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Circuits and structure in matroids and graphs

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CIRCUITS AND STRUCTURE IN MATROIDS AND GRAPHS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by

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Abstract

This dissertation consists of several results on matroid and graph structure and is organized into three main parts. The main goal of the first part, Chapters 1-3, is to produce a unique decomposition of 3-connected matroids into more highly connected pieces. In Chapter 1, we review the definitions and main results from the previous work of Hall, Oxley, Semple, and Whittle. In Chapter 2, we introduce operations that allow us to decompose a 3-connected matroid $M$ into a pair of 3-connected pieces by breaking the matroid apart at a 3-separation. We also generalize a result of Akkari and Oxley. In Chapter 3, we produce the decomposition. We analyze the properties of equivalent 3-separations and then use these properties to create a decomposition tree that is labeled by subsets of $M$. We then use the decomposition tree as a guide to show us where to break $M$ apart.

The second part, Chapter 4, specializes the results of the first part to graphs. Given a simple 3-connected graph $G$, we first analyze the properties of $G$ that come from the presence of crossing 3-separations in its cycle matroid. We prove that specializing the decomposition in Chapter 3 gives us a decomposition of $G$ into graphs whose cycle matroids are sequentially 4-connected. We show that every simple 3-connected graph whose cycle matroid is sequential is a minor of certain special planar graphs that we call sequential 3-paths. We also prove that a sequential 3-connected binary matroid is a 3-connected minor of the cycle matroid of a sequential 3-path.

The third part, Chapter 5, explores the presence of circuits of various sizes in simple representable matroids. We prove that a sufficiently large, simple binary matroid either has circuits of all sizes or is isomorphic to one of two exceptions.
We also show that, up to isomorphism, all but six sufficiently large, simple binary matroids have Hamiltonian circuits through each element.
Chapter 1
Introduction

1.1 Overview
This dissertation consists of several results in matroid structure, concentrating on 3-separations in 3-connected matroids and cycles in representable matroids. In this first chapter, we introduce the notation, foundational results, and history of the problems considered in the rest of this work. We assume basic knowledge of matroids and graphs, and we will generally follow the notation in Oxley’s *Matroid Theory* [19] unless otherwise specified. The next three chapters apply results of Hall, Oxley, Semple, and Whittle’s project on 3-separations in matroids (see [23], [24], [12], and [13]) to obtain a unique decomposition of 3-connected matroids and graphs and to characterize a special class of graphs. In Chapter 5, motivated by a result of Bondy [3], we explore Hamiltonian circuits in matroids.

In Section 1.2 we will review important basic matroid definitions and theorems. In Section 1.3 we will review matroid connectivity and the equivalence of 3-separations used in [23]. Finally, in Section 1.4 we review the relevant definitions and results on 3-separations in the work of Hall, Oxley, Semple, and Whittle.

1.2 Basic Matroid Definitions and Results
Let $M$ be a matroid and $X \subseteq E(M)$. Let $\mathcal{J}$ be the collection of independent sets of $M$ that are contained in $X$. Then $\mathcal{J}$ is collection of the independent sets of a matroid on $X$ that we call the *restriction* of $M$ to $X$, or the *deletion* of $E(M) - X$ from $M$. We denote deletion of $X$ from $M$ by $M \setminus X$ and the restriction of $M$ to $X$ by $M|X$. If we delete $E(M) - X$ in the dual matroid, $M^*$, we obtain a matroid that we call the the *contraction of $M$ to $X$* or the *contraction of $M$ by $E(M) - X$*. 
We denote the contraction of $M$ by $X$ by $M/X$ and the contraction of $M$ to $X$ by $M.X$. Moreover, deletion and contraction in matroids are generalizations of the corresponding operations on edges of graphs; that is, if $G$ is a graph and $e$ is in $E(G)$, then $M(G)\setminus e = M(G'\setminus e)$ and $M(G)/e = M(G/e)$.

If $M'$ and $M$ are matroids such that $M'\setminus X = M$ for some subset $X$ of $E(M')$, then we say that $M'$ is an extension of $M$ by $X$. If $M'$ and $M$ are matroids such that $M'/X = M$, then we say that $M'$ is a coextension of $M$ by $X$. One way to obtain an extension of $M$ is by the use of modular cuts. A pair of flats $F_1$ and $F_2$ is modular in $M$, or $(F_1, F_2)$ is a modular pair of flats, if $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$.

A flat $F$ is a modular flat of a matroid $M$ if $(F, X)$ is a modular pair of flats for all flats $X$ of $M$. A set $\mathcal{F}$ of flats of $M$ is a modular cut if the members of $\mathcal{F}$ satisfy the following two properties:

(i) If $F \in \mathcal{F}$ and $F'$ is a flat of $M$ containing $F$, then $F' \in \mathcal{F}$.

(ii) If $F_1, F_2 \in \mathcal{F}$ and $(F_1, F_2)$ is a modular pair, then $F_1 \cap F_2 \in \mathcal{F}$.

We have the following standard result relating the single-element extensions of $M$ and the modular cuts of $M$:

**Theorem 1.2.1.** Let $\mathcal{M}$ be a modular cut of a matroid $M$ on a set $E$. Then there is a unique extension $N$ of $M$ on $E \cup e$ such that $\mathcal{M}$ consists of those flats $F$ of $M$ for which $F \cup e$ is a flat of $N$ having the same rank as $F$. Moreover, for all subsets $X$ of $E$, we have the following:

(i) $r_N(X) = r_M(X)$.

(ii) $r_N(X \cup e) = r_M(X)$, if $\text{cl}_M(X) \in \mathcal{M}$.

(iii) $r_N(X \cup e) = r_M(X) + 1$, if $\text{cl}_M(X) \notin \mathcal{M}$. 

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If $M$ is a modular cut of $M$, then we denote the single-element extension $M'$ of $M$ from this modular cut by $M +_M e$, where $\{e\} = E(M') - E(M)$. A nice feature of modular cuts is that the intersection of two modular cuts is also a modular cut. If we want to add an element to several flats $F_1, F_2, \ldots, F_n$, then we can consider the modular cut that is the intersection of all the modular cuts that contain $\{F_1, F_2, \ldots, F_n\}$. We say that this intersection is the modular cut generated by $\{F_1, F_2, \ldots, F_n\}$, which we denote by $\langle F_1, F_2, \ldots, F_n \rangle$. A modular cut is said to be principal if it is the modular cut generated by a single flat $F$. Modular cuts can be difficult to work with, but they might be the only way to proceed when we want an extension that has certain properties. We use modular cuts to obtain special extensions in Chapter 2.

If $G_1$ and $G_2$ are two graphs having a common set $V$ of vertices, then, by identifying the common vertices, we obtain a graph $G$ having $G_1$ and $G_2$ as subgraphs. Let $M_1$ and $M_2$ be matroids having a common restriction $N$; we would like have an analogous procedure for obtaining a matroid $M$ that has $M_1$ and $M_2$ as restrictions. Unfortunately, this is not always possible. However, there is such a matroid when $N$ is a modular flat of $M_1$ or $M_2$ (see Section 12.4 in [19]). This matroid $M$ is called the generalized parallel connection of $M_1$ and $M_2$ across $N$, which we write as $P_N(M_1, M_2)$. The following proposition, which is Proposition 12.4.13 in [19], is a characterization of the flats of a generalized parallel connection.

**Proposition 1.2.2.** The set $F$ is a flat of $P_N(M_1, M_2)$ if and only if $F \cap E(M_1)$ is a flat of $M_1$ and $F \cap E(M_2)$ is a flat of $M_2$.

In Section 2.5, we will use the preceding proposition to prove that a certain matroid is the generalized parallel connection of two matroids, and, in Section 2.3,
we will use the following result, which follows from Proposition 12.4.15 in [19], for a similar purpose.

**Proposition 1.2.3.** Let $M$ be a simple matroid on a set $E$ and suppose that, for some subset $T$ of $E(M)$, the matroid $M/T = M_1 \oplus M_2$. If $T$ is a modular flat of $M \setminus E(M_2)$, then $M = P_{M|T}(M \setminus E(M_2), M \setminus E(M_1))$.

### 1.3 Matroid Connectivity

Let $M$ be a matroid. A partition $(A, B)$ of the ground set $E(M)$ is $k$-separating if $r(A) + r(B) - r(M) < k$. A $k$-separating partition $(A, B)$ is a $k$-separation if $|A|, |B| \geq k$. For $k \geq 2$, a matroid is $k$-connected if it has no $j$-separations for all $j < k$. A subset $A$ of $E(M)$ is $k$-separating if $(A, E(M) - A)$ is $k$-separating. For all subsets $A$ of $E(M)$, the **connectivity function** $\lambda$ of $M$ is defined by $\lambda(A) = r(A) + r(E(M) - A) - r(A)$. It is straightforward to prove that $A$ is $k$-separating in $M$ if and only if $A$ is $k$-separating in $M^*$. A $k$-separating set $A$, or a $k$-separating partition $(A, E(M) - A)$, or $k$-separation $(A, E(M) - A)$ is *exact* if $\lambda(A) = k - 1$.

Note that we are using an ordered pair for $k$-separating partitions; we do this in order to simplify notation in later results.

A result that is often useful when dealing with separations in extensions is the following, which is Lemma 8.2.10 in [19]:

**Lemma 1.3.1.** Let $X_1, X_2, Y_1, \text{ and } Y_2$ be subsets of the ground set of a matroid $M$. If $X_1 \supseteq Y_1$ and $X_2 \supseteq Y_2$, then

$$r(X_1) + r(X_2) - r(X_1 \cup X_2) \geq r(Y_1) + r(Y_2) - r(Y_1 \cup Y_2).$$

A $3$-separating set $A$ of a $3$-connected matroid $M$ is *sequential* if $A$ has an ordering $(a_1, a_2, \ldots, a_n)$ such that the set $\{a_1, a_2, \ldots, a_i\}$ is $3$-separating for all $i \in \{1, 2, \ldots, n\}$. If $(A, B)$ is $3$-separating in $M$, then we say $(A, B)$ is *sequential*
if $A$ or $B$ is sequential. If all 3-separations of $M$ are sequential, then we say that $M$ is \textit{sequentially 4-connected}. A 3-connected matroid $M$ is \textit{sequential} if $E(M)$ has an ordering $(x_1, x_2, \ldots, x_n)$ such that $\{x_1, x_2, \ldots, x_i\}$ is 3-separating for all $i \in \{3, 4, \ldots, n-3\}$.

The \textit{coclosure} $\text{cl}^*(A)$ of $A$ is the closure of $A$ in the dual matroid $M^*$. We say that $A$ is \textit{fully closed} in $M$ if $A$ is closed in both $M$ and $M^*$. The \textit{full closure} $\text{fcl}(A)$ is the intersection of all fully closed sets containing $A$. This is a well-defined closure operator, and one way to obtain the full closure is by starting with $A$ and alternately taking the closure and coclosure of the resulting set until no new elements are added. If we have two exactly 3-separating sets $A$ and $B$, then $A$ is \textit{equivalent} to $B$ if $\text{fcl}(A) = \text{fcl}(B)$. Given an exact 3-separating partition $(A, B)$, an element $e \in \text{fcl}(A) \cap \text{fcl}(B)$ is said to be \textit{sequential} in $(A, B)$. Two 3-separating partitions $(A_1, A_2)$ and $(B_1, B_2)$ are \textit{equivalent} if $A_1$ is equivalent to $B_1$ and $A_2$ is equivalent to $B_2$. We define $\text{cl}^{(*)}(A)$ to be $\text{cl}(A) \cup \text{cl}^*(A)$; we have defined $\text{cl}^{(*)}$ for convenience, but it should be noted that $\text{cl}^{(*)}$ is not a true closure operator.

The following two lemmas, proved in [23], are particularly useful:

\textbf{Lemma 1.3.2.} Let $M$ be a 3-connected matroid with $A$ and $B$ be 3-separating subsets of $E(M)$.

(i) If $|A \cap B| \geq 2$, then $A \cup B$ is 3-separating.

(ii) If $|E(M) - (A \cup B)| \geq 2$, then $A \cap B$ is 3-separating.

\textbf{Lemma 1.3.3.} Let $(A, B)$ be exactly 3-separating in a matroid $M$.

(i) For $e \in E(M)$, the partition $(A \cup e, B - e)$ is 3-separating if and only if $e \in \text{cl}^{(*)}(A)$. 

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(ii) For $e \in B$, the partition $(A \cup e, B - e)$ is exactly 3-separating if and only if $e$ is in exactly one of $\text{cl}(A) \cap \text{cl}(B)$ and $\text{cl}^*(A) \cap \text{cl}^*(B)$.

(iii) The elements of $\text{fcl}(A) - A$ can be ordered $(a_1, a_2, \ldots, a_n)$ in such a way that $A \cup \{a_1, a_2, \ldots, a_n\}$ is 3-separating for all $i \in \{1, 2, \ldots, n\}$.

We define a 3-sequence in $M$ to be an ordered partition $(A, x_1, x_2, \ldots, x_n, B)$ of $E(M)$ such that $|A|, |B| \geq 2$ and $(A \cup \{x_1, x_2, \ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots, x_n\} \cup B)$ is exactly 3-separating for all $i \in \{0, 1, \ldots, n\}$. If $(A, B)$ is a 3-separation and $B$ is sequential with ordering $(b_1, b_2, \ldots, b_n)$, then $(A, b_1, b_2, \ldots, b_{n-2}, \{b_{n-1}, b_n\})$ is a 3-sequence, and we will also call $(A, b_1, b_2, \ldots, b_n)$ a 3-sequence in this case. Note that, since a set $C$ is 3-separating in $M$ if and only if it is 3-separating in $M^*$, a 3-sequence of $M$ is also a 3-sequence in $M^*$.

For a particular ordering $\vec{X} = (x_1, x_2, \ldots, x_n)$, we may write the 3-sequence as $(A, \vec{X}, B)$. Sometimes we write $\vec{X}_i$ for $(x_1, x_2, \ldots, x_i)$ and $\vec{X}_i^\rightarrow$ for $(x_i, x_{i+1}, \ldots, x_n)$. If $Y \subseteq X$, then $\vec{X}$ induces an order on $Y$ and $X - Y$. We write $\vec{X} \cap Y$ and $\vec{X} - Y$ for the ordered sets with this induced ordering. Note that whenever we discuss a 3-sequence $(A, X, B)$, it will be implicit that there is an ordering on $X$. Similarly, when we write $(A, X_1, X_2, \ldots, X_n, B)$ for a 3-sequence $(A, X, B)$, we mean that the sets $X_i$ are a partition of $X$ into subsets of elements that are consecutive in the ordering on $X$ and that inherit an ordering from the ordering on $X$.

By Lemma 1.3.3, an element $e_i$ in $\vec{X}$ is in the closure or coclosure of $A \cup \vec{X}_{i-1}$. If $e_i \in \text{cl}(A \cup \vec{X}_{i-1})$, then we say that $e_i$ is a guts element. If $e_i \in \text{cl}^*(A \cup \vec{X}_{i-1})$, then we say that $e_i$ is a coguts element. By [12, Lemma 4.6], if $e_i$ is a guts element of $\vec{X}$, then $e_i$ is a guts element in every reordering of $\vec{X}$ that gives us a 3-sequence. Similarly, if $e_i$ is a coguts element of $\vec{X}$, then $e_i$ is a coguts element in every reordering of $\vec{X}$ that gives us a 3-sequence.
Finally, we will define three structures that frequently appear in 3-sequences. Given a 3-sequence \((A, \overline{X}, B)\) in a 3-connected matroid \(M\), let \(S\) be a subset of \(X\). Suppose first that \(|S| \geq 4\). We call \(S\) a segment if every three-element subset of it is a triangle. Dually, \(S\) is a cosegment if every three-element subset is a triad. These definitions call for \(S\) to have at least four elements. We now extend this to two- and three-element sets. If \(|S| = 3\) and its three elements are all guts elements or all coguts elements, then \(S\) is a segment or a cosegment, respectively. If \(|S| = 2\), if the two elements of \(S\) occur consecutively in some 3-sequence, and if both are guts elements or both are coguts elements, then we will call \(S\) a degenerate segment or a degenerate cosegment, respectively.

If \(|S| \geq 4\), we call \(S\) a fan if there is an ordering \((s_1, s_2, \ldots, s_n)\) of the elements of \(S\) such that, for all \(i \in \{1, 2, \ldots, n-2\},\)

(i) the triple \(\{s_i, s_{i+1}, s_{i+2}\}\) is a triangle or a triad, and

(ii) if \(\{s_i, s_{i+1}, s_{i+2}\}\) is a triangle, then \(\{s_{i+1}, s_{i+2}, s_{i+3}\}\) is a triad; if \(\{s_i, s_{i+1}, s_{i+2}\}\) is a triad, then \(\{s_{i+1}, s_{i+2}, s_{i+3}\}\) is a triangle.

We now define three-element and two-element fans. If \(|S| = 3\) and (i) holds, then \(S\) is a fan if \(S\) consists of two guts and one coguts element, or two coguts and one guts element. If \(|S| = 2\), then \(S\) is a fan if its two elements \(s_1\) and \(s_2\) occur consecutively in some 3-sequence, if \(s_1\) precedes \(s_2\) in every 3-sequence, and if exactly one of \(s_1\) and \(s_2\) is a guts element. A segment, cosegment, or fan \(S\) is maximal if no other segment, cosegment, or fan properly contains \(S\). By results in [12], we may assume the elements of a segment, cosegment, or fan \(S\) occur consecutively in some 3-sequence of \(M\).
1.4 Flowers and 3-Trees in Matroids

The next three chapters use the structure of the 3-separations in a matroid $M$ to obtain information about the structure of $M$. A useful technique in graph theory and matroid theory is to use separations in the object to obtain structural properties or to reduce the cases in a proof. Tutte [28] introduced the abstract definition of separations in matroids, and Cunningham and Edmonds [7] produced a unique decomposition of 2-connected matroids based on 2-separations. Coullard, Gardner and Wagner [6] produced a unique decomposition of 3-connected graphs into pieces in which all the 3-separations cross, and Leo [17] produced a unique decomposition of binary matroids which generalizes the decomposition of Coullard, Gardner, and Wagner.

Our goal is to produce a labeled tree for a 3-connected matroid that displays its inequivalent 3-separations and, using these displayed 3-separations, decompose the matroid into sequentially 4-connected pieces and certain special 3-connected pieces. To do so, we use many concepts and results from Hall, Oxley, Semple, and Whittle; we include several of these results in this section and elsewhere in the dissertation.

Coullard, Gardner, and Wagner [6] decompose 3-connected graphs into pieces in which all 3-separations cross. That is, if $(A, B)$ and $(C, D)$ are 3-separations of $G$, then $A \cap C$, $A \cap D$, $B \cap C$, and $B \cap D$ are nonempty. Oxley, Semple, and Whittle [23] were able to describe the interactions of such crossing separations by using the equivalence of 3-separations and structures called flowers.

Let $n$ be a positive integer and $M$ be a 3-connected matroid. Then the partition $(P_1, P_2, \ldots, P_n)$ of $E(M)$ is a flower of $M$ with petals $P_1, P_2, \ldots, P_n$ if each $P_i$ has at least two elements and is 3-separating, and if each $P_i \cup P_{i+1}$ is 3-separating, where all subscripts are interpreted modulo $n$. Note that, by interpreting the subscripts
modulo $n$, we have a cyclic ordering of the petals. A petal $P_i$ of a flower $\Phi = (P_1, P_2, \ldots, P_n)$ is \emph{tight} if $P_i \notin \mathrm{fcl}(P_{i-1}) \cup \mathrm{fcl}(P_{i+1})$, and a flower is \emph{tight} if all of its petals are tight. An element $e$ is \emph{loose} if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal $P_i$ of $\Phi$, and $e$ is \emph{tight} in the petal $P_i$ if it is not loose.

We say that a flower $\Phi$ of $M$ is an \emph{anemone} if any union of petals is 3-separating. We say that $\Phi$ is a \emph{daisy} if unions of petals that are consecutive in the cyclic ordering are 3-separating and if these are the only unions of petals that are 3-separating. The \emph{local connectivity function} $\sqcap$ on subsets $A$ and $B$ of $E(M)$ is defined by $\sqcap(A, B) = r(A) + r(B) - r(A \cup B)$. We are able to further classify flowers based on the local connectivity of the petals.

For $n \geq 3$, an anemone $(P_1, P_2, \ldots, P_n)$ is called

(i) a \emph{paddle} if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in \{1, 2, \ldots, n\}$;

(ii) a \emph{copaddle} if $\sqcap(P_i, P_j) = 0$ for all distinct $i, j \in \{1, 2, \ldots, n\}$; and

(iii) \emph{spike-like} if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

Similarly, a daisy $(P_1, P_2, \ldots, P_n)$ is called

(i) \emph{swirl-like} if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive $i$ and $j$, while $\sqcap(P_i, P_j) = 0$ for all nonconsecutive $i$ and $j$; and

(ii) \emph{Vámos-like} if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive $i$ and $j$, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

Finally, if $n = 3$ and $\sqcap(P_i, P_j) = 1$ for all distinct $i$ and $j$, then we could think of the flower as being swirl-like or spike-like, depending on the properties of its loose elements; however, in this situation we say that the flower is \emph{unresolved}.

The following are two main results, Theorem 4.1 and Proposition 4.2, in [23]:
Theorem 1.4.1. If \( \Phi = (P_1, P_2, \ldots, P_n) \) is a flower, then \( \Phi \) is either an anemone or a daisy. Moreover, if \( n \geq 3 \), then \( \Phi \) is either a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or is unresolved.

