1971

The Modeling and Time-Optimal Control of Chemical Processes.

John Nelson Beard Jr
Louisiana State University and Agricultural & Mechanical College

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THE MODELING AND TIME-OPTIMAL CONTROL
OF CHEMICAL PROCESSES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
The Department of Chemical Engineering

by

John Nelson Beard, Jr.
B.S., University of South Carolina, 1958
M.S., Louisiana State University, 1970
May 1971

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This dissertation describes a Modeling Time-Optimal Controller which uses state variable feedback to carry out a time-optimal setpoint change, for systems that can be described by a second-order lag plus dead time model, without prior knowledge of the unsteady-state model parameters. A steady-state model of the system is required. In addition, the parameters of a second-order lag plus dead time model describing the system can be obtained as a by-product of the setpoint change. The controller is easily tuned and is relatively insensitive to changes in the process dynamics. It can be implemented with either analog or digital components. An analog version would allow the time-optimal control of processes not having a digital computer available. It has been demonstrated on both analog and digital computers in controlling and modeling linear second-order overdamped and underdamped systems, linear third- and fourth-order systems with dead time, and a highly nonlinear exothermic continuous stirred-tank reactor with dead
time. All demonstrations are of computer simulated systems.

Correlations, in the time domain, are presented which may be used with a second-order lag plus dead time model of the system to determine the switching times for time-optimal setpoint changes.
CHAPTER I

INTRODUCTION

The time-optimal control problem is basically one of determining the control law that will drive a process from an initial state to a specified final state in minimum time. From optimal control theory we know that the optimum control action will be "on-off" or "bang-bang" control (1). Figure I-1 illustrates the temperature response of a system to the three control actions which are shown in Figure I-2; a simple step change, a PID controller tuned to minimize the integral of the absolute error, and time-optimal control. The data below show that the 5% settling time of the time-optimal trajectory is much shorter than for the other two trajectories.

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For time-optimal control, full heating is applied at time $t=0$, a switch to full cooling is made at $t_1$, and a
FIGURE I-1. System Responses.
FIGURE 1-2. Control Actions.
switch to conventional setpoint control is made at $t_2$.
The problem is to determine the times at which to make the
switches, $t_1$ and $t_2$. Time-optimal control can be "open-
loop" or "programmed" which uses precomputed switching
times or it can be "closed-loop" or "feedback" which
uses state variables to compute the switching times after
the setpoint change has started.

The system in Figures I-1 and I-2 can be
represented in the time domain by the differential
equation:

$$\frac{d^2 T}{dt^2} + 0.1667 \frac{dT}{dt} + 0.00667T = 0.00667M \quad (1-1)$$

with 5 minutes delay time. In Figure I-1, Temperature =
140. + 25T. The system can be represented in the Laplace
domain by the transfer function:

$$G(s) = \frac{e^{-5s}}{(15s + 1)(10s + 1)}$$

Throughout this work we often refer to control with $M=2$
or $M=3$, etc. $M=2$ implies that the initial forcing function,
for the time-optimal control of Equation I-1, is
$(0.00667)(M)$ where $M=+2$. After $t_1$, $M$ equals -2. At
$M=+2$, the system would line out at $T=+2$ if the initial
forcing function were applied until the system reached
steady-state. The initial value of M is sometimes referred to as $M_1$ and the value of M after $t_1$ as $M_2$.

We have restricted this research to the case of $M_1 = -M_2 = M$. The extension of this work to other ratios of $M_1$ to $M_2$ should be straightforward.

In attempting to apply time-optimal control to a process, we encounter two rather formidable obstacles:

1. An unsteady-state mathematical model of the system is required, and
2. A nonlinear multi-point boundary-value problem, with unspecified final time, must usually be solved in order to determine the switching times.

If a second-order lag plus dead time model is used, the equations can be solved analytically and the multi-point boundary-value problem can be avoided (2); however, an unsteady-state model of the system is still required.

A major accomplishment of the research described herein is the invention of a feedback controller that carries out the time-optimal control of second-order systems and suboptimal* control of higher-order systems.

*Suboptimal control, as used here, means that the controller will switch only once between the extremes of the manipulated variable. True time-optimal control for linear systems requires $n-1$ switches between extremes, where $n$ is the order of the system (1). Throughout this work the term "time-optimal" is used in a generic sense to include true time-optimal, suboptimal, and other "bang-bang" controls which approximate true time-optimal control.
which can be approximated by second-order lag plus dead time models, without prior knowledge of unsteady-state model parameters. A steady-state model of the system is required. The higher-order systems are controlled as if they were second-order systems with dead time.

In addition, the controller provides the parameters of a second-order plus dead time model of the system as a by-product of a time-optimal setpoint change. This model closely approximates the step responses as well as the trajectories and switching times of the process during time-optimal control.

Since the Modeling Time-Optimal Controller does not require an unsteady-state model of the process, it could be particularly useful in controlling processes in which the equations describing the system are subject to change. An analog version of the controller would also allow time-optimal control of processes not having a digital computer available.

The controller was implemented on both the IBM 360/65 digital computer and the EAI 680 Analog (Parallel Hybrid) Computer. A detailed analog flowplan may be found in Appendix F; detailed digital computer programs in Appendices G and H. All of these programs are large due to process simulation and the number of options included. Implementation of either the analog or digital
version of the controller itself is simple and does not require a large computer. The XDS Sigma 5 digital computer was used to calculate analog computer pot settings and amplifier gains and to set the pots on the analog computer. The hybrid computer program used to do this is included in Appendix F.

All demonstrations are of computer simulated systems. No attempt was made to control experimental equipment.

All plots were drawn by a Calcomp Plotter. The computer program used to make the plots may be found in Appendix I.

All digital computer programs are written in Fortran IV.

Chapter II discusses briefly the two methods used to solve the time-optimal control problem; the quasilinearization algorithm (3, 4, 5, 6), and a phase plane technique (2, 7, 8). The origin of the dimensionless groups \( t_1 \frac{dx}{dt} \) and \((t_2-t_1) \frac{dx}{dt}\) is given. These groups are key variables in the correlations used in later chapters with the Modeling Time-Optimal Controller.

In Chapter III we discuss correlations in the time domain that can be used for the open-loop time-optimal control of systems where the parameters of a second-order plus dead time model describing the system
are known. The correlations are demonstrated on under-
damped and overdamped second-order systems without dead
time and a third-order overdamped system with dead time.
The frequency domain correlations presented by Latour (9)
are limited to use with overdamped systems.

Chapter IV describes a Modeling Time-Optimal
Controller which uses state variable feedback to carry
out a time-optimal setpoint change without prior know­
ledge of unsteady-state model parameters. In addition,
the parameters of a second-order lag plus dead time model
describing the system can be obtained as a by-product
of the setpoint change. The time-optimal control and
modeling of underdamped and overdamped second-order
systems is demonstrated.

Latour, et al. (10) and Douglas and Denn (11)
describe feedback time-optimal controllers, however,
they both rely on a predetermined model of the system.
Javinsky and Kadlec (12) compared the time-optimal
control of an experimental laboratory sized reactor with
the time-optimal control of an analog computer
simulation of the reactor. Overall agreement was found
to be good, however most differences that they observed
were attributed to uncertainties in the model parameters.

In Chapter V the demonstration of the Modeling
Time-Optimal Controller is extended to third- and fourth-
order systems with dead time. The controllers
insensitivity to changes in process dynamics is demonstrated. A simple analog configuration is shown.

In Chapter VI the Modeling Time-Optimal Controller is used to control the highly nonlinear continuous stirred-tank reactor which was studied by Koppel (2) and Latour, et al. (7). This reactor is a modified version of the reactor studied by Aris and Amundson (13) and by Grethlein and Lapidus (14).

Throughout the work we switched to the new steady-state forcing function at $t_2$, instead of using a more complicated control action, in order to avoid confounding the results of the "bang-bang" control.
Nomenclature

G(s)                Transfer Function
M, M_1, M_2         Forcing Function Multipliers
n                   Order of the system
t                   Time
t_1                 Time of switch between extremes of the manipulated variable
t_2                 Time of switch to conventional setpoint control
T, T_0, T_f         Controlled variable; initial value, final value
X                   Fraction of change completed; X = \frac{T-T_0}{T_f-T_0}
LITERATURE CITED


CHAPTER II

DETERMINATION OF OPTIMUM SWITCHING TIMES

The two methods used to determine optimum switching times are discussed briefly below. The quasilinearization algorithm (1, 2, 3) solves the multi-point boundary-value problem directly and can be used for nonlinear systems and systems of higher than second-order. The phase plane technique (4, 5, 6) avoids the boundary-value problem and is simpler; however, it is limited to linear systems of second-order. The quasilinearization algorithm was used initially and led directly to the key correlations discussed in Chapter IV. The phase plane technique was used to fill in data for systems where the switching times were changing very rapidly and the quasilinearization algorithm had difficulty in converging.

A. Quasilinearization

Let us consider a linear second-order differential equation:

\[
\frac{d^2 X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = U(t)
\]
This equation can be put into dimensionless form by letting \( \tilde{t} = B_1 t, \) \( A = \frac{B_2}{B_1^2}, \) and \( \tilde{U}(t) = U(t)/B_1^2. \)

\[
\frac{d^2 X}{dt^2} + \frac{dX}{dt} + A X = \tilde{U}(t)
\]

Here \( B_1, B_2, \) and \( A \) are constants and \( \tilde{U}(t) \) is the control law to be determined. We define \( X \) so that at \( t=0., X=0., \) and at \( t=t_2, X=1. \)

Using Pontryagin's Minimum Principle (7, 8, 9), it is possible to obtain the following set of equations* which, when solved simultaneously, will specify the switching times for time-optimal control:

\[
\frac{dX}{d\tilde{t}} = \frac{dX_1}{dt} = \dot{X}_1 = X_2
\]

\[
\frac{d^2X}{d\tilde{t}^2} = \frac{dX_2}{dt} = \ddot{X}_2 = -X_2 - AX_1 + \tilde{U}(t)
\]

(II-1)

\[
\frac{d\lambda_1}{dt} = \dot{\lambda}_1 = A\lambda_2
\]

\[
\frac{d\lambda_2}{dt} = \dot{\lambda}_2 = -\lambda_1 + \lambda_2
\]

If we define the vector \( \mathbf{X} = [X_1 \ X_2 \ \lambda_1 \ \lambda_2]^T \) then we have the following boundary conditions:

*These equations are derived in Appendix B.
Note that there are six variables in this set of equations: \( X_1, X_2, \lambda_1, \lambda_2, t_1, t_2 \).

The generalized Newton-Raphson or quasilinearization algorithm with a modification to take care of the unknown switching times, was used to solve this multipoint boundary-value problem. In the presentation below, the generalized algorithm (quasilinearization) and the familiar Newton-Raphson procedure for determining the root of an equation are compared.

### Newton-Raphson

See Figure II-1

\[
\frac{df}{dX} \bigg|_{X_n} = \frac{f(X_{n+1}) - f(X_n)}{X_{n+1} - X_n}
\]

This is usually seen in the form:

\[
X_{n+1} = X_n - \frac{df}{dX} \bigg|_{X_n} (X_n - f(X_n))
\]

since the extrapolation is to \( f(X_{n+1}) = 0 \). This is repeated until

### Quasilinearization

See Figure II-2

\[
\overline{J}X_n = \frac{f_{n+1} - f_n}{X_{n+1} - X_n}
\]

where \( \overline{J} \) is the Jacobian Matrix and the subscripts \( n \) and \( n+1 \) represent iterations. The problem is solved at each value of \( t \), the extrapolation being from the value at one
FIGURE II-1. Newton-Raphson Procedure.
FIGURE II-2. Quasilinearization.
I^n+i - Xn| < ε, a small number. For our purposes we will express it as

\[ f(X_{n+1}) = f(X_n) + \frac{df}{dX} (X_{n+1} - X_n) \]

iteration to the value at the next iteration.

Expanding for \( f_1 \):

\[ f_{n+1}^{1} = f_1 + \frac{\partial f_1}{\partial X_1} (X_{1,n+1} - X_1) \]

\[ + \frac{\partial f_1}{\partial X_2} (X_{2,n+1} - X_2) + \ldots \]

Repeat until

\[ |X_{n+1} - X_n| < \varepsilon, \text{ a small number.} \]

In our case \( \bar{X} = X \) therefore

\[ \bar{X}_{n+1} = X + \bar{J}_n (X_{n+1} - X_n) \]  \hspace{1cm} (II-2)

In order to take care of the unspecified switching times, we make the following substitutions, as suggested by Long (10):
Our time-optimal equations II-1 then become:

\[
\frac{dX_1}{dS} = C_5 X_2
\]

\[
\frac{dX_2}{dS} = -C_5 X_2 - C_5 A X_1 + C_5 \bar{U}
\]

\[
\frac{d\lambda_1}{dS} = C_5 A \lambda_2
\]

\[
\frac{d\lambda_2}{dS} = -C_5 \lambda_1 + C_5 \lambda_2
\]

for the interval \(0 \leq S \leq 1\). During the interval \(1 \leq S \leq 2\), the optimal equations become:

\[
\frac{dX_1}{dS} = C_6 X_2
\]

\[
\frac{dX_2}{dS} = -C_6 X_2 - C_6 A X_1 + C_6 \bar{U}
\]

\[
\frac{d\lambda_1}{dS} = C_6 A \lambda_2
\]

\[
\frac{d\lambda_2}{dS} = -C_6 \lambda_1 + C_6 \lambda_2
\]
The unknowns $C_5$ and $C_6$ are incorporated into the algorithm so that now we redefine $\bar{X}$:

$$X = [X_1 \ X_2 \ \lambda_1 \ \lambda_2 \ C_5 \ C_6]^T$$

Since $C_5$ and $C_6$ are constant with respect to time and vary only from one iteration to the next, $\dot{C}_5 = \dot{C}_6 = 0$. The integration is from $S = 0$ to $S = 2$. The first switch $t_1$ occurs at $S = 1$ and $t_2$ occurs at $S = 2$. When the quasilinearization algorithm converges:

$$t_1 = C_5$$
$$t_2 = t_1 + C_6$$

For our case, Equation II-2 becomes:

$$\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{\lambda}_1 \\
\dot{\lambda}_2 \\
\dot{C}_5 \\
\dot{C}_6 \\
\end{bmatrix}_{n+1} = 
\begin{bmatrix}
0 & C_5 & 0 & 0 & X_2 & 0 \\
0 & -C_5 & 0 & 0 & -X_2 & -A X_1 \\
0 & 0 & C_5 & 0 & 0 & A\lambda_2 \\
0 & 0 & 0 & -C_5 & C_5 & -\lambda_1 + \lambda_2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_n 
\begin{bmatrix}
X_1 \\
X_2 \\
\lambda_1 \\
\lambda_2 \\
C_5 \\
C_6 \\
\end{bmatrix}_n + 
\begin{bmatrix}
X_1 \\
X_2 \\
\lambda_1 \\
\lambda_2 \\
C_5 \\
C_6 \\
\end{bmatrix}_{n+1}$$ (II-5)

for the interval $0 \leq S \leq 1$. During the interval $1 \leq S \leq 2$ we have a similar set of equations with $C_6$ replacing $C_5$.

Since these equations are linear, the general solution can be written as the sum of one particular and six homogeneous solutions. At each value of $t$,
where $\bar{X}$ is a vector of constants, $X_P$ is the particular solution and $X_H$ is a $6 \times 6$ matrix containing the homogeneous solutions. We solved Equation II-5, using a fourth-order Runge-Kutta routine, with arbitrary initial conditions to obtain the particular and homogeneous solutions. We know the value of $X_{n+1}$ at six points from the boundary conditions, therefore we can form a set of six algebraic equations and solve for the only unknowns, the six values of $\bar{X}$. Knowing $\bar{X}$ we can calculate the entire trajectory for $X_{n+1}$ from Equation II-6.

We continue until

$$\left| \bar{X}_{n+1} - \bar{X}_n \right| < \varepsilon, \text{ a small number.}$$

The computer program and a flowplan of the algorithm can be found in Appendix D. Since the algorithm is capable of handling nonlinear systems, provisions are included in the program for two switches between extremes of the manipulated variable. The program is written for the nonlinear system:

$$\frac{d^2X}{dt^2} + \frac{dX}{dt} + AX + A_1X^2 = U$$
however it could be easily modified to handle other systems. Since we allow for the possibility of two switches, there are seven homogeneous solutions, three switching times ($t_1$, $t_2$, $t_3$), and three unknowns ($C_5$, $C_6$, $C_7$). The integration is from $S = 0$ to $S = 3$.

Since time-optimum control of linear second-order systems requires only one switch between extremes of the manipulated variable, the algorithm in each case found $t_1 = t_2$, i.e., there was no change in the value of $X$ between $S = 1$ and $S = 2$. Figures II-3 and II-4 are exaggerated illustrations showing trajectories in the $S$ (modified-time) domain and time domains respectively.

It should be noted that, at convergence, the slope of the trajectory in Figure II-3 at $S = l^-$ is:

$$\frac{dx}{ds}\bigg|_{S=1^-} = \frac{dx}{dt/C_5} = C_5 \frac{dx}{dt} = t_1 \frac{dx}{dt}$$

Since there is no change in $X$ between $S = 1$ and $S = 2$:

$$\frac{dx}{dt}\bigg|_{S=1^-} = \frac{dx}{dt}\bigg|_{S=2^+}$$

At $S = 2^+$:

$$\frac{dx}{ds}\bigg|_{S=2^+} = \frac{dx}{dt} = \frac{t - C_5}{C_6} = C_6 \frac{dx}{dt} = (t_2 - t_1) \frac{dx}{dt}$$

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FIGURE II-3. Quasilinearization in the Modified Time Domain.
FIGURE II-4. Quasilinearization in the Time Domain.
The two dimensionless groups \( t_1 \frac{dx}{dt} \) and \( (t_2-t_1) \frac{dx}{dt} \) will be key variables in the correlations used with the Modeling Time-Optimal Controller which is described in Chapter IV.

For the time-optimal control equations, a fairly close first approximation of the trajectory of each state variable is required in order for the quasilinearization algorithm to converge. Therefore, we started with an initial value solution for the system:

\[
\ddot{x} + \dot{x} = 1
\]

using switching times obtained using the phase plane technique discussed later in this chapter. An initial value of \( \lambda_1 \), that was close enough so that the quasilinearization algorithm would converge, was found by trial and error. Once we obtained convergence with this system, we proceeded in stepwise fashion to the system of interest.

B. Phase Plane Technique

The phase plane technique used here is essentially the same as the one used by Koppel (4, 5), except that the differential equations are in slightly different form. The phase plane trajectories before
### Nomenclature for Discussion of Quasilinearization

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Coefficient of $X$ in second-order differential equation</td>
</tr>
<tr>
<td>$A_1$</td>
<td>Coefficient of $X^2$ in nonlinear second-order differential equation</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>Vector of unknown constants</td>
</tr>
<tr>
<td>$B_1, B_2$</td>
<td>Coefficients of second-order differential equation</td>
</tr>
<tr>
<td>$C_5, C_6, C_7$</td>
<td>Variables in the modified time domain</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>A small number</td>
</tr>
<tr>
<td>$f$</td>
<td>Function in the Newton-Raphson demonstration</td>
</tr>
<tr>
<td>$\bar{f}$</td>
<td>Vector function in the Quasilinearization demonstration</td>
</tr>
<tr>
<td>$f_1$</td>
<td>An element of $\bar{f}$</td>
</tr>
<tr>
<td>$\bar{J}$</td>
<td>Jacobian matrix</td>
</tr>
<tr>
<td>$\lambda_1, \lambda_2$</td>
<td>Defined by Equation II-1</td>
</tr>
<tr>
<td>$\dot{\lambda}_1, \dot{\lambda}_2$</td>
<td>Defined by Equation II-1</td>
</tr>
<tr>
<td>$n$</td>
<td>Iterations in the Newton-Raphson and Quasilinearization algorithms</td>
</tr>
<tr>
<td>$S$</td>
<td>Modified time variable</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$t_1$</td>
<td>Time of first switch between extremes of the manipulated variable</td>
</tr>
</tbody>
</table>
\( t_2 \) Time of second switch between extremes of the manipulated variable if two switches are allowed. If not, time of switch to new steady-state value.

\( t_3 \) Time of switch to new steady-state value if two switches between extremes of the manipulated variable are allowed.

\( \bar{\tau} \) Dimensionless time

\( u \) Forcing function

\( \bar{u} \) Dimensionless forcing function

\( \bar{X} \) Fraction of setpoint change completed

\( X_1, X_2 \) Defined by Equation II-1

\( \dot{X}_1, \dot{X}_2 \) Defined by Equation II-1

\( \bar{X} \) Vector of variables in the Quasilinearization algorithm

\( \bar{X}_0, \bar{X}_{t_1}, \bar{X}_{t_2} \) Boundary conditions at \( t = 0, t_1, \) and \( t_2 \)

\( \bar{X}_p \) Vector of particular solutions

\( \bar{X}_H \) Matrix of homogeneous solutions

\( \bar{\bar{X}} \) Defined by Equation II-2
and after \( t_1 \) are determined analytically and \( t_1 \) and \( t_2 \)
are selected so that the two trajectories coincide at
\( t_1 \), as shown in Figure II-5.

**Trajectory for \( 0 \leq t < t_1 \)**

\[
\ddot{X} + \dot{X} + AX = AM_1 
\]

**Boundary Conditions:**

\( \theta t = 0, \ X = 0, \ \dot{X} = 0 \)

has the solution

\[
X = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} + M_1 
\]

where

\[
\lambda_1 = \frac{-1 - \sqrt{1-4A}}{2} 
\]

\[
\lambda_2 = \frac{-1 + \sqrt{1-4A}}{2} 
\]

\[
C_1 = \frac{-M_1}{1 - \lambda_1/\lambda_2} 
\]

\[
C_2 = \frac{-\lambda_1}{\lambda_2} C_1 
\]

**Trajectory for \( t_1 \leq t < t_2 \)**

\[
\ddot{X} + \dot{X} + AX = AM_2 
\]

**Boundary Conditions:**

\( \theta t = t_2, \ X = 1, \ \dot{X} = 0 \)

has the solution
FIGURE II-5. Phase Plane.
\[ X = C_3e^{-\lambda_1 t} + C_4e^{-\lambda_2 t} + M_2 \]

where \( \lambda_1 \) and \( \lambda_2 \) are as previously defined,

\[ C_3 = \frac{(1 - M_2)e^{\lambda_1 t_2}}{(1 - \lambda_1/\lambda_2)} \]

\[ C_4 = \frac{-(1 - M_2)e^{\lambda_2 t_2}}{(1 - \lambda_2/\lambda_1)} \]

At \( t = t_1 \), \( X \) for the \( M = +2 \) trajectory must equal \( X \) for the \( M = -2 \) trajectory or, eliminating \( X \):

\[ C_1e^{-\lambda_1 t_1} + C_2e^{-\lambda_2 t_1} + M_1 = C_3e^{-\lambda_1 t_1} + C_4e^{-\lambda_2 t_1} + M_2 \quad (II-7) \]

Likewise the values of \( \dot{X} \) must be equal, or:

\[ -\lambda_1 C_1e^{-\lambda_1 t_1} - \lambda_2 C_2e^{-\lambda_2 t_1} = -\lambda_1 C_3e^{-\lambda_1 t_1} - \lambda_2 C_4e^{-\lambda_2 t_1} \quad (II-8) \]

The pattern search routine (11) found in Appendix E was used to solve Equations II-7 and II-8 for \( t_1 \) and \( t_2 \).

