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# A user's guide to fortran programs for Wigner and Racah coefficients of SU3

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# A USER'S GUIDE TO FORTRAN PROGRAMS FOR WIGNER AND RACAH COEFFICIENTS OF $SU_3$ *

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## PROGRAM SUMMARY

*Title of program*: SU3 WIGNER & RACAH COEFFICIENTS

*Catalogue number*: ACRM

*Program obtainable from*: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue).

*Computer*: IBM 360/67; *Installation*: The University of Michigan, Ann Arbor, Michigan, USA

*Operating system*: MTS/360

*Programming language used*: FORTRAN IV

*High speed storage required*: $SU_3 \supset SU_2 \times U_1$ Wigner coefficients, 13 008 words

$SU_3$ Racah coefficients, 14 654 words;

$SU_3 \supset R_3$ Wigner coefficients, 14 202 words

$SU_3$ Racah coefficients, 14 654 words. $SU_3 \supset R_3$ Wigner coefficients, 14 202 words

*No. of bits in a word*: 32

*Is the program overlaid?* No.

*No. of magnetic tapes required*: None.

*Other peripherals used*: Card reader, line printer

*No. of cards in combined program and test deck*: 2046

*Card punching code*: EBCDIC.

*Keywords*: $SU_3$, Wigner coefficient, Racah coefficient, Clebsch–Gordan coefficient, Recoupling coefficient, Isoscalar factor, U-function, Unitary coupling, Unitary recoupling, K-band projection, Hypercharge.

*Nature of physical problem*

$SU_3 \supset SU_2 \times U_1$ and $SU_3 \supset R_3$ Wigner coefficients as well as $SU_3$ Racah coefficients are calculated for arbitrary couplings and multiplicity.

*Method of solution*

A build-up process based on the Biedenharn–Louck prescription for specifying the outer multiplicity is employed to generate $SU_3 \supset SU_2 \times U_1$ Wigner coefficients [1]. $SU_3$ Racah coefficients follow through standard recoupling formulae [2]. $SU_3 \supset R_3$ Wigner coefficients are obtained from the corresponding $SU_3 \supset SU_2 \times U_1$ Wigner coefficients via unitary transformation coefficients relating $SU_3 \supset SU_2 \times U_1$ and $SU_3 \supset R_3$ basis states [3].

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*Restrictions on the complexity of the problem*

Factorials $M!$, $M \leq M_{max} = 32$, and binomial coefficients $\binom{N}{M}$, $M \leq N \leq N_{max} = 32$, are stored in common. Typically for $SU_3 \supset SU_2 \times U_1$ Wigner coefficients $\Lambda_1 + \Lambda_2 + \Lambda_3 \leq M_{max}$ whereas for $SU_3 \supset R_3$ Wigner coefficients $\lambda + \mu + L \leq N_{max}$. The limits $M_{max}$ and $N_{max}$ may be altered by modifying one and only one subprogram.

*Typical running time*

Running time is a critical function of the complexity of the couplings involved. $SU_3 \supset R_3$ Wigner coefficients are evaluated as a weighted sum over $SU_3 \supset SU_2 \times U_1$ Wigner coefficients and hence are the most costly to calculate.

*Unusual features of the program*

Four of the fifteen subprograms contain internally dimensioned arrays. To conserve high speed storage the sizes of these arrays should be set at a large enough value to accomodate but not over-accomodate the needs of a particular user. A prescription for fixing the array size parameters in terms of representation labels is spelled out within each subprogram with specific examples given as a part of the Long Write-Up.

*References*

[1] J.P. Draayer and Yoshimi Akiyama J. Math. Phys., in press.
[2] K.T. Hecht, Nuclear Physics 62 (1965) 1.
[3] J.P. Draayer, Nuclear Physics A129 (1969) 647.

# LONG WRITE-UP

## 1. Introduction

The FORTRAN IV deck with which this write-up is concerned consists of two main parts. The first part (labelled PART 1: SU3 PACKAGE) consists of codes which can be used to generate $SU_3 \supset SU_2 \times U_1$ and $SU_3 \supset R_3$ Wigner coefficients as well as $SU_3$ Racah coefficients for *arbitrary* couplings and multiplicities. The second part (labelled PART 2: SAMPLE CONTROL ROUTINES) is comprised of three control routines (one for each type of coefficient) which illustrate usage of the $SU_3$ package subprograms and provide a standard input–output format for the casual user. The purpose of this guide is to (1) define the function served by each of the fifteen subprograms in the $SU_3$ package, (2) illustrate simple alterations that can be made in adapting the programs to a specific need, (3) organize the subprograms into sets for efficient calculation of either $SU_3 \supset SU_2 \times U_1$ Wigner coefficients, $SU_3$ Racah coefficients or $SU_3 \supset R_3$ Wigner coefficients, and (4) tabulate for all three coefficient types output from the sample control routines which illustrates both special and general symmetry properties as well as program checks versatility. A detailed description of the algebraic foundations upon which the codes are based is available in ref. [1].

