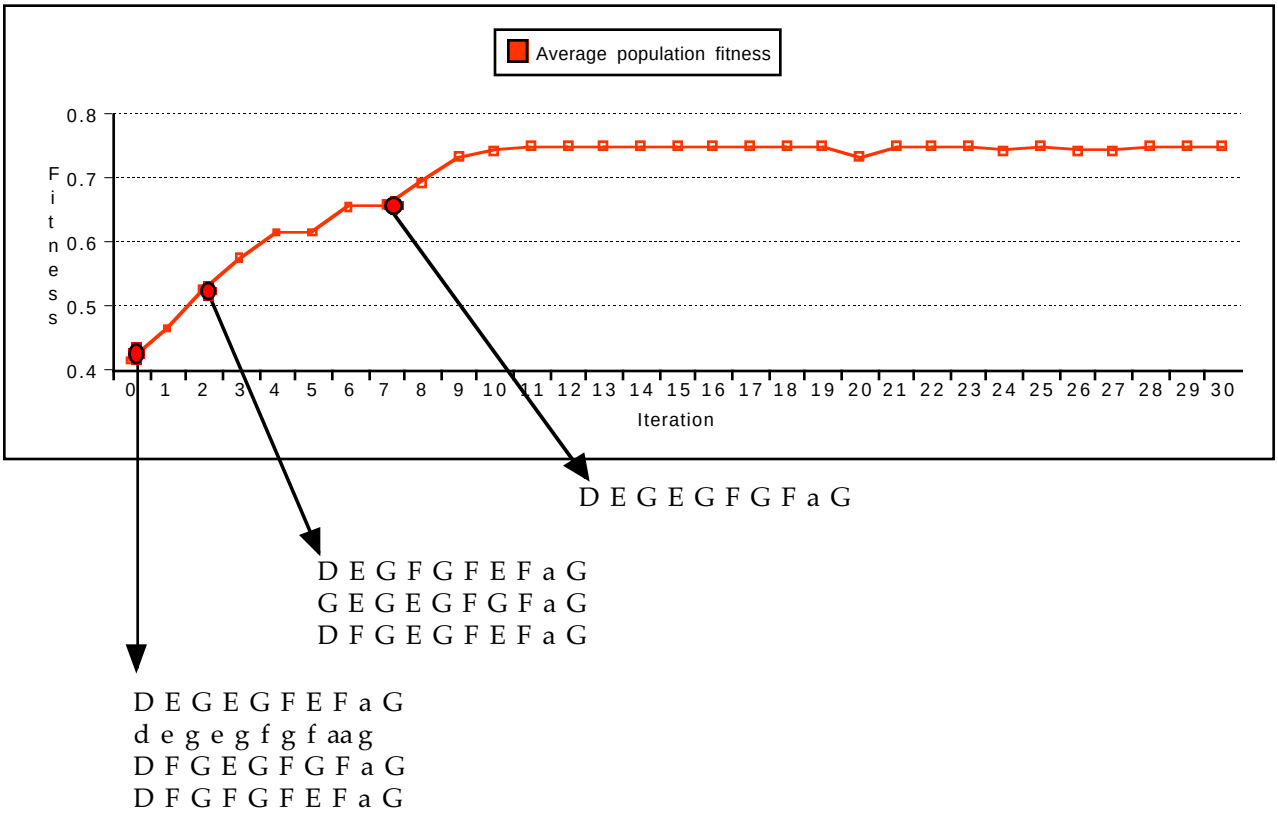


Fig. 6.5. Generation of optimal countermelodies by the genetic algorithm to cantus firmus <d, c, h, c, h, a, h, a, F#, G>. The average fitness of the population was derived by the same process of "grade-point averages" used in schools and colleges to convert letter grades to numerical scores.

In using a trained neural network as the fitness evaluator, we are limited to chromosomes of length equal to the length of the cantus firmi used in the training of the network, since the architecture of the network was initially determined by this length. Furthermore, for the best predictive ability, we need to use cantus firmi that are not too different from the ones used in the training of the network. Further experiments with more cantus firmi can extend the utility of this evaluation method.

We compared the performance of the genetic algorithm with a quasi-random algorithm that at each iteration randomly generated a new population. While this random algorithm succeeded in producing some reasonably good countermelodies, it took a long time to do so, since for each randomly produced population a considerable number of attempts were necessary to ensure that the constituent chromosomes represented valid countermelodies. The genetic algorithm, on the other hand, (quasi-) randomly generated a population only at the beginning; for the rest of the time, its evolutionary processes took over and in a relatively short time produced a variety of good countermelodies. Also, the genetic algorithm always produced populations with higher average fitness than did the randomized algorithm.

Example 6.10. The following is output of the random algorithm for the same cantus firmus as for example 6.7 (<D, F, E, D, G, F, a, G, F, E, D>). The random algorithm ran for 200 iterations, generating one countermelody per iteration, and was allowed 20 attempts per countermelody. Weighted rules gave preference for stepwise motion. Shown are the progressively better countermelodies generated in the course of the algorithm: only one optimal countermelody was found (iteration 124).

Countermelody	**Grade**	**Iteration**
1. aa d g f g d f e d c# d	0.74	2
2. aa f g f e a c h d c# d	0.75	8
3. d d c f g d f e d c# d	0.78	13
4. a a G F C D F E D C# D	0.85	64
5. aa aa g f e d c e d c# d	0.89	124

■

In the above outputs of the genetic algorithm, it will be observed that we began with the benefit of reasonably fit initial populations, thanks to the multiple initialization attempts at creating valid chromosomes. Even so, there is no guarantee that all initial populations have a reasonable level of fitness. Unless we spend an inordinate amount of time in generating an initial population by increasing the number of initialization attempts, it may still be possible that initial populations have very low fitness. This is especially true of more complex species (like second species), or of species with more than one added part, where for each cantus firmus, they are very more possible countermelodies compared to our simple first species with one added part.

Example 6.11. We performed two runs of the genetic algorithm to cantus firmus <D, a, G, F, E, D, F, E, D> (Jeppesen 1931, 117) to produce counterpoint in second species with one added part, and in first species with two added parts. The initial populations for both runs had very low fitness, and in both cases contained only a single chromosome that actually had a non-zero fitness. Yet, with the elitist model, the valid initial chromosomes survived and thus seized the opportunity to foster more and more valid and fit offspring.

In the run to produce second species with one added part, despite an initial population of low average fitness (for more details see appendix E, section E.2.1), the genetic algorithm succeeded in finding several near-optimal countermelodies. Several rules had to be fuzzified in order to render

Jeppesen's solution valid (Lewin's, Recovery, Outline Tritone in Four, Harmonic Range, Unisons, and Simultaneous Leaps Rules); these rules returned an aggregate score of 0.99 for Jeppesen's solution. The following shows the final population, and Jeppesen's solution. Figure 6.6 illustrates the generation of the optimal countermelodies.

Countermelody	**Grade**	**Iteration**
1. a f c d e E a b c G F a D F E C# D	0.99	16
2. d f c d e E a b c G F a D F E C# D	0.99	22
3. a f c d e h a b c G F a D F E C# D	0.99	22
4. a f c d e E a h c G F a D F E C# D	0.99	26
5. a f c d e d a b c G F a D F E C# D	0.99	27
6. d f c d e h a b c G F a D F E C# D	0.99	29
7. a d c d e E a b c G F a D F E C# D	0.99	30
8. a f c d e h a h c G F a D F E C# D	0.99	32
9. a b c f e E a b c G F a D F E C# D	0.99	34
10. d f c d e E a h c G F a D F E C# D	0.99	40
11. a d c d e h a h c G F a D F E C# D	0.99	41
12. d f c d e d a b c G F a D F E C# D	0.99	41
13. d f c d e h a h c G F a D F E C# D	0.99	44
14. a f c d e h a d c G F a D F E C# D	0.99	46
15. d b c f e E a b c G F a D F E C# D	0.99	46
16. d f c d e d a h c G F a D F E C# D	0.99	46
17. a h c d e E a b c G F a D F E C# D	0.99	48
18. a b c f e E a h c G F a D F E C# D	0.99	49
19. a b c d e E a b c G F a D F E C# D	0.99	49
20. a d c d e h a b c G F a D F E C# D	0.99	50

Jeppesen's solution

a h c d e d f F G a b F a h c# - d	0.99	

In the run to produce counterpoint in first species with two added parts, the genetic algorithm again produced several near-optimal countermelodies, despite a poor start (for more details see appendix E, section E.2.2). Here also, several rules had to be fuzzified in order to render Jeppesen's solution valid (Lewin's, Stepwise Motion, Recovery, Unisons, and Simultaneous Leaps Rules); these rules returned an aggregate score of 0.91 for Jeppesen's solution. Figure 6.7 illustrates the generation of the optimal countermelodies.

Countermelody pair	**Grade**	**Iteration**
1. D D G a G F D C# D d d e f c d aa a d	0.92	18
2. D D G a G F D C# D d d e d c d aa a d	0.92	34

Jeppesen's solution

d c e f g aa d c# d D F E D E F a A D	0.91	

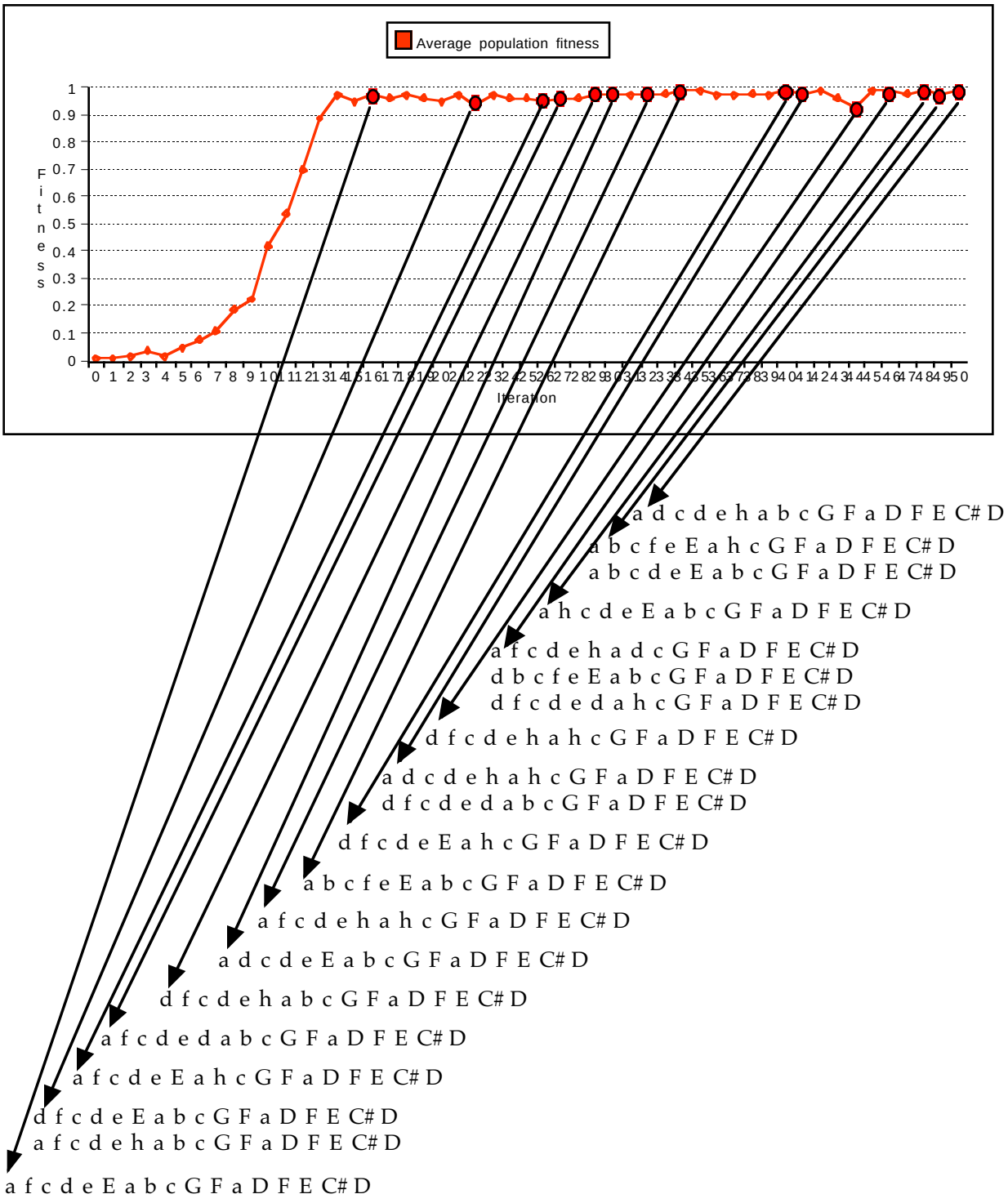


Fig. 6.6. Generation of optimal countermelodies by the genetic algorithm to cantus firmus <D, a, G, F, E, D, F, E, D> (second species, two parts)

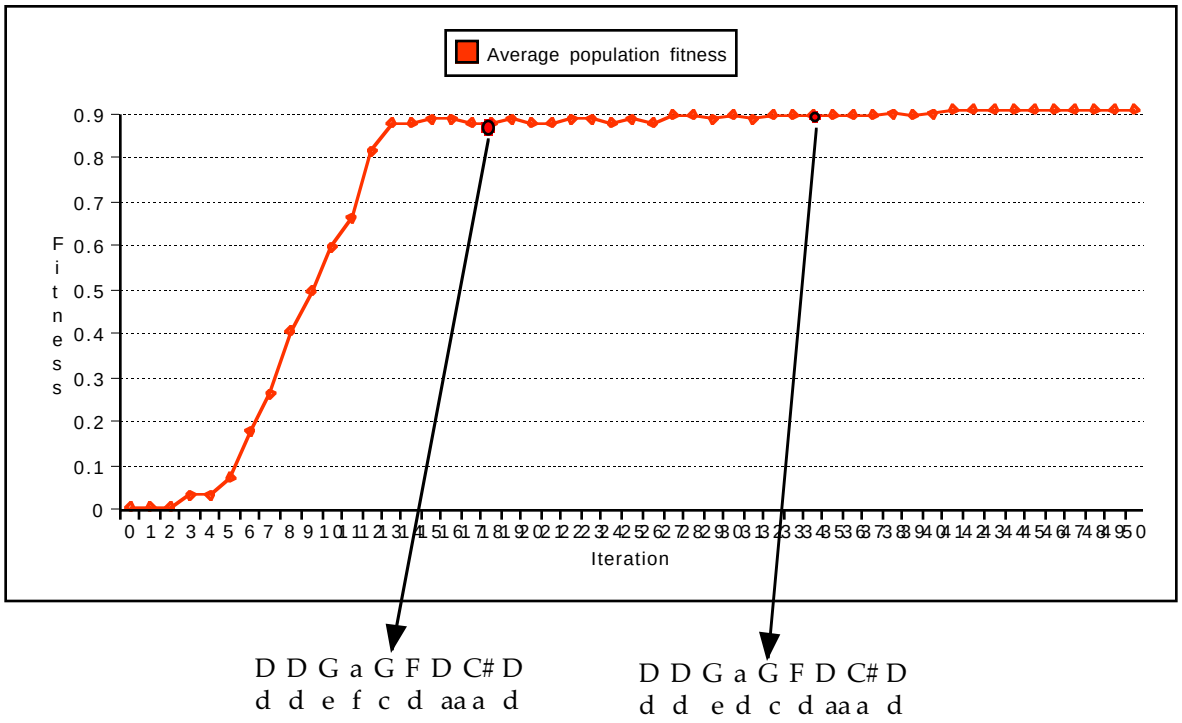


Fig. 6.7. Generation of near-optimal countermelodies by the genetic algorithm to cantus firmus <D, a, G, F, E, D, F, E, D> (first species, three parts)

6.2.1. Musical Gestures and Schemata

Applying the genetic algorithm to music composition requires justification. We have tried to provide some by comparing its performance with a random generation of countermelodies (example 6.10). In this section, we consider how the *Schema Theorem* (Holland 1975) might provide a mathematical characterization of the process of countermelody generation by, and hence another justification for, the genetic algorithm.

Consider again the countermelodies generated in example 6.6 (cantus firmus <D, F, E, D, G, F, a, G, F, E, D>, with rules weighted to favor Jeppesen's preferences). Examining the various substrings that constitute the optimal countermelody found in the third generation (<a, aa, g, f, e, d, c, h, d, c#, d>), we observe that the following substrings figure prominently in previous generations: "a aa …", "… g f e …", and "… h d c# d". They are present, not only in the above ancestral strings, but also in other highly fit strings, as can be verified from appendix E, listing E.2.

These observations are quite consistent with the Schema Theorem, in which context the three substrings can be characterized respectively as *schemata* (similarity templates) (a) "a aa * * * * * * *

* *", (b) "* * g f e * * * * * *", and (c) "* * * * * * * h d c# d", where the symbol '*' indicates "don't care." These schemata represent well-known musical *gestures* (short, easily recognizable phrases): schema (a) is the opening octave leap; schema (b) is simply the descending scale pattern outlining a third; and schema (c) is a common cadential pattern in the dorian mode. These three schemata qualify as *building blocks* (*fit* schemata [based on the average fitness of all its representative strings] with short *defining lengths* [distance between the first and last specific string position]). Of these, schema (a) is among the fittest and shortest of the schemata throughout the first three generations; its length is in fact the shortest for schemata that represent meaningful musical gestures. The fittest schema with the longest defining length and highest *order* (number of fixed positions) present in these generations is "a aa g f e * * h d c# d". This schema can be described musically as a pattern beginning with an opening octave leap followed by stepwise motion in the opposite direction (as per the "strict" form of the Recovery Rule), and closing with a well-known dorian mode cadential pattern: most musicians will agree that this makes for a very satisfying melody.

The Schema Theorem estimates, for a particular schema, a lower bound on the number of chromosomes representing this schema at a particular generation. This estimate is based on several factors: crossover and mutation rates, and, in the previous generation, the number of chromosomes representing this schema, average fitness of all chromosomes representing this schema, and average fitness of all chromosomes. Thus:

$$\text{Est}_{\text{lower}}\,[m\,(s,\,t)] \geq (f_s\,(t-1)/f_{\text{pop}}\,(t-1)\,)\,m\,(s,\,t-1)\,[1-p_c\,(\text{defLen}\,(s)/(\text{len}-1)\,)]\,(1-p_m)^{\text{order}\,(s)} \tag{6.1}$$

where
$\text{Est}_{\text{lower}}\,[m\,(s,\,t)]$ is the lower-bound estimate of the number of chromosomes representing schema s at generation t,
$f_s\,(t-1)$ is the average fitness of all chromosomes representing schema s at generation $t-1$,
$f_{\text{pop}}\,(t-1)$ is the average fitness of all chromosomes in the population at generation $t-1$,
$m\,(s,\,t-1)$ is the number of chromosomes representing schema s at generation $t-1$,
p_c is the crossover rate ($0 \leq p_c \leq 1$),
defLen (s) is the defining length of schema s,
len is the length of the chromosome,
p_m is the mutation rate ($0 \leq p_m \leq 1$),
order (s) is the order of schema s.

The Schema Theorem estimates only the lower bound because it considers only the destructive aspects of the crossover and mutation operations. By considering otherwise, we can estimate the upper bound as follows:

Est_{upper} [m (s, t)] =

(a) estimate of the number of chromosomes representing s selected into generation t
+ (b) estimate of the number of chromosomes resulting from the crossover operation involving chromosomes representing s
+ (c) estimate of the number of chromosomes resulting from the crossover operation involving chromosomes not representing s
+ (d) estimate of the number of chromosomes resulting from the mutation operation involving chromosomes not representing s

In (b), we assume that the crossover operation does not destroy any representative of schema s. We also assume that the mutation operation does not destroy any representative of s surviving into the current generation. On the other hand, we shall ignore the relatively low probability that the crossover and mutation operations will produce a chromosome representing schema s ([c] and [d]). Then,

$$(a) = (f_s\ (t - 1)/f_{pop}\ (t - 1)\ )\ m\ (s, t - 1)\ (1 - p_c)$$
$$(b) = (f_s\ (t - 1)/f_{pop}\ (t - 1)\ )\ m\ (s, t - 1)\ (p_c)$$

Thus, we arrive at an estimate of the upper bound:

$$Est_{upper}\ [m\ (s, t)] \leq (f_s\ (t - 1)/f_{pop}\ (t - 1)\ )\ m\ (s, t - 1) \qquad (6.2)$$

where
Est_{upper} [m (s, t)] is the upper-bound estimate of the number of chromosomes representing schema s at generation t,
f_s (t - 1) is the average fitness of all chromosomes representing schema s at generation t - 1,
f_{pop} (t - 1) is the average fitness of all chromosomes in the population at generation t - 1,
m (s, t - 1) is the number of chromosomes representing schema s at generation t - 1.

Example 6.12. Figures 6.8–10 compare lower- and upper-bound estimates of the number of occurrences of schema (a) "a aa * * * * * * * * *", (b) "* * g f e * * * * * *", and (c) "* * * * * * * h d c# d", during the first 20 iterations of a genetic algorithm run to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D> (example 6.6). For each of the schemata, the actual number of representatives compares quite closely to the lower- and upper-bound estimates.

Figure 6.11 compares average fitnesses of the population, and the three schemata. The increase of average fitness of representatives of the schemata corresponds to the increase of the average population fitness. Notice that in this instance, the schema with the shortest defining length and order ("a aa * * * * * * * * *") also tends to have the highest average fitness.

An unfortunate mutation occurring in the ninth generation produced a zero-fitness chromosome ("a aa g f e d c e d c# d"). The mutation operation apparently altered the fourth gene from the end (from "h" to "e"), which resulted in a chromosome that represented the first two schemata but excluded the third. This explains the dip in the average fitnesses of the first two schemata in the ninth generation, from which recovery was immediate (fig. 6.11). Compare also the effect on the upper- and lower-bound estimates for these schemata (figs. 6.8–10).

See appendix E, table E.1 for the schemata fitnesses, population fitness, and schemata occurrences used to derive the bounds estimates (equations 6.1–2). ■

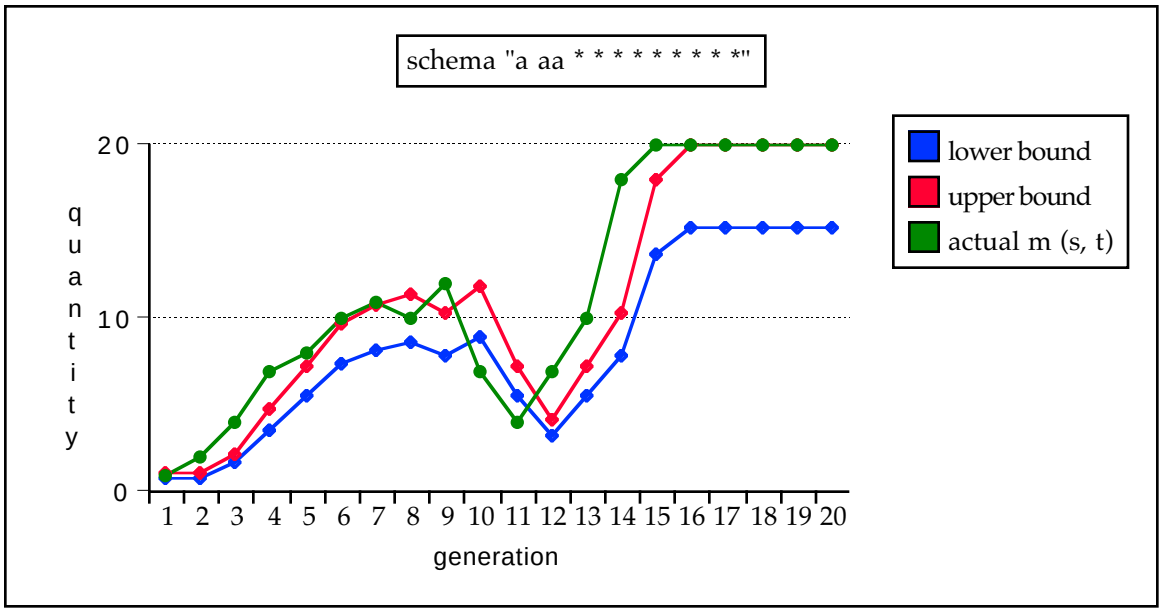


Fig. 6.8. Lower- and upper-bound estimates of the number of occurrences of schema "a aa * * * * * * * * * *" during the first 20 iterations of a genetic algorithm run to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

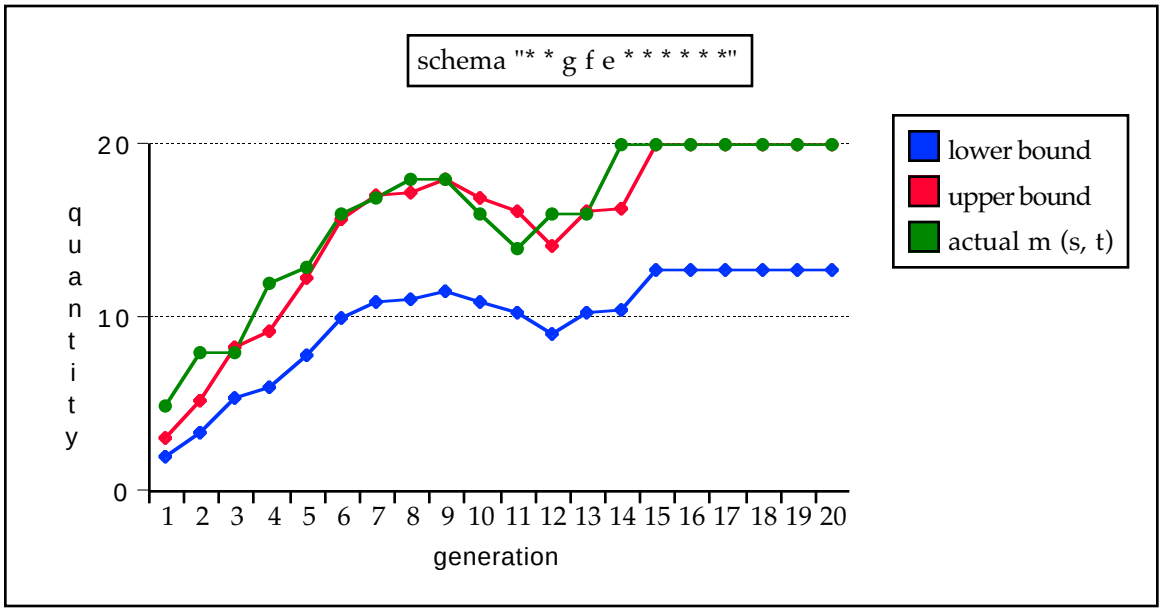


Fig. 6.9. Lower- and upper-bound estimates of the number of occurrences of schema "* * g f e * * * * * *" during the first 20 iterations of a genetic algorithm run to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

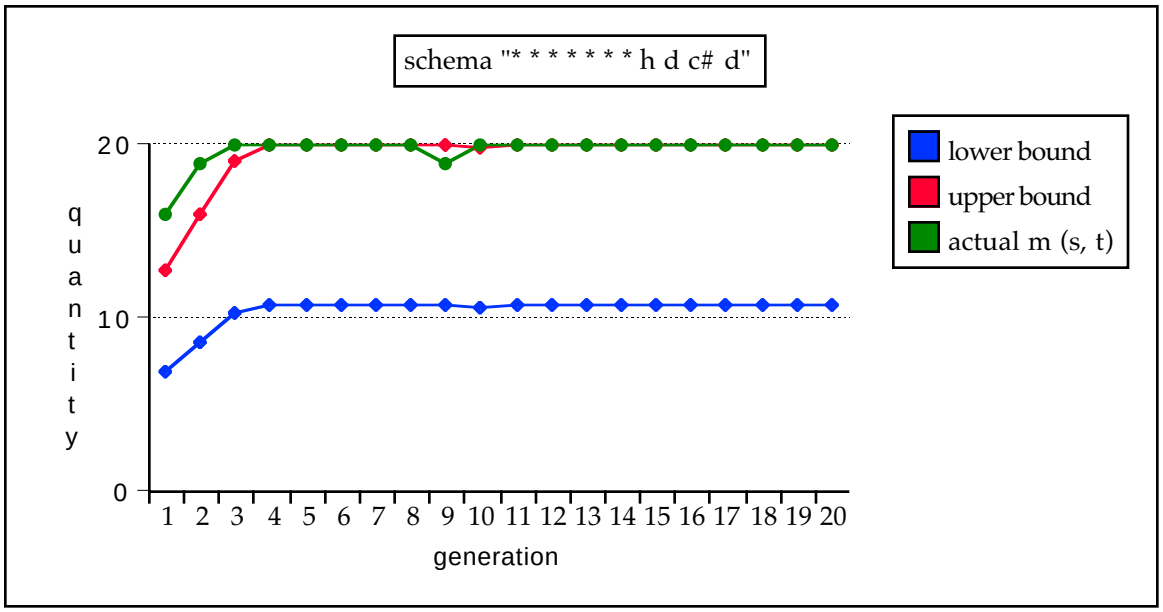


Fig. 6.10. Lower- and upper-bound estimates of the number of occurrences of schema "* * * * * * * h d c# d" during the first 20 iterations of a genetic algorithm run to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

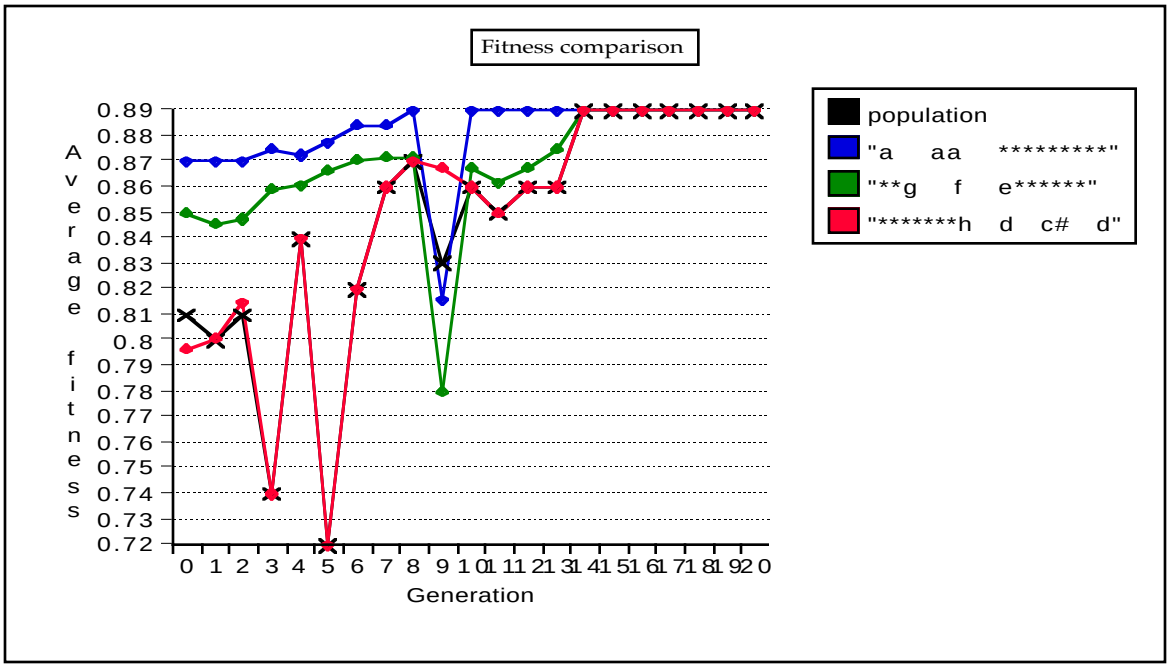


Fig. 6.11. Comparison of the average fitnesses of the population and of schema (a) "a aa * * * * * * * * *", (b) "* * g f e * * * * * *", and (c) "* * * * * * * h d c# d" during the first 20 iterations of a genetic algorithm run to the cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

The Schema Theorem was one of the first attempts to justify mathematically the efficacy of genetic algorithms. Here, we have observed how it may characterize the creation of countermelodies. We believe that these albeit brief observations warrant further research on the relationship between musical gestures and schemata.