These trajectories in the \( X \) vs \( t \) plane are shown in Figure II-6. Prior to \( t_1 \), the time-optimal
FIGURE II-6. Time-Optimal Trajectory.
trajectory follows the \( M = +2 \) curve. Between \( t_1 \) and \( t_2 \) it follows the \( M = -2 \) curve, and after \( t_2 \) it leaves these two curves and is controlled by the new steady-state forcing function.
Nomenclature for Discussion of Phase Plane Technique

A \quad \text{Coefficient of dimensionless second-order differential equation}

C_1, C_2 \quad \text{Constants in analytical solution; } 0 \leq t \leq t_1

C_3, C_4 \quad \text{Constants in analytical solution; } t_1 \leq t \leq t_2

\lambda_1, \lambda_2 \quad \text{Roots of the characteristic equation}

M_1, M_2 \quad \text{Forcing function multipliers}

t \quad \text{Time}

t_1 \quad \text{Time of switch between extremes of the manipulated variable}

t_2 \quad \text{Time of switch to new steady-state value of the manipulated variable}

X \quad \text{Fraction of setpoint change completed}

\dot{X} \quad \frac{dX}{dt}

\ddot{X} \quad \frac{d^2X}{dt^2}
LITERATURE CITED


CHAPTER III

CORRELATIONS IN THE TIME DOMAIN

The Origin and Use of the Correlations

The time-optimal control problem was solved for various values of $A$ and $M$ in the dimensionless equation

$$\frac{d^2X}{dt^2} + \frac{dX}{dt} + AX = AM$$

(III-1)

using the methods discussed in Chapter II. The dimensionless switching times $\bar{t}_1$ and $\bar{t}_2$, and the percentage of the step change completed, $X$ at $t_1$, were used to generate Figures III-1, III-2, and III-3. These figures can be used to time-optimally control systems where the parameters of a second-order plus dead time model describing the system are known. The frequency domain correlations presented by Latour (1) are only applicable to overdamped systems however, Figures III-1, III-2, and III-3 can be used with underdamped or overdamped systems. In the following examples we demonstrate the use of these figures in controlling several different systems. In some cases the trajectory obtained is not "perfect," i.e., there may be slight
FIGURE III-1. Percent of Setpoint Change Completed at $t_1$. 

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FIGURE III-2. Dimensionless $t_1$. 

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FIGURE III-3. Dimensionless $t_2$. 

$M = 0.8$
$M = 1.$
$M = 2.$
$M = 3.$
$M = 4.$
$M = 5.$
overshoot or undershoot. However, in most cases this is less than three percent. In all of the examples we must have the equation describing the system in the form of Equation III-1. Examples III-1 and III-2 assume that we know the parameters describing a second-order overdamped system without dead time. Examples III-3 and III-4 are similar except that the system is underdamped. In Example III-5 a third-order system with dead time is fitted with a second-order lag plus dead time model using the slope-intercept method (2). Other methods could have been used to determine the model time constants (3, 4, 5). The model is used to determine $t_1$ and $t_2$ using Figures III-2 and III-3. Example III-6 is similar except that the model is a least squares fit of the step response using a pattern search routine similar to that used in Chapter II. Thus in each example we have a second-order equation of the form,

$$\frac{d^2X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = B_2 M$$ (III-2)

where $X$ is defined to range between 0 and 1. In order to transform Equation III-2 into the form of Equation III-1 we let $\bar{t} = B_1 t$ and $A = B_2 / B_1^2$. 

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Enlarged versions of Figures III-1, III-2, and III-3 were used in these examples in order to obtain more accuracy. Data used in generating these figures may be found in Appendix C.

Example III-1

Time-optimal control of the system:

\[
\frac{d^2X}{dt^2} + .1667 \frac{dX}{dt} + .00667 X = .00667 M \quad (III-3)
\]

or in the Laplace domain:

\[
G(s) = \frac{1}{(15s + 1)(10s + 1)}
\]

using Figure III-1.

In order to put this into dimensionless form let \( \bar{t} = B_1 t = .1667 t \) and \( A = B_2/B_1^2 = .24 \) which yields.

\[
\frac{d^2X}{dt^2} + \frac{dX}{dt} + .24 X = .24 M \quad (III-4)
\]

Entering Figure III-1 with \( A = .24 \) and \( M = 2 \), we determine that \( t_1 \), the time for switching between the extremes of the manipulated variable, should occur when \( X = .945 \). Since this is a second-order system without dead time, \( t_2 \) will occur when \( \frac{dX}{dt} = 0 \). This example and Example III-3, which is a similar treatment of an underdamped system, are probably not very practical since it would be difficult to incorporate a real or
model dead time into the procedure. This more or less limits its use to the specific case treated, i.e., second-order systems without dead time. Figure III-1 is put to more practical use in later chapters.

Figure III-4 shows the step response and time-optimal responses of Equation III-3 for \( M = 2, 3, \) and \( 4 \) using this procedure. For \( M = 3 \) and \( M = 4 \), \( t_1 \) occurs at \( X = .884 \) and \( X = .840 \) respectively.

**Example III-2**

Time-optimal control of Equation III-3 using Figures III-2 and III-3. The original differential equation is put into dimensionless form, as in Example III-1:

\[
\frac{d^2\bar{X}}{d\bar{t}^2} + \frac{d\bar{X}}{d\bar{t}} + .24 \bar{X} = .24 M
\]

where \( \bar{t} = .1667 t \).

Entering Figure III-2 with \( A = .24 \) and \( M = 2 \), we determine that the dimensionless switching time \( \bar{t}_1 = 3.29 \). From Figure III-3 we similarly determine that \( \bar{t}_2 = 3.65 \). Converting to non-dimensionless time:

\[
t_1 = \frac{\bar{t}_1}{.1667} = 19.75
\]

\[
t_2 = \frac{\bar{t}_2}{.1667} = 21.90
\]
FIGURE III-4. Time-Optimal Control of an Overdamped Second-Order System Using Figure III-1.
Figure III-5 shows the time-optimal trajectories for $M = 2, 3, \text{ and } 4$ for this system. Dimensionless switching times for the other trajectories are:

- $M = 3, \quad \bar{t}_1 = 2.25, \quad \bar{t}_2 = 2.69$ and
- $M = 4, \quad \bar{t}_1 = 1.75, \quad \bar{t}_2 = 2.23$.

Time-optimal control using Figures III-2 and III-3 is not limited to systems without dead time, as is demonstrated in Examples III-5 and III-6.

**Example III-3**

This example is similar to Example III-1 except that the system is underdamped:

\[
\frac{d^2X}{dt^2} + .40 \frac{dX}{dt} + X = M \tag{III-5}
\]

or in the Laplace domain

\[ G(s) = \frac{1}{(s^2 + .4s + 1)} \]

The dimensionless form of this equation is

\[
\frac{d^2X}{d\bar{t}^2} + \frac{dX}{d\bar{t}} + 6.25 \ X = 6.25 \ M \tag{III-6}
\]

where $\bar{t} = .40 \ t$. 

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We enter Figure III-1 with \( A = 6.25 \) and determine the value of \( X \) at which \( t_1 \) occurs for a specified value of \( M \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.709</td>
</tr>
<tr>
<td>3</td>
<td>.650</td>
</tr>
<tr>
<td>4</td>
<td>.616</td>
</tr>
</tbody>
</table>

As in Example III-1, \( t_2 \) occurs when \( \frac{dX}{dt} = 0 \).

In Example III-1, the overdamped case, time-optimal control with \( M \leq 1 \) does not have much meaning. For the underdamped case, however, it is possible to obtain time-optimal control with \( M \leq 1 \) since there is overshoot during the step response. From Figure III-1 we obtain:

<table>
<thead>
<tr>
<th>( M )</th>
<th>( X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>.8735</td>
</tr>
<tr>
<td>.8</td>
<td>.940</td>
</tr>
</tbody>
</table>

Figure III-6 shows the step response and time-optimal trajectories discussed in this example.

**Example III-4**

Time-optimally control Equation III-5 using Figure III-2 and III-3. The original differential equation is put into the dimensionless form

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Figure III-6. Time-Optimal Control of an Underdamped Second-Order System using Figure III-1.
\[ \frac{d^2 X}{dt^2} + \frac{dX}{dt} + 6.25 \, X = 6.25 \, M \]

where \( \bar{t} = 0.40 \, t \).

We enter Figures III-2 and III-3 with \( A = 6.25 \) and read off the dimensionless switching times for the desired value of \( M \). The dimensionless times are converted to real time as in Example III-2.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>2.080</td>
<td>2.340</td>
</tr>
<tr>
<td>1.0</td>
<td>1.630</td>
<td>1.988</td>
</tr>
<tr>
<td>2.0</td>
<td>0.926</td>
<td>1.355</td>
</tr>
<tr>
<td>3.0</td>
<td>0.702</td>
<td>1.118</td>
</tr>
<tr>
<td>4.0</td>
<td>0.590</td>
<td>0.971</td>
</tr>
</tbody>
</table>

Time-optimal trajectories for these cases are shown in Figure III-7.

Example III-5

Time-optimal control of the third-order system:

\[ \frac{d^3 X}{dt^3} + 0.367 \frac{d^2 X}{dt^2} + 0.040 \frac{dX}{dt} + 0.00133 \, X = 0.00133 \, M \]  

with 5 minutes dead time using Figures III-2 and III-3.

The transfer function describing the system is:

\[ G(s) = \frac{e^{-5s}}{(15s + 1)(10s + 1)(5s + 1)} \]
In this example, a second-order lag plus dead time model of the system was obtained using the slope-intercept method (2). Figure III-8 shows the step response of the system. From Figure III-9, the slope-intercept diagram, we obtain a model dead time $\theta = \tau_1 = 8.$, and time constants $\tau_1 = 18.$ and $\tau_2 = 8.$. The transfer function of the model is therefore

$$G_m(s) = \frac{e^{-8s}}{(18s + 1)(8s + 1)}$$

which corresponds to the differential equation:

$$\frac{d^2X}{dt^2} + 0.1805 \frac{dX}{dt} + 0.00695 X = 0.00695 M$$  \hspace{1cm} (III-8)

with dead time $\theta = 8.$.. The step responses of this model and of the third-order system are compared in Figure III-10. In dimensionless form the model is

$$\frac{d^2X}{d\tau^2} + \frac{dX}{d\tau} + 0.2135 X = 0.2135 M$$  \hspace{1cm} (III-9)

with dead time $\Theta = 1.71.$ Using Figures III-2 and III-3, as in the previous examples, we determine the switching times for various values of $M$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20.10</td>
<td>22.10</td>
</tr>
<tr>
<td>3</td>
<td>13.69</td>
<td>16.08</td>
</tr>
<tr>
<td>4</td>
<td>10.51</td>
<td>13.30</td>
</tr>
</tbody>
</table>
FIGURE III-10. Comparison of Third-Order System and Slope-Intercept Model Step Responses.

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Figure III-11 shows the time-optimal responses for these cases.

Example III-6

This example is similar to Example III-5 except that the second-order plus dead time model is obtained by using a pattern search routine to find the parameters $B_1, B_2,$ and $\theta$ which give a least squares fit to the step response. The model obtained for Equation III-7 is:

$$\frac{d^2X}{dt^2} + .129 \frac{dX}{dt} + .00477 X = .00477 M \quad (III-10)$$

with model dead time $\theta = 8$. Figure III-12 compares the step responses of the system and model. The following switching times were obtained as in Example III-5:

<table>
<thead>
<tr>
<th>M</th>
<th>t1</th>
<th>t2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>21.95</td>
<td>25.01</td>
</tr>
<tr>
<td>3</td>
<td>15.11</td>
<td>18.60</td>
</tr>
<tr>
<td>4</td>
<td>12.01</td>
<td>15.50</td>
</tr>
</tbody>
</table>

Figure III-13 shows the time-optimal responses for these cases.
FIGURE III-11. Time-Optimal Control of a Third-Order System using $t_1$ and $t_2$ for Slope-Intercept Model.
FIGURE III-13. Time-Optimal Control of a Third-Order System using $t_1$ and $t_2$ for Least Squares Model.
Nomenclature

\( A \) Coefficient in Equation III-1

\( B_1, B_2 \) Coefficients in Equation III-2

\( G(s) \) System transfer function

\( G_M(s) \) Model transfer function

\( N \) Forcing function multiplier

\( s \) Laplace transform variable

\( t \) Time

\( t_1 \) Time of switch between extremes of the manipulated variable

\( t_2 \) Time of switch to new steady-state value of the manipulated variable

\( \bar{t} \) Dimensionless time

\( \bar{t}_1 \) Dimensionless \( t_1 \)

\( \bar{t}_2 \) Dimensionless \( t_2 \)

\( \tau_1, \tau_2 \) Model time constants

\( \theta \) Model dead time

\( \bar{\theta} \) Dimensionless model dead time

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CHAPTER IV

TIME-OPTIMAL CONTROL AND MODELING OF SECOND-ORDER SYSTEMS

The Modeling Time-Optimal Controller

The differential equations describing the time-optimal control of a linear second-order system are:

\[
\frac{d^2X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = B_2 M_2 \quad 0 \leq t \leq t_1
\]

\[
\begin{align*}
\theta & t = 0, \ X = 0, \ & \frac{dX}{dt} = 0 \\
\frac{d^2X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = -B_2 M_2 \quad t_1 \leq t \leq t_2 \\
\theta & t = t_2, \ X = 1, \ & \frac{dX}{dt} = 0
\end{align*}
\]

where \( B_1, B_2, M_1, \) and \( M_2 \) are constants. As shown in Chapter II, these equations can be expressed as:

\[
\frac{d^2X}{dt^2} + \frac{dX}{dt} + A X = AM
\]

In the traditional phase plane representation of the time-optimal control problem, the switching curves that determine \( t_1 \) are functions of \( B_1, B_2, \) and \( M. \)
Therefore each system requires a different switching curve. The switching curves for two different systems, described by Equations IV-1 and IV-2 are shown in Figures IV-1 and IV-2 respectively.

\[
\frac{d^2X}{dt^2} + 1.667 \frac{dX}{dt} + 0.00667 X = 0.00667 M \quad (IV-1)
\]

or in the Laplace domain

\[
G(s) = \frac{1}{(15s + 1)(10s + 1)}
\]

and

\[
\frac{d^2X}{dt^2} + 1.2 \frac{dX}{dt} + 0.2 X = 0.2M \quad (IV-2)
\]

or in the Laplace domain

\[
G(s) = \frac{1}{(5s + 1)(s + 1)}
\]

Latour (1) and Douglas and Denn (2) describe feedback time-optimal controllers which use switching curves for fixed, predetermined values of \(B_1\) and \(B_2\). While studying the solution of the time-optimal control boundary-value problem using quasilinearization, we made the single most important discovery of this research: that if a dimensionless phase plane is used in which
FIGURE IV-1. Phase Plane for Equation IV-1.
Figure IV-2. Phase Plane for Equation IV-2.
Instead of $\frac{dX}{dt}$, is plotted v.s. $X$, a single switching curve can be obtained which is applicable for all second-order systems.

This curve and the time-optimal trajectories of Equations IV-1 and IV-2 are shown in Figure IV-3. The switching curve was obtained by solving the time-optimal control problem for various values of $A$ and $M$. We can measure $t$, and $X$, and approximate $\frac{dX}{dt}$ since it is not changing very fast. The controller simply compares $t \frac{dX}{dt}$ with the switching curve, $f(X)$, and switches between the extremes of the manipulated variable when $t \frac{dX}{dt} > f(X)$. The switching curve is a slight function of $M$, the forcing function multiplier, but the curves are so close that a curve through the $M = 2$ points can be used for other values of $M$. The system used to approximate $\frac{dX}{dt}$ (3) limited the analog implementation to a maximum of about $M = 3$. A digital algorithm, however, does not have this limitation. In Figure IV-3 the $M = 2$ and $M = 5$ data points are shown but the switching curve is drawn through the $M = 2$ points only. The limitation of $M = 3$ or less is not a severe one since the improvement of time-optimal control over other methods shown in Figure I-1 was with $M = 2$.

*See Chapter II for the origin of this dimensionless group.
FIGURE IV-3. Dimensionless Phase Plane.

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The time for switching to conventional setpoint control, $t_2$, is a function of $X$ and $\frac{dX}{dt}$ at $t_1$, as shown in Figure IV-4. Here, as in Figure IV-3, the lines are through the $M = 2$ points only. Knowing $t_1$, $X$, and $\frac{dX}{dt}$ at $t_1$ we can compute $t_2$, using the correlation in Figure IV-4.*

$$t_2 = 
\begin{bmatrix}
1 - X \\
b \frac{dX}{dt}
\end{bmatrix}
+ t_1$$

where:

\[ b = .490 \quad .811 \leq X \leq 1. \]

\[ b = .545 \quad .5 \leq X \leq .811 \]

The data points used in plotting Figures IV-3 and IV-4 are tabulated in Appendix A.

The two examples which follow demonstrate the use of this algorithm. Example IV-1 is the time-optimal control and modeling of an overdamped linear second-order system. Example IV-2 is similar except that the system is underdamped.

*See Chapter II for the origin of the dimensionless group $(t_2 - t_1) \frac{dX}{dt}$. 

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FIGURE IV-4. $t_2$ Correlation.

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Example IV-1. Time-Optimal Control and Modeling of an Overdamped Second-order System.

The time-optimal trajectory of the second-order system:

\[
\frac{d^2X}{dt^2} + 0.1667 \frac{dX}{dt} + 0.00667 X = 0.00667 M, \quad M = 2 \quad (IV-3)
\]

is shown in Figures IV-5 and IV-6. Note that in this example the differential equation describing the system was not used in accomplishing the time-optimal control. In fact a good approximation of the differential equation describing the system can be obtained as a by-product of the time-optimal control.

For the "unknown" system of Equation IV-3 we desire to determine \( B_1 \) and \( B_2 \) in the equation:

\[
\frac{d^2X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = B_2 M
\]

or in dimensionless form:

\[
\frac{d^2\bar{X}}{d\bar{t}^2} + \frac{d\bar{X}}{d\bar{t}} + AX = AM
\]

where \( \bar{t} = B_1 t \) and \( A = B_2 / B_1^2 \). When carrying out a time-optimal setpoint change with our system, we can measure \( t_1 = 19.86 \) and \( X = 0.945 \) at \( t_1 \). Knowing \( X \) at \( t_1 \), we can determine that \( A = 0.24 \) from Figure III-1. Knowing \( A \) and \( M \), we can get the dimensionless switching times \( \bar{t}_1 = 3.29 \)
FIGURE IV-5. Time-Optimal Trajectory of Equation IV-3.
FIGURE IV-6. Dimensionless Phase Plane for Equation IV-3.

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from Figure III-2 and $T_2 = 3.65$ from Figure III-3. We can now calculate $B_1$ and $B_2$.

$$B_1 = \frac{T_1}{t_1} = \frac{3.29}{19.86} = .1657$$

$$B_2 = AB_1^2 = (.24)(.1657)^2 = .00658$$

For the system

$$\frac{d^2X}{dt^2} + .1667 \frac{dX}{dt} + .00667 X = .00667 M$$

we obtained the model

$$\frac{d^2X}{dt^2} + .1657 \frac{dX}{dt} + .00658 X = .00658 M$$

Figure IV-7 shows that the step and time-optimal responses of this system and its model are almost identical.

**Example IV-2. Time-Optimal Control and Modeling of an Underdamped Second-order System.**

This example is similar to Example IV-1 except that the system is underdamped.

$$\frac{d^2X}{dt^2} + .40 \frac{dX}{dt} + 1. X = 1. M \quad (IV-4)$$

Figure IV-8 shows the step and time-optimal responses of this system. Figure IV-9 is a dimensionless phase plane showing the time-optimal trajectory of the system.
FIGURE IV-7. Comparison of Second-Order System and Model Trajectories.

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We measure $t_1 = .9204$ and $X = .705$ at $t_1$. From Figure III-1 we determine that $A = 6.75$ and from Figure III-2 we obtain the dimensionless switching time $\bar{t}_1 = .356$. With this information we can calculate $B_1$ and $B_2$.

$$B_1 = \frac{t_1}{\bar{t}_1} = \frac{.356}{.9204} = .386$$

$$B_2 = AB_1^2 = (6.75)(.386)^2 = 1.008$$

Our model is therefore:

$$\frac{d^2X}{dt^2} + .386 \frac{dX}{dt} + 1.008 X = 1.008 M$$

The time-optimal trajectories of the system and its model are shown in Figure IV-10. Their step responses are compared in Figure IV-11. In both cases the fit is very good.
FIGURE IV-10. Underdamped System and Model Time-Optimal Trajectories.
FIGURE IV-11. Underdamped System and Model Step Responses.
Nomenclature

A  Coefficient in dimensionless second-order differential equation
b  Parameter in Figure IV-4
B_1, B_2  Coefficients in second-order differential equation
f(X)  Value of switching curve in the dimensionless phase plane
G(s)  System transfer function
M, M_1, M_2  Forcing function multipliers
s  Laplace transform variable
t  Time
t_1  Time of switch between extremes of the manipulated variable
t_2  Time of switch to new steady-state value of the manipulated variable
\bar{t}  Dimensionless time
\bar{t}_1  Dimensionless t_1
\bar{t}_2  Dimensionless t_2
X  Fraction of setpoint change completed
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CHAPTER V

TIME-OPTIMAL CONTROL AND MODELING OF HIGHER-ORDER SYSTEMS

Application to a Third-Order System with Dead Time

The results described in Chapter IV can be extended to higher-order systems by treating them as second-order systems with dead time, $\Theta$. When a predicted trajectory $(t - \Theta) \frac{dX}{dt} + \Theta \frac{dX}{dt} = t \frac{dX}{dt}$ plotted vs. $X + \Theta \frac{dX}{dt} = X + \Delta X$ crosses the switching curve in the dimensionless phase plane, the switch is made between the extremes of the manipulated variable. Figure V-1 shows the actual and predicted trajectories for the third-order system:

$$\frac{d^3X}{dt^3} + .367 \frac{d^2X}{dt^2} + .040 \frac{dX}{dt} + .00133 X = .00133 M \quad (V-1)$$

with 5 minutes dead time and $M = 2$. The transfer function describing this system is:

$$G(s) = \frac{e^{-5s}}{(15s + 1)(10s + 1)(5s + 1)}$$

Model parameters can be obtained in the same manner as for second-order systems if $(X + \Delta X)$ at $t_1$ is used

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FIGURE V-1. Dimensionless Phase Plane for a Third-Order System.
instead of $X$ at $t_1$. During the time-optimal setpoint change we measure:

$$t_1 = 20.00$$
$$t_2 = 22.26$$
$$(X + \Delta X) \text{ at } t_1 = .942$$

This gives the second-order model:

$$\frac{d^2X}{dt^2} + .1585 \frac{dX}{dt} + .0063 X = .0063 M$$

with a model dead time $\theta = 10.\ldots$ The determination of $\theta$ is discussed in the next section on tuning. System and model trajectories are shown in Figure V-2.

The model obtained here can be used with Figures III-2 and III-3 to determine switching times for the system at other values of $M$. Figure V-3 shows these trajectories for $M = 2$, $3$, and $4$.

**Tuning**

A single tuning adjustment is required when the controller is put into service; a model dead time, $\theta$, is determined that gives nearly equal overshoot and undershoot* about the new steady state value, $X = 1.\ldots$

---

*The overshoot and undershoot should be at most only about one or two percent. If one should happen to obtain equal overshoot and undershoot of greater value, say three or four percent, one might suspect that large nonlinearities exist, such as those discussed in Chapter VI.
FIGURE V-2. Comparison of Third-Order System and Second-Order Model Trajectories.
This controller essentially simplifies the parameter identification of a second-order lag plus dead time model from a three-dimensional search to a one-dimensional search. Here, as before, it is assumed that we have a steady-state model of the system.