## 2. Subprogram description

The fifteen subprograms which comprise Part 1 of the FORTRAN IV deck (carrying identification labels A, B, ..., O, respectively) have been written in a notation which, insofar as practical, parallels that used in ref. [1]. Common equivalences (less subscripts) are as follows:

| | |
|---|---|
| $\lambda$ | ~ LAM, LM, L |
| $\mu$ | ~ MU, M |
| $\rho$ | ~ KRO, KO, K |
| $\kappa$ | ~ KAP, KA, K |
| $\epsilon$ | ~ IE |
| $2\Lambda$ | ~ JT |
| $2M_\Lambda$ | ~ MT |
| K | ~ K |
| L | ~ L |
| M | ~ M |

The abbreviated forms ($\lambda \sim$ L, $\mu \sim$ M, etc.) have only been used when a proper interpretation can be readily inferred from context. Caution: IP(IQ) has been used interchangably for $p(q)$ and $\widetilde{p} = \mu - q$ ($\widetilde{q} = \lambda - p$). A running index $\kappa$ has been introduced to distinguish $\mathcal{X}$-bands.

$$\kappa = (\mathcal{X} - \mathcal{X}_{\min})/2 + 1.$$

The statement function KSTART (LAM, MU, L) [KSTART (MU, LAM, L)] defines $\mathcal{X}_{\min}$, the minimum $\mathcal{X}$-value for each L as given by eq. (4a) [eq. (4b)] of ref. [1]. Other statement functions include MULT (LAM, MU, L) for determining the number of occurrences of a given L in the $(\lambda\mu)$ irreducible representation of $SU_3$ (zero if L does not occur), IDM (LAM, MU) which gives the dimension of the $(\lambda\mu)$ irreducible representation of $SU_3$, ND(M, N) for a linear reference to the binomial coefficients $\binom{M}{N}$, $N \leqslant M \leqslant M_{\max}$, and finally INDEX (J1TD, LAM1, J1T, J2TD, LAM2, J2T) for a linear reference to the coefficients

$$\langle(\lambda_1\mu_1)\epsilon_1\Lambda_1, (\lambda_2\mu_2)\epsilon_2\Lambda_2 \| (\lambda_3\mu_3)E\rangle_\rho$$

with

$$\mathrm{JTD} = |\epsilon - \epsilon_E|/3.$$

The name of each subprogram, the function it performs, and a reference to where the equivalent algebraic result can be found is given in table 1.

The definition of special parameters used in the calling sequence is given by means of comment statements within each subprogram. In particular for

CEWU3 (fixed $\lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3$):

$$\langle(\lambda_1\mu_1)\epsilon_1\Lambda_1; (\lambda_2\mu_2)\epsilon_2\Lambda_2\|(\lambda_3\mu_3)E\rangle_\rho = \text{DEWU3 (KRO, IND)},$$

$$\rho = \text{KRO}, \qquad 1 \leqslant \text{KRO} \leqslant \text{KROMAX},$$

$$\epsilon_1 = \epsilon_{3E} - \epsilon_2,$$

$$\left.\begin{array}{l} \epsilon_2 = \text{IEA (IND)} \\ 2\Lambda_1 = \text{J1TA (IND)} \\ 2\Lambda_2 = \text{J2TA (IND)} \end{array}\right\} \quad 1 \leqslant \text{IND} \leqslant \text{INDMAX};$$

CWU3 (fixed $\lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3\epsilon_3\Lambda_3$):

$$\langle(\lambda_1\mu_1)\epsilon_1\Lambda_1; (\lambda_2\mu_2)\epsilon_2\Lambda_2\|(\lambda_3\mu_3)\epsilon_3\Lambda_3\rangle_\rho = \text{DWU3 (KRO, IND)};$$

$$\rho = \text{KRO}, \qquad 1 \leqslant \text{KRO} \leqslant \text{KROMAX},$$

$$\epsilon_1 = \epsilon_3 - \epsilon_2,$$

Table 1
Subprograms in the $SU_3$ package.

| | Name | Function | Ref. |
|---|---|---|---|
| A. | BLOCKS | Generate common *blocks* (binomial coefficients and factorials) | [2] |
| B. | $CEWU_3$ | Calculate extremal Wigner coefficients for $SU_3 \supset SU_2 \times U_1$ coupling | [1], eq.(17), eq. (20) |
| C. | $CEWU_3S$ | $CEWU_3$ Support routine | [1], eq. (18), eq. (21) |
| D. | $MULTU_3$ | Calculate multiplicity for the $SU_3$ Coupling $(\lambda_1\mu_1) \times (\lambda_2\mu_2) \to (\lambda_3\mu_3)$ | [3] |
| E. | MULTHY | Calculate multiplicity (theory) for the $SU_3$ coupling $(\lambda_1\mu_1) \times (\lambda_2\mu_2) \to (\lambda_3\mu_3)$ | |
| F. | $CWU_3$ | Calculate Wigner coefficients for $SU_3 \supset SU_2 \times U_1$ coupling | [1] eq. (19) |
| G. | $CRU_3$ | Calculate Racah coefficients for $SU_3$ | [1] eq. (22) |
| H. | DBSR | Double back substitution routine for solving simultaneous equations | [4] |
| I. | DLUT | Decompose a real matrix into the product of a lower and an upper triangular matrix | [4] |
| J. | DELTA | Calculate *Delta* for $R_3$ routines $[\Delta(j_1j_2j_3)]$ | [5] |
| K. | $DRR_3$ | Calculate Racah coefficients for $R_3$ $[W(j_1j_2l_2l_1;j_3l_3)]$ | [5] |
| L. | $CWU_3R_3$ | Calculate Wigner coefficients for $SU_3 \supset R_3$ Coupling | [1] eq. (31) |
| M. | $DTU_3R_3$ | Calculate transformation coefficients for $SU_3 \supset R_3$ Reduction $(\langle G\|(G_E)KLM\rangle)$ | [1], eq. (26) |
| N. | CONMAT | Calculate orthonormalization matrix for $SU_3 \supset R_3$ reduction | [1], eq. (6) |
| O. | $DWR_3$ | Calculate Wigner coefficients for $R_3$ $(\langle j_1m_1;j_2m_2\|j_3m_3\rangle)$ | [6] |

$$\left.\begin{array}{ll} \epsilon_2 = \text{IE2MAX-3(IESMAX-IES)} & 1\leqslant \text{IES}\leqslant \text{IESMAX}, \\ 2\Lambda_1 = \text{J1TMAX(IES, J2S)-2(J1S-1)} & 1\leqslant \text{J2S}\leqslant \text{J2SMAX(IES)}, \\ 2\Lambda_2 = \text{J2TMAX(IES)-2(J2S-1)} & 1\leqslant \text{J1S}\leqslant \text{JISMAX(IES, J2S)}. \end{array}\right\}$$

$$\text{IND} = [\text{INDMAT(IES, J2S)-J1T}]/2, \quad 1\leqslant \text{IND}\leqslant \text{INDMAX};$$