6.3. Exhaustive and Heuristic Algorithms Compared

The exhaustive generate-and-test method suffers from two drawbacks: it is impractical for long cantus firmi, and even for short ones, one has to wait for the algorithm to produce the entire set of correct countermelodies before being able to evaluate their relative merit. The best-first search with branch-and-bound pruning algorithm addresses both drawbacks by its selective exploration of the search tree. This algorithm, in conjunction with fuzzy rule application, produces countermelodies in order of merit (based on preference criteria of a particular listener), and avoids wasting time searching for inferior countermelodies. At the worst, however, this algorithm degenerates to an exhaustive search: it may turn out that the heuristics are not effective to begin with, or else a particular cantus firmus may defeat all attempts at smart searching. Also, we have seen that its selective search ability is seriously undermined if its evaluation function (such as a trained neural network) fails to produce non-increasing scores along any path down the search tree.

Both the exhaustive search and the best-first algorithms guarantee optimal solutions, but the latter still takes a very long time to process cantus firmi of any significant length. The exhaustive search technique quickly yields a considerable number of solutions, but since it does not produce them in order of merit, one must still wait for the entire set of countermelodies before sorting them. The genetic algorithm often appears the best compromise, producing near-optimal solutions in a relatively short time. On the other hand, some runs may not produce even a single successful counterpoint. See appendix E for sample outputs of the best-first and genetic algorithms, using various evaluation methods (trained neural network, and explicitly weighted rules).

Implementations of these methods will not only be useful in helping the musician by producing high quality countermelodies that can be subsequently assessed or verified according to criteria of interest to the expert, but can also be used to solve other complexity-related problems. For example, the rules and gamut can easily be modified to solve, among other problems, the Three-Dimensional Matching problem. We do so along the lines of the transformation we described (chapter 4, section 4.1.2) in the NP-completeness proof: we write a system to create first species with two added parts, with an

appropriate gamut, and create rules that ensure all notes are distinct. We can even have an optimization version of the matching problem, where various matchings can have different "happiness" ratings, and the goal is to find the happiest matching.

6.4. Summary

Exhaustive searches for valid countermelodies are practical only for short cantus firmi, owing to the intractability of the problem. Two heuristic algorithms to solve the species counterpoint optimization problem are the best-first branch-and-bound, and genetic algorithms. In these algorithms, fuzzy rule application provides the basis for heuristic searches.

Experiments with various rule weightings, including that produced by a simplified application of the Jeppesen Experiment, produced promising results. So also with the output of a trained neural network. If its output is to be used in the best-first algorithm, however, the issue of non-increasing scores of partial countermelodies in the network output needs to be observed.

CHAPTER 7. CONCLUSION

In this dissertation, we investigated machine intelligence applied to music. Using species counterpoint as our musical model, we examined both theoretical and practical aspects of algorithmic music composition. We related general music problems to a vast realm of difficult problems by deriving their complexity classes. We formalized various species counterpoint problems as decision, enumeration, number, and optimization problems. We then proved NP-completeness, #P-completeness, "strong sense" NP-completeness, and NP-equivalence of these formalized problems, inferring that efficient solutions of these music problems give us efficient solutions of many other problems, including NP and NP-complete problems. On the other hand, the addition of these music problems to the growing lists of NP-complete and related problems lends further weight to the strong belief among researchers that $P \neq NP$.

We showed how restricted versions of the decision problem can be solved efficiently by using the finite state machine model. Straightforward extensions to the basic finite state model enabled us to solve enumeration, number, and even optimization versions of the decision problem, and also to output desired countermelodies. We then investigated what specific aspects render these problems unsolvable not only by finite state machines, but also insolvable efficiently by any computational model at all.

Producing even one correct countermelody for a cantus firmus is thus a non-trivial problem. But even more interesting is the challenge of producing human-like music. To meet this challenge, we formulated new rules for species counterpoint, called "artistry" rules, and proposed fuzzy rule application to represent more accurately the intent and nature of these and other rules. We argued that this extension to crisp rule application enabled us to approximate human musical preferences, and we supported our claim by performing experiments to tune the weights of fuzzy rules directly by means of the "Jeppesen Experiment" and indirectly by backpropagation neural network learning. We thereby obtained heuristic measures, which can then be used in solving more general and intractable problems.

To solve the intractable music optimization problem, we developed programs using two algorithms: a branch-and-bound optimizing algorithm, and an adaptation of genetic algorithms as an approximating alternative to the branch-and-bound algorithm. In these algorithms, the tuned fuzzy rules guide the search process.

We now summarize the key ideas expressed, and contributions made, in this dissertation:
a) Fuzzy application of existing rules as a basis of artistic machine composition of music. Such application of certain rules is truer to their spirit than their traditional crisp treatment.

b) New "artistry rules" to reflect modern tastes in music. These rules are especially suitable for fuzzy application and for use in machine learning of human musical tastes.
c) Efficient solution of restricted decision, enumeration, number, and optimization problems using the finite state machine model. We showed how to construct finite state machines to implement certain types of rules that are amenable to efficient solutions.
d) Intractability of the general compositional problem and related problem classes. We distinguished which kinds of problems are intractable, and which are not. For those that are intractable, we proved their intractability, thus showing their close connection with interesting non-musical problems.
e) Intractability of the generalized rule redundancy and rule indispensability problem. Simple but suboptimal rule elimination is tractable, but the general optimization problem is not.
f) Methods to reflect human musical preferences. We outlined a procedure to capture the rule preferences of an expert by an iterative process of rule weight tuning. The general problem of weight tuning is non-trivial, and in some formulations, even intractable. On the other hand, artificial neural network training offers a more efficient but suboptimal solution. We outlined a procedure to train a backpropagation neural network to reflect an expert's preferences through fuzzy rule input.
g) Heuristic algorithms to produce human-like musical compositions, using possibly the above methods of reflecting human musical preferences. We explored the use of genetic algorithms and found that its performance compared favorably with exhaustive and best-first search algorithms.

7.1. Future Work

Results from the application of our artistry rules suggest that several more need to be identified. This would enable a more accurate description of human musical preferences, and promote more comprehensive feature input for neural network training. In addition to rules whose weightings can be adjusted to reflect human preferences, we could also devise methods to tune these and other rules to reflect the element of surprise or thwarted expectation that characterize great musical works; apparently the desire to be surprised is deeply rooted in nature (Werner and Todd 1997).

From a music theoretic standpoint, we might consider whether any of the numerous rules are redundant, or what would be a sufficient subset of the rules. As we have indicated, this problem in general is intractable. As shown for the intractable compositional problems, we could perhaps be less ambitious here, too. In chapter 2, we showed a way to measure the selectivity of rules, and we could use this data to optimize the order of rule application to reduce somewhat the computational demands in

evaluating candidate countermelodies. What would be desirable are more correctness rules that eliminate shorter partial candidate countermelodies that would in any case be eliminated further down in the search tree.

Our theoretical discussions have largely centered around time-complexity issues. There remains, however, a vast amount of space-complexity issues still waiting to be explored with respect to the music compositional, and perhaps the rule redundancy, problems. Specifically, the issues involve log space and P-space completeness. This is quite significant because P-space completeness is an even stronger measure of intractability than is NP-completeness. Also interesting is the question whether the transformations we used in the NP-completeness and related proofs are also log space transformations. Jones (1973) and Stockmeyer and Meyer (1973) have observed that the polynomial transformations used in most NP-completeness proofs are also log-space transformations.

We have presently developed application programs for a subset of the instructional corpus of species counterpoint, namely, first and second species: two and three parts, fourth species: two parts. Although they illustrate satisfactorily the hypotheses we made, it would be interesting to see how these algorithms scale up to the more complex species, especially fifth species with more than two parts.

Species counterpoint is a pedagogical basis for more concrete musical styles. We would like to see how our analysis applies to these other styles, in particular, four-part common-practice harmony, and two- and three-part counterpoint in the style of J. S. Bach. The exponential difficulty should be even more apparent in these more complicated styles. The overriding factor remains human artistry.

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APPENDIX A. RULES OF SPECIES COUNTERPOINT

In section A.1 we describe the rules commonly used in first species counterpoint with one added part. In section A.2 we show the output of the algorithm designed to discover indispensable rules, with respect to specific cantus firmi.

A.1. Rules of First Species Counterpoint in Two Parts

We categorize these rules as "melody," "harmony," and "counterpoint." We also list here under the heading "Artistry Rules" four additional rules that we have proposed. The rules are described both informally, and using if-then syntax.

A.1.1. Melody Rules

These rules govern the countermelody only: the cantus firmus is not affected by these rules. Some of the rules prescribe what notes are allowed at certain positions in the countermelody, others describe the relationship between adjacent notes, yet others affect notes with respect to the entire span of the countermelody.

Table A.1. Melody rules of first species counterpoint in two parts

	Rule	**Description**
1.	Rests	Disallow rests. If *note* of the countermelody is *p* Then *p* is not *rest*.
2.	Final Note	The last note of a melody must be at the interval of either an octave, perfect fifth, or unison with the last note of the cantus firmus. If *note* of the cantus firmus is *p* and *note'* of the countermelody is *q* and *last* (*p*) is true and *last* (*q*) is true Then *interval* (*p, q*) is *perfect octave* or *interval* (*p, q*) is *perfect fifth* or *interval* (*p, q*) is *perfect unison*.
3.	Allow Certain Ficta Notes	Allow C#, c#, cc#, Eb, eb, eeb, FF#, F#, f#, ff#, Bb. If *note* of the countermelody is *p* Then *vera* (*p*) is *true* or *p* is C# or *p* is c# or *p* is cc# or *p* is Eb or *p* is eb or *p* is eeb or *p* is FF# or *p* is F# or *p* is f# or *p* is ff# or *p* is Bb.

(table A.1 continued)

4.	Ficta Notes	In the dorian mode, the notes FF#, F#, f#, ff# are allowed if they lead by downward step to the penultimate note of the melody. If the *mode* of the cantus firmus is *dorian* and *antepenultimate* (countermelody) is FF# | F# | f# | ff# Then *penultimate* (countermelody) is EE | E | e | ee.
5.	B flats	Allow Bb, b, and bb only in the dorian mode. If *note* of the countermelody is *p* and the *mode* of the cantus firmus is not *dorian* Then *p* is not Bb and *p* is not b and *p* is not bb.
6.	Lewin's	For every non-final note *p* of a melody, there should exist a note *q* between *p* and the final note *r* of the melody such that the musical interval *p–q* is a descending step if *p* is higher than *r*, and an ascending step if *p* is lower than *r*. If $note_n$ of the countermelody is *p* and *final* (countermelody) is *r* and *higher than* (*p*, *r*) is *true* Then *exists* (*q* = $note_{n+i}$ of the countermelody) and *lower than* (*q*, *p*) and *interval size* (*p*, *q*) is *second*. If $note_n$ of the countermelody is *p* and *final* (countermelody) is *r* and *lower than* (*p*, *r*) is *true* Then *exists* (*q* = $note_{n+i}$ of the countermelody) and *higher than* (*q*, *p*) and *interval size* (*p*, *q*) is *second*.
7.	Melodic Range	Disallow a melodic range exceeding a 13th. If the *lowest note* of the countermelody is *p* and the *highest note* of the countermelody is *q* Then *greater than* (*interval size* (*p*, *q*), *thirteenth*) is *false*.
8.	Dissonant Melodic Intervals	Disallow augmented or diminished intervals. If $note_n$ of the countermelody is *p* and $note_{n+1}$ of the countermelody is *q* Then the *interval type* (*p*, *q*) is *major* or the *interval type* (*p*, *q*) is *minor*.
9.	Other Inadmissible Intervals	Disallow compound intervals, 7ths, or ascending leaps of a major 6th. If $note_n$ of the countermelody is *p* and $note_{n+1}$ of the countermelody is *q* Then *less than* (*interval size* (*p*, *q*), *ninth*) and not *interval size* (*p*, *q*) is *seventh* and not (*interval* (*p*, *q*) is *major sixth* and *ascending* (*p*, *q*)).

(table A.1 continued)

10.	Repeated Notes Once Only	Allow repeated notes only once in the course of a melody. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_{m}$ of the countermelody is r and $note_{m+1}$ of the countermelody is s and $p = q$ and $r = s$ Then n = m.
11.	Repeated Notes	Stronger version of Rule 9: disallow all repeated notes. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q Then $p \neq q$.
12.	Successive Minor Seconds	Disallow successive minor 2nds. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_{n+2}$ of the countermelody is r and *interval* (p, q) is *minor second* Then *interval* (q, r) is not *minor second*.
13.	Stepwise Motion	Stepwise motion should predominate. (see also Note Variety rule) If *number of steps* (countermelody) is *steps* and *number of leaps* (countermelody) is *leaps* Then *steps* >> *leaps*.
14.	Recovery	A leap in one direction should be followed by motion in the opposite direction. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_{n+2}$ of the countermelody is r Then if *ascending* (p, q) is *true* then *descending* (q, r) is *true* and if *descending* (p, q) is *true* then *ascending* (q, r) is *true*.
15.	More Than Two Leaps in Succession	Disallow more than two ascending/descending leaps in succession. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_{n+2}$ of the countermelody is r and $note_{n+3}$ of the countermelody is s and ((*ascending* (p, q) and *ascending* (q, r) and *ascending* (r, s)) or (*descending* (p, q) and *descending* (q, r) and *descending* (r, s)) and *greater than* (*interval size* (p, q), *second*) and *greater than* (*interval size* (q, r), *second*) Then *greater than* (*interval size* (r, s), *second*) is *false*.

(table A.1 continued)

16. Two Leaps in Succession

Disallow two ascending/descending leaps in succession, in which the second is larger than the first.
If $note_{n}$ of the countermelody is p
and $note_{n+1}$ of the countermelody is q
and $note_{n+2}$ of the countermelody is r
and ((*ascending* (p, q) and *ascending* (q, r))
or (*descending* (p, q) and *descending* (q, r)))
and *greater than* (*interval size* (p, q), *second*)
and *greater than* (*interval size* (q, r), *second*)
Then *greater than* (*interval size* (q, r), *interval size* (p, q)) is *false*.

17. Outline Tritone in Three

Disallow successions of three notes that outline the tritone.
If $note_{n}$ of the countermelody is p
and $note_{n+2}$ of the countermelody is q
Then *interval* (p, q) is not *augmented fourth*.

18. Outline Tritone in Four

Disallow successions of four notes that outline the tritone.
If $note_{n}$ of the countermelody is p
and $note_{n+3}$ of the countermelody is q
Then *interval* (p, q) is not *augmented fourth*.

19. Two Fourths or Fifths

Disallow successions of fourths or fifths.
If $note_{n}$ of the countermelody is p
and $note_{n+1}$ of the countermelody is q
and $note_{n+2}$ of the countermelody is r
Then not (*interval size* (p, q) is *fourth* and *interval size* (q, r) is *fourth*)
and not (*interval size* (p, q) is *fifth* and *interval size* (q, r) is *fifth*)

20. Approach to Cadence

Governs the correct approach to cadences in the respective modes.
If *antepenultimate* (countermelody) is p and *penultimate* (countermelody) is q and *final* (countermelody) is r
and *interval* (q, r) is *minor second*
Then *interval size* (p, q) is *second* or (*interval size* (p, q) is *third* and *descending* (p, q) is *true* and *ascending* (q, r) is *true*) or (*interval size* (p, q) is *third* and *ascending* (p, q) is *true* and *descending* (q, r) is *true*).

21. Climax

Disallow more than one occurrence of the highest note of the melody.
If $note_{n}$ of the countermelody is p
and $note_{m}$ of the countermelody is q
and *highest* (p) is *true* and *highest* (q) is *true*
Then n = m.

(table A.1 continued)

22.	Chromatic Resolution	Chromatic notes must resolve. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q Then if *sharped* (p) then *ascending* (p, q) is *true* and *interval* (p, q) is *minor second* and if *flatted* (p) then *descending* (p, q) is *true* and *interval* (p, q) is *minor second*.
23.	Diatonic Resolution	Sharped notes must resolve by ascending step. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q Then if *sharped* (p) then *ascending* (p, q) is *true* and *interval* (p, q) is *minor second*.
24.	Symmetric Leaps	A leap from a note in one direction should not be followed by leap in the opposite direction to that same note. If $note_{n}$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_{n+2}$ of the countermelody is r and *greater than* (*interval size* (p, q), *second*) is *true* Then $p \neq r$.

A.1.2. Harmony Rules

Harmony rules govern the relationships between notes of the cantus firmus and the countermelody. Only notes of both parts that sound simultaneously are affected.

Table A.2. Harmony rules of first species counterpoint in two parts

	Rule	**Description**
1.	Harmonic Range	Disallow more than one occurrence of the harmonic interval of the twelfth. If $note_{n}$ of the cantus firmus is p and $note_{n}$ of the countermelody is q and $note_{m}$ of the cantus firmus is r and $note_{m}$ of the countermelody is s and $n \neq m$ Then *not* (*interval size* (p, q) is *twelfth* and *interval size* (r, s) is *twelfth*).

(table A.2 continued)

2.	Dissonant Harmonic Intervals	Disallow augmented and diminished harmonic intervals, and intervals of the 2nd, 4th, 7th, 9th, and 11th. If $note_n$ of the cantus firmus is p and $note_n$ of the countermelody is q and *interval type* (p, q) is pq and *interval size* (p, q) is *pq_size* Then pq is not *augmented* and pq is not *diminished* and *pq_size* is not *second* and *pq_size* is not *fourth* and *pq_size* is not *seventh* and *pq_size* is not *ninth* and *pq_size* is not *eleventh*.
3.	Unisons	Disallow harmonic intervals of the unison except at the beginning or end. If $note_n$ of the cantus firmus is p and $note_n$ of the countermelody is q and *interval* (p, q) is *perfect unison* Then n = 1 or n = *length* (cantus firmus).
4.	Preserve Pentachord	The first harmonic interval should be either a unison, perfect fifth, or octave. If $note_1$ of the cantus firmus is p and $note'_1$ of the countermelody is q and *interval* (p, q) is pq Then pq is *perfect unison* of pq is *perfect fifth* or pq is *perfect octave*.

A.1.3. Counterpoint Rules

Counterpoint rules govern both the horizontal movement and vertical relationships of the countermelody and the cantus firmus. These rules are local in nature, in that they govern a finite number of notes at a time.

Table A.3. Counterpoint rules of first species counterpoint in two parts

	Rule	**Description**
1.	Parallel Fifths and Octaves	Disallow parallel perfect fifths and octaves. If $note_n$ of the cantus firmus is p and $note_n$ of the countermelody is q and $note_{n+1}$ of the cantus firmus is r and $note_{n+1}$ of the countermelody is s and the *interval* (p, q) is *perfect fifth* Then the *interval* (r, s) is not *perfect fifth*. If $note_n$ of the cantus firmus is p and $note_n$ of the countermelody is q and $note_{n+1}$ of the cantus firmus is r and $note_{n+1}$ of the countermelody is s and the *interval* (p, q) is *perfect octave* Then the *interval* (r, s) is not *perfect octave*.

(table A.3 continued)

2.	Exposed Fifths and Octaves	Disallow exposed fifths and octaves. If $note_n$ of the cantus firmus is p and $note_n$ of the countermelody is q and *interval size* (p, q) is *fifth* or *interval size* (p, q) is *octave* Then *exposed* (p, q) is *false*.
3.	Parallel Thirds and Sixths	Disallow more than five parallel thirds and sixths. If $note_{n+i}$ of the cantus firmus is p_{i+1} and $note_{n+i}$ of the countermelody is q_{i+1}, $0 \le i \le 5$ and *interval size* (p_i, q_i) is *third*, $1 \le j \le 5$ Then not (*interval size* (p_6, q_6) is *third*). If $note_{n+i}$ of the cantus firmus is p_{i+1} and $note_{n+i}$ of the countermelody is q_{i+1}, $0 \le i \le 5$ and *interval size* (p_i, q_i) is *sixth*, $1 \le j \le 5$ Then not (*interval size* (p_6, q_6) is *sixth*).
4.	Simultaneous Leaps	Disallow simultaneous leaps in the same direction in both the cantus firmus and the added melody. If $note_n$ of the cantus firmus is p and $note_n$ of the cantus firmus is q and $note_{n+1}$ of the countermelody is r and $note_{n+1}$ of the countermelody is s and *greater than* (*interval size* (p, q), *second*) is *true* and *greater than* (*interval size* (r, s), *second*) is *true* Then not (*ascending* (p, q) is *true* and *ascending* (r, s) is *true*) and not (*descending* (p, q) is *true* and *descending* (r, s) is *true*).
5.	Simultaneous Fourths	Disallow simultaneous leaps of the fourth in both the cantus firmus and the added melody. If $note_n$ of the countermelody is p and $note_{n+1}$ of the countermelody is q and $note_n$ of the cantus firmus is r and $note_{n+1}$ of the cantus firmus is s Then not (*interval size* (p, q) is *fourth* and *interval size* (r, s) is *fourth*).

(table A.3 continued)

6.	Final Cadence	Enforces the cadential formulae for the various modes. If *interval* (*penultimate* (countermelody), *penultimate* (cantus firmus)) is *major sixth* Then *interval* (*final* (countermelody), *final* (cantus firmus)) is *perfect octave*. If *interval* (*penultimate* (countermelody), *penultimate* (cantus firmus)) is *minor third* Then *interval* (*final* (countermelody), *final* (cantus firmus)) is *perfect unison*. If *interval* (*penultimate* (countermelody), *penultimate* (cantus firmus)) is *minor tenth* Then *interval* (*final* (countermelody), *final* (cantus firmus)) is *perfect octave*. If *interval* (*penultimate* (countermelody), *penultimate* (cantus firmus)) is *major third* Then *interval* (*final* (countermelody), *final* (cantus firmus)) is *perfect fifth*.

A.1.4. Artistry Rules

We present here four rules we have formulated to reflect preferences of an expert who contributed to experiments we carried out to reflect human musical preferences (appendices B, C). These rules hold no claim to historical accuracy, therefore.

Table A.4. Artistry rules of first species counterpoint in two parts

	Rule	**Description**
1.	Tonality	Melodies in the dorian and lydian modes should prefer b and bb over h and hh. If the *mode* of the cantus firmus is *dorian* or the *mode* is *lydian* Then any *note* of the countermelody is not *h* and is not *hh*.
2.	Final in Leaps of Fourths and Fifths	Leaps of fourths and fifths should preferably involve the final or its octave equivalent. If $note_n$ of the countermelody is *p* and $note_{n+1}$ of the countermelody is *q* and *interval size* (*p, q*) is *fourth* or *interval size* (*p, q*) is *fifth* Then *p* is *final* (cantus firmus) or *q* is *final* (cantus firmus) or (*interval* (*p, final* (cantus firmus)) is *perfect octave*) or (*interval* (*q, final* (cantus firmus)) is *perfect octave*).

(table A.4 continued)

3.	False Relations	In the dorian mode, notes C and c are best avoided if they occur close to the cadence. In the mixolydian mode, notes F and f are best avoided if they occur close to the cadence. If the *mode* of the cantus firmus is *dorian* and note of the countermelody is *n* and *close to cadence* (*n*) is *true* Then *n* is *not* C and *n* is *not* c. If the *mode* of the cantus firmus is *mixolydian* and note of the countermelody is *n* and *close to cadence* (*n*) is *true* Then *n* is *not* F and *n* is *not* f.
4.	Note Variety	The same note should not be heard too many times in a melody. If for any note *n* of countermelody *m*, *degree of repetition* (*n*) is *low* Then merit $\mu_{note\ variety}$ of *m* is *good*. If for any note *n* of countermelody *m*, *degree of repetition* (*n*) is *average* Then merit $\mu_{note\ variety}$ of *m* is *average*. If for any note *n* of countermelody *m*, *degree of repetition* (*n*) is *high* Then merit $\mu_{note\ variety}$ of *m* is *poor*.

A.2. Indispensable Rules with respect to Cantus Firmi <D, E, C#, D>, <a, D, F, E, D, C#, D>, and <D, F, E, D, G, F, a, G, F, E, D>

Using the algorithm described in chapter 2, section 2.1, we discovered the indispensable rules with respect to the following three cantus firmi: <D, E, C#, D>, <a, D, F, E, D, C#, D>, and <D, F, E, D, G, F, a, G, F, E, D>. The indispensable rules are those that, without which, at least one more countermelody would have been admitted. Listings A.1–3 show the output of algorithm 2.1 applied on the three cantus firmi.