Once $\theta$ is determined, the controller is rather insensitive to changes in the system dynamics. This is demonstrated in a later section. Even though the dynamics of the system may change, the controller compensates for much of the change and does not require retuning. This is because the tuning itself is a correction for deviation from second-order behavior.

The controller was tuned in the following manner for the third-order system described by Equation V-1. A time-optimal setpoint change was begun with $M = 2$. As the trajectory, shown in Figure V-4 as Curve 1, started to rise, we observed an apparent dead time of about $\theta = 8$, which was quickly entered into the controller so that it could be used to compute the switching times for Curve 1. A one-dimensional search (1) on $\theta$ was then made to determine the $\theta$ that gave nearly equal overshoot and undershoot about the new steady-state value, $X = 1$. In this case we simply made two additional setpoint changes with $\theta = 9$ and $\theta = 10$ which produced Curves 2 and 3 respectively. A model dead time, $\theta = 10$, produced a
FIGURE V-4. The Tuning of a Third-Order System.
satisfactory trajectory. These responses are shown in Figure V-4.

**Insensitivity to Process Changes**

Control schemes which use a predetermined model and precalculated switching times (2, 3) cannot compensate for changes in the process dynamics. Examples V-1, V-2, and V-3 demonstrate that the Modeling Time-Optimal Controller is insensitive to process dynamics changes and compensates for them. In these examples we compare the time-optimal responses obtained with our controller (Case I) with precalculated switching times (Case II). In all three examples the Modeling Time-Optimal Controller, Case I, is tuned for the system:

\[
\frac{d^3X}{dt^3} + 0.367 \frac{d^2X}{dt^2} + 0.040 \frac{dX}{dt} + 0.00133 X = 0.00133 M \quad (V-1)
\]

with dead time of 5 minutes. The transfer function describing this system is

\[
G(s) = \frac{e^{-5s}}{(15s + 1)(10s + 1)(5s + 1)}
\]

For Case II we obtained a least squares fit of a second-order plus dead time model to a step response of the system described by Equation V-1. The model obtained was
\[
\frac{d^2X}{dt^2} + 0.129 \frac{dX}{dt} + 0.00477 X = 0.00477 M \quad (V-2)
\]

with model dead time, \( \theta = 8 \). Switching times for this model were computed, using the phase plane technique described in Chapter II, to be:

\[
t_1 = 22.10
\]
\[
t_2 = 24.90
\]

All three examples use these switching times for Case II.

**Example V-1**

In this example we assume that the process dynamics of the system are known, i.e., that Equation V-1 describes the system. Figure V-5 shows that both cases give fairly good time-optimal responses.

**Example V-2**

In this example, we assume that the system has changed so that it is now described by:

\[
\frac{d^3X}{dt^3} + 0.40 \frac{d^2X}{dt^2} + 0.05 \frac{dX}{dt} + 0.002 X = 0.002 M \quad (V-3)
\]

or

\[
G(s) = \frac{e^{-5s}}{(10s + 1)(10s + 1)(5s + 1)}
\]
FIGURE V-5. Time-Optimal Control of the System Described by Equation V-1.
Figure V-6 demonstrates the superiority of the Modeling Time-Optimal Controller if we know only the new steady-state model.

**Example V-3**

This example is similar to Example V-2 except that the system is described by:

\[
\frac{d^3X}{dt^3} + .50 \frac{d^2X}{dt^2} + .08 \frac{dX}{dt} + .004 X = .004 M \quad (V-4)
\]

or

\[
G(s) = \frac{e^{-5s}}{(5s + 1)(10s + 1)(5s + 1)}
\]

The control of this system is shown in Figure V-7.

Figures V-5, V-6, and V-7 demonstrate that the Modeling Time-Optimal Controller effectively compensates for changes in the process dynamics. This can also be clearly seen in the following tabulation of switching times used in these three examples:

<table>
<thead>
<tr>
<th>Example</th>
<th>Time</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td>V-1</td>
<td>t_1</td>
<td>20.00</td>
<td>22.10</td>
</tr>
<tr>
<td></td>
<td>t_2</td>
<td>22.26</td>
<td>24.90</td>
</tr>
<tr>
<td>V-2</td>
<td>t_1</td>
<td>16.74</td>
<td>22.10</td>
</tr>
<tr>
<td></td>
<td>t_2</td>
<td>18.35</td>
<td>24.90</td>
</tr>
<tr>
<td>V-3</td>
<td>t_1</td>
<td>13.16</td>
<td>22.10</td>
</tr>
<tr>
<td></td>
<td>t_2</td>
<td>14.11</td>
<td>24.90</td>
</tr>
</tbody>
</table>

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FIGURE V-6. Time-Optimal Control of the System Described by Equation V-3.

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Application to a Fourth-Order System with Dead Time

The time-optimal control and modeling of the fourth-order system:

\[
\frac{d^4X}{dt^4} + 0.416 \frac{d^3X}{dt^3} + 0.0583 \frac{d^2X}{dt^2} + 0.0033 \frac{dX}{dt} + 0.000067 X = 0.000067 M \tag{V-5}
\]

with 5 minutes dead time is demonstrated below. This system is described by the transfer function:

\[
G(s) = \frac{e^{-5s}}{(20s + 1)(15s + 1)(10s + 1)(5s + 1)}
\]

Figure V-8 shows the tuning of the controller for this system. Figure V-9 shows the dimensionless phase plane after tuning. Applying the same method used earlier in this chapter for the third-order system we obtain the model:

\[
\frac{d^2X}{dt^2} + 0.08325 \frac{dX}{dt} + 0.00206 X = 0.00206 M \tag{V-6}
\]

with model dead time \(\Theta = 15\).

Figure V-10 compares the system and model trajectories. Figure V-11 demonstrates the control of
FIGURE V-8. Tuning of the Fourth-Order System.

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the fourth-order system at $M = 2, 3, \text{ and } 4$ using $t_1$ and $t_2$ for the model from Figures III-2 and III-3.

**Implementation**

The Modeling Time-Optimal Controller has been implemented on both digital and analog (parallel hybrid) computers. Both implementations are simple and neither requires a large computer. A detailed analog computer flowplan may be found in Appendix F. The digital programs used in this research may be found in Appendices G and H. A simplified analog flowplan is shown in Figure V-12. Inputs to the controller are $t, (T - T_0), \frac{1}{T_f - T_0}, \text{ and } \theta$. Here $T_0$ and $T_f$ are the initial and final values of the controlled variable $T$. The outputs from the controller are two logic signals. The switch between extremes of the manipulated variable is made when comparator 1 becomes high. The switch to conventional setpoint control is made when comparator 2 becomes high.
Nomenclature

*b* Parameter in Figure IV-4

**COMP** Analog comparator unit

**DIF** Analog differentiation circuit, see Ref. 4

\( f(X) \) Value of switching curve in the dimensionless phase plane

\( G(s) \) System transfer function

**M** Forcing function multiplier

**MULT** Analog multiplier unit

**s** Laplace transform variable

**t** Time

\( t_1 \) Time of switch between extremes of the manipulated variable

\( t_2 \) Time of switch to new steady-state value of the manipulated variable

**T** Controlled variable

**T_0** Initial value of **T**

**T_f** Final value of **T**

**T/S** Analog track/store unit

**θ** Model dead time

**VDFG** Analog Variable Diode Function Generator unit

**X** Fraction of setpoint change completed

\( ΔX \) Extrapolation of **X** in the dimensionless phase plane
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CHAPTER VI

TIME-OPTIMAL CONTROL AND MODELING OF A HIGHLY NONLINEAR SYSTEM

Introduction

The Modeling Time-Optimal Controller will probably control mildly nonlinear systems as satisfactorily as it does linear systems with no modification to the algorithm. For highly nonlinear systems it may be possible to find a simple modification to the controller that will allow it to be used satisfactorily. The modification, of course, will depend on the nature of the process and its nonlinearities. For the exothermic Continuous Stirred-Tank Reactor (CSTR) which is discussed below, a simple modification, which gave good control, was easily found.

Latour, et al. (1) studied the modeling and time-optimal control of Orent's (2) simulation of a modified version of the Aris-Amundson (3), Grethlein-Lapidus (4) continuous stirred-tank reactor. This is a highly nonlinear system with an irreversible exothermic reaction \( \text{A} \xrightarrow{k} \text{B} \) and first-order kinetics. The reactor simulation, which is described in Appendix J, consists
of three ordinary differential equations which were obtained from mass and energy balances on the reactor and an energy balance on the cooling coil. The reactor temperature is the controlled variable and the coolant flowrate is the manipulated variable.

The reactor simulation used in our work is identical to that used by Latour, et al. except that:

1) they used an analog computer whereas we use a digital computer and a Runga-Kutta integration routine, and

2) they delayed the output temperature by six seconds and we delayed the coolant rate by six seconds.

Steady-state Temperature Model

The steady-state temperature model shown in Figure VI-1 was obtained by operating at different coolant rates until steady-state was reached.

Tuning for $M = 2$

Figure VI-2 shows the tuning for a time-optimal setpoint decrease from $460^\circ K$ to $455^\circ K$ with $M = 2$. The initial estimate of the model dead time, $\theta = 10$, caused the first switch, at $t_1$, to occur too early. Subsequent runs indicated that the model dead time for equal overshoot and undershoot about $X = 1$ lies between
FIGURE VI-1. Steady-State Temperature Model of CSTR.
FIGURE VI-2. Tuning of CSTR for Time-Optimal Setpoint Decrease, 460°K to 455°K, with $M = 2$. 

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\( \theta = 4 \) and \( \theta = 5 \). The value of \( \theta \) is not critical so that although \( \theta = 5 \) was arbitrarily chosen to be used in the discussion to follow, \( \theta = 4 \) would have done just as well.

Figure VI-3 shows an attempt at similar tuning for a time-optimal setpoint increase from \( 460^\circ K \) to \( 465^\circ K \) with \( M = 2 \). A model dead time of \( \theta = 2 \) gives nearly equal overshoot and undershoot of about 4\%. This is larger than desired and caused us to look for a simple modification to the controller that would decrease it.

The curves in Figure VI-3 indicate that the interval \( (t_2 - t_1) \) is too long, that is they reach a peak and instead of lining out they dip down. An examination of the coolant temperature, shown in Figure VI-4 for \( \theta = 5 \), shows what is causing the dip; the fact the coolant temperature takes about twenty seconds to reach the new steady-state value when the temperature is increasing. This is many times the interval \( (t_2 - t_1) \). We found that if we divided the interval \( (t_2 - t_1) \) by 3 for time-optimal setpoint increase, we could obtain much better control.

Figure VI-5 shows the improved trajectories obtained during the tuning for a setpoint increase.
FIGURE VI-3. First Attempt to Tune CSTR for Time-
Optimal Setpoint Increase, 460°K to
465°K, with \( M = 2 \).
FIGURE VI-4. Coolant Temperature Trajectory Corresponding to \( \theta=5 \). for Figure VI-3.
FIGURE VI-5. Modified Tuning of CSTR for Time-Optimal Setpoint Increase, 460°K to 465°K, with \( M = 2 \).
using this modified algorithm. In this case a model dead time of $\theta = 5$. was obtained. Figure VI-6 shows the improved coolant temperature trajectory for the modified algorithm with $\theta = 5$.

**Tuning for $M \geq 2$**

Since the system is highly nonlinear, one might expect different tuning, model dead times, and models for different values of $M$. In order to check this the controller was tuned and the system was modeled for setpoint increases, 460°K to 465°K, at $M = 2, 3, \text{ and } 4$. Similar runs were made for setpoint decreases, 460°K to 455°K. Figures VI-7, VI-8, and VI-9 show the tuned setpoint increases, and Figures VI-10, VI-11, and VI-12 the tuned setpoint decreases for $M = 2, 3, \text{ and } 4$ respectively. These figures also show the time-optimal control of the model using the same switching times used for the CSTR. The step responses of the CSTR and model are also shown. The models obtained are as follows:
FIGURE VI-6. Coolant Temperature Trajectory Corresponding to $\theta = 5$. for Figure VI-5.
FIGURE VI-7. CSTR and Model Trajectories for Setpoint Increase, 460°K to 465°K, with M = 2.
FIGURE VI-8. CSTR and Model Trajectories for Setpoint Increase, $460^\circ K$ to $465^\circ K$, with $M = 3$.

$T_1 = 33.4$

$T_2 = 34.6$
FIGURE VI-9. CSTR and Model Trajectories for Setpoint Increase, 460°K to 465°K, with M = 4.
FIGURE VI-10. CSTR and Model Trajectories for Setpoint Decrease, 460°K to 455°K, with M = 2.

$t_1 = 50.7$
$t_2 = 53.9$
FIGURE VI-11. CSTR and Model Trajectories for Setpoint Decrease, 460°K to 455°K, with M = 3.
FIGURE VI-12. CSTR and Model Trajectories for Setpoint Decrease, 460°K to 455°K, with M = 4.

\[ t_1 = 22.5 \]
\[ t_2 = 24.1 \]
<table>
<thead>
<tr>
<th>Model Number</th>
<th>Setpoint Change</th>
<th>M</th>
<th>Model</th>
<th>θ</th>
<th>t₂-t₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>460°K-465°K</td>
<td>2</td>
<td>$\ddot{X} + 0.1118 \dot{X} + 0.00153 X = 0.00153 M$</td>
<td>5</td>
<td>$(t₂-t₁)/3.$</td>
</tr>
<tr>
<td>2</td>
<td>460°K-465°K</td>
<td>3</td>
<td>$\ddot{X} + 0.1250 \dot{X} + 0.00172 X = 0.00172 M$</td>
<td>5</td>
<td>$(t₂-t₁)/3.$</td>
</tr>
<tr>
<td>3</td>
<td>460°K-465°K</td>
<td>4</td>
<td>$\ddot{X} + 0.1025 \dot{X} + 0.00144 X = 0.00144 M$</td>
<td>4</td>
<td>$(t₂-t₁)/3.$</td>
</tr>
<tr>
<td>4</td>
<td>460°K-455°K</td>
<td>2</td>
<td>$\ddot{X} + 0.1025 \dot{X} + 0.00150 X = 0.00150 M$</td>
<td>5</td>
<td>$(t₂-t₁)/1.$</td>
</tr>
<tr>
<td>5</td>
<td>460°K-455°K</td>
<td>3</td>
<td>$\ddot{X} + 0.1632 \dot{X} + 0.00235 X = 0.00235 M$</td>
<td>7</td>
<td>$(t₂-t₁)/1.$</td>
</tr>
<tr>
<td>6</td>
<td>460°K-455°K</td>
<td>4</td>
<td>$\ddot{X} + 0.3200 \dot{X} + 0.00450 X = 0.00450 M$</td>
<td>10</td>
<td>$(t₂-t₁)/1.$</td>
</tr>
</tbody>
</table>
Therefore, as expected, we do obtain different models for setpoint changes at different values of $M$. However, since there is only one tuning parameter other than the $(t_2 - t_1)$ correction, the variable $\theta$ should be easy to implement for time-optimal control at different values of $M$.

**Use of the Time Domain Correlations**

The time domain correlations in Figure III-2 and III-3 can be used with the models obtained in the previous section to control the CSTR. Figure VI-13 shows the time-optimal control of the CSTR for a setpoint increase from 460°K to 465°K using Model Number 1 and the time domain correlations in Figures III-2 and III-3. Figure VI-14 shows trajectories for a setpoint decrease from 460°K to 455°K also using Model Number 1 (the model for a setpoint increase) and Figures III-2 and III-3. In Figure VI-14 the interval $(t_2 - t_1)/1.$ was used. Although the trajectories in Figure VI-14 are not perfect, Figures VI-13 and VI-14 do demonstrate that models obtained using the Modeling Time-Optimal Controller can be used to produce good control of even a highly nonlinear system.

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FIGURE VI-14. Time-Optimal Control of CSTR for Setpoint Decrease using Time Domain Correlations with Model for Setpoint Increase, except \((t_2-t_1)/1\).
Larger Setpoint Changes

Although the CSTR is a nonlinear system, the Modeling Time-Optimal Controller will control the CSTR for setpoint changes of different size than the one used to tune the controller. Figure VI-15 shows a time-optimal setpoint increase from 460°K to 470°K with $M = 2$. Since $M = 2$ for a 10°K change corresponds to $M = 4$ for a 5°K change, the tuning for 460°K to 465°K with $M = 4$ was used. Figure VI-16 shows a similar trajectory for a setpoint decrease, 460°K to 450°K, tuned for 460°K to 455°K with $M = 4$. 

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FIGURE VI-15. Time-Optimal Setpoint Increase From 460°K to 470°K.
FIGURE VI-16. Time-Optimal Setpoint Decrease from 460°K to 450°K.
Nomenclature

A  Reactant
B  Product
k  Reaction rate
M  Forcing function multiplier
t  Time, sec.
t_1  Time of switch between extremes of the manipulated variable, sec.
t_2  Time of switch to the new steady state forcing function, sec.
θ  Model dead time, sec.
X  Fraction of the setpoint change completed
LITERATURE CITED


APPENDIX A

Data for Modeling Time-Optimal Controller Correlations

Data for Figure IV-3
Data for Figure IV-4
Data for Switching Curve in Figure IV-3

<table>
<thead>
<tr>
<th>X @ t1</th>
<th>( \frac{dx}{dt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\[ M = 2 \]

<table>
<thead>
<tr>
<th>X @ t1</th>
<th>( \frac{dx}{dt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99690</td>
<td>0.71330</td>
</tr>
<tr>
<td>0.99570</td>
<td>0.72440</td>
</tr>
<tr>
<td>0.99380</td>
<td>0.73600</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\[ M = 5 \]

<table>
<thead>
<tr>
<th>X @ t1</th>
<th>( \frac{dx}{dt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.89500</td>
<td>1.17900</td>
</tr>
<tr>
<td>0.85600</td>
<td>1.22200</td>
</tr>
<tr>
<td>0.82600</td>
<td>1.23900</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

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Data for Correlation in Figure IV-4

<table>
<thead>
<tr>
<th>X @ t1</th>
<th>(t₂ - t₁) dX/dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99690</td>
<td>0.00602</td>
</tr>
<tr>
<td>0.99570</td>
<td>0.00923</td>
</tr>
<tr>
<td>0.99380</td>
<td>0.01261</td>
</tr>
<tr>
<td>0.99230</td>
<td>0.01615</td>
</tr>
<tr>
<td>0.989060</td>
<td>0.01989</td>
</tr>
<tr>
<td>0.98870</td>
<td>0.02383</td>
</tr>
<tr>
<td>0.98670</td>
<td>0.02799</td>
</tr>
<tr>
<td>0.98470</td>
<td>0.03236</td>
</tr>
<tr>
<td>0.98250</td>
<td>0.03695</td>
</tr>
<tr>
<td>0.97050</td>
<td>0.06242</td>
</tr>
<tr>
<td>0.95600</td>
<td>0.09175</td>
</tr>
<tr>
<td>0.94200</td>
<td>0.11930</td>
</tr>
<tr>
<td>0.88900</td>
<td>0.22900</td>
</tr>
<tr>
<td>0.85400</td>
<td>0.30100</td>
</tr>
<tr>
<td>0.82900</td>
<td>0.35000</td>
</tr>
<tr>
<td>0.81100</td>
<td>0.38600</td>
</tr>
<tr>
<td>0.79600</td>
<td>0.41400</td>
</tr>
<tr>
<td>0.78500</td>
<td>0.43600</td>
</tr>
<tr>
<td>0.77500</td>
<td>0.45500</td>
</tr>
<tr>
<td>0.74800</td>
<td>0.50600</td>
</tr>
<tr>
<td>0.73100</td>
<td>0.53700</td>
</tr>
<tr>
<td>0.71900</td>
<td>0.55900</td>
</tr>
<tr>
<td>0.68900</td>
<td>0.61300</td>
</tr>
<tr>
<td>0.67500</td>
<td>0.63700</td>
</tr>
<tr>
<td>0.66600</td>
<td>0.65200</td>
</tr>
</tbody>
</table>

\[ M = 2 \]

<table>
<thead>
<tr>
<th>X @ t1</th>
<th>(t₂ - t₁) dX/dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.89500</td>
<td>0.22500</td>
</tr>
<tr>
<td>0.85600</td>
<td>0.30700</td>
</tr>
<tr>
<td>0.82600</td>
<td>0.36700</td>
</tr>
<tr>
<td>0.80400</td>
<td>0.41300</td>
</tr>
<tr>
<td>0.78500</td>
<td>0.45000</td>
</tr>
<tr>
<td>0.75800</td>
<td>0.50500</td>
</tr>
<tr>
<td>0.73700</td>
<td>0.54400</td>
</tr>
<tr>
<td>0.64100</td>
<td>0.72200</td>
</tr>
<tr>
<td>0.61000</td>
<td>0.77500</td>
</tr>
<tr>
<td>0.58200</td>
<td>0.82200</td>
</tr>
</tbody>
</table>

\[ M = 5 \]
APPENDIX B

Derivation of Optimal Equations
Derivation of the Time-Optimal Control Equations

Statement of the Problem: Determine the forcing function $U(t)$ which will drive the system $\ddot{P} + \dot{P} + AP = U(t)$ from steady state at $P = 0$ to steady state at $P = 1$ in minimum time, i.e.

minimize:

$$S = \int_0^{TF} dt$$

Solution

Formulate in terms of state variables

$$X_1 = P$$
$$X_2 = \dot{P} = \dot{X}_1$$
$$X_3 = S = \int_0^{TF} dt$$

$$\dot{X}_1 = X_2$$
$$\dot{X}_2 = -X_2 - AX_1 + U(t)$$
$$\dot{X}_3 = 1.$$

Form the Hamiltonian

$$H = \lambda_1 X_2 + \lambda_2 ( -X_2 - AX_1 + U(t) ) + \lambda_3 (1)$$

Conditions for optimal control

$$\frac{d\bar{X}}{dt} = \frac{\partial H}{\partial \bar{X}} \quad \text{and} \quad \frac{d\bar{\lambda}}{dt} = - \frac{\partial H}{\partial \bar{X}}$$
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - Ax_1 + u(t) \\
\dot{x}_3 &= 1 \\
\lambda_1 &= \lambda_2 \\
\lambda_2 &= -\lambda_1 + \lambda_2 \\
\lambda_3 &= 0 \\
\lambda_3(T) &= \frac{\partial S}{\partial x_3} = 1 \Rightarrow \lambda_3 = 1 \text{ for } 0 \leq t \leq TF
\end{align*}
\]

Since \( \frac{\partial H}{\partial u} \) is not a function of \( u \) the control lies on a boundary

\[\lambda_2 > 0, \quad u(t) = \text{greatest negative value}\]

\[\lambda_2 < 0, \quad u(t) = \text{greatest positive value}.\]

Since final time is unspecified, \( H = 0 \) along the entire trajectory.

\[\text{at } t = 0, \quad H = 0 = \lambda_2 u(t) + 1 \]

\[\lambda_2 = -\frac{1}{u(t)}\]

\[\text{at } t_1 \text{ and } t_2, \quad \lambda_2 = 0\]

Therefore our optimal equations are:

\[\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - Ax_1 + u(t)
\end{align*}\]
\[
\begin{align*}
\dot{\lambda}_1 &= A\lambda_2 \\
\dot{\lambda}_2 &= -\lambda_1 + \lambda_2
\end{align*}
\]

\[
X_0 = \begin{bmatrix} 0 \\ 0 \\ \text{free} \\ 1 \\ \frac{1}{U(t)} \end{bmatrix}; 
X_{t_1} = \begin{bmatrix} \text{free} \\ \text{free} \\ \text{free} \\ 0 \end{bmatrix}; 
X_{t_2} = \begin{bmatrix} 1 \\ 0 \\ \text{free} \\ \text{free} \end{bmatrix}
\]
APPENDIX C

Data for Time Domain Correlations

Data for Figure III-1.
Data for Figure III-2.
Data for Figure III-3.
## Data for Figure III-1

<table>
<thead>
<tr>
<th>$X \times t_1$</th>
<th>$A$</th>
<th>$X \times t_1$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$M = 0.8$</strong></td>
<td></td>
<td><strong>$M = 3.0$</strong></td>
<td></td>
</tr>
<tr>
<td>0.991</td>
<td>2.00</td>
<td>0.992</td>
<td>0.02</td>
</tr>
<tr>
<td>0.974</td>
<td>3.00</td>
<td>0.988</td>
<td>0.03</td>
</tr>
<tr>
<td>0.950</td>
<td>5.00</td>
<td>0.984</td>
<td>0.04</td>
</tr>
<tr>
<td>0.917</td>
<td>10.00</td>
<td>0.979</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.953</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.925</td>
<td>0.15</td>
</tr>
<tr>
<td><strong>$M = 1.0$</strong></td>
<td></td>
<td><strong>$M = 4.0$</strong></td>
<td></td>
</tr>
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<td>0.977</td>
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<td>0.75</td>
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<td>0.914</td>
<td>3.00</td>
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<td>0.885</td>
<td>5.00</td>
<td>0.667</td>
<td>4.00</td>
</tr>
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<td>0.850</td>
<td>10.00</td>
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<td>10.00</td>
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<td><strong>$M = 4.0$</strong></td>
<td></td>
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<td>0.9957</td>
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<td>0.9923</td>
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<td>0.9906</td>
<td>0.06</td>
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<td>0.05</td>
</tr>
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<td>0.9887</td>
<td>0.07</td>
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<td>0.9867</td>
<td>0.08</td>
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</tr>
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<td>0.09</td>
<td>0.770</td>
<td>0.50</td>
</tr>
<tr>
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$M = 5$. 