Theorem 1.4.2. If \( \Phi = (P_1, P_2, \ldots, P_n) \) is a flower, then it is also a flower in \( M^* \). Moreover, for \( n \geq 3 \),

(i) if \( \Phi \) is either spike-like, swirl-like, Vámos-like, or unresolved, the \( \Phi \) has the same type in \( M^* \) as in \( M \); and

(ii) if \( \Phi \) is a paddle in \( M \), then \( \Phi \) is a copaddle in \( M^* \).

We say that a flower \( \Phi \) displays a 3-separating set \( A \) or a 3-separation \((A, B)\) if \( A \) is a union of petals of \( \Phi \). There is a natural quasi order \( \preceq \) on the flowers of \( M \) where \( \Phi_1 \preceq \Phi_2 \) if every non-sequential 3-separation displayed by \( \Phi_1 \) is equivalent to one displayed by \( \Phi_2 \). Two flowers \( \Phi_1 \) and \( \Phi_2 \) are equivalent if \( \Phi_1 \preceq \Phi_2 \) and \( \Phi_2 \preceq \Phi_1 \).

We say that a 3-separation \((A, B)\) of \( M \) conforms to the flower \( \Phi \) if either \((A, B)\) is equivalent to a 3-separation that is displayed by \( \Phi \), or \((A, B)\) is equivalent to a 3-separation \((A', B')\) with the property that \( A' \) or \( B' \) is contained in a petal of \( \Phi \). Tight maximal flowers are precisely what are needed to describe crossing inequivalent non-sequential 3-separations because of the following result, which is Theorem 8.1 in [23]:

Theorem 1.4.3. Let \( M \) be a 3-connected matroid with at least 9 elements and let \( \Phi \) be a tight maximal flower in \( M \). Then every non-sequential 3-separation of \( M \) conforms with \( \Phi \).

Let \( \pi \) be a partition of a finite set \( E \). Let \( T \) be a tree such that every member of \( \pi \) labels a vertex of \( T \); some vertices may be unlabeled and no vertex is labeled by more than one member of \( \pi \). We say that \( T \) is a \( \pi \)-labeled tree; labeled vertices are
called *bag vertices* and members of π are called *bags*. Let $T'$ be a subtree of $T$. The union of those bags that label vertices of $T'$ is the subset of $E$ displayed by $T'$. Let $e$ be an edge of $T$. The *partition of $E$ displayed by $e$* is the partition displayed by the components of $T\setminus e$. The *partition of $E$ displayed by $v$* is the partition displayed by the components of $T - v$. The edges incident with $v$ are in natural one-to-one correspondence with the components of $T - v$, and hence with the members of the partition displayed by $v$. In what follows, if a cyclic ordering $(e_1, e_2, \ldots, e_n)$ is imposed on the edges incident with $v$, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by $v$. Let $M$ be a 3-connected matroid with ground set $E$. An *almost partial 3-tree* $T$ for $M$ is a $\pi$-labelled tree, where $\pi$ is a partition of $E$ such that the following conditions hold:

**(i)** For each edge $e$ of $T$, the partition $(X,Y)$ of $E$ displayed by $e$ is 3-separating, and, if $e$ is incident with two bag vertices, then $(X,Y)$ is a non-sequential 3-separation.

**(ii)** Every non-bag vertex $v$ is labeled either $D$ or $A$. Moreover, if $V$ is labeled $D$, then there is a cyclic ordering on the edges incident with $v$.

**(iii)** If a vertex $v$ is labeled $A$, then the partition of $E$ displayed by $v$ is a tight maximal anemone of order at least three.

**(iv)** If a vertex $v$ is labeled $D$, then the partition of $E$ displayed by $v$, with the cyclic order induced by the cyclic ordering on the edges incident with $v$, is a tight maximal daisy of order at least three.

A vertex labeled by $D$ or $A$ corresponds to a flower of $M$. We say that the 3-separations displayed by this flower are the 3-separations *displayed* by $v$ and that
$v$ is a flower vertex of $T$. A 3-separation is displayed by an almost partial 3-tree $T$ if it is displayed by some edge or some flower vertex of $T$. A 3-separation $(A, B)$ of $M$ conforms with an almost partial 3-tree if either $(A, B)$ is equivalent to a 3-separation that is displayed by the flower vertex or an edge of $T$, or if $(A, B)$ is equivalent to a 3-separation $(A', B')$ with the property that either $A'$ or $B'$ is contained in a bag of $T$. An almost partial 3-tree for $M$ is a partial 3-tree if every non-sequential 3-separation of $M$ conforms with $T$.

We now define a quasi order on the set of partial 3-trees for $M$. Let $T_1$ and $T_2$ be two partial 3-trees for $M$. Then $T_1 \preceq T_2$ if all of the non-sequential 3-separations displayed by $T_1$ are displayed by $T_2$. If $T_1 \preceq T_2$ and $T_2 \preceq T_1$, then $T_1$ is equivalent to $T_2$. A partial 3-tree is maximal if it is maximal with respect to this quasi order.

The following is the main result, Theorem 9.1, of [23].

**Theorem 1.4.4.** Let $M$ be a 3-connected matroid such that $|E(M)| \geq 9$, and let $T$ be a maximal partial 3-tree for $M$. Then every non-sequential 3-separation of $M$ is equivalent to a 3-separation displayed by $T$.

Regretfully, maximal partial 3-trees are not “unique” enough for our purposes. Certain tight flowers may be displayed in one maximal partial 3-tree for $M$ but not in another. Results of Oxley, Semple, and Whittle in [24] allows us to gain some uniqueness in what is called a 3-tree, in which the underlying tree is unique for a given matroid, up to the existence of certain degree-2 vertices. Call two edges in a maximal partial 3-tree twins if they are incident with a common bag vertex and display equivalent 3-separations.

A maximal partial 3-tree is a 3-tree if

(i) every edge incident with a degree-3 flower vertex displays a non-sequential 3-separation; and
(ii) if two twins, $e$ and $f$, are incident with a bag vertex $v$, then the other ends of $e$ and $f$ are flower vertices of degree at least four, $v$ has degree two, and $v$ labels a non-empty bag.

For a 3-tree $T$, let $R(T)$, the reduction of $T$, be obtained by contracting one edge from each pair of twins. The following combines the main results of [24]:

**Theorem 1.4.5.** If $M$ is a 3-connected matroid with at least nine elements, then $M$ has a 3-tree. Moreover, if $T_1$ and $T_2$ are 3-trees for $M$, then $R(T_1)$ is isomorphic to $R(T_2)$ and there is an isomorphism $\phi$ from $V(R(T_1))$ onto $V(R(T_2))$ such that

(i) $\phi$ maps the vertices of $T_1$ of degree at least three bijectively onto the vertices of $T_2$ of degree at least three so that each flower vertex is mapped to an equivalent one of the same type and each bag vertex is mapped to a bag vertex of the same degree; and

(ii) if $\phi$ maps an edge $u_1v_1$ of $R(T_1)$ to an edge $u_2v_2$ of $R(T_2)$, then the equivalent 3-separations displayed by the one or two edges of $T_1$ corresponding to $u_1v_1$ are equivalent to the 3-separations displayed by the one or two edges of $T_2$ corresponding to $u_2v_2$.

In addition, if $\phi$ maps adjacent flower vertices $u_1$ and $v_1$ of $T_1$ onto non-adjacent vertices $u_2$ and $v_2$ of $T_2$, then every element in the bag vertex $w_2$ of $T_2$ that is adjacent to $u_2$ and $v_2$ is loose in the flower displayed by $u_2$ or in the flower displayed by $v_2$, and is also loose in the flower displayed by $u_1$ or the flower displayed by $v_1$.

To summarize, a 3-connected matroid $M$ has special $\pi$-labeled trees called 3-trees which display all of the non-sequential 3-separations of $M$, up to equivalence, and different 3-trees for $M$ have isomorphic reductions. The uniqueness we gain by considering a 3-tree of $M$ is still not enough for a unique decomposition because
the partition $\pi$ of elements for a 3-tree is not unique. However, in Chapter 3, we will construct the decomposition tree for $M$, which will provide the unique subsets of $E(M)$ that we need for a unique decomposition.
Chapter 2
Decomposition Operations on 3-Connected Matroids

2.1 Overview
In this chapter we will examine how to decompose a 3-connected rank-$r$ matroid $M$ into 3-connected pieces by using a 3-separation $(A, B)$ of $M$. We are motivated by the clique-sum operation in graphs. A graph $G$ is the clique-sum of two graphs $G_1$ and $G_2$ if $G$ is obtained by identifying a common $K_n$-subgraph $N$ of $G_1$ and $G_2$ and then deleting the identified edges. If $G_1$ and $G_2$ are simple and $n$-connected and if $G_1, G_2 \neq N$, then the vertices of $N$ form a vertex cut in $G$. The clique-sum operation is related to the generalized parallel connection by the fact that $N$ is a rank-$(n - 1)$ modular flat of $M(G_1)$ and $M(G_2)$. Hence $M(G)$ is isomorphic to $P_N(M(G_1), M(G_2)) \setminus N$ and has an $n$-separation $(E(G_1) - E(N), E(G_2) - E(N))$. We can recover $G_1$ and $G_2$ from $G$ by adding the edges that were deleted and then by viewing $G_1$ and $G_2$ as subgraphs of the resulting extension. We approach the problem of decomposing a 3-connected matroid into 3-connected pieces with the same strategy.

Since $(A, B)$ is a 3-separation, the rank of $\text{cl}(A) \cap \text{cl}(B)$ is at most 2. Whether or not $\text{cl}(A) \cap \text{cl}(B)$ is truly a rank-2 flat, or line, there is a potential for a rank-2 intersection. We call $\text{cl}(A) \cap \text{cl}(B)$ the line of separation of $(A, B)$ in $M$, and we say that $x$ is in the guts of the separation $(A, B)$ if $x \in \text{cl}(A) \cap \text{cl}(B)$.

If $M$ is representable over a field $F$, then $M$ can be embedded in the projective geometry $PG(r - 1, F)$. Hence, there is a line $L_P$ in the projective geometry that is the intersection of the closures of $A$ and $B$ in the projective geometry. Let $L'$ be a finite subset of $L_P$ and let $M'$ be the restriction of $PG(r - 1, F)$ to $E(M) \cup L'$. If
we choose \( L' \) with enough elements, then \( M' \) is \( F \)-representable, \( M' \) has an exact 3-separation \((A \cup L', B)\), and \( \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \) is a line. However, if \( M \) is not representable, there is no such projective geometry, and hence no embedding. In order to obtain an extension \( M' \) of \( M \) in which \( \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \) is a line, we must add a set \( L' \) of elements to \( M \) by using special modular cuts. In Section 2.2, we prove that elements can be added in the way we desire.

Once we have that the line of separation \( L \) of \((A \cup L', B)\) in \( M' \) is truly a line, in Section 2.3, we prove that we can obtain a decomposition from \( M' \) and \((A \cup L', B)\) in both the representable and non-representable cases:

**Theorem 2.1.1.** Let \( M \) be a 3-connected matroid and let \((A, B)\) be a 3-separation of \( M \). Then there is a matroid \( M' \) that is either \( M \) or an extension of \( M \) by the set \( L' \) with the following properties:

**(i)** \( M' \) has a rank-2 dependent flat \( L \) such that \( L' \subseteq L \).

**(ii)** \( M' \) has an exact 3-separation \((A \cup L', B)\) and \( L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \).

**(iii)** \( M' \), \( M'| (A \cup L) \), and \( M'| (B \cup L) \) are 3-connected.

**(iv)** If \( M \) is representable over a field \( F \), then \( M' \) is representable over \( F \) and \( M' = P_L(M'| (A \cup L), M'| (B \cup L)) \).

For a matroid \( M' \) we obtain by Theorem 2.1.1, we say that we may decompose \( M' \) at the line of separation \( L \) and that the two restrictions \( M'| (A \cup L) \) and \( M'| (B \cup L) \) are the pieces of the decomposition.

Again, let \( M \) be a 3-connected matroid. Let \((A, B)\) be a 3-separation of \( M \) with \( r(B) \geq 3 \), and let \( L = \text{cl}(A) \cap \text{cl}(B) \). If \( L \) is nonempty and \( X \subseteq L \), then \((A \cup X, B - X)\) is a 3-separation of \( M \) and has the same line of separation as \((A \cup L, B - L)\). We can add elements to the line of separation, if necessary, and
then decompose. However, consider the case where \( \text{cl}(A) \cap \text{cl}(B) \) is empty and \( \text{cl}^*(A) \cap \text{cl}^*(B) \) is not empty. Then, we have at least one element in the line of separation of \((A, B)\) in the dual matroid \(M^*\). In this case, it would be natural to consider decomposing across the line of separation in the dual, and in Section 2.4 we will consider this dual operation. However, we will only use lines of separation in our decomposition, so we will also consider an operation that allows us to use two lines of separation instead of duality. Finally, in Section 2.5, we will prove a result that generalizes work of Akkari and Oxley [1] and provides another decomposition.

### 2.2 Adding Points to the Guts of a 3-Separation

Let \( M \) be a 3-connected matroid and let \((A, B)\) be an exact 3-separating partition of \( M \). In this section we will produce an extension \( M' \) of \( M \) such that \( \text{cl}_M'(A) \cap \text{cl}_M'(B) \) has rank 2. There are three cases depending on the representability of \( M \): (i) \( M \) is representable over the finite field \( GF(q) \) for some \( q \), (ii) \( M \) is representable over an infinite field \( F \), or (iii) \( M \) is non-representable.

Suppose \( M \) is representable over \( GF(q) \). Choose a specific representation for \( M \); this representation embeds \( M \) in \( PG(r - 1, q) \). Let \( P = PG(r - 1, q) \). Then \( \text{cl}_P(A) \cap \text{cl}_P(B) \) is a line \( L \) in \( P \) containing \( q + 1 \) points. Let \( L' = L - E(M) \), and we may consider the matroids \( M' = P|(E(M) \cup L) \), \( A' = P|(A \cup L) \), and \( B' = P|(B \cup L) \). Then \( M' \) is a \( GF(q) \)-representable extension of \( M \) and \( M' \) has a decomposition with two pieces \( A' \) and \( B' \). In particular, we have the following:

**Lemma 2.2.1.** Let \( M \) be a 3-connected matroid representable over \( GF(q) \) and let \((A, B)\) be a 3-separation of \( M \). Then there is a \( GF(q) \)-representable matroid \( M' \), which is either \( M \) or an extension of \( M \) by the set \( L' \), with the following properties:

(i) \( M' \) has a \((q + 1)\)-point line \( L \) with \( L' \subseteq L \).

(ii) \((A \cup L', B)\) is a 3-separation of \( M' \) and \( L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \).

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If $M$ is representable over an infinite field $F$, then, by a result of Rado [8] (see Oxley [19, Section 6.8]), $M$ is representable over $GF(p)$, for all sufficiently large primes $p$. Hence we may choose a sufficiently large prime number and apply Lemma 2.2.1. At this stage we should note that because a representable matroid can be represented over different finite fields, our decomposition depends on our choice of a field in which to work. Moreover, because we will not address issues of inequivalent representations, the decomposition also depends on fixing a particular representation for $M$ within this field.

Finally, suppose $M$ is non-representable. In this case, we no longer have the ability to view $M$ as a restriction of a projective space. This fact means that producing extensions of $M$ with certain properties can be difficult. In particular, we need to use modular cuts. We want to break $M$ apart along a line that corresponds to the intersection of the closures of $A$ and $B$. However, in $M$, this intersection could be empty or consist of just a single element. Using modular cuts, we shall initially add a set $X$ of two elements to $M$ to produce a matroid $M'$ such that $r_{M'}(A \cup X) = r_M(A)$, $r_{M'}(B \cup X) = r_M(B)$, and $cl_{M'}(A) \cap cl_{M'}(B)$ is a line spanned by $X$. Once we have produced this line, then we can add arbitrarily many points to it. Note that if a matroid is representable over a field $F$, then we can still use this same procedure, but, in the process of doing so, we may not preserve $F$-representability nor be able to reverse the decomposition.

Further discussion will require some preliminary definitions. Elements $x$ and $y$ in a matroid $M$ are **clones** if the bijection on $E(M)$ that interchanges $x$ and $y$ and fixes every other element is an isomorphism. We say that the clones $e$ and $e'$ are **independent** if $\{e, e'\}$ is independent. Let $F$ be a flat of $M$, let $e \in F$, and let $M_F$ be the extension of $M$ obtained by adding the point $e'$ via the principal modular cut generated by $F$. We say that $e$ is **free** in $F$ if $e$ and $e'$ are clones in $M_F$. In
general, an element \( x \) of \( M \) is \textit{fixed} in \( M \) if there is no single-element extension \( M' \) of \( M \) by \( x' \) such that \( x \) and \( x' \) are independent clones.

At this point, let us fix a 3-connected matroid \( M \) and a 3-separation \((A, B)\) of \( M \). We will next show how to extend \( M \) to a matroid \( M' \) by adding a set \( L' \) of elements that are free in the line of separation \( L = \text{cl}_{M'}(A \cup L') \cap \text{cl}_{M'}(B) \). In the end, we will have a decomposition of \( M' \) into two pieces \( A' \) and \( B' \), where \( A' = M'|(A \cup L') \) and \( B' = M'|(B \cup L') \), as in the representable case.

**Proposition 2.2.2.** Let \( M \) be a 3-connected matroid and \((A, B)\) be a 3-separation of \( M \). Then there is an extension \( M' \) of \( M \) by \( L' \) with \( |L'| \geq 1 \) having the following properties:

(i) \( L' \subseteq \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \).

(ii) \((A \cup L', B)\) is a 3-separation of \( M' \).

(iii) Let \( L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B) \). If \( |L| \geq 2 \), then the members of \( L' \) are free in \( L \).

We will prove this proposition by showing that a special modular cut lets us add elements to \( \text{cl}(A) \cap \text{cl}(B) \) and that these elements have the desired properties. We say that elements added to \( M \) in this way have been \textit{added freely to the line of separation} of \((A, B)\). Geelen, Gerards, and Whittle [10] first proved the existence of such a modular cut, and hence of such an extension of \( M \). We include the following lemma, which follows from [10, Lemma 6.3].

**Lemma 2.2.3.** Let \( N \) be a matroid, let \((C, D)\) be a partition of \( E(N) \), and let \( \mathcal{F} \) be the collection of all flats \( F \) of \( N \) such that \( r_{N/F}(C - F) + r_{N/F}(D - F) = r(N/F) \). Then \( \mathcal{F} \) is a modular cut of \( N \).

We say that a flat \( F \) of \( M \) \textit{spans the guts} of the 3-separation \((A, B)\) if \( r_{M/F}(A - F) + r_{M/F}(B - F) = r(M/F) \); that is, \( M/F \) is the direct sum of the elements left
in $A$ and the elements left in $B$ after contracting $F$. Thus, Lemma 2.2.3 says that set of flats that span the guts of $(A, B)$ is a modular cut. Let $M$ be this modular cut and let $M_1 = M +_M e$. In the next lemma we show that any member of $M$ has rank at least two and contains $\text{cl}_M(A) \cap \text{cl}_M(B)$. After showing that $e$ is in $\text{cl}_{M_1}(A) \cap \text{cl}_{M_1}(B)$, it follows that $M_1$ is 3-connected. We then show that $e$ is not fixed and that the element $f$ added to $M_1$ by the modular cut of flats that span the guts of the 3-separation $(A \cup e, B)$ of $M_1$ is an independent clone of $e$. We collect these results in Proposition 2.2.8, from which Proposition 2.2.2 follows easily. The proofs of some of the following lemmas use terminology and results on freedom in matroids from [9] and [11], except that we call a flat that is a union of circuits a cyclic flat instead of a fully dependent flat.

**Lemma 2.2.4.** If $F \in \mathcal{M}$, then $r(F) \geq 2$ and $\text{cl}_M(A) \cap \text{cl}_M(B) \subseteq F$.

**Proof.** Let $F$ be a flat of $M$ in $\mathcal{M}$. Then,

$$0 = r_{M/F}(A - F) + r_{M/F}(B - F) - r(M/F) = r_M(A \cup F) + r_M(B \cup F) - r(M) - r(F).$$

Thus $r(M) + r(F) = r(A \cup F) + r(B \cup F) \geq r(A) + r(B) = r(M) + 2$. Therefore $r(F) \geq 2$. Suppose $f \in (\text{cl}_M(A) \cap \text{cl}_M(B)) - F$. Then $f \in \text{cl}_{M/F}(A - F) \cap \text{cl}_{M/F}(B - F)$ and $r_{M/F}(f) = 0$. Hence $f \in \text{cl}_M(F) = F$, a contradiction. Therefore, $\text{cl}_M(A) \cap \text{cl}_M(B) \subseteq F$. \qed

**Lemma 2.2.5.** The added element $e$ is in $\text{cl}_{M_1}(A) \cap \text{cl}_{M_1}(B)$ and $M_1$ is 3-connected.

**Proof.** To show that $M_1$ is 3-connected, by [19, Proposition 8.1.10], we need only show that $e$ was not added as a loop or a coloop, nor was added in parallel to another element. Since each flat $F$ in $\mathcal{M}$ has rank at least 2, $e$ was not added as a loop or in parallel with another element. Since $\text{cl}(A)$ and $\text{cl}(B)$ span the
guts, we have that \( e \in \text{cl}_{M_1}(A) \) and \( e \in \text{cl}_{M_1}(B) \). Then \( e \) is not a coloop because
\[ e \in \text{cl}_{M_1}(A) \cap \text{cl}_{M_1}(B). \]
Therefore, \( M_1 \) is 3-connected.