Listing A.1. Output of algorithm 2.1 for cantus firmus <D, E, C#, D>

Cantus firmus: <D, E, C#, D>
Gamut: {R, FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
Alpha cut: 0

	Rule	ρ	**w**
	Tonality	1.0	1.0
	Final in Leaps of 4ths and 5ths	1.0	1.0
	False Relations	1.0	1.0
	Note Variety	1.0	1.0

With all rules applied,
No. countermelodies accepted: 2
No. countermelodies analyzed: 184

	Rule disabled	**countermelodies analyzed**	**countermelodies accepted**	**uniquely rejected**
1.	Rests	184	2	0
2.	Final Note	391	2	0

(listing A.1 continued)

3.	Allow Certain Ficta Notes	184	2	0
4.	Ficta Notes	230	4	2
5.	B flats	184	2	0
6.	Lewin's	322	7	5
7.	Melodic Range	184	2	0
8.	Dissonant Melodic Intervals	184	2	0
9.	Other Inadmissible Intervals	184	2	0
10.	Repeated Notes Once Only	184	2	0
11.	Repeated Notes	207	2	0
12.	Successive Minor Seconds	184	2	0
13.	Stepwise Motion	184	2	0
14.	Recovery	184	2	0
15.	More Than Two Leaps in Succession	184	2	0
16.	Two Leaps in Succession	184	2	0
17.	Outline Tritone in Three	184	2	0
18.	Outline Tritone in Four	184	2	0
19.	Two Fourths or Fifths	184	2	0
20.	Approach to Cadence	184	2	0
21.	Climax	184	2	0
22.	Chromatic Resolution	184	2	0
23.	Diatonic Resolution	184	2	0
24.	Symmetric Leaps	184	2	0
25.	Harmonic Range	184	2	0
26.	Dissonant Harmonic Intervals	276	4	2
27.	Unisons	184	2	0
28.	Preserve Pentachord	184	6	4
29.	Parallel Fifths and Octaves	184	2	0
30.	Exposed Fifths and Octaves	184	2	0
31.	Parallel Thirds and Sixths	184	2	0
32.	Simultaneous Leaps	184	2	0
33.	Simultaneous Fourths	184	2	0
34.	Final Cadence	253	2	0

Indispensable rules for cantus firmus <D, E, C#, D>:
4. Ficta Notes
6. Lewin's
26. Dissonant Harmonic Intervals
28. Preserve Pentachord

Listing A.2. Output of algorithm 2.1 for cantus firmus <a, D, F, E, D, C#, D>

Cantus firmus: **<**a, D, F, E, D, C#, D**>**
Gamut: {R, FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
Alpha cut: 0

Rule	ρ	**w**
Tonality	1.0	1.0
Final in Leaps of 4ths and 5ths	1.0	1.0
False Relations	1.0	1.0
Note Variety	1.0	1.0

(listing A.2 continued)

With all rules applied,
No. countermelodies accepted: 12
No. countermelodies analyzed: 1012

	Rule disabled	countermelodies analyzed	countermelodies accepted	uniquely rejected
1.	Rests	1012	12	0
2.	Final Note	1219	12	0
3	Allow Certain Ficta Notes	1012	12	0
4.	CheckForFictaNotes	1679	25	13
5.	B flats	1012	12	0
6.	Lewin's	5060	160	148
7.	Melodic Range	1012	12	0
8.	Dissonant Melodic Intervals	1012	12	0
9.	Other Inadmissible Intervals	1035	14	2
10.	Repeated Notes Once Only	1012	12	0
11.	Repeated Notes	1380	23	11
12.	Successive Minor Seconds	1012	12	0
13.	Stepwise Motion	1012	12	0
14.	Recovery	1012	12	0
15.	More Than Two Leaps in Succession	1012	12	0
16.	Two Leaps in Succession	1012	12	0
17.	Outline Tritone in Three	1012	12	0
18.	Outline Tritone in Four	1012	12	0
19.	Two Fourths or Fifths	1012	12	0
20.	Approach to Cadence	1012	12	0
21.	Climax	1012	18	6
22.	Chromatic Resolution	1104	14	2
23.	Diatonic Resolution	1012	12	0
24.	Symmetric Leaps	1035	14	2
25.	Harmonic Range	1012	12	0
26.	Dissonant Harmonic Intervals	4738	97	85
27.	Unisons	1357	13	1
28.	Preserve Pentachord	1012	25	13
29.	Parallel Fifths and Octaves	1219	14	2
30.	Exposed Fifths and Octaves	1012	12	0
31.	Parallel Thirds and Sixths	1012	12	0
32.	Simultaneous Leaps	1127	16	4
33.	Simultaneous Fourths	1012	12	0
34.	Final Cadence	1702	23	11

Indispensable rules for cantus firmus <a, D, F, E, D, C#, D>:
4. Ficta Notes
6. Lewin's
9. Other Inadmissible Intervals
11. Repeated Notes
21. Climax
22. Chromatic Resolution
24. Symmetric Leaps
26. Dissonant Harmonic Intervals

(listing A.2 continued)

27. Unisons
28. Preserve Pentachord
29. Parallel Fifths and Octaves
32. Simultaneous Leaps
34. Final Cadence

Listing A.3. Output of algorithm 2.1 for cantus firmus <D, F, E, D, G, F, a, G, F, E, D>

Cantus firmus: **<**D, F, E, D, G, F, a, G, F, E, D**>**
Gamut: {R, FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
Alpha cut: 0

Rule	ρ	**w**
Tonality	1.0	1.0
Final in Leaps of 4ths and 5ths	1.0	1.0
False Relations	1.0	1.0
Note Variety	1.0	1.0

With all rules applied,
No. countermelodies accepted: 76
No. countermelodies analyzed: 4853

			countermelodies	
	Rule disabled	**analyzed**	**accepted**	**uniquely rejected**
1.	Rests	4853	76	0
2.	Final Note	5060	76	0
3.	Allow Certain Ficta Notes	4853	76	0
4.	CheckForFictaNotes	9499	206	130
5.	B flats	4853	76	0
6.	Lewin's	50324	1828	1752
7.	Melodic Range	4853	76	0
8.	Dissonant Melodic Intervals	6808	129	53
9.	Other Inadmissible Intervals	5152	103	27
10.	Repeated Notes Once Only	4853	76	0
11.	Repeated Notes	11730	129	53
12.	Successive Minor Seconds	4853	76	0
13.	Stepwise Motion	4853	76	0
14.	Recovery	4853	76	0
15.	More Than Two Leaps in Succession	4853	76	0
16.	Two Leaps in Succession	4853	76	0
17.	Outline Tritone in Three	4853	76	0
18.	Outline Tritone in Four	4853	76	0
19.	Two Fourths or Fifths	4853	76	0
20.	Approach to Cadence	4853	76	0
21.	Climax	4853	117	41
22.	Chromatic Resolution	5842	107	31
23.	Diatonic Resolution	4899	76	0
24.	Symmetric Leaps	6693	93	17
25.	Harmonic Range	4853	76	0
26.	Dissonant Harmonic Intervals	1260032	31103	31027
27.	Unisons	12420	169	93
28.	Preserve Pentachord	4853	135	59

(listing A.3 continued)

29.	Parallel Fifths and Octaves	7015	118	42
30.	Exposed Fifths and Octaves	6647	90	14
31.	Parallel Thirds and Sixths	4853	76	0
32.	Simultaneous Leaps	9177	153	77
33.	Simultaneous Fourths	4853	76	0
34.	Final Cadence	13041	248	172

Indispensable rules for cantus firmus <D, F, E, D, G, F, a, G, F, E, D>:
4. Ficta Notes
6. Lewin's
8. Dissonant Melodic Intervals
9. Other Inadmissible Intervals
11. Repeated Notes
21. Climax
22. Chromatic Resolution
24. Symmetric Leaps
26. Dissonant Harmonic Intervals
27. Unisons
28. Preserve Pentachord
29. Parallel Fifths and Octaves
30. Exposed Fifths and Octaves
32. Simultaneous Leaps
34. Final Cadence

APPENDIX B. COUNTERPOINTS IN THE JEPPESEN EXPERIMENT

To the cantus firmus <D, F, E, D, G, F, a, G, F, E, D> one of Jeppesen's solutions was <a, aa, g, f, e, d, c, h, d, c#, d>, which we used to generate alternative countermelodies (see chapter 5, section 5.1). We applied the rule weight vector <w_1, w_2, w_3, w_4> to the four artistry rules: <Tonality, Final in Leaps of Fourths and Fifths, False Relations, Note Variety>. Shown below are the outputs for data sets comprising partially filled countermelodies starting with one blank at each note position, followed successively by partially filled countermelodies with two blanks, and so on. For each countermelody generated, the output also shows the respective rule scores (1. Tonality; 2. Final in Leaps of Fourths and Fifths; 3. False Relations; 4. Note Variety), followed by the aggregate scores. In all instances, Jeppesen's solution appears, and in most cases, as first in a list of alternatives, where more than one alternative exists for the blank/s.

Listing B.1. Jeppesen's countermelody and alternatives

Jeppesen's Countermelody: <a, aa, g, f, e, d, c, h, d, c#, d>
Cantus firmus: <D, F, E, D, G, F, a, G, F, E, D>

Output for data set with *num blanks* = 1, weight vector = <1, 1, 1, 1>:

Countermelody:	**Rule scores:**	1.	2.	3.	4.	**Aggregate score**
** aa g f e d c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
2. aa - g f e d c h d c# d		0.91	1.00	0.91	0.73	0.89
Number of solutions: 2.						
*a * g f e d c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
Number of solutions: 1.						
*a aa * f e d c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
Number of solutions: 1.						
*a aa g * e d c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
Number of solutions: 1.						
*a aa g f * d c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
Number of solutions: 1.						
*a aa g f e * c h d c# d*						
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.91
Number of solutions: 1.						

(listing B.1 continued)

```
   a aa g f e d * h d c# d
1. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.91
2. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.91
Number of solutions: 2.

   a aa g f e d c * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.91
Number of solutions: 1.

   a aa g f e d c h * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.91
Number of solutions: 1.

   a aa g f e d c h d *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.91
Number of solutions: 1.

   a aa g f e d c h d c# *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.91
Number of solutions: 1.
```

Output for data set with *num blanks* = 1, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * aa g f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
Number of solutions: 2.

   a *  g f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa * f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g * e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f * d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e d * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 2.

   a aa g f e d c * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

(listing B.1 continued)

```
   a aa g f e d c h * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e d c h d *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e d c h d c# *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

Output for data set with *num blanks* = 2, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  g f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
3. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
4. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
6. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
Number of solutions: 6.

   a *  * f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
Number of solutions: 2.

   a aa * * e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g * * d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f * * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e d * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 2.

   a aa g f e d c * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

(listing B.1 continued)

```
   a aa g f e d c h * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e d c h d *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

Output for data set with *num blanks* =3, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * f e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
3. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
4. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
6. d a  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
7. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
9. d c  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. d - e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
11. d - c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
Number of solutions: 11.

   a *  * * e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
3. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
4. a - h d e d c h d c# d           0.82   1.00   0.91   0.55   0.76
Number of solutions: 4.

   a aa * * * d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g * * * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f * * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e d * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 2.
```

(listing B.1 continued)

```
   a aa g f e d c * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g f e d c h * *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

Output for data set with *num blanks* = 4, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * e d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
3. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
4. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
6. d a  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
7. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
9. d -  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. d a c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
11. d c e f e d c h d c# d   0.91   1.00   0.91   0.55   0.78
12. a - c d e d c h d c# d   0.91   1.00   0.91   0.55   0.78
13. a - h d e d c h d c# d   0.82   1.00   0.91   0.55   0.76
14. d a h d e d c h d c# d   0.82   1.00   0.91   0.55   0.76
15. d - c f e d c h d c# d   0.91   0.91   0.91   0.55   0.76
16. d - c d e d c h d c# d   0.91   1.00   0.91   0.45   0.75
Number of solutions: 16.

   a *  * * * d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
3. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
4. a -  h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
Number of solutions: 4.

   a aa * * * * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.

   a aa g * * * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f * * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

(listing B.1 continued)

```
   a aa g f e d * * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 2.

   a aa g f e d c * * *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
Number of solutions: 1.
```

Output for data set with *num blanks* = 5, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * * d c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
3. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
4. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
6. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
7. d a  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
9. d c  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. d - e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
11. d a c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
12. a - c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
13. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
14. d a h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
15. d - c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
16. d - c d e d c h d c# d          0.91   1.00   0.91   0.45   0.75
Number of solutions: 16.

   a *  * * * * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
3. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
4. a -  h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
Number of solutions: 4.

   a aa * * * * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g * * * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f * * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

(listing B.1 continued)

```
   a aa g f e * * * * * d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e d * * * *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 2.
```

Output for data set with *num blanks* = 6, weight vector = <1, 1, 1, 2>:

```
Countermelody:       Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * * * c h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
3. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
4. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
6. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
7. d a  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
9. d c  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. d - e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
11. d a c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
12. a - c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
13. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
14. d a h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
15. d - c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
16. d - c d e d c h d c# d          0.91   1.00   0.91   0.45   0.75
Number of solutions: 16.

   a *  * * * * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
5. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
6. a -  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
7. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
8. a -  h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
9. a -  h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
Number of solutions: 9.

   a aa * * * * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g * * * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

(listing B.1 continued)

```
   a aa g f * * * * * * d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f e * * * * * *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

Output for data set with *num blanks* = 7, weight vector = <1, 1, 1, 2>:

```
Countermelody:       Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * * * * h d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
6. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
7. d a  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
8. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
9. aa - g f e d e h d c# d          0.91   1.00   1.00   0.64   0.84
10. aa - g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
11. aa d g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
12. aa f g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
13. aa d g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
14. aa f g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
15. d c  g f e f e h d c# d         0.91   0.91   1.00   0.64   0.82
16. d c  g f e d e h d c# d         0.91   0.91   1.00   0.64   0.82
17. d -  g f e d c h d c# d         0.91   1.00   0.91   0.64   0.82
18. d a  e f e d c h d c# d         0.91   0.91   0.91   0.64   0.80
19. a -  e f e d c h d c# d         0.91   0.91   0.91   0.64   0.80
20. d c  g f e d c h d c# d         0.91   0.91   0.91   0.64   0.80
21. d c  e f e d e h d c# d         0.91   1.00   1.00   0.55   0.80
22. d -  c d e f e h d c# d         0.91   1.00   1.00   0.55   0.80
23. d -  g f e f e h d c# d         0.91   1.00   1.00   0.55   0.80
24. d -  g f e d e h d c# d         0.91   1.00   1.00   0.55   0.80
25. d a  e f e d e h d c# d         0.91   0.91   1.00   0.55   0.78
26. d -  c f e d e h d c# d         0.91   0.91   1.00   0.55   0.78
27. a -  e f e d e h d c# d         0.91   0.91   1.00   0.55   0.78
28. d c  e f e d c h d c# d         0.91   1.00   0.91   0.55   0.78
29. a -  c d e d c h d c# d         0.91   1.00   0.91   0.55   0.78
30. d a  c d e d c h d c# d         0.91   1.00   0.91   0.55   0.78
31. d -  e f e d c h d c# d         0.91   1.00   0.91   0.55   0.78
32. d a  h d e f e h d c# d         0.82   1.00   1.00   0.55   0.78
33. a -  h d e f e h d c# d         0.82   1.00   1.00   0.55   0.78
34. d -  e f e d e h d c# d         0.91   1.00   1.00   0.45   0.76
35. d -  c f e d c h d c# d         0.91   0.91   0.91   0.55   0.76
36. a -  h d e d c h d c# d         0.82   1.00   0.91   0.55   0.76
37. d a  h d e d c h d c# d         0.82   1.00   0.91   0.55   0.76
38. d -  c d e d c h d c# d         0.91   1.00   0.91   0.45   0.75
Number of solutions: 38.
```

(listing B.1 continued)

```
   a *  * * * * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
5. a -  h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
6. a -  G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
7. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
9. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. a - h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
11. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
Number of solutions: 11.

   a aa * * * * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g * * * * * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.

   a aa g f * * * * * *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

Output for data set with *num blanks* = 8, weight vector = <1, 1, 1, 2>:

```
Countermelody:       Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * * * * * d c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
6. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
7. aa - g f e d c e d c# d          1.00   1.00   0.91   0.64   0.84
8. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
9. d a  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
10. aa - g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
11. aa - g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
12. aa - g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
13. aa d g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
14. aa f g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
15. aa d g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
16. aa f g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
17. aa d g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
18. aa f g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
19. aa d g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
20. aa f g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
21. a - G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
22. d c G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
```

(listing B.1 continued)

```
23. d a G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
24. a - h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
25. d c g f e d e h d c# d          0.91   0.91   1.00   0.64   0.82
26. d - G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
27. d c g f e f e h d c# d          0.91   0.91   1.00   0.64   0.82
28. d a h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
29. d - g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
30. d a e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
31. d c g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
32. a - e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
33. d - g f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
34. d c e f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
35. d - g f e f c e d c# d          1.00   1.00   0.91   0.55   0.80
36. d - g f e f e h d c# d          0.91   1.00   1.00   0.55   0.80
37. d - c d e f e h d c# d          0.91   1.00   1.00   0.55   0.80
38. d - g f e d c e d c# d          1.00   1.00   0.91   0.55   0.80
39. d a e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
40. a - e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
41. d - c f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
42. d c g f e d c e d c# d          1.00   0.91   0.91   0.55   0.78
43. d c g f e f c e d c# d          1.00   0.91   0.91   0.55   0.78
44. d a c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
45. a - c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
46. d - e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
47. d c e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
48. a - h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
49. d a h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
50. d c e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
51. d - e f e d e h d c# d          0.91   1.00   1.00   0.45   0.76
52. d - e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
53. d - c d e f c e d c# d          1.00   1.00   0.91   0.45   0.76
54. d a h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
55. d - c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
56. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
57. d - c d e d c h d c# d          0.91   1.00   0.91   0.45   0.75
58. d - c f e d c e d c# d          1.00   0.91   0.91   0.45   0.75
Number of solutions: 58.

   a *  * * * * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
5. a -  h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
6. a -  G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
7. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
9. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. a - h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
11. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
Number of solutions: 11.

   a aa * * * * * * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

(listing B.1 continued)

```
   a aa g * * * * * * * *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
Number of solutions: 3.
```

Output for data set with *num blanks* = 9, weight vector = <1, 1, 1, 2>:

```
Countermelody:        Rule scores:  1.     2.     3.     4.     Aggregate score

   * *  * * * * * * * c# d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
6. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
7. aa - g f e d c e d c# d          1.00   1.00   0.91   0.64   0.84
8. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
9. d a  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
10. aa - g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
11. aa - g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
12. aa - g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
13. aa d g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
14. aa f g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
15. aa d g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
16. aa f g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
17. aa d g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
18. aa f g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
19. aa d g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
20. aa f g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
21. a - G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
22. d c G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
23. d a G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
24. a - h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
25. d c g f e d e h d c# d          0.91   0.91   1.00   0.64   0.82
26. d - G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
27. d c g f e f e h d c# d          0.91   0.91   1.00   0.64   0.82
28. d a h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
29. d - g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
30. d a e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
31. d c g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
32. a - e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
33. d - g f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
34. d c e f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
35. d - g f e f c e d c# d          1.00   1.00   0.91   0.55   0.80
36. d - g f e f e h d c# d          0.91   1.00   1.00   0.55   0.80
37. d - c d e f e h d c# d          0.91   1.00   1.00   0.55   0.80
38. d - g f e d c e d c# d          1.00   1.00   0.91   0.55   0.80
39. d a e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
40. a - e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
41. d - c f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
42. d c g f e d c e d c# d          1.00   0.91   0.91   0.55   0.78
43. d c g f e f c e d c# d          1.00   0.91   0.91   0.55   0.78
44. d a c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
45. a - c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
46. d - e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
47. d c e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
48. a - h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
```

(listing B.1 continued)

```
49. d a h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
50. d c e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
51. d - e f e d e h d c# d          0.91   1.00   1.00   0.45   0.76
52. d - e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
53. d - c d e f c e d c# d          1.00   1.00   0.91   0.45   0.76
54. d a h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
55. d - c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
56. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
57. d - c d e d c h d c# d          0.91   1.00   0.91   0.45   0.75
58. d - c f e d c e d c# d          1.00   0.91   0.91   0.45   0.75
```
Number of solutions: 58.

```
   a *  * * * * * * * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
5. a -  h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
6. a -  G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
7. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
8. a -  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
9. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
10. a - h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
11. a - h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
```
Number of solutions: 11.

```
   a aa * * * * * * * * *  *
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
```
Number of solutions: 3.

Output for data set with *num blanks* = 10, weight vector = <1, 1, 1, 2>:

Countermelody: **Rule scores:** 1. 2. 3. 4. **Aggregate score**

```
   * *  * * * * * * * * *  d
1. a aa g f e d c h d c# d          0.91   1.00   0.91   0.82   0.89
2. a aa g f e f e h d c# d          0.91   1.00   1.00   0.73   0.87
3. a aa g f e d e h d c# d          0.91   1.00   1.00   0.73   0.87
4. aa - g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
5. aa d g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
6. aa f g f e d c h d c# d          0.91   1.00   0.91   0.73   0.85
7. aa - g f e d c e d c# d          1.00   1.00   0.91   0.64   0.84
8. a -  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
9. d a  c d e f e h d c# d          0.91   1.00   1.00   0.64   0.84
10. aa - g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
11. aa - g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
12. aa - g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
13. aa d g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
14. aa f g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
15. aa d g f e f c e d c# d         1.00   1.00   0.91   0.64   0.84
16. aa f g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
17. aa d g f e d c e d c# d         1.00   1.00   0.91   0.64   0.84
18. aa f g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
19. aa d g f e d e h d c# d         0.91   1.00   1.00   0.64   0.84
20. aa f g f e f e h d c# d         0.91   1.00   1.00   0.64   0.84
21. a -  G a h d c e d c# d         0.91   1.00   0.91   0.64   0.82
22. d c  G a h d c e d c# d         0.91   1.00   0.91   0.64   0.82
```

(listing B.1 continued)

```
23. d a  G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
24. a -  h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
25. d c  g f e d e h d c# d          0.91   0.91   1.00   0.64   0.82
26. d -  G a h d c e d c# d          0.91   1.00   0.91   0.64   0.82
27. d c  g f e f e h d c# d          0.91   0.91   1.00   0.64   0.82
28. d a  h d e f c e d c# d          0.91   1.00   0.91   0.64   0.82
29. d -  g f e d c h d c# d          0.91   1.00   0.91   0.64   0.82
30. d a  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
31. d c  g f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
32. a -  e f e d c h d c# d          0.91   0.91   0.91   0.64   0.80
33. d -  g f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
34. d c  e f e d e h d c# d          0.91   1.00   1.00   0.55   0.80
35. d -  g f e f c e d c# d          1.00   1.00   0.91   0.55   0.80
36. d -  g f e f e h d c# d          0.91   1.00   1.00   0.55   0.80
37. d -  c d e f e h d c# d          0.91   1.00   1.00   0.55   0.80
38. d -  g f e d c e d c# d          1.00   1.00   0.91   0.55   0.80
39. d a  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
40. a -  e f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
41. d -  c f e d e h d c# d          0.91   0.91   1.00   0.55   0.78
42. d c  g f e d c e d c# d          1.00   0.91   0.91   0.55   0.78
43. d c  g f e f c e d c# d          1.00   0.91   0.91   0.55   0.78
44. d a  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
45. a -  c d e d c h d c# d          0.91   1.00   0.91   0.55   0.78
46. d -  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
47. d c  e f e d c h d c# d          0.91   1.00   0.91   0.55   0.78
48. a -  h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
49. d a  h d e f e h d c# d          0.82   1.00   1.00   0.55   0.78
50. d c  e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
51. d -  e f e d e h d c# d          0.91   1.00   1.00   0.45   0.76
52. d -  e f e d c e d c# d          1.00   1.00   0.91   0.45   0.76
53. d -  c d e f c e d c# d          1.00   1.00   0.91   0.45   0.76
54. d a  h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
55. d -  c f e d c h d c# d          0.91   0.91   0.91   0.55   0.76
56. a -  h d e d c h d c# d          0.82   1.00   0.91   0.55   0.76
57. d -  c d e d c h d c# d          0.91   1.00   0.91   0.45   0.75
58. d -  c f e d c e d c# d          1.00   0.91   0.91   0.45   0.75
Number of solutions: 58.

   a *  * * * * * * * *  *
1. a aa g f e d c h d c# d           0.91   1.00   0.91   0.82   0.89
2. a -  G F E D A B D C# d           0.91   1.00   1.00   0.73   0.87
3. a D  G F E D A B D C# d           0.91   1.00   1.00   0.73   0.87
4. a aa g f e f e h d c# d           0.91   1.00   1.00   0.73   0.87
5. a aa g f e d e h d c# d           0.91   1.00   1.00   0.73   0.87
6. a -  G F E D C B D C# d           0.91   1.00   0.91   0.73   0.85
7. a D  G F E D C B D C# d           0.91   1.00   0.91   0.73   0.85
8. a -  G F C D C E D C# d           1.00   1.00   0.91   0.64   0.84
9. a -  c d e f e h d c# d           0.91   1.00   1.00   0.64   0.84
10. a - G F E D C E D C# d           1.00   1.00   0.91   0.64   0.84
11. a D G F C D C E D C# d           1.00   1.00   0.91   0.64   0.84
12. a D G F E D C E D C# d           1.00   1.00   0.91   0.64   0.84
13. a - h d e f c e d c# d           0.91   1.00   0.91   0.64   0.82
14. a - G a h d c e d c# d           0.91   1.00   0.91   0.64   0.82
15. a - e f e d c h d c# d           0.91   0.91   0.91   0.64   0.80
16. a - e f e d e h d c# d           0.91   0.91   1.00   0.55   0.78
17. a - c d e d c h d c# d           0.91   1.00   0.91   0.55   0.78
18. a - h d e f e h d c# d           0.82   1.00   1.00   0.55   0.78
19. a - h d e d c h d c# d           0.82   1.00   0.91   0.55   0.76
Number of solutions: 19.
```

(listing B.1 continued)

Output for data set with *num blanks* = 11, weight vector = <1, 1, 1, 2>:

Countermelody:	**Rule scores:**	1.	2.	3.	4.	**Aggregate score**
	* * * * * * * * * * * *					
1. a aa g f e d c h d c# d		0.91	1.00	0.91	0.82	0.89
2. a - G F E D A B D C# d		0.91	1.00	1.00	0.73	0.87
3. a D G F E D A B D C# d		0.91	1.00	1.00	0.73	0.87
4. a aa g f e f e h d c# d		0.91	1.00	1.00	0.73	0.87
5. a aa g f e d e h d c# d		0.91	1.00	1.00	0.73	0.87
6. D Bb C F E D A B D C# d		0.91	0.91	1.00	0.73	0.85
7. a - G F E D C B D C# d		0.91	1.00	0.91	0.73	0.85
8. aa - g f e d c h d c# d		0.91	1.00	0.91	0.73	0.85
9. aa f g f e d c h d c# d		0.91	1.00	0.91	0.73	0.85
10. aa d g f e d c h d c# d		0.91	1.00	0.91	0.73	0.85
11. a D G F E D C B D C# d		0.91	1.00	0.91	0.73	0.85
12. aa - g f e d e h d c# d		0.91	1.00	1.00	0.64	0.84
13. D - G F E D A B D C# d		0.91	1.00	1.00	0.64	0.84
14. a - c d e f e h d c# d		0.91	1.00	1.00	0.64	0.84
15. a - G F E D C E D C# d		1.00	1.00	0.91	0.64	0.84
16. a - G F C D C E D C# d		1.00	1.00	0.91	0.64	0.84
17. aa - g f e f e h d c# d		0.91	1.00	1.00	0.64	0.84
18. d a c d e f e h d c# d		0.91	1.00	1.00	0.64	0.84
19. aa - g f e f c e d c# d		1.00	1.00	0.91	0.64	0.84
20. aa - g f e d c e d c# d		1.00	1.00	0.91	0.64	0.84
21. aa d g f e d c e d c# d		1.00	1.00	0.91	0.64	0.84
22. a D G F C D C E D C# d		1.00	1.00	0.91	0.64	0.84
23. a D G F E D C E D C# d		1.00	1.00	0.91	0.64	0.84
24. aa d g f e d e h d c# d		0.91	1.00	1.00	0.64	0.84
25. aa f g f e f c e d c# d		1.00	1.00	0.91	0.64	0.84
26. aa d g f e f c e d c# d		1.00	1.00	0.91	0.64	0.84
27. aa f g f e f e h d c# d		0.91	1.00	1.00	0.64	0.84
28. aa d g f e f e h d c# d		0.91	1.00	1.00	0.64	0.84
29. aa f g f e d e h d c# d		0.91	1.00	1.00	0.64	0.84
30. aa f g f e d c e d c# d		1.00	1.00	0.91	0.64	0.84
31. D - C F E D A B D C# d		0.91	0.91	1.00	0.64	0.82
32. d - G a h d c e d c# d		0.91	1.00	0.91	0.64	0.82
33. a - h d e f c e d c# d		0.91	1.00	0.91	0.64	0.82
34. d a G a h d c e d c# d		0.91	1.00	0.91	0.64	0.82
35. d c G a h d c e d c# d		0.91	1.00	0.91	0.64	0.82
36. d a h d e f c e d c# d		0.91	1.00	0.91	0.64	0.82
37. a - G a h d c e d c# d		0.91	1.00	0.91	0.64	0.82
38. d c g f e d e h d c# d		0.91	0.91	1.00	0.64	0.82
39. d c g f e f e h d c# d		0.91	0.91	1.00	0.64	0.82
40. d - g f e d c h d c# d		0.91	1.00	0.91	0.64	0.82
41. D - A F E D C B D C# d		0.91	1.00	0.91	0.64	0.82
42. D - G F E D C B D C# d		0.91	1.00	0.91	0.64	0.82
43. d a e f e d c h d c# d		0.91	0.91	0.91	0.64	0.80
44. a - e f e d c h d c# d		0.91	0.91	0.91	0.64	0.80
45. d c g f e d c h d c# d		0.91	0.91	0.91	0.64	0.80
46. D - G F C D C E D C# d		1.00	1.00	0.91	0.55	0.80
47. D - G F E D C E D C# d		1.00	1.00	0.91	0.55	0.80
48. D - A F E D A B D C# d		0.91	1.00	1.00	0.55	0.80
49. d - g f e d e h d c# d		0.91	1.00	1.00	0.55	0.80
50. d - c d e f e h d c# d		0.91	1.00	1.00	0.55	0.80
51. d - g f e f e h d c# d		0.91	1.00	1.00	0.55	0.80
52. d - g f e f c e d c# d		1.00	1.00	0.91	0.55	0.80
53. d c e f e d e h d c# d		0.91	1.00	1.00	0.55	0.80
54. d - g f e d c e d c# d		1.00	1.00	0.91	0.55	0.80

(listing B.1 continued)

```
55. d -  c f e d e h d c# d        0.91   0.91   1.00   0.55   0.78
56. d c  g f e f c e d c# d        1.00   0.91   0.91   0.55   0.78
57. a -  e f e d e h d c# d        0.91   0.91   1.00   0.55   0.78
58. d c  g f e d c e d c# d        1.00   0.91   0.91   0.55   0.78
59. d a  e f e d e h d c# d        0.91   0.91   1.00   0.55   0.78
60. a -  c d e d c h d c# d        0.91   1.00   0.91   0.55   0.78
61. d c  e f e d c h d c# d        0.91   1.00   0.91   0.55   0.78
62. d a  c d e d c h d c# d        0.91   1.00   0.91   0.55   0.78
63. d -  e f e d c h d c# d        0.91   1.00   0.91   0.55   0.78
64. a -  h d e f e h d c# d        0.82   1.00   1.00   0.55   0.78
65. d a  h d e f e h d c# d        0.82   1.00   1.00   0.55   0.78
66. d -  c d e f c e d c# d        1.00   1.00   0.91   0.45   0.76
67. d -  e f e d c e d c# d        1.00   1.00   0.91   0.45   0.76
68. d -  e f e d e h d c# d        0.91   1.00   1.00   0.45   0.76
69. d c  e f e d c e d c# d        1.00   1.00   0.91   0.45   0.76
70. d a  h d e d c h d c# d        0.82   1.00   0.91   0.55   0.76
71. a -  h d e d c h d c# d        0.82   1.00   0.91   0.55   0.76
72. D -  C F E D C B D C# d        0.91   0.91   0.91   0.55   0.76
73. d -  c f e d c h d c# d        0.91   0.91   0.91   0.55   0.76
74. d -  c d e d c h d c# d        0.91   1.00   0.91   0.45   0.75
75. D -  C F E D C E D C# d        1.00   0.91   0.91   0.45   0.75
76. d -  c f e d c e d c# d        1.00   0.91   0.91   0.45   0.75
Number of solutions: 76.
```

APPENDIX C. COUNTERPOINTS IN ARTIFICIAL NEURAL NETWORK TRAINING

We show here the data and configurations used in neural network training (chapter 5, section 5.2):

C.1. The 100 training and validation exemplars.

C.2. The above data reconfigured as partially filled countermelodies for the training of networks to produce non-increasing scores with respect to partial countermelody lengths.