CRU3 (fixed $\lambda_1\mu_1\lambda_2\mu_2\lambda\mu\lambda_3\mu_3\lambda_{12}\mu_{12}\lambda_{23}\mu_{23}$):

$$U((\lambda_1\mu_1),(\lambda_2\mu_2),(\lambda\mu),(\lambda_3\mu_3);(\lambda_{12}\mu_{12})\rho_A\rho_B(\lambda_{34}\mu_{34})\rho_C\rho_D) = \text{DRU3 (KA, KB, KC, KD)}$$

$$\begin{array}{lll} \rho_A & = \text{KA}, & 1\leqslant \text{KA}\leqslant \text{KROA}, \\ \rho_B & = \text{KB}, & 1\leqslant \text{KB}\leqslant \text{KROB}, \\ \rho_C & = \text{KC}, & 1\leqslant \text{KC}\leqslant \text{KROC}, \\ \rho_D & = \text{KD}, & 1\leqslant \text{KD}\leqslant \text{KROD}; \end{array}$$

CWU3R3 (fixed $\lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3L_1L_2L_3$):

$$\langle(\lambda_1\mu_1)\mathcal{K}_1L_1;(\lambda_2\mu_2)\mathcal{K}_2L_2\|(\lambda_3\mu_3)\mathcal{K}_3L_3\rangle_\rho = \text{DWU3R3(KRO, KA1, KA2, KA3)},$$

$$\begin{array}{lll} \rho & = \text{KRO}, & 1\leqslant \text{KRO}\leqslant \text{KROMAX}, \\ \mathcal{K}_1 & = \mathcal{K}_{1\min} + 2(\text{KA1-1}), & 1\leqslant \text{KA1}\leqslant \text{KA1MAX}, \\ \mathcal{K}_2 & = \mathcal{K}_{2\min} + 2(\text{KA2-1}), & 1\leqslant \text{KA2}\leqslant \text{KA2MAX}, \\ \mathcal{K}_3 & = \mathcal{K}_{3\min} + 2(\text{KA3-1}), & 1\leqslant \text{KA3}\leqslant \text{KA3MAX}. \end{array}$$

Parameters not explicitly defined (for example, NEC $= \nu_3 = \{(\lambda_1 + \lambda_2 - \lambda_3) + 2(\mu_1 + \mu_2 - \mu_3)\}/3$ in CEWU3, CEWU3S, CWU3) are arguments transferred between subprograms so as to reduce redundant numerical calculation.

### 3. Subprogram modification

Of the fifteen subprograms in the $SU_3$ package, routines (C–E, H–K, M–O) (see table 1) are completely general in the sense that they perform operations on arrays whose sizes are fixed externally to the routines themselves (accomplished by reducing all multiply subscripted variables to linear form, the proper indexing being done within each subprogram). Routines (B, F–G, L), on the other hand, contain internally dimensioned arrays the sizes of which must be set at a large enough value to accomodate the needs of a particular user. A prescription for fixing the sizes of these arrays is given by suitable comment statements withing each subprogram. Table 2 gives the values of the relevant parameters (external and internal) for four cases of special interest in nuclear structure studies; namely, standard shell model calculations assuming full usage of $SU_3$ technology (e.g., see ref. [7]) and general two-body effective interactions (e.g., see ref. [8]) in each of the $N = 1, 2, 3, 4$ shells of the harmonic oscillator. (Note

Table 2
Array size parameters.

| Subprogram | Parameter | Oscillator Shell Number | | | |
|---|---|---|---|---|---|
| | | $N$=1 | $N$=2 | $N$=3 | $N$=4 |
| CEWU3 | External–N1 | 3 | 5 | 7 | 9 |
| | N2 | 35 | 165 | 455 | 969 |
| | Internal– X1 | 15 | 45 | 91 | 153 |
| | X2 | 5 | 9 | 13 | 17 |
| CWU3* | External–N1 | 3 | 5 | 7 | 9 |
| | N2 | 5(3) | 9(5) | 13(7) | 17(9) |
| | N3 | 27(6) | 125(15) | 343(35) | 729(63) |
| | NA | 35(10) | 165(35) | 455(84) | 969(165) |
| | NB | 5(3) | 9(5) | 13(7) | 17(9) |
| | Internal– X1 | 81(18) | 625(75) | 2401(245) | 6561(540) |
| | X2 | 5(3) | 9(5) | 13(7) | 17(9) |
| | X3 | 25(9) | 81(25) | 169(49) | 289(81) |
| CRU3 | External–NA | 1 | 2 | 3 | 3 |
| | NB | 1 | 2 | 3 | 3 |
| | NC | 1 | 2 | 3 | 3 |
| | ND | 3 | 5 | 7 | 9 |
| | Internal– X1 | 1 | 2 | 3 | 3 |
| | X2 | 1 | 2 | 3 | 3 |
| | X3 | 1 | 2 | 3 | 3 |
| | X4 | 3 | 5 | 7 | 9 |
| | X5 | 10 | 35 | 84 | 165 |
| | X6 | 10 | 35 | 84 | 165 |
| | X7 | 10 | 35 | 84 | 165 |
| | X8 | 35 | 165 | 455 | 969 |
| | X9 | 10 | 70 | 252 | 495 |
| | X10 | 10 | 70 | 252 | 495 |
| | X11 | 10 | 70 | 252 | 495 |
| | X12 | 105 | 825 | 3185 | 8721 |
| | X13 | 5 | 9 | 13 | 17 |
| | X14 | 15 | 45 | 91 | 153 |
| | X15 | 6 | 15 | 35 | 63 |
| | X16 | 6 | 30 | 105 | 189 |
| | X17 | 3 | 5 | 7 | 9 |
| | X18 | 9 | 25 | 49 | 81 |
| CWU3R3** | External–N0 | 3 | 5 | 7 | 9 |
| | N1 | 2 | 5 | 11 | 21 |
| | N2 | 2 | 2 | 2 | 2 |
| | N3 | 2 | 5 | 11 | 21 |
| | NA | 35 | 165 | 455 | 969 |
| | Internal– X1 | 2 | 5 | 11 | 21 |
| | X2 | 2 | 2 | 2 | 2 |
| | X3 | 8 | 50 | 242 | 882 |
| | X4 | 4 | 10 | 22 | 42 |