The next lemma will show that \( e \) can be independently cloned. In \( M_1 \), we have the 3-separation \((A \cup e, B)\). Let \( M_1 \) be the modular cut of flats that span the guts of \((A \cup e, B)\) and let \( M_2 = M_1 +_{M_1} f \). Then, we will show that \( e \) and \( f \) are independent clones. In other words, once we add one element to the guts of \((A, B)\) by \( M \), adding other elements to the guts is the same as independently cloning \( e \).

**Lemma 2.2.6.** The added element \( e \) is not fixed in \( M_1 \).

**Proof.** By [9, Corollary 3.5] (see [11, Proposition 4.1]), we need only show that \( \text{cl}(e) \) is not in the modular cut generated by the cyclic flats of \( M_1 \) containing \( e \). If \( F \) is a cyclic flat of \( M_1 \) containing \( e \), then \( r_{M_1}((F-e) \cup e) = r_{M_1}(F-e) \). Thus \( F-e \in \mathcal{M} \), and so \( F-e \) spans the guts of \((A, B)\). Let \( \mathcal{F} = \{F-e \mid F \text{ is a cyclic flat of } M_1 \text{ containing } e\} \). Then \( \mathcal{F} \subseteq \mathcal{M} \), and so the modular cut generated by \( \mathcal{F} \) is contained in \( \mathcal{M} \). Suppose \( \text{cl}(e) \) is in the modular cut generated by the cyclic flats of \( M_1 \) containing \( e \). Then \( \text{cl}(e) - e \in \mathcal{M} \). By Lemma 2.2.4, \( r(\text{cl}(e) - e) \geq 2 \), a contradiction.

**Lemma 2.2.7.** The pair \( \{e, f\} \) is a set of independent clones in \( M_2 \).

**Proof.** Since \( \{e\} \) does not span the guts, \( \{e, f\} \) is independent. By [11, Proposition 4.9], we need only show that a cyclic flat \( F \) of \( M_2 \) contains \( f \) if and only if \( F \) contains \( e \). Let \( F \) be a cyclic flat of \( M_2 \) that contains \( f \). Then \( r(F-f) = r(F) \), and so \( F-f \in \mathcal{M}_1 \). Thus \( F-f \) spans the guts of \((A \cup e, B)\). By Lemma 2.2.4, \( e \in \text{cl}_{M_1}(A \cup e) \cap \text{cl}_{M_1}(B) \), and so \( e \in F-f \). Hence \( e \in F \). Let \( F \) be a cyclic flat of \( M_2 \) that contains \( e \); then, \( r_{M_2}(F-e) = r_{M_2}(F) \). Suppose \( f \notin F \); then, \( F \) does not span the guts of \((A \cup e, B)\). However, \( F \) is a cyclic flat of \( M_1 \), and
so $F - e$ is a flat of $M$ that spans the guts of $(A, B)$. Now $e$ is in $\text{cl}_{M^1}(F - e)$ and, by Lemma 2.2.5, in $\text{cl}_{M^1}(A) \cap \text{cl}_{M^1}(B)$. Since $r_{M/(F-e)}(e) = 0$, we have that $r_{M/(F-e)}(F - e) = r_{M/F}(F)$, that $r_{M/(F-e)}((A \cup e) - (F)) = r_{M/F}(A - F)$, and that $r_{M/(F-e)}(B - (F - e)) = r_{M/F}(B - F)$. Hence $F$ spans the guts of $(A \cup e, B)$, a contradiction. Thus $f \in F$. Therefore, $F$ is a cyclic flat of $M_2$ containing $e$ if and only if $F$ also contains $f$. 

**Proposition 2.2.8.** Let $M$ be a 3-connected matroid and let $(A, B)$ be a 3-separation of $M$. Let $M'$ be obtained from $M$ by adding $e$ via the modular cut of flats that span the guts of $(A, B)$. Let $M''$ be obtained from $M'$ by independently cloning $e$. Then $M'$ and $M''$ are 3-connected and $(A \cup \{e, e'\}, B)$ is an exact 3-separation of $M''$. Moreover, the line of separation $\text{cl}_{M''}(A \cup \{e, e'\}) \cap \text{cl}_{M''}(B)$ is a rank-2 flat $L$ of $M''$ spanned by $\{e, e'\}$ and $e$ and $e'$ are free in $L$.

**Proof.** By Lemma 2.2.5, $M'$ is 3-connected. Since $\{e, e'\}$ is a set of independent clones, $M''$ is 3-connected. Let $L = \text{cl}_{M''}(A \cup \{e, e'\}) \cap \text{cl}_{M''}(B)$. By construction, $\{e, e'\} \subseteq L$ and $L$ is a line of separation. Since $\{e, e'\}$ is independent, we have that $L$ is spanned by $\{e, e'\}$. Since $L$ spans the guts of $(A \cup \{e, e'\}, B)$, the modular cut of flats that span the guts is equal to the principal modular cut of $M''$ generated by $L$. Thus cloning $e$ is the same as freely adding to line of separation of $(A \cup \{e, e'\}, B)$. Hence, $e$ is free in $L$, and, as $e$ and $e'$ are clones, $e'$ is free in $L$. 

**2.3 Decomposing with a Line of Separation**

Let $M$ be a 3-connected matroid and let $(A, B)$ be a 3-separation of $M$. We have defined two different ways to obtain an extension $M'$ of $M$ such that $L' = E(M') - E(M)$ and $(A \cup L', B)$ is an exact 3-separation of $M'$. Also, in each case, $M'$ has a line $L$ and two restrictions $A'$ and $B'$ of $M'$ defined by $A' = E(M)|(A \cup L)$, $B' = E(M)|(B \cup L)$. These extensions are what are needed to prove Theorem 2.1.1.
If $M$ is not representable, then, by Proposition 2.2.2, we can add elements freely to the line of separation of $(A, B)$ via the modular cut of flats that span the guts. If $M$ is representable, then $M$ is representable over a finite field $GF(q)$. When we view $M$ as a restriction of the projective geometry $PG(r - 1, q)$, we can add up to $q + 1$ points to the line of separation, by Lemma 2.2.1. We say that the extension $M'$ of $M$ is obtained by completing the line of separation of $(A, B)$ in either of two cases: (i) if $M$ is non-representable and we add three points freely to the line of separation; or (ii) if $M$ is $GF(q)$-represented and we add up to $q + 1$ points to the line of separation. Moreover, we say that $L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$ is the completed line of separation.

In this section, we will first prove a useful lemma and then Theorem 2.1.1. After that, we will show that if $(A, B)$ is non-sequential, then when we add elements to the line of separation, the sequential and non-sequential 3-separations of $M'$ come from sequential and non-sequential 3-separations of $M$, respectively. Finally, while not all 3-separations of $M$ are preserved in $M'$, we will examine some important 3-separations that are preserved in $M'$.

**Lemma 2.3.1.** Let $M$ be a 3-connected matroid and let $(A, B)$ be a 3-separation of $M$. Suppose $M'$ is an extension of $M$ by a set $L'$ of elements such that $L' \subseteq \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$ and the restriction $L$ defined by $L = M|_{(\text{cl}_{M'}(A) \cap \text{cl}_{M'}(B))}$ is a simple matroid.

(i) $M'$ is 3-connected and $(A \cup L', B)$ is a 3-separation of $M'$.

(ii) If $r(A) \geq 3$, then $(A - L, B \cup L)$ is a 3-separation of $M$ and

$L = \text{cl}_{M'}(A - L) \cap \text{cl}_{M'}(B)$.

(iii) If $r(A), r(B) \geq 3$, then $L = \text{cl}_{M'}(A - L) \cap \text{cl}_{M'}(B - L)$. 
Proof. We prove that $M'$ is 3-connected by induction on $|L'|$. Note that, since $(A,B)$ is a 3-separation, $r(L) \leq 2$. Suppose $|L'| = 1$. Then $M'$ is a single-element extension of $M$ by an element $e$. By [19, Proposition 8.1.10], we need only show that $e$ is not a loop, not a coloop, and in a trivial parallel class. Since $e \in \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$, $e$ is not a coloop. Since $L$ is simple, $e$ is neither a loop nor in a non-trivial parallel class. Suppose that $M'$ is an extension of $M$ with $|L'| > k$ and that the lemma is true for $|L'| \leq k$. Let $e \in L'$ and $M^− = M'|(E(M') − e)$. Then $M^−$ is 3-connected by the induction hypothesis. Moreover, $e$ is not a loop, is not a coloop, and is in a trivial parallel class. Therefore, $M'$ is 3-connected.

Since $r(A) = r(A \cup L')$, we have that $(A \cup L', B)$ is a 3-separation of $M'$. Suppose $r(A) \geq 3$, and let $e \in \text{cl}_M(A) \cap \text{cl}_M(B)$. Then $(A − e, B \cup e)$ is 3-separating, and so $r(A − e) = r(A)$. Hence $r(A) \geq 3$ and

$$r_M(A − L) + r_M(B \cup L) − r(M) = r_M(A) + r_M(B) − r(M) = 2.$$ 

Thus $(A − L, B \cup (L − L'))$ is a 3-separation in $M$. We have that $r(M) = r(M')$, that $r_M(A) = r_M(A − L) = r_M'(A − L)$, and that $r_M(B) = r_M'(B \cup (L − L')) = r_M'(B \cup L)$. Thus $(A − L, B \cup L)$ is a 3-separation in $M'$. If $r(B) \geq 3$, then, similarly, $r_M(B) = r_M'(B − L)$.

Let $e \in L$. Since $r_M'(A − L) − r_M(A)$ and $r_M'(B) = r_M(B)$, we deduce that $e \in \text{cl}_{M'}(A − L) \cap \text{cl}_{M'}(B)$. If $r(B) \geq 3$, then $r_M(B) = r_M'(B − L) = r_M'(B − L)$, and so $e \in \text{cl}_{M'}(A − L) \cap \text{cl}_{M'}(B − L)$. \hfill \qed

Proof of Theorem 2.1.1. If $M$ is not representable, then let $M'$ be an extension of $M$ from Proposition 2.2.2 with $|L'| \geq 3$. If $M$ is representable over $GF(q)$, then let $M'$ be an extension of $M$ from Lemma 2.2.1. In each case, let $L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$ and let $L' = E(M') − E(M)$. Let $A' = M'|(A \cup L)$, and let $B' = M'|(B \cup L)$.
First, by Lemma 2.3.1, $M'$ is 3-connected. Next we show that $A'$ and $B'$ are 3-connected. It suffices to prove the former. Suppose $A'$ has a 1- or 2-separation $(X,Y)$. Since $|L| \geq 3$, $L$ is spanned by at least one of $X$ or $Y$. Without loss of generality, we may assume $L \subseteq \text{cl}_{M'}(Y)$. Consider $(X,Y \cup (B' - X))$, a partition of $M'$. We obtain the following contradiction:

\[
2 \leq r_{M'}(X) + r_{M'}(Y \cup (B' - X)) - r(M') \\
\leq r_{M'}(X) + r_{M'}(Y) + r_{M'}(B') - r_{M'}(L) - r(M') \\
= r_{M'}(X) + r_{M'}(Y) - r_{M'}(A') = r_{A'}(X) + r_{A'}(Y) - r_{A'}(A') \\
< 2.
\]

Therefore, $A'$ is 3-connected.

Finally, if $M$ is representable over $GF(q)$, then we prove that $M'$ is the generalized parallel connection of $A'$ and $B'$. We may use the fact that a $(q+1)$-point line is modular in a simple $GF(q)$-representable matroid (see [19, Proposition 6.9.1]) and hence is modular in $A'$. Then, by Proposition 1.2.3, we need to show that $M'/L = A'/L \oplus B'/L$. Since $(A \cup L, B - L)$ is exactly 3-separating in $M'$, we know that $r(M') + 2 = r(A \cup L) + r(B - L)$. Thus $r(M'/L) = r(A'/L) + r(B'/L)$; that is, $M'/L$ is the direct sum of $A'/L$ and $B'/L$, as required.

Next, as we mentioned earlier in the section, we will prove some results on how the 3-separations of $M$, $M'$, $A'$, and $B'$ are related. The following three lemmas will be proved for an arbitrary 3-connected matroid $N$, and then we will return to our fixed matroid $M$.

**Lemma 2.3.2.** Let $(C,D)$ be a non-sequential 3-separation of a 3-connected matroid $N$. Let $N'$ be a simple single-element extension of $N$ by an element $e$ such
that \( e \in \text{cl}_{N'}(C) \cap \text{cl}_{N'}(D) \). If \((X, Y)\) is a non-sequential 3-separation of \(N'\) and \(e \in X\), then \((X - e, Y)\) is a non-sequential 3-separation of \(N\).

**Proof.** As \((X, Y)\) is non-sequential, \(|X|, |Y| \geq 4\) by [23, Lemma 3.4]. Thus \(|X - e| \geq 3\). By Lemma 1.3.1,

\[
2 = r_{N'}(X) + r_{N'}(Y) - r(N') \geq r_N(X - e) + r_N(Y) - r(N) \geq 2.
\]

Hence \((X - e, Y)\) is an exact 3-separation of \(N\). Since \(r(N) = r(N')\) and \(r_N(Y) = r_{N'}(Y)\), it follows that \(r_N(X - e) = r_{N'}(X - e) = r_N(X)\); that is, \(e \in \text{cl}_{N'}(X - e)\). Suppose \((X - e, Y)\) is sequential in \(N\). Then either (i) \(fcl_N(X - e) = E(N)\) or (ii) \(fcl_N(Y) = E(N)\). Recall that we write \(\overrightarrow{Z}\) for an ordering \((z_1, z_2, z_3, \ldots, z_n)\) of the elements in a set \(Z\) and we write \(\overrightarrow{Z}_i\) for the set consisting of the first \(i\) elements in such an ordered set.

In case (i), we have an ordering \((y_1, y_2, \ldots, y_n)\) of the set \(Y\) such that \((X - e) \cup \overrightarrow{Y}_i\) is 3-separating for all \(i \in \{1, 2, \ldots, n\}\). Then, for each \(i\), either \(y_{i+1} \in \text{cl}_N((X - e) \cup \overrightarrow{Y}_i)\) or \(y_{i+1} \in \text{cl}^*_N((X - e) \cup \overrightarrow{Y}_i)\). Now \(e \in \text{cl}_{N'}(X)\). Hence, if \(y_{i+1} \in \text{cl}_N((X - e) \cup \overrightarrow{Y}_i)\), then \(y_{i+1} \in \text{cl}_{N'}(X \cup \overrightarrow{Y}_i)\). On the other hand, note that \(N^* = (N' \setminus e)^* = (N')^*/e\).

If \(y_{i+1} \in \text{cl}^*_N((X - e) \cup \overrightarrow{Y}_i)\), then \(y_{i+1} \in \text{cl}^*_N(X \cup \overrightarrow{Y}_i)\). We conclude that \((X, Y)\) is a sequential 3-separation of \(N'\), a contradiction.

Consider case (ii). Then there is an ordering \(\overrightarrow{X} = (x_1, x_2, \ldots, x_m)\) of \(X - e\) such that, for all \(i\), either \(x_{i+1} \in \text{cl}_N(Y \cup \overrightarrow{X}_i)\) or \(x_{i+1} \in \text{cl}^*_N(Y \cup \overrightarrow{X}_i)\). Suppose \(e \in \text{cl}_{N'}(Y)\). Then either \(x_{i+1} \in \text{cl}_{N'}(Y \cup e \cup \overrightarrow{X}_i)\) or \(x_{i+1} \in \text{cl}^*_N(Y \cup e \cup \overrightarrow{X}_i)\), as above. So if \(e \in \text{cl}_{N'}(Y)\), then \((Y, e, x_1, x_2, \ldots, x_m)\) is a sequence showing that \((Y, X)\) is sequential, a contradiction. Hence we may assume that \(e \notin \text{cl}_{N'}(Y)\), and so \(\text{cl}_{N'}(Y) = \text{cl}_N(Y)\).

Now, by Lemma 1.3.2, \(\overrightarrow{X}\) can be chosen so that \(\text{cl}_{N'}(Y) = Y \cup \overrightarrow{X}_i\) for some \(i \in \{1, 2, \ldots, m\}\). Then \((Y, X)\) is equivalent to \((\text{cl}_{N'}(Y), X - \text{cl}_{N'}(Y))\); that is,
Thus we may assume that $Y$ is closed in $N'$ with $e \in X$, $e \in \text{cl}_{N'}(X - e)$, and $e \notin \text{cl}_{N'}(Y)$. We also know that $e \in \text{cl}_{N'}(C)$ and $e \in \text{cl}_{N'}(D)$. Now $(X - e) \cap C \neq \emptyset$; otherwise, $e \in \text{cl}_{N'}(Y \cap C) = \text{cl}_{N'}(C) \subseteq \text{cl}_{N'}(Y)$. Similarly, $(X - e) \cap D \neq \emptyset$.

If $|D \cap Y| = 0$, then, as $|Y| \geq 4$ and $Y$ is 3-separating, we may use Lemma 1.3.2 to reorder $\overrightarrow{X}$ so that the members of $C \cap (X - e)$ come first in the order and the members of $B \cap (X - e)$ come last. Then we have that $D$ is sequential, a contradiction. Therefore, $|D \cap Y| \geq 1$ and, similarly, $|C \cap Y| \geq 1$.

If $|D \cap Y| = 1$, then let $D \cap Y = \{d\}$. Thus $|D \cap (X - e)| \geq 2$, and so $Y \cup C$ is 3-separating; that is, $(C \cup d, D - d)$ is 3-separating. Since we have the 3-sequence $(Y, x_1, x_2, \ldots, x_m)$, that $|C \cap Y| \geq 2$, and that $D - d \subseteq X - e$, we deduce that $(D - d) \cap \{x_j, \ldots, x_m\}$ is 3-separating for all $j \in \{1, 2, \ldots, m\}$, by Lemma 1.3.2. Thus, we have the 3-sequence $(C, d, x_{m_1}, x_{m_2}, \ldots, x_{m_k})$, where $\{x_{m_1}, x_{m_2}, \ldots, x_{m_k}\}$ is the set $D \cap \{x_1, x_2, \ldots, x_m\}$ with the ordering induced by $\overrightarrow{X}$. Thus, $D$ is sequential, a contradiction. Hence $|D \cap Y| \geq 2$, and, similarly, $|C \cap Y| \geq 2$.

We will now show that $|C \cap (X - e)| \geq 2$. Suppose $C \cap (X - e) = \{c\}$. We know that $e \notin \text{cl}_N(Y)$. Since $C - c \subseteq Y$ and $e \in \text{cl}_N(C)$, we deduce that $c \notin \text{cl}_N(Y)$, and so $c \notin \text{cl}_N(C - c)$. By Lemma 1.3.2, $(C, D)$ and $(C - c, D \cup c)$ are 3-separating in $N$, and so $c \in \text{cl}_{N'}(D) \cap \text{cl}_{N'}(C - c)$. Recall that, for some ordering $x_1, x_2, \ldots, x_m$ of $X - e$, we have $(Y, x_1, x_2, \ldots, x_m)$ is a 3-sequence for $N$. Let $c = x_k$. By Lemma 1.3.2, $D \cap \{x_j, \ldots, x_m\}$ is 3-separating for all $j \in \{1, 2, \ldots, m\}$. Thus $(Y, c, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$ is a sequential ordering of $N$. Since $c \notin \text{cl}_N(Y)$, it follows that $c \in \text{cl}_{N'}(Y) \cap \text{cl}_{N'}(X - \{e, c\})$. Suppose that $c \in \text{cl}_{(N')}^*(Y)$. Then $(Y \cup c, X - c)$ is 3-separating in $N'$. Hence, as $e \in \text{cl}(C) \subseteq \text{cl}(Y \cup c)$, it follows that $(Y, c, e, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$ is a 3-sequence in $N'$. Thus we have the contradiction that $X$ is sequential. Now we may assume that $c \notin \text{cl}_{(N')}^*(Y)$. Thus, $c \in \text{cl}_{N'}(X - c)$.
$\text{cl}_{N'}(Y)$, and so $c \in \text{cl}_{(N')^*}(Y)$. Hence $c \in \text{cl}_{(N')^*}(Y \cup e)$. By the Mac Lane-Steinitz exchange property, $e \in \text{cl}_{(N')^*}(Y \cup c)$. By Lemma 1.3.2, $Y \cup C$, which equals $Y \cup c$, is a 3-separation in $N'$. Thus $(Y \cup c, e, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$ is a 3-sequence $N'$. Since $X$ is 3-separating, we have that $(Y, c, e, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$ is 3-sequence in $N'$, and this 3-sequence shows that $X$ is sequential, a contradiction.

Thus $|C \cap (X - e)| \geq 2$, and, similarly, $|D \cap (X - e)| \geq 2$.