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<th>A</th>
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| M = 1.0 |      |
| 1.00  | 2.34300 |
| 2.00  | 1.37300 |
| 3.00  | 1.04000 |
| 5.00  | 0.74900 |
| 10.00 | 0.49300 |

| M = 2.0 |      | M = 4.0 |      |
| 0.02  | 34.82849 | 0.02 | 14.90000 |
| 0.03  | 23.29869 | 0.03 | 10.11600 |
| 0.04  | 17.52409 | 0.04 | 7.73200  |
| 0.05  | 14.07270 | 0.05 | 6.30400  |
| 0.06  | 11.77200 | 0.10 | 3.46000  |
| 0.07  | 10.13230 | 0.25 | 1.70000  |
| 0.08  | 8.90590  | 0.50 | 1.05300  |
| 0.09  | 7.99510  | 1.50 | 0.52800  |
| 0.10  | 7.19740  | 1.75 | 0.48200  |
| 0.15  | 4.94790  | 2.00 | 0.44500  |
| 0.20  | 3.83500  | 3.00 | 0.35200  |
| 0.25  | 3.17600  | 4.00 | 0.30000  |
| 0.50  | 1.85000  | 10.00| 0.18100  |
| 0.75  | 1.38400  | 15.00| 0.14560  |
| 1.00  | 1.13800  |      |        |
| 1.50  | 0.87300  |      |        |
| 1.75  | 0.79100  |      |        |
| 2.00  | 0.72800  |      |        |
| 3.00  | 0.56800  |      |        |
| 4.00  | 0.47900  |      |        |
| 5.00  | 0.42100  |      |        |
| 10.00 | 0.28400  |      |        |
| 15.00 | 0.22700  |      |        |
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A & t1 \\
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0.02 & 11.71000 \\
0.03 & 8.01550 \\
0.04 & 6.16600 \\
0.05 & 5.05900 \\
0.10 & 2.83800 \\
0.20 & 1.68300 \\
0.40 & 1.04600 \\
0.60 & 0.80500 \\
0.80 & 0.67200 \\
1.00 & 0.58700 \\
2.00 & 0.39000 \\
5.00 & 0.23300 \\
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APPENDIX D

Computer Program for Quasilinearization

Flowplan of the Quasilinearization Algorithm
Main Program
Subroutine RUNGØ6
Subroutine DXDTØ6
Subroutine DXDTØ2
Input Data Set
Final Output for Above Data
Starting Trajectories in Modified Time Domain
Trajectories in Modified Time Domain After 1 Iteration
Trajectories in Modified Time Domain After the Last Iteration
FLOWPLAN OF QUASILINEARIZATION ALGORITHM

Start

Read Input Data

Initialize Variables

Generate Starting Values and Store in Prior Iteration Array

Write Starting Values

Obtain Particular Solution and Store in Particular Solution Array

Obtain Homogeneous Solutions and Store in Homogeneous Solution Arrays

Form Array Using Points From Homogeneous Solution Arrays Corresponding to Known Boundary Conditions, Put into Column Vector Form

Invert Homogeneous Solution Array

Calculate Quasilinearization Coefficients

Calculate New Values of the Variables

Another Iteration Needed?

F T

Store Solution in Prior Iteration Array

Step Iteration Counter

Write and Plot Answers

End

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This program calculates time-optimum switching times for a second-order differential equation of the form
\[ \frac{d^2x}{dt^2} + \frac{dx}{dt} + a_1x + a_2x^2 = 0 \]
using the quasilinearization algorithm modified for undetermined end points as discussed in Chapter I.

In this program, \( x, y, u, \) and \( v \) correspond to \( x_1, x_2, \lambda_1, \) and \( \lambda_2, \) respectively in the text.

**Definition of Input Variables**

- \( N_k \): Number of problems in the data set
- \( N_t \): Maximum number of quasilinearization iterations
- \( N_1 \): Number of integration steps to be taken per unit value of \( S \) in the modified time domain.
- \( L_0 \): Factor to make the number of steps in the initial value solution equal to the number of steps in the quasilinearization solution.
- \( D(1) \): Coefficient of the \( \frac{dx}{dt} \) term above, usually \( = 1. \)
- \( D(6) \): Initial value of the controlled variable \( u. \)
- \( D(7) \): Final value of the controlled variable \( v. \)
- \( D(8) \): Coefficient of \( x \) in the differential equation.
- \( D(9) \): Coefficient of \( x^2 \) in the differential equation.
- \( D(13) \): Causes writing of intermediate answers.
- \( D(14) \): Quasilinearization epsilon.
- \( D(15) \): Causes elimination of full final printout and plots.
- \( D(18) \): Causes no cards to be punched.
- \( P_R \): Problem number.
- \( K_N \): Terminal value of \( k, \) see statement 1000 below.
- \( T_S 1 \): Estimate of the first switching time, \( T_1. \)
- \( T_S 2 \): Estimate of the second switching time if there are three; if there are only two then \( T_S 2 = T_S 1. \)
- \( T_S 3 \): Estimate of the final switching time.
- \( X(1) \): Initial \( x_1. \)
- \( Y(1) \): Initial \( x_2. \)
U(1) = ESTIMATED INITIAL VALUE OF LAMBDA 1

TYPE DECLARATIONS
REAL *4X610, Y(610), U(610), V(610), XR(610), YR(610), UR(610), VR(610),
1XH(610, 4), YH(610, 4), VH(610, 4), AH(100), D(100), B1, B2, A2, A3,
2A5, A6, T(610), CDUML(4), DUMM(4), DUMD, PR1, PR2, LM(20), PX(610), PY(610)
4, B3, B3P, B3H3, B3H4, B3H5, B3H6, PB3, A7, YT, TS1, TS2, TS3
5, C1(100), C2(100), C3(100), C4(100), RCF, RC2
REAL *8uble
INTEGER *4N, NQIC, JM1, N1, N2, N3, NM1, RUN, PR8B, N4, K, KP, KN, IK, NK, IT, JN
1, L, LL, TSM1, N2P1, TJ, PN1

READ INPUT DATA
READ(5, 75) N, IT, N1, LL, RUN
READ(5, 79) D(1), D(6), D(7), D(8), D(9), D(13), D(14), D(15), D(18)
P=1=N1
D970IXK=1, NK
READ(5, 67) PR8B, KN, TS1, TS2, TS3, D(8), X(1), Y(1), U(1), V(1)
CONTINUE
90 CONTINUE

1000 D954K=LL, KN

THE INTEGER K MAY BE USED TO SPECIFY THE VALUE OF ONE OF THE
PARAMETERS. FOR INSTANCE WE MIGHT SPECIFY THAT D(8) = *01*K
AND SOLVE FOR MANY VALUES OF D(8) IN STEPWISE FASHION.
THE SOLUTION TRAJECTORY OF ONE RUN WILL BE THE STARTING
TRAJECTORY OF THE NEXT RUN.

QIC=0
D(13)=D(8)*K
C3(K)=D(10)
D(11)=C3(K)
D(2) = (L2*D(1))**2
D(3) = (L2*D(1))**2/(D(7) - D(6))
D(4) = O(L1)*O(1))**2*(D(7) - D(6))
D(5) = O(L1)*O(1))**2*(D(7) - D(6))

C
C CALCULATE T
IF(K = NE + LL) GET089
D(17) = 0*
N1 = L(N1 - 1) + 1
89 C9NTINUE
C
88 N2 = (N1 - 1)*2 + 1
N3 = (N1 - 1)*3 + 1
N = N3
D91J = 1, N
1 T(J) = 0*
D62J = 2, N
J'M1 = J = 1
D(12) = 1000* / (N1 - 1)
T(J) = T(JM1) + D(12)
2 C9NTINUE
D98OJ = 1, N
80 T(J) = T(J)/1000*
DT = D(12)/1000*

C
C SPECIFY VALUES AT ITERATION PRIOR TO FIRST ONE
IF(D(17) = EQ.1) GET082
IF(K = NE + LL) GET082
C
C USE INITIAL VALUE INTEGRATION TO GENERATE STARTING VALUES
V(1) = 1/D(1)/D(10)
PH1 = T51
PH2 = T52 - PH1
PH3 = T53 - PH1 - PH2
CALL RUN306(X,Y,U,V,B1,B2,L,PX,PU,PV,T,A,D,N,P31,P32,N1,N2,
1N3,N3,P34,DT)
D(17)=1.

C STORE SOLUTION IN PRIOR ITERATION ARRAYS
D974J=L,N3,L
PX(J/L)=X(J)
PY(J/L)=Y(J)
PU(J/L)=U(J)
PV(J/L)=V(J)
X(J/L)=PX(J/L)
Y(J/L)=PY(J/L)
U(J/L)=PU(J/L)
V(J/L)=PV(J/L)
N1=PN1
G0T088

C 62 CONTINUE
C IF(D(13)=NE.1.*)G0T061
C WRITE INITIAL VALUES
55 WRITE(6,58)QIC
56 WRITE(6,57)(O(J),J=1,28)
D599J=1,N
59 WRITE(6,15)(J),X(J),Y(J),U(J),V(J)
C SET QUASILINEARIZATION COUNTER TO ONE
61 QIC=1
C G0TAIN PARTICULAR SOLUTION
22 X(1)=0.
Y(1)=0
U(1)=0
V(1)=-1/D(1)/D(10)
B1=0
B2=C+C
B3=0
C=1
CALL RUNG06(X,Y,U,V,B1,B2,C,2,5,BX,PX,PY,PV,T,A,N,B1,B2,N1,N2,
1N3,B3,P83,DT)
C
STORE PARTICULAR SOLUTION IN ARRAY
D96J=1,N
XP(J)=X(J)
YP(J)=Y(J)
UP(J)=U(J)
VP(J)=V(J)
B1P=B1
B2P=B2
B3P=B3
FOR THIS PROBLEM WE NEED HOMOGENEOUS SOLUTIONS NO 3,4,5,6

OBTAIN HOMOGENEOUS SOLUTION NO 3
X(1)=0
Y(1)=0
U(1)=1
V(1)=0
B1=0
B2=0
B3=0
C=0
CALL RUNG06(X,Y,U,V,B1,B2,C,2,5,BX,PX,PY,PV,T,A,N,B1,B2,N1,N2,
1N3,B3,P83,DT)
C STORE SOLUTION IN ARRAY
D97J=1;
XH(J,1)=X(J)
YH(J,1)=Y(J)
UH(J,1)=U(J)
VH(J,1)=V(J)
B1H2=31
B2H2=32
B3H2=33

C OBTAIN HOMOGENEOUS SOLUTION NO 5
X(1)=0·
Y(1)=0·
U(1)=0·
V(1)=0·
B1=1·
B2=0·
B3=0·
C=0·
CALL RUNGO6(X,Y,U,V,B1,B2,C,PX,PY,PV,T,N,NP1,PB2,N1,N2,IN3,P3,DT)

C STORE SOLUTION IN ARRAY
D98J=1;
XH(J,2)=X(J)
YH(J,2)=Y(J)
UH(J,2)=U(J)
VH(J,2)=V(J)
R1H3=R1
B2H3=32
B3H3=33

C OBTAIN HOMOGENEOUS SOLUTION NO 6
X(1)=0·
Y(1)=.
!!(1)=0.
V(1)=.
B1=.
F=.
B3=.
C=0.
CALL RUNG06(X,Y,U,V,B1,B2,C,PX,PU,PY,PV,T,X,N,PB1,PB2,N1,N2,
1N3,3,P3,DT)

STORE SOLUTION IN ARRAY
D89J=1,N
XH(J,3)=X(J)
YH(J,3)=Y(J)
UH(J,3)=U(J)
VH(J,3)=V(J)
B1H5=B1
B2H5=B2
B3H5=B3

OBTAIN HOMOGENEOUS SOLUTION NO 7
X(1)=0.
Y(1)=0.
U(1)=0.
V(1)=0.
B1=0.
B2=0.
B3=1.
C=0.
CALL RUNG06(X,Y,U,V,B1,B2,C,PX,PU,PY,PV,T,X,N,PB1,PB2,N1,N2,
1N3,3,P3,DT)

STORE SOLUTION IN ARRAY
D910J=1,N
X4(J,4) = x(J)
Y4(J,4) = y(J)
U4(J,4) = u(J)
V4(J,4) = v(J)
B1H4 = 11
B2H6 = 32
B3H6 = 43

SET UP MATRIX LM
LM(1) = XH(N3,1)
LM(2) = YH(N3,1)
LM(3) = VH(N1,1)
LM(4) = VH(N2,1)
LM(5) = XH(N3,2)
LM(6) = YH(N3,2)
LM(7) = VH(N1,2)
LM(8) = VH(N2,2)
LM(9) = XH(N3,3)
LM(10) = YH(N3,3)
LM(11) = VH(N1,3)
LM(12) = VH(N2,3)
LM(13) = XH(N3,4)
LM(14) = YH(N3,4)
LM(15) = VH(N1,4)
LM(16) = VH(N2,4)

INVERT MATRIX USING IRM SCIENTIFIC SUBR PACKAGE SSP
CALL MINV(LM(4), DUMD, DUML, DUMM)

CALCULATE QUASILINEARIZATION COEFFICIENTS
A2 = LM(1) * (1 - XP(N3))
1 + LM(5) * (-YP(N3))
2 + LM(9) * (-VP(N1))
3 + LM(13) * (-VP(N2))
A3 = \text{YPE}(2) \ast (1 \ast \text{XP}(N3))
1 + \text{LM}(6) \ast (\text{YP}(N3))
2 + \text{LM}(10) \ast (\text{VP}(N1))
3 + \text{LM}(14) \ast (\text{VP}(N2))
A5 = \text{LM}(3) \ast (1 \ast \text{XP}(N3))
1 + \text{LM}(7) \ast (\text{YP}(N3))
2 + \text{LM}(11) \ast (\text{VP}(N1))
3 + \text{LM}(15) \ast (\text{VP}(N2))
A6 = \text{LM}(4) \ast (1 \ast \text{XP}(N3))
1 + \text{LM}(8) \ast (\text{YP}(N3))
2 + \text{LM}(12) \ast (\text{VP}(N1))
3 + \text{LM}(16) \ast (\text{VP}(N2))

C C
CALCULATE VALUES OF X, Y, U, V, B1, B2 AFTER QIC PLUS 1 ITERATIONS
D91 J = 1, N
X(J) = \text{XP}(J) \ast \text{DBLE}(A2 \ast \text{YH}(J, 1)) \ast \text{DBLE}(A3 \ast \text{XH}(J, 2)) \ast \text{DBLE}(A5 \ast \text{YH}(J, 3))
1 + \text{DBLE}(A6 \ast \text{YH}(J, 4))
Y(J) = \text{YP}(J) \ast \text{DBLE}(A2 \ast \text{YH}(J, 1)) \ast \text{DBLE}(A3 \ast \text{YH}(J, 2)) \ast \text{DBLE}(A5 \ast \text{YH}(J, 3))
1 + \text{DBLE}(A6 \ast \text{YH}(J, 4))
U(J) = \text{UP}(J) \ast \text{DBLE}(A2 \ast \text{UH}(J, 1)) \ast \text{DBLE}(A3 \ast \text{UH}(J, 2)) \ast \text{DBLE}(A5 \ast \text{UH}(J, 3))
1 + \text{DBLE}(A6 \ast \text{UH}(J, 4))
11 V(J) = \text{VP}(J) \ast \text{DBLE}(A2 \ast \text{VH}(J, 1)) \ast \text{DBLE}(A3 \ast \text{VH}(J, 2)) \ast \text{DBLE}(A5 \ast \text{VH}(J, 3))
1 + \text{DBLE}(A6 \ast \text{VH}(J, 4))
B1 = \text{SIP} \ast \text{DBLE}(A2 \ast \text{B1H2}) \ast \text{DBLE}(A3 \ast \text{B1H3}) \ast \text{DBLE}(A5 \ast \text{B1H5}) \ast \text{DBLE}(A6 \ast \text{B1H6})
B2 = \text{SIP} \ast \text{DBLE}(A2 \ast \text{B2H2}) \ast \text{DBLE}(A3 \ast \text{B2H3}) \ast \text{DBLE}(A5 \ast \text{B2H5}) \ast \text{DBLE}(A6 \ast \text{B2H6})
B3 = \text{SP} \ast \text{DBLE}(A2 \ast \text{B3H2}) \ast \text{DBLE}(A3 \ast \text{B3H3}) \ast \text{DBLE}(A5 \ast \text{B3H5}) \ast \text{DBLE}(A6 \ast \text{B3H6})

C C
IF (J13) .NE. 1* G0T 016

C WRITE (6, 13) QIC, B1, B2, B3, A2, A3, A5, A6
WRITE (6, 12)
D91 J = 1, N
14 WRITE (6, 15) T(J), X(J), Y(J), U(J), V(J)
C TEST TO SEE IF ANOTHER ITERATION IS NEEDED
16 D917J=1,J*
   IF(ABS(X(J)-PX(J))*GT*D(14))GOT818
   IF(ABS(Y(J)-PY(J))*GT*D(14))GOT918
   IF(ABS(U(J)-PU(J))*GT*D(14))GOT918
   IF(ABS(V(J)-PV(J))*GT*D(14))GOT818
17 CONTINUE
   GOT819
C
C STORE THIS ITERATION IN PRIOR ITERATION ARRAY
19 D920J=1,J*
   PB1=PB
   PB2=PB
   PB3=PB
   PX(J)=X(J)
   PY(J)=Y(J)
   PU(J)=U(J)
   PV(J)=V(J)
20 C STEP QUASILINEARIZATION COUNTER
21 QIC=QIC+1

C LIMIT NUMBER OF ITERATIONS
   IF(QIC.GE.IT)GOT821

C BEGIN NEW ITERATION
   GOT822

C WRITE (6,23) IT
   GOT870

C WRITE FULL OUTPUT
10 CONTINUE
   TS1=TS1*T(N1)
TS2 = 3 + 2 \times (T(N2) - 1^*)
TS3 = 3 + 2 \times (T(N3) - 2^*)

C
C COMPUTE RATE OF CHANGE OF FRACTION COMPLETE AT SWITCH TIME
TSM1 = \#1 - 1
RCF = \frac{(X(N1) - X(TSM1))}{(T(N1) - T(TSM1))}
N2P1 = \#2 + 1
RC2 = \frac{(X(N2P1) - X(N2))}{(T(N2P1) - T(N2))}

C
C WRITE SWITCHING TIMES
IF(RUN \#E \#1) G0T064
WRITE(6,78) D(14)
WRITE(6,82) DT
WRITE(6,65)
64 WRITE(6,63) PR8B, RUN, D(8), D(9), D(10), D(11), TS1, TS2, TS3, 0IC, J(1)
1, X(N1), RCF, RC2
IF(D(18) \#EQ \#1) G0T076
WRITE(7,77) D(8), D(10), D(11), TS1, TS3, X(N1), RCF, RC2
76 CONTINUE
IF(D(15) \#EQ \#1) G0T054
WRITE(6,37) PR8B, RUN, 0IC
WRITE(6,33)
WRITE(6,34) D(1), D(2), D(3)
WRITE(6,83) D(4), D(5)
WRITE(6,35)
WRITE(6,36) D(8), D(9)
WRITE(6,84) D(10), D(11)

C
C CONVERT TO DIMENSIONLESS TIME DOMAIN FROM S DOMAIN
D93[R] = 1, \#1
38 T(J) = 31 \times T(J)
TS1 = T(N1)
N4 = \#1 + 1
D93[R] = N4, \#2
39 \[ T(J)=31+32*(T(J)-1) \]
TS2=T(N2)
N4\#2+1
D94\#J=\#N4,N3
40 \[ T(J)=31+32+33*(T(J)-2) \]
TS3=T(N3)
WRITE(6,50)TS1,TS2,TS3
WRITE(6,41)
WRITE(6,42)
D94\#J=1,N
WRITE(6,43)T(J),X(J),Y(J),U(J),V(J)
IF(N\#LE\#200)GRTB86
WRITE(6,85)
GRTB54
86 CONTINUE

C REDUCE DATA POINTS TO 100 OR LESS
IF(N\#LE\#100)GRTB49
N\#1=N\#1
N=N\#1/2
D95CJ=1/N
T(J)*=J
T(J)*=T(J)
X(J)*=X(TJ)
Y(J)*=Y(TJ)
U(J)*=U(TJ)
V(J)*=V(TJ)
50 C PLOT DIMENSIONLESS VARIABLES ON BULL-LINE PLOTTER
CALL PL1(T,X,-N,2,)
136HPL1 T IF X1 VS DIMENSIONLESS TIME
2,36: DIMENSIONLESS TIME
3,36: X1
CALL PL11(T,Y,-N,2,)

D951J=1,
WRITE(6,48)T(J),X(J),Y(J)

PL0T ORIGINAL VARIABLES ON 9N-LINE PLOTTER
CALL PLO11(T,X,-N,2,136H PLOT 9F P VS TIME
2,36h TIME
3,36h P )
CALL PLOT1(T,Y,-N,2,136H PLOT 9F DP/DT VS TIME
2,36h TIME
3,36h DP/DT )
CALL PLOT1(X,Y,-N,2,136H PLOT 9F DP/DT VS P
2,36h P
3,36h DP/DT )
CONTINUE
CONTINUE

FORMAT STATEMENTS
FORMAT(45H S X1 X2 L1 L2)
FORMAT(7H1QIC I5,7E10.3)
FORMAT(1H SE10.3)
FORMAT(25H1QIC ITERATIONS EXCEEDED 15)
FORMAT(31H 8RIGINAL DIFFERENTIAL EQUATION)
FORMAT(10H DP/DT2 +F10.5,8H*DP/DT +F10.5,4H*P +F10.5,19H*P**2 = M)
FORMAT(36H DIMENSIONLESS DIFFERENTIAL EQUATION)
FORMAT(18H DP/DT2 + DP/DT +F10.5,4H*P +F10.5,11H*P**2 = U )
FORMAT(15H PROBLEM NUMBER 15,20H RUN NUMBER 15,
115H ITERATIONS 15)
FORMAT(44H RESULTS IN TERMS OF DIMENSIONLESS VARIABLES)
FORMAT(45H X1=P X2=DP/DT LAMBDA1 LAMBDA2 )
FORMAT(1H SE10.3)
FORMAT(3SH RESULTS IN TERMS OF ORIGINAL VARIABLES)
47 FORMAT(3SH T P DP/DT )
48 FORMAT(1H 3E10•3)
57 FORMAT(1H 10E10•3)
66 FORMAT(7H QIC I5)
66 FORMAT(3H TS1 = F10•5,11H TS2 = F10•5,11H TS3 = F10•5)
63 FORMAT(1H 2I5,4F6•3,3F8•4,14,4F 9•5)
65 FORMAT(93HO PNO RUN C1 C2 U1 U2 TS1 TS2 T
1S1 QIC L1(1) R0CFC R0CI )
67 FORMAT(215,4F10•5 /4F10•5)
75 FORMAT(A15)
77 FORMAT(F6•2,4F8•4,3F8•5)
78 FORMAT(29H1QUASILINEARIZATION EPSILON =F10•5)
79 FORMAT(7F10•3)
82 FORMAT(5H DT =F10•5)
83 FORMAT(5H M = F10•5,5H 0R F10•5)
84 FORMAT(5H U = F10•5,5H OR F10•5)
35 FORMAT(52H1CANNOT USE ONLINE PLOTTER WITH MORE THAN 200 POINTS)
36 STOP
38 END
SUBROUTINE RUNG06(X1,X2,L1,L2,C1,C2,B,XP1,XP2,LP1,Lp2,T,A,D,N, 
PC1,PC2,N1,N2,N3,C3,PC3,DT)

THIS SUBROUTINE SOLVES FOUR FIRST-ORDER DIFFERENTIAL EQUATIONS 
simultaneously using a fourth-order Runge-Kutta scheme.