* The values in ( ) apply if CWU3 is used in conjunction with CRU3 only.
** Appropriate for representations of $SU_3$ with $\lambda + 2\mu \leqslant N(N+1)(N+2)$.

that within each shell smaller limits are possible if one is dealing with less than mid-shell nuclei.) The routines in the test deck correspond to the case $N = 2$. Adaptations to other usages can be made in an equally straightforward fashion.

Routine A, BLOCKS, together with the block data statements, generates the common blocks required by the other subprograms. These include the binomial coefficients $\binom{M}{N}$, $N \leqslant M \leqslant M_{max}$ (= 32) and factorials $N!$, $N \leqslant N_{max}$ (= 32). The upper limit on $M_{max}$ and $N_{max}$ are machine dependent (e.g. $\binom{32}{16} = 601\,080\,390$ as opposed to $\binom{33}{16} = 1\,166\,803\,110$ does not yield a fixed point (integer) overflow condition). They imply restrictions on the "arbitrariness" of the coupling to which the programs apply (e.g. $\lambda + \mu + L \leqslant M_{max}$ as can be seen from eq. (26) of ref. [1]). For all practical purposes, however, the limits $M_{max} = N_{max} = 32$ are sufficiently high so as to impose no serious limitations. In any case, should a need arise there exist machines (e.g., the IBM 360 series for which $N_{max} = 50$ is possible) and/or techniques (e.g., the use of logarithms for large number operations) which allow these limits to be extended. Note, however, that the limits are fixed in BLOCKS and extensions can be made by modifying this and only this subprogram.

A conversion to double precision arithmetic may be required for complex couplings involving large $\rho$-multiplicities ($\rho_{max} \gtrsim 5$) and/or large $\lambda + \mu$ values ($\lambda + \mu \gtrsim 16$). This is particularly true for the $SU_3 \supset R_3$ Wigner coefficients because of the alternating nature of the sums required for an evaluation of the transformation coefficients from the $SU_3 \supset SU_2 \times U_1$ to the $SU_3 \supset R_3$ scheme. The conversion can be made by simply removing the "C" from column 1 of the statement "IMPLICIT REAL *8(D)" in each subprogram and replacing

SQRT → DSQRT,
FLOAT → DFLOAT,
ABS → DABS,
.E-06 → .D-12,
.EO → .DO,

throughtout, It may also be necessary (depending upon the machine and/or compiler) to change the type declarations "IMPLICIT REAL *8(D)" to "DOUBLE PRECISION(D)" and "IMPLICIT INTEGER(X)" to "INTEGER(X)" throughout.

## 4. Subprogram usage

The three sample control routines which comprise Part 2 of the FORTRAN IV deck (carrying identification lables, X, Y, Z, respectively) can be used in conjunction with the fifteen subprograms of the $SU_3$ package to generate output in the format as given by the illustrative tables X, Y, Z of Test Run Output. For this purpose it is convenient to group the subprograms of the $SU_3$ package into the four sets given in table 3.

Table 3
Subprogram for the $SU_3$ package.

| SET 1 | SET 2 | SET 3 | SET 4 |
|---|---|---|---|
| A. BLOCKS | F. CWU3 | G. CRU3 | L. CWU3R3 |
| B. CEWU3 | | H. DDBS | M. DTU3R3 |
| C. CEWU3S | | I. DLUT | N. CONMAT |
| D. MULTU3 | | J. DELTA | O. DWR3 |
| E. MULTHY | | K. DRR3 | J. (DELTA) |

Note that sets 3 and 4 both contain DELTA. A simple combination of 1, 2, 3, 4 together with X, Y, Z can then be used to calculate the desired coefficients:

$SU_3 \supset SU_2 \times U_1$ Wigner coefficients (X+1+2),
$SU_3$ Racah coefficients (Y+1+2+3),
$SU_3 \supset R_3$ Wigner coefficients (Z+1+4).

Since the size of a particular array is normally set at the limit first encountered in the compilation process, it is important that the main program (X, Y, Z) be compiled first and BLOCKS second with the remaining subprograms (1 + 2, 1 + 2 + 3, 1 + 4) in any convenient order. Once properly compiles, the following data cards can be used to obtain required Wigner and Racah coefficients ($N$ = number of cards):

$SU_3 \supset SU_2 \times U_1$ Wigner coefficients
Card(s) 1 – N Format (16I5)
$\lambda_1\ \mu_1\ \lambda_2\ \mu_2\ \lambda_3\ \mu_3\ \epsilon_3\ 2\Lambda_3$
...
ENDFILE (= negative integer)

$SU_3$ Racah coefficients
Card(s) 1 – N Format (16I5)
$\lambda_1 \mu_1 \lambda_2 \mu_2 \lambda \mu \lambda_3\ \mu_3\ \lambda_{12}\ \mu_{12}\ \lambda_{23}\ \mu_{23}$
...
ENDFILE (= negative integer)

$SU_3 \supset R_3$ Wigner coefficients

Card 1 Format (16I5)

$L_{1\,\mathrm{min}}\ L_{1\,\mathrm{max}}\ L_{3\,\mathrm{min}}\ L_{3\,\mathrm{max}}$

Card(s) 2 – N Format (16I5)

$\lambda_1\ \mu_1\ \lambda_2\ \mu_2\ \lambda_3\ \mu_3\ I_1\ J_1\ I_2\ J_2\ I_3\ J_3\ L_2$

...