Now, by Lemma 1.3.2, $C \cap Y, D \cap Y, C \cap X, (C \cap X) \cup e, D \cap X,$ and $(D \cap X) \cup e$ are 3-separating in $N'$. Thus $(C \cup Y \cup e, D \cap X)$ and $(C \cup Y, (D \cap X) \cup e)$ are 3-separating. Hence $e$ is in exactly one of $e \in \text{cl}_{N'}(C \cup Y) \cap \text{cl}_{N'}(D \cap X)$ and $\text{cl}_{N'}(C \cup Y) \cap \text{cl}_{N'}(D \cap X)$. However, $e \in \text{cl}_{N'}(C)$, and so $e \in \text{cl}_{N'}(D \cap X)$ and $e \notin \text{cl}_{N'}(D \cap X)$. By symmetry, $e \in \text{cl}_{N'}(C \cap X)$. As $(Y, x_1, x_2, \ldots, x_m)$ is a 3-sequence in $N$, we have, by Lemma 1.3.2, that $D \cap \{x_j, \ldots, x_m\}$ is 3-separating in $N$ for all $j$. Moreover, $Y \cup (C \cap \{x_1, \ldots, x_j\})$ is 3-separating in $N$ for all $j$.

Therefore, we can reorder the 3-sequence $(Y, x_1, x_2, \ldots, x_m)$ to

$$(Y, [(x_1, x_2, \ldots, x_m) - \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}], x_{i_1}, x_{i_2}, \ldots, x_{i_k}),$$

where $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ is the ordering induced on $D \cap \{x_1, x_2, \ldots, x_m\}$ by $X$. Because $e \in \text{cl}_{N'}(C \cap X)$, there is no cocircuit that contains $e$ and is contained in $(D \cap X) \cup e$. Now, $x_{i_t} \in \text{cl}_{N'}(\{x_{i_{t+1}}, \ldots, x_{i_k}\})$ or $x_{i_t} \in \text{cl}_{N'}(\{x_{i_{t+1}}, \ldots, x_{i_k}\})$. In the first case, $x_{i_t} \in \text{cl}_{(N')^*}(\{x_{i_{t+1}}, \ldots, x_{i_k}\})$. In the second case, $x_{i_t} \in \text{cl}_{(N')^*}(\{x_{i_{t+1}}, \ldots, x_{i_k}\} \cup e)$. But the cocircuit containing $x_{i_t}$ in $(D \cap X) \cup e$ does not contain $e$, and so $x_{i_t} \in \text{cl}_{(N')^*}(\{x_{i_{t+1}}, \ldots, x_{i_k}\})$. Hence, as $e \in \text{cl}(C \cap X) \cap \text{cl}(D \cap X)$,

$$(Y, [(x_1, x_2, \ldots, x_m) - \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}], e, x_{i_1}, \ldots, x_{i_k})$$

is a 3-sequence in $N'$. Therefore, $X$ is sequential in $N'$, a contradiction. \qed
Lemma 2.3.3. Let $(C, D)$ be a 3-separation of a 3-connected matroid $N$. Let $N'$ be a simple single-element extension of $N$ by an element $e$ such that $e \in \text{cl}_{N'}(C) \cap \text{cl}_{N'}(D)$. If $(X, Y)$ is a sequential 3-separation of $N'$ and $e \in X$, then $(X - e, Y)$ is a sequential 3-separation of $N$ or $|X - e| = 2$.

Proof. Suppose $Y$ is sequential and $(X, y_1, y_2, \ldots, y_n)$ is a 3-sequence. Then, since $r_N((X - e) \cup \overrightarrow{Y}_i) \leq r_{N'}(X \cup \overrightarrow{Y}_i)$, we have $r_M((N - e) \cup \overrightarrow{Y}_i) + r_N(\overrightarrow{Y}_i) - r(N) \leq 2$. Since $N$ is 3-connected, equality must hold. Now suppose that $X$ is sequential and $(Y, x_1, x_2, \ldots, x_m)$ is a 3-sequence. Then $e = x_j$ for some $j$. Since $r_N((Y \cup \overrightarrow{X}_i) - x_j) \leq r_{N'}(Y \cup \overrightarrow{X}_i)$, we have $r_N((Y \cup \overrightarrow{X}_i) - e) + r_N(\overrightarrow{X}_i - e) - r(N) \leq 2$. Since $N$ is 3-connected, equality must hold. Therefore, $X$ is sequential in $N$. \[\square\]

Lemma 2.3.4. Let $(C, x_1, x_2, \ldots, x_n, D)$ be a 3-sequence in a 3-connected matroid $N$ and $\overrightarrow{X} = (x_1, x_2, \ldots, x_n)$. If $N \setminus D$ is 3-connected, then $C \cup \overrightarrow{X}_i$ is 3-separating in $N \setminus D$ for all $i$.

Proof. By Lemma 1.3.1,

$$2 = r_N(C \cup \overrightarrow{X}_i) + r_N(D \cup \overrightarrow{X}_i) - r(N) \geq r_{N \setminus D}(C \cup \overrightarrow{X}_i) + r_{N \setminus D}(\overrightarrow{X}_i) - r(N \setminus D).$$

We deduce that $C \cup \overrightarrow{X}_i$ is 3-separating in $N \setminus D$. \[\square\]

We have already fixed $M$ and $(A, B)$. For the rest of the results in this section, we also fix an extension $M'$ of $M$ satisfying the conclusion of Theorem 2.1.1 with $L' = E(M') - E(M)$, $L = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$, $A' = M'|(A \cup L)$, and $B' = M'|(B \cup L)$.

Lemma 2.3.5. Suppose $(A, B)$ is non-sequential. If $(C, D)$ is a 3-separation of $M'$ and $(C - L', D - L')$ is a 3-separation of $M$, then $(C, D)$ is sequential in $M'$ if and only if $(C - L', D - L')$ is sequential in $M$. 

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Proof. Let \( k = |L'| \), let \( M_0 = M \), and let \( M_k = M' \). Choose an order \((l_1, l_2, \ldots, l_k)\) on \( L' \) and let \( M_i \) be the restriction of \( M' \) to \( E(M) \cup \overrightarrow{L}_i \). The result is clearly true if \( M' = M \) and is true for \( M_1 \) by Lemmas 2.3.2 and 2.3.3. Moreover, if the proposition is true for \( M_i \), then it is true for \( M_{i+1} \) by Lemmas 2.3.2 and 2.3.3. The result then follows by induction on \( k \).

\[
\text{Lemma 2.3.6. Let } (C, D) \text{ be exactly } 3\text{-separating in } M' \text{ with } C \subseteq A. \text{ If } r(C) \geq 3, \text{ then } (C - L, D \cup L) \text{ is an exactly } 3\text{-separating partition of } M' \text{ and } (C - L, (D \cap A) \cup L) \text{ is an exactly } 3\text{-separating partition of } A'.
\]

Proof. First, note that \( L \subseteq \text{cl}_{M'}(B) \subseteq \text{cl}_M(D) \) and

\[
r_{M'}(C - L) + r_{M'}(D \cup L) = r_M(C - L) + r_M(D) = r(M) + 2 = r(M') + 2.
\]

Since \( r(C) \geq 3 \), we have \( |C - L| \geq 1 \). If \( |C - L| = 1 \), then \( \text{cl}_M(D) \) is a hyperplane of \( M \), and so the element of \( C - L \) is a coloop of \( M \), a contradiction. Hence \( |C - L| \geq 2 \), and so \((C - L, D \cup L)\) is exactly 3-separating in \( M' \). We have that \((C - L, (D \cap A) \cup L)\) is exactly 3-separating in \( A' \) because

\[
2 = r_{M'}(C - L) + r_{M'}(D \cup L) - r(M') \geq r_{A'}(C - L) + r_{A'}((D \cap A) \cup L) - r(A') \geq 2,
\]

where the first inequality follows by Lemma 1.3.1.

\[
\text{Lemma 2.3.7. If } (C, D) \text{ is exactly } 3\text{-separating in } A' \text{ with } L \subseteq \text{cl}_{A'}(D), \text{ then } (C, D \cup (B - L)) \text{ is exactly } 3\text{-separating in } M'.
\]

Proof. We know that \( r(A') + r(B') - r(M') = 2 \), that \( r(C) + r(D) - r(A') = 2 \) and that \( r(C) + r(B \cup L) - r(C \cup B \cup L) \leq r(A') + r(B') - r(M') = 2 \). But, as \( r((B \cup L) \cap \text{cl}_{A'}(C)) = r(L) = 2 \), we have \( r(C) + r(B') - r(C \cup B \cup L) = 2 \). Hence \( r(A') + r(B') - r(M') = r(C) + r(B') - r(C \cup B \cup L) \), and so \( r(A') + r(C \cup B \cup L) - r(M') = r(C) \). Thus \( r(C) + r(D) - 2 + r(C \cup B \cup L) - r(M') = r(C) \),
and so \( r(D) + r(C \cup B \cup L) - r(M') = 2 \). Then, since \( \text{cl}_{M'}(C) \subseteq L \), we have that \( r(D) + r(C \cup (B - L)) - r(M') = 2 \).

**Lemma 2.3.8.** If \((C, D)\) is a non-sequential 3-separation of \( A' \) with \( L \subseteq \text{cl}_{A'}(D) \), then \((C, D \cup (B - L))\) is a non-sequential 3-separation in \( M' \).

**Proof.** By Lemma 2.3.7, \((C, D \cup (B - L))\) is a 3-separation in \( M' \). Suppose that \((C, D \cup (B - L))\) is sequential in \( M' \). If \( D \cup (B - L) \) is sequential with 3-sequence \((C, d_1, d_2, \ldots, d_n)\), then, by Lemma 1.3.2, we can move the elements of \( B - L \) to the end of the sequence. Then, by Lemma 2.3.4, \((C, D)\) is sequential. If \( C \) is sequential in \( M' \) and \( \overrightarrow{C} \) is a sequential ordering of \( C \), then \((B - L) \cup D \cup \overrightarrow{C}_i\) is 3-separating, for all \( i \), and, as \( A' \) is 3-connected,

\[
2 = r((B - L) \cup D \cup \overrightarrow{C}_i) + r(\overrightarrow{C}_i^C) - r(M) \geq r(D \cup \overrightarrow{C}_i) + r(\overrightarrow{C}_i^C) - r(A').
\]

Thus \( C \) is sequential in \( A' \), a contradiction. Hence \((C, D \cup (B - L))\) is non-sequential in \( M' \).

### 2.4 Multiple Lines of Separation

Let \( M \) be a 3-connected matroid and let \((A, B)\) be a 3-separation of \( M \). Suppose that there is an element \( x \) in \( B \) such that \((A \cup x, B - x)\) is also a 3-separation. By Lemma 1.3.3, \( x \) is in the closure or coclosure of both \( A \) and \( B - x \). If \( x \) is in the closure of both, then \( x \) is in the line of separation of \((A, B)\). Moreover, if \( x \) is on the coclosure of both, then the line of separation of \((A \cup x, B - x)\) is different from the line of separation of \((A, B)\). However, in the dual matroid \( M^* \), the element \( x \) is in the closure of \( A \) and \( B - x \). The following corollary is the dual of Theorem 2.1.1.

**Corollary 2.4.1.** Let \( M \) be a 3-connected matroid and let \((A, B)\) be a 3-separation of \( M^* \). Then there is a matroid \( M' \) that is either \( M \) or a coextension of \( M \) by the set \( L' \) of elements with the following properties:
(i) \( (M')^* \) has a rank-2 dependent flat \( L \) such that \( L' \subseteq L \).

(ii) \( M' \) has an exact 3-separation \((A \cup L', B)\) and \( L = \text{cl}_{M'}^*(A) \cap \text{cl}_{M'}^*(B) \).

(iii) \( M', M'.(A \cup L), \) and \( M'.(B \cup L) \) are 3-connected.

(iv) If \( M \) is representable over a field \( F \), then \( M' \) is representable over \( F \) and 
\[ (M')^* = P_L((M')^*|(A \cup L), (M')^*|(B \cup L)) . \]

If \( x \in \text{cl}_M^*(A - x) \cap \text{cl}_M^*(B) \), then we say that \( x \) is in the coline of separation or in the coguts of the separation, and we call the operation in Corollary 2.4.1 decomposing at a coline of separation. If we complete a line of separation in \( M^* \) to obtain \( M' \), then we say that \( M' \) was obtained by completing the coline of separation.

The following lemma indicates how decomposing at a coline of separation is related to decomposing at a line of separation.

**Lemma 2.4.2.** Let \( M \) be a 3-connected matroid and let \( (A, B) \) be a 3-separation of \( M \). Let \( M_1 \) be obtained from \( M \) by completing the line of separation of \( (A, B) \) and let \( M_2 \) be obtained from \( M \) by completing the coline of separation of \( (A, B) \).

Let \( L_1 = \text{cl}_{M_1}(A) \cap \text{cl}_{M_1}(B) \), \( L_1' = E(M_1) - E(M) \), \( L_2 = \text{cl}_{M_2}(A) \cap \text{cl}_{M_2}(B) \), and \( L_2' = E(M_2) - E(M) \). Let \( A_1' = M_1|(A \cup L_1) \), \( B_1' = M_1|(B \cup L_1) \), \( A_2' = M_2.(A \cup L_2) \), and \( B_2 = M_2.(B \cup L_2) \).

(i) \( M_1|E(M) = M = M_2.E(M) \).

(ii) \( A_1 \setminus L_1' = A_2 \setminus L_2' \) and \( B_1 \setminus L_1' = B_2 \setminus L_2' \).

(iii) \( L_1 \) is a segment in \( A_1 \) and \( B_1 \), and \( L_2 \) is a cosegment in \( A_2 \) and \( B_2 \).

**Proof.** By construction, \( M_1 \) is an extension of \( M \) and \( M_2 \) is a coextension of \( M \), and so \( M_1|E(M) = M = M_2.E(M) \). Also, \( A_1 \setminus L_1' = M_1|A = M|A = M_2.A = A_2 \setminus L_2' \).

Thus, \( A_1 \setminus L_1' = A_2 \setminus L_2' \), and, similarly, \( B_1 \setminus L_1' = B_2 \setminus L_2' \). By construction, \( L_1 \) is a
rank-2 flat with \( |L_1| \geq 3 \) in \( A_1 \) and \( B_2 \), and so \( L_1 \) is a segment in \( A_1 \) and \( B_1 \). By duality, \( L_2 \) is a cosegment in \( A_2 \) and \( B_2 \).

Coullard, Gardner, and Wagner used colines of separation in their decomposition of 3-connected graphs in [6], but Leo [17] used lines of separation in his decomposition of binary matroids. While decomposing at a coline of separation has many useful properties, we will not use this operation in our decomposition. Hence we need another way to deal with elements on the coline of separation. We first prove two general lemmas that will be useful in Chapter 3.

**Lemma 2.4.3.** Let \((A, B)\) and \((C, D)\) be 3-separations of \( M \) and \( M_e \) be an extension of \( M \) by the element \( e \) such that \( e \in cl_{M_e}(A) \cap cl_{M_e}(B) \). If \( C \supseteq A \), then \( e \in cl_{M_e}(C) \) and \((C \cup e, D)\) is a 3-separation of \( M_e \).

**Proof.** As \( C \supseteq A \), we have that \( e \in cl_{M_e}(C) \). Thus \( r_{M_e}(C \cup e) + r_{M_e}(D) - r(M_e) = r_M(C) + r_M(D) - r(M) \).

**Lemma 2.4.4.** Let \( M \) be a 3-connected matroid and, for some \( n \geq 2 \), let \( S = \{(A_i, B_i) \mid 1 \leq i \leq n\} \) be a set of 3-separations of \( M \) such that \( A_i \cap A_j = \emptyset \) for all \( i \neq j \). For each \( i \), let \( M_i \) be the extension of \( M \) obtained by completing the line of separation of \((A_i, B_i)\). Then there is a unique 3-connected extension \( M' \) of \( M \) such that \( E(M') = \bigcup_{i=1}^{n} (E(M_i)) \) and each \( M_i \) is a restriction of \( M' \). Moreover, if \( L' = E(M') - E(M) \), then \( S' = \{(A_i, B_i \cup L') \mid 1 \leq i \leq n\} \) is a set of 3-separations of \( M' \).

**Proof.** We will prove the lemma first when \( M \) is a \( GF(q) \)-represented matroid, and then when \( M \) is a non-representable matroid.

Suppose \( M \) is represented over \( GF(q) \). Then, for each \( i \), the extension \( M_i \) is unique. Let \( L_i = E(M_i) - E(M) \) and \( M' = PG(r-1, q)[E(M) \cup L_1 \cup L_2 \cup \cdots \cup L_n] \).
Evidently $E(M') = \bigcup_{i=1}^{n} (E(M_i))$ and $M_i = M'(E(M) \cup L_i)$ for all $i$. Now suppose $e \in L_1$. Then $e \in \text{cl}_{M_1}(A_1) \cap \text{cl}_{M_1}(B_1)$, and so $e \in \text{cl}_{M'}(A_1) \cap \text{cl}_{M'}(B_1)$. Suppose $i \geq 2$. Then $A_1 \subseteq B_i$, and so $e \in \text{cl}_{M'}(B_i)$. Hence $L_1 \subseteq \text{cl}_{M'}(B_i)$, we deduce, by symmetry, that $(A_i, B_i \cup L_1 \cup L_2 \cup \cdots \cup L_n)$ is a 3-separation of $M'$.

By symmetry, this is also true for $i = 1$.

Suppose $M$ is non-representable. First we show that the order of adding exactly two lines freely does not matter. Let $M_1$ be obtained by completing the line of separation $(A_1, B_1)$ and let $L_1' = E(M_1) - E(M)$. By Lemma 2.4.3, $(A_2, B_2 \cup L_1')$ is a 3-separation of $M_1$. Let $M_2$ be obtained by completing the line of separation of $(A_2, B_2 \cup L_1')$ in $M_1$ and let $L_2'$ be the set of elements added in the process. Add the element $e$ to the guts of the 3-separation $(A_1, B_1 \cup L_1' \cup L_2')$ in $M_2$ to obtain $M_2'$ and add $f$ to the guts of the separation $(A_2, B_2 \cup L_1' \cup L_2')$ in $M_2$ to obtain $M_2''$. By a result of Cheung [5] (see [19, p. 259]), since we are adding $e$ and $f$ by principal modular cuts, there is a unique extension $M'''$ of $M_2$ by the elements $e$ and $f$ such that $M''' \setminus e = M_2'$ and $M''' \setminus f = M_2''$. Clearly $M''\setminus(L_2' \cup f)$ is equal to the matroid obtained from $M$ by adding the elements of $L_1' \cup e$ freely to the line of separation $(A_1, B_1)$ in $M$. Consider the matroid $N = M''\setminus(L_1' \cup e)$. Then each member of $L_2' \cup f$ is free in $\text{cl}_{N}(A_2) \cap \text{cl}_{N}(B_2)$. Thus we may view $M_2$ as having been obtained by completing the lines of separation in either order. A straightforward induction completes the proof.

We may use the previous lemmas in the case of having two lines of separation on each side of a coguts element to obtain the following:

**Lemma 2.4.5.** Let $M$ be a 3-connected matroid and let $(A, B)$ be a 3-separation of $M$. Suppose there is an element $e$ in $B$ such that $e \in \text{cl}^*(A) \cap \text{cl}^*(B - e)$. Then
there is a matroid $M'$ that is either $M$ or a extension of $M$ by the set of elements $L'$ with the following properties:

(i) $M'$ has two rank-2 dependent flats $L_1$ and $L_2$ such that $L' \subseteq L_1 \cup L_2$ and $|L_1 \cap L_2| \leq 1$.

(ii) $M'$ has exact $3$-separations $(A \cup (L_1 \cap L'), B \cup (L' - L_1))$ and $(A \cup (L_1 \cap L') \cup e, B \cup (L' - L_1))$.

(iii) $L_1 = \text{cl}_{M'}(A) \cap \text{cl}_{M'}(B)$ and $L_2 = \text{cl}_{M'}(A \cup e) \cap \text{cl}_{M'}(B - e)$.

(iv) $M', M'|((A \cup L_1), M'|((B-e) \cup L_2), M|(L_1 \cup L_2 \cup e), M|(A \cup L_2 \cup e), M|(B \cup L_1), \ M|(A \cup L_1 \cup L_2 \cup e)$, and $M|(B \cup L_1 \cup L_2)$ are $3$-connected.

(v) If $M$ is represented over $GF(q)$, then $M'$ is represented over $GF(q)$ and $M' = P_{L_2}(P_{L_1}(M'|((A \cup L_1), M'|((B-e) \cup L_2), M|(L_1 \cup L_2 \cup e)), M'|((B-e) \cup L_2))$.

Proof. If $M$ is represented over $GF(q)$, then let $P = PG(r - 1, q)$. Then, by Lemma 2.4.4, there is $GF(q)$-represented matroid $M'$ which can be considered to be obtained by completing these two lines of separation in either order. We may apply Lemma 2.1.1 to $M', M' \setminus (L_2 - L_1)$, and $M' \setminus (L_1 - L_2)$ to show that $M'$ has the other desired properties. If $M$ is non-representable, then, by Lemma 2.4.4, there is a matroid $M'$ which can be considered to be obtained by completing these two lines of separation in either order. We may apply Lemma 2.1.1 to $M', M' \setminus (L_2 - L_1)$, and $M' \setminus (L_1 - L_2)$ to show that $M'$ has the other desired properties. □

From (iv) in the previous lemma, we have many different pieces to choose from. We will call the decomposition associated with $M'$ having pieces $M'|((A \cup L_2 \cup e)$ and $M'|((B \cup L_1)$ decomposing at a plane with overlap.