C.3. Comparisons between the expert's and the neural network's outputs on the validation exemplars, within various grading schemes.

C.1. The 100 Training and Validation Exemplars

In training the artificial neural network to compute a function of rule scores, we used 100 counterpoints. Using the Palestrina program, we first generated ten distinct cantus firmi, two for each of five modes, ending on the notes D, E, F, G, and a, respectively. For each of the cantus firmi, we further generated ten grammatically correct countermelodies in first species. All the melodies had a length of ten notes. Listing C.1 groups the countermelodies by their respective modes (dorian, phrygian, lydian, mixolydian, and aeolian), and presents the original evaluations by the expert.

Listing C.1. Training and validation exemplars

TRAINING DATA

Dorian mode

Counterpoint	**Expert evaluation**
d d c d e f e d c# d D F E D C# D E F E D	0.72
d d g f e d c d c# d D F E D C# D E F E D	0.5
d d c d e d c d c# d D F E D C# D E F E D	0.4
d d e f e h c d c# d D F E D C# D E F E D	0.55
d d g f e h c d c# d D F E D C# D E F E D	0.55
D D E A Bb A B D C# D D a G F G a G F E D	0.4

(listing C.1 continued)

```
D C B A B C E D C# D
D a G F G a G F E  D          0.65

D C E D B C B D C# D
D a G F G a G F E  D          0.5

D C Bb A B C E D C# D
D a G  F G a G F E  D         0.5

D D E D C A B D C# D
D a G F G a G F E  D          0.67

d d g f e  f e d c# d
D F E D C# D E F E  D         0.7

D D G F E B C D C# D
D F E D C# D E F E D          0.4

d c g f e d c d c# d
D F E D C# D E F E D          0.5

D D E A B A B D C# D
D a G F G a G F E  D          0.3

D D E D B C B D C# D
D a G F G a G F E  D          0.45

D D GG A B C E D C# D
D a G  F G a G F E  D         0.5
```

Phrygian mode

Counterpoint	**Expert evaluation**
e d e c d e f e d e E F G a h G a G F E	0.97
e d e c d g f e d e E F G a h G a G F E	0.75
E D E C D E F C D E E F G a h G a G F E	0.45
e d h c d g f e d e E F G a h G a G F E	0.7
E D E F G d c h a h E F G a h G a G F E	0.5
e d e hh aa g f e d e E F G G a h a G F E	0.4

(listing C.1 continued)

```
h a h E F G c h a h
E F G G a h a G F E          0.65

e a h d c d c h a h
E F G G a h a G F E          0.45

h a e d c d c h a h
E F G G a h a G F E          0.32

e d e d c d f e d e
E F G G a h a G F E          0.8
E D E C D E F E D E
E F G a h G a G F E          0.69

e a h c d g f e d e
E F G a h G a G F E          0.8

E A B C B C F E D E
E F G a h G a G F E          0.5

h a h d c d f e d e
E F G G a h a G F E          0.77

e d e h c d f e d e
E F G G a h a G F E          0.82

E D E B C D F E D E
E F G G a h a G F E          0.65
```

Lydian mode

Counterpoint	**Expert evaluation**

```
f e d f e d e f g f
F c b a G G C D E F          0.6

f e d f e d e aa g f
F c b a G G C D  E F         0.6

FF FF GG A C B A FF GG FF
F  c  b  a G G C D  E  F     0.45

f e g f g d e aa g f
F c b a G G C D  E F         0.57

FF FF GG FF C B A FF GG FF
F  c  b  a  G G C D  E  F    0.5

c d e f g cc hh aa g f
c F G a b a  G  F  E F       0.47

g aa hh aa g cc hh aa g f
c F  G  a  b a  G  F  E F    0.3
```

(listing C.1 continued)

```
c f e c d f e f g f
c F G a b a G F E F          0.62

g cc hh aa g f e aa g f
c F  G  a  b a G F  E F      0.5

g aa hh aa g aa e f g f
c F G a b a G F E F          0.28

F E D C B GG A FF GG FF
F c b a G G  C D  E  F       0.3

f e d c e d e aa g f
F c b a G G C D  E F         0.55

f e d c e d e f g f
F c b a G G C D E F          0.6

c d h c d f e f g f
c F G a b a G F E F          0.55

g f e f g f e aa g f
c F G a b a G F  E F         0.75

g cc hh aa g aa e f g f
c F  G  a  b a  G F E F      0.45
```

Mixolydian mode

Counterpoint	**Expert evaluation**

```
g aa ee dd cc hh aa g f# g
G d  c  h  a  G  a  h a  G   0.85

g f e d e h c d c# d
G d c h a G a h a  G         0.45

g f aa g f e c d c# d
G d c  h a G a h a  G        0.45

g f g d e hh aa g f# g
G d c h a G  a  h a  G       0.45

GG GG A D C B A GG FF# GG
G  d  c h a G a h  a  G      0.8

D E D E D F E F a  G
d c h c h a h a F# G         0.57

d e d e d e g f aa g
d c h c h a h a F# G         0.38
```

(listing C.1 continued)

```
d e hh aa d e g f aa g
d c h  c  h a h a F# G        0.35

D E D E G F E F a  G
d c h c h a h a F# G          0.65

D E D C D F E F a  G
d c h c h a h a F# G          0.55

GG GG A GG C B A GG FF# GG
G  d  c h  a G a h  a  G      0.67

g f aa g cc hh aa g f# g
G d c  h a  G  a  h a  G      0.55

G F a G F E F E F# G
G d c h a G a h a  G          0.5

d e d e g f g f aa g
d c h c h a h a F# G          0.45

aa aa hh aa g aa g f aa g
d  c  h  c  h a  h a F# G     0.25

d e hh aa g aa g f aa g
d c h  c  h a  h a F# G       0.3
```

Aeolian mode

Counterpoint	**Expert evaluation**
aa d e f g aa g aa hh aa a h c a h a E F# G# a	0.62
A GG FF A GG D C A B A a h c a h a E F# G# a	0.6
aa g e f g aa g aa hh aa a h c a h a E F# G# a	0.75
e d e f g aa g aa hh aa a h c a h a E F# G# a	0.52
a G a e d c h a h a a h c a h a E F# G# a	0.42
a G h a G c h a h a a e d c h a E F# G# a	0.85
aa g f aa g cc hh aa hh aa a e d c h a E F# G# a	0.5

(listing C.1 continued)

```
A A B A GG D C A  B  A
a e d c h  a E F# G# a        0.45

aa g hh aa hh cc hh aa hh aa
a  e d  c  h  a  E  F# G# a   0.7

e a h a d c h a  h  a
a e d c h a E F# G# a         0.67

A GG A C B D C A  B  A
a h  c a h a E F# G# a        0.92

e d e f g cc hh aa hh aa
a h c a h a  E  F# G# a       0.7

a D E F E F G a  h  a
a h c a h a E F# G# a         0.47

a a h a d c h a  h  a
a e d c h a E F# G# a         0.45

aa g hh aa g cc hh aa hh aa
a  e d  c  h a  E  F# G# a    0.45

aa g f aa g aa g aa hh aa
a  e d c  h a  E F# G# a      0.6
```

VALIDATION DATA

Dorian mode

Counterpoint	**Expert evaluation**
D D C F E B C D C# D D F E D C# D E F E D	0.45
d d c h e d c d c# d D F E D C# D E F E D	0.55
D D E A B C B D C# D D a G F G a G F E D	0.65
D A Bb A B C E D C# D D a G F G a G F E D	0.65

Phrygian mode

Counterpoint	**Expert evaluation**
e d h c d e f e d e E F G a h G a G F E	0.77

(listing C.1 continued)

```
E D E D B C F E D E
E F G a h G a G F E          0.57

e aa g hh aa g f e d e
E F  G G  a  h a G F E       0.6

e a h d c d f e d e
E F G G a h a G F E          0.95
```

Lydian mode

Counterpoint	**Expert evaluation**
FF FF GG C B GG A FF GG FF F c b a G G C D E F	0.35
f e g f e d e aa g f F c b a G G C D E F	0.77
C D E D G F E a G F c F G a b a G F E F	0.52
g aa hh aa dd cc hh aa g f c F G a b a G F E F	0.3

Mixolydian mode

Counterpoint	**Expert evaluation**
G G a G F E F E F# G G d c h a G a h a G	0.58
g f e d c d c d c# d G d c h a G a h a G	0.75
D E G F G F G F a G d c h c h a h a F# G	0.58
D E G F G F E F a G d c h c h a h a F# G	0.6

Aeolian mode

Counterpoint	**Expert evaluation**
aa d e f g cc hh aa hh aa a h c a h a E F# G# a	0.57
aa g e f g cc hh aa hh aa a h c a h a E F# G# a	0.55

(listing C.1 continued)

```
a G h a d c h a  h  a
a e d c h a E F# G# a        0.3

aa g f e d c h a  h  a
a  e d c h a E F# G# a       0.9
```

C.2. Partially Filled Countermelodies to Satisfy the Assumption of the Best-First Algorithm

Listing C.2 shows the mixolydian exemplars reconfigured as partially filled countermelodies for the training of networks to produce non-increasing scores of partial countermelodies with respect to length. Expert's scores are shown as percentages, as 1-of-10 bit vectors (000000001 represents an expert's score of 0.9 and above, 0000000010 a score between 0.8 and 0.89, etc.), and as 1-of-5 bit vectors (based on the second grade partition shown in table 5.1: 00001 represents an expert's scores of 0.8 and above, 00010 a score between 0.6 and 0.79, 00100 between 0.40 and 0.59, 01000 between 0.2 and 0.39, and 10000 between 0.0 and 0.19). Partial countermelodies receive the same score as their full counterparts; where identical partial countermelodies have full counterparts with different scores, these partial countermelodies receive the higher/highest of these different scores, with subsequent duplicate entries removed (e.g., see partial countermelodies nos. 41–52 in the training set, and nos. 15–17 in the validation set).

Listing C.2. Partially filled countermelodies for network training to produce non-increasing scores

Cantus firmus: <d, c, h, c, h, a, h, a, F#, G>

Mixolydian mode training data

	(Partial) countermelody	Expert's score	1-of-10 representation of expert's score	1-of-5 representation of expert's score
1.	D E D E G F E F a G	0.65	0000001000	00010
2.	d e d e g f g f aa g	0.45	0000100000	00100
3.	D E D C D F E F a G	0.55	0000010000	00100
4.	D E D E D F E F a G	0.57	0000010000	00100
5.	d e hh aa g aa g f aa g	0.3	0001000000	01000
6.	aa aa hh aa g aa g f aa g	0.25	0010000000	01000
7.	d e hh aa d e g f aa g	0.35	0001000000	01000
8.	d e d e d e g f aa g	0.38	0001000000	01000
9.	- E D E G F E F a G	0.65	0000001000	00010
10.	- e d e g f g f aa g	0.45	0000100000	00100
11.	- E D C D F E F a G	0.55	0000010000	00100
12.	- E D E D F E F a G	0.57	0000010000	00100
13.	- e hh aa g aa g f aa g	0.3	0001000000	01000
14.	- aa hh aa g aa g f aa g	0.25	0010000000	01000
15.	- e hh aa d e g f aa g	0.35	0001000000	01000
16.	- e d e d e g f aa g	0.38	0001000000	01000

(listing C.2 continued)

17.	- - D E G F E F a G	0.65	0000001000	00010
18.	- - d e g f g f aa g	0.45	0000100000	00100
19.	- - D C D F E F a G	0.55	0000010000	00100
20.	- - D E D F E F a G	0.57	0000010000	00100
21.	- - hh aa g aa g f aa g	0.3	0001000000	01000
22.	- - hh aa g aa g f aa g	0.25	0010000000	01000
23.	- - hh aa d e g f aa g	0.35	0001000000	01000
24.	- - d e d e g f aa g	0.38	0001000000	01000
25.	- - - E G F E F a G	0.65	0000001000	00010
26.	- - - e g f g f aa g	0.45	0000100000	00100
27.	- - - C D F E F a G	0.55	0000010000	00100
28.	- - - E D F E F a G	0.57	0000010000	00100
29.	- - - aa g aa g f aa g	0.3	0001000000	01000
30.	- - - aa g aa g f aa g	0.25	0010000000	01000
31.	- - - aa d e g f aa g	0.35	0001000000	01000
32.	- - - e d e g f aa g	0.38	0001000000	01000
33.	- - - - G F E F a G	0.65	0000001000	00010
34.	- - - - g f g f aa g	0.45	0000100000	00100
35.	- - - - D F E F a G	0.55	0000010000	00100
36.	- - - - D F E F a G	0.57	0000010000	00100
37.	- - - - g aa g f aa g	0.3	0001000000	01000
38.	- - - - g aa g f aa g	0.25	0010000000	01000
39.	- - - - d e g f aa g	0.35	0001000000	01000
40.	- - - - d e g f aa g	0.38	0001000000	01000
41.	- - - - - F E F a G	0.65	0000001000	00010
42.	- - - - - f g f aa g	0.45	0000100000	00100
43.	- - - - - aa g f aa g	0.3	0001000000	01000
44.	- - - - - e g f aa g	0.38	0001000000	01000
45.	- - - - - - E F a G	0.65	0000001000	00010
46.	- - - - - - g f aa g	0.45	0000100000	00100
47.	- - - - - - - F a G	0.65	0000001000	00010
48.	- - - - - - - f aa g	0.45	0000100000	00100
49.	- - - - - - - - a G	0.65	0000001000	00010
50.	- - - - - - - - aa g	0.45	0000100000	00100
51.	- - - - - - - - - G	0.65	0000001000	00010
52.	- - - - - - - - - g	0.45	0000100000	00100

Mixolydian mode validation data

	(Partial) countermelody	**Expert's score**	**1-of-10 representation of expert's score**	**1-of-5 representation of expert's score**
1.	D E G F G F E F a G	0.6	0000001000	00010
2.	D E G F G F G F a G	0.58	0000010000	00100

(listing C.2 continued)

```
3.    - E G F G F E F a G     0.6         0000001000        00010
4.    - E G F G F G F a G     0.58        0000010000        00100

5.    - - G F G F E F a G     0.6         0000001000        00010
6.    - - G F G F G F a G     0.58        0000010000        00100

7.    - - - F G F E F a G     0.6         0000001000        00010
8.    - - - F G F G F a G     0.58        0000010000        00100

9.    - - - - G F E F a G     0.6         0000001000        00010
10.   - - - - G F G F a G     0.58        0000010000        00100

11.   - - - - - F E F a G     0.6         0000001000        00010
12.   - - - - - F G F a G     0.58        0000010000        00100

13.   - - - - - - E F a G     0.6         0000001000        00010
14.   - - - - - - G F a G     0.58        0000010000        00100

15.   - - - - - - - F a G     0.6         0000001000        00010

16.   - - - - - - - - a G     0.6         0000001000        00010

17.   - - - - - - - - - G     0.6         0000001000        00010
```

C.3. Comparisons between the Expert's and the Neural Network's Evaluations of the Validation Exemplars, within Various Grading Schemes

We show here the detailed comparisons between the expert's grading and the neural network's trained output, based on the grading schemes described in chapter 5, section 5.2, and illustrated in table 5.1. For each grading scheme, we compare the grades imposed on each of the twenty validation exemplars. Section C.3.1 shows the outputs of neural networks with 40 input nodes, and section C.3.2 of networks with 60 input nodes. The specifications of the networks trained in the respective sections are given in table C.1 (both specifications are identical except for the number of input nodes).

Table C.1. Specifications of the neural networks trained with various grading schemes. The number of output nodes specified here corresponds to the network trained on the 1-of-5 partitions; the network trained on the pass-fail grading scheme would have 2 output notes, and the network trained on the 1-of-10 grading scheme would have 10 output nodes. Output of networks (a) and (b) are described in sections C.3.1 and C.3.2, respectively.

network type:	**(a)** **back propagation** **with validation and epoch training,** **nodes with bias term**	**(b)** **back propagation** **with validation and epoch training,** **nodes with bias term**
learning rate	0.1	0.1
momentum term	0.1	0.1
error threshold	0.1	0.1
no. iterations	until error threshold is achieved or until validation error increases	until error threshold is achieved or until validation error increases
no. layers	3	3
no. inputs	40	60
no. middle layer nodes	8	15
no. outputs	5	5
layer connections	maximal	maximal

C.3.1. Neural Network with 40 Input Nodes

The 40 input nodes correspond to the errors detected by the four artistry rules, indicated in a 1-of-10 vector: one for each note of the countermelody, whose length is 10. The network architecture of all networks trained in this section is the one shown in figure 5.2. The following table (C.2) shows the neural network's evaluation of the 20 validation exemplars, represented as seven grading schemes (indicated as "1." through "7.") corresponding to those shown in table 5.1. For each grading scheme, the output reflects the composite evaluations of ten separately trained networks, one for each of the ten different cantus firmi used in the training and validation sets.

Table C.2. Comparison of expert's and network's grades of 20 validation exemplars under various grading schemes (40 input nodes)

	1.		2.		3.		4.		5.		6.		7.		8.	
	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN
1.	F	P	C	C	F	D	D	C	D	C	D	C	D	C	0.4	0.5
2.	P	P	C	C	D	D	C	D	C	C	C	C	C	C	0.5	0.5
3.	P	P	B	C	D	D	C	C	B	C	B	B	B	C	0.6	0.5
4.	P	P	B	C	D	D	C	D	B	C	B	B	B	C	0.6	0.5
5.	P	P	B	A	C	D	B	D	A	B	A	C	A	A	0.7	0.9
6.	P	P	C	C	D	D	C	D	C	C	C	B	B	C	0.5	0.5
7.	P	F	B	C	D	F	C	D	B	D	B	D	B	F	0.6	0.6
8.	P	P	A	B	A	C	A	C	A	A	A	A	A	A	0.9	0.7
9.	F	F	D	D	F	F	F	D	F	F	F	D	F	C	0.3	0.3
10.	P	P	B	C	C	D	B	F	A	C	A	C	A	B	0.7	0.5
11.	P	P	C	B	D	C	D	F	C	A	C	B	C	A	0.5	0.7
12.	F	F	D	D	F	F	F	C	F	F	F	B	F	F	0.3	0.3
13.	P	P	C	C	D	D	C	B	C	C	C	A	B	C	0.5	0.5
14.	P	F	B	C	C	F	B	B	A	D	A	B	A	C	0.7	0.4
15.	P	F	C	C	D	F	C	C	C	F	C	C	B	F	0.5	0.3
16.	P	P	B	B	D	D	C	C	B	C	B	C	B	B	0.6	0.5
17.	P	P	C	B	D	C	C	C	C	A	C	A	B	A	0.5	0.7
18.	P	P	C	B	D	C	C	B	C	A	C	A	C	A	0.5	0.7
19.	F	F	D	B	F	F	F	D	F	A	F	D	F	A	0.3	0.4
20.	P	P	A	B	A	D	A	D	A	B	A	D	A	B	0.9	0.6

C.3.2. Neural Network with 60 Input Nodes

In addition to having 40 input nodes as for the network used in section C.3.1, the networks here consist of additional input nodes giving information on each of the ten notes of the cantus firmus and of the countermelody, for a total of 60 input nodes. The network architecture for all networks trained in this section is the one shown in figure 5.3. As did table C.2, the following table (C.3) shows the evaluations according to the seven grading schemes shown in table 5.1. For each grading scheme, the output reflects the evaluations of a single network, trained on all 80 training and 20 validation exemplars.

Table C.3. Comparison of expert's and network's grades of 20 validation exemplars under various grading schemes (60 input nodes)

	1.		2.		3.		4.		5.		6.		7.		8.	
	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN	Exp	ANN
1.	F	P	C	C	F	D	D	C	D	D	D	C	D	C	0.4	0.5
2.	P	P	C	C	D	D	C	D	C	C	C	C	C	C	0.5	0.5
3.	P	P	B	C	D	D	C	D	B	C	B	C	B	C	0.6	0.5
4.	P	P	B	C	D	D	C	D	B	C	B	C	B	C	0.6	0.5
5.	P	P	B	B	C	D	B	B	A	A	A	B	A	A	0.7	0.5
6.	P	P	C	B	D	D	C	D	C	B	C	B	B	B	0.5	0.4
7.	P	P	B	C	D	D	C	C	B	D	B	B	B	A	0.6	0.5
8.	P	P	A	C	A	D	A	D	A	C	A	C	A	C	0.9	0.5
9.	F	P	D	B	F	D	F	C	F	B	F	B	F	A	0.3	0.5
10.	P	P	B	C	C	D	B	C	A	A	A	C	A	B	0.7	0.5
11.	P	P	C	B	D	D	D	C	C	C	C	C	C	B	0.5	0.4
12.	F	F	D	C	F	F	F	F	F	C	F	C	F	F	0.3	0.4
13.	P	P	C	C	D	D	C	D	C	C	C	C	B	C	0.5	0.5
14.	P	F	B	B	C	F	B	D	A	D	A	B	A	A	0.7	0.4
15.	P	F	C	C	D	F	C	D	C	F	C	D	B	D	0.5	0.5
16.	P	F	B	C	D	F	C	C	B	C	B	C	B	B	0.6	0.4
17.	P	P	C	C	D	F	C	D	C	D	C	D	B	A	0.5	0.4
18.	P	P	C	B	D	F	C	D	C	A	C	B	C	A	0.5	0.4
19.	F	F	D	C	F	F	F	D	F	D	F	D	F	D	0.3	0.4
20.	P	F	A	C	A	F	A	D	A	D	A	D	A	D	0.9	0.4

APPENDIX D. IMPLEMENTATION OF MUSIC COMPOSITION ALGORITHMS

We discuss here the implementation of the two music composition algorithms discussed in the text, including the exhaustive algorithm (chapter 6, sections 6.1–3). All algorithms are modules of the overall program we call "Palestrina," after the great sixteenth-century composer whose style species counterpoint emulates. We programmed Palestrina in C++ using object-oriented design and implementation, and show here the key objects and methods used.

D.1. Inheritance Hierarchies and Navigability of the Classes

In this section, we outline also inheritance hierarchies and describe the navigability between the main classes in the Palestrina application. We have omitted a considerable amount of detail and have shown only what is relevant to the discussion in the main text.

D.1.1. General Interactions

The main classes in the Palestrina program are the following (base classes):
CApplication, CAnalyzer, CRulesManager, CRules, CBaseSearchEngine. In the following, whenever a base class is referenced, the reader is asked to add the phrase "and its subclasses …" where appropriate. The CApplication class and its subclasses provide the external interface. These application classes invoke the CAnalyzer class to convert external data into internal form and set up the required data structures. The CAnalyzer calls in turn the various algorithms to generate and diagnose countermelodies (fig. D.1).

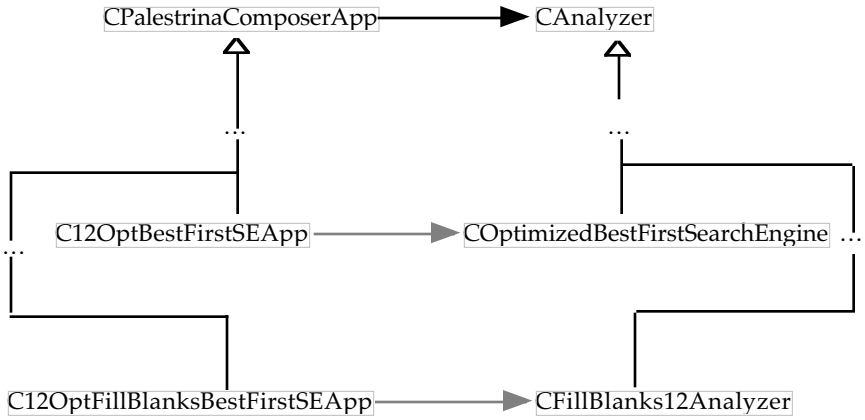

Fig. D.1. Navigability between the application and analyzer classes

The CAnalyzer directly runs the generate-and-test algorithm, and uses CBaseSearchEngine to run the best-first search algorithm and to interface with the genetic algorithm classes (fig. D.2).

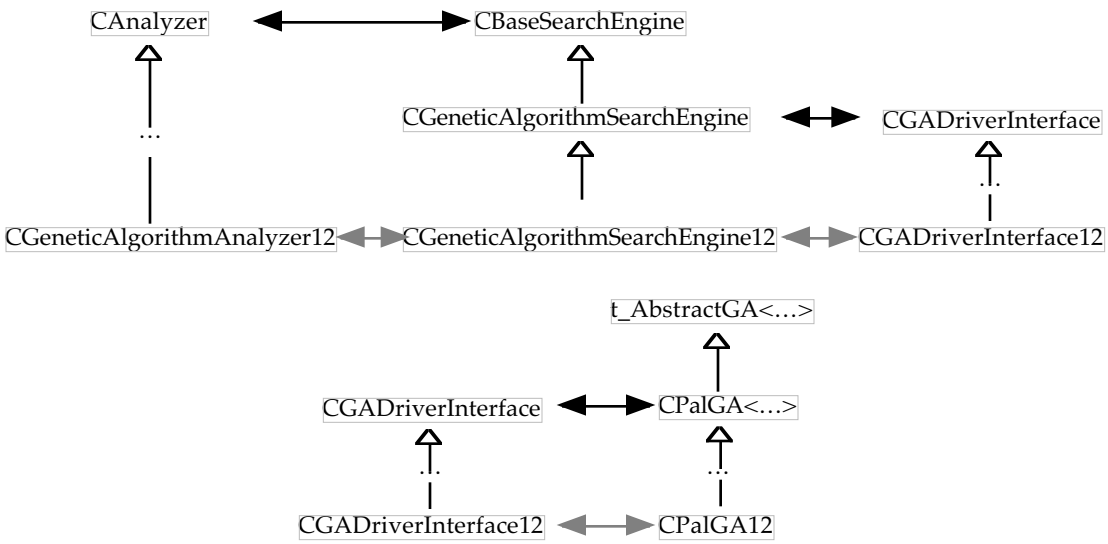

Fig. D.2. Navigability between the analyzer, search engine, and genetic algorithm classes

Both the analyzer and search engine classes rely on the CRulesManager to obtain the various CRules's responses to the various countermelodies submitted for approval (fig. D.3).

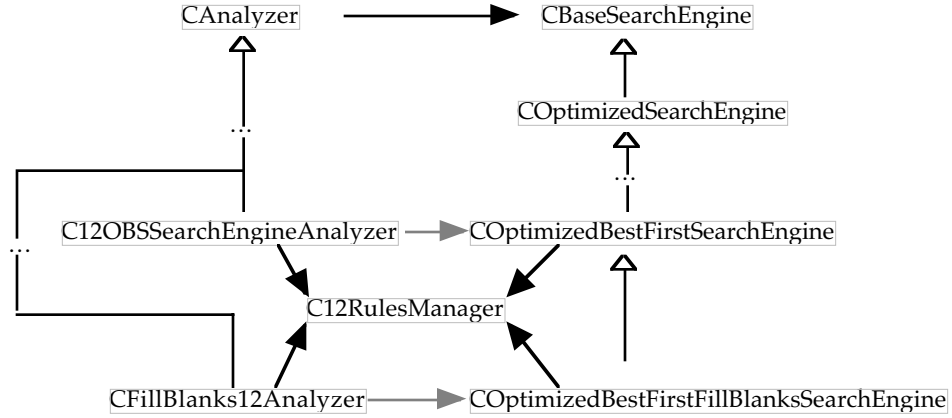

Fig. D.3. Navigability between analyzer, search engine, and rules manager classes

Further details of the class hierarchies are quite complex: here we shall only attempt to describe other classes used by the classes mentioned in the main text and in the subsequent sections (CAnalyzer, COptimizedSearchEngine, COptimizedBestFirstSearchEngine, COptimizdBestFirstFillBlanksSearchEngine, and t_CAbstractGA).

D.1.2. Best-First Search Engine and Genetic Algorithm Classes

The best-first search engine class implements the best-first search algorithm using the priority queue and a dynamic list to retain information on the partial countermelodies in the process of conducting the search. Figure D.4 shows the classes used by the best-first search engine class.