ENDFILE (= negative integer).

Note that for Elliott-like $SU_3 \supset R_3$ projection (IJ) = (01) if $\mu \leqslant \lambda$ and (IJ) = (01) if $\mu > \lambda$.

The examples shown as part of the Test Run Output include numerical checks on all the symmetry properties of the Wigner coefficients. Also illustrated are a number of coefficients which vanish due to the Biedenharn–Louck prescription for specifying the $\rho$-multiplicity. The statement NON-EXIST denotes an $SU_3$ coupling violation while PAR–CHECK indicates that the coupling, although allowed, requires higher limits on the size parameters for internally dimensioned arrays. The statement SUB-LABEL indicates an incorrectly specified subgroup label ($\epsilon_3$, $\Lambda_3$, $L_2$, etc.).

Routines X, Y, Z by no means represent the most efficient use of the subprogram; they simply serve to illustrate the required call sequences and provide a standard input-output format for the casual user. More creative users should incorporate the routines into their own programs for more efficient evaluation of the required coefficients.

## 5. Conclusion

The programs have been written in a form which the authors found convenient for performing nuclear shell model calculations. It is our hope that other users may find the routines equally helpful in their studies of the usefulness of $SU_3$ in describing physical phenomena.

## References


[1] J.P. Draayer and Yoshimi Akiyama, J. Math. Phys., in press.

[2] M. Abramowitz and I.A. Stegun, Handb. of Mathem. Functions, Nat. Bur. of Stand., 1964.

[3] C.K. Chew and R.T. Sharp, Can. J. Phys. 44 (1966) 2789.

[4] G. Forsythe and C. Moler, COmputer Solution of Linear Algebraic Systems, Prentice Hall.

[5] M. Rotenberg, R. Bivins, N. Metropolis, and J.K. Wooten, The 3 – $J$ and 6 – $J$ Symbols, MIT Press.

[6] M.E. Rose, Elementary Theory of Angular Momentum, Wiley.

[7] R.D. Ratna Raju, J.P. Draayer, and K.T. Hecht, Nucl. Phys. A202 (1973) 433.

[8] T.T. Huo and G.E. Brown, Nucl. Phys. 85 (1966) 40.