Suppose $M$ has a 3-sequence $(C, x, y, D)$ such that $x \in \text{cl}(C) \cap \text{cl}(D \cup y)$, $x \in \text{cl}(C \cup y) \cap \text{cl}(D)$, $y \in \text{cl}^*(C \cup x) \cap \text{cl}^*(D)$, and $y \in \text{cl}^*(C) \cap \text{cl}^*(D \cup x)$. There is
no easy choice as to which line or coline of separation to use. However, this is a special case of the situation in the previous lemma, where we did not require the existence of the guts element $x$, and we shall not consider this case any further.

2.5 Detaching a Plane from a 3-Connected Matroid

Akkari and Oxley [1] proved the following:

**Theorem 2.5.1.** Let $e$ and $f$ be distinct elements of a 3-connected matroid $M$. Suppose that

(i) \{e, f\} is both in a triangle \{e, f, a\} and a triad \{e, f, z\} of $M$. Then either

(ii) $M$ is isomorphic to $U_{2,4}$, or

(iii) $M$ is isomorphic to $P_{\Delta}(M(K_4), M/z)\setminus\{e, f\}$, where $\Delta = \{e, f, a\}$.

In particular, if both $M\setminus e, f$ and $M/e, f$ are disconnected, then (i) holds, hence so does one of (ii) and (iii).

If $M$ is a 3-connected matroid to which we may apply Theorem 2.5.1, then we can view $M$ as having a decomposition into pieces isomorphic to $M(K_4)$ and $M/z$. By Theorem 2.1.1, both of these pieces are 3-connected.

The goal of this section is to prove the following theorem, which is a generalization of Theorem 2.5.1.

**Theorem 2.5.2.** Let $M$ be a 3-connected matroid with $r(M) \geq 3$. Let $x$ be an element of $E(M)$ and let $Z$ be a subset of $E(M)$ of size at least two that does not contain $x$. If $Z \cup x$ is a cocircuit of $M$ and $Z$ is a segment in $M$, then there is a matroid $M_y$, which is either $M$ or a 3-connected single-element extension of $M$ by $y$, that has the following properties:
(i) There is an element $y$ in $E(M_y)$ such that $Z \cup y$ is a segment in $M_y$ and $Z \cup x$ is a cocircuit of $M_y$.

(ii) There is rank-3 matroid $N$ containing $Z \cup y$ such that $M_y$ is isomorphic to $P_S(N, M_y/x) \setminus Z$, where $S = M_y|(Z \cup y) = N|(Z \cup y)$.

(iii) If $M_y/x$ is representable over a field $F$, then $M_y$ is representable over $F$.

There are two main steps to proving this theorem: (a) proving the existence of $M_y$ and (b) proving a generalization of Theorem 2.5.1 which we may apply to $M_y$.

Let $M$ be a 3-connected matroid with rank $r \geq 3$. Suppose $Z$ is a segment in $M$ and $Z \cup x$ is a cocircuit of $M$. Let $H_0$ be the hyperplane $E(M) - (Z \cup x)$. Then, as $Z$ is a line in $M$ that is not spanned by $H_0$, we know that $cl_M(Z)$ meets $H$ in at most one point. Assume that $cl(Z) \cap H_0$ is empty. Now $r(Z \cup x) = 3$, and $Z \cup x$ is a cocircuit. Thus $(Z \cup x, H_0)$ is 3-separating, and we may add two elements freely to the line of separation in order to obtain a 3-connected matroid $N$ where $cl_N(Z \cup x) \cap cl_N(H_0)$ has at least two elements. Let $S = cl_N(Z \cup x)$, $H = cl_N(H_0)$, and $L = S \cap H$. We will proceed by producing a modular cut that adds an element to $cl_N(Z) \cap cl_N(H)$. First, notice that neither of the two points we freely added is in $cl(Z) \cap cl(H)$.

**Lemma 2.5.3.** $cl_N(Z) \cap H = \emptyset$.

*Proof.* Let the two points that are freely added be $a$ and $b$. Since $a$ and $b$ are independent clones, if $a \in cl_N(Z) \cap H$, then $b \in cl_N(Z) \cap H$. But as $r(cl_N(Z) \cap H) \leq 1$, we have the contradiction that $r(\{a, b\}) = 1$. Therefore, neither $a$ nor $b$ is in $cl_N(Z) \cap H$. \qed

The next three lemmas give us more information about flats $F$ of $N$ which contain $Z$ or $L$. 

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Lemma 2.5.4. \( S = Z \cup x \cup L \).

Proof. First, \( L \subseteq \text{cl}(Z \cup x) \). If \( h \in H - L \), then \( r(Z \cup x \cup h) = 4 \). So \( h \notin \text{cl}(Z \cup x) \).

Therefore, \( S = Z \cup x \cup L \). \( \square \)

Lemma 2.5.5. Let \( F \) be a flat of \( N \) containing \( Z \).

(i) If \( x \in F \), then \( S \subseteq F \).

(ii) If \( x \notin F \), then \( F \cap L = \emptyset \).

Proof. If \( x \in F \), then \( Z \cup x \subseteq F \), and so, \( S = \text{cl}(Z \cup x) \subseteq \text{cl}(F) = F \). Assume \( x \notin F \). If \( |F \cap L| \geq 2 \), then \( Z \cup L \subseteq F \) and so \( x \in F \), a contradiction. If \( L \cap F = \{l\} \), then \( r(Z \cup l) = 3 = r(S) \). Thus \( x \in F \), a contradiction. Therefore, \( L \cap F = \emptyset \). \( \square \)

Lemma 2.5.6. Let \( F \) be a flat of \( N \) containing \( L \).

(i) If \( x \in F \), then \( S \subseteq F \).

(ii) If \( x \notin F \), then \( F \cap (Z \cup x) \) is empty, \( r(F \cup Z) = r(F) + 1 \), and \( x \in \text{cl}(F \cup Z) \).

(iii) If \( F \) does not contain \( Z \), then \( F \cap S = L \).

Proof. If \( x \in F \), then, since \( r(L \cup x) = 3 = r(S) \), we have that \( S \subseteq \text{cl}(L \cup x) \subseteq \text{cl}(F) = F \). Now assume \( x \notin F \). Suppose \( |F \cap Z| \geq 2 \). Then \( Z \subseteq F \), and, as \( x \in \text{cl}(Z \cup L) \), we have the contradiction that \( x \in F \). Suppose \( F \cap Z = \{z\} \). Then \( F \supseteq L \cup z \). But \( L \cup z \) spans \( S \), and so \( x \in F \), a contradiction. Thus \( Z \cap F \) is empty.

By submodularity, \( r(F \cup S) + r(L) \leq r(F) + r(S) \); that is, \( r(F \cup S) \leq r(F) + 1 \). Thus \( r(F \cup Z) \leq r(F \cup S) \leq r(F) + 1 \). If \( r(F \cup Z) = r(F) \), then \( Z \subseteq F \), a contradiction. Thus \( r(F \cup Z) = r(F \cup S) = r(F) + 1 \) and \( x \in \text{cl}(F \cup Z) \).

If \( F \cap (Z \cup x) \neq \emptyset \), then \( x \) is in this intersection or not. If \( x \) is in the intersection, then, by (i), \( Z \subseteq F \), a contradiction. If \( x \) is not in this intersection, then, by (ii), \( F \cap (Z \cup x) = \emptyset \); that is, \( F \cap S = L \). \( \square \)
We will now consider the flats in the modular cut $M$ of $N$ generated by $\{L, Z\}$.

**Lemma 2.5.7.** Let $F_1$ and $F_2$ be a modular pair of flats of $N$ with $Z \subseteq F_1$, $L \subseteq F_2$. If $x \in F_1 \cup F_2$, then $F_1 \cap F_2$ contains $Z$ or $L$. If $x \notin F_1 \cup F_2$, then $F_1 \cap F_2 \cap S = \emptyset$ and $r((F_1 \cap F_2) \cup Z) = r(F_1 \cap F_2) + 1$.

**Proof.** If $x \in F_2$, then, by Lemma 2.5.6, $S \subseteq F_2$ and $F_1 \cap F_2 \supseteq Z$. If $x \in F_1$, then, by Lemma 2.5.5, $S \subseteq F_1$ and $F_1 \cap F_2 \supseteq L$. Finally, suppose $x \notin F_1 \cup F_2$. Thus $F_1 \cap L = \emptyset$ and $F_2 \cap Z = \emptyset$. Hence $F_1 \cap F_2 \cap S = \emptyset$. Since $F_1$ and $F_2$ are a modular pair, $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$. Moreover,

$$r(F_1) + r(F_2 \cup Z) \geq r(F_1 \cup F_2) + r(F_1 \cap \text{cl}(F_2 \cup Z)) \geq r(F_1 \cup F_2) + r((F_1 \cap F_2) \cup Z).$$

Also, $r(F_2 \cup Z) = r(F_2) + 1$. Thus,

$$r(F_1 \cup F_2) + r(F_1 \cap F_2) + 1 \geq r(F_1 \cup F_2) + r((F_1 \cap F_2) \cup Z)$$

and $r(F_1 \cap F_2) + 1 \geq r((F_1 \cap F_2) \cup Z)$. If $r(F_1 \cap F_2) = r((F_1 \cap F_2) \cup Z)$, then $Z \subseteq F_1 \cap F_2$, a contradiction. Hence $r(F_1 \cap F_2) + 1 = r((F_1 \cap F_2) \cup Z)$. □

**Lemma 2.5.8.** Let $F$ be a flat of $N$ containing a flat $F'$ such that $F' \cap S = \emptyset$ and $r(F' \cup Z) = r(F') + 1$. Then,

(i) $Z \subseteq F$, or

(ii) $r(F \cup Z) = r(F) + 1$ and

(a) $F \cap S = L$;

(b) $F \cap S = \emptyset$; or

(c) $F \cap S = \{x\}$ and $r((F - x) \cup Z) = r(F - x) + 1$. 

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Proof. First, observe the following:

2.5.9. If $F_1 \supseteq F'$, then \( r(F_1) \leq r(F_1 \cup Z) \leq r(F_1) + 1 \).

To see the second inequality, note that, by submodularity,

\[
    r(F' \cup Z) + r(F_1) \geq r(F_1 \cup Z) + r(F'),
\]

and so

\[
    r(F_1 \cup Z) - r(F_1) \leq r(F' \cup Z) - r(F') \leq 1.
\]

Hence 2.5.9 holds.

Now assume that \( Z \) is not contained in \( F \). Then \( |F \cap Z| \leq 1 \). Moreover, as \( F \) and \( F \cap H \) are flats containing \( F' \) but not containing \( Z \), we have, by 2.5.9, that

\[
    r(F \cup Z) = r(F) + 1 \tag{2.1}
\]

and

\[
    r((F \cap H) \cup Z) = r(F \cap H) + 1. \tag{2.2}
\]

Assume that \( |F \cap L| \geq 2 \); then, \( L \subseteq F \). Hence, by Lemma 2.5.6, \( F \cap Z = \emptyset \) and (a) holds.

We may now suppose that \( |F \cap L| \leq 1 \). Suppose \( F \cap Z = \{z\} \). Then, using (2.2), we get

\[
    r((F \cap H) \cup z) = r(F \cap H) + 1 = r((F \cap H \cup Z)).
\]

Hence \( Z \subseteq \text{cl}((F \cap H) \cup z) \subseteq F \), a contradiction. Thus \( F \cap Z = \emptyset \). Suppose \( F \cap L = \{l\} \). Since \( Z \cap H = \emptyset \), we deduce that \( l \notin Z \). By (2.2), if \( z \in Z \), then \((F \cap H) \cup Z\) is spanned by \((F \cap H) \cup z\). Thus \( Z \cup l \), and hence \( L \), is spanned by \((F \cap H) \cup z\). Now, \( F \cap L = \{l\} \). If \( l' \in L - l \), then \( l' \notin F \cap H \) and \( l' \in \text{cl}((F \cap H) \cup z) \).

Thus, by the Mac Lane-Steinitz exchange property, \( z \in \text{cl}((F \cap H) \cup l') \subseteq H \), a
contradiction. We conclude that $F \cap L = \emptyset$, and so either $F \cap S = \emptyset$ and (b) holds, or $F \cap S = \{x\}$. In the latter case, $F - x \supseteq F'$, and so

$$1 \leq r((F - x) \cup Z) - r(F - x) \leq r(F' \cup Z) - r(F') = 1.$$ 

Hence $r((F - x) \cup Z) = r(F - x) + 1$ and (c) holds.

From the previous lemma, $\mathcal{M}$ contains at least four types of flats $F$:

**Z-type:** $Z \subseteq F$;

**L-type:** $F \cap S = L$ and $r(F \cup Z) = r(F) + 1$;

**H-type:** $F \cap S = \emptyset$ and $r(F \cup Z) = r(F) + 1$; or

**xH-type:** $F \cap S = \{x\}$, $r(F \cup Z) = r(F) + 1$, and $r((F - x) \cup Z) = r(F - x) + 1$.

Note that if $F$ is an xH-type flat, then $F - x$ is an H-type flat. We prove that these four types are also sufficient to describe all the flats in the modular cut $\mathcal{M}$.

**Lemma 2.5.10.** If $F_1$ and $F_2$ are a modular pair of flats in $\mathcal{N}$ and each is of one of the four types listed above, then $F_1 \cap F_2$ is also a flat of one of the four types above.

**Proof.** We will frequently use the following inequality:

$$r(F_1 \cup F_2) + r(F_1 \cap F_2) + 2 \geq r(F_1 \cup F_2 \cup Z) + r((F_1 \cap F_2) \cup Z). \quad (2.3)$$

To see that this is true, note that $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$ and that if $F$ is a flat of one of the four types and $Z$ is not contained in $F$, then $r(F \cup Z) = r(F) + 1$. Thus (2.3) follows from submodularity and

$$r(F_1 \cup Z) + r(F_2 \cup Z) \leq r(F_1) + r(F_2) + 2 \leq r(F_1 \cup F_2) + r(F_1 \cap F_2) + 2.$$
If both $F_1$ and $F_2$ contain $Z$, then $F_1 \cap F_2$ contains $Z$. If $F_1$ contains $Z$ but $F_2$ does not, then $r(F_1) + r(F_2 \cup Z) \geq r(F_1 \cup F_2) + r((F_1 \cap F_2) \cup Z)$ and $r(F_2 \cup Z) = r(F_2) + 1$. Then, using modularity, $r(F_1 \cap F_2) + 1 \geq r((F_1 \cap F_2) \cup Z)$. If $r(F_1 \cap F_2) = r((F_1 \cap F_2) \cup Z)$, then $Z \subseteq F_1 \cap F_2$, a contradiction. Thus $r(F_1 \cap F_2) + 1 = r((F_1 \cap F_2) \cup Z)$. If $F_2$ is L-type, then, by Lemma 2.5.7, $F_1 \cap F_2$ is Z-type, L-type, or H-type. If $F_2$ is H-type, then $F_1 \cap F_2 \cap S = \emptyset$, and so we have that $F_1 \cap F_2$ is H-type. If $F_2$ is xH-type, then $F_1 \cap F_2 \cap S$ is the empty set or $\{x\}$. In the former case, $F_1 \cap F_2$ is H-type. In the latter case, we will show that $r((F_1 \cap F_2) - x) \cup Z) = r((F_1 \cap F_2) - x) + 1$, and hence, $F_1 \cap F_2$ is xH-type. To simplify notation, let $F = F_1 \cap F_2$. As $F \cap S = \{x\}$, $F - x$ is contained in $H$. Since $F_2$ is xH-type, $r((F_2 - x) \cup Z) = r(F_2 - x) + 1$. Thus, by submodularity, $r(F_1 \cup F_2) + r(F) + 2 \geq r((F_1 \cup F_2)) + r((F - x) \cup Z)$, and so $r(F - x) + 1 \geq r((F - x) \cup Z)$. If $r(F - x) + 1 = r((F - x) \cup Z)$, then $Z$ is contained in $F$, a contradiction. Thus $r(F - x) + 1 = r((F - x) \cup Z)$.

Assume that neither $F_1$ nor $F_2$ contains $Z$. If both $F_1$ and $F_2$ contain $L$, then $F_1 \cap F_2$ contains $L$, and so $F_1 \cap F_2$ is an L-type flat. So we may assume $F_2$ is an H-type or xH-type flat. Then $F_2$ or $F_2 - x$ is an H-type flat, and so $cl(F_1 \cup F_2)$ is a flat that contains an H-type flat. Thus, by Lemma 2.5.8, either $Z \subseteq cl(F_1 \cup F_2)$ or $r(F_1 \cup F_2) \cup Z) = r(F_1 \cup F_2) + 1$.

If $r(F_1 \cup F_2 \cup Z) = r(F_1 \cup F_2) + 1$, then, by (2.3), $r(F_1 \cap F_2) + 1 \geq r((F_1 \cap F_2) \cup Z)$. If $r(F_1 \cap F_2) = r((F_1 \cap F_2) \cup Z)$, then $Z \subseteq F_1 \cap F_2$, a contradiction. Thus $r(F_1 \cap F_2) + 1 = r((F_1 \cap F_2) \cup Z)$. Now $x$ is not in $F_1$; otherwise, by Lemma 2.5.6, $Z \subseteq F_1$, a contradiction. Thus $F_1 \cap F_2 \cap S$ is the empty set. Therefore, $F_1 \cap F_2$ is H-type.

If $Z \subseteq cl(F_1 \cup F_2)$, then $F_1$ is L-type and $F_2$ is xH-type. Thus $F_1 \cap F_2 \cap S = \emptyset$. We need only show that $r(F_1 \cap F_2 \cup Z) = r(F_1 \cap F_2) + 1$ to have that $F_1 \cap F_2$ is H-type. If $r(F_1 \cap F_2 \cup Z) = r(F_1 \cap F_2)$, then $Z \subseteq F_1 \cap F_2$, a contradiction. Note that, since $F_2$ is xH-type, $(F_1, F_2 - x)$ is a modular pair and $cl(F_1 \cup (F_2 - x)) \subseteq H$. Thus
cl\((F_1 \cup (F_2 - x))\) contains the H-type flat \(F_2 - x\) and contains no member of \(Z\). Hence, by Lemma 2.5.8, \(r(F_1 \cup (F_2 - x)) \cup Z = r(F_1 \cup (F_2 - x)) + 1\). Thus, by (2.3), \(r(F_1 \cap (F_2 - x)) + 1 \geq r((F_1 \cap (F_2 - x)) \cup Z)\). If \(r(F_1 \cap (F_2 - x)) = r((F_1 \cap (F_2 - x)) \cup Z)\), then \(Z \subseteq F_1 \cap (F_2 - x)\), a contradiction. Thus \(r(F_1 \cap (F_2 - x)) + 1 = r((F_1 \cap (F_2 - x)) \cup Z)\).

Suppose \(x \in \text{cl}(F_1 \cap (F_2 - x) \cup z)\) for some \(z\) in \(Z\). Then as \(r(F_2) = r(F_2 - x) + 1\), we have that \(r(F_1 \cap F_2) = r(F_1 \cap (F_2 - x)) + 1\), and hence \(x \notin \text{cl}(F_1 \cap (F_2 - x))\). By the Mac Lane-Steinitz exchange property, \(z \in \text{cl}(F_1 \cap F_2) \subseteq H\), a contradiction. Therefore, \(r(F_1 \cap F_2) + 1 = r((F_1 \cap F_2) \cup Z)\), and so \(F_1 \cap F_2\) is H-type.

Finally, we may assume that \(F_1\) and \(F_2\) are H-type or xH-type flats. Since \(\text{cl}(F_1 \cup F_2)\) contains an H-type flat, Lemma 2.5.8 and (2.3) give us that \(r(F_1 \cap F_2) + 1 \geq r((F_1 \cap F_2) \cup Z)\). If \(r(F_1 \cap F_2) = r((F_1 \cap F_2) \cup Z)\), then \(Z \subseteq F_1 \cap F_2\), a contradiction. Thus, \(r(F_1 \cap F_2) + 1 = r((F_1 \cap F_2) \cup Z)\). If \(x \notin F_1 \cap F_2\), then \(F_1 \cap F_2\) is H-type. If \(x \in F_1 \cap F_2\), then we must show that \(r(((F_1 \cap F_2) - x) \cup Z) = r((F_1 \cap F_2) - x) + 1\).

Note that \(F_1 - x\) and \(F_2 - x\) are a modular pair of H-type flats and \((F_1 \cup F_2) - x\) is a flat containing an H-type flat. Thus, by Lemma 2.5.8 and (2.3), we have that \(r(((F_1 \cap F_2) - x) \cup Z) \leq r((F_1 \cap F_2) - x) + 1\). Again, if \(r(((F_1 \cap F_2) - x) \cup Z) = r((F_1 \cap F_2) - x)\), then \(Z \subseteq (F_1 \cap F_2) - x\), a contradiction. Thus \(r(((F_1 \cap F_2) - x) \cup Z) = r((F_1 \cap F_2) - x) + 1\), and so \(F_1 \cap F_2\) is an xH-type flat. \(\square\)

We now collect the preceding lemmas to show that the set of Z-type, L-type, H-type, and xH-type flats are a modular cut.

**Proposition 2.5.11.** Let \(\mathcal{M}\) be the modular cut generated by \(Z\) and \(L\) and let \(F \in \mathcal{M}\). Then \(F\) is Z-type, L-type, H-type, or xH-type, and \(r(F) \geq 2\).

**Proof.** Consider how \(\mathcal{M}\) is formed. Certainly \(\mathcal{M}\) contains all flats containing \(Z\) and all flats containing \(L\). Such flats are of Z-type or L-type. By 2.5.7, if \(F_1\) is a Z-type flat, \(F_2\) is an L-type flat, and \((F_1, F_2)\) is a modular pair, then \(F_1 \cap F_2\)
is a Z-type, L-type, or H-type flat. A flat containing a Z-type or L-type flat is of Z-type or L-type, while one containing an H-type flat is of Z-type, L-type, H-type or xH-type.