SOLVES FOUR FIRST-ORDER DIFFERENTIAL EQUATIONS SIMULTANEOUSLY 
USING FOURTH-ORDER RUNGA-KUTTA SCHEME

TYPE DECLARATIONS

REAL*4X1(N),X2(N),L1(N),L2(N),XP1(N),XP2(N),LP1(N),LP2(N),M11,M12, 
1M13,M14,M21,M22,M23,M24,M31,M32,M33,M34,M41,M42,M43,M44, X1C, 
2X2C,L1C,L2C,XP1C,XP2C,LP1C,LP2C,F1,F2,F3,F4,D(100),A(100),T(N) 
3,C1,C2,PC1,PC2,DC1,DC2,DCP1,DCP2,C3,PC3,DC3,DCP3,DTT 
INTEGER*4 JM1,TS1

INITIALIZE VARIABLES 
JM1=1

CALCULATE RUNGA KUTTA CONSTANTS

D02J=P,N 
DC1=C1 
DCP1=PC1 
IF(J.LE.N1)G0T83 
DC1=C2 
DCP1=PC2 
IF(J.LE.N2)G0T83 
DC1=C3 
DCP1=PC3 
X1C=X1(JM1) 
X2C=X2(JM1) 
L1C=L1(JM1) 
L2C=L2(JM1) 
IF(D(17).EQ.1.)G0T84
SIMPLE INITIAL VALUE SOLUTION USING ESTIMATED TS1, TS2, TS3, AND U(1) WHICH WERE IN THE INPUT DATA. THIS IS USED TO GET STARTING TRAJECTORIES
A(1) = b(10)
IF (J > 3E-2) A(1) = D(11)
DTT = DT * PC1
IF (J > GT * N1) DTT = DT * PC2
IF (J > GT * N2) DTT = DT * PC3

CALL DXDT02(X1C, X2C, L1C, L2C, D, F1, F2, F3, F4, A)
M11 = DTT * F1
M12 = DTT * F2
M13 = DTT * F3
M14 = DTT * F4
CALL DXDT02(X1C + M11/2, X2C + M12/2, L1C + M13/2, L2C + M14/2, D, F1, F2, F3, F4, A)
M21 = DTT * F1
M22 = DTT * F2
M23 = DTT * F3
M24 = DTT * F4
CALL DXDT02(X1C + M21/2, X2C + M22/2, L1C + M23/2, L2C + M24/2, D, F1, F2, F3, F4, A)
M31 = DTT * F1
M32 = DTT * F2
M33 = DTT * F3
M34 = DTT * F4
CALL DXDT02(X1C + M31, X2C + M32, L1C + M33, L2C + M34, D, F1, F2, F3, F4, A)
M41 = DTT * F1
M42 = DTT * F2
M43 = DTT * F3
M44 = DTT * F4
G97A5
C  \begin{verbatim}
C JACOBI EEARIZATION ALGORITHM: INTEGRATION
C
XP1C=XP1(JM1) NP2C=XP2(JM1)
LV1C=LP1(JM1) LP2C=LP2(JM1)
A(1)=D(10)
IF(LP2C.GEQ.O.)A(1)*D(11)
CALL DXXPT06(X1C,X2C,L1C,L2C,XP1C,XP2C,LP1C,LP2C,D,B,F1,F2,F3,F4,A,DC1, DPC1 )
M11=DT*F1
M12=DT*F2
M13=DT*F3
M14=DT*F4
CALL DXXPT06(X1C+M11/2.,X2C+M12/2.,L1C+M13/2.,L2C+M14/2.,XP1C,XP2C, LP1C,LP2C,D,B,F1,F2,F3,F4,A,DC1, DPC1 )
M21=DT*F1
M22=DT*F2
M23=DT*F3
M24=DT*F4
CALL DXXPT06(X1C+M21/2.,X2C+M22/2.,L1C+M23/2.,L2C+M24/2.,XP1C,XP2C, LP1C,LP2C,D,B,F1,F2,F3,F4,A,DC1, DPC1 )
M31=DT*F1
M32=DT*F2
M33=DT*F3
M34=DT*F4
CALL DXXPT06(X1C+M31, X2C+M32,L1C+M33,L2C+M34,XP1C,XP2C,LP1C,LP2C,D, B,F1,F2,F3,F4,A,DC1, DPC1 )
M41=DT*F1
M42=DT*F2
M43=DT*F3
M44=DT*F4
C  \end{verbatim}
C CALCULATE NEW VALUES OF X1, X2, L1, L2
5
X1(J)=X1(JM1)+((M11+2*M21+2*M31+M41)/6.)
X2(J)=X2(JM1)+((M12+2*M22+2*M32+M42)/6.)
\end{verbatim}
JM1 = JM1 + 1
RETURN
END
SUBROUTINE DXOT06(X1,X2,L1,L2,PX1,PX2,PL1,PL2,C,B,F1,F2,F3,F4,A,
1(C5,PC5))

THIS SUBROUTINE CALCULATES DERIVATIVES FOR RUN306 SUBROUTINE

TYPE DECLARATIONS
REAL*4X1,X2,L1,L2,PX1,PX2,PL1,PL2,D(100),A(100),F1,F2,F3,F4,C5,
1PC5,C1,C2

C1*D(8)
C2*D(9)
F1=PC5*PX2*B
1+(X2-PX2*B)*PC5
2+(C5*PC5*B)*PX2
F2=PC5*(-PX2=C1*PX1=C2*PX1**2+A(1))*B
1+(X1-PX1*B)*PC5*(-C1=2*C2*PX1)
2=(X2-PX2*B)*PC5
3+(C5*PC5*B)*(-PX2=C1*PX1=C2*PX1**2+A(1))
F3=PC5*(C1=PL2+2*C2*PL2*PX1)*B
1+(X1-PX1*B)**2*PC5*C2*PL2
2+(L2-PL2*B)*PC5*(C1+C2*PX1)
3+(C5*PC5*B)*(C1*PL2+2*C2*PL2*PX1)
F4=PC5*(-PL1+PL2)*B
1=(L1-PL1*B)*PC5
2=(L2-PL2*B)*PC5
3+(C5*PC5*4)*(-PL1+PL2)
RETURN
END
SUBROUTINE DXDT02(X1,X2,L1,L2, D,F1,F2,F3,F4,A)

THIS SUBROUTINE CALCULATES DERIVATIVES FOR RUNG06 SUBROUTINE

TYPE DECLARATIONS
REAL*4X1,X2,L1,L2, D(100), A(100), F1,F2,F3,F4,C1,C2

C1=0.8
C2=0.9
F1=X2
F2=-X2+C1*X1-C2*(X1)**2+A(1)
F3=C1*X2+2*C2*L2*X1
F4=-L1+L2
RETURN
END
1   100   51   1   4   13
0   0   1   .25   0   1   .0001  INPUT DATA SET

|   2   | 3.176 | 3.176 | 3.545 | .20 |
|   0   | 0     | -1.62961 | -2   |
PROBLEM NUMBER 2  RUN NUMBER 13  ITERATIONS 8

ORIGINAL DIFFERENTIAL EQUATION
\[ \frac{D^2P}{DT^2} + 1.00000 \frac{DP}{DT} + 0.20000P + 0.0 \ast P^{**2} = M \]
\[ M = 0.40000 \text{ OR } -0.40000 \]

DIMENSIONLESS DIFFERENTIAL EQUATION
\[ \frac{D^2P}{DT^2} + \frac{DP}{DT} + 0.20000P + 0.0 \ast \frac{P^{**2}}{U} = 0.40000 \text{ OR } -0.40000 \]
\[ U = 0.40000 \text{ OR } -0.40000 \]

\[ TS1 = 3.83472 \quad TS2 = 3.83472 \quad TS3 = 4.19596 \]

RESULTS IN TERMS OF DIMENSIONLESS VARIABLES

<table>
<thead>
<tr>
<th>X1=P</th>
<th>X2=DP/DT</th>
<th>LAMBDA1</th>
<th>LAMBDA2</th>
</tr>
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<td>-0.205E01</td>
<td>-0.250E01</td>
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<tr>
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<td>0.115E-02</td>
<td>0.295E-01</td>
<td>0.293E01</td>
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<td>0.568E-01</td>
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<td>0.981E-02</td>
<td>0.821E-01</td>
<td>0.217E01</td>
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<tr>
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<td>0.223E00</td>
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<tr>
<td>0.920E00</td>
<td>0.126E00</td>
<td>0.234E00</td>
<td>0.255E01</td>
</tr>
<tr>
<td>0.997E00</td>
<td>0.144E00</td>
<td>0.244E00</td>
<td>0.259E01</td>
</tr>
<tr>
<td>0.107E01</td>
<td>0.163E00</td>
<td>0.253E00</td>
<td>0.264E01</td>
</tr>
<tr>
<td>0.115E01</td>
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STARTING TRAJECTORIES IN
THE MODIFIED TIME DOMAIN

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STARTING TRAJECTORIES IN THE MODIFIED TIME DOMAIN
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0.720E 00 0.911E 00 0.293E 00-0.483E 01-0.143E 01
0.740E 00 0.940E 00 0.288E 00-0.490E 01-0.116E 01
0.760E 00 0.969E 00 0.283E 00-0.497E 01-0.090E 01
0.780E 00 0.998E 00 0.277E 00-0.503E 01-0.064E 01
0.800E 00 0.103E 01 0.272E 00-0.510E 01-0.038E 01
0.820E 00 0.105E 01 0.266E 00-0.517E 01-0.012E 01
0.840E 00 0.108E 01 0.260E 00-0.524E 01-0.000E 01
0.860E 00 0.111E 01 0.255E 00-0.531E 01-0.000E 01
0.880E 00 0.113E 01 0.249E 00-0.538E 01-0.000E 01
0.900E 00 0.116E 01 0.243E 00-0.545E 01-0.000E 01
0.920E 00 0.119E 01 0.237E 00-0.552E 01-0.000E 01
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0.960E 00 0.124E 01 0.225E 00-0.565E 01-0.000E 01
0.980E 00 0.126E 01 0.219E 00-0.572E 01-0.000E 01
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1.08E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
1.10E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
1.12E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
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1.16E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
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1.24E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
1.26E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
1.28E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
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1.32E 01 0.129E 01 0.213E 00-0.578E 01-0.000E 01
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TRAJECTORIES IN THE MODIFIED
TIME DOMAIN AFTER 1 ITERATION

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**TRAJECTORIES IN THE MODIFIED TIME DOMAIN AFTER 1 ITERATION**
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0.282E 01 0.101E 01 0.724E-01-0.163E 01 0.120E 02
0.284E 01 0.101E 01 0.649E-01-0.151E 01 0.124E 02
0.286E 01 0.101E 01 0.573E-01-0.140E 01 0.127E 02
0.288E 01 0.101E 01 0.496E-01-0.128E 01 0.131E 02
0.290E 01 0.101E 01 0.417E-01-0.116E 01 0.134E 02
0.292E 01 0.101E 01 0.337E-01-0.103E 01 0.138E 02
0.294E 01 0.100E 01 0.255E-01-0.912E 00 0.142E 02
0.296E 01 0.100E 01 0.171E-01-0.788E 00 0.145E 02
0.298E 01 0.100E 01 0.863E-02-0.664E 00 0.149E 02
0.300E 01 0.100E 01 0.119E-06-0.539E 00 0.153E 02

TRAJECTORIES IN THE MODIFIED TIME DOMAIN AFTER 1 ITERATION
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Trajectories in the modified time domain after the last iteration.
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| 0.136E 01 | 0.148E 01 | 0.160E 01 | 0.172E 01 | 0.184E 01 | 0.196E 01 | 0.208E 01 | 0.220E 01 | 0.232E 01 | 0.244E 01 | 0.256E 01 | 0.268E 01 | 0.280E 01 | 0.292E 01 | 0.304E 01 | 0.316E 01 | 0.328E 01 | 0.340E 01 | 0.352E 01 | 0.364E 01 | 0.376E 01 | 0.388E 01 | 0.400E 01 |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.140E 01 | 0.152E 01 | 0.164E 01 | 0.176E 01 | 0.188E 01 | 0.200E 01 | 0.212E 01 | 0.224E 01 | 0.236E 01 | 0.248E 01 | 0.260E 01 | 0.272E 01 | 0.284E 01 | 0.296E 01 | 0.308E 01 | 0.320E 01 | 0.332E 01 | 0.344E 01 | 0.356E 01 | 0.368E 01 | 0.380E 01 | 0.392E 01 |
| 0.142E 01 | 0.154E 01 | 0.166E 01 | 0.178E 01 | 0.190E 01 | 0.202E 01 | 0.214E 01 | 0.226E 01 | 0.238E 01 | 0.250E 01 | 0.262E 01 | 0.274E 01 | 0.286E 01 | 0.298E 01 | 0.310E 01 | 0.322E 01 | 0.334E 01 | 0.346E 01 | 0.358E 01 | 0.370E 01 | 0.382E 01 |
| 0.144E 01 | 0.156E 01 | 0.168E 01 | 0.180E 01 | 0.192E 01 | 0.204E 01 | 0.216E 01 | 0.228E 01 | 0.240E 01 | 0.252E 01 | 0.264E 01 | 0.276E 01 | 0.288E 01 | 0.300E 01 | 0.312E 01 | 0.324E 01 | 0.336E 01 | 0.348E 01 | 0.360E 01 | 0.372E 01 |
| 0.146E 01 | 0.158E 01 | 0.170E 01 | 0.182E 01 | 0.194E 01 | 0.206E 01 | 0.218E 01 | 0.230E 01 | 0.242E 01 | 0.254E 01 | 0.266E 01 | 0.278E 01 | 0.290E 01 | 0.302E 01 | 0.314E 01 | 0.326E 01 | 0.338E 01 | 0.350E 01 | 0.362E 01 |

Trajectories in the Modified Time Domain After the Last Iteration.
TRAJECTORIES IN THE MODIFIED TIME DOMAIN AFTER THE LAST ITERATION
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<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
<th>Value 5</th>
<th>Value 6</th>
<th>Value 7</th>
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**Trajectories in the Modified Time Domain After the Last Iteration**

2 13 0.200 0.0 0.400-0.400 3.8347 3.8347 4.1960

8 -2.05440 0.95580 0.98117 0.09175
APPENDIX E

Computer Program for Pattern Search

Main Program
Subroutine PATRN
Subroutine PTRB
Subroutine EVAL
Input Data Set
Output for Above Input Data
THIS PROGRAM CALCULATES THE TIME-OPTIMAL SWITCHING TIMES FOR
A LINEAR SECOND-ORDER EQUATION OF THE FORM

\[ \frac{d^2x}{dt^2} + \frac{dx}{dt} + AA*x = AA*y \]

USING THE PHASE PLANE TECHNIQUE

DESCRIPTED IN CHAPTER II. THE PROGRAM WILL NOT WORK FOR AA = 0.25

DEFINITION OF INPUT VARIABLES

N = DIMENSION OF THE SEARCH* IN THIS CASE N = 2
N1 = NUMBER OF PROBLEMS IN THE DATA SET
B1(1) = INITIAL ESTIMATE OF T1
B1(2) = INITIAL ESTIMATE OF T2
AA = PARAMETER IN DIFFERENTIAL EQUATION
W = 0 CAUSES WRITING OF ONLY THE FINAL ANSWERS
    = 1 CAUSES WRITING OF SOME OF THE INTERMEDIATE ANSWERS
    = 2 CAUSES WRITING OF ALL INTERMEDIATE ANSWERS

COMMON B1, B2, B, N, C1, C2, D, EPSI, FEV, W, T, TM1, TT, P, AA

TYPE DECLARATIONS

INTEGER*4 B, N, FEV, W, N1, K1, K2

WRITE(6,8)

READ INPUT DATA
READ(5,1) N, N1, W
D97K=1, N1
READ(5,2) (B1(I), I=1,N)
READ(5,14) AA

SET INITIAL STEP SIZE
D99=1.,
D(1)=.001
C SET FUNCTION EVALUATION COUNTER TO ZERO AND SPECIFY CONSTANTS
FEV = 0
EPSI = .00000000000001
C WRITE INPUT DATA
WRITE(6,12) K
WRITE(6,10) N
WRITE(6,11)( B(I), I=1,N )
C BEGIN SEARCH
CALL PATRN
C WRITE ANSWERS
WRITE(6,6)( B(I), I=1,N )
WRITE(6,4) B
WRITE(6,3) FEV
WRITE(6,5) C2
WRITE(6,13) AA
WRITE(6,15) B(3)
WRITE(6,16) B(4)
WRITE(6,17) B(5)
CONTINUE
C FORMAT STATEMENTS
1 FORMAT(3I5)
2 FORMAT(5F10.5)
3 FORMAT(24H FUNCTION EVALUATIONS = I10)
4 FORMAT(15H BASE POINTS = I10)
5 FORMAT(23H FINAL COST FUNCTION = E10.3)
6 FORMAT(18H STOPPING POINT = 5F15.6)
8 FORMAT(1H1)
10 FORMAT(14H DIMENSIONS = I5)
11 FORMAT(18H STARTING POINT = 5F15.6)
12 FORMAT(1140,PRBL*EM = I5)
13 FORMAT(19H D2X/DT2 + DX/DT + F10.5,6H X = U)
14 FORMAT(F10.4)
15 FORMAT(11H X AT T1 = F10.5)
16 FORMAT(12H T1*DX/DT = F10.5)
17 FORMAT(17H (T2-T1)*DX/DT = F10.5)
       CALL RELCE
       STOP
       END
SUBROUTINE PATRN

THIS SUBROUTINE CONDUCTS A PATTERN SEARCH AS PER EQUATIONS
OF OPTIMIZATION BY WILDE AND HEICHTLER, PAGE 307

COMMON B1, B2, B, N, C1, C2, D, EPSI, FEY, W, T, TM1, TT, P, AA

TYPE DECLARATIONS
INTEGER B, N, FEY, W, BIT1, BIT2
LOGICAL*4 AND

BEGIN SEARCH
B=1
1 D92I=1 , N
2 T(I)=B1(I)

INTERMEDIATE WRITE STATEMENTS
IF(W*LT*2) G0T93
WRITE(6,4)B, FEY, C2, 01, BIT1, BIT2
WRITE(6,5)(B1(I), I=1, N)
WRITE(6,5)(T(I), I=1, N)
CONTINUE

EVALUATE C2 AT T(I)
D96I=1, N
6 TT(I)=T(I)
CALL EVAL

P E T U R B A B O U T T(I)
CALL PTR8

D97I=1, N
7 R2(I)=T(I)
C TEST TO SEE IF WE HAVE AN IMPROVED BASE POINT
D981=1,N
IF(ABS(ABS(B2(I))-ABS(B1(I)))*GE_EPSI*1,N)G0T9
C9NTINUE
C IF BASE POINT HAS NOT IMPROVED, DECREASE STEP SIZE
D922I=1,N
22 D(I)=-5*D(I)
C TEST STEP SIZE
D910I=1,N
IF(-D(I)*GT.000000000001)G0T81
10 C9NTINUE
IF(A-LT.1)RETURN
WRITE(6,5)(D(I),I=1,N)
RETURN
C ACCELERATION STEP
9 C1=C2
D911I=1,N
11 T(I)=2*B2(I)-B1(I)
C INTERMEDIATE WRITE STATEMENTS
IF(A-LT.2)G0T012
WRITE(6,4)B,F,EV,C2,C1,B1,T,B2
WRITE(6,5)(B1(I),I=1,N)
WRITE(6,5)(T(I),I=1,N)
WRITE(6,5)(D(I),I=1,N)
12 C9NTINUE
C EVALUATE C2 AT T(I)
D913I=1,N
13 TT(I)=T(I)
CALL EVAL

TEST TO SEE IF ACCELERATION STEP IS AN IMPROVEMENT
IF(C2.LT.C1)G0T014

IF ACCELERATION STEP IS NOT AN IMPROVEMENT, RETURN ABOUT THE
ACCELERATED POINT TO SEE IF THERE IS AN IMPROVEMENT NEAR BY
C2=C1
D915I=1,N

TM1(I)=T(I)
CALL PTRB
D916I=1,N
IF(ABS(ABS(T(I))*ABS(TM1(I)))*GE*EPSI*1*)G0T019

CONTINUE
D917I=1,N

B1(I)=B2(I)

STEP BASE POINT COUNTER
B=B+1
BIT1=BIT1+1

INTERMEDIATE WRITE STATEMENTS
IF(N.LT.1)G0T018
KD1=B/10
KD2=(B-1)/10
KDOUM=KD1-KD2
IF(KDOUM.EQ.0.AND.B.GT.20.AND.B.NE.1)G0T018
WRITE(6,4)B,FEV,C2,C1,BIT1,BIT2
WRITE(6,5)(B1(I),I=1,N)
WRITE(6,5)(B2(I),I=1,N)
WRITE(6,5)(T(I),I=1,N)
WRITE(6,5)(T+1(I),I=1,N)
IF ACCELERATION STEP IS AN IMPROVEMENT, CONTINUE

CALL PTRB

D9201=1,N

R1(I)=R2(I)

STEP BASEPOINT COUNTER

B=B+1

BIT2=BIT2+1

INTERMEDIATE WRITE STATEMENTS

IF (*LT*1)G8T623

KD1=B/10

KD2=(B-1)/10

KDUM=KD1-KD2

IF (KDUM*EQ.0.*AND.B*GT.20.*AND.B*NE.1)G8T623

WRITE(6,4)B,FEV,C2,C1,BIT1,BIT2

WRITE(6,5)(B1(I),I=1,N)