# TEST RUN OUTPUT

## TABLE X: $SU_3 \supset SU_2 \times U_1$ Wigner coefficients

```
<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>

  4  3  11  3    2  0  -2  2    5  2   9  1      -0.4472130
         8  2            1  1          9  1       0.6708192
         5  1            4  0          9  1       0.5916078

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>

  3  4  -5  1    0  2  -4  0    2  5  -9  1       0.5916081
        -8  2           -1  1         -9  1      -0.6708204
       -11  3            2  2         -9  1      -0.4472136

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>

  2  5  -9  3    2  0  -2  2    3  4 -11  3       0.4714043
        -9  1           -2  2        -11  3      -0.3333331
       -12  2            1  1        -11  3       0.8164963

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>

  2  0   1  1    2  5 -12  2    3  4 -11  3       0.8164966
        -2  2           -9  3        -11  3      -0.4714044
        -2  2           -9  1        -11  3      -0.3333333

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 5

  8  5  15  5    4  4 -12  4    8  5   3  1       0.1175695   0.1423146   0.1096674   0.0443789   0.0018869
        15  3          -12  4          3  1      -0.3450901  -0.1365644  -0.0184610   0.0167523   0.0074424
        12  6           -9  5          3  1       0.0000000   0.1655406   0.2816388   0.2310127   0.0701511
        12  4           -9  5          3  1       0.0564783  -0.2752138  -0.2112512  -0.0084270   0.0499113
        12  4           -9  3          3  1      -0.1844581  -0.1531486  -0.0091124   0.0894425   0.0532419
        12  2           -9  3          3  1       0.3320628  -0.0200952  -0.0829114  -0.0072025   0.0239624
         9  7           -6  6          3  1       0.0000000   0.0000002   0.2022517   0.4024984   0.2813261
         9  5           -6  6          3  1       0.0000000   0.1051989  -0.1794031  -0.2076343   0.0249458
         9  5           -6  4          3  1       0.0996186  -0.0904384  -0.0876961   0.1535112   0.2342139
         9  3           -6  4          3  1      -0.2065165   0.1963400  -0.0400202  -0.1430856   0.0352273
         9  3           -6  2          3  1      -0.0640483  -0.1125946  -0.1315216   0.0252074   0.1306762
         9  1           -6  2          3  1       0.1634971   0.1579462  -0.0337038  -0.0678923   0.0367159
         6  8           -3  7          3  1       0.0000001   0.0000001   0.0000002   0.2424927   0.4396580
         6  6           -3  7          3  1      -0.0000001  -0.0000002   0.1374027  -0.1435394  -0.2045853
         6  6           -3  5          3  1      -0.0000003   0.2231607   0.1218487   0.0779112   0.2948456
         6  4           -3  5          3  1       0.0621235  -0.2892222   0.0844727  -0.0347559  -0.1428876
         6  4           -3  3          3  1      -0.1079206  -0.1062852  -0.1136591  -0.0208115   0.2714724
         6  2           -3  3          3  1       0.0668970  -0.0197446   0.1448354  -0.0704561  -0.0628080
         6  2           -3  1          3  1       0.1466495   0.0044746  -0.1198090  -0.0951678   0.1840702
         6  0           -3  1          3  1      -0.1337924   0.0996605   0.0942513  -0.1028597   0.0295924
         3  9            0  8          3  1       0.0000001   0.0000000   0.0000000   0.0000001   0.2867444
         3  7            0  8          3  1      -0.0000001  -0.0000000   0.0000000   0.1426699  -0.2797034
         3  7            0  6          3  1      -0.0000004  -0.0000000   0.3238611   0.2165026  -0.0146409
         3  5            0  6          3  1       0.0000005   0.1487740  -0.2625284   0.0667632  -0.1226110
         3  5            0  4          3  1       0.1141281  -0.0478189   0.0754315   0.1197792   0.0571560
         3  3            0  4          3  1      -0.1414666   0.1091161  -0.1482300   0.1253772  -0.1449734
         3  3            0  2          3  1      -0.1884385  -0.0769747  -0.0762355  -0.0198023   0.1454155
         3  1            0  2          3  1       0.1751731  -0.0377186  -0.0605741   0.1637287  -0.1494466
         3  1            0  0          3  1       0.1824744   0.1209746  -0.0895329  -0.1307549   0.1614395
         0  8            3  7          3  1      -0.0000002   0.0000001   0.0         0.3526897  -0.2040735
         0  6            3  7          3  1       0.0000002  -0.0000002   0.1869814  -0.3143120   0.1319153
         0  6            3  5          3  1       0.0000007   0.3036838   0.1118390   0.1888314  -0.1579979
         0  4            3  5          3  1       0.0842883  -0.2367578   0.1572902  -0.2032034   0.0893505
         0  4            3  3          3  1      -0.0225273  -0.1080241  -0.0137818   0.2023816  -0.1064696
         0  2            3  3          3  1      -0.0625755   0.0099035   0.1338661  -0.1473395   0.0495516
         0  2            3  1          3  1       0.0236511  -0.1794822  -0.0638043   0.1302010  -0.0474012
        -3  7            6  6          3  1       0.0000001  -0.0000004   0.4003033  -0.1438671  -0.0236966
        -3  5            6  6          3  1       0.0000002   0.1942371  -0.2852132   0.0664855   0.0256631
        -3  5            6  4          3  1       0.1839325   0.1027606   0.2410648  -0.1085403  -0.0099629
```

```
          -3  3        6  4         3  1     -0.0919667  0.0984762 -0.1452513  0.0353649  0.0132181
          -3  3        6  2         3  1     -0.3251516  0.0090789  0.1224855 -C.0406722 -C.0062435
          -6  6        9  5         3  1      0.0000000  0.4445478 -0.0733276 -0.0560823  C.0244341
          -6  4        9  5         3  1      0.1560720 -0.2264612  0.0242500  0.0308638 -0.0119579
          -6  4        9  3         3  1      0.1911489  0.1552420 -0.0416569 -0.0167740  C.0091319
          -9  5       12  4         3  1      0.4725329 -0.0439957 -0.0324188  0.0124972 -C.0009368

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 2

   6  1  -5  7   2  1  -4  2   5  2  -9  5    0.2390458  0.8366600
         -5  5         -4  2         -9  5   -0.3779646  0.0
         -8  6         -1  3         -9  5    0.8164969 -0.4082484
         -8  6         -1  1         -9  5    0.3651483  0.3651485

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 2

   1  6   8  6   1  2   1  3   2  5   9  5    0.8164959  0.4082484
          8  6          1  1          9  5   -0.3651482  0.3651479
          5  7          4  2          9  5    0.2390453 -0.8366588
          5  5          4  2          9  5    0.3779625 -0.0000018

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 2

   2  5   9  5   2  1  -1  3   1  6   8  6    0.6666661  0.3333331
          9  5         -1  1          8  6   -0.2981412  0.2981421
          6  6          2  2          8  6   -0.2981417 -0.4472139
          6  4          2  2          8  6    0.3333327  C.0000001
          6  6          2  0          8  6    0.3651480 -0.3651479
          3  7          5  1          8  6   -0.1951799  0.6831298
          3  5          5  1          8  6   -0.3086062  0.0000012

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 2

   2  1   5  1   2  5   3  7   1  6   8  6    0.5413314  0.4601318
          5  1          3  5          8  6   -0.2567768  0.1711834
          2  2          6  6          8  6   -0.0000005 -0.5374836
          2  0          6  6          8  6    0.5063686  0.1012735
          2  2          6  4          8  6    0.2773494 -0.1848992
         -1  3          9  5          8  6   -0.3697998  0.6471494
         -1  1          9  5          8  6   -0.4134487 -0.0826893

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 4

   9  3          5  5          6  6               PAR-CHECK

<(LM1MU1)E1,J1;(LM?MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 2

   9  3          3  3          6  6   9  4        SUB-LABEL

<(LM1MU1)E1,J1;(LM2MU2)E2,J2::(LM3MU3)E3,J3>  RO = 1,..., 0

   9  3          1  1          6  6               NON-EXIST
```