Finally, if \( F_1 \) and \( F_2 \) are a modular pair of flats and each is of one of the four specified types, then, by Lemma 2.5.10, so is \( F_1 \cap F_2 \). We deduce that every flat in \( \mathcal{M} \) is of one of the four specified types. In particular, if \( F \in \mathcal{M} \), then \( r(F) \geq 2 \). To see this, suppose that \( r(F) \leq 1 \). Then \( 2 \leq r(F \cup Z) = r(F) + 1 \leq 2 \). As \( Z \cap H = \emptyset \), we must have \( F = \{z\} \subseteq Z \), so \( F \) is not any of the four types, a contradiction. \( \square \)

By the last proposition, we have an extension \( N' \) of \( N \) obtained by adding a point \( y \) to the intersection of the flats \( Z \) and \( L \) using the modular cut \( \mathcal{M} \). Delete the elements on \( L \) which were freely added to obtain the matroid \( M_y \). Lemma 2.3.1 gives us the following:

**Corollary 2.5.12.** The matroid \( M_y \) is 3-connected.

We have the segment \( Z \cup y \) and the cocircuit \( Z \cup x \) of \( M_y \). Thus we have obtained the desired extension \( M_y \) that is necessary for the proof of Theorem 2.5.2. All that remains is to generalize Theorem 2.5.1, and we will closely mirror its proof.

At this point we will change notation. We relabel \( M_y \) as \( M' \) and \( E(M) - (Z \cup x) \) as \( H \). Since \( M'/x \) has a line \( Z \cup y \), let \( M_2 \) be the matroid obtained from \( M'/x \) by relabelling \( Z \) to \( Z' \), and, if \( z \in Z \), then we relabel \( z \) as \( z' \) in \( M'/x \). We now describe a rank-3 matroid \( M_1 \). This matroid has two \((|Z|+1)\)-point lines, \( Z \cup y \) and \( Z' \cup y \), meeting in the single point \( y \) and has one point \( x \) on neither line. Moreover, for each \( z \) in \( Z \), the point \( z' \) in \( Z' \) is in a triangle with \( x \) and \( z \). See Figure 2.1 for an illustration of \( M_1 \) and \( M_2 \). Recall that, in the statement of Theorem 2.5.2, \( S = M'| (Z \cup y) \), and, similarly, we define \( S' = M'| (Z' \cup y) \). We will prove that
FIGURE 2.1. The matroids $M_1$ and $M_2$

$M' = P_{S'}(M_1, M_2)\setminus Z'$ by showing that the flats of $M'$ are flats of $P_{S'}(M_1, M_2)\setminus Z'$, and then complete the proof of Theorem 2.5.2.

Lemma 2.5.13. Let $X$ be a flat of $M'$ that avoids $Z \cup x$. Then $cl_{M'}(X \cup x) \subseteq X \cup (Z \cup x)$.

Proof. Since $E(M') - (Z \cup x)$ is a hyperplane of $M'$, it follows that $X$ is a flat of $M'\setminus (Z \cup x)$. But $x$ is a coloop of $M'\setminus Z$, so $M'/x \setminus Z = M'\setminus (Z \cup x)$. Hence $X$ is a flat of $(M'/x)\setminus Z$. Thus $cl_{M'/x}(X) \subseteq X \cup Z$, so $cl_{M'}(X \cup x) \subseteq X \cup (Z \cup x)$. 

Lemma 2.5.14. The set $Z \cup x$ is a cocircuit of $P_{S'}(M_1, M_2)\setminus Z'$.

Proof. Clearly $Z \cup x$ is a cocircuit of $P_{S'}(M_1, M_2)$. Suppose that $Z \cup x$ is not a cocircuit of $P_{S'}(M_1, M_2)\setminus Z'$. In that case, $Z \cup x$ is a union of cocircuits of $P_{S'}(M_1, M_2)\setminus Z'$. We deduce that $Z$ and $\{x\}$ are cocircuits of $P_{S'}(M_1, M_2)\setminus Z'$ by considering intersections with the circuits contained in $Z \cup y$. Hence,

$$r(P_{S'}(M_1, M_2)\setminus (Z \cup Z')) \leq r(P_{S'}(M_1, M_2)) - 2 = r(M') - 2.$$

But,

$$P_{S'}(M_1, M_2)\setminus (Z \cup Z' \cup x) = M_2 \setminus Z' = M'/x \setminus Z = M'\setminus (Z \cup x).$$
Since $Z \cup x$ is a cocircuit of $M'$, it follows that

$$r(P_{\mathcal{S}'}(M_1, M_2) \setminus (Z \cup Z')) = r(M') - 1,$$

which is a contradiction. Thus $Z \cup x$ is a cocircuit of $P_{\mathcal{S}'}(M_1, M_2) \setminus Z'$.

\[\square\]

**Lemma 2.5.15.** Every flat of $M'$ is a flat of $P_{\mathcal{S}'}(M_1, M_2) \setminus Z'$.

**Proof.** Let $F$ be a flat of $M'$. Then one of the following five possibilities must occur:

(i) $x \in F$ and $y \in F$;

(ii) $x \in F$ and $y \notin F$;

(iii) $x \notin F$ and $|Z \cap F| = 0$;

(iv) $x \notin F$ and $|Z \cap F| = 1$;

(v) $x \notin F$ and $|Z \cap F| \geq 2$

We need to show that, in each case, $F$ is a flat of $P_{\mathcal{S}'}(M_1, M_2) \setminus Z'$. In the rest of the argument, if $X$ is a subset of $E(M'/z)$, then $X'$ will denote the corresponding subset of $E(M_2)$. Hence $X'$ is obtained from $X$ by, if necessary, relabeling a member $z$ as $z'$.

(i) $F - x$ is a flat $F_1$ of $M'/x$. Since $y \in F$ and $Z \cup y$ is a rank-2 flat, $F_1$ either contains or avoids $Z$. Consider $x \cup F_1 \cup F_1'$. This set meets $E(M_2)$ in $F_1'$, a flat of $M_2$. Moreover, $x \cup F_1 \cup F_1'$ meets $E(M_1)$ in either $E(M_1)$ or $\{x, y\}$, depending on whether $F$ does or does not contain $Z$. Hence $x \cup F_1 \cup F_1'$ meets $E(M_1)$ in a flat of $M_1$. We conclude that $z \cup F_1 \cup F_1'$ is a flat of $P_{\mathcal{S}'}(M_1, M_2)$. Thus $(z \cup F_1 \cup F_1') - Z'$ is a flat of $P_{\mathcal{S}'}(M_1, M_2) \setminus Z'$; that is, $F$ is a flat of $P_{\mathcal{S}'}(M_1, M_2) \setminus Z'$. 

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(ii) $F - x$ is again a flat $F_1$ of $M' / x$ and $z \cup F_1 \cup F'_1$ again meets $E(M_2)$ in $F'_1$, a flat of $M_2$. But, since $y \notin F$, at most one member of $Z$ is in $F$, so $x \cup F_1 \cup F'_1$ meets $E(M_1)$ in $\{z, z', x\}$ for some member $z$ of $Z$, or $\{x\}$. Each of these is a flat of $M_1$. Thus $z \cup F_1 \cup F'_1$ is a flat of $P_{S'}(M_1, M_2)$, and so, as in case (i), $F$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$.

(iii) $\text{cl}_{M'}(F \cup x)$ is a flat of $M'$ containing $x$. Thus, by case (i) or (ii), $\text{cl}_{M'}(F \cup x)$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$. But, by Lemma 2.5.13, $\text{cl}_{M'}((F \cup x) \subseteq F \cup Z \cup x$, and by Lemma 2.5.14, $Z \cup x$ is a cocircuit of $P_{S'}(M_1, M_2) \setminus Z'$. Thus $\text{cl}_{M'}((F \cup z) - (Z \cup x)$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$; that is, $F$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$.

(iv) Let $F \cap Z = \{z\}$. Since $Z \cup x$ is a cocircuit of $M'$ meeting the flat $F$ in $\{z\}$, we deduce that $F - z$ is a flat of $M'$. Then, by Lemma 2.5.13, $\text{cl}_{M'}((F - z) \cup x) \subseteq (F - z) \cup Z \cup x$. Certainly $z \notin \text{cl}_{M'}(F - z)$ and $x \notin \text{cl}_{M'}((F - z) \cup z)$. Hence by the Mac Lane-Steinitz exchange property, $z \notin \text{cl}_{M'}((F - z) \cup x)$. Further, if $z_1$ and $z_2$ are in $Z \cap \text{cl}_{M'}((F - z) \cup x)$ and $z, z_1$ and $z_2$ are all different, then $z \in Z \cap \text{cl}_{M'}((F - z) \cup x)$, a contradiction. Thus $\text{cl}_{M' / z}((F - z) \subseteq (F - z) \cup z_0$ for some member $z_0$ of $Z$ other than $z$, and so $\text{cl}_{M_2}(F - z)$ is $F - z$ or $(F - z) \cup z'_0$, and meets $E(M_2)$ in $\text{cl}_{M_2}(F - z)$, a flat of $M_2$, and meets $E(M_1)$ in $\{z\}$ or $\{z, z'_0\}$, each of which is a flat of $M_1$. Thus $F \cup \text{cl}_{M_2}(F - z)$ is a flat of $P_{S'}(M_1, M_2)$, and so $[F \cup \text{cl}_{M_2}(F - e)]$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$; that is, $F$ is a flat of $P_{S'}(M_1, M_2) \setminus Z'$.

(v) $F \supseteq Z$, so $y \in F$. Moreover, $F - Z$ is a flat of $M'$, so, by Lemma 2.5.13, $\text{cl}_{M'((F - Z) \cup x) \subseteq (F - Z) \cup (Z \cup x)$. Now $z \notin \text{cl}_{M'}(F - Z)$ and $x \notin \text{cl}_{M'}((F - Z) \cup z)$. Then, by the Mac Lane-Steinitz exchange property, $z \notin \text{cl}_N((F - Z) \cup x)$. Hence $\text{cl}_{M' / z}(F - Z) = F - Z$. Thus $F \cap E(M_2)$, which equals
$F - Z$, is a flat of $M_2$; and $F \cap E(M_1)$, which equals $Z \cup y$, is a flat of $M_1$.

Therefore, $F$ is a flat of $P_S(M_1, M_2)$, and so $F$ is a flat of $P_S(M_1, M_2) \setminus Z'$.

\[ \square \]

\textit{Proof of Theorem 2.5.2.} If needed, we have shown the existence of the appropriate extension $M_y$ of $M$. By Lemma 2.5.15, we have that $M$ or $M_y$ is a quotient of $P_S(M_1, M_2) \setminus Z'$. Since these two matroids have the same rank and the same ground set, by [19, Corollary 7.3.4], they are equal, which completes the proof. \[ \square \]
Chapter 3
A Unique Decomposition of 3-Connected Matroids

3.1 Overview

In this chapter we will prove the following two theorems:

**Theorem 3.1.1.** Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$. Then $M$ has a decomposition into a unique set of sequentially 4-connected matroids and certain special 3-connected matroids.

**Theorem 3.1.2.** Let $M$ be a 3-connected matroid that is represented over $GF(q)$ with $|E(M)| \geq 9$. Then $M$ has a decomposition into a unique set of sequentially 4-connected $GF(q)$-represented matroids and certain special 3-connected $GF(q)$-represented matroids. Moreover, $M$ can be recovered by identifying identical elements and then deleting certain labeled elements.

To do this, we associate with $M$ a *decomposition tree* with the vertices of the tree being labeled by subsets of $E(M)$. This decomposition tree will guide the way in which $M$ is broken apart. As mentioned in Chapter 1, Oxley, Semple, and Whittle have used a 3-tree to describe the structure of the non-sequential 3-separations of $M$. However, $M$ can have different 3-trees. Decomposing at lines of separation displayed by the edges of a given 3-tree of $M$ is sufficient to produce a decomposition of $M$ similar to that which we will prove, but without uniqueness.

In Section 3.2, we analyze the behavior of elements whose positions in the decomposition are potentially ambiguous. Then, in Section 3.3, we obtain a natural representation for each equivalence class of 3-separations in $M$; this representation displays all separations equivalent to a given separation. In Section 3.4, we will
combine the preceding results to obtain the decomposition tree for \( M \). We will describe how to obtain the pieces of the decomposition and prove the existence and uniqueness of the decomposition in Section 3.5.

### 3.2 Equivalence in 3-Separations

Flowers describe the complicated interactions of crossing 3-separations of a 3-connected matroid \( M \), up to equivalence. The equivalence of 3-separations gives us certain elements in flowers, called loose elements, that can move to different petals by the equivalence of 3-separations. The analysis of loose elements allows us to describe how the equivalence works in flowers. In this section, we consider how the equivalence works in pairs of inequivalent non-sequential 3-separations in \( M \).

**Lemma 3.2.1.** Let \( M \) be a 3-connected matroid and let \((A, B)\) and \((C, D)\) be inequivalent non-sequential 3-separations of \( M \). If there is an element \( x \) in \( \text{fcl}(A) \cap \text{fcl}(B) \cap \text{fcl}(C) \cap \text{fcl}(D) \), then at least one of the following holds:

(i) \( x \) is loose in some flower \( \Phi \) of \( M \); or

(ii) there are non-crossing inequivalent non-sequential 3-separations \((A', B')\) and \((C', D')\) of \( M \) such that \((A', B')\) and \((C', D')\) are equivalent to \((A, B)\) and \((C, D)\), respectively.

**Proof.** Let \( T \) be a 3-tree for \( M \). Then \((A, B)\) and \((C, D)\) are equivalent to 3-separations \((A', B')\) and \((C', D')\), respectively, displayed by \( T \). If \((A', B')\) and \((C', D')\) are not displayed by the same flower vertex, then, without loss of generality, there is an edge \( x \) of \( T \) and its induced partition \((X, Y)\) of \( E(M) \) such that \( A' \subseteq X \) and \( D' \subseteq Y \). Thus \( A' \cap D' \) is empty, and so \((A', B')\) and \((C', D')\) do not cross. If \((A', B')\) and \((C', D')\) are displayed by the same vertex displaying the flower \( \Phi \), then \( A, B, C, \) and \( D \) are unions of petals of \( \Phi \). Thus \( e \) is loose in \( \Phi \). \( \square \)
The next three results, which are Theorems 6.1, 7.1, and 7.4 of [23], describe properties of loose elements in the various types of flowers.

**Theorem 3.2.2.** Let $\Phi$ be a Vámos-like flower. Then $\Phi$ has no loose elements. Hence any flower equivalent to $\Phi$ is equal to $\Phi$ up to a permutation of the petals.

**Theorem 3.2.3.** Let $M$ be a 3-connected matroid and let $\Phi$ be a tight flower of $M$ of order $n \geq 3$ that is a paddle, a copaddle, or is spike-like. Let $T$ and $L$ denote the sets of tight and loose elements of $\Phi$, respectively. For each petal $P_i$ of $\Phi$, let $T_i = P_i \cap T$.

(i) If $\Phi$ is a paddle, then $L$ is a segment, and $L \subseteq \text{cl}(T_i)$ for each $i \in \{1,2,\ldots,n\}$;

(ii) if $\Phi$ is a copaddle, then $L$ is a cosegment, and $L \subseteq \text{cl}^*(T_i)$ for each $i \in \{1,2,\ldots,n\}$; and

(iii) if $\Phi$ is spike-like, then $|L| \leq 2$. If $L$ contains a single element, then that element is either in the closure of $T_i$, for each $i$, or is in the coclosure of $T_i$, for each $i$. If $|L| = 2$, then one member of $L$ is contained in the closure of each $T_i$, while the other member is contained in the coclosure of each $T_i$.

If $F = (f_1, f_2, \ldots, f_n)$ is a fan, and $i, j \in \{1,2,\ldots,n\}$, then we will say that $\{f_1, f_2, \ldots, f_i\}$ is an initial section of $F$, and that $\{f_i, f_{i+1}, \ldots, f_n\}$ is a terminal section of $F$.

**Theorem 3.2.4.** In a matroid $M$, let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight swirl-like flower of order at least 3 with set $T$ of tight elements and $L$ of loose elements. Let $T_i = P_i \cap T$ for all $i$. Then there is a partition $(F_1, F_2, \ldots, F_n)$ of $L$ into fans, some of which may be empty, with the following property: a partition $(Q_1, Q_2, \ldots, Q_n)$ of $E(M)$ is a tight swirl-like flower equivalent to $\Phi$ if and only if $Q_i = F_{i-1}^- \cup T_i \cup F_i^+$.
for all $i \in \{1, 2, \ldots, n\}$, where $F_{i-1}^-$ is a terminal section of $F_{i-1}$, and $F_i^+$ is an initial section of $F_i$.

The next two theorems are the main results of this section.

**Theorem 3.2.5.** Let $M$ be a 3-connected matroid and let $(A, B)$ and $(C, D)$ be inequivalent non-sequential 3-separations of $M$ with $A \cap D = \emptyset$. If $|\text{fcl}(A) \cap \text{fcl}(D)| > 1$, then $(\text{fcl}(A) - \text{fcl}(D), (B \cap C) - (\text{fcl}(A) \cup \text{fcl}(D)), \text{fcl}(D))$ and $(\text{fcl}(A), (B \cap C) - (\text{fcl}(A) \cup \text{fcl}(D)), \text{fcl}(D) - \text{fcl}(A))$ are flowers in $M$.

**Theorem 3.2.6.** Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$, and let $\Phi = (P_1, P_2, \ldots, P_n)$ and $\Psi = (Q_1, Q_2, \ldots, Q_m)$ be tight maximal flowers in $M$ of orders $n$ and $m$, respectively. Let $P_1 = \bigcup_{i=2}^{n} Q_i$, let $Q_1 = \bigcup_{i=2}^{n} P_i$, and let $L$ be the set of elements that are loose in both $\Phi$ and $\Psi$.

1. If $\Phi$ or $\Psi$ is a Vámos-like flower, then $L = \emptyset$.

2. If $\Phi$ and $\Psi$ are neither swirl-like nor unresolved, then $|L| \leq 1$. Moreover, if $L = \{l\}$, then $l \in \text{cl}(P_i)$ and $l \in \text{cl}(Q_j)$ for all $i$ and $j$, or $l \in \text{cl}^*(P_i)$ and $l \in \text{cl}^*(Q_j)$ for all $i$ and $j$.

3. If $\Phi$ is a swirl-like or unresolved flower, then $|L| \leq 2$. Moreover, at most one of the elements of $L$ is in $\text{fcl}(P_2)$ and at most one is in $\text{fcl}(P_n)$.

Let $M$ be a 3-connected matroid. We say that a partition $(A, B, C)$ of the elements of $E(M)$ is an $ABC$-partition if $\lambda(A) = \lambda(C) = 2$, $\lambda(B) \geq 2$, $A \not\subseteq \text{fcl}(B \cup C)$, and $C \not\subseteq \text{fcl}(A \cup B)$. We will prove a series of results on $ABC$-partitions that will be used to prove Proposition 3.2.19, from which Theorem 3.2.5 follows easily, and to prove Theorem 3.2.6.
Lemma 3.2.7. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\) and let \(A_0 = A - \text{fcl}(B \cup C)\). Then \(M\) has a 3-sequence

\[(A_0, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_l, (B \cup C) - \text{fcl}(A))\]

for some non-negative integers \(j, k,\) and \(l\), where \(a_i \in A\) for all \(i \in \{1, 2, \ldots, j\}\), \(b_i \in B\) for all \(i \in \{1, 2, \ldots, k\}\), and \(c_i \in C\) for all \(i \in \{1, 2, \ldots, l\}\).

Proof. Since \(C \notin \text{fcl}(A \cup B)\), the partition \((\text{fcl}(A \cup B), C - \text{fcl}(A \cup B))\) is a non-sequential 3-separation. Thus, by \([23, \text{Lemma 3.4(i)}]\), \(|C - \text{fcl}(A \cup B)| \geq 4\). By symmetry, \(|A_0| \geq 4\). By Lemma 1.3.3, \(M\) has a 3-sequence of the form 

\[(A_0, x_1, x_2, \ldots, x_n, (B \cup C) - \text{fcl}(A))\]

where \(\text{fcl}(A) = A_0 \cup \{x_1, x_2, \ldots, x_n\}\). Let \(\overrightarrow{A_i} = A_0 \cup \{x_1, x_2, \ldots, x_i\}\) for \(i \geq 0\). Then, both \(\overrightarrow{A_i}\) and \(A\) are 3-separating, and \(|E(M) - (\overrightarrow{A_i} \cup A)| \geq |C - \text{fcl}(A \cup B)| \geq 2\). Thus, we may apply Lemma 1.3.2. Hence \(\overrightarrow{A_i} \cap A\) is 3-separating for all \(i \geq 0\). By \([12, \text{Lemma 4.4}]\), we can reorder the original 3-sequence to obtain the 3-sequence

\[(A_0, a_1, a_2, \ldots, a_j, y_1, y_2, \ldots, y_m, (B \cup C) - \text{fcl}(A))\]

where \(a_i \in A\) for \(i \in \{1, 2, \ldots, j\}\) and \(y_i \in B \cup C\) for \(i \in \{1, 2, \ldots, m\}\). Redefine \(\overrightarrow{A_i} = A \cup \{y_1, y_2, \ldots, y_i\}\) for \(i \geq 0\). As \(|E(M) - (\overrightarrow{A_i} \cup (A \cup B))| \geq |C - \text{fcl}(A \cup B)| \geq 2\), we may apply Lemma 1.3.2. Thus \(\overrightarrow{A_i} \cap (A \cup B)\) is 3-separating for all \(i \geq 0\). Then by \([12, \text{Lemma 4.4}]\), we may reorder to obtain the following 3-sequence:

\[(A_0, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_l, (B \cup C) - \text{fcl}(A))\]

where \(a_i \in A\) for all \(i \in \{1, 2, \ldots, j\}\), \(b_i \in B\) for all \(i \in \{1, 2, \ldots, k\}\), and \(c_i \in C\) for all \(i \in \{1, 2, \ldots, l\}\). \(\square\)

Lemma 3.2.8. If \((A, B, C)\) is an ABC-partition of \(M\) and \(\lambda(B) = 2\), then the partition \((A, B, C)\) is a flower in \(M\). If \((A, B, C)\) is an ABC-partition of \(M\) and \(\lambda(B) > 2\), then exactly one of the following holds:
(i) \( \cap(A, B) = 1 \) and \( \cap(A, C) = 0 \);

(ii) \( \cap(A, B) = 2 \) and \( \cap(A, C) = 1 \); or

(iii) \( \cap(A, B) = 2 \) and \( \cap(A, C) = 0 \).