WRITE(6,5)(B2(I),I=1,N)

WRITE(6,5)(D(I),I=1,N)

WRITE(6,5)(T(I),I=1,N)

WRITE(6,5)(TM1(I),I=1,N)

D9211=1,N

B2(I)=T(I)

G9T92

F9RMAT STATEMENTS

F9RMAT(1H 5D20*14)

F9RMAT(1H 2I10,2D20*14,2I10)
SUBROUTINE PTRB

THIS SUBROUTINE PERTURBS ABOUT A POINT FOR SUBROUTINE PATRN

C91M98B1,B2,M,N,C1,C2,D,EPPI,FEV,W,T,TM1,T,T,P,AA

TYPE DECLARATIONS
INTEGER*N,FEV,W

D91I=1,N
C1=C2

EVALUATE C2 AT T(I) + D(I)
D92K=1,N
TT(K)=T(K)
T(I)=T(I)+D(I)
IF(TT(I)*LT.0.0) TT(I)=0.
CALL EVAL
IF(C2.LT.C1) GOTO 93
T(I)=T(I)+D(I)
G9T91

EVALUATE C2 AT T(I)-D(I)
D94K=1,N
TT(K)=T(K)
T(I)=T(I)-D(I)
IF(TT(I)*LT.0.0) TT(I)=0.
CALL EVAL
IF(C2.LT.C1) GOTO 95
T(I)=T(I)-D(I)
G9T91

IF NEITHER PERTURBATION IS AN IMPROVEMENT
SUBROUTINE EVAL
THIS SUBROUTINE EVALUATES THE COST FUNCTION FOR SUBROUTINES
PATRN AND PTRB USING ANALYTICAL TRAJECTORIES OF THE SECOND-ORDER
DIFFERENTIAL EQUATION D2X/DT2 + DX/DT + AA*X = AA*M FOR AA *NE*25

COMMON B1(5), B2(5), C1, C2, D, EPSI, FEV, W, T, TM1, TT, P, AA

TYPE DECLARATIONS
REAL AA, M1, AT(100), R, THETA, AAA, AB, RX(100), AA1, AA2
REAL T2, M2, TH2, RX2(100), A2, AB2, R2, TDX, T2DX
REAL DR1, DR2, DA1, DA2, DB1, DB2, DTH1, DTH2, DTX1(100), DTX2(100)
COMPLEX BL1, L2, AC1, AC2, X, L1T, L2T, CEXP, C*4PLX
COMPLEX C3, C4, L3T, L4T, X2, DX1, DX2
INTEGER NFEV, W, I
LOGICAL AND, OR

STEP FUNCTION EVALUATION COUNTER
FEV = FEV + 1

EVALUATE C2
DO6 J = 1, 1

SPECIFY VARIABLES
M1 = 2
M2 = -M1
AT(1) = TT(1)
T2 = TT(2)

DO1 I = 1, 1
AA1 = 1 - 4 * AA
IF (AA1 * LT * 0.) GT 0.3
L1 = .5 * SORT(1 - 4 * AA)
L2 = .5 * SORT(1 - 4 * AA)
L3T=L1*T2
L4T=L2*T2
G3'T'4

3
CONTINUE
AA2=*5+8:RT(-AA1)
L1=CMPLX(*5,AA2)
L2=CMPLX(*5,-AA2)
L3T=L1*T2
L4T=L2*T2

4
AC1=+M1/(1+L1/L2)
AC2=*1/(L2/L1-1)
C3=(1:*2)*CEXP(L3T)/(1+L1/L2)
C4=(1:*2)*(-L1/L2)*CEXP(L4T)/(1+L1/L2)
L1T=L1*AT(I)
L2T=L2*AT(I)
X=AC1/CEXP(L1T)+AC2/CEXP(L2T)+M1
XF=C3/CEXP(L1T)+C4/CEXP(L2T)+M2
DX1=-L1*AC1/CEXP(L1T)-L2*AC2/CEXP(L2T)
DX2=-L1*C3/CEXP(L1T)-L2*C4/CEXP(L2T)
R=CABS(X)
R2=CABS(DX2)
DR1=CABS(DX1)
DR2=CABS(DX2)
AAA=REAL(X)
A2=REAL(X2)
DA1=REAL(DX1)
DA2=REAL(DX2)
AB=AIMAG(X)
AB2=AIMAG(X2)
DB1=AIMAG(DX1)
DB2=AIMAG(DX2)
THETA=ATAN2(AB,AAA)
TH2=ATAN2(AB2,A2)
DTH1=ATAN2(DB1,DA1)
C  
C   DATA: Z(1:2)
C  
C   RX(I) = C(BT(I) + SIN(BT(I)))
C   RX2(I) = C2(BT(I) + SIN(BT(I)))
C   DX1(I) = DR1(C(BT1(I) + SIN(BT1(I)))
C   DX2(I) = DR2(C(BT2(I) + SIN(BT2(I)))
C   T1DX = AT(I) * DX1(I)
C   T2DX = (T2 - AT(I)) * DX1(I)
C
C   COMPUTE COST FUNCTION
C   C2 = ABS(RX(I) - RX2(I)) + ABS(DX1(I) - DX2(I))
C
C   B1(3) = RX(I)
C   B1(4) = T1DX
C   B1(5) = T2DX
C
C   1 CONTINUE
C   6 CONTINUE
C
C   FORMAT STATEMENTS
C   2 FORMAT(1H, 10F10.5)
C   RETURN
C   END
PROBLEM = 1
DIMENSIONS = 2
STARTING POINT = 3.000000
STOPPING POINT = 3.842098
BASE POINTS = 96
FUNCTION EVALUATIONS = 1008
FINAL COST FUNCTION = 4.14E-04
D2X/DT2 + DX/DT + *20000 X =

X AT T1 = .95768
T1*DX/DT = .97513
(T2-T1)*DX/DT = .08985

OUTPUT
APPENDIX F

Analog Computer Flowplan
and
Associated Hybrid Computer Programs

Analog Flowplan
Main Hybrid Program
Subroutine JNB5
Subroutine JNB6
Subroutine JNB3
Subroutine JNB4

Input Data for Control of Third-Order System with Dead Time
Pot Settings for Above Input Data
Analog Plot From Above Settings
THIS HYBRID PROGRAM CALCULATES PHT SETTINGS AND GAINS AND SETS PHTS ON THE ANALOG COMPUTER. IT ALSO READS DATA FROM THE ANALOG COMPUTER AND PRINTS IT OUT.

TYPE DECLARATIONS
REAL*4D(50),A(50),B(50),B1,B2,PM,TM,DT,TS1,TS2,M,B3,DT
REAL*4C(6,100),ATS1,ATS2,BETA
INTEGER*4N,I,ST,RUN,K1,9RD,K1
LOGICAL*4 TRSL,SS
COMMON BETA

DEFINITION OF INPUT VARIABLES
N = NUMBER OF PROBLEMS; NOTE THAT PHTS WILL BE SET FOR THE LAST PROBLEM ONLY.
B1 = COEFFICIENT OF DX/DT FOR SECOND-ORDER SYSTEM
COEFFICIENT OF D2X/DT2 FOR THIRD-ORDER SYSTEM
B2 = COEFFICIENT OF DX/DT FOR THIRD-ORDER SYSTEM
COEFFICIENT OF D2X/DT2 FOR THIRD-ORDER SYSTEM
B3 = COEFFICIENT OF X FOR THIRD-ORDER SYSTEM
PM = MAXIMUM VALUE OF THE CONTROLLED VARIABLE
TM = MAXIMUM VALUE OF TIME
TS1 = TIME OF FIRST SWITCH
TS2 = TIME OF SECOND SWITCH
MDT = MODEL DEADTIME
RUN = RUN NUMBER
M = FORCING FUNCTION MULTIPLIER
DT = SYSTEM DEAD TIME
K1 = 1 CAUSES THE PROGRAM TO SKIP THE PHT CALCULATIONS AND PHT SETTING OPERATIONS AND GO DIRECTLY TO READ DATA
ORD = ORDER OF THE SYSTEM

INITIALIZE SENSE LINE
SS=TRSL(1)
J=1

READ INPUT DATA
READ(5,4)N
K851=1
READ(5,6)B1,B2,B3,PM, TM, TS1, TS2, M, DT
READ(5,15)K1,B1,B2,B3
IF(K1.GT.0)G9T816
IF(B1.GE.2)G9T819

COMPUTE SETTINGS FOR SECOND-ORDER SYSTEM
CALL JN33(B1,B2,PM, TM, D, TS1, TS2, M, RUN, MDT, DT)
WRITE(102,3)
WRITE(102,22)
WRITE(102,23)

PAUSE FOR THE OPERATOR TO CHECK GAIN PATCHING
PAU5E6

SET POTS FOR SECOND-ORDER SYSTEM
CALL JN34(D)
G9T816
CONTINUE

COMPUTE SETTINGS FOR THIRD-ORDER SYSTEM
CALL JN53(B1,B2,B3,PM, TM, D, TS1, TS2, M, RUN, MDT, DT)
CONTINUE
WRITE(102,24)
WRITE(102,25)
WRITE(102,26)

PAUSE FOR OPERATOR TO CHECK GAIN PATCHING
PAU5E1
SET PTS FOR THIRD-ORDER SYSTEM
CALL JN35(D)

CONTINUE

PAUSE?

SET ANALOG MODE TO 'IC'
CALL SAM9('IC')

READ INITIAL CONDITIONS
CALL CRAC(0,C(1,1))
D87I=2,5
CALL CRAC(I,C(I,1))
CALL CRAC(7,C(6,1))

SET ANALOG MODE TO 'PC'
CALL SAM9('PC')

PAUSE SO THAT OPERATOR CAN MANUALLY START THE PROBLEM BY
SETTING THE ANALOG MODE TO 'PP'.

PAUSE5

DETERMINE IF IT IS TIME TO READ
IF(TRL(1))G0T08
G0T09

J=J+1
JM1=J-1
IF(J*GE*300)G0T01
IF(C(I,JM1)*GE*9)G0T01

READ DATA
CALL CRAC(0,C(1,J))
D810J=2,5
CALL CRAC(I,C(I,J))
CALL CRAC(7,C(6, J))

RESET SENSE LINE LOW BECAUSE MONO STABLE IS HOLDING IT HIGH

SS=TRSL(1)
G9799
C
CONTINUE
READ ATS1 AND ATS2
CALL SACS('A',71)
CALL DVMR(ATS1)
CALL SACS('A',41)
CALL DVMR(ATS2)

C
SET ANALOG MODE TO 'PC'
CALL SAMS('PC')

C
CONVERT TO ORIGINAL PROBLEM VARIABLES
D911K=1,JM1
C(1,K)=C(1,K)*TM
C(2,K)=C(2,K)*PM
C(3,K)=C(3,K)*2*
C(4,K)=C(4,K)*2*
C(5,K)=C(5,K)*2*
C(6,K)=C(6,K)*BETA

11 CONTINUE
ATS1=ATS1*TM
ATS2=ATS2*TM

WRITE ANSWERS
WRITE(6,12)ATS1,ATS2
WRITE(6,17)R1,R2,R3,M,MDT,PM,PDM,PD2,PD,T,S1,T2,T4,DT
WRITE(6,18)K1,KRD
D913J=1,JM1
13 WRITE(6,14)(C(I,J),I=1,6)
C
CALL SAD9('PC!')
C
PAUSE FOR THE OPERATOR TO DECIDE IF HE WANTS TO PUNCH THE OUTPUT
C
ON PAPER TAPE. IF SO, HE PUTS SELECT SWITCH 1 ON.
PAUSE7
C
PUNCH PAPER TAPE IF SWITCH 1 IS ON
CALL SSWTCH(1,IS1)
IF(IS1.NE.1)G6T020
WRITE(69,12)ATS1,ATS2
WRITE(69,17)S1,B2,B3,M,MDT,PM,PDM,PD2M,PDT,TS1,TS2,TM,DT
WRITE(69,18)K1,8RD
DB21J=1,JM1
21 WRITE(69,14)(C(I,J),I=1,6)
20 CONTINUE
C
FORMAT STATEMENTS
3 FORMAT(46H SECOND-ORDER SYSTEM. PATCH INLET OF A55 FROM)
4 FORMAT( 15)
6 FORMAT(8F7.4,I5,2F7.4)
12 FORMAT(1H12F10.5/)
14 FORMAT(1H 6F10.5/)
15 FORMAT(2I5)
17 FORMAT(1H 13F8.4/)
18 FORMAT(1H 2I5)
22 FORMAT(47H X JUNCTION : THE OUTPUT OF A15 TO THE INPUT)
23 FORMAT(53H OF A30, AND THE OUTPUT OF A02 TO THE X JUNCTION *)
24 FORMAT(50H THIRD-ORDER SYSTEM. PATCH OUTPUT OF A102 TO THE)
25 FORMAT(47H INPUT OF A55, THE OUTPUT OF A15 TO THE INPUT)
26 FORMAT(53H OF A110, AND THE OUTPUT OF A102 TO THE X JUNCTION *)
CALL RELFC
END
SUBROUTINE UNR5(B1,B2,B3,PM,TM,D(12),G(40),M,DMAX,DMIN,TS1,TS2,MDT,PDT,BETA)

THIS SUBROUTINE COMPUTES PGT SETTINGS AND AMPLIFIER INPUT GAINS FOR THE THIRD-ORDER SYSTEM.

TYPE DECLARATIONS
REAL*4 PDM,PM,B1,B2,TM,D(40),G(40),M,DMAX,DMIN,TS1,TS2,MDT,PDT,BETA
INTEGER*4 DUM,I,NN,N1,RUN
COMMON BETA

COMPUTE (DP/DT)MAX = PDM
PDM=PM*B3**3333335/M/2

COMPUTE (D2P/DT2)MAX = PD2M
PD2M=PM*B3**6666665/M/2

COMPUTE INTEGRATOR INPUT PGT SETTINGS TIMES BETA
D(1)=B2*PDM/PD2M
D(2)=B3*PM/PD2M
D(3)=R1
D(4)=PD2M/PDM
D(5)=PD4/PM
D(6)=1/TM
D(7)=B3*M/PD2M
D(8)=B3*M/PD2M*2
D(9)=B3/PD2M+B3*M/PD2M

DETERMINE BETA
DMAX=0.
DMIN=D(1)
NN=1
N1=33
D93I=1,NN.
IF(C(I) .LT. DMIN) DMIN = D(I)
CONTINUE
IF(DMIN .GE. 1.0) G0T85
D051 = 1.0
D(I) = D(I) * 1.0 / DMIN
CONTINUE
D052 = 1.0
IF(DMAX .GT. D(I)) G0T82
D02I = 1.0
D(I) = D(I) * 5.0 / DMAX
CONTINUE
BETA = B1 / D(3)

C
C COMPUTE INTEGRATOR INPUT PUT SETTINGS
C
CONTINUE
D(1) = B2 * PDM / PDM / BETA
D(2) = B3 * PDM / PDM / BETA
D(3) = B1 / BETA
D(4) = PDM / PDM / BETA
D(5) = PDM / PDM / BETA
D(6) = 1.0 / TM / BETA
D(7) = B3 * M / PDM / BETA
D(8) = B3 * M / PDM / BETA
D(9) = B3 / PDM / BETA * B3 / PDM / BETA
IF(D(7) .LT. 5.0) G0T815
BETA = BETA / 5.0 * D(7) + 0.01 * BETA
G0T817
CONTINUE
IF(ABS(D(7) - D(8)) .LT. 5.0) G0T816
BETA = BETA / 5.0 * (ABS(D(7) - D(8))) + 0.01 * BETA
G0T817
CONTINUE
IF (ABS(67)-68+69) LT 5 BETBETABETABETA = 5*(ABS(67)-68+69)) + 01 BETA
G97F17

C9NTINUE

C9M9UTE OTHER P9T SETTINGS
D(10) = 0005
D(11) = TS1/TM
D(12) = TS2/TM
D(13) = 2/TM/PDM
D(14) = 5
D(15) = 1
D(16) = 1
D(17) = 1
D(18) = 8
D(19) = 2
DTI = 1/DT
D(20) = (MDT)/TM
D(21) = 1/(D(17)*500*PDM*TM)
D(22) = (MDT)*PDM/2
D(23) = DTI/BETA/10
D(24) = 2*DTI/BETA
D(25) = 7*DTI/15/BETA
D(26) = 9*DTI/BETA
D(27) = DTI/BETA/10
D(28) = DTI/BETA/10
D(29) = 0
D(30) = 1/BETA
D(31) = P/PDM
D(32) = 5
D(33) = 2

C9M9UTE FINAL P9T SETTINGS AND INPUT GAINS
D97I = 1
\N1
G(I)=1.
CONTINUE
D(I)=1.
IF(D(I)<99)G8T86
D(I)=D(I)/10.
G(I)=10.
CONTINUE

WRITE P8T SETTINGS AND GAINS
WRITE(6,11)RUN
WRITE(6,14)BETA
WRITE(6,12)B1,B2,B3,M,MDT,PM,PDM,PD2M,PDT,TS1,TS2,TM,DT
WRITE(6,10)
WRITE(6,13)
WRITE(6,21)D(21),G(21)
WRITE(102,21)D(21),G(21)
WRITE(6,9)F(D(I),G(I)),I=1,N1)

FORMAT STATEMENTS
9 FORMAT(1H 2F10.4)
10 FORMAT(63H P8T$77,75,110,115,80,05,07,37,67,35,64,94,90,41,95,101
1,100, )
11 FORMAT(8H1RUN N8.$15)
12 FORMAT(1H 13F8.4)
13 FORMAT(50H 70,16,11,004,45,36,47,42,60,93,91,55,27,57,105,29)
14 FORMAT( 8H BETA = F10.5)
21 FORMAT(15H SET P8T Q04 = F10.4,17H WITH GAIN = F10.5)
RETURN
END
SUBROUTINE JNB6(D)

THIS SUBROUTINE SETS POTS FOR THE THIRD-ORDER SYSTEM

TYPE DECLARATIONS
REAL*4D(40)

SET POTS
CALL SPOT( 77, D( 1))
CALL SPOT( 75, D( 2))
CALL SPOT(110, D( 3))
CALL SPOT(115, D( 4))
CALL SPOT( 80, D( 5))
CALL SPOT( 5, D( 6))
CALL SPOT( 7, D( 7))
CALL SPOT( 37, D( 8))
CALL SPOT( 67, D( 9))
CALL SPOT( 35, D(10))
CALL SPOT( 64, D(11))
CALL SPOT( 90, D(13))
CALL SPOT( 44, D(14))
CALL SPOT( 95, D(15))
CALL SPOT(101, D(16))
CALL SPOT(100, D(17))
CALL SPOT( 70, D(18))
CALL SPOT( 1, D(19))
CALL SPOT( 11, D(20))
CALL SPOT( 45, D(22))
CALL SPOT( 36, D(23))
CALL SPOT( 47, D(24))
CALL SPOT( 42, D(25))
CALL SPOT( 60, D(26))
CALL SPOT( 93, D(27))
CALL SPOT( 91, D(28))

210
SUBROUTINE WNS3 (B1, B2, PM, TM, DT, TS1, TS2, M, RUN, MDT, DT)

THIS SUBROUTINE COMPUTES PUT SETTINGS AND AMPLIFIER GAINS FOR THE SECOND-ORDER SYSTEM

TYPE DECLARATIONS
REAL*4 PDM, PM, B1, B2, TM, D(40), G(40), M, DMAX, DMIN, TS1, TS2, MDT, DT, BETA
REAL*4 DT, DTI
INTEGER*4 DUM, I, NN, N1, RUN
COMMON BETA

COMPUTE (DP/DT)MAX = PDM
PDM = PM * SQRT(B2) * M/2
DUM = PDM
IF (DUM EQ 0) DUM = 1
PDM = DUM

COMPUTE INTEGRATOR INPUT PUT SETTINGS TIMES BETA
D(1) = B2 * PM / PDM
D(2) = B1
D(3) = PDM / PM
D(4) = 1 / TM
D(5) = B2 * M / PDM
D(6) = B2 * M / PDM * 2

DETERMINE BETA
DMAX = C*
DMIN = D(1)
NN = 7
N1 = 33
D93I = 1, NN
IF (D(I) LT DMIN) DMIN = D(I)

CONTINUE
COMPUTE OTHER POT SETTINGS
D(8) = 0.0005
D(9) = TS1/TM
D(10) = TS2/TM
D(11) = PM/TM/PDM
D(12) = 0.5
D(13) = 1
D(14) = 1
D(15) = 1
D(16) = 8
D(17) = 2
IF(DT > GT + 1) GOTO 19
DTI = 0
GOTO 20
19
DTI = 1.0 / DT
20 CONTINUE
D(18) = (MDT) / TM
D(19) = 1.0 / (D(15) * 500 * PDM * TM)

D(20) AND D(21) ARE JUST DUMMY FILLERS
D(20) = 0.1
D(21) = 1
D(22) = (MDT) * PDM / 2
D(23) = DTI / BETA / 10
D(24) = 2 * DTI / BETA
D(25) = 7 * DTI / 15 / BETA
D(26) = 9 * DTI / BETA
D(27) = DTI / BETA / 10
D(28) = DTI / BETA / 10
D(29) = 0
D(30) = 1 / BETA
D(31) = PM / PDM
D(32) = 0.5
D(33) = 2.

C COMPUTE FINAL POT SETTINGS AND INPUT GAINS
D97I = 1 \( \times 1 \)
G(I) = 1.
7 CONTINUE
D96I = 1 \( \times 1 \)
IF(C(I) LT 99) G8T06
D(I) = D(I)/10.
G(I) = 10.
6 CONTINUE
C WRITE POT SETTINGS AND GAINS
WRITE(6,11) RUN
WRITE(6,14) BETA
WRITE(6,12) B1, B2, M, MDT, PM, PDM, POT, TS1, TS2, TM, DT
WRITE(6,10)
WRITE(6,13)
WRITE(6,21) D(19), G(19)
WRITE(102,21) D(19), G(19)
WRITE(6,9) ((D(I), G(I)), I = 1, N1)

C C FORMAT STATEMENTS
9 FORMAT(1H 2F10.4)
10 FORMAT(53H POTS: 00, 52, 03, 05, 07, 37, 67, 35, 64, 94, 90, 41, 95, 101, 100, )
11 FORMAT(RH1RUN N8, 15)
12 FORMAT(1H 11F9.4)
13 FORMAT(50H 70, 16, 51, Q04, 45, 36, 47, 42, 60, 93, 91, 55, 95, 72, 57, 105, 20)
14 FORMAT(8H BETA = F10.5)
21 FORMAT(15H SET POT Q04 = F10.4, 17H WITH GAIN = F10.5)
RETURN
END
SUBROUTINE JV54(D)

THIS SUBROUTINE SETS POTS FOR THE SECOND-ORDER SYSTEM

TYPE DECLARATIONS
REAL*4D(40)

SET POTS
CALL SPT(D, 1)
CALL SPT(D, 2)
CALL SPT(D, 3)
CALL SPT(D, 4)
CALL SPT(D, 5)
CALL SPT(D, 6)
CALL SPT(D, 7)
CALL SPT(D, 8)
CALL SPT(D, 9)
CALL SPT(D, 10)
CALL SPT(D, 11)
CALL SPT(D, 12)
CALL SPT(D, 13)
CALL SPT(D, 14)
CALL SPT(D, 15)
CALL SPT(D, 16)
CALL SPT(D, 17)
CALL SPT(D, 18)
CALL SPT(D, 19)
CALL SPT(D, 20)
CALL SPT(D, 21)
CALL SPT(D, 22)
CALL SPT(D, 23)
CALL SPT(D, 24)
CALL SPT(D, 25)
CALL SPT(D, 26)
CALL SPT(D, 27)
CALL SPT(D, 28)
CALL SPT(D, 29)
CALL SPT(D, 30)
\begin{align*}
\text{POT SETTINGs} \\
\text{SET PST 104 = 9091} \quad \text{WITH GAIN = 1.00000}
\end{align*}
APPENDIX G

Computer Program for Simulation and Control of Linear Systems

Main Program
Subroutine DXDT02
Subroutine JNB2
Subroutine JNB3

Input Data for System Time-Optimal Trajectory in Figure V-2

Output for Above Input Data
THIS PROGRAM IS USED TO SIMULATE AND CONTROL LINEAR SYSTEMS OF ORDER UP TO FOUR. FIVE CONTROL MODES ARE PROVIDED. SEE DEFINITION OF DA(12) BELOW. A FOURTH-ORDER RUNGE-KUTTA SCHEME IS USED TO INTEGRATE THE DIFFERENTIAL EQUATIONS.