## TABLE Y: $SU_3$ Racah coefficients

```
U((L1,M1)(L2,M2)(LM,MU)(L3,M3);(LA,MA)RA,RB(LC,MC)RC,RD)

     6  4   2  0   5  5   2  0   6  5  1  1  4  0  1  1      0.4513354

     6  4   2  0   5  5   2  0   6  5  1  1  0  2  1  1      0.5340269

     6  4   2  0   5  5   2  0   6  5  1  1  2  1  1  1     -0.0610819
                                       1  1        1  2      0.7123057

     6  4   2  0   5  5   2  0   5  4  1  1  4  0  1  1      0.6938879

     6  4   2  0   5  5   2  0   5  4  1  1  0  2  1  1      0.2932209

     6  4   2  0   5  5   2  0   5  4  1  1  2  1  1  1      0.0283791
                                       1  1        1  2     -0.6570641

     6  4   2  0   5  5   2  0   7  3  1  1  4  0  1  1      0.4082477

     6  4   2  0   5  5   2  0   7  3  1  1  0  2  1  1     -0.5520518
```

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 4 | 2 | 0 | 5 | 5 | 2 | 0 | 7 | 3 | 1 | 1 | 2 | 1 | 1 | 1 | 0.6945825 |
| | | | | | | | | | | 1 | 1 | | | 1 | 2 | 0.2147684 |
| 6 | 4 | 2 | 0 | 5 | 5 | 2 | 0 | 4 | 6 | 1 | 1 | 4 | 0 | 1 | 1 | 0.3849001 |
| 6 | 4 | 2 | 0 | 5 | 5 | 2 | 0 | 4 | 6 | 1 | 1 | 0 | 2 | 1 | 1 | -0.5692743 |
| 6 | 4 | 2 | 0 | 5 | 5 | 2 | 0 | 4 | 6 | 1 | 1 | 2 | 1 | 1 | 1 | -0.7162519 |
| | | | | | | | | | | 1 | 1 | | | 1 | 2 | 0.1214912 |
| 6 | 6 | 0 | 4 | 6 | 6 | 4 | 0 | 8 | 3 | 1 | 1 | 4 | 4 | 1 | 1 | -0.0231100 |
| | | | | | | | | | | 1 | 1 | | | 1 | 2 | -0.1105537 |
| | | | | | | | | | | 1 | 1 | | | 1 | 3 | -0.1472464 |
| | | | | | | | | | | 1 | 1 | | | 1 | 4 | -0.0984493 |
| | | | | | | | | | | 1 | 1 | | | 1 | 5 | 0.4153821 |
| 9 | 3 | 1 | 2 | 8 | 2 | 2 | 1 | 8 | 3 | 1 | 1 | 2 | 2 | 1 | 1 | 0.2168387 |
| | | | | | | | | | | 1 | 1 | | | 1 | 2 | -0.0399911 |
| | | | | | | | | | | 1 | 1 | | | 2 | 1 | -0.2052838 |
| | | | | | | | | | | 1 | 1 | | | 2 | 2 | 0.1964977 |
| | | | | | | | | | | 1 | 2 | | | 1 | 1 | -0.0382751 |
| | | | | | | | | | | 1 | 2 | | | 1 | 2 | 0.1965769 |
| | | | | | | | | | | 1 | 2 | | | 2 | 1 | 0.2263088 |
| | | | | | | | | | | 1 | 2 | | | 2 | 2 | 0.4226531 |
| | | | | | | | | | | 2 | 1 | | | 1 | 1 | 0.1321138 |
| | | | | | | | | | | 2 | 1 | | | 1 | 2 | -0.0243655 |
| | | | | | | | | | | 2 | 1 | | | 2 | 1 | 0.4405778 |
| | | | | | | | | | | 2 | 1 | | | 2 | 2 | -0.0707791 |
| | | | | | | | | | | 2 | 2 | | | 1 | 1 | 0.0422222 |
| | | | | | | | | | | 2 | 2 | | | 1 | 2 | 0.4751489 |
| | | | | | | | | | | 2 | 2 | | | 2 | 1 | 0.0972563 |
| | | | | | | | | | | 2 | 2 | | | 2 | 2 | -0.2836112 |
| 7 | 4 | 1 | 2 | 5 | 5 | 0 | 2 | 4 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 0.4360309 |
| 5 | 5 | 2 | 0 | 7 | 4 | 2 | 1 | 4 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 0.4360307 |
| 9 | 3 | 1 | 1 | 6 | 6 | 2 | 2 | 9 | 3 | 1 | 1 | 3 | 3 | 1 | 1 | 0.4364350 |
| | | | | | | | | | | 1 | 1 | | | 1 | 2 | 0.0000003 |
| | | | | | | | | | | 2 | 1 | | | 1 | 1 | 0.1390705 |
| | | | | | | | | | | 2 | 1 | | | 1 | 2 | 0.5916793 |
| 9 | 3 | 1 | 1 | 6 | 6 | 3 | 3 | 9 | 3 | | | 4 | 4 | | | PAR-CHECK |
| 8 | 2 | 0 | 4 | 5 | 3 | 2 | 1 | 7 | 2 | 0 | 0 | 2 | 5 | 0 | 0 | NON-EXIST |

## TABLE Z: $SU_3 \supset R_3$ Wigner coefficients

<((LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 01 | 1 | 2 | 2 | 0 | 01 | 1 | 2 | 5 | 2 | 01 | 1 | 2 | 0.3055046 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 1 | 2 | -0.3243206 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 1 | 2 | -0.6856724 |
| | | | 1 | 2 | | | | 1 | 2 | | | | 1 | 3 | -0.6117334 |
| | | | 1 | 2 | | | | 1 | 2 | | | | 2 | 3 | 0.1279683 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 1 | 3 | -0.3646375 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 1 | 3 | -0.0000002 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 2 | 3 | 0.2080802 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 2 | 3 | -0.6205428 |