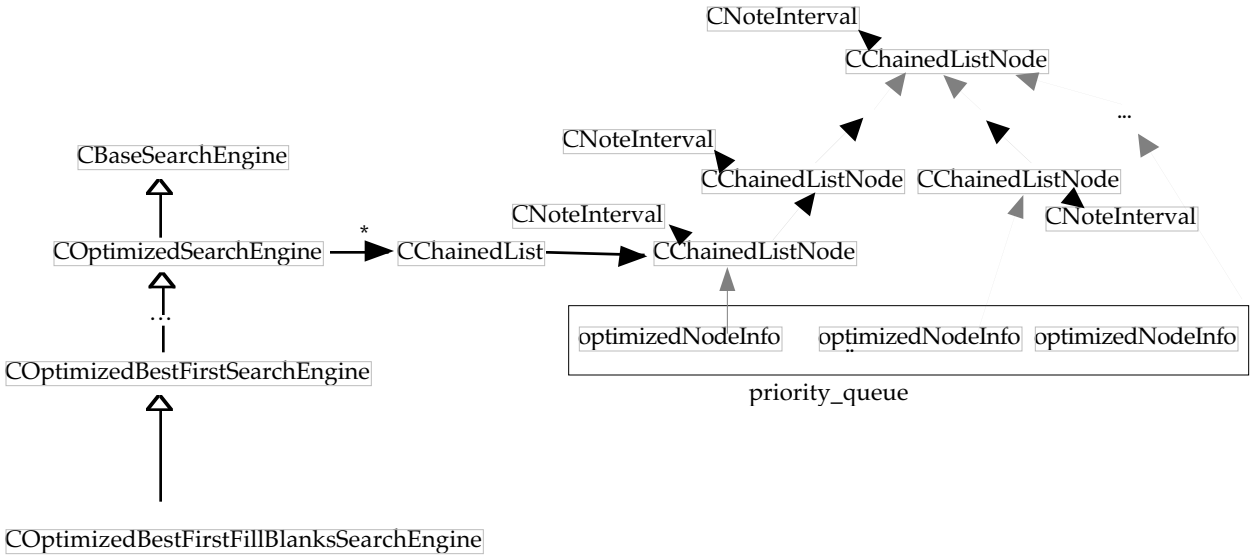

Fig. D.4. Classes used by the best-first search engine class

The genetic algorithm controls a population class that in turn maintains a population of hypotheses, consisting of chromosomes composed of genes. The population class also regenerates itself at each iteration of the algorithm with the probabilistic assistance of a roulette wheel class. Figure D.5 shows the objects used by the genetic algorithm class:

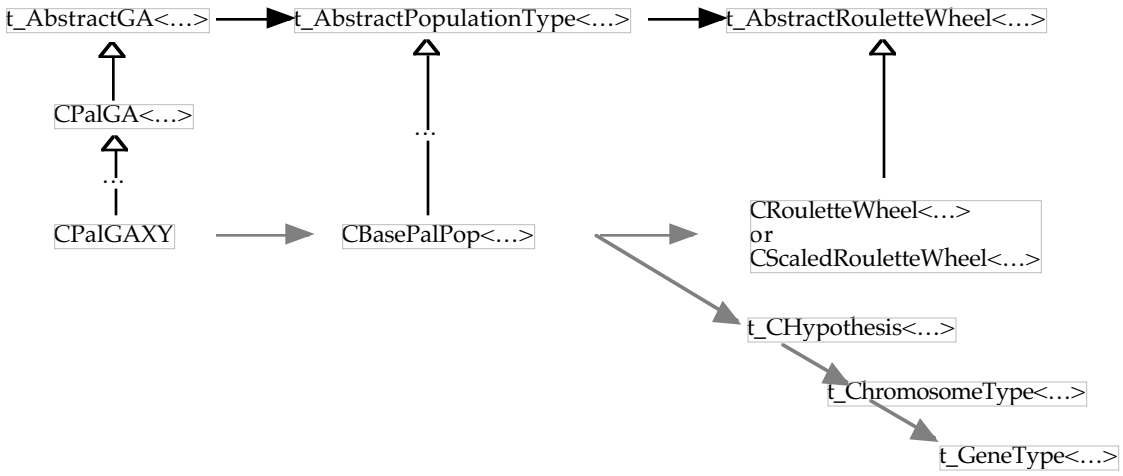

Fig. D.5. Classes used by the genetic algorithm class

D.1.3. Interactions with the Backpropagation Artificial Neural Network

One choice of evaluation function for the best-first search and genetic algorithms is the output of a trained neural network. We made the connection with the neural network through the rules manager class, which communicates with the neural network through a driver class (see below). The best-first search classes then call upon the rules manager to evaluate candidate countermelodies, while the genetic algorithm classes invoke the rules manager indirectly through the analyzer (figs. D.6–7).

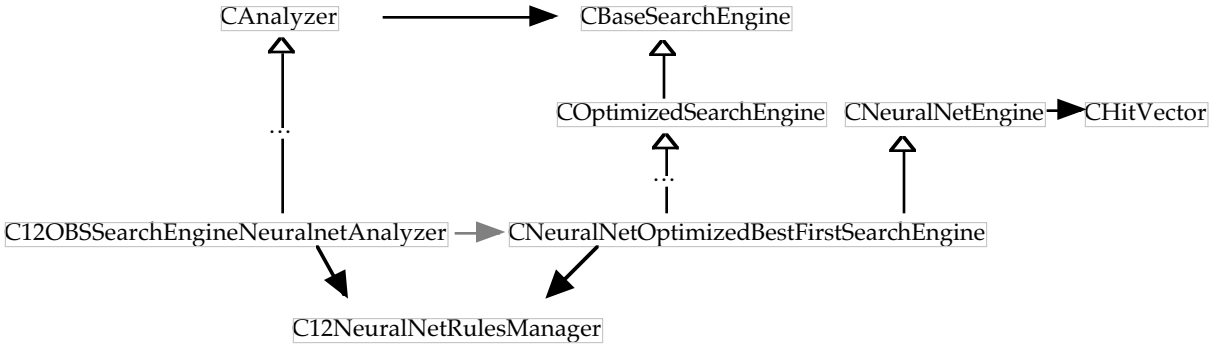

Fig. D.6. Navigability between the neural network based analyzer, best-first search engine, and rules manager classes. Compare figure D.7.

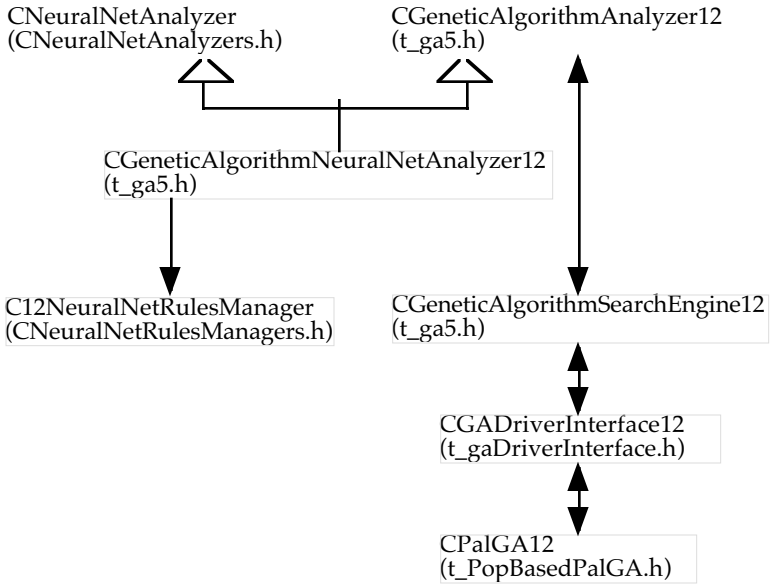

Fig. D.7. Navigability between the neural network based analyzer, genetic algorithm search engine, and rules manager classes.

Note that in both figures D.6 and 7, it is the neural net rules manager that communicates with the neural network, through CBPWithValidationEpochBiasedFunction, a driver class that acts as a go-between for the neural network (fig. D.8).

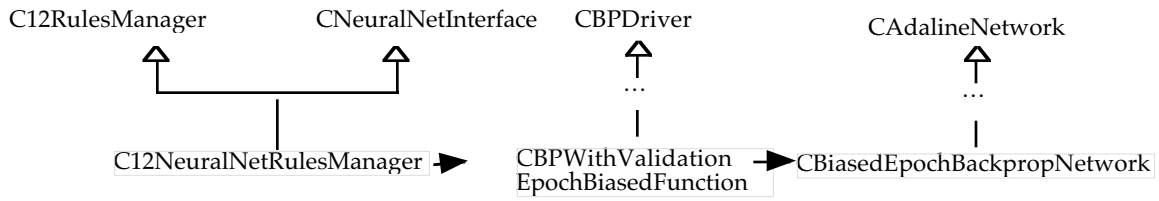

Fig. D.8. Interface between the neural net rules manager and the neural network

D.2. Source Code for Generate-and-Test, Best-First Search, and Fill-in-the-Blanks Algorithms

We next present class methods used by the various classes (described above) to implement the key algorithms used in this dissertation: Generate-and-Test, Best-First Search with Branch-and-Bound Pruning, and the Fill-in-the-Blanks algorithms. As with the design descriptions in the above sections, we have confined ourselves to the higher level methods, and have omitted lower level details.

D.2.1. Generate-and-Test

For the psuedocode of this algorithm, see algorithm 6.1. The algorithm to generate and test candidate melodies is a method in CAnalyzer (algorithm D.1).

```
void CAnalyzer::GenerateAndTest (void) {
    short curLevel;
    fuzzyBoolean result;
    short initVal = 0;
    vector <short> index (fNumCantusFirmusNotes, initVal);
    curLevel = 0;
    while (curLevel >= 0) {
        AddToLists (fNoteArray.at (index.at (curLevel)));
        result = fRulesManager->AnalyzeData ();
        if ((result == kError) || (curLevel == (fNumCantusFirmusNotes - 1))) {
            if (result != kError)
                PrintGoodMelody ();
            RemoveLastItemFromLists ();
            while ((curLevel >= 0) && (index.at (curLevel) == fUpperSearchLimitIndex)) {
                if (curLevel > 0)
                    RemoveLastItemFromLists ();
                index.at (curLevel) = 0;
                --curLevel;
            }
            if (curLevel >= 0)
                index.at (curLevel) = index.at (curLevel) + 1;
        } // if ((result == kError) || (...))
```

(algorithm D.1 continued)

```
            else
                ++curLevel;
        } // while
} // GenerateAndTest
```

Algorithm D.1. Implementation of the generate-and-test algorithm

D.2.2. Best-First Search with Branch-and-Bound Pruning

While the exhaustive search algorithm outputs countermelodies ordered lexicographically, the best-first algorithm outputs countermelodies ordered by merit. In our implementation, the algorithm accomplishes this with the aid of the priority queue from the standard template library. We implemented the best-first search algorithm as methods in the class CBaseSearchEngine and its descendants (algorithm D.2).

```
void COptimizedSearchEngine::Search (void) {
    ClearPriorityQueue ();
    InsertPossibleFirstLayerNotes ();
    PerformBestFirstBranchAndBoundSearch ();
    ClearPriorityQueue ();
} // COptimizedSearchEngine::Search

void COptimizedBestFirstSearchEngine::InsertPossibleFirstLayerNotes (void) {
    for (short it = 0; it < fSearchNoteArray.size (); ++it) {
        noteType newNote = fSearchNoteArray.at (it);
        SetUpChainedLists (newNote);
        fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
        if (theResult != kError) { // we only want grammatically correct melodies
            double hiMu, loMu;
            fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
            optimizedNodeInfo node;
            node.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
            node.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
                fFAPMelodicIntervalsChainedList->GetHeadNode (),
                fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
            if (Promising (node)) {
                fPriorityQueue.push (node);
                StoreChainedListNodePtrs (chainedListPtrVector);
            }
            else
                ReleaseLatestChainedListsNodes ();
        } // if (theResult != kError)
        else
            ReleaseLatestChainedListsNodes ();
    } // for
} // COptimizedBestFirstSearchEngine::InsertPossibleFirstLayerNotes
```

(algorithm D.2 continued)

```
void COptimizedBestFirstSearchEngine::PerformBestFirstBranchAndBoundSearch (void) {
    fNumMelodiesFound = 0;
    fTiePossible = true; // used to allow tied melodies to be included
    fPrevMu = 0;
    fGoOn = true;
    while (! fPriorityQueue.empty () && fGoOn) {
        optimizedNodeInfo parent;
        parent = fPriorityQueue.top ();
        fPriorityQueue.pop ();
        if (Leaf (parent))
            ProcessLeaf (parent);
        else
            ProcessInternalNode (parent);
    } // while
    DeleteChainedListNodePtrs (chainedListPtrVector);
} // COptimizedBestFirstSearchEngine::PerformBestFirstBranchAndBoundSearch

void COptimizedBestFirstSearchEngine::ProcessLeaf (const optimizedNodeInfo& leafNode) {
    ++fNumMelodiesFound;
    AnalyzeCompleteMelody (leafNode);
    double curMu;
    leafNode.GetMu (curMu);
    if (curMu >= fAlphaCut) {
        if (fNumMelodiesFound <= fNumMelodiesRequired)
            PrintGoodMelody ();
        else
            if (Equal (curMu, fPrevMu))
                PrintGoodMelody ();
            else
                fGoOn = false;
    } // if curMu >= fAlphaCut
    else
        fGoOn = false;
    fPrevMu = curMu;
} // COptimizedBestFirstSearchEngine::ProcessLeaf

void COptimizedBestFirstSearchEngine::ProcessInternalNode (const optimizedNodeInfo& parent) {
    for (short it = 0; it < fSearchNoteArray.size (); ++it) {
        noteType newNote = fSearchNoteArray.at (it);
        SetUpPartialChainedLists (parent, newNote);
        fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
        if (theResult != kError) {
            double hiMu, loMu;
            fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
            optimizedNodeInfo child;
            child = parent;
            child.SetLevel (parent.GetLevel () + 1);
            child.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
            child.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
```

(algorithm D.2 continued)

```
                fFAPMelodicIntervalsChainedList->GetHeadNode (),
                fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
            if (Promising (child)) {
                fPriorityQueue.push (child);
                StoreChainedListNodePtrs (chainedListPtrVector);
            } // if (Promising (child))
            else
                ReleaseLatestChainedListsNodes ();
        } // if (theResult != kError)
        else
            ReleaseLatestChainedListsNodes ();
    } // for it
} // COptimizedBestFirstSearchEngine::ProcessInternalNode
```

Algorithm D.2. Implementation of the best-first branch-and-bound algorithm

D.2.3. Fill-in-the-Blanks Algorithm Used in the Jeppesen Experiment

The implementation of the algorithm to find alternative solutions at various points in the countermelody, indicated by blanks, uses the best-first algorithm to generation these alternatives. We created COptimizedBestFirstFillBlanksSearchEngine, a subclass of the class that handled the best-first search (see above), COptimizedBestFirstSearchEngine. Only two protected methods had to be overridden for this subclass: InsertPossibleFirstLayerNotes and ProcessInternalNode (algorithm D.3). For the description of the Jeppesen Experiment that uses this algorithm, see chapter 5, section 5.1.

```
void COptimizedBestFirstFillBlanksSearchEngine::InsertPossibleFirstLayerNotes (void) {
    noteType noteSpecified;
    if ((noteSpecified = ((CNote *) (fGivenFirstAddedPartList->GetNthItem (1)))->GetNoteType ())
        != unknownNote) { // not a wildcard note
        SetUpChainedLists (noteSpecified);
        fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
        if (theResult != kError) { // we only want grammatically correct melodies
            double hiMu, loMu;
            fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
            optimizedNodeInfo node;
            node.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
            node.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
                fFAPMelodicIntervalsChainedList->GetHeadNode (),
                fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
            if (Promising (node)) {
                fPriorityQueue.push (node);
                StoreChainedListNodePtrs (chainedListPtrVector);
            }
            else
                ReleaseLatestChainedListsNodes ();
        } // if
```

(algorithm D.3 continued)

```
            else
                ReleaseLatestChainedListsNodes ();
        } // if not a wildcard note
        else { // wildcard note
            for (short it = 0; it < fSearchNoteArray.size (); ++it) {
                noteType newNote = fSearchNoteArray.at (it);
                SetUpChainedLists (newNote);
                fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
                if (theResult != kError) { // we only want grammatically correct melodies
                    double hiMu, loMu;
                    fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
                    optimizedNodeInfo node;
                    node.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
                    node.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
                        fFAPMelodicIntervalsChainedList->GetHeadNode (),
                        fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
                    if (Promising (node)) {
                        fPriorityQueue.push (node);
                        StoreChainedListNodePtrs (chainedListPtrVector);
                    }
                    else
                        ReleaseLatestChainedListsNodes ();
                } // if
                else
                    ReleaseLatestChainedListsNodes ();
            } // for
        } // else wildcard note
} // COptimizedBestFirstFillBlanksSearchEngine::InsertPossibleFirstLayerNotes

void COptimizedBestFirstFillBlanksSearchEngine::ProcessInternalNode (const optimizedNodeInfo&
parent) {
        noteType nextNoteSpecified;
        if ((nextNoteSpecified = ((CNote *)
            (fGivenFirstAddedPartList->GetNthItem (parent.GetLevel () + 1)))->GetNoteType ())
            != unknownNote) { // not a wildcard note
            SetUpPartialChainedLists (parent, nextNoteSpecified);
            fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
            if (theResult != kError) {
                double hiMu, loMu;
                fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
                optimizedNodeInfo child;
                child = parent;
                child.SetLevel (parent.GetLevel () + 1);
                child.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
                child.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
                    fFAPMelodicIntervalsChainedList->GetHeadNode (),
                    fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
                if (Promising (child)) {
                    fPriorityQueue.push (child);
```

(algorithm D.3 continued)

```
                    StoreChainedListNodePtrs (chainedListPtrVector);
                } // if (Promising (child))
                else
                    ReleaseLatestChainedListsNodes ();
            } // if (theResult != kError)
            else
                ReleaseLatestChainedListsNodes ();
        } // if not wildcard note
        else { // wildcard note
            for (short it = 0; it < fSearchNoteArray.size (); ++it) {
                noteType newNote = fSearchNoteArray.at (it);
                SetUpPartialChainedLists (parent, newNote);
                fuzzyBoolean theResult = fRulesManager->AnalyzeDataWithoutHistory ();
                if (theResult != kError) {
                    double hiMu, loMu;
                    fRulesManager->CalculateFuzzyTotal (hiMu, loMu);
                    optimizedNodeInfo child;
                    child = parent;
                    child.SetLevel (parent.GetLevel () + 1);
                    child.SetMu (hiMu, loMu, fNumCantusFirmusNotes);
                    child.SetPartialMelody (fFirstAddedPartChainedList->GetHeadNode (),
                        fFAPMelodicIntervalsChainedList->GetHeadNode (),
                        fCFandFAPHarmonicIntervalsChainedList->GetHeadNode ());
                    if (Promising (child)) {
                        fPriorityQueue.push (child);
                        StoreChainedListNodePtrs (chainedListPtrVector);
                    } // if (Promising (child))
                    else
                        ReleaseLatestChainedListsNodes ();
                } // if (theResult != kError)
                else
                    ReleaseLatestChainedListsNodes ();
            } // for it
        } // else wildcard note
} // COptimizedBestFirstFillBlanksSearchEngine::ProcessInternalNode
```

Algorithm D.3. Implementation of the algorithm used to find alternative countermelodies for the Jeppesen Experiment

D.3. Genetic Algorithm

We implement the genetic algorithm as a central object that evolves collections of strings. Thus, the genetic algorithm itself, populations, hypotheses, chromosomes, and genes are all distinct interacting objects that respond appropriately to messages. For greater generality, we wrote the application using templates. The following class methods (algorithm D.4) illustrate the main mechanism of our implementation of the genetic algorithm.

```
template < typename T = default_type, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_CAbstractGA < T, T2, T3 >::MainEventLoop (void) {
    unsigned long numIterations = 0;
    Evaluate (0);
    while (fAllTimeBestHypothesis < fFitnessThreshold && numIterations < fMaxIterations) {
        fPopulation->CreateNewGeneration ();
        Evaluate (++numIterations);
    }
}

template < typename T = default_type, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_CAbstractGA < T, T2, T3 >::Evaluate (unsigned long iteration) {
    vector < t_CHypothesis < T, T2 > >::iterator h;
    for (h = fPopulation->begin (); h != fPopulation->end (); ++h) {
        if (fAllTimeBestHypothesis < * h) {
            fAllTimeBestHypothesis = * h;
        }
    }
}

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::CreateNewGeneration (void) {
    SetUpCycle ();
    Select ();
    Crossover ();
    Mutate ();
    Update ();
}

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::SetUpCycle (void) {
    fWheel->SetUpWheel (*this);
    fNewGeneration.clear ();
    fNewGeneration.reserve (fNumHypotheses);
} // t_AbstractPopulationType::SetUpCycle

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::Select (void) {
    for (short i = 0; i < fNumElites; ++i) {
        DoSelect ();
    }
} // t_AbstractPopulationType::Select

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
inline void t_AbstractPopulationType < T, T2, T3 >::DoSelect (void) {
    fNewGeneration.push_back (at (fWheel->ProbabilisticallySelectedIndex ()));
} // t_AbstractPopulationType::DoSelect
```

(algorithm D.4 continued)

```
template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::Crossover (void) {
    for (short i = 0; i < fNumCrossoverPairs; ++i) {
        short index, index2;
        index = fWheel->ProbabilisticallySelectedIndex (); // choose two chromosomes for crossover
        index2 = fWheel->ProbabilisticallySelectedIndex ();
        t_CHypothesis < T, T2 > h3 (fChromosomeSize), h4 (fChromosomeSize);
        DoCrossover (at (index).GetChromosome (), at (index2).GetChromosome (),
                h3.GetChromosome (), h4.GetChromosome ());
        fNewGeneration.push_back (h3);
        fNewGeneration.push_back (h4);
    }
} // t_AbstractPopulationType::Crossover

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::Mutate (void) {
    for (short i = 0; i < fNumMutations; ++i) {
        DoMutate (fNewGeneration.at (rand () % fNumHypotheses).GetChromosome ());
    } // for
    if (BiasedToss (fProbabilityOfSingleMutation)) {
        DoMutate (fNewGeneration.at (rand () % fNumHypotheses).GetChromosome ());
    } // if
} // t_AbstractPopulationType::Mutate

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
void t_AbstractPopulationType < T, T2, T3 >::Update (void) {
    std::vector < t_CHypothesis < T, T2 > >::iterator h;
    for (h = fNewGeneration.begin (); h != fNewGeneration.end (); ++h) {
        h->SetFitness (CalculateFitness (Decode (h->GetChromosome ())));
    }
    * this = fNewGeneration;
    stable_sort (begin (), end (), greater < t_CHypothesis < T, T2 > >());
} // t_AbstractPopulationType::Update

template < typename T, typename T2 = t_ChromosomeType < T >, typename T3 = T >
inline T2 t_AbstractPopulationType < T, T2, T3 >::Decode (const T2& chromosome) {
    return chromosome;
} // t_AbstractPopulationType::Decode

template < typename T, typename F, typename T2 = t_ChromosomeType < T >, typename T3 = T >
inline double t_BasePopulationType < T, F, T2, T3 >::CalculateFitness (const T2& c) {
    return fFitnessFunction (c);
} // t_BasePopulationType::CalculateFitness
```

(algorithm D.4 continued)

```
template < typename T, typename F, typename P = CTautology, typename P2 = P, typename P3 = P,
            typename T2 = t_ChromosomeType < T >, typename T3 = T >
void CBasePalPop < T, F, P, P2, P3, T2, T3 >::Crossover (void) {
    // Probabilistic pick couples, for each apply DoCrossover and add offspring to fNewGeneration
    for (short i = 0; i < fNumCrossoverPairs; ++i) {
        short crossoverIndex;
        t_CHypothesis < T, T2 > h3 (fChromosomeSize), h4 (fChromosomeSize);
        unsigned long numFailedAttempts = 0;
        do {
            short index, index2;
            index = fWheel->ProbabilisticallySelectedIndex ();
            index2 = fWheel->ProbabilisticallySelectedIndex ();
            crossoverIndex = DoCrossover (at (index).GetChromosome(), at(index2).GetChromosome (),
                                h3.GetChromosome (), h4.GetChromosome ());
        } while ((! fPredicate2 (h3.GetChromosome (), crossoverIndex)
            || ! fPredicate2 (h4.GetChromosome (), crossoverIndex))
            && (++numFailedAttempts < fMaxCrossoverTries));
        if (numFailedAttempts < fMaxCrossoverTries) {
            fNewGeneration.push_back (h3);
            fNewGeneration.push_back (h4);
        }
        else {
            short index, index2;
            index = fWheel->ProbabilisticallySelectedIndex ();
            index2 = fWheel->ProbabilisticallySelectedIndex ();
            fNewGeneration.push_back (at (index));
            fNewGeneration.push_back (at (index2));
        }
    }
} // CBasePalPop::Crossover

template < typename T, typename F, typename P = CTautology, typename P2 = P, typename P3 = P,
            typename T2 = t_ChromosomeType < T >, typename T3 = T >
inline short CBasePalPop < T, F, P, P2, P3, T2, T3 >::DoCrossover (const T2& c, const T2& c2, T2& c3,
T2& c4) {
    return c.Crossover (c2, c3, c4);
} // CBasePalPop::DoCrossover

template < typename T, typename F, typename P = CTautology, typename P2 = P, typename P3 = P,
            typename T2 = t_ChromosomeType < T >, typename T3 = T >
inline void CBasePalPop < T, F, P, P2, P3, T2, T3 >::DoMutate (T2& c) {
    c.MutateWithMaxNumAttempts (fPredicate3);
} // CBasePalPop::DoMutate
```

Algorithm D.4. Class methods of an object-oriented implementation of the genetic algorithm. DoCrossover, DoMutate, and CalculateFitness are null methods in the abstract class t_AbstractPopulationType, and are defined in descendant classes t_BasePopulationType and CBasePalPop.

In an attempt to prevent premature convergence, we employed the technique of fitness scaling (chapter 6, section 6.2). We designed roulette wheel classes that determined which chromosomes were to be selected for the next generation. In the subclass "CScaledRouletteWheel," we overrode the method "SetUpWheel" (see method "SetUpCycle" in algorithm D.4) to use fitness scaling (algorithm D.5); this part of the algorithm is based on Goldberg (1989, 79).

```
template < typename T, typename T2 >
void CScaledRouletteWheel < T, T2 >::SetUpWheel (const populationType& population) {
        double rawMin, rawMax, rawAvg;
        FindPopMinMaxAvg (population, rawMin, rawMax, rawAvg);
        double gradient, intercept;
        CalculateScalingCoefficients (rawMin, rawMax, rawAvg, gradient, intercept);
        double frontier = fMin = 0;
        for (short i = 0; i < population.size (); ++i) {
                frontier += (gradient * ( population [ i ].GetFitness () ) + intercept);
                fWheel [i] = frontier;
        } / / for
        fMax = frontier;
} / / CScaledRouletteWheel::SetUpWheel

template < typename T, typename T2 >
void CScaledRouletteWheel < T, T2 >::FindPopMinMaxAvg (const populationType& population,
double& rawMin, double& rawMax, double& rawAvg) {
        rawMin = numeric_limits < double >::max ();
        rawMax = 0;
        double totalFitness = 0;
        populationType::const_iterator h;
        for (h = population.begin (); h != population.end (); ++h) {
                double fitness = h->GetFitness ();
                if (fitness < rawMin) rawMin = fitness;
                if (fitness > rawMax) rawMax = fitness;
                totalFitness += fitness;
        }
        rawAvg = (population.size () ? totalFitness / population.size () : 0);
} / / CScaledRouletteWheel::FindPopMinMaxAvg

template < typename T, typename T2 >
void CScaledRouletteWheel < T, T2 > :: CalculateScalingCoefficients (double rawMin, double
rawMax, double rawAvg, double& gradient, double& intercept ) {
        double delta;
        if (rawMin > ( fFitnessMultiple * rawAvg - rawMax) / (fFitnessMultiple - 1.0) ) {
                delta = rawMax - rawAvg;
                gradient = (fFitnessMultiple - 1.0) * rawAvg / delta;
                intercept = rawAvg * (rawMax - fFitnessMultiple * rawAvg) / delta;
        }
        else {
                delta = rawAvg - rawMin;
                gradient = rawAvg / delta;
                intercept = -rawMin * rawAvg / delta;
```

(algorithm D.5 continued)

```
        }
} // CScaledRouletteWheel::CalculateScalingCoefficients
```

Algorithm D.5. Fitness scaling in the CScaledRouletteWheel class. The method "SetUpWheel" prepares for its fitness-scaled probabilistic spin by calculating the gradient and intercept through the private method "CalculateScalingCoefficients".

APPENDIX E. OUTPUT OF PROGRAMS USING GENERATE-AND-TEST, BRANCH-AND-BOUND, AND GENETIC ALGORITHMS

We list here the output of the Palestrina application to various cantus firmi, using the exhaustive, best-first branch-and-bound, and genetic algorithms. These outputs have been referred to in chapters 5 and 6. They are as follows: E.1. Two genetic algorithm outputs to cantus firmus <D, F, E, D, G, F, a, G, F, E, D>. with rules weighted to favor Jeppesen's preferences; E.2.1. Genetic algorithm output to cantus firmus <D, a, G, F, E, D, F, E, D> in second species with one added part; E.2.2. Genetic algorithm output to cantus firmus <D, a, G, F, E, D, F, E, D> in first species with two added parts; E.3.1. Best-first algorithm to cantus firmus <d, c, h, c, h, a, h, a, F#, G>, with trained neural network evaluation that is not non-increasing with respect to partial countermelody length; E.3.2. Best-first algorithm to cantus firmus <d, c, h, c, h, a, h, a, F#, G>, with trained neural network evaluation that is (almost) non-increasing with respect to partial countermelody length.

E.1. Two Genetic Algorithm Outputs with Rules Weighted to Favor Jeppesen's Preferences

Listing E.1 shows the output of the genetic algorithm based on rules weighted according to the instance of the Jeppesen experiment described in chapter 5, section 5.1, giving preference to the Note Variety rule. The genetic algorithm produced the optimal countermelody in just seven iterations; this optimal countermelody is also Jeppesen's.