**DEFINITION OF INPUT VARIABLES**

- $K(3)$: number of problems in the data set
- $DA(1)$: coefficient of $\frac{\partial x}{\partial t}$ for second-order system
  - coefficient of $\frac{\partial^2 x}{\partial t^2}$ for third-order system
  - coefficient of $\frac{\partial^3 x}{\partial t^3}$ for fourth-order system
- $DA(2)$: coefficient of $x$ for second-order system
  - coefficient of $\frac{\partial x}{\partial t}$ for third-order system
  - coefficient of $\frac{\partial^2 x}{\partial t^2}$ for fourth-order system
- $DA(3)$: time of first switch
- $DA(4)$: time of second switch if there are three
- $DA(5)$: time of last switch
- $DA(6)$: coefficient of $x$ for third-order system
  - coefficient of $\frac{\partial x}{\partial t}$ for fourth-order system
- $DA(7)$: initial forcing function
- $DA(8)$: second forcing function
- $DA(9)$: fraction of change complete at first switch
- $DA(10)$: final value of the controlled variable
- $DA(11)$: divide by this constant for real time
- $DA(12)$: 1 causes proportional control
  - 2 causes bang-bang control with switches at precalculated times
  - 3 causes bang-bang control with initial switch at specified fraction of change completed. The second switch occurs when $\frac{\partial x}{\partial t} = 0$.
  - 4 causes time-optimal control by the modeling time-optimal controller
  - 5 causes a simple step change
- $DA(13)$: not used
DA(14) = PROBLEM NUMBER
DA(15) = NOT USED
DA(16) = SPECIFIES THE MODE (SEE DA(12)) TO BE USED AFTER THE FINAL SWITCH
DA(17) = FINAL STEADY-STATE FORCING FUNCTION
DA(18) = INTEGRATION STEP SIZE
DA(19) = TIME DELAY
DA(20) = DIVIDE FORCING FUNCTION BY THIS FOR BETTER SCALING ON PLOT
DA(21) = MODEL DEAD TIME
DA(22) = 1. CAUSES CARDS TO BE PUNCHED
DA(23) = NOT USED
DA(24) = ORDER OF DIFFERENTIAL EQUATION
DA(25) = COEFFICIENT OF X FOR FOURTH-ORDER SYSTEM
DA(26) = NOT USED
DA(27) = NOT USED
DA(28) = NOT USED
DA(29) = NOT USED
DA(30) = NOT USED

TYPE DECLARATIONS
REAL A(100), D(100), X1(4500), X2(4500), L1(4500), L2(4500), T(4500),
M11, M12, M13, M14, M21, M22, M23, M24, M31, M32, M33, M34, M41, M42, M43, M44,
2X1C, X2C, L1C, L2C, F1, F2, F3, F4, DT; U(4500), TS1, TS2, TS3, H(4500)
3,
DA(100), DX, DXX, B1(100), B2(100), DUM, TXD(4500)
INTEGER JM1, IM1, N, KN, JTS1, K(100), K3, MF, PRL3, KN
REAL RD3LE
COMMON TXD

READ INPUT COMMON TO ALL PROBLEMS
READ(5,25)R(3)
K3=K(3)

READ PROBLEM INPUT DATA AND BEGIN PROBLEM SOLUTION
D927F$=1, K3
READ(6,26)(DA(I),I=1,30)
DA(2)=DA(7)
DA(3)=DA(2)
D911KNN=1,1
K(4)=DA(12)
K(5)=DA(15)
K(6)=DA(16)
PR9B=DA(14)

C C DETERMINE IF THE SETPOINT CHANGE IS AN INCREASE OR A DECREASE
IF(DA(10)*GT.0*)GOTO16
DUM=DA(7)
DA(7)=DA(8)
DA(8)=DUM
16 CONTINUE

C C WRITE INPUT DATA
WRITE(6,2)PR9B
D934I=1,30
34 WRITE(6,31)I,DA(I)
WRITE(6,33)K(3),K(4),K(5),K(6)

C C INITIALIZE VARIABLES
J91=1
N=4499
D(1)=1.
A(4)=1.
A(5)=0.
K(2)=1
K(7)=0
K(8)=0
DX=0
DXX=0
C SET UP TIME SCALE
D03I=1;
T(I)=0;
D04I=2;
IM1=I-1;
DT=DA(18);
T(I)=T(IM1)+DT
C CONTINUE
1 CONTINUE
19 CONTINUE
C CARRY OUT INTEGRATION
CALL JNB2(N,T,DA,X1,X2,U,X,K,L1,L2,A,H,D,MF,ME,DX,DXX)
C C WRITE ANSWERS
12 WRITE(6,8)DA(3),DA(5)
WRITE(6,13)
D014I=1,N I45
IF(DA(22)*NE.1*)G8T8123
C C PUNCH CARDS
WRITE(7,17)T(I),X1(I)
123 CONTINUE
14 WRITE(6,15)T(I),X1(I),X2(I),U(I)
7 CONTINUE
C C PUT ANSWERS IN A, U, AND D ARRAYS FOR PLOTTING
D92P1=1,100
D(I)=C
22 A(I)=C
\( u = 0 \)
\[ J = J + 1 \]
\[ A(J) = A(I) \]
\[ X? (J) = A(J) / DA(10) \]
\[ L1(J) = L1(I) \]
\[ TXD(J) = TXD(I) \]
\[ U(J) = J(I) \]
\[ IF (DA(20) \cdot NE.0) U(J) = U(J) / DA(20) \]
\[ D(J) = T(I) / DA(11) \]

23

C

PLBT X VS T AND THE FORCING FUNCTION VS T ON THE ON-LINE PLATTER
CALL PLTSC(T0, A, D, U, 99, 99, 0, 136H TIME=OPTIMAL CONTROL
2,36H TIME
3,36H P AND U

C

DATA FOR PLOTTING THE SWITCHING CURVE
B1(1) = .5
B1(2) = .666
B1(3) = .675
B1(4) = .689
B1(5) = .719
B1(6) = .731
B1(7) = .748
B1(8) = .775
B1(9) = .785
B1(10) = .796
B1(11) = .811
B1(12) = .829
B1(13) = .854
B1(14) = .889
B1(15) = .942
B1(16) = .956
C
PLOT DIMENSIONLESS PHASE PLANE ON THE ON-LINE PLOTTER
CALL PLOTSCT(X2,TXD,B1,B2,99,-20,0,
136HDIMENSIONLESS PHASE PLANE
2,36H X
3,36H T X DX/DT)
C
11 CONTINUE
37 CONTINUE
FORMAT STATEMENTS

2 FORMAT(16H1, PROBLEM NUMBER 15)
3 FORMAT(7H1TS1 = F10.5, 6HTS2 = F10.5)
13 FORMAT(39H T P DP/DT U)
15 FORMAT(1H 2F10.6, 2E10.3)
17 FORMAT(7F10.5)
25 FORMAT(15)
26 FORMAT(10F7.3)
31 FORMAT(4H DA(12, 4H) = F10.7)
33 FORMAT(1H 415)
END
SUBROUTINE DXDT02(X1, X2, L1, L2, DA, D, F1, F2, F3, F4, A)

THIS SUBROUTINE CALCULATES DERIVATIVES FOR THE RUNG01 SUBROUTINE

TYPE DECLARATIONS
REAL*4 X1, X2, L1, L2, D(100), A(100), F1, F2, F3, F4, C1, C2, DA(100)

IF(DA(24) GT 2) GOTO 81
F1 = X2
F2 = A(1) - DA(1) * X2 - DA(2) * X1
F3 = 0
F4 = 0
GOTO 83
1 CONTINUE
IF(DA(24) GT 3) GOTO 82
F1 = X2
F2 = L2
F3 = 0
F4 = A(1) - DA(1) * L2 - DA(2) * X2 - DA(6) * X1
GOTO 83
2 CONTINUE
F1 = X2
F2 = L1
F3 = L2
F4 = A(1) - DA(1) * L2 - DA(2) * X1 - DA(6) * X2 - DA(25) * X1
CONTINUE
3 CONTINUE
RETURN
END
SUBROUTINE JNB2(N,T,DA,X1,X2,U,K,L1,L2,A,H,DF,ME,DX,DXA)

THIS SUBROUTINE CARRIES OUT A FOURTH-ORDER RUNGE-KUTTA INTEGRATION

TYPE DECLARATIONS
REAL*4 A(100),D(100),X(4500),X2(4500),L1(4500),L2(4500),T(4500),
1111,1213,1412121231212341214121412442
2X1C,X2C,L1C,L2C,F1,F2,F3,F4,DT,U(4500),TS1,TS2,TS3
REAL*8 DBL
COMMON TD

INITIAL CONDITIONS
X1(1)=0
X2(1)=0
L1(1)=0
U(1)=DA(7)
L2(1)=0
U(1)=0
DT=DA(18)
DB2J=2,N
JM1=J=1
X1C=X1(JM1)
X2C=X2(JM1)
L1C=L1(JM1)
L2C=L2(JM1)
A(1)=0

SELECT TYPE FORCING FUNCTION
CALL JNB3(K,T,DA,A,X1C,J,JM1,X1,DX,DXA,DF,ME)

CARRY OUT FOURTH ORDER RUNGE KUTTA INTEGRATION
```c
C INTRODUCE TIME DELAY
U(J) = A(1)
K4 = DA(19)/DA(18)
K(8) = K4
K5 = J = K4
IF (K5 .LE. 0) K5 = 1
A(1) = U(K5)
C
CALL DXDT02(X1C,X2C,L1C,L2C,DA,D,F1,F2,F3,F4,A)
M11 = DT*F1
M12 = DT*F2
M13 = DT*F3
M14 = DT*F4
CALL DXDT02(X1C+M11/2.,X2C+M12/2.,L1C+M13/2.,L2C+M14/2.,
1DA,D,F1,F2,F3,F4,A)
M21 = DT*F1
M22 = DT*F2
M23 = DT*F3
M24 = DT*F4
CALL DXDT02(X1C+M21/2.,X2C+M22/2.,L1C+M23/2.,L2C+M24/2.,
1DA,D,F1,F2,F3,F4,A)
M31 = DT*F1
M32 = DT*F2
M33 = DT*F3
M34 = DT*F4
CALL DXDT02(X1C+M31,X2C+M32,L1C+M33,L2C+M34,
1DA,D,F1,F2,F3,F4,A)
M41 = DT*F1
M42 = DT*F2
M43 = DT*F3
M44 = DT*F4
C C
C CALCULATE NEW VALUES OF X1,X2,L1,L2
```
\[ x_2(j) = x_1(j) + \frac{(M_1 + 2) \cdot (M_2 + 2) \cdot (M_3 + 2) \cdot (M_4 + 2) \cdot (M_5 + 2) \cdot (M_6 + 2)}{6} \]

\[ L_1(j) = L_2(j) = L_3(j) = L_4(j) = x_1(j) + \frac{1}{2} \]

\[ D_X = x_4(j) - x_5(j) - x_6(j) \]

\[ T_X = x_7(j) - x_{21}(j) \]

\[ A = x_8(j) - x_{21}(j) \]

RETURN

END
SUBROUTINE JN33(K,T,DA,A,X1C,J,JM1,X1,DX,DXX,DA2,DU4)

THIS SUBROUTINE COMPUTES THE PROPER FORCING FUNCTION FOR THE
RUNGE-KUTTA INTEGRATION ROUTINE

TYPE DECLARATIONS
REAL*4 A(100),DA(100),T(4500),X1C,X1(4500),DX,DXX,DA2,DU4
REAL*4 MDT, TXD, FCN, XP
INTEGER*4 K(100), J, JM1, MF, ME

SELECT TYPE FORCING FUNCTION
IF(K(4)*EQ*1)G0T824
IF(K(4)*EQ*2)G0T821
IF(K(4)*EQ*3)G0T832
IF(K(4)*EQ*4)G0T81
IF(K(4)*EQ*5)G0T84

IF K(4) = 2 USE BANG-BANG CONTROL, SWITCHING AT SPECIFIED TIMES
   CONTINUE
   A(1)*=DA(7)
   IF(T(J)*GE*DA(3))A(1)*=DA(8)
   IF(T(J)*GE*DA(5))K(4)*=K(6)
   G0T929

IF K(4) = 3 USE BANG-BANG CONTROL, SWITCHING FIRST AT SPECIFIED
   PERCENTAGE COMPLETE, THEN AT CHANGE OF SLOPE
   CONTINUE
   A(1)*=DA(7)
   IF(ABS(X1C/DA(10))*GE*DA(9)*AND*K(7)*EQ*0)*DA(3)*=T(J)
   IF(ABS(X1C/DA(10))*GE*DA(9))*K(7)*=1
   IF(K(7)*EQ*1)*A(1)*=DA(8)
   IF((ABS(X1C/DA(10)))*GE*DA(9))G0T928
   IF(A(5)*LT*0*AND*K(4)*NE*K(6))DA(5)*=T(J)
   IF(A(5)*LT*0*)K(4)*=K(6)
IF (XP GT .715 AND XP LE .745) FCN = 1.234 + 03C*(XP - .715)
IF (XP GT .745 AND XP LE .785) FCN = 1.247 + 005/040*(XP - .745)
IF (XP GT .785 AND XP LE .810) FCN = 1.252 + 005/025*(XP - .785)
IF (XP GT .810 AND XP LE .855) FCN = 1.247 + 027/045*(XP - .810)
IF (XP GT .855 AND XP LE .890) FCN = 1.220 + 050/035*(XP - .855)
IF (XP GT .890 AND XP LE .930) FCN = 1.170 + 083/040*(XP - .890)
IF (XP GT .930 AND XP LE .960) FCN = 1.087 + 117/030*(XP - .930)
IF (XP GT .960 AND XP LT 1.00) FCN = 80 + 30/020*(XP - .98)
IF (XP GE .1) FCN = 0
IF (TXDLT FCN AND DA(10) LE 0) A(1) = DA(7)
IF (TXD LT FCN AND DA(10) LE 0) A(1) = DA(8)
IF (ME EQ 1) G$T847
IF (TXDGE FCN AND DA(10) LE 0) A(1) = DA(8)
IF (TXDGE FCN AND DA(10) LE 0) A(1) = DA(7)
IF (TXDGE FCN AND MF EQ 0) DA(3) = T(J)
IF (TXDGE FCN AND MF EQ 0 AND XP GE .811)
1 DA(5) = DA(3) + (1 - XP)/.490/DX
IF (TXDGE FCN AND MF EQ 0 AND XP LT .811)
1 DA(5) = DA(3) + (1 - XP)/.545/DX
IF (TXDGE FCN AND MF EQ 0) WRITE (6,3) DA(3), DA(5), XP, TXD
IF (TXDGE FCN AND MF EQ 0) K(4) = 2
IF (TXDGE FCN) MF = 1
47 CONTINUE
IF (ME NE 1) G$T820
K(4) = K(6)
G$T924
2 CONTINUE
C FOR STEP CHANGE OR NEW STEADY STATE FORCING FUNCTION
4 CONTINUE
A(1) = DA(17)
20 CONTINUE
C
C FORMAT STATEMENTS

3 FORMAT(6H T1 = F10.5, 10H T2 = F10.5, 15H X AT T1 = F10.5, 114H TDX/DT = F10.5)

10 FORMAT(1H 9E10.3, 215)
RETURN
END
INPUT DATA SET

367
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69
2000
36
6.02

0.002665
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67

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APPENDIX H

Computer Program for Simulation and Control of Continuous Stirred-Tank Reactor

Main Program
Subroutine DXDT
Subroutine JB121
Subroutine JB124
Input Data for Figure VI-5, θ = 5
Output for Above Input Data
INITIAL VALUE SOLUTION TO KAPPEL'S CSTR

THIS PROGRAM SIMULATES AND CONTROLS A CONTINUOUS STIRRED-TANK
REACTOR. FOUR MODES OF CONTROL ARE PROVIDED: SEE DEFINITION
OF INPUT VARIABLES BELOW.

DEFINITION OF INPUT VARIABLES

KM  = NUMBER OF PROBLEMS IN DATA SET
MODE = MODES OF CONTROL
   1 CAUSES STEP CHANGE
   2 CAUSES BANG-BANG CONTROL USING PRECALCULATED
      SWITCHING TIMES
   3 CAUSES PROPORTIONAL CONTROL
   4 CAUSES BANG-BANG CONTROL BY MODELING TIME-OPTIMAL
      CONTROLLER
PROB  = PROBLEM NUMBER
B( 1) = FREQUENCY FACTOR, 1/SEC
B( 2) = ACTIVATION ENERGY, CAL/MOLE
B( 3) = GAS CONSTANT, CAL/MOLE/DEG F
B( 4) = REACTOR VOLUME, CC
B( 5) = FEED CONCENTRATION, MOLE/CC
B( 6) = FEED RATE, CC/SEC
B( 7) = FEED DENSITY, GRAMS/CC
B( 8) = FEED HEAT CAPACITY, CAL/GRAM/DEG F
B( 9) = FEED INLET TEMPERATURE, DEG K
B(10) = EXOTHERMIC HEAT OF REACTION, CAL/MOLE
B(11) = UA, OVERALL HEAT TRANSFER COEFFICIENT TIMES AREA OF
        COOLING COIL
B(12) = INLET COOLANT TEMPERATURE, DEG K
B(13) = COOLING COIL VOLUME, CC
B(14) = COOLANT DENSITY, GRAMS/CC
B(15) = COOLANT HEAT CAPACITY, CAL/GRAM/DEG K
B(16) = COOLANT FLOW RATE, CC/SEC
B(17) = INITIAL REACTOR TEMPERATURE
C
B(12) = INITIAL COOLANT TEMPERATURE, DEG K
B(13) = INITIAL CONCENTRATION, MOLE/CC
B(20) = INITIAL COOLANT FLOW RATE, CC/SEC
B(21) = INTEGRATION STEP SIZE
B(22) = NOT USED
B(23) = MINIMUM ALLOWABLE COOLING RATE, CC/SEC
B(24) = MAXIMUM ALLOWABLE COOLANT RATE, CC/SEC
B(25) = NEW STEADY-STATE TEMPERATURE
B(26) = TIME OF FIRST SWITCH
B(27) = TIME OF SECOND SWITCH IF THERE ARE THREE
B(28) = TIME OF LAST SWITCH
B(29) = NEW STEADY STATE FORCING FUNCTION
B(30) = PROPORTIONAL CONTROLLER GAIN
B(31) = FRACTION COMPLETE AT FIRST SWITCH
B(32) = NOT USED
B(33) = NOT USED
B(34) = SYSTEM DEAD TIME, SEC
B(35) = MODEL DEAD TIME, SEC
B(36) = DIVIDE (T2-T1) BY THIS
B(37) = NOT USED
B(38) = NOT USED
B(39) = NOT USED
B(40) = NOT USED

TYPE DECLARATIONS
REAL*4A(100), B(100), X1(1000), X3(1000), T(1000), XT(100),
1M11, M12, M13, M21, M22, M23, M31, M32, M33, M41, M42, M43, XX1(100), RST, RT,
2XIC, X2C, X3C, F1, F2, F3, DT, U(1000), TS1, TS2, TS3, ATS1, ATS2, ATS3
3DF(1000), DX, HT, FX, X4(100), X5(100), X6(100)
INTEGER*4JM1, J1M1, N, KN, IN, KM, M, ME, MF, K, K1, K2, K3, K4, K5, K6, <7, PR9B
1, JP, JJ, TI, JJD10, JJD11, X0, XX, K, K9, X10, JM2
REAL*8D, LE, X2(1000), DX
LOGICAL*4 OR
C C
C COMMON DX
C COMMON A, B, X1, X2, X3, T, XT, Y11, Y12, Y13, Y21, Y22, Y23, Y31, Y32, Y33,
1M41, M42, M43, X1, RST, RT, X1C, X2C, X3C, F1, F2, F3, DT, U, TS1, TS2, TS3,
2ATS1, ATS2, ATS3, DF, DXX, HT, FXX, X4, X5, X6, JM1, J1, IM1, N, KN, I, KM, MDE,
3ME, IF, K, K1, K2, K3, K4, K5, K6, K7, PR9B, JPK4, J1, JJD10, JJD11
C
C READ INPUT DATA
C READ(5, 18) KM
C D91KN=1, KM
C READ(5, 20) MDE, PR9B
C READ(5, 2) (B(I), I=1, 40)
C IF(B(29) .LE. 8) B(21) = 8
C IF(B(29) .GT. 8 .AND. B(29) .LE. 6 + 2) B(21) = 8 .B(29)
C IF(B(29) .GT. 6 + 2) B(21) = 5 + 0
C B(21) = 1
C D81KK=I, 1
C IF(MDE .EQ. 1) GBT88
C IF(KK .EQ. 1) B(21) = B(21) / 10
C IF(KK .EQ. 2) B(21) = B(21) / 5
C IF(KK .EQ. 3) B(21) = B(21) / 2
C IF(KK .EQ. 4) B(21) = B(21)
C CONTINUE
C
C WRITE INPUT DATA
C WRITE(6, 50) PR9B
C WRITE(6, 27)
C WRITE(6, 21)(B(I), I=1, 40)
C WRITE(6, 29) MDE
C
C C
C INITIALIZE VARIABLES
C ME = 0
C IF = C
C RST = 0
C DX = 0
DX=0.
FXX=0.
J11=1
N=999
A(1)=MDE*1.
XX1(1)=3(17)
XT(1)=0.
T(1)=0.
JJD10=1
JJD11=1
B(43)=0.
B(44)=0.
B(41)=0.