Proof. The first part of the lemma follows from the definition of a flower in \( M \). By [23, Corollary 2.4(iv)],

\[
\lambda(A) + \cap(B, C) = \lambda(C) + \cap(A, B) = \lambda(B) + \cap(A, C).
\]

We have \( \lambda(A) = \lambda(C) = 2 \) and \( \lambda(B) > 2 \). Thus \( \cap(A, B) = \cap(B, C) \geq 1 \). Moreover, by Lemma 1.3.1,

\[
\cap(A, B) = r(A) + r(B) - r(A \cup B) \leq r(A) + r(B \cup C) - r(M) = \lambda(A) = 2.
\]

Hence \( \lambda(C) + \cap(A, B) \leq 4 \). This leaves exactly the three possibilities noted in (i)-(iii).

By the previous lemma, there are three \( ABC \)-types for ABC-partitions that are not flowers. If \( (A, B, C) \) satisfies Lemma 3.2.8(i), then we say that \( (A, B, C) \) is \( ABC \)-coguts. If \( (A, B, C) \) satisfies Lemma 3.2.8(ii), then we say that \( (A, B, C) \) is \( ABC \)-guts. If \( (A, B, C) \) satisfies Lemma 3.2.8(iii), then we say that that \( (A, B, C) \) is \( ABC \)-null. Table 3.1 summarizes all of the permitted values for \( \cap(A, B), \cap(B, C), \cap(A, C) \), and \( \lambda(B) \) in an ABC-partition. The first column of values indicates a co-paddle, the second indicates an unresolved flower, and the fourth column indicates a paddle. The third, fifth, and sixth columns indicate ABC-coguts, ABC-guts, and ABC-null, respectively.

<table>
<thead>
<tr>
<th>( \cap(A, B), \cap(B, C) )</th>
<th>( \cap(A, C) )</th>
<th>( \lambda(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 1 2 2 2 2 1 0 0 3 2 3 4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Observe that if \((A, B, C)\) is an ABC-partition of \(M\), then it is an ABC-partition of \(M^*\). The following lemma characterizes this duality:

**Lemma 3.2.9.** Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\).

(i) \((A, B, C)\) is ABC-guts in \(M\) if and only if \((A, B, C)\) is ABC-coguts in \(M^*\).

(ii) \((A, B, C)\) is ABC-null in \(M\) if and only if \((A, B, C)\) is ABC-null in \(M^*\).

**Proof.** By Lemma 2.6 in [23], if \(X\) and \(Y\) are disjoint sets in \(M\), then

\[
\cap_M(X, Y) + \cap_{M^*}(X, Y) = \lambda(X) + \lambda(Y) - \lambda(X \cup Y).
\]

Specializing to our situation, we have that \(\cap_M(A, C) + \cap_{M^*}(A, C) = 4 - \lambda(B)\) and \(\cap_M(A, B) + \cap_{M^*}(A, B) = \lambda(B) - 2\). Then, \((\cap_M(A, B), \cap_M(A, C), \lambda(B)) = (1, 0, 3)\) if and only if \((\cap_{M^*}(A, B), \cap_{M^*}(A, C), \lambda(B)) = (2, 1, 3)\), which is (i). In the other case, \((\cap_M(A, B), \cap_M(A, C), \lambda(B)) = (\cap_{M^*}(A, B), \cap_{M^*}(A, C), \lambda(B)) = (2, 0, 4)\), which is (ii). \(\square\)

Over the next several lemmas, we will show that any non-flower ABC-partition has at most a single element in \(\text{fcl}(A) \cap C\). Moreover, we will see that (i) if \((A, B, C)\) is ABC-null, then \(\text{fcl}(A) \cap C\) is empty; (ii) if \((A, B, C)\) is ABC-guts and \(c \in \text{fcl}(A) \cap C\), then \(c\) is in \(\text{cl}(A)\); and (iii) if \((A, B, C)\) is ABC-coguts and \(c \in \text{fcl}(A) \cap C\), then \(c\) is in \(\text{cl}^*(A)\).

**Lemma 3.2.10.** Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\) and let \(b \in \text{cl}^*(A)\). Suppose \(|B - \text{cl}^*(A)| \geq 2\).

If \(b \in \text{cl}(A)\), then exactly one of the following is true:

(i) \((A, B, C)\) and \((A \cup b, B - b, C)\) are flowers;

(ii) \((A, B, C)\) and \((A \cup b, B - b, C)\) are the same ABC-type;
(iii) \((A, B, C)\) is ABC-null and \((\cup b, B - b, C)\) is ABC-coguts; or

(iv) \((A, B, C)\) is ABC-guts or ABC-coguts and \((\cup b, B - b, C)\) is a flower.

If \(b \in \text{cl}^\ast(A)\), then exactly one of the following is true:

(i) \((A, B, C)\) and \((\cup b, B - b, C)\) are flowers;

(ii) \((A, B, C)\) and \((\cup b, B - b, C)\) are the same ABC-type;

(iii) \((A, B, C)\) is ABC-null and \((\cup b, B - b, C)\) is ABC-guts; or

(iv) \((A, B, C)\) is ABC-guts or ABC-coguts and \((\cup b, B - b, C)\) is a flower.

Proof. Since \(|B - \text{cl}^\ast(A)| \geq 2\), we have that \((\cup b, B - b, C)\) is an ABC-partition and we may apply Lemma 3.2.8 to it.

Suppose \(b \in \text{cl}(A)\). Then \(r(\cup b) = r(A)\), and so

\[
\cap(\cup b, C) = r(\cup b) + r(C) - r(\cup C) = r(A) + r(C) - r(\cup C) = \cap(A, C).
\]

Now, either \(r(B) = r(B - b)\) or \(r(B) = r(B - b) + 1\). In the first case, \(\cap(\cup b, B - b) = \cap(A, B)\). Thus \((A, B, C)\) and \((\cup b, B - b, C)\) are the same ABC-type or both are flowers. In the other case, \(\cap(\cup b, B - b) = \cap(A, B) - 1\). Thus \((A, B, C)\) is ABC-null and \((\cup b, B - b, C)\) is ABC-coguts; or \((A, B, C)\) is ABC-guts and \((\cup b, B - b, C)\) is a flower; or \((A, B, C)\) is ABC-coguts and \((\cup b, B - b, C)\) is a flower. The rest of the result follows by duality. \(\square\)

Lemma 3.2.11. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\) and let \(B' = \text{fcl}(A) \cap B\). If \(|B - B'| \geq 2\), then exactly one of the following holds:

(i) \((\cup B', B - B', C)\) is a flower;

(ii) \((A, B, C)\) is ABC-null and \((\cup B', B - B', C)\) is ABC-coguts;
(iii) \((A, B, C)\) is ABC-null and \((A \cup B', B - B', C)\) is ABC-guts; or

(iv) \((A, B, C)\) and \((A \cup B', B - B', C)\) are the same ABC-type.

Moreover, if \((A, B, C)\) is a flower, then \((A \cup B', B - B', C)\) is a flower.

Proof. By Lemma 3.2.7, we have a 3-sequence \((A, b_1, b_2, \ldots, b_n, (B - B') \cup C)\) with \(B' = \{b_1, b_2, \ldots, b_n\}\). Let \(\overrightarrow{A_i} = A \cup \{b_1, b_2, \ldots, b_i\}\). We may apply Lemma 3.2.10 to the ABC-partition \((A, B, C)\) and the element \(b_1\), and then recursively to \((\overrightarrow{A_i}, B - \overrightarrow{A_i}, C)\) and \(b_{i+1}\) for each \(i \in \{1, 2, \ldots, n - 1\}\). If \((A, B, C)\) is a flower in \(M\), then \((A \cup B', B - B', C)\) is a flower in \(M\). If \((A, B, C)\) is ABC-guts or ABC-coguts, then \((A \cup B', B - B', C)\) is the same ABC-type or is a flower. If \((A, B, C)\) is ABC-null, then \((A \cup B', B - B', C)\) can be any ABC-type or a flower.

\(\square\)

Lemma 3.2.12. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\)
with \(fcl(A) \cap B = \emptyset\). If \(c \in cl^{(e)}(A)\), then \((A, B, C)\) and \((A \cup c, B, C - c)\)
are the same ABC-type, or both are flowers.

Proof. Suppose \(c \in cl(A)\). Then \((A \cup B, C)\) and \((A \cup B \cup c, C - c)\) are equivalent 3-separations with \(c \in cl(A \cup B)\). Thus \(c \in cl(C - c)\), and so \(\cap(A \cup c, C - c) = \cap(A, C)\).
Moreover, \(\cap(A \cup c, B) = \cap(A, B)\). Therefore, both \((A, B, C)\) and \((A \cup c, B, C - c)\)
are the same ABC-type, or both are flowers. The rest of the result follows by
duality. \(\square\)

Lemma 3.2.13. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\)
and let \(C' = fcl(A) \cap C\). If \(fcl(A) \cap B = \emptyset\), then \((A, B, C)\) and \((A \cup C', B, C - C')\)
are the same ABC-type, or both are flowers.

Proof. By Lemma 3.2.7, we have a 3-sequence \((A, c_1, c_2, \ldots, c_n, B, C - fcl(A))\) and
\(C' = \{c_1, c_2, \ldots, c_n\}\). Let \(\overrightarrow{A_i} = A \cup \{c_1, c_2, \ldots, c_i\}\). We may apply Lemma 3.2.12 to
the ABC-partition \((A, B, C)\) and \(c_1\), and then recursively to \((\overline{A_i}, B, C - \overline{A_i})\) and \(c_{i+1}\) for each \(i \in \{1, 2, \ldots, k\}\). Then, both \((A, B, C)\) and \((A \cup C', B, C - C')\) are the same ABC-type, or both are flowers.

\[\text{Corollary 3.2.14.} \text{ If } (A, B, C) \text{ is a flower and } |B - \text{fcl}(A)| \geq 2, \text{ then } (\text{fcl}(A), B - \text{fcl}(A), C - \text{fcl}(A)) \text{ is a flower.}\]

\[\text{Proof.} \text{ By Lemma 3.2.11, } (\text{fcl}(A) - C, B - \text{fcl}(A), C) \text{ is a flower. By Lemma 3.2.13, we deduce that } (\text{fcl}(A), B - \text{fcl}(A), C - \text{fcl}(A)) \text{ is a flower.}\]

\[\text{Lemma 3.2.15.} \text{ Let } (A, B, C) \text{ be an ABC-partition of a 3-connected matroid } M. \text{ If } (A, B, C) \text{ is ABC-guts and } \text{fcl}(A) \cap B = \emptyset, \text{ then } \text{fcl}(A) \cap C \subseteq \text{cl}(A) \text{ and } |\text{fcl}(A) \cap C| \leq 1.\]

\[\text{Proof.} \text{ Let } c \in C \text{ and suppose } c \in \text{cl}^*(\text{cl}(A)). \text{ Then } r(B \cup (C - (\text{cl}(A) \cup c))) = r(B \cup C) - 1. \text{ Since } \cap(A, B) = 2, \text{ we have that } r(\text{cl}(A) \cup B) = r(A) + r(B) - 2. \text{ Since } \lambda(A) = 2, \text{ we have that } r(M) = r(A) + r(B \cup C) - 2. \text{ Now, by submodularity and substitution, } r(M \setminus c) \leq r(\text{cl}(A) \cup B) + r(B \cup (C - (\text{cl}(A) \cup c))) - r(B) = [r(A) + r(B) - 2] + [r(B \cup C) - 1] - r(B) = r(M) - 1. \text{ Thus } c \text{ is a coloop of } M, \text{ a contradiction. By Lemma 3.2.13, the ABC-partition } (\text{cl}(A), B, C - \text{cl}(A)) \text{ is ABC-guts, and, since we can add no elements to cl}(A) \text{ by coclosure, cl}(A) = \text{cl}(A). \text{ If } |\text{cl}(A) \cap C| \geq 2, \text{ then } r(\text{cl}(A) \cap C) \leq \cap(A, C) = 1, \text{ a contradiction. Therefore, } \text{fcl}(A) \cap C \subseteq \text{cl}(A) \text{ and } |\text{fcl}(A) \cap C| \leq 1.\]

\[\text{Lemma 3.2.16.} \text{ Let } (A, B, C) \text{ be an ABC-partition of a 3-connected matroid } M. \text{ If } (A, B, C) \text{ is ABC-coguts and } \text{fcl}(A) \cap B = \emptyset, \text{ then } \text{fcl}(A) \cap C \subseteq \text{cl}^*(A) \text{ and } |\text{fcl}(A) \cap C| \leq 1.\]

\[\text{Proof.} \text{ By Lemma 3.2.9, } (A, B, C) \text{ is ABC-guts in } M^*. \text{ Thus, the result follows from Lemma 3.2.15 by duality.}\]
Lemma 3.2.17. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\). If \((A, B, C)\) is ABC-null and \(\text{fcl}(A) \cap B = \emptyset\), then \(\text{fcl}(A) \cap C = \emptyset\).

Proof. Let \(c \in C \cap \text{cl}(A)\). Then \(r(c) \leq \cap(A, C) = 0\), a contradiction. Hence, by Lemma 3.2.9 and duality, the intersection \(C \cap \text{cl}^*(A)\) is also empty. Thus \(\text{fcl}(A) \cap C = \emptyset\).

\[
\square
\]

Corollary 3.2.18. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\). If \((A, B, C)\) is not a flower and \(\text{fcl}(A) \cap B = \emptyset\), then \(|\text{fcl}(A) \cap C| \leq 1\).

We can now collect the previous lemmas to prove the following proposition and Theorem 3.2.5.

Proposition 3.2.19. Let \((A, B, C)\) be an ABC-partition of a 3-connected matroid \(M\) with \(|B - (\text{fcl}(A) \cup \text{fcl}(C))| \geq 2\). If \((\text{fcl}(A) - \text{fcl}(C), B - (\text{fcl}(A) \cup \text{fcl}(C)), \text{fcl}(C))\) or \((\text{fcl}(A), B - (\text{fcl}(A) \cup \text{fcl}(C)), \text{fcl}(C) - \text{fcl}(A))\) is not a flower in \(M\), then \(|\text{fcl}(A) \cap \text{fcl}(C)| \leq 1\).

Proof. Let \((\text{fcl}(A) - \text{fcl}(C), B - (\text{fcl}(A) \cup \text{fcl}(C)), \text{fcl}(C)) = (A', B', C')\). We may assume that \((A', B', C')\) is not a flower. Then \(\text{fcl}(A') \cap B' = \emptyset\), \(\text{fcl}(A') = \text{fcl}(A)\), and, by Corollary 3.2.18, \(|\text{fcl}(A') \cap C'| \leq 1\). Therefore, \(|\text{fcl}(A) \cap \text{fcl}(C)| \leq 1\). If \((\text{fcl}(A), B - (\text{fcl}(A) \cup \text{fcl}(C)), \text{fcl}(C) - \text{fcl}(A))\) is not a flower, then we may apply the same argument to \((\text{fcl}(C) - \text{fcl}(A), B - (\text{fcl}(A) \cup \text{fcl}(C)), \text{fcl}(A))\), which completes the proof.

\[
\square
\]

Proof of Theorem 3.2.5. Consider the partition \((A, B \cap C, D)\). If \(\lambda(B \cap C) < 2\), then \(|B \cap C| = 1\), contradicting the assumption that \((A, B)\) and \((C, D)\) are inequivalent. Thus \((A, B \cap C, D)\) is an ABC-partition. Now, if \(|(B \cap C) - \text{fcl}(A)| < 2\), or \(|(B \cap C) - \text{fcl}(D)| < 2\), or \(|(B \cap C) - (\text{fcl}(A) \cup \text{fcl}(D))| < 2\), then we would also have
that \((A, B)\) and \((C, D)\) are equivalent, a contradiction. Thus, we may apply the contrapositive of Proposition 3.2.19, and the result follows.

Before we prove Proposition 3.2.6, we state [23, Lemma 7.5], which we need in the proof.

**Lemma 3.2.20.** Let \(P_i\) and \(P_j\) be petals of a tight swirl-like flower \(\Phi\) of order at least 3 in a 3-connected matroid \(M\) with \(|E(M)| \geq 9\).

(i) \(|\text{cl}(P_i) \cap \text{cl}(P_j)| \leq 1\), and, if \(P_i\) and \(P_j\) are not consecutive, then \(\text{cl}(P_i) \cap \text{cl}(P_j) = \emptyset\).

(ii) \(|\text{cl}^*(P_i) \cap \text{cl}^*(P_j)| \leq 1\), and, if \(P_i\) and \(P_j\) are not consecutive, then \(\text{cl}^*(P_i) \cap \text{cl}^*(P_j) = \emptyset\).

(iii) If \(\text{cl}(P_i) \cap P_j \neq \emptyset\), then \(\text{cl}^*(P_i) \cap P_j = \emptyset\).

**Proof of Theorem 3.2.6.** By Theorem 3.2.2, Vámos-like flowers have no loose elements, and so \(L\) is empty. Suppose \(\Phi\) and \(\Psi\) are neither swirl-like nor unresolved and \(|L| \geq 1\). If \(l \in L\), then \(l \in \text{cl}^*(P_i)\) and \(l \in \text{cl}^*(Q_j)\) for all \(i\) and \(j\), by Theorem 3.2.3. Moreover, \(L = \text{fcl}(P_i) \cap \text{fcl}(Q_j)\) for some \(i, j \neq 1\). Now, for \(i, j \neq 1\), \((P_i, E(M) - (P_i \cup Q_j), Q_j)\) is an ABC-partition and not a flower. Thus \(|L| = |\text{fcl}(P_i) \cap \text{fcl}(Q_j)| \leq 1\), and if \(l = \text{fcl}(P_i) \cap \text{fcl}(Q_j)\), then, by Lemmas 3.2.15 and 3.2.16, \(l \in \text{cl}(P_i) \cap \text{cl}(Q_j)\) or \(x \in \text{cl}^*(P_i) \cap \text{cl}^*(Q_j)\). Therefore, (ii) holds.

Suppose neither \(\Phi\) nor \(\Psi\) is unresolved and suppose \(\Phi\) is a swirl-like flower. If \(l \in L\), then \(l\) is loose in \(\Phi\), and so \(l\) is in exactly one of \(\text{fcl}(P_i) \cap \text{fcl}(P_2)\) and \(\text{fcl}(P_1) \cap \text{fcl}(P_n)\), by Theorem 3.2.4. Moreover, \(l\) is loose in no other petals of \(\Phi\). Since \(l\) is loose in \(\Psi\), we have that \(l \in \text{fcl}(Q_2)\) or \(l \in \text{fcl}(Q_m)\). We also have the following ABC-partitions that are not flowers: \((P_2, E(M) - (P_2 \cup Q_2), Q_2)\), \((P_2, E(M) - (P_2 \cup Q_m), Q_m)\), \((P_n, E(M) - (P_n \cup Q_2), Q_2)\), and \((P_n, E(M) - (P_n \cup Q_2), Q_2)\).
$Q_m)$. Suppose $|L| \geq 3$ and $\Psi$ is a paddle, a copaddle, or spike-like. Since $|L| \geq 3$, we may assume that $\text{fcl}(P_1) \cap \text{fcl}(P_2)$ has at least two elements. Since $\Psi$ is an anemone, by Theorem 3.2.3, we deduce that exactly one of the following is true: $L \subseteq \text{cl}(Q_2)$ or $L \subseteq \text{cl}^*(Q_2)$. In either case, $\text{fcl}(P_2) \cap \text{fcl}(Q_2) \geq 2$, which is a contradiction since $(P_2, E(M) - (P_2 \cup Q_2), Q_2)$ is an ABC-partition and is not a flower.