Listing E.1. Genetic algorithm output to cantus firmus <D, F, E, D, G, F, a, G, F, E, D> with rules weighted to favor Jeppesen's choice: optimal countermelodies found in seven iterations

```
Cantus firmus: D F E D G F a G F E D
    crossover rate: 0.6
    mutation rate: 0.1
    max num init tries: 50
    max num crossover tries: 20
    max num mutation tries: 20
    fitness multiple: 1.2
    population size: 50
    num iterations: 50
    elitist model, max num elites: 50
Initial hypotheses:
chromosome                              fitness
a  a  G  F  E  D  A  B  D  C# D   0.872727
a  aa g  f  e  d  e  h  d  c# d   0.872727
a  a  G  F  E  D  A  B  D  C# D   0.872727
a  a  G  F  E  D  C  B  D  C# D   0.854545
aa aa g  f  e  d  c  h  d  c# d   0.854545
aa aa g  f  e  d  c  h  d  c# d   0.854545
a  a  G  F  C  D  C  E  D  C# D   0.836364
a  a  c  d  e  f  e  h  d  c# d   0.836364
a  a  G  F  C  D  C  E  D  C# D   0.836364
aa d  g  f  e  f  c  e  d  c# d   0.836364
a  a  G  F  C  D  C  E  D  C# D   0.836364
```

(listing E.1 continued)

```
aa aa g  f  e  f  c  e  d  c# d   0.836364
aa d  g  f  e  d  e  h  d  c# d   0.836364
a  D  G  F  C  D  C  E  D  C# D   0.836364
a  D  G  F  C  D  C  E  D  C# D   0.836364
D  D  A  F  E  D  C  B  D  C# D   0.818182
d  d  G  a  h  d  c  e  d  c# d   0.818182
d  a  G  a  h  d  c  e  d  c# d   0.818182
d  c  G  a  h  d  c  e  d  c# d   0.818182
a  a  G  a  h  d  c  e  d  c# d   0.818182
D  D  C  F  E  D  A  B  D  C# D   0.818182
a  a  h  d  e  f  c  e  d  c# d   0.818182
d  d  G  a  h  d  c  e  d  c# d   0.818182
a  a  G  a  h  d  c  e  d  c# d   0.818182
d  d  c  d  e  f  e  h  d  c# d   0.8
d  d  g  f  e  d  c  e  d  c# d   0.8
d  a  e  f  e  d  c  h  d  c# d   0.8
d  c  g  f  e  d  c  h  d  c# d   0.8
d  d  g  f  e  d  c  e  d  c# d   0.8
d  c  e  f  e  d  e  h  d  c# d   0.8
d  c  g  f  e  d  c  h  d  c# d   0.8
d  d  c  d  e  f  e  h  d  c# d   0.8
D  D  G  F  E  D  C  E  D  C# D   0.8
a  a  h  d  e  f  e  h  d  c# d   0.781818
d  c  g  f  e  f  c  e  d  c# d   0.781818
a  a  h  d  e  f  e  h  d  c# d   0.781818
d  d  c  d  e  f  c  e  d  c# d   0.763636
d  d  c  f  e  d  c  h  d  c# d   0.763636
d  a  h  d  e  d  c  h  d  c# d   0.763636
d  d  c  f  e  d  c  h  d  c# d   0.763636
D  D  C  F  E  D  C  B  D  C# D   0.763636
d  d  c  f  e  d  c  h  d  c# d   0.763636
d  a  h  d  e  d  c  h  d  c# d   0.763636
d  d  c  f  e  d  c  h  d  c# d   0.763636
d  d  c  f  e  d  c  h  d  c# d   0.763636
d  d  c  d  e  d  c  h  d  c# d   0.745455
d  d  c  f  e  d  c  e  d  c# d   0.745455
D  D  C  F  E  D  C  E  D  C# D   0.745455
d  d  c  f  e  d  c  e  d  c# d   0.745455
C  a  G  F  C  D  C  GG D  GG a   0
```

Average fitness: 0.789455

Performing genetic algorithm: Please wait!

iteration	**fitness**		
	average	**best**	
		local	**global**
0.	0.789455	0.872727	0.872727
1.	0.750545	0.872727	0.872727
2.	0.803636	0.872727	0.872727
3.	0.8	0.872727	0.872727
4.	0.802182	0.872727	0.872727
5.	0.805455	0.872727	0.872727
6.	0.796364	0.872727	0.872727
7.	0.789091	0.890909	0.890909
8.	0.802545	0.890909	0.890909
9.	0.808	0.890909	0.890909
10.	0.794182	0.890909	0.890909
11.	0.803273	0.890909	0.890909
12.	0.802182	0.890909	0.890909

(listing E.1 continued)

```
13.     0.798545        0.890909        0.890909
14.     0.814545        0.890909        0.890909
15.     0.814545        0.890909        0.890909
16.     0.818545        0.890909        0.890909
17.     0.818909        0.890909        0.890909
18.     0.818909        0.890909        0.890909
19.     0.819636        0.890909        0.890909
20.     0.816727        0.890909        0.890909
21.     0.822182        0.890909        0.890909
22.     0.829455        0.890909        0.890909
23.     0.836727        0.890909        0.890909
24.     0.816727        0.890909        0.890909
25.     0.824727        0.890909        0.890909
26.     0.823636        0.890909        0.890909
27.     0.841455        0.890909        0.890909
28.     0.846182        0.890909        0.890909
29.     0.849455        0.890909        0.890909
30.     0.853818        0.890909        0.890909
31.     0.856364        0.890909        0.890909
32.     0.844           0.890909        0.890909
33.     0.864           0.890909        0.890909
34.     0.868           0.890909        0.890909
35.     0.871636        0.890909        0.890909
36.     0.875636        0.890909        0.890909
37.     0.862182        0.890909        0.890909
38.     0.876364        0.890909        0.890909
39.     0.879636        0.890909        0.890909
40.     0.881455        0.890909        0.890909
41.     0.885091        0.890909        0.890909
42.     0.890909        0.890909        0.890909
43.     0.890909        0.890909        0.890909
44.     0.890909        0.890909        0.890909
45.     0.890909        0.890909        0.890909
46.     0.890909        0.890909        0.890909
47.     0.890909        0.890909        0.890909
48.     0.890182        0.890909        0.890909
49.     0.890182        0.890909        0.890909
50.     0.890182        0.890909        0.890909
```

Number of iterations performed: 50

Final hypotheses:

chromosome	**fitness**
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909
a aa g f e d c h d c# d	0.890909

(listing E.1 continued)

```
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
aa aa g  f  e  d  c  h  d  c# d   0.854545

Average fitness: 0.890182
Fittest population found on iteration #42:
chromosome                        fitness
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
```

(listing E.1 continued)

```
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
a  aa g  f  e  d  c  h  d  c# d   0.890909
Its average fitness: 0.890909
List of successively fitter hypotheses as the population evolved:
a  a  G  F  E  D  A  B  D  C# D   0.872727      0
a  aa g  f  e  d  c  h  d  c# d   0.890909      7
Best hypothesis first found:
a  aa g  f  e  d  c  h  d  c# d   0.890909      7
```

Another run of the genetic algorithm, this time using a smaller population size (20), also produced the optimal countermelody. Here we also asked the program to append information on the chromosomes to enable us to track the evolution of the individual chromosomes (INITIAL: chromosome created in the initial population; SELECTION: chromosome probabilistically selected from the previous generation [followed by the number of the chromosome in that previous generation]; CROSSOVER: chromosome created by crossover [ordinal number of the parents also indicated]; MUTATION: chromosome created by mutation). See chapter 6, section 6.2 for an outline of the evolution of the optimal countermelody (Jeppesen's choice: <a, aa, g, f, e, d, c, h, d, c#, d>).

Listing E.2. Genetic algorithm output to cantus firmus <D, F, E, D, G, F, a, G, F, E, D> with rules weighted to favor Jeppesen's choice: optimal countermelodies found in three iterations. Shown also is additional information to enable tracking of ancestry of individual chromosomes.

```
Cantus firmus: D F E D G F a G F E D
    crossover rate: 0.6
    mutation rate: 0.1
    max num init tries: 50
```

(listing E.2 continued)

```
    max num crossover tries: 20
    max num mutation tries: 20
    fitness multiple: 1.2
    population size: 20
    num iterations: 50
    elitist model, max num elites: 20
Initial hypotheses:
```

chromosome	**fitness**	**operation**
a aa g f e f e h d c# d	0.872727	INITIAL
a D G F C D C E D C# D	0.836364	INITIAL
aa f g f e f e h d c# d	0.836364	INITIAL
d a c d e f e h d c# d	0.836364	INITIAL
aa d g f e f e h d c# d	0.836364	INITIAL
a a G F C D C E D C# D	0.836364	INITIAL
d c G a h d c e d c# d	0.818182	INITIAL
d a G a h d c e d c# d	0.818182	INITIAL
d a G a h d c e d c# d	0.818182	INITIAL
d a G a h d c e d c# d	0.818182	INITIAL
D D A F E D A B D C# D	0.8	INITIAL
d a e f e d c h d c# d	0.8	INITIAL
a a e f e d c h d c# d	0.8	INITIAL
a a h d e f e h d c# d	0.781818	INITIAL
d c e f e d c h d c# d	0.781818	INITIAL
d a c d e d c h d c# d	0.781818	INITIAL
d d c f e d c h d c# d	0.763636	INITIAL
d a h d e d c h d c# d	0.763636	INITIAL
a a h d e d c h d c# d	0.763636	INITIAL
d d c d e d c h d c# d	0.745455	INITIAL

Average fitness: 0.805455

Performing genetic algorithm: Please wait!

iteration	**fitness**		
	average	**best**	
		local	**global**
0.	0.81	0.87	0.87
1.	0.80	0.87	0.87
2.	0.81	0.87	0.87
3.	0.74	0.89	0.89
4.	0.84	0.89	0.89
5.	0.72	0.89	0.89
6.	0.82	0.89	0.89
7.	0.86	0.89	0.89
8.	0.87	0.89	0.89
9.	0.83	0.89	0.89
10.	0.86	0.89	0.89
11.	0.85	0.89	0.89
12.	0.86	0.89	0.89
13.	0.86	0.89	0.89
14.	0.89	0.89	0.89
15.	0.89	0.89	0.89
16.	0.89	0.89	0.89
17.	0.89	0.89	0.89
18.	0.89	0.89	0.89
19.	0.89	0.89	0.89
20.	0.89	0.89	0.89
21.	0.89	0.89	0.89
22.	0.89	0.89	0.89
23.	0.89	0.89	0.89

(listing E.2 continued)

```
24.     0.89    0.89    0.89
25.     0.89    0.89    0.89
26.     0.89    0.89    0.89
27.     0.89    0.89    0.89
28.     0.89    0.89    0.89
29.     0.85    0.89    0.89
30.     0.89    0.89    0.89
31.     0.89    0.89    0.89
32.     0.89    0.89    0.89
33.     0.85    0.89    0.89
34.     0.89    0.89    0.89
35.     0.85    0.89    0.89
36.     0.89    0.89    0.89
37.     0.89    0.89    0.89
38.     0.85    0.89    0.89
39.     0.89    0.89    0.89
40.     0.89    0.89    0.89
41.     0.89    0.89    0.89
42.     0.89    0.89    0.89
43.     0.89    0.89    0.89
44.     0.85    0.89    0.89
45.     0.89    0.89    0.89
46.     0.89    0.89    0.89
47.     0.89    0.89    0.89
48.     0.89    0.89    0.89
49.     0.89    0.89    0.89
50.     0.89    0.89    0.89
```

Number of iterations performed: 50

Final hypotheses:

chromosome	**fitness**	**operation**	**ancestry**		
a aa g f e d c h d c# d	0.89	SELECTION	8		
a aa g f e d c h d c# d	0.89	SELECTION	12		
a aa g f e d c h d c# d	0.89	SELECTION	18		
a aa g f e d c h d c# d	0.89	SELECTION	4		
a aa g f e d c h d c# d	0.89	SELECTION	15		
a aa g f e d c h d c# d	0.89	SELECTION	5		
a aa g f e d c h d c# d	0.89	SELECTION	8		
a aa g f e d c h d c# d	0.89	SELECTION	18		
a aa g f e d c h d c# d	0.89	CROSSOVER	14	3	
a aa g f e d c h d c# d	0.89	CROSSOVER	14	3	MUTATION
a aa g f e d c h d c# d	0.89	CROSSOVER	13	19	
a aa g f e d c h d c# d	0.89	CROSSOVER	13	19	
a aa g f e d c h d c# d	0.89	CROSSOVER	5	17	
a aa g f e d c h d c# d	0.89	CROSSOVER	5	17	MUTATION
a aa g f e d c h d c# d	0.89	CROSSOVER	2	14	
a aa g f e d c h d c# d	0.89	CROSSOVER	2	14	
a aa g f e d c h d c# d	0.89	CROSSOVER	7	4	
a aa g f e d c h d c# d	0.89	CROSSOVER	7	4	
a aa g f e d c h d c# d	0.89	CROSSOVER	8	16	
a aa g f e d c h d c# d	0.89	CROSSOVER	8	16	

Average fitness: 0.89
Fittest population found on iteration #15:

chromosome	**fitness**	**operation**	**ancestry**
a aa g f e d c h d c# d	0.89	SELECTION	8
a aa g f e d c h d c# d	0.89	SELECTION	12
a aa g f e d c h d c# d	0.89	SELECTION	10
a aa g f e d c h d c# d	0.89	SELECTION	5

(listing E.2 continued)

```
a  aa g  f  e  d  c  h  d  c# d   0.89     SELECTION     0
a  aa g  f  e  d  c  h  d  c# d   0.89     SELECTION     14
a  aa g  f  e  d  c  h  d  c# d   0.89     SELECTION     17
a  aa g  f  e  d  c  h  d  c# d   0.89     SELECTION     14
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     8      0
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     8      0
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     16     9
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     16     9
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     2      0
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     2      0
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     11     10
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     11     10
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     9      7      MUTATION
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     9      7
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     10     7
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     10     7
Its average fitness: 0.89
List of successively fitter hypotheses as the population evolved:
a  aa g  f  e  f  e  h  d  c# d   0.87     INITIAL     0
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     17     1          3
Best hypothesis first found:
a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     17     1          3

****0. 0.81, 0.87, 0.87
       chromosome                          fitness   operation
 0.    a  aa g  f  e  f  e  h  d  c# d   0.87     INITIAL
 1.    a  D  G  F  C  D  C  E  D  C# D   0.84     INITIAL
 2.    aa f  g  f  e  f  e  h  d  c# d   0.84     INITIAL
 3.    d  a  c  d  e  f  e  h  d  c# d   0.84     INITIAL
 4.    aa d  g  f  e  f  e  h  d  c# d   0.84     INITIAL
 5.    a  a  G  F  C  D  C  E  D  C# D   0.84     INITIAL
 6.    d  c  G  a  h  d  c  e  d  c# d   0.82     INITIAL
 7.    d  a  G  a  h  d  c  e  d  c# d   0.82     INITIAL
 8.    d  a  G  a  h  d  c  e  d  c# d   0.82     INITIAL
 9.    d  a  G  a  h  d  c  e  d  c# d   0.82     INITIAL
10.    D  D  A  F  E  D  A  B  D  C# D   0.80     INITIAL
11.    d  a  e  f  e  d  c  h  d  c# d   0.80     INITIAL
12.    a  a  e  f  e  d  c  h  d  c# d   0.80     INITIAL
13.    a  a  h  d  e  f  e  h  d  c# d   0.78     INITIAL
14.    d  c  e  f  e  d  c  h  d  c# d   0.78     INITIAL
15.    d  a  c  d  e  d  c  h  d  c# d   0.78     INITIAL
16.    d  d  c  f  e  d  c  h  d  c# d   0.76     INITIAL
17.    d  a  h  d  e  d  c  h  d  c# d   0.76     INITIAL
18.    a  a  h  d  e  d  c  h  d  c# d   0.76     INITIAL
19.    d  d  c  d  e  d  c  h  d  c# d   0.75     INITIAL
***1. 0.80, 0.87, 0.87
       chromosome                          fitness   operation     ancestry
 0.    a  aa g  f  e  f  e  h  d  c# d   0.87     CROSSOVER     0      4
 1.    a  a  G  F  E  D  C  E  D  C# D   0.84     SELECTION     5      MUTATION
 2.    aa d  g  f  e  f  e  h  d  c# d   0.84     SELECTION     4
 3.    aa f  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     4      2
 4.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     4      2
 5.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     0      4
 6.    d  a  c  d  e  f  e  h  d  c# d   0.84     CROSSOVER     3      12
 7.    d  a  G  a  h  d  c  e  d  c# d   0.82     CROSSOVER     8      16
 8.    D  D  A  F  E  D  A  B  D  C# D   0.80     SELECTION     10
 9.    D  D  A  F  E  D  A  B  D  C# D   0.80     SELECTION     10
10.    a  a  e  f  e  d  c  h  d  c# d   0.80     SELECTION     12
11.    d  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     19     11
12.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     3      12
```

(listing E.2 continued)

```
13.    a  a  h  d  e  f  e  h  d  c# d   0.78     SELECTION     13
14.    d  c  e  f  e  d  c  h  d  c# d   0.78     CROSSOVER     14      18
15.    d  a  h  d  e  d  c  h  d  c# d   0.76     SELECTION     15      MUTATION
16.    a  a  h  d  e  d  c  h  d  c# d   0.76     CROSSOVER     14      18
17.    d  d  c  f  e  d  c  h  d  c# d   0.76     CROSSOVER     8       16
18.    d  d  c  d  e  d  c  h  d  c# d   0.75     SELECTION     19
19.    d  d  c  d  e  d  c  h  d  c# d   0.75     CROSSOVER     19      11
****2. 0.81, 0.87, 0.87
       chromosome                                 fitness   operation     ancestry
 0.    a  aa g  f  e  f  e  h  d  c# d   0.87     SELECTION     0       MUTATION
 1.    a  aa g  f  e  f  e  h  d  c# d   0.87     CROSSOVER     2       0
 2.    aa d  g  f  e  f  e  h  d  c# d   0.84     SELECTION     2
 3.    aa d  g  f  e  f  e  h  d  c# d   0.84     SELECTION     5
 4.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     2       0
 5.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     5       10
 6.    a  a  c  d  e  f  e  h  d  c# d   0.84     CROSSOVER     10      6       MUTATION
 7.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     5       4
 8.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     5       4
 9.    d  a  e  f  e  d  c  h  d  c# d   0.80     SELECTION     11
10.    D  D  A  F  E  D  A  B  D  C# D   0.80     SELECTION     9
11.    d  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     14      11
12.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     5       10
13.    d  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     10      6
14.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     16      12
15.    a  a  h  d  e  f  e  h  d  c# d   0.78     SELECTION     13
16.    a  a  h  d  e  f  e  h  d  c# d   0.78     SELECTION     13
17.    d  c  e  f  e  d  c  h  d  c# d   0.78     CROSSOVER     14      11
18.    d  a  h  d  e  d  c  h  d  c# d   0.76     SELECTION     15
19.    a  a  h  d  e  d  c  h  d  c# d   0.76     CROSSOVER     16      12
***3. 0.74, 0.89, 0.89
       chromosome                                 fitness   operation     ancestry
 0.    a  aa g  f  e  d  c  h  d  c# d   0.89     CROSSOVER     17      1
 1.    a  aa g  f  e  f  e  h  d  c# d   0.87     SELECTION     0       MUTATION
 2.    a  aa g  f  e  f  e  h  d  c# d   0.87     CROSSOVER     1       14
 3.    a  aa g  f  e  f  e  h  d  c# d   0.87     CROSSOVER     7       0
 4.    aa d  g  f  e  d  c  h  d  c# d   0.85     CROSSOVER     12      3
 5.    aa d  g  f  e  f  e  h  d  c# d   0.84     SELECTION     5
 6.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     7       0
 7.    aa d  g  f  e  f  e  h  d  c# d   0.84     CROSSOVER     12      2
 8.    a  a  e  f  e  d  c  h  d  c# d   0.80     SELECTION     12
 9.    a  a  e  f  e  d  c  h  d  c# d   0.80     SELECTION     14
10.    a  a  e  f  e  d  c  h  d  c# d   0.80     SELECTION     12      MUTATION
11.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     1       14
12.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     12      2
13.    a  a  e  f  e  d  c  h  d  c# d   0.80     CROSSOVER     12      16
14.    d  c  e  f  e  d  c  h  d  c# d   0.78     SELECTION     17
15.    a  a  h  d  e  f  e  h  d  c# d   0.78     SELECTION     15
16.    a  a  h  d  e  f  e  h  d  c# d   0.78     SELECTION     15
17.    a  a  h  d  e  f  e  h  d  c# d   0.78     CROSSOVER     12      16
18.    d  c  e  f  e  f  e  h  d  c# d   0.00     CROSSOVER     17      1
19.    a  a  e  f  e  f  e  h  d  c# d   0.00     CROSSOVER     12      3
```

E.1.1. Average Schema and Population Fitness, and Estimated Bounds on Occurrences of Instances

The following tables (E.1–3) shows the average schema and population fitness, derived from the second run of the genetic algorithm to cantus firmus <D, F, E, D, G, F, a, G, F, E, D> as described in section E.1. We used this to estimate the lower- and upper-bounds on occurrences of representatives of schemata "a aa * * * * * * * * *", "* * g f e * * * * * *", and "* * * * * * * h d c# d" (chapter 6, section 6.2.1).

Table E.1. Average schema and population fitness, and estimated bounds of schema (a) "a aa * * * * * * * * *"; (b) "* * g f e * * * * * *"; (c) "* * * * * * * h d c# d"

(a) Schema "a aa * * * * * * * * *"

t	schema fitness (s, t-1)	pop-ulation fitness (t-1)	schema occurrence (s, t-1)	lower bound estimate [schema occurrence (s, t)]	upper bound estimate [schema occurrence (s, t)]	schema occurrence (s, t)
1	0.87	0.81	1	0.82	1.07	1
2	0.87	0.80	1	0.83	1.09	2
3	0.87	0.81	2	1.64	2.15	4
4	0.88	0.74	4	3.60	4.73	7
5	0.87	0.84	7	5.54	7.27	8
6	0.88	0.72	8	7.42	9.75	10
7	0.88	0.82	10	8.21	10.78	11
8	0.89	0.86	11	8.61	11.31	10
9	0.89	0.87	10	7.79	10.23	12
10	0.82	0.83	12	8.98	11.80	7
11	0.89	0.86	7	5.52	7.24	4
12	0.89	0.85	4	3.19	4.19	7
13	0.89	0.86	7	5.52	7.24	10
14	0.89	0.86	10	7.88	10.35	18
15	0.89	0.89	18	13.71	18.00	20
16	0.89	0.89	20	15.23	20.00	20
17	0.89	0.89	20	15.23	20.00	20
18	0.89	0.89	20	15.23	20.00	20
19	0.89	0.89	20	15.23	20.00	20
20	0.89	0.89	20	15.23	20.00	20

(table E.1 continued)

(b) Schema "* * g f e * * * * * *"

t	schema fitness (s, t-1)	population fitness (t-1)	schema occurrence (s, t-1)	lower bound estimate [schema occurrence (s, t)]	upper bound estimate [schema occurrence (s, t)]	schema occurrence (s, t)
1	0.85	0.81	3	2.02	3.15	5
2	0.85	0.80	5	3.39	5.29	8
3	0.85	0.81	8	5.37	8.37	8
4	0.86	0.74	8	5.96	9.28	12
5	0.86	0.84	12	7.89	12.30	13
6	0.87	0.72	13	10.04	15.65	16
7	0.87	0.82	16	10.91	17.00	17
8	0.87	0.86	17	11.06	17.23	18
9	0.87	0.87	18	11.57	18.03	18
10	0.78	0.83	18	10.85	16.92	16
11	0.87	0.86	16	10.35	16.14	14
12	0.86	0.85	14	9.10	14.19	16
13	0.87	0.86	16	10.35	16.14	16
14	0.88	0.86	16	10.44	16.28	20
15	0.89	0.89	20	12.83	20.00	20
16	0.89	0.89	20	12.83	20.00	20
17	0.89	0.89	20	12.83	20.00	20
18	0.89	0.89	20	12.83	20.00	20
19	0.89	0.89	20	12.83	20.00	20
20	0.89	0.89	20	12.83	20.00	20

(table E.1 continued)

(c) Schema "* * * * * * * h d c# d"

t	schema fitness (s, t-1)	pop-ulation fitness (t-1)	schema occurrence (s, t-1)	lower bound estimate [schema occurrence (s, t)]	upper bound estimate [schema occurrence (s, t)]	schema occurrence (s, t)
1	0.80	0.81	13	6.88	12.80	16
2	0.80	0.80	16	8.61	16.01	19
3	0.81	0.81	19	10.28	19.11	20
4	0.74	0.74	20	10.76	20.0	20
5	0.84	0.84	20	10.76	20.0	20
6	0.72	0.72	20	10.76	20.0	20
7	0.82	0.82	20	10.76	20.0	20
8	0.86	0.86	20	10.76	20.0	20
9	0.87	0.87	20	10.76	20.0	20
10	0.87	0.83	19	10.69	19.87	20
11	0.86	0.86	20	10.76	20.0	20
12	0.85	0.85	20	10.76	20.0	20
13	0.86	0.86	20	10.76	20.0	20
14	0.86	0.86	20	10.76	20.0	20
15	0.89	0.89	20	10.76	20.0	20
16	0.89	0.89	20	10.76	20.0	20
17	0.89	0.89	20	10.76	20.0	20
18	0.89	0.89	20	10.76	20.0	20
19	0.89	0.89	20	10.76	20.0	20
20	0.89	0.89	20	10.76	20.0	20

E.2. Genetic Algorithm Performance in Second Species, and First Species in Three Parts

We assess the performance of the genetic algorithm in the following two sections in more complex forms of species counterpoint, namely, second species in two parts, and first species in three parts. The number of possible solutions is considerably increased; for example, there are thousands of optimal solutions to the first instance below.

E.2.1. Output to Cantus Firmus <D, a, G, F, E, D, F, E, D> in Second Species with One Added Part

Despite an initial population with very low average fitness, the genetic algorithm succeeded in finding several near-optimal countermelodies within 50 iterations (compared to Jeppesen's solution, which the fuzzified rules gave a score of 0.99: see example 6.9). Notice that the single valid countermelody in the

initial population is not optimal, so the optimal countermelodies all must have evolved from either crossover or mutation.

Listing E.3. Genetic algorithm output to cantus firmus <D, a, G, F, E, D, F, E, D> in second species, one added part

```
Cantus firmus: D a G F E D F E D
Alpha cut: 0
Lewin's rule (second species) ro1: 1 w: 1 bonus: 0
Law of recovery ro1: 1 w: 1 bonus: 0
Outline tritone in four ro1: 1 w: 1 bonus: 0
Harmonic range ro1: 1 w: 1 bonus: 0
Unisons (second species) ro1: 1 w: 1 bonus: 0
Simul skip same direction ro1: 1 w: 1 bonus: 0
Unisons on weak beats approached by leap (second species) ro1: 1 w: 1 bonus: 0
    crossover rate: 0.6
    mutation rate: 0.1
    max num init tries: 50
    max num crossover tries: 20
    max num mutation tries: 20
    fitness multiple: 1.2
    population size: 100
    num iterations: 50
    max num no change before stopping: 3
    elitist model
Initial hypotheses:
chromosome                                                  fitness
a  f  c  d  e  E  a  b  c  G  F  a  c  F  E  C# D          0.97
h  g  c  g  e  E  c  h  e  E  F  D  d  f  e  c# d          0
c# h  E  aa A  R  h  a  h  a  G  F  C  Bb F  D  a          0
GG c# GG R  h  R  aa g  Bb Bb e  a  G  F  d  c# d          0
FF d  D  d  d  c  a  h  G  F  FF c  F  aa A  C# D          0
A  f# a  D  C# g  E  D  Bb D  B  e  d  GG d  e  a          0
B  g  C  FF B  f  E  F  G  F  D  d  f  d  e  aa a          0
GG C  c  a  Bb c  F  f# a  A  D  FF E  e  c# c# d          0
c  F  h  f  B  F  b  G  c  B  A  e  a  d  e  c# d          0
E  A  Bb B  h  c  d  h  c  F  b  G  B  D  C# C# D          0
c# FF R  FF FF E  R  G  FF a  aa C  g  f  e  c# d          0
A  c  h  F  FF f# G  a  G  F  e  f  E  a  a  c  a          0
e  G  d  R  G  a  C# aa h  h  A  Bb G  a  e  R  a          0
D  G  F  D  F  G  E  c  h  c# g  aa C  f# b  f  a          0
f# A  GG d  g  f# e  g  c  b  a  h  a  e  a  G  D          0
e  R  Bb A  B  h  a  R  FF C  F  E  a  h  c# c# d          0
a  d  B  A  C  B  c# G  h  c  h  B  D  C  A  FF a          0
C# E  D  F  D  b  F  G  d  c# E  a  c# C  a  h  a          0
d  Bb h  c  h  E  d  C  F  g  F  Bb B  E  e  a  a          0
R  c# B  A  GG A  C  FF A  C  GG D  A  D  E  C# D          0
B  b  FF F  E  a  D  aa Bb F  f  d  f  aa g  aa a          0
FF C  g  Bb e  FF Bb FF A  GG FF A  FF C  R  Bb a          0
GG B  R  GG G  C  D  E  F  G  d  G  C  Bb d  c# d          0
f# C  C  Bb C# Bb b  a  D  E  G  C  e  c  C  c  a          0
E  F  F  C# FF Bb c# h  FF h  h  b  c  g  B  C# D          0
R  G  e  d  E  C# e  C  F  C  B  h  C# c# C  c  a          0
g  R  C  Bb R  FF c  R  c# FF g  c  h  d  e  c# d          0
F  f  h  F  f# h  aa aa B  R  G  f  D  E  FF f# a          0
FF R  aa f# C# F  c# GG C# a  FF G  c# G  GG aa a          0
R  a  R  B  R  FF Bb B  G  F  E  C  A  D  C# C# D          0
R  f  h  f# g  D  Bb E  c  C  F  FF F  D  C  d  d          0
c# F  C  c  h  a  F  c  B  h  B  A  G  F  C  C  a          0
E  F  aa FF Bb G  d  aa B  A  E  h  E  c  g  R  a          0
```