C

INITIAL CONDITIONS

16 CONTINUE
X1(1)=E(19)
X2(1)=E(17)
X3(1)=E(18)

6 CONTINUE
X4(1)=FXX
X5(1)=0Xn
X6(1)=136.

C

BEGIN INTEGRATION
DO 10 JJ=2,N
J=2
J1=J-1
J2=J-2
CONTINUE
B(42)=0.
T(J)=T(J1)+R(21)
X1C=X1(J1)
X2C=X2(J1)
X3C=X3(J+1)

C
C COMPUTE FORCING FUNCTION
C IF(JJ.EQ.2)CALL JB124
C CARRY OUT FOURTH-ORDER Runga kutta integration
CALL JB121
CALL JB124

C
C IF(B(42).NE.1.0)GOT010
B(21)=B(21)/10.
GOT14

C CONTINUE
X1(JM2)=X1(JM1)
X2(JM2)=X2(JM1)
X3(JM2)=X3(JM1)
T(JM2)=T(JM1)
X1(JM1)=X1(J)
X2(JM1)=X2(J)
X3(JM1)=X3(J)
T(JM1)=T(J)

5 WRITE ANSWERS
WRITE(6,49)B(26),B(28)
WRITE(6,48)B(26),B(41)

C CONTINUE
C F R M A T STATEMENTS
? FORMAT(10F7.3)
11 FORMAT(1H 80Y, 3F10.5)
12 FORMAT(I5)
20 FORMAT(2I5)
21 FORMAT(1H 5E10.3)
27 FORMAT(11H INPUT DATA)
29 FORMAT(1H I5)
43 FORMAT(16H0ACTUAL TS1 = F10.5, 11H TS2 = F10.5)
49 FORMAT(16H0COMPUTED TS1 = F10.5, 11H TS2 = F10.5)
50 FORMAT(18H1PROBLEM NUMBER = I5)
RETURN
SUBROUTINE DXDT(X1,X2,X3,B,F1,F2,F3,A)

THIS SUBROUTINE COMPUTES DERIVATIVES FOR THE FOURTH-ORDER
RUNGA KUTTA ROUTINE IN SUBROUTINE JR121

TYPE DECLARATIONS
REAL*4X1,X2,X3,B(100),A(100),F1,F2,F3,D(20)
INTEGER*4 MDE

A(16)=B(1)*EXP(-B(2)/B(3)/X2)
IF(X3.LT.X2)A(17)=B(11)*((X3-B(12))/ALOG((X2-B(12))/(X2-X3)))
IF(X3.GE.X2)A(17)=0.
F1=B(6)*B(5)/B(4)-(B(6)/B(4)+A(16))*X1
F2=B(6)*((B(9)-X2)/B(4)+B(12)*A(16)*X1/B(7)/B(9)-A(17)/B(4)/B(7)
1/B(8)
F3=2*(A(17)/B(13)/B(14)/B(15)-A(1)*((X3-B(12))/A(1))
RETURN
END
SUBROUTINE JB121

This subroutine carries out a fourth-order Runge-Kutta integration for the main program.

TYPE DECLARATIONS
REAL*4 A(100), B(100), X1(1000), X3(1000), T(1000), XT(100),
M11, M13, M21, M22, M31, M32, M33, M41, M42, M43, X1(100), RST, RT,
2X1C, X2C, X3C, F1, F2, F3, DT, U(1000), TS1, TS2, TS3, ATS1, ATS2, ATS3
3DF(1000), DXX, HT, FXX, X4(100), X5(100), X6(100), FR
INTEGER*4 JM1, JM1N, KN, I, KM, MDE, ME, MF, K, K1, K2, K3, K4, K5, K6, K7
PR8B
1JPK4, JJ, TI, JD10, JD11
REAL*8D0LE, X2(1000), DX
LOGICAL*4 AND

COMMON DX
COMMON A, B, X1, X2, X3, T, XT, M11, M12, M13, M21, M22, M31, M32, M33,
M41, M42, M43, X3C, RST, RT, X1C, X2C, X3C, F1, F2, F3, DT, U, TS1, TS2, TS3,
2ATS1, ATS2, ATS3, DF, DXX, HT, FXX, X4, X5, X6, JM1, JM1N, KN, I, KM, MDE,
3ME, MF, K, K1, K2, K3, K4, K5, K6, K7, PR8B, JPK4, JJ, JD10, JD11

INTRODUCE TIME DELAY
U(JJ)=A(1)
K4=K(34)/B(21)
K5=JJ*K4
IF(K5.LE.0) K5=1
A(1)=U(K5)

DT=R(21)
call dxot(X1C, X2C, X3C, B, F1, F2, F3, A)
M11=DT*F1
M12=DT*F2
M13=DT*F3
call dxot(X1C+M11/2, X2C+M12/2, X3C+M13/2, B, F1, F2, F3, A)
\[ M1 = DT * F1 \\
M2 = DT * F2 \\
M3 = DT * F3 \\
\text{CALL DXDT(X1C+M21/2\*, x2C+M22/2\*, x3C+M23/2\*, b, F1, F2, F3, A)} \\
M31 = DT * F1 \\
M32 = DT * F2 \\
M33 = DT * F3 \\
\text{CALL DXDT(X1C+M31, x2C+M32, x3C+M33, b, F1, F2, F3, A)} \\
M41 = DT * F1 \\
M42 = DT * F2 \\
M43 = DT * F3 \\
\text{CALCULATE NEW VALUES OF X1, X2, X3} \\
X1(J) = X1(JM1) + (M11 + 2*M21 + 2*M31 + M41) / 6 \\
X2(J) = X2(JM1) + (M12 + 2*M22 + 2*M32 + M42) / 6 \\
X3(J) = X3(JM1) + (M13 + 2*M23 + 2*M33 + M43) / 6 \\
\text{IF (MDE = EQ + 2) G60T62} \\
DX = (X2(J) - X2(JM1)) / (T(J) - T(JM1)) / (B(25) - B(17)) \\
FXX = (B(17) - X2(J)) / (B(17) - B(25)) \\
JPK4 = J + K4 \\
DXX = DX*(T(J) - B(35)) \\
X4(J) = A(3) \\
X5(J) = DX \\
X6(J) = B(30) \\
\text{IF (N LT 39900) JP = 400} \\
\text{IF (N LT 19900) JP = 200} \\
\text{IF (N LT 9950) JP = 100} \\
\text{IF (N LT 4950) JP = 50} \\
\text{IF (N LT 995) JP = 10} \\
\text{IF (N LT 95) JP = 1} \\
JJD11 = (JJ + JJD11 - 1) / (JP + 1) + 1 \\
JJD10 = JJ / JP + 1 \\
\text{IF (JJD11 NE JJD10) XT(JJD10) = T(J)} \\
\text{IF (JJD11 NE JJD10) XX1(JJD10) = X2(J)} \\
F3 = (X2(J) - B(17)) / (B(25) - B(17)) \]
IF (JJ < 11 . NE. JJ < JUD10)
WRITE (6,3) T(J), FR, X1(J), X2(J), X3(J), A(1), JJ, MF, B(21)
C
IF (T(J) < GT.65.) B(21) = 1.
C
C FORMAT STATEMENTS
1 FORMAT(14,2F10.4)
3 FORMAT(1H6,2,F7.4,F10.7,2F10.3,2F6.2,I5,4F10.5)
4 FORMAT(2F10.4)
RETURN
END
SUBROUTINE JB124

TYPE DECLARATIONS
REAL*4 A(100), B(100), X1(100), X3(1000), T(1000), XT(100),
M11, M12, M13, M21, M22, M23, M31, M32, M33, M41, M42, M43,
XX1(100), RST, RT,
2X1C, X2C, X3C, F1, F2, F3, XT, U(1000), TS1, TS2, TS3, ATS1, ATS2, ATS3
3, OF(1000), DX, HT, FX, X4(100), X5(100), X6(100)
REAL*8 D, D(100), DX
REAL*8 DBLE, XP(1000), DX
INTEGER*4 JM1, JM1, N1, N2, KN, I, KM, MDE, ME, MF, K, K1, K2, K3, K4, K5, K6, K7, PR88
1, JP4, JJ, JJ10, JJD10, JJD11
LOGICAL*4 AND

COMMON DX
COMMON A, B, X1, X2, X3, T, XT, M11, M12, M13, M21, M22, M23, M31, M32, M33,
M41, M42, M43, XX1, RST, RT, X1C, X2C, X3C, F1, F2, F3, XT, U, TS1, TS2, TS3,
2ATS1, ATS2, ATS3, OF, DX, HT, FX, X4, X5, X6, JM1, JM1, N1, NK, I, KM, MDE,
3ME, MF, K, K1, K2, K3, K4, K5, K6, K7, PR88, JP4, JJ, JJ10, JJD11

2 CONTINUE
U(1)*B(20)
IF (MDE.EQ.3) U(1)=B(20)
IF (MDE.EQ.2) G0T017
IF (MDE.EQ.3) G0T08
IF (MDE.EQ.4) G0T028

F9R STEP CHANGE MDE=1

A(1) = B(29)
U(1) = B(29)
IF (F.EQ.2) B(26) = 0
IF (F.EQ.2) B(28) = 0
G0T39

SWITCH AT SPECIFIED TIMES

254
17 CONTINUE 
IF(T(J)·LT·B(22))G0T96
MDE=1
R(41)=T(J)
G0T619
6 CONTINUE 
A(1)·B(23)
IF(B(17)·GT·B(25))A(1)=B(24)
IF(T(J)·LT·B(26))G0T9
A(1)=B(24)
IF(B(17)·GT·B(25))A(1)=3(23)
G0T99

C FOR PROPORTIONAL CONTROL
8 CONTINUE 
49 RST=RST+B(35)·X2C
A(1)·B(29)=B(30)·((B(25)·X2C)+RST·RT·B(21))
IF(MDE·EQ·3)A(1)=B(20)·B(30)·((B(25)·X2C)+RST·RT·B(21))
G0T9

C FOR MODELING TIME=OPTIMAL CONTROLLER
28 CONTINUE 
M0T=B(35)
A(3)=FX+DX·MDT

C COMPUTE SWITCHING POINT

C COMPUTE PREDICTED X
XP=A(3)

C COMPUTE PREDICTED T DX/DT
TXD=DX+DX·B(35)

C COMPUTE FCN
IF (.P=FT =5000) FCN=1
IF (.P=GT =500 AND .P=LE =665) FCN=1 + 2/165*(.P=5)
IF (.P=GT =665 AND .P=LE =715) FCN=1 + 034/050*(.P=665)
IF (.P=GT =715 AND .P=LE =745) FCN=1 + 013/030*(.P=715)
IF (.P=GT =745 AND .P=LE =785) FCN=1 + 005/040*(.P=745)
IF (.P=GT =785 AND .P=LE =810) FCN=1 + 005/025*(.P=785)
IF (.P=GT =810 AND .P=LE =855) FCN=1 + 027/045*(.P=810)
IF (.P=GT =855 AND .P=LE =890) FCN=1 + 80/035*(.P=855)
IF (.P=GT =890 AND .P=LE =930) FCN=1 + 083/040*(.P=890)
IF (.P=GT =930 AND .P=LE =960) FCN=1 + 117/030*(.P=930)
IF (.P=GT =960 AND .P=LE =980) FCN=97/017/020*(.P=96)
IF (.P=GT =980 AND .P=LE =100) FCN=80/030/020*(.P=98)
IF (.P=GT =100) FCN=0
B(43) = FCN
B(44) = TXD
B(45) = XP
11 CONTINUE
IF (TXD*LT = FCN AND (B(17) = B(25) = GE = 0) A(1) = B(24)
IF (TXD*LT = FCN AND (B(17) = B(25) = LE = 0) A(1) = B(23)
G9187
IF (.P=EQ = 1) G0188
IF (TXD*GE = FCN AND (B(17) = B(25) = GE = 0) A(1) = 3(23)
IF (TXD*GE = FCN AND (B(17) = B(25) = LE = 0) A(1) = B(24)
7 CONTINUE
IF (TXD*GE = FCN AND MF = EQ = 0) B(26) = T(J)
IF (TXD*GE = FCN AND MF = EQ = 0) M(28) = B(26) = (1 = XP) = 500/8X /B(36)
IF (TXD*GE = FCN AND MF = EQ = 0) MDE = 2
IF (TXD*GE = FCN) MF = 1
S(30) = FCN
CONTINUE
C
C FORMAT STATEMENTS
1 FORMAT(3,15.5)
2 FORMAT(1H F10.3, F10.8, 3F10.3, 5F10.5)

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**COMPUTED**  
**TS1 = 53.19650**  
**TS2 = 54.16261**  

**ACTUAL**  
**TS1 = 53.19650**  
**TS2 = 54.19641**  

**OUTPUT**
APPENDIX I

Computer Program to Plot Figures on Calcomp Plotter

Main Program

Input Data for Figure III-2
THIS PROGRAM PLOTS GRAPHS ON THE CALCOPM PLOTTER FROM DATA READ IN ON CARDS

NOTE THAT AS IT IS NOW PROGRAMMED THE CALCOPM PLOTTER DOES NOT LABEL LOGARITHMIC AXES CORRECTLY IF THE STARTING POINT IS NOT A POWER OF TEN. IT WILL BE OFF BY ONE POWER OF TEN FOR INSTANCE IF THE SCALE STARTS AT .2, .2 WILL BE LABELED 2.

DEFINITION OF INPUT VARIABLES

NN = NUMBER OF PROBLEMS IN THE DATA SET
N = TOTAL NUMBER OF DATA POINTS
N2 = NUMBER OF CURVES TO BE PLOTTED
N5 = -1 CAUSES SEMI-LOG PLOT WITH X BEING THE LOG AXIS
0 CAUSES LOG-LOG PLOT
1 CAUSES SEMI-LOG PLOT WITH Y BEING THE LOG AXIS
2 CAUSES LINEAR PLOT
N6 = 1 CAUSES READING OF SHORT DATA SET
N1(I) = NUMBER OF POINTS ON THE ITH CURVE
D(1) =
D(2) = } FORCES PEN AGAINST LIMIT SWITCH
D(3) =
D(4) =
D(5) = } MOVES PEN ACROSS PAPER D(5) INCHES AND SETS ITS ORIGIN THERE
D(6) =
D(7) = 1* PARAMETER IN UNUSED DO LOOP
D(8) = 1* PARAMETER IN UNUSED DO LOOP
D(9) = MINIMUM VALUE ON THE X AXIS
D(10) = INCREMENT ON X AXIS, VALUE PER INCH
D(11) = MINIMUM VALUE ON Y AXIS
D(12) = INCREMENT ON THE Y AXIS, VALUE PER INCH
D(13) =
D(14) = } STARTING COORDINATES FOR THE Y AXIS
D(15) = NO. OF CHARACTERS IN ORDINATE LABEL, + = CLOCKWISE
D(16) = Y AXIS LENGTH, INCHES
D(17) = Y AXIS ANGLE, DEGREES
D(18) = }
D(19) = }
D(20) = }
D(21) = HEADER HEIGHT, INCHES
D(22) = HEADER ANGLE, DEGREES
D(23) = NO. OF CHARACTERS IN HEADER
D(24) = }
D(25) = NO. OF CHARACTERS IN ABSCISA, = COUNTERCLOCKWISE
D(26) = X AXIS LENGTH, INCHES
D(27) = X AXIS ANGLE, DEGREES
D(28) = }
D(29) = MOVE TO TOP OF Y AXIS WITH PEN UP
D(30) = }
D(31) = }
D(32) = DRAW TOP OF RECTANGLE
D(33) = }
D(34) = }
D(35) = DRAW RHS OF RECTANGLE
D(36) = }
D(37) = }
D(38) = MOVE PEN TO STARTING POINT OF NEXT PLAT, WITH PEN UP
D(39) = }
D(40) = REPEAT CYCLE, 1 MEANS USE EVERY POINT
D(41) = 1 PLACES A SYMBOL AT EVERY POINT
  = 0 PLACES NO SYMBOLS
D(42) = Specifies type of symbol
  AB = Label for X Axis
  BRD = Label for Y Axis
  HEAD = Title
C3(J) = Data point value along the X Axis
T1(J) = Data point value along the Y Axis

TYPE DECLARATIONS
DIMENSION BUFFER(8000)
REAL*4T3(500),D(100),X(102),Y(102),C3(500),T1(500),T2(500)
REAL*4L1(500)
INTEGER*4V1(50),FX,N,N,KJ,D3,D6,D7,D8,D15,D22,D25,D30,D33,D36,
1D39,D40,D41,D42,N2,N3,N4,N5,N6,N1,KK
INTEGER*4AH(9),BRD(9),READ(9)
LOGICAL*4GR,AND

C C READ INPUT DATA
READ(5,23)NN
D924KK=1,NN
KJ=KK
READ(5,10)(N1(J),J=1,N6)
IF(N6.EQ.1)G078101

C C LONg DATA SET (N6.NE.1)
READ(5,26)(D(I),I=1,42)
G078100

C C SHORT DATA SET (N6.EQ.1)
101 READ(5,102)D(9),D(10),D(11),D(12),D(5),D(16),D(26),D(41)
D(1)=0.*
D(2)=40.*
D(3)=3.*
D(4)=0.*
IF(D(5).EQ.0.)D(5)=1.5
D(6)=3.*
D(7)=1.*
D(8)=1.*
D(13)=0.*
D(14)=0.*
D(15)=3.*
IF(D(16).EQ.0.)D(16)=8.5
C(17) = 9.0
C(18) = 1.
D(11) = D(16) - 1.
D(20) = 1
D(21) = 0.
D(22) = 36.
D(23) = 0.
D(24) = 0.
D(25) = -36.
IF(C(26) EQ 0.) D(26) = 5.5
D(27) = 0.
D(28) = 0.
D(29) = D(16).
D(30) = 3.
D(31) = D(26).
D(32) = D(16).
D(33) = 2.
D(34) = D(26).
D(35) = 0.
D(36) = 2.
D(37) = 0.
D(38) = 0.
D(39) = 3.
D(40) = 1.
D(42) = 1.
100 CONTINUE

READ LABELS
READ(5,65)AB
READ(5,25)RD
READ(5,25)HEAD
D92 = 1.
READ(5,23)C3(J), T1(J)
WRITE INPUT DATA
WRITE (6,150) C3(J), T1(J)
CONTINUE
WRITE (6,151) N1, N2, N5, N6
WRITE (6,152) (N1(J), J=1, 12)
WRITE (6,152) (N1(I), I=1, 42)
WRITE (6,28) AR
WRITE (6,28) BRC
WRITE (6,28) HEAD
WRITE (6,152)

CONVERT SELECTED SUBSCRIPTED VARIABLES TO NON-SUBSCRIPTED INTEGERS
D3=D(3)
D5=D(5)
D7=D(7)
D8=D(3)
D15=D(15)
D22=D(22)
D25=D(25)
D30=D(30)
D33=D(33)
D36=D(36)
D39=D(39)
D40=D(40)
D41=D(41)
D42=D(42)

INITIALIZE PLOT ROUTINE
CALL PLTS(HUFFER(1), 8000)

SET PEN AT STARTING POINT
CALL PLT(D(1), D(2), D3)
CALL PLT(D(4), D(5), D6)
SCALE CURVES TO INCHES FOR PLOTTING, MANUALLY

KX=1(1)
X(KX+1)=D(9)
X(KX+2)=D(10)
Y(KX+1)=D(11)
Y(KX+2)=D(12)

VALUES OF X AND Y DO NOT REQUIRE SCALING

PL0T AXES
IF(N5.EQ.2.OR.N5.EQ.1)
1CALL AXIS(D(13),D(14),BRD,D15,D(16),D(17),Y(KX+1),Y(KX+2))
IF(N5.EQ.1.OR.N5.EQ.0)
1CALL LGAXIS(D(13),D(14),BRD,D15,D(16),D(17),Y(KX+1),Y(KX+2))
CALL SYMPOL(D(18),D(19),D(20),HEA,D(21),D22)
CONTINUE
15
IF(N5.EQ.2.OR.N5.EQ.1)
1CALL AXIS(D(23),D(24),AR,D25,D(26),D(27),X(KX+1),X(KX+2))
IF(N5.EQ.0.OR.N5.EQ.1)
1CALL LGAXIS(D(23),D(24),AR,D25,D(26),D(27),X(KX+1),X(KX+2))
CALL PL0T(D(28),D(29),D30)
CALL PL0T(D(31),D(32),D33)
CALL PL0T(D(34),D(35),D36)
D(43)=D(26)+1*1
CALL PL0T(D(43),0*3)
CALL PL0T(D(43),D(16),2)
154
CONTINUE

CREATE TABLE OF VALUES TO BE PLOTTED

K=0
10 CONTINUE:
D95=1,i:2
D42=1
1  K=111
2  IF(I:E=1)K=111(I-1)+K
3     X(K:K+1)=D(I)
4     X(K+?)=D(110)
5     Y(K:K+1)=D(111)
6     Y(K+?)=D(112)
7  94 J=1,<X
8     X(J)=C3(J+K)
9     Y(J)=T1(J+K)

20  CALL PL0T(D(37),D(38),D39)
21     IF(NS=NE.2)
22       CALL L3ILINE(X,Y,XX,DX,DXI,DX2,N5)
23         IF(N5.EQ.2)
24           CALL FLINE(X,Y,-XX,DX,DXI,DX2)
25          CONTINUE

21  MOVE PEN TO START OF NEW PLAT
22  CALL PL0T(0.,0.3)
23  CALL PL0T(12.,0.3)
24  CONTINUE

24  TER'\:ATE JOB BY EMPTYING BUFFER
25  CALL PL0T(6.,1.,3)
26     CALL PL0T(0.,0.,999)

1  FORMAT STATEMENTS
11  FORMAT(415)
12  FORMAT(1415)
13  FORMAT(15)
14  FORMAT(124)
15  FORMAT(*F10.4)
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APPENDIX J

Simulation of the Continuous Stirred-Tank Reactor
Simulation of the CSTR

Mass balance on the reactor assuming perfect mixing:

$$V \frac{dC}{dt} = FC_0 - FC - VkC$$

where

- $V$ = Material volume; 1000 cc
- $C$ = Concentration of component A at the reactor outlet; mol/cc
- $C_0$ = Concentration of component A at the reactor inlet; $6.5 \times 10^{-3}$ mole/cc
- $F$ = Feed rate; 10 cc/sec
- $k$ = Anhenius reaction rate constant; sec$^{-1}$
  $$= k_0 \exp \left(\frac{-E}{RT}\right)$$
- $k_0$ = Frequency factor; $7.86 \times 10^{12}$ sec$^{-1}$
- $E$ = Activation energy; 28,000 cal/mol
- $R$ = Gas constant; 1.987 cal/mol °K
- $T$ = Reactor temperature; °K

Energy balance on the reactor assuming constant physical properties and no heat transfer to the surroundings

$$V \rho \sigma \frac{dT}{dt} = F \rho \sigma (T_0 - T) - \Delta HVkC - \frac{UA (T_c - T_{Co})}{\ln[(T-T_{Co})/(T-T_c)]}$$

where

- $\rho$ = Material density; 1 gm/cc
- $\sigma$ = Material heat capacity; 1 cal/gm °K
$T_0 = \text{Feed temperature; } 350^\circ K$

$-\Delta H = \text{Exothermic heat of reaction; } 27,000 \text{ cal/mol}$

$UA = \text{Overall heat transfer coefficient times cooling coil area; } 7 \text{ cal/sec } \circ K$

$T_{co} = \text{Inlet coolant temperature; } 300^\circ K$

**Energy balance on cooling coil**

\[
\frac{V_C \rho_c c_c}{2} \frac{dT_c}{dt} = \frac{UA (T_c - T_{co})}{\ln \left[ \frac{(T - T_{co})}{(T - T_c)} \right]} - F_C \rho_c c_c (T_c - T_{co})
\]

where

$V_C = \text{coil volume; } 100 \text{ cc}$

$\rho_c = \text{coolant density; } 1 \text{ gm/cc}$

$c_c = \text{coolant heat capacity; } 1 \text{ cal/gm } \circ K$

$F_C = \text{coolant flow rate; cc/sec.}$
VITA

The author was born in Columbia, South Carolina on November 21, 1935. He attended public elementary and secondary schools in Columbia, graduating from Dreher High School in May, 1953. He did his undergraduate work at the University of South Carolina, graduating with a Bachelor of Science Degree in Chemical Engineering in January, 1958. After graduation he spent three years on active duty as an officer in the U.S. Navy and seven years full-time employment with Esso Research Laboratories in Baton Rouge, Louisiana. He entered Louisiana State University on a full-time basis in September, 1968 and received a Master of Science Degree in Chemical Engineering.

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Candidate: John Nelson Beard, Jr.

Major Field: Chemical Engineering

Title of Thesis: The Modeling and Time-Optimal Control of Chemical Processes

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination: May 10, 1971