<((LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 10 | 1 | 2 | 0 | 2 | 10 | 1 | 2 | 2 | 5 | 10 | 1 | 2 | 0.3055046 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 1 | 2 | -0.3243206 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 1 | 2 | -0.6856724 |
| | | | 1 | 2 | | | | 1 | 2 | | | | 1 | 3 | -0.6117334 |
| | | | 1 | 2 | | | | 1 | 2 | | | | 2 | 3 | 0.1279683 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 1 | 3 | -0.3646375 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 1 | 3 | -0.0000002 |
| | | | 1 | 3 | | | | 1 | 2 | | | | 2 | 3 | 0.2080802 |
| | | | 2 | 3 | | | | 1 | 2 | | | | 2 | 3 | -0.6205428 |

<((LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>

```
  2  5  10  1  2    2  0  01  1  2    3  4  10  1  2      0.3220300
            1  3              1  2                1  2      0.7629643
            2  3              1  2                1  2     -0.1596035
            1  2              1  2                1  3      0.2889273
            1  2              1  2                2  3      0.6108462
            1  3              1  2                1  3     -0.3843608
            2  3              1  2                1  3      0.2193345
            1  3              1  2                2  3     -0.0000009
            2  3              1  2                2  3     -0.6541088

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>

  2  0  01  1  2    2  5  10  1  3    3  4  10  1  2     -0.7629647
            1  2              2  3                1  2      0.1596033
            1  2              1  3                1  3     -0.3843611
            1  2              2  3                1  3      0.2193345
            1  2              1  3                2  3     -0.0000009
            1  2              2  3                2  3     -0.6541088

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 5

  8  5  01  1  2    4  4  01  1  2    8  5  01  1  2      0.0373651   0.0439388   0.0066230   0.0500954  -0.0197199
            1  2              2  2                1  2     -0.0715490  -0.0841364  -0.0126820  -0.0959254   0.0377607
            1  3              1  2                1  2      0.0878139   0.0663584  -0.0058050   0.0696993  -0.0043999
            2  3              1  2                1  2     -0.0458331  -0.0572700  -0.0797430  -0.1007875  -0.1361247
            1  3              2  2                1  2      0.0348150  -0.0435452   0.0030094  -0.0660166  -0.0728580
            2  3              2  2                1  2      0.0780247   0.0302061  -0.0994340  -0.0086091  -0.1148131
            1  2              1  2                1  3      0.0665561   0.0018454  -0.0007162  -0.0121268  -0.0526576
            1  2              2  2                1  3      0.0440922   0.0670548  -0.0054795   0.0802245   0.0321348
            1  2              1  2                2  3      0.0319812  -0.0067081  -0.1074766  -0.0546460  -0.1453726
            1  2              2  2                2  3     -0.0694705  -0.0543092  -0.0072868  -0.0657465  -0.0389647
            1  3              1  2                1  3     -0.0301820  -0.0161238   0.0304046   0.0073204   0.0376334
            2  3              1  2                1  3      0.0045619   0.0181886  -0.0662000  -0.0334105  -0.0775405
            1  3              2  2                1  3      0.0577942   0.0308752  -0.0582204  -0.0140175  -0.0720625
            2  3              2  2                1  3      0.0475206   0.0149339  -0.1066674   0.0305866  -0.1196150
            1  3              1  2                2  3     -0.0416047  -0.0226490   0.1253651  -0.0060092   0.1424709
            2  3              1  2                2  3      0.0637214  -0.0163902   0.0600707   0.0712655   0.0523129
            1  3              2  2                2  3      0.0234109  -0.0063930  -0.0066255   0.0448966  -0.0047176
            2  3              2  2                2  3     -0.1220173   0.0313848  -0.1150265  -0.1364630  -0.1001715

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RC = 1,..., 2

  6  1  01  1  2    2  1  01  1  3    5  2  01  1  2     -0.2285706   0.5428551
            1  3              1  3                1  2      0.6408313   0.2746418
            1  2              1  3                1  3     -0.1975533  -0.0238425
            1  2              1  3                2  3     -0.4630282  -0.3367475
            1  3              1  3                1  3      0.2612962  -0.2228701
            1  3              1  3                2  3     -0.0652961  -0.5223668

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 2

  1  6  10  1  2    1  2  10  1  3    2  5  10  1  2     -0.2285706  -0.5428551
            1  3              1  3                1  2      0.6408313  -0.2746418
            1  2              1  3                1  3     -0.1975533   0.0238425
            1  2              1  3                2  3     -0.4630282   0.3367475
            1  3              1  3                1  3      0.2612962   0.2228701
            1  3              1  3                2  3     -0.0652961   0.5223668

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 2

  2  5  10  1  2    2  1  01  1  3    1  6  10  1  2      0.2015804   0.4787537
            1  3              1  3                1  2     -0.2061467   0.0248799
            2  3              1  3                1  2     -0.4831694   0.3513960
            1  2              1  3                1  3      0.4776473  -0.2047058
            1  3              1  3                1  3     -0.2304413  -0.1965532
            2  3              1  3                1  3      0.0575853  -0.4606845

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 2

  2  1  01  1  2    2  5  10  1  3    1  6  10  1  2     -0.4276024   0.0038171
            1  2              2  3                1  2      0.0471822  -0.2359120
            1  3              1  3                1  2     -0.1853252   0.0936484
            1  3              2  3                1  2     -0.5969408  -0.0243648
            1  2              1  3                1  3     -0.2970074   0.0798906
            1  2              2  3                1  3     -0.1277711  -0.1916558
            1  3              1  3                1  3      0.0827109  -0.2913679
            1  3              2  3                1  3     -0.3034552  -0.3513695

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 4

  9  3              5  5        1  1  6  6                PAR-CHECK

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 2

  9  3              3  3        0  0  6  6                SUB-LABEL

<(LM1MU1)IJ,K1,L1;(LM2MU2)IJ,K2,L2::(LM3MU3)IJ,K3,L3>  RO = 1,..., 0

  9  3              1  1              6  6                NON-EXIST
```