(listing E.3 continued)

```
B  G  b  a  d  F  a  c  B  GG FF FF A  B  d  c# d        0
e  c  aa e  a  G  R  Bb D  g  R  c# f# e  A  C# D        0
a  d  G  D  b  F  c  h  a  FF F  b  B  A  C# D  a        0
f  D  b  a  b  F  FF GG C  Bb A  D  GG G  d  c# d        0
a  F  GG g  GG d  C  A  GG E  f  A  a  c# D  C# D        0
Bb e  c  d  D  F  c  f  c  b  a  f  b  C  E  C# D        0
d  G  b  a  h  B  E  D  C# d  D  E  F  E  C# C# D        0
c  e  h  g  d  a  h  Bb d  c# G  F  C  c  d  e  a        0
f# g  e  GG G  GG A  FF C  GG Bb GG A  B  C# C# D        0
D  a  E  G  C  F  a  E  c  b  a  F  Bb b  B  C# D        0
Bb A  g  g  FF A  c# C  A  F  a  G  F  FF a  c# d        0
Bb c  A  G  Bb GG B  C  f  a  d  c  h  c# d  D  a        0
c  c  G  h  b  f  c# GG d  a  F  D  h  E  e  c# d        0
c  d  e  d  R  G  A  g  b  f  c  F  e  f# a  c# d        0
R  e  C# f  f# d  aa C  A  C  FF F  h  a  E  h  d        0
R  D  A  C  GG A  Bb FF F  E  E  D  GG A  D  C# D        0
b  A  F  f  c# F  D  C# B  C  B  h  B  A  E  C# D        0
C# C  c  b  A  B  G  d  Bb D  a  A  Bb B  E  C# D        0
f# g  F  R  Bb aa B  B  f# D  R  C# E  E  C# C# D        0
D  E  F  D  C  E  E  g  G  c  d  aa f  c  e  c# d        0
c  c  C  c  g  f  b  G  aa g  c  C  C  G  h  c# d        0
E  a  A  A  A  E  D  F  C  d  G  a  B  B  GG D  a        0
a  h  c  E  C  c  a  f  e  c# e  h  c  G  c  R  a        0
C# GG f# D  b  E  a  D  e  B  E  g  f  B  E  C# D        0
R  G  R  f# FF c  c  d  a  f  G  G  C# e  B  F  a        0
Bb GG g  d  b  FF aa C# h  Bb C  c# B  c# aa f# a        0
b  F  g  D  G  f  B  GG B  A  C  A  h  c  h  c# d        0
B  C  b  aa R  F  A  g  f# B  f  c# F  f  Bb B  a        0
A  B  b  Bb Bb Bb c  B  f  g  f  C# E  g  D  C# D        0
D  B  A  E  a  h  c  C  A  C  D  a  GG R  D  C# D        0
c# aa h  g  D  h  e  e  d  D  GG FF C  B  A  C# D        0
c# e  C# C# C  f# FF c# f# c# a  C  B  b  C# f  a        0
F  FF f# b  D  c# f# b  B  c  GG a  B  C# Bb f# a        0
C  C  c  a  b  Bb D  F  D  C# A  a  A  B  C# C# D        0
a  d  h  G  R  g  GG C  h  GG e  e  h  c# d  c# d        0
g  g  c  f# B  FF g  e  f# c  D  FF A  E  aa FF a        0
a  g  c  D  e  a  F  FF GG D  b  F  Bb G  E  Bb a        0
GG g  aa R  aa g  g  R  R  a  g  A  C# FF aa f# a        0
d  aa e  aa e  Bb e  FF f# F  D  d  g  c  b  f  a        0
f  d  E  b  D  C# c# B  B  f  E  a  F  h  A  C# D        0
b  C  c# R  FF F  b  e  f  f  G  F  B  E  h  e  a        0
E  a  c# g  f  aa f  c  E  B  g  FF E  e  c  F  a        0
a  g  E  c  h  e  d  D  E  D  d  e  c  a  h  c# d        0
C  D  C  d  G  d  a  h  c  b  a  h  a  F  E  C# D        0
a  C# E  c  A  Bb B  A  D  G  h  B  D  Bb A  C# D        0
Bb aa a  c# c  C  G  E  b  a  f  aa h  c  d  c# d        0
GG G  Bb Bb c# g  A  C  C  E  d  f# B  D  A  C# D        0
d  D  A  Bb C  G  F  FF C  A  F  E  D  G  E  C# D        0
FF F  FF c  Bb Bb G  g  d  h  d  h  c  h  a  c# d        0
h  g  aa a  c# GG e  A  FF b  D  h  F  f  g  Bb a        0
F  b  c# B  g  b  R  C# c  g  b  e  h  aa A  C# D        0
D  a  h  e  d  g  aa f  a  c# E  h  d  D  E  C# D        0
C  FF R  A  C  FF FF R  a  FF e  a  d  c  e  c# d        0
E  B  d  a  h  d  a  G  R  f# C  C  E  c  e  c# d        0
h  B  e  g  F  a  C  c  h  c  d  D  d  B  F  a  a        0
c# c# GG F  C# G  g  E  c  a  C  B  D  C  B  C# D        0
aa R  g  c# Bb h  C# A  aa d  h  E  h  a  e  c# a        0
F  e  a  e  h  c  d  b  F  R  g  e  aa e  F  f# a        0
c# E  D  a  d  FF C  R  F  a  F  D  E  a  A  R  a        0
f  e  b  h  d  C# E  h  c  D  G  F  h  d  c# c# d        0
```

(listing E.3 continued)

```
FF G  C  R  F  G  F  D  G  D  e  h  g  d  c# c# d        0
h  F  e  f# R  FF f  FF b  d  E  e  D  f# f# B  a        0
aa f  c  E  FF Bb aa C  Bb e  R  A  e  e  E  e  a        0
D  F  GG R  A  Bb GG FF A  Bb C  Bb E  h  G  FF a        0
b  GG F  a  GG B  e  B  a  E  h  B  A  B  C# C# D        0
aa FF h  c# g  D  b  e  FF c# f  D  b  f  B  C# D        0
a  B  d  f  C# C# e  F  c# f  a  A  Bb e  c# c# d        0
```

Average fitness: 0.0097

Optimal countermelodies	**Grade**	**Iteration**
1. a f c d e E a b c G F a D F E C# D	0.99	16
2. d f c d e E a b c G F a D F E C# D	0.99	22
3. a f c d e h a b c G F a D F E C# D	0.99	22
4. a f c d e E a h c G F a D F E C# D	0.99	26
5. a f c d e d a b c G F a D F E C# D	0.99	27
6. d f c d e h a b c G F a D F E C# D	0.99	29
7. a d c d e E a b c G F a D F E C# D	0.99	30
8. a f c d e h a h c G F a D F E C# D	0.99	32
9. a b c f e E a b c G F a D F E C# D	0.99	34
10. d f c d e E a h c G F a D F E C# D	0.99	40
11. a d c d e h a h c G F a D F E C# D	0.99	41
12. d f c d e d a b c G F a D F E C# D	0.99	41
13. d f c d e h a h c G F a D F E C# D	0.99	44
14. a f c d e h a d c G F a D F E C# D	0.99	46
15. d b c f e E a b c G F a D F E C# D	0.99	46
16. d f c d e d a h c G F a D F E C# D	0.99	46
17. a h c d e E a b c G F a D F E C# D	0.99	48
18. a b c f e E a h c G F a D F E C# D	0.99	49
19. a b c d e E a b c G F a D F E C# D	0.99	49
20. a d c d e h a b c G F a D F E C# D	0.99	50

E.2.2. Output to Cantus Firmus <D, a, G, F, E, D, F, E, D> in First Species with Two Added Parts

As with E.2.1, the initial population has very low average fitness, also containing only one valid countermelody. Again, the near-optimal countermelodies produced in this run of the genetic algorithm must have evolved from either crossover or mutation.

Listing E.4. Genetic algorithm output to cantus firmus <D, a, G, F, E, D, F, E, D> in first species, two added parts

```
Cantus firmus: D a G F E D F E D
Alpha cut: 0
Lewin's rule ro1: 1 w: 1 bonus: 0
Stepwise motion should predominate ro1: 1 w: 1 bonus: 0
Law of recovery ro1: 1 w: 1 bonus: 0
Unisons ro1: 1 w: 1 bonus: 0
Simul skip same direction ro1: 1 w: 1 bonus: 0
    crossover rate: 0.6
    mutation rate: 0.1
    max num init tries: 50
    max num crossover tries: 20
    max num mutation tries: 20
    fitness multiple: 1.2
    population size: 100
    num iterations: 50
```

(listing E.4 continued)

```
    max num no change before stopping: 3
    elitist model
Initial hypotheses:
chromosome                      fitness
d  D  G  a  G  F  D  C# D
a  d  e  f  c  d  aa a  d   0.89
b  f# e  F  f  C  G  e  f
B  C  C  C# c  GG GG c# c#  0
g  C  A  g  FF D  D  h  b
f  d  C  FF d  c  g  c# h   0
FF D  b  C# GG g  R  aa R
c# A  G  d  e  h  C# f  GG  0
c# aa g  Bb a  f  aa R  C
FF f  C  Bb A  a  F  D  FF  0
G  aa C  c  a  G  b  A  a
G  h  E  A  C  B  C  c# d   0
E  f# g  d  E  F  c  e  d
c# D  G  C  G  c  A  a  E   0
a  FF b  e  d  g  G  D  GG
D  R  f# E  B  B  a  R  c   0
GG f  e  h  c  C# C# B  g
f  f  R  Bb aa GG C# f  d   0
f  E  h  R  c# GG D  C# C#
C# e  e  a  C  h  a  c# c#  0
c  aa f  A  e  G  a  f  d
e  E  Bb F  f  Bb D  Bb D   0
f# R  E  R  Bb g  aa g  d
G  c# R  E  f  Bb a  d  a   0
c  h  a  b  C# GG c  b  F
E  aa B  R  f  E  aa R  d   0
D  R  F  C  C# D  b  e  c
aa c# e  B  aa FF B  C  R   0
A  e  d  a  C  b  f  A  a
c# Bb FF d  R  E  h  C# D   0
G  e  h  f# f# F  c  c# D
h  c  b  d  B  a  F  B  a   0
h  aa h  D  e  GG GG d  C#
R  F  A  D  B  C  F  e  f   0
B  A  B  FF d  F  c  G  Bb
FF C  g  aa aa d  aa R  h   0
D  F  b  c  C  R  a  G  d
A  c  G  A  a  G  A  B  D   0
GG F  h  c# e  f  F  R  b
a  h  g  aa G  a  f  C# h   0
e  f  h  aa h  c# aa g  h
E  h  C  E  e  aa f# E  C   0
D  g  b  E  R  E  F  Bb D
a  C  c  E  FF f  a  D  GG  0
R  d  b  a  c  F  b  C  a
D  F  G  d  e  f  b  c  D   0
D  c  C  Bb B  C# c  F  F
f# A  F  f# FF A  a  d  G   0
c# aa A  f# d  f  B  Bb aa
F  c# a  E  GG B  b  F  a   0
C# A  E  B  C# R  GG A  D
B  E  C  E  A  C# A  c# d   0
D  c# GG G  G  c  E  FF a
aa E  d  R  GG G  c  D  d   0
Bb F  D  D  F  d  c# C# a
GG d  f# G  Bb E  e  A  D   0
```

(listing E.4 continued)

```
R  f  FF aa D  e  Bb c# d
FF B  FF FF g  aa f  C# a  0
R  c# GG h  c# b  c  a  a
A  FF R  B  f# F  f  c  d  0
A  aa D  D  d  aa F  C  Bb
aa C# F  FF b  c# a  c# A  0
h  F  G  D  d  F  c  e  C
GG d  e  c  h  b  a  F  g  0
a  d  R  GG C  FF A  F  d
f  C# GG aa G  GG C  GG D  0
C  A  C# FF R  GG D  C# d
h  A  E  a  F  B  D  R  a  0
E  h  c# R  aa aa f# f# D
d  c# e  f# G  f# C  D  d  0
C  F  f  g  E  e  g  F  C
F  F  R  c# c# d  a  C# F  0
c# FF h  c  h  e  B  R  D
F  GG e  GG R  Bb e  GG d  0
e  G  FF f  h  G  R  f# g
aa C  A  c# b  F  aa E  a  0
e  h  A  B  C  e  F  b  e
g  A  C# f# A  f# c# B  FF 0
A  C  b  G  A  F  e  c# d
f# GG a  c# B  h  FF A  D  0
C# h  F  C# h  aa FF A  Bb
FF a  Bb g  Bb g  C  D  FF 0
f  R  A  g  D  c# e  G  Bb
Bb f  FF B  Bb aa FF e  d  0
R  A  f  f  A  a  D  a  D
c  E  A  h  e  c# d  f  a  0
c  e  E  a  c  c# C# E  a
FF A  E  F  b  D  Bb c# d  0
c# C  E  D  C  A  A  B  E
f# FF C  D  G  F  A  C# C  0
G  A  F  E  R  a  G  A  d
C# D  C# g  A  C  h  aa R  0
G  f  A  aa a  G  C  E  D
E  R  A  D  a  C# b  e  C  0
d  C# D  Bb G  A  R  C# C#
g  E  B  B  d  FF C  A  B  0
C# h  C  c# c  f# aa g  A
c# f# R  f# c# G  c  c# h  0
D  R  g  D  A  FF GG Bb a
e  G  F  d  GG F  a  FF D  0
G  R  F  b  a  D  C# c# g
F  Bb FF B  a  R  f  b  g  0
FF d  e  c# g  c  D  E  d
F  F  E  c  e  G  b  G  a  0
aa e  f# C# a  f# e  a  d
f# g  FF C  c# C  aa FF B  0
D  Bb Bb c# aa c# c  c# D
C# e  b  C  c# a  c  c# a  0
b  a  c  E  R  a  GG A  e
B  C# E  g  c# h  C  G  GG 0
aa R  e  C  Bb b  R  G  F
f# G  C  C# C# F  FF B  f  0
G  FF A  f  B  A  F  FF d
Bb aa Bb f# GG G  a  R  D  0
h  D  c# f# d  D  FF D  E
B  C# f  Bb g  Bb b  E  g  0
```

(listing E.4 continued)

```
A  D  E  a  h  R  B  c# d
f# FF GG A  GG D  A  E  a  0
d  C# B  f# G  h  a  e  d
GG GG g  f# e  aa FF C  D  0
h  h  D  E  C  D  d  e  a
f  d  E  aa c  b  a  c# d  0
C  c# g  g  b  C# A  GG FF
A  Bb GG C  GG g  b  d  c# 0
D  A  D  B  A  a  F  C  c#
b  f  Bb A  C  D  a  G  b  0
C  b  f# aa E  a  G  GG d
A  B  R  a  f# G  A  e  a  0
E  a  G  F  f# F  c  a  aa
FF a  d  FF c  FF A  C# E  0
A  F  e  d  f# Bb A  A  d
f  Bb FF h  FF C# F  f# D  0
B  D  B  GG B  G  D  a  a
A  F  D  a  GG B  D  C# D  0
e  R  c# e  f  h  F  C# c#
GG aa Bb f# C# C# A  R  FF 0
F  f# e  aa f# aa a  e  a
G  FF h  a  h  d  c# C# D  0
C# e  E  F  f  e  a  GG c#
R  Bb A  c# f# B  E  E  g  0
E  b  f  C# aa aa a  b  d
G  FF G  h  d  d  e  FF a  0
d  Bb Bb h  FF F  aa f# a
E  e  GG D  G  GG h  f  d  0
E  f# G  C# C  g  h  F  d
A  g  E  c  C  f# G  aa a  0
d  G  FF d  aa Bb D  d  f#
h  Bb E  G  e  c# a  aa C# 0
FF e  F  e  F  aa f# F  D
f  c# d  Bb f  A  C# GG d  0
f  G  g  B  c  C  e  d  a
GG g  G  f# D  Bb G  D  D  0
FF f  e  G  B  G  FF c# d
f  E  E  G  f# h  a  A  D  0
b  R  c  c  f# B  aa E  c
c# A  A  c# E  e  FF c# e  0
f# aa c  f  G  F  D  B  a
C  d  GG f  c  b  d  a  d  0
C  FF C  D  d  D  a  c# d
b  c  e  d  e  b  c  e  a  0
R  c# A  D  B  e  D  Bb a
Bb C  G  R  c  Bb FF R  d  0
c# b  Bb A  E  F  D  GG c
F  h  b  c  G  a  D  E  f  0
FF d  aa a  h  d  c  C# d
f  F  a  D  G  b  R  c  a  0
f# FF C# Bb b  F  f# A  D
D  A  Bb FF f  b  B  c# d  0
GG d  G  R  c# f# b  FF F
d  d  c  C# F  c# C# b  A  0
G  GG aa F  e  G  C  f# f
FF C# f  C  f  c  A  a  R  0
h  Bb Bb f# d  C# FF A  D
F  G  B  g  C# GG R  c# a  0
b  A  h  A  h  B  A  FF d
Bb D  c  e  f  B  C  FF D  0
```

(listing E.4 continued)

```
a  C  g  aa B  Bb F  Bb G
G  b  f  c  GG C# Bb F  g   0
g  D  aa aa h  c# C  GG e
E  A  R  C  D  d  R  g  d   0
f  Bb R  f  d  D  Bb B  F
d  f  f  d  c# GG b  Bb R   0
aa FF d  c# h  B  F  G  D
c  Bb e  D  e  f# c  a  a   0
E  B  Bb GG FF R  F  aa C
f# C# aa A  b  F  f# C# f   0
f  a  R  aa C  Bb a  d  R
G  A  G  C  E  c  f  aa f   0
F  F  FF b  D  GG C# A  d
h  a  a  GG e  h  a  d  a   0
a  e  g  d  e  f  c  e  d
D  c  b  a  A  F  Bb h  a   0
R  h  C  g  aa g  f# b  D
f  f# E  a  C# c  B  aa d   0
C# e  B  D  C  GG FF F  C#
D  b  h  a  c  h  a  b  F   0
c# D  b  c  C  c# A  FF h
c# d  c# F  G  f  c# c# c   0
R  f# C# G  g  b  c  b  c#
g  FF R  R  C  f  D  h  g   0

Average fitness: 0.0089
```

Optimal countermelodies	**Grade**	**Iteration**
1. D D G a G F D C# D d d e f c d aa a d	0.92	18
2. D D G a G F D C# D d d e d c d aa a d	0.92	34

E.3. Effect of Trained Neural Networks on the Execution of the Best-First Algorithm

The best-first algorithm assumes that the score of an individual partial countermelody does not increase when that partial countermelody is subsequently filled. With trained neural network output this is often difficult to satisfy. In the first place, the countermelodies used for training the neural network are always completely filled. In chapter 5, section 5.2.1 we described a method of training the network on partial countermelodies, by configuring the training data in such a way that the partial form of the countermelodies always have scores at least as high as do the complete countermelodies. We show here the effects of such training.

E.3.1. Best-First Algorithm Output to Cantus Firmus <d, c, h, c, h, a, h, a, F♯, G>, with Trained Neural Network Evaluation That Is Not Non-Increasing with respect to Partial Melody Length

The network was trained on 80 training exemplars, using 1-of-5 representation of the expert's scores, based on the second grading scheme shown in table 5.1. This was the same network used for the genetic

algorithm output in example 6.4. This network was unsatisfactory for use by the best-first search algorithm, as scores were not non-increasing with respect to partial countermelody length. This misled the algorithm to output several optimal countermelodies after suboptimal ones. In listing E.5, we show the partial countermelodies stored and their scores maintained by the priority queue, as well as the complete countermelodies. Note that the first of the eight optimal countermelodies appears only after 32 suboptimal countermelodies appear.

Listing E.5. Best-first algorithm output to cantus firmus <d, c, h, c, h, a, h, a, F#, G>, without non-increasing trained network evaluation. Also shown is the state of the priority queue with respect to partial countermelodies and their scores. Next to each valid countermelody are the individual scores of the four artistry rules (Tonality, Final in Leaps of Fourths and Fifths, False Relations, and Note Variety, respectively) and the neural network evaluation (to be interpreted as follows: 0.75: B, 0.5: C, 0.25: D).

```
Cantus firmus: d c h c h a h a F# G
Gamut: {R, FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
 1.  0.5    GG
 2.  0.5    G
 3.  0.5    d
 4.  0.5    g
 5.  0.5    A  GG
 6.  0.5    FF A  GG
 7.  0.5    GG FF A  GG
 8. 0.25    FF GG FF A  GG
 9. 0.25    A  GG FF A  GG
10.  0.5    a  G
11.  0.5    F  a  G
12.  0.5    E  F  a  G
13.  0.5    G  F  a  G
14.  0.5    F  E  F  a  G
15.  0.5    D  F  E  F  a  G
16.  0.5    E  F  E  F  a  G
17.  0.5    G  F  E  F  a  G
18.  0.5    C  D  F  E  F  a  G
19.  0.5    E  D  F  E  F  a  G
20.  0.5    F  D  F  E  F  a  G
21.  0.5    a  D  F  E  F  a  G
22.  0.5    D  C  D  F  E  F  a  G
23.  0.5    E  C  D  F  E  F  a  G
24.  0.5    G  C  D  F  E  F  a  G
25.  0.5    C  D  C  D  F  E  F  a  G
26.  0.5    E  D  C  D  F  E  F  a  G
27.  0.5    F  D  C  D  F  E  F  a  G
28.  0.5    a  D  C  D  F  E  F  a  G
29.  0.5    G  C  D  C  D  F  E  F  a  G
*** 1.
  G  C  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
30.  0.5    D  F  D  C  D  F  E  F  a  G
*** 2.
  D  F  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
31.  0.5    D  E  D  C  D  F  E  F  a  G
32.  0.5    G  E  D  C  D  F  E  F  a  G
*** 3.
  D  E  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
```

(listing E.5 continued)

```
*** 4.
  G  E  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
33.  0.5    C  E  C  D  F  E  F  a  G
34.  0.5    G  C  E  C  D  F  E  F  a  G
*** 5.
  G  C  E  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
35.  0.5    C  G  C  D  F  E  F  a  G
36.  0.5    E  G  C  D  F  E  F  a  G
37.  0.5    F  G  C  D  F  E  F  a  G
38.  0.5    a  G  C  D  F  E  F  a  G
39.  0.5    G  C  G  C  D  F  E  F  a  G
*** 6.
  G  C  G  C  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
40.  0.5    D  E  G  C  D  F  E  F  a  G
41.  0.5    G  E  G  C  D  F  E  F  a  G
*** 7.
  D  E  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
*** 8.
  G  E  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
42.  0.5    D  F  G  C  D  F  E  F  a  G
*** 9.
  D  F  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
43.  0.5    D  E  D  F  E  F  a  G
44.  0.5    G  E  D  F  E  F  a  G
45.  0.5    C  D  E  D  F  E  F  a  G
46.  0.5    E  D  E  D  F  E  F  a  G
47.  0.5    F  D  E  D  F  E  F  a  G
48.  0.5    a  D  E  D  F  E  F  a  G
49.  0.5    G  C  D  E  D  F  E  F  a  G
*** 10.
  G  C  D  E  D  F  E  F  a  G     1   1   0.8 0.6 0.5
50.  0.5    D  E  D  E  D  F  E  F  a  G
51.  0.5    G  E  D  E  D  F  E  F  a  G
*** 11.
  D  E  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 12.
  G  E  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
52.  0.5    D  F  D  E  D  F  E  F  a  G
*** 13.
  D  F  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
53.  0.5    C  G  E  D  F  E  F  a  G
54.  0.5    E  G  E  D  F  E  F  a  G
55.  0.5    F  G  E  D  F  E  F  a  G
56.  0.5    a  G  E  D  F  E  F  a  G
57.  0.5    G  C  G  E  D  F  E  F  a  G
*** 14.
  G  C  G  E  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
58.  0.5    D  F  G  E  D  F  E  F  a  G
*** 15.
  D  F  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
59.  0.5    D  E  G  E  D  F  E  F  a  G
60.  0.5    G  E  G  E  D  F  E  F  a  G
*** 16.
  D  E  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 17.
  G  E  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
61.  0.5    D  a  D  F  E  F  a  G
62.  0.5    E  a  D  F  E  F  a  G
63.  0.5    G  a  D  F  E  F  a  G
64.  0.5    C  D  a  D  F  E  F  a  G
```

(listing E.5 continued)

```
65. 0.25    E  D  a  D  F  E  F  a  G
66. 0.25    F  D  a  D  F  E  F  a  G
67. 0.25    a  D  a  D  F  E  F  a  G
68.  0.5    a  E  a  D  F  E  F  a  G
69.  0.5    C  G  a  D  F  E  F  a  G
70.  0.5    E  G  a  D  F  E  F  a  G
71.  0.5    F  G  a  D  F  E  F  a  G
72.  0.5    a  G  a  D  F  E  F  a  G
73.  0.5    D  F  D  F  E  F  a  G
74.  0.5    G  F  D  F  E  F  a  G
75.  0.5    C  D  F  D  F  E  F  a  G
76.  0.5    E  D  F  D  F  E  F  a  G
77.  0.5    F  D  F  D  F  E  F  a  G
78.  0.5    a  D  F  D  F  E  F  a  G
79.  0.5    G  C  D  F  D  F  E  F  a  G
*** 18.
  G  C  D  F  D  F  E  F  a  G     1   1   0.8 0.6 0.5
80.  0.5    D  F  D  F  D  F  E  F  a  G
*** 19.
  D  F  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
81.  0.5    D  E  D  F  D  F  E  F  a  G
82.  0.5    G  E  D  F  D  F  E  F  a  G
*** 20.
  D  E  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 21.
  G  E  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
83.  0.5    C  G  F  D  F  E  F  a  G
84.  0.5    E  G  F  D  F  E  F  a  G
85.  0.5    F  G  F  D  F  E  F  a  G
86.  0.5    a  G  F  D  F  E  F  a  G
87.  0.5    G  C  G  F  D  F  E  F  a  G
*** 22.
  G  C  G  F  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
88.  0.5    D  E  G  F  D  F  E  F  a  G
89.  0.5    G  E  G  F  D  F  E  F  a  G
*** 23.
  D  E  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 24.
  G  E  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
90.  0.5    D  F  G  F  D  F  E  F  a  G
*** 25.
  D  F  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
91. 0.25    a  E  F  E  F  a  G
92. 0.25    E  G  F  E  F  a  G
93. 0.25    F  G  F  E  F  a  G
94. 0.25    a  G  F  E  F  a  G
95.  0.5    F  G  F  a  G
96.  0.5    E  F  G  F  a  G
97. 0.25    G  F  G  F  a  G
98. 0.25    a  E  F  G  F  a  G
99.  0.5    aa g
100.  0.5   f  aa g
101.  0.5   g  f  aa g
102.  0.5   e  g  f  aa g
103.  0.5   f  g  f  aa g
104. 0.25   aa g  f  aa g
105.  0.5   d  e  g  f  aa g
106. 0.25   e  d  e  g  f  aa g
107. 0.25   g  d  e  g  f  aa g
108. 0.25   aa d  e  g  f  aa g
```

(listing E.5 continued)

```
109. 0.25   g  f  g  f  aa g
110. 0.25   D  a  E  F  E  F  a  G
111. 0.25   E  a  E  F  E  F  a  G
112. 0.25   G  a  E  F  E  F  a  G
113. 0.25   C  D  a  E  F  E  F  a  G
114. 0.25   E  D  a  E  F  E  F  a  G
115. 0.25   F  D  a  E  F  E  F  a  G
116. 0.25   a  D  a  E  F  E  F  a  G
117. 0.25   a  E  a  E  F  E  F  a  G
118. 0.75   E  G  a  E  F  E  F  a  G
119. 0.75   F  G  a  E  F  E  F  a  G
120. 0.75   a  G  a  E  F  E  F  a  G
121. 0.25   d  e  d  e  g  f  aa g
122. 0.25   g  e  d  e  g  f  aa g
123. 0.25   e  d  e  d  e  g  f  aa g
124. 0.25   g  d  e  d  e  g  f  aa g
125. 0.25   aa d  e  d  e  g  f  aa g
126.  0.5   d  e  d  e  d  e  g  f  aa g
*** 26.
  d  e  d  e  d  e  g  f  aa g     1   1   0.9 0.5 0.5
127.  0.5   d  g  d  e  d  e  g  f  aa g
*** 27.
  d  g  d  e  d  e  g  f  aa g     1   1   0.9 0.5 0.5
128. 0.25   e  g  e  d  e  g  f  aa g
129. 0.25   aa g  e  d  e  g  f  aa g
130.  0.5   d  e  g  e  d  e  g  f  aa g
*** 28.
  d  e  g  e  d  e  g  f  aa g     1   1   0.9 0.5 0.5
131. 0.25   d  aa d  e  g  f  aa g
132. 0.25   g  aa d  e  g  f  aa g
133. 0.25   e  d  aa d  e  g  f  aa g
134. 0.25   g  d  aa d  e  g  f  aa g
135. 0.25   aa d  aa d  e  g  f  aa g
136. 0.25   e  g  aa d  e  g  f  aa g
137. 0.25   aa g  aa d  e  g  f  aa g
138. 0.25   D  F  G  F  E  F  a  G
139. 0.25   G  F  G  F  E  F  a  G
140. 0.25   C  D  F  G  F  E  F  a  G
141. 0.25   E  D  F  G  F  E  F  a  G
142. 0.25   F  D  F  G  F  E  F  a  G
143. 0.25   a  D  F  G  F  E  F  a  G
144.  0.5   G  C  D  F  G  F  E  F  a  G
*** 29.
  G  C  D  F  G  F  E  F  a  G     1   1   0.8 0.6 0.5
145.  0.5   D  E  D  F  G  F  E  F  a  G
146.  0.5   G  E  D  F  G  F  E  F  a  G
*** 30.
  D  E  D  F  G  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 31.
  G  E  D  F  G  F  E  F  a  G     1   1   0.8 0.5 0.5
147.  0.5   D  F  D  F  G  F  E  F  a  G
*** 32.
  D  F  D  F  G  F  E  F  a  G     1   1   0.8 0.5 0.5
148. 0.75   E  G  F  G  F  E  F  a  G
149. 0.75   F  G  F  G  F  E  F  a  G
150. 0.75   a  G  F  G  F  E  F  a  G
151. 0.75   D  E  G  F  G  F  E  F  a  G
152.  0.5   G  E  G  F  G  F  E  F  a  G
*** 33.
  D  E  G  F  G  F  E  F  a  G     1   1   0.8 0.5 0.75
```

(listing E.5 continued)

```
153. 0.75   D  F  G  F  G  F  E  F  a  G
*** 34.
  D  F  G  F  G  F  E  F  a  G     1   1   0.8 0.5 0.75
*** 35.
  G  E  G  F  G  F  E  F  a  G     1   1   0.8 0.4 0.5
154. 0.25   D  E  G  F  E  F  a  G
155. 0.25   G  E  G  F  E  F  a  G
156. 0.25   C  D  E  G  F  E  F  a  G
157. 0.25   E  D  E  G  F  E  F  a  G
158. 0.25   F  D  E  G  F  E  F  a  G
159. 0.25   a  D  E  G  F  E  F  a  G
160.  0.5   G  C  D  E  G  F  E  F  a  G
*** 36.
  G  C  D  E  G  F  E  F  a  G     1   1   0.8 0.6 0.5
161.  0.5   D  F  D  E  G  F  E  F  a  G
*** 37.
  D  F  D  E  G  F  E  F  a  G     1   1   0.8 0.5 0.5
162.  0.5   D  E  D  E  G  F  E  F  a  G
163.  0.5   G  E  D  E  G  F  E  F  a  G
*** 38.
  D  E  D  E  G  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 39.
  G  E  D  E  G  F  E  F  a  G     1   1   0.8 0.5 0.5
164. 0.75   E  G  E  G  F  E  F  a  G
165. 0.75   F  G  E  G  F  E  F  a  G
166. 0.75   a  G  E  G  F  E  F  a  G
167. 0.75   D  E  G  E  G  F  E  F  a  G
168.  0.5   G  E  G  E  G  F  E  F  a  G
*** 40.
  D  E  G  E  G  F  E  F  a  G     1   1   0.8 0.5 0.75
169. 0.75   D  F  G  E  G  F  E  F  a  G
*** 41.
  D  F  G  E  G  F  E  F  a  G     1   1   0.8 0.5 0.75
*** 42.
  G  E  G  E  G  F  E  F  a  G     1   1   0.8 0.4 0.5
170. 0.25   D  a  E  F  G  F  a  G
171. 0.25   E  a  E  F  G  F  a  G
172. 0.25   G  a  E  F  G  F  a  G
173. 0.25   C  D  a  E  F  G  F  a  G
174. 0.25   E  D  a  E  F  G  F  a  G
175. 0.25   F  D  a  E  F  G  F  a  G
176. 0.25   a  D  a  E  F  G  F  a  G
177. 0.25   a  E  a  E  F  G  F  a  G
178. 0.25   E  G  a  E  F  G  F  a  G
179. 0.25   F  G  a  E  F  G  F  a  G
180. 0.25   a  G  a  E  F  G  F  a  G
181. 0.25   D  a  G  F  E  F  a  G
182. 0.25   E  a  G  F  E  F  a  G
183. 0.25   G  a  G  F  E  F  a  G
184. 0.25   C  D  a  G  F  E  F  a  G
185. 0.25   E  D  a  G  F  E  F  a  G
186. 0.25   F  D  a  G  F  E  F  a  G
187. 0.25   a  D  a  G  F  E  F  a  G
188. 0.75   E  G  a  G  F  E  F  a  G
189. 0.75   F  G  a  G  F  E  F  a  G
190. 0.75   a  G  a  G  F  E  F  a  G
191. 0.25   a  E  a  G  F  E  F  a  G
192. 0.25   e  g  f  g  f  aa g
193. 0.25   aa g  f  g  f  aa g
194. 0.25   d  e  g  f  g  f  aa g
```

(listing E.5 continued)

```
195. 0.25   g  e  g  f  g  f  aa g
196. 0.25   e  d  e  g  f  g  f  aa g
197. 0.25   g  d  e  g  f  g  f  aa g
198. 0.25   aa d  e  g  f  g  f  aa g
199.  0.5   d  e  d  e  g  f  g  f  aa g
*** 43.
  d  e  d  e  g  f  g  f  aa g      1   1   0.8 0.5 0.5
200.  0.5   d  g  d  e  g  f  g  f  aa g
*** 44.
  d  g  d  e  g  f  g  f  aa g      1   1   0.8 0.5 0.5
201. 0.75   e  g  e  g  f  g  f  aa g
202. 0.75   aa g  e  g  f  g  f  aa g
203. 0.75   d  e  g  e  g  f  g  f  aa g
*** 45.
  d  e  g  e  g  f  g  f  aa g      1   1   0.8 0.5 0.75
204. 0.25   E  G  F  G  F  a  G
205. 0.25   F  G  F  G  F  a  G
206. 0.25   a  G  F  G  F  a  G
207. 0.25   D  E  G  F  G  F  a  G
208. 0.25   G  E  G  F  G  F  a  G
209. 0.25   C  D  E  G  F  G  F  a  G
210. 0.25   E  D  E  G  F  G  F  a  G
211. 0.25   F  D  E  G  F  G  F  a  G
212. 0.25   a  D  E  G  F  G  F  a  G
213.  0.5   G  C  D  E  G  F  G  F  a  G
*** 46.
  G  C  D  E  G  F  G  F  a  G      1   1   0.8 0.6 0.5
214. 0.25   D  E  D  E  G  F  G  F  a  G
215. 0.25   G  E  D  E  G  F  G  F  a  G
*** 47.
  D  E  D  E  G  F  G  F  a  G      1   1   0.8 0.5 0.25
*** 48.
  G  E  D  E  G  F  G  F  a  G      1   1   0.8 0.5 0.25
216. 0.25   D  F  D  E  G  F  G  F  a  G
*** 49.
  D  F  D  E  G  F  G  F  a  G      1   1   0.8 0.5 0.25
217. 0.75   E  G  E  G  F  G  F  a  G
218. 0.75   F  G  E  G  F  G  F  a  G
219. 0.75   a  G  E  G  F  G  F  a  G
220. 0.75   D  E  G  E  G  F  G  F  a  G
221. 0.75   G  E  G  E  G  F  G  F  a  G
*** 50.
  D  E  G  E  G  F  G  F  a  G      1   1   0.8 0.5 0.75
*** 51.
  G  E  G  E  G  F  G  F  a  G      1   1   0.8 0.4 0.75
222. 0.75   D  F  G  E  G  F  G  F  a  G
*** 52.
  D  F  G  E  G  F  G  F  a  G      1   1   0.8 0.5 0.75
223. 0.25   E  a  G  F  G  F  a  G
224. 0.25   a  E  a  G  F  G  F  a  G
225. 0.25   G  F  G  F  G  F  a  G
226. 0.25   E  G  F  G  F  G  F  a  G
227. 0.25   F  G  F  G  F  G  F  a  G
228. 0.25   a  G  F  G  F  G  F  a  G
229. 0.25   D  E  G  F  G  F  G  F  a  G
230. 0.25   G  E  G  F  G  F  G  F  a  G
*** 53.
  D  E  G  F  G  F  G  F  a  G      1   1   0.8 0.5 0.25
*** 54.
  G  E  G  F  G  F  G  F  a  G      1   1   0.8 0.4 0.25
```

(listing E.5 continued)

```
231. 0.25   g  aa g  f  aa g
232. 0.25   e  g  aa g  f  aa g
233. 0.25   aa g  aa g  f  aa g
234. 0.25   d  e  g  aa g  f  aa g
235. 0.25   g  e  g  aa g  f  aa g
236. 0.25   e  d  e  g  aa g  f  aa g
237. 0.25   g  d  e  g  aa g  f  aa g
238. 0.25   aa d  e  g  aa g  f  aa g
239. 0.75   e  g  e  g  aa g  f  aa g
240. 0.75   aa g  e  g  aa g  f  aa g
241. 0.25   g  aa g  aa g  f  aa g
242. 0.25   e  g  aa g  aa g  f  aa g
243. 0.25   aa g  aa g  aa g  f  aa g
244. 0.25   GG FF GG FF A  GG
245. 0.25   FF GG FF GG FF A  GG
246. 0.25   A  GG FF GG FF A  GG
247. 0.25   GG FF GG FF GG FF A  GG
248. 0.25   A  GG FF GG FF GG FF A  GG
249. 0.25   B  A  GG FF GG FF A  GG
250. 0.25   GG A  GG FF A  GG
251. 0.25   FF GG A  GG FF A  GG
252. 0.25   A  GG A  GG FF A  GG
253. 0.25   GG FF GG A  GG FF A  GG
254. 0.25   A  GG FF GG A  GG FF A  GG
255. 0.25   GG A  GG A  GG FF A  GG
256. 0.25   B  A  GG A  GG FF A  GG
257. 0.25   FF GG A  GG A  GG FF A  GG
258. 0.25   A  GG A  GG A  GG FF A  GG
259. 0.25   A  B  A  GG A  GG FF A  GG
260. 0.25   GG A  B  A  GG A  GG FF A  GG
*** 55.
  GG A  B  A  GG A  GG FF A  GG
  d  c  h  c  h  a  h  a  F# G
    1   1   1   0.4 0.25
maxQSize: 18
Priority queue size: 0
chainedListPtrVector size: 0
Number of solutions:  55.
```

E.3.2. Best-First Algorithm Output to Cantus Firmus <d, c, h, c, h, a, h, a, F#, G>, with Trained Neural Network Evaluation That Is (Almost) Non-Increasing with respect to Partial Melody Length

The network was trained on data shown in appendix C, section C.1, using 1-of-5 representation of the expert's scores, again based on the second grading scheme shown in table 5.1. This resulted in output by the neural network that almost always had non-increasing scores (listing E.6; exceptions noted in the listing), with most of the optimal countermelodies output before suboptimal ones. Countermelodies 13 and 40 are optimal but they appear after suboptimal countermelodies; see the 65th and 9th entries in the priority queue for the cause of these anomalies (the partial countermelodies for these two optimal countermelodies both at these points in the processing received a score of 0.5, which is less than the

score of the complete countermelodies). Notice that this trained network differs from the one used in the previous output (E.3.1), and the optimal countermelodies output are also different.

Listing E.6. Best-first algorithm output to cantus firmus <d, c, h, c, h, a, h, a, F#, G>, with almost always non-increasing trained network evaluation. Also shown is the state of the priority queue with respect to partial countermelodies and their scores. Next to each valid countermelody are the individual scores of the four artistry rules (Tonality, Final in Leaps of Fourths and Fifths, False Relations, and Note Variety, respectively) and the neural network evaluation (to be interpreted as follows: 0.75: B, 0.5: C, 0.25: D).

```
Cantus firmus:      d c h c h a h a F# G
Gamut: { R, FF, GG, A, Bb, B, C, C#, D, E, F, G, a, b, h, c, c#, d, e, f, f#, g, aa}
```

(listing E.6 continued)

```
 1. 0.75    GG
 2. 0.75    G
 3. 0.75    d
 4.  0.5    g
 5. 0.75    A  GG
 6. 0.75    FF A  GG
 7. 0.75    GG FF A  GG
 8.  0.5    FF GG FF A  GG
 9.  0.5    A  GG FF A  GG
10. 0.75    a  G
11. 0.75    F  a  G
12. 0.75    E  F  a  G
13.  0.5    G  F  a  G
14.  0.5    F  E  F  a  G
15.  0.5    D  F  E  F  a  G
16. 0.75    E  F  E  F  a  G
17. 0.75    G  F  E  F  a  G
18. 0.75    a  E  F  E  F  a  G
19. 0.75    D  a  E  F  E  F  a  G
20.  0.5    E  a  E  F  E  F  a  G
21.  0.5    G  a  E  F  E  F  a  G
22. 0.75    C  D  a  E  F  E  F  a  G
23. 0.75    E  D  a  E  F  E  F  a  G
24. 0.75    F  D  a  E  F  E  F  a  G
25. 0.75    a  D  a  E  F  E  F  a  G
26. 0.75    E  G  F  E  F  a  G
27. 0.75    F  G  F  E  F  a  G
28. 0.75    a  G  F  E  F  a  G
29. 0.75    D  E  G  F  E  F  a  G
30.  0.5    G  E  G  F  E  F  a  G
31. 0.75    C  D  E  G  F  E  F  a  G
32. 0.75    E  D  E  G  F  E  F  a  G
33. 0.75    F  D  E  G  F  E  F  a  G
34. 0.75    a  D  E  G  F  E  F  a  G
35. 0.75    G  C  D  E  G  F  E  F  a  G
*** 1.
  G  C  D  E  G  F  E  F  a  G      1   1   0.8 0.6 0.75
36. 0.75    D  E  D  E  G  F  E  F  a  G
37. 0.75    G  E  D  E  G  F  E  F  a  G
*** 2.
  D  E  D  E  G  F  E  F  a  G      1   1   0.8 0.5 0.75
*** 3.
  G  E  D  E  G  F  E  F  a  G      1   1   0.8 0.5 0.75
38. 0.75    D  F  D  E  G  F  E  F  a  G
*** 4.
  D  F  D  E  G  F  E  F  a  G      1   1   0.8 0.5 0.75
```

(listing E.6 continued)

```
39. 0.75    D  F  G  F  E  F  a  G
40.  0.5    G  F  G  F  E  F  a  G
41. 0.75    C  D  F  G  F  E  F  a  G
42. 0.75    E  D  F  G  F  E  F  a  G
43. 0.75    F  D  F  G  F  E  F  a  G
44. 0.75    a  D  F  G  F  E  F  a  G
45. 0.75    G  C  D  F  G  F  E  F  a  G
*** 5.
  G  C  D  F  G  F  E  F  a  G      1   1   0.8 0.6 0.75
46. 0.75    D  E  D  F  G  F  E  F  a  G
47. 0.75    G  E  D  F  G  F  E  F  a  G
*** 6.
  D  E  D  F  G  F  E  F  a  G      1   1   0.8 0.5 0.75
*** 7.
  G  E  D  F  G  F  E  F  a  G      1   1   0.8 0.5 0.75
48. 0.75    D  F  D  F  G  F  E  F  a  G
*** 8.
  D  F  D  F  G  F  E  F  a  G      1   1   0.8 0.5 0.75
49. 0.75    D  a  G  F  E  F  a  G
50.  0.5    E  a  G  F  E  F  a  G
51.  0.5    G  a  G  F  E  F  a  G
52. 0.75    C  D  a  G  F  E  F  a  G
53. 0.75    E  D  a  G  F  E  F  a  G
54. 0.75    F  D  a  G  F  E  F  a  G
55. 0.75    a  D  a  G  F  E  F  a  G
56.  0.5    E  G  a  G  F  E  F  a  G
57.  0.5    F  G  a  G  F  E  F  a  G
58.  0.5    a  G  a  G  F  E  F  a  G
59. 0.75    E  G  E  G  F  E  F  a  G
60. 0.75    F  G  E  G  F  E  F  a  G
61. 0.75    a  G  E  G  F  E  F  a  G
62.  0.5    D  E  G  E  G  F  E  F  a  G
63. 0.75    G  E  G  E  G  F  E  F  a  G
*** 9.
  G  E  G  E  G  F  E  F  a  G      1   1   0.8 0.4 0.75
64.  0.5    D  F  G  E  G  F  E  F  a  G
*** 10.
  D  E  G  E  G  F  E  F  a  G      1   1   0.8 0.5 0.5
*** 11.
  D  F  G  E  G  F  E  F  a  G      1   1   0.8 0.5 0.5
65.  0.5    E  G  F  G  F  E  F  a  G
66. 0.75    F  G  F  G  F  E  F  a  G
67. 0.75    a  G  F  G  F  E  F  a  G
68.  0.5    D  F  G  F  G  F  E  F  a  G
*** 12.
  D  F  G  F  G  F  E  F  a  G      1   1   0.8 0.5 0.5
69.  0.5    D  E  G  F  G  F  E  F  a  G
70. 0.75    G  E  G  F  G  F  E  F  a  G
*** 13.
  G  E  G  F  G  F  E  F  a  G      1   1   0.8 0.4 0.75
*** 14.
  D  E  G  F  G  F  E  F  a  G      1   1   0.8 0.5 0.5
71.  0.5    a  E  a  G  F  E  F  a  G
72.  0.5    E  G  a  E  F  E  F  a  G
73.  0.5    F  G  a  E  F  E  F  a  G
74. 0.75    a  G  a  E  F  E  F  a  G
75.  0.5    a  E  a  E  F  E  F  a  G
76.  0.5    C  D  F  E  F  a  G
77.  0.5    E  D  F  E  F  a  G
78.  0.5    F  D  F  E  F  a  G
```

(listing E.6 continued)

```
79.  0.5    a  D  F  E  F  a  G
80.  0.5    D  C  D  F  E  F  a  G
81.  0.5    E  C  D  F  E  F  a  G
82.  0.5    G  C  D  F  E  F  a  G
83.  0.5    C  D  C  D  F  E  F  a  G
84.  0.5    E  D  C  D  F  E  F  a  G
85.  0.5    F  D  C  D  F  E  F  a  G
86.  0.5    a  D  C  D  F  E  F  a  G
87.  0.5    G  C  D  C  D  F  E  F  a  G
*** 15.
  G  C  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
88.  0.5    D  F  D  C  D  F  E  F  a  G
*** 16.
  D  F  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
89.  0.5    D  E  D  C  D  F  E  F  a  G
90.  0.5    G  E  D  C  D  F  E  F  a  G
*** 17.
  D  E  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
*** 18.
  G  E  D  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
91.  0.5    C  E  C  D  F  E  F  a  G
92.  0.5    G  C  E  C  D  F  E  F  a  G
*** 19.
  G  C  E  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
93.  0.5    C  G  C  D  F  E  F  a  G
94.  0.5    E  G  C  D  F  E  F  a  G
95.  0.5    F  G  C  D  F  E  F  a  G
96.  0.5    a  G  C  D  F  E  F  a  G
97.  0.5    G  C  G  C  D  F  E  F  a  G
*** 20.
  G  C  G  C  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
98.  0.5    D  F  G  C  D  F  E  F  a  G
*** 21.
  D  F  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
99.  0.5    D  E  G  C  D  F  E  F  a  G
100.  0.5   G  E  G  C  D  F  E  F  a  G
*** 22.
  D  E  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
*** 23.
  G  E  G  C  D  F  E  F  a  G     1   1   0.8 0.6 0.5
101.  0.5   D  F  D  F  E  F  a  G
102.  0.5   G  F  D  F  E  F  a  G
103.  0.5   C  D  F  D  F  E  F  a  G
104.  0.5   E  D  F  D  F  E  F  a  G
105.  0.5   F  D  F  D  F  E  F  a  G
106.  0.5   a  D  F  D  F  E  F  a  G
107.  0.5   G  C  D  F  D  F  E  F  a  G
*** 24.
  G  C  D  F  D  F  E  F  a  G     1   1   0.8 0.6 0.5
108.  0.5   D  F  D  F  D  F  E  F  a  G
*** 25.
  D  F  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
109.  0.5   D  E  D  F  D  F  E  F  a  G
110.  0.5   G  E  D  F  D  F  E  F  a  G
*** 26.
  D  E  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 27.
  G  E  D  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
111.  0.5   C  G  F  D  F  E  F  a  G
112.  0.5   E  G  F  D  F  E  F  a  G
```

(listing E.6 continued)

```
113.  0.5   F  G  F  D  F  E  F  a  G
114.  0.5   a  G  F  D  F  E  F  a  G
115.  0.5   G  C  G  F  D  F  E  F  a  G
*** 28.
  G  C  G  F  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
116.  0.5   D  E  G  F  D  F  E  F  a  G
117.  0.5   G  E  G  F  D  F  E  F  a  G
*** 29.
  D  E  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 30.
  G  E  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
118.  0.5   D  F  G  F  D  F  E  F  a  G
*** 31.
  D  F  G  F  D  F  E  F  a  G     1   1   0.8 0.5 0.5
119.  0.5   D  a  D  F  E  F  a  G
120.  0.5   E  a  D  F  E  F  a  G
121.  0.5   G  a  D  F  E  F  a  G
122.  0.5   C  D  a  D  F  E  F  a  G
123.  0.5   E  D  a  D  F  E  F  a  G
124.  0.5   F  D  a  D  F  E  F  a  G
125.  0.5   a  D  a  D  F  E  F  a  G
126.  0.5   C  G  a  D  F  E  F  a  G
127.  0.5   E  G  a  D  F  E  F  a  G
128.  0.5   F  G  a  D  F  E  F  a  G
129.  0.5   a  G  a  D  F  E  F  a  G
130.  0.5   a  E  a  D  F  E  F  a  G
131.  0.5   D  E  D  F  E  F  a  G
132.  0.5   G  E  D  F  E  F  a  G
133.  0.5   C  D  E  D  F  E  F  a  G
134.  0.5   E  D  E  D  F  E  F  a  G
135.  0.5   F  D  E  D  F  E  F  a  G
136.  0.5   a  D  E  D  F  E  F  a  G
137.  0.5   G  C  D  E  D  F  E  F  a  G
*** 32.
  G  C  D  E  D  F  E  F  a  G     1   1   0.8 0.6 0.5
138.  0.5   D  E  D  E  D  F  E  F  a  G
139.  0.5   G  E  D  E  D  F  E  F  a  G
*** 33.
  D  E  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 34.
  G  E  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
140.  0.5   D  F  D  E  D  F  E  F  a  G
*** 35.
  D  F  D  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
141.  0.5   C  G  E  D  F  E  F  a  G
142.  0.5   E  G  E  D  F  E  F  a  G
143.  0.5   F  G  E  D  F  E  F  a  G
144.  0.5   a  G  E  D  F  E  F  a  G
145.  0.5   G  C  G  E  D  F  E  F  a  G
*** 36.
  G  C  G  E  D  F  E  F  a  G     1   0.9 0.8 0.6 0.5
146.  0.5   D  F  G  E  D  F  E  F  a  G
*** 37.
  D  F  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
147.  0.5   D  E  G  E  D  F  E  F  a  G
148.  0.5   G  E  G  E  D  F  E  F  a  G
*** 38.
  D  E  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
*** 39.
  G  E  G  E  D  F  E  F  a  G     1   1   0.8 0.5 0.5
```

(listing E.6 continued)

```
149. 0.75   GG A  GG FF A  GG
150. 0.75   FF GG A  GG FF A  GG
151. 0.75   A  GG A  GG FF A  GG
152. 0.75   GG FF GG A  GG FF A  GG
153. 0.75   A  GG FF GG A  GG FF A  GG
154. 0.75   GG A  GG A  GG FF A  GG
155. 0.75   B  A  GG A  GG FF A  GG
156. 0.75   FF GG A  GG A  GG FF A  GG
157. 0.75   A  GG A  GG A  GG FF A  GG
158. 0.75   A  B  A  GG A  GG FF A  GG
159. 0.75   GG A  B  A  GG A  GG FF A  GG
*** 40.
  GG A  B  A  GG A  GG FF A  GG     1   1   1   0.4 0.75
160. 0.75   GG FF GG FF A  GG
161. 0.75   FF GG FF GG FF A  GG
162. 0.75   A  GG FF GG FF A  GG
163. 0.75   GG FF GG FF GG FF A  GG
164. 0.75   A  GG FF GG FF GG FF A  GG
165. 0.75   B  A  GG FF GG FF A  GG
166.  0.5   F  G  F  a  G
167.  0.5   E  F  G  F  a  G
168. 0.75   G  F  G  F  a  G
169.  0.5   E  G  F  G  F  a  G
170.  0.5   F  G  F  G  F  a  G
171.  0.5   a  G  F  G  F  a  G
172.  0.5   D  E  G  F  G  F  a  G
173.  0.5   G  E  G  F  G  F  a  G
174.  0.5   C  D  E  G  F  G  F  a  G
175.  0.5   E  D  E  G  F  G  F  a  G
176.  0.5   F  D  E  G  F  G  F  a  G
177.  0.5   a  D  E  G  F  G  F  a  G
178.  0.5   G  C  D  E  G  F  G  F  a  G
*** 41.
  G  C  D  E  G  F  G  F  a  G     1   1   0.8 0.6 0.5
179.  0.5   D  E  D  E  G  F  G  F  a  G
180.  0.5   G  E  D  E  G  F  G  F  a  G
*** 42.
  D  E  D  E  G  F  G  F  a  G     1   1   0.8 0.5 0.5
*** 43.
  G  E  D  E  G  F  G  F  a  G     1   1   0.8 0.5 0.5
181.  0.5   D  F  D  E  G  F  G  F  a  G
*** 44.
  D  F  D  E  G  F  G  F  a  G     1   1   0.8 0.5 0.5
182.  0.5   E  G  E  G  F  G  F  a  G
183.  0.5   F  G  E  G  F  G  F  a  G
184.  0.5   a  G  E  G  F  G  F  a  G
185.  0.5   D  E  G  E  G  F  G  F  a  G
186.  0.5   G  E  G  E  G  F  G  F  a  G
*** 45.
  D  E  G  E  G  F  G  F  a  G     1   1   0.8 0.5 0.5
*** 46.
  G  E  G  E  G  F  G  F  a  G     1   1   0.8 0.4 0.5
187.  0.5   D  F  G  E  G  F  G  F  a  G
*** 47.
  D  F  G  E  G  F  G  F  a  G     1   1   0.8 0.5 0.5
188.  0.5   E  a  G  F  G  F  a  G
189.  0.5   a  E  a  G  F  G  F  a  G
190.  0.5   G  F  G  F  G  F  a  G
191.  0.5   E  G  F  G  F  G  F  a  G
192.  0.5   F  G  F  G  F  G  F  a  G
```

(listing E.6 continued)

```
193.  0.5   a  G  F  G  F  G  F  a  G
194.  0.5   D  E  G  F  G  F  G  F  a  G
195.  0.5   G  E  G  F  G  F  G  F  a  G
*** 48.
  D  E  G  F  G  F  G  F  a  G      1   1   0.8 0.5 0.5
*** 49.
  G  E  G  F  G  F  G  F  a  G      1   1   0.8 0.4 0.5
196.  0.5   a  E  F  G  F  a  G
197.  0.5   D  a  E  F  G  F  a  G
198.  0.5   E  a  E  F  G  F  a  G
199.  0.5   G  a  E  F  G  F  a  G
200.  0.5   C  D  a  E  F  G  F  a  G
201.  0.5   E  D  a  E  F  G  F  a  G
202.  0.5   F  D  a  E  F  G  F  a  G
203.  0.5   a  D  a  E  F  G  F  a  G
204.  0.5   E  G  a  E  F  G  F  a  G
205.  0.5   F  G  a  E  F  G  F  a  G
206.  0.5   a  G  a  E  F  G  F  a  G
207.  0.5   a  E  a  E  F  G  F  a  G
208.  0.5   aa g
209.  0.5   f  aa g
210.  0.5   g  f  aa g
211. 0.25   e  g  f  aa g
212.  0.5   f  g  f  aa g
213. 0.25   aa g  f  aa g
214.  0.5   g  f  g  f  aa g
215.  0.5   e  g  f  g  f  aa g
216.  0.5   aa g  f  g  f  aa g
217.  0.5   d  e  g  f  g  f  aa g
218.  0.5   g  e  g  f  g  f  aa g
219.  0.5   e  d  e  g  f  g  f  aa g
220.  0.5   g  d  e  g  f  g  f  aa g
221.  0.5   aa d  e  g  f  g  f  aa g
222.  0.5   d  e  d  e  g  f  g  f  aa g
*** 50.
  d  e  d  e  g  f  g  f  aa g      1   1   0.8 0.5 0.5
223.  0.5   d  g  d  e  g  f  g  f  aa g
*** 51.
  d  g  d  e  g  f  g  f  aa g      1   1   0.8 0.5 0.5
224.  0.5   e  g  e  g  f  g  f  aa g
225.  0.5   aa g  e  g  f  g  f  aa g
226.  0.5   d  e  g  e  g  f  g  f  aa g
*** 52.
  d  e  g  e  g  f  g  f  aa g      1   1   0.8 0.5 0.5
227. 0.25   g  aa g  f  aa g
228. 0.25   e  g  aa g  f  aa g
229. 0.25   aa g  aa g  f  aa g
230. 0.25   d  e  g  aa g  f  aa g
231. 0.25   g  e  g  aa g  f  aa g
232. 0.25   e  d  e  g  aa g  f  aa g
233. 0.25   g  d  e  g  aa g  f  aa g
234. 0.25   aa d  e  g  aa g  f  aa g
235. 0.25   e  g  e  g  aa g  f  aa g
236. 0.25   aa g  e  g  aa g  f  aa g
237. 0.25   g  aa g  aa g  f  aa g
238. 0.25   e  g  aa g  aa g  f  aa g
239. 0.25   aa g  aa g  aa g  f  aa g
240. 0.25   d  e  g  f  aa g
241. 0.25   e  d  e  g  f  aa g
242. 0.25   g  d  e  g  f  aa g
```

(listing E.6 continued)

```
243. 0.25   aa d  e  g  f  aa g
244. 0.25   d  e  d  e  g  f  aa g
245. 0.25   g  e  d  e  g  f  aa g
246. 0.25   e  d  e  d  e  g  f  aa g
247. 0.25   g  d  e  d  e  g  f  aa g
248. 0.25   aa d  e  d  e  g  f  aa g
249. 0.25   d  e  d  e  d  e  g  f  aa g
*** 53.
  d  e  d  e  d  e  g  f  aa g      1   1   0.9 0.5 0.25
250. 0.25   d  g  d  e  d  e  g  f  aa g
*** 54.
  d  g  d  e  d  e  g  f  aa g      1   1   0.9 0.5 0.25
251. 0.25   e  g  e  d  e  g  f  aa g
252. 0.25   aa g  e  d  e  g  f  aa g
253. 0.25   d  e  g  e  d  e  g  f  aa g
*** 55.
  d  e  g  e  d  e  g  f  aa g      1   1   0.9 0.5 0.25
254. 0.25   d  aa d  e  g  f  aa g
255. 0.25   g  aa d  e  g  f  aa g
256. 0.25   e  d  aa d  e  g  f  aa g
257. 0.25   g  d  aa d  e  g  f  aa g
258. 0.25   aa d  aa d  e  g  f  aa g
259. 0.25   e  g  aa d  e  g  f  aa g
260. 0.25   aa g  aa d  e  g  f  aa g
maxQSize: 15
Priority queue size: 0
chainedListPtrVector size: 0
Number of solutions:  55.
```

VITA

Nigel Gwee received a bachelor of music (piano) degree from the University of Western Australia in 1979. He came to the United States to pursue graduate studies and in 1989 completed a master of music (voice) degree from Drake University, Iowa, and in 1998 a master of science (systems science) degree from Louisiana State University, Baton Rouge.

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