Assume that $\Phi$ and $\Psi$ are swirl-like flowers. Each member of $L$ is in exactly one of $\text{fcl}(Q_1) \cap \text{fcl}(Q_2)$ and $\text{fcl}(Q_1) \cap \text{fcl}(Q_m)$. Suppose $l_1 \in \text{fcl}(Q_2) \cap \text{fcl}(P_2)$. Without loss of generality, we may assume that $l_1 \in Q_2$. Suppose $l_2 \in \text{fcl}(P_2) \cap Q_m$ and $l_1 \neq l_2$. By Lemma 1.3.3(iii), the elements of $\text{fcl}(P_2) - P_2$ can be ordered $(p_1, p_2, \ldots, p_n)$ in such a way that $P \cup \{p_1, p_2, \ldots, p_n\}$ is 3-separating for all $i \in \{1, 2, \ldots, n\}$. Then, by Lemma 3.2.7, we may assume that there is an ordering of $\text{fcl}(P_2) - P_2$ in which $l_1 = p_{n-1}$ and $l_2 = p_n$ and that there is an ordering of $\text{fcl}(P_2) - P_2$ in which $l_1 = p_{n-1}$ and $l_2 = p_n$. Hence we have a contradiction to Lemma 3.2.20. If $l_2 \in Q_2$, then we have a contradiction to Proposition 3.2.19.

If $l_1 \in \text{fcl}(P_2) \cap Q_2$ and $l_2 \in \text{fcl}(P_n) \cap Q_2$, then, by Lemma 3.2.7, we may assume that there is an ordering of $\text{fcl}(P_2) - P_2$ in which $l_1$ is the last element and that there is an ordering of $\text{fcl}(P_n) - P_n$ in which $l_2$ is the last element. Then $Q_2 - l_1$ and $Q_2 - l_2$ are 3-separating. Thus $Q_2 - \{l_1, l_2\}$ is 3-separating, $l_1 \in \text{cl}^*(Q_2 - \{l_1, l_2\})$, and $l_2 \in \text{cl}^*(Q_2 - \{l_1, l_2\})$, which contradicts the fact that $l_1$ and $l_2$ are in a fan.

Finally, suppose $l_1 \in P_2$, $l_2 \in Q_2$, and $\{l_1, l_2\} \subseteq \text{fcl}(P_2) \cap \text{fcl}(Q_2)$. Then we have a contradiction to Proposition 3.2.19. Thus if $|L| > 2$, we have a contradiction.

The remaining case is to consider unresolved flowers. An unresolved flower can be thought of as a swirl-like flower or a spike-like flower depending on the properties of its loose elements. If $\Phi$ has an element $l$ such that $l \in \text{cl}(P_1) \cap \text{cl}(P_2) \cap \text{cl}(P_3)$ or $l \in \text{cl}^*(P_1) \cap \text{cl}^*(P_2) \cap \text{cl}^*(P_3)$, then we may treat $\Phi$ as a spike-like flower and apply
previous arguments. Otherwise, we may treat $\Phi$ as a swirl-like flower, and apply previous arguments. In either case, $|L| \leq 2$. \hfill $\square$

3.3 Representations of Equivalent Separations

Each non-sequential 3-separation of a 3-connected matroid $M$ is contained in an equivalence class of 3-separations. At most two members of this class are displayed by a particular 3-tree for $M$. Let $e$ be an edge of a 3-tree $T$ of $M$, and let $(X, Y)$ be the 3-separation displayed by $e$. Let $S_e$ be the set of 3-separations of $M$ equivalent to $(X, Y)$. If $f$ is a twin of $e$ in $T$, then $S_e = S_f$. By Theorem 1.4.5, $S = \{S_e | e \in T\}$ does not depend on $T$; that is, if $T_1$ and $T_2$ are 3-trees of $M$ and $e \in E(T_1)$, then either there is a unique edge $g$ in $T_2$ such that $S_e = S_g$, or there is a unique pair of twins $\{g, h\}$ in $T_2$ such that $S_e = S_g = S_h$. Thus there is a one-to-one correspondence between elements of $S$ and edges of $R(T_1)$, the reduction of $T_1$, that depends only on the matroid and not on the tree used to obtain $R(T_1)$. In the rest of the section, we will prove a series of lemmas that give us a natural way to represent different types of equivalence classes of 3-separations. The following result is noted in [12], and its proof is given here for completeness.

**Lemma 3.3.1.** Let $(X_0, Y_0)$ be a non-sequential 3-separation of $M$ and let $S$ be the set of 3-separations of $M$ equivalent to $(X_0, Y_0)$. Then there is a partition $(X, S_{XY}, Y)$ of $E(M)$ such that any element of $S$ can be written in the form $(X \cup Z, Y \cup (S_{XY} - Z))$ for some $Z \subseteq S_{XY}$.

**Proof.** Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be in $S$. Then $fcl(X_0) = fcl(X_1) = fcl(X_2)$ and $fcl(Y_0) = fcl(Y_1) = fcl(Y_2)$. Let $X = E(M) - fcl(Y_0)$, $Y = E(M) - fcl(X_0)$, and $S_{XY} = fcl(X_0) \cap fcl(Y_0)$. Thus the partition $(X, S_{XY}, Y)$ does not depend on any particular 3-separation in $S$. Suppose $e \in X - X_1$. Then $e \notin fcl(Y_0) = fcl(Y_1)$. But, as $e \notin X_1$, we have that $e \in Y_1 \subseteq fcl(Y_1)$, a contradiction. Thus $X \subseteq X_1$. Let
e \in X_1 - X. If e \in Y, then e \in \text{fcl}(Y) and e \in \text{fcl}(X_1), so e \in S_{XY}, a contradiction. Thus e \in S_{XY}, and so X_1 = X \cup Z for some Z \subseteq S_{XY} and Y_1 = Y \cup (S_{XY} - Z). \qed

Lemma 3.3.2. Let \((X_0, Y_0)\) be a non-sequential 3-separation of \(M\) equivalent to a 3-separation \((P_1, \bigcup_{i=2}^n P_i)\) displayed by a tight maximal flower \(\Phi = (P_1, \ldots, P_n)\) in \(M\). Let \(S\) be the set of 3-separations of \(M\) equivalent to \((X_0, Y_0)\). Then there is a partition \((X, T_X, L_X, Y)\) of \(E(M)\) such that \(X, X \cup T_X, \text{ and } X \cup T_X \cup L_X\) are 3-separating and every 3-separation in \(S\) can be written in the form \((X \cup Z, Y \cup ((T_X \cup L_X) - Z))\) for some \(Z \subseteq T_X \cup L_X\). Moreover, \(T_X\) contains only tight elements of the petal \(P_1\), and \(L_X\) contains only loose elements of the flower \(\Phi\).

Proof. By Lemma 3.3.1, we have a partition \((X, S_{XY}, Y)\) of \(E(M)\) with \(X \subseteq P_1, Y \subseteq \bigcup_{i=2}^n P_i, \text{ and } S_{XY} = \text{fcl}(X) \cap \text{fcl}(Y)\). By [24, Lemma 3.9], \(\text{fcl}(P_1) = \text{fcl}(X) = \text{fcl}(X_0)\). Thus we may partition \(\text{fcl}(X_0)\) into \(X, \text{ the tight elements } S_X \text{ of } P_1 \text{ outside of } X, \text{ and the loose elements } L_X \text{ of } \Phi \text{ in } \text{fcl}(P_1)\). The rest follows by Lemma 3.3.1. \qed

Lemma 3.3.3. Let \((X_0, Y_0)\) a non-sequential 3-separation of \(M\) equivalent such that \(M\) has two tight maximal flowers \(\Phi = (P_1, \ldots, P_n)\) and \(\Psi = (Q_1, \ldots, Q_m)\) with \(\text{fcl}(X_0) = \text{fcl}(P_1)\) and \(\text{fcl}(Y_0) = \text{fcl}(Q_1)\). Let \(S\) be the set of 3-separations equivalent to \((X_0, Y_0)\). Then there is a partition \(\{X, L, T_{XY}, Y\}\) of \(E(M)\) and an ordered 5-tuple \((X, L_X, T_{XY}, L_Y, Y)\) of subsets of \(E(M)\) whose union is \(E(M)\) such that any 3-separation \((X_1, Y_1)\) in \(S\) can be written in the form \((X \cup Z, Y \cup ((L \cup T_{XY}) - Z))\) for some \(Z \subseteq L \cup T_{XY}\). Moreover, \(L_X\) is a set of loose elements in \(\Psi\), \(L_Y\) is a set of loose elements in \(\Phi\), and \(T_{XY}\) consists of all of the elements that are tight in both \(P_1\) and \(Q_1\).
Proof. By Lemma 3.3.1, we have a partition \((X, S_{XY}, Y)\) of \(E(M)\) with \(X \subseteq P_1\), 
\(Y \subseteq Q_1\), and \(S_{XY} = \text{fcl}(X) \cap \text{fcl}(Y)\). Let \(L_X\) be the set of elements that are loose in \(\Psi\) and contained in \(S_{XY}\). Let \(L_Y\) be the set of elements that are loose in \(\Phi\) and contained in \(S_{XY}\). Let \(T_{XY} = E(M) - (X \cup L_X \cup L_Y \cup Y)\). An element of \(T_{XY}\) is tight in both flowers since it is loose in neither flower, and the rest of the result follows by Lemma 3.3.1.

Lemma 3.3.4. Let \((X_0, Y_0)\) be a sequential 3-separation displayed by some 3-tree \(T\) of \(M\), with \(X_0\) being sequential. Then there is a tight flower \(\Phi = (P_1, P_2, \ldots, P_n)\) in \(M\) with \(\text{fcl}(X_0) = \text{fcl}(P_1)\). Then \((X, L_X, Y)\) is a partition of \(E(M)\) such that if \((X_1, Y_1)\) is equivalent to \((X_0, Y_0)\), then \(X_1 \subseteq X \cup L_X\), and \(L_X\) is a set of loose elements in \(\Phi\).

Proof. Since \((X_0, Y_0)\) is sequential and is displayed by \(T\), by Lemma 5.10 in [24], there is a flower vertex \(v\) of \(T\) displaying a tight flower \(\Phi\) such that \(X_0\) labels a bag vertex incident with \(v\). Let \(\Phi = (P_1, P_2, \ldots, P_n)\) with \(X_0 = P_1\). The set \(X\) of tight elements of \(P_1\) is 3-separating. Let \(L_X\) be the set of loose elements in \(\Phi\) that are in \(\text{fcl}(P_1)\). Then, \(X \cup L_X = \text{fcl}(P_1)\) is 3-separating. We obtain the desired partition \((X, L_X, E(M) - \text{fcl}(P_1))\). If \((X_1, Y_1)\) is equivalent to \((X_0, Y_0)\), then \(\text{fcl}(X_1) = \text{fcl}(X_0) = \text{fcl}(X)\), and \(X_1 \subseteq X \cup L_X = \text{fcl}(X_0)\).

Let \(M\) be a 3-connected matroid and let \(T\) be a 3-tree for \(M\). Each edge \(e\) of the reduction \(R(T)\) corresponds to an equivalence class \(S_e\) of 3-separations of \(M\). We now classify each such edge into one of five different types based on the corresponding equivalence classes of 3-separations. This classification uses Lemmas 3.3.1-3.3.4.

(1) Let \(e\) have endpoints that are bag vertices of \(T\). By Lemma 3.10 in [24], every 3-tree for \(M\) has a unique edge displaying a member of \(S_e\) and both
endpoints of this edge are bag vertices. We have a non-sequential separation displayed by $e$, which can be represented by a partition $(X, S_{XY}, Y)$, as in Lemma 3.3.1. We call such an edge a **bag edge**.

(2) Let $e$ have one end vertex that is a bag and one end vertex that is a flower vertex of degree $n$. Then $e$ is not a twin edge in $T$ and every 3-tree of $M$ has a unique edge displaying a member of $S_e$ such that one endpoint of this edge is a bag and the other is a degree-$n$ flower vertex. We say $e$ is a **petal edge**, and there are two further possibilities.

(a) The edge $e$ displays a sequential 3-separation. Then $S_e$ can be represented by $(X, L_X, Y)$, as in Lemma 3.3.4. We call $e$ a **sequential petal edge**.

(b) The edge $e$ displays a non-sequential 3-separation. Then $S_e$ can be represented by $(X, T_X, L_X, Y)$, as in Lemma 3.3.2. We call $e$ a **non-sequential petal edge**.

(3) Let both end vertices of $e$ be flower vertices. By Lemma 3.10 in [24], every 3-tree of $M$ has exactly one edge or exactly two (twin) edges that display members of $S_e$. Then $S_e$ can be represented by $(X, L_X, T_{XY}, L_Y, Y)$, as in Lemma 3.3.3. We may refine this further into two types.

(a) $T_{XY} = \emptyset$. We say $e$ is an **unspli t twin edge**. In this case, $M$ has a 3-tree $T'$ in which there is a unique edge displaying a member of $S_e$ and this edge joins two flower vertices of $T'$.

(b) $T_{XY} \neq \emptyset$. We say $e$ is a **split twin edge**. In this case, every 3-tree $T'$ of $M$ has exactly two edges displaying members of $S_e$. These edges are twins and their non-common vertices are both flower vertices.
3.4 Decomposition Tree

Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$. A 3-tree $T$ for $M$ gives us a partition of the ground set that displays all of the non-sequential 3-separations of $M$, up to equivalence. The difficulty with obtaining a unique partition of the ground set that displays all the non-sequential 3-separations of $M$ is the presence of elements that appear in different bags in different 3-trees for $M$. Thus we need more information about such elements before we can produce a unique decomposition.

To analyze the behavior of these elements that can move, we create a labeled tree that will allow us to display these elements along with the non-sequential 3-separations that correspond to their movements. We call this new tree an expanded 3-tree of $M$. Given a 3-tree $T$ of $M$, we show how to construct an expanded 3-tree from $T$ and the reduction $R(T)$, and then show that this construction is well-defined and does not depend on the particular choice of $T$. A bag of $T$ is normal if it does not label a degree-2 vertex that is incident with a pair of twin edges.

Recall that, in a 3-tree for a matroid $M$, the vertices are either bag vertices or flower vertices. Each bag vertex is labeled by a (possibly empty) subset of $E(M)$ such that every element appears in exactly one of these subsets. The flower vertices have no associated subset of $E(M)$. In defining an expanded 3-tree $T'$ based on $T$, we shall modify the labeling on $T$ by adding labels to flower vertices, by allowing the sets labeling vertices to overlap, and by adding or removing labeled vertices by subdividing or contracting edges. Formally, we proceed as follows:

**Step 0:** Assign the empty set as the label of each flower vertex of $T$. Call these flower bags.

**Step 1:** Let $\{e, f\}$ be a pair of twin edges in $T$. Let $v$ be their common vertex, and let $e = v_x v$ and $f = vv_y$. Let $V$ be the set labeling $v$, let $V_x$ be the set labeling
the flower vertex $v_x$, and let $V_y$ be the set labeling the flower vertex $v_y$. Let $g$ be the edge in $R(T)$ corresponding to $e$. The equivalence class $S_g$ can be represented by $(X, L_X, T_{XY}, L_Y, Y)$. If $g$ is an unsplit twin edge, then $T_{XY}$ is empty. In $T$, contract the edge, labeling the resulting vertex $v_x$. Replace $V_x$ by $V_x \cup L_X$ as the set labeling $v_x$ and replace $V_y$ by $V_y \cup L_Y$ as the set labeling $v_y$. Delete members of $L_X \cup L_Y$ from all sets that label normal bag vertices. If $g$ is a split twin edge, then replace $V$ by $T_{XY}$ as the set labeling $v$. Replace $V_x$ by $V_x \cup L_X$ as the set labeling $v_x$ and replace $V_y$ by $V_y \cup L_Y$ as the set labeling $v_y$. Delete members of $L_X \cup L_Y$ from all sets that label normal bag vertices. Apply Step 1 to every other pair of twins in $T$.

**Step 2:** Let $e$ be an edge of $T$ that is adjacent to two flower vertices. Then there is an unsplit twin edge $f$ in $R(T)$ such that the 3-separation induced by $e$ in $T$ is in $S_f$. Let $e = v_xv_y$, where $v_x$ and $v_y$ are flower vertices labeled by $V_x$ and $V_y$, respectively. Then $S_f$ can be represented by $(X, L_X, T_{XY}, L_Y, Y)$ with $T_{XY}$ empty. Replace $V_x$ by $V_x \cup L_X$ as the set labeling $v_x$ and replace $V_y$ by $V_y \cup L_Y$ as the set labeling $v_y$. Delete members of $L_X \cup L_Y$ from all sets that label normal bag vertices. Apply Step 2 to every other edge adjacent to two flower vertices.

**Step 3:** Let $e$ be an edge of $T$ such that the corresponding edge $f$ in $R(T)$ is a petal edge. If $f$ is a sequential petal edge, then we can represent $S_f$ by $(X, L_X, Y)$. Replace the label of the pendant vertex by $X$, replace the set $V$ labeling the flower vertex by $V \cup L_X$, and delete members of $L_X$ from all sets that label normal bag vertices. If $f$ is a non-sequential petal edge, then we can represent the separation displayed by $e$ by $(X, T_X, L_X, Y)$. Let $e = v_xv_y$, where $v_x$ is labeled by a normal bag $V_x$ and $v_y$ is a flower vertex.
labeled by $V_y$. If $T_X$ is not empty, subdivide $e$ to obtain the two-edge path $v_xvv_y$. Label $v_x$ by $V_x$, label $v$ by $T_X$, and replace $V_y$ by $V_y \cup L_X$ as the set labeling $V_y$. Delete members of $T_X \cup L_X$ from all sets that label normal bag vertices, including the set labeling $v_x$. Apply Step 3 to every other edge that corresponds to a petal edge.

**Step 4:** Let $e$ be an edge of $T$ such that the corresponding edge $f$ in $R(T)$ is a bag edge. Then we can represent $S_f$ by $(X, S_{XY}, Y)$. Let $e = v_xv_y$ where $v_x$ and $v_y$ are bag vertices labeled by $V_X$ and $V_Y$, respectively. If $S_{XY}$ is not empty, then subdivide $e$ to obtain the two-edge path $v_xvv_y$. Label $v_x$ by $V_X$, $v_y$ by $V_Y$, and $v$ by $S_{XY}$. Delete members of $S_{XY}$ from all sets that label normal bag vertices, including the sets labeling $v_x$ and $v_y$. Apply Step 4 to every other edge that corresponds to a bag edge.

In the resulting expanded 3-tree $T'$, call a bag a *sequential bag* if it labels one of the following types of vertices in $T'$: a flower vertex, a vertex obtained by subdividing an edge in Step 3 or Step 4, a degree-2 vertex incident with a pair of twin edges, or a pendant vertex whose label is sequential in $M$. Call all other bags *non-sequential bags*. Similarly, call a vertex *sequential* or *non-sequential* if it is labeled by a sequential or non-sequential bag, respectively. Note that we have not yet determined whether the above procedure produces a unique expanded 3-tree.

**Lemma 3.4.1.** The expanded 3-tree $T'$ obtained from $M$ and a 3-tree $T$ of $M$ is well-defined. Moreover, the expanded 3-trees $T'_1$ and $T'_2$ obtained from two different 3-trees $T_1$ and $T_2$ of $M$ are isomorphic and corresponding vertices are labeled by the same set.

**Proof.** Clearly, the tree $T'$ is unique for a given 3-tree $T$. Let $v$ be a vertex of $T'$ and let $B$ be its label. If $v$ is a flower vertex, then, in Steps 1-3, all the loose
elements of the flower displayed by \( v \) in \( T \) are added to \( B \) without duplication. If \( v \) is a sequential vertex that is not a flower vertex, then the label of \( v \) is created or modified only when a particular edge or pair of twin edges is considered. If \( v \) is a non-sequential vertex, then let \( B' \) be its label in \( T \). By construction, \( B \subseteq B' \). Let \( n \) be the degree of \( v \) in \( T \) and let \( \{e_1, e_2, \ldots, e_n\} \) be the edges of \( T \) adjacent to \( v \). If \( B' \) is not empty, then, for each \( i \), let \((A_i, B_i)\) be the 3-separation of \( M \) induced by \( e_i \) with \( B' \subseteq B_i \). Note that \( B' = \bigcap_{i=1}^{n} B_i \). Then, by construction, either \( B' \) and \( B \) are empty, or \( B = \bigcap_{i=1}^{n} (B_i - \text{fcl}(A_i)) \). In either case, \( B \) is uniquely determined. Therefore, \( T' \) is well-defined.

In the procedure for constructing an expanded 3-tree \( T \), we consider each edge \( e \) of \( R(T) \). By Theorem 1.4.5, \( R(T_1) \) is isomorphic to \( R(T_2) \), and, since we decide to contract or subdivide \( e \) based on a representation for \( S_e \), which is the same for \( T_1 \) and \( T_2 \), the trees \( T'_1 \) and \( T'_2 \) are isomorphic. Let \( \phi \) be an isomorphism from \( T'_1 \) onto \( T'_2 \). By Theorem 1.4.5, if \( v \) is a flower vertex, then \( v \) and \( \phi(v) \) display equivalent flowers. Thus, by the construction of \( T'_1 \) and \( T'_2 \), \( v \) and \( \phi(v) \) are labeled by the same set of loose elements. If \( v \) is a sequential vertex that is not a flower vertex, then the labels of \( v \) and \( \phi(v) \) were determined by a representation for the same equivalence class of 3-separations, and so are equal. Finally, suppose \( v \) is a non-sequential vertex and let \( B \) be the label of \( v \) in \( T_1 \). Let \( n \) be the degree of \( v \) in \( T_1 \) and let \( \{e_1, e_2, \ldots, e_n\} \) be the edges of \( T_1 \) adjacent to \( v \). If \( B \) is not empty, then, for each \( i \), let \((A_i, B_i)\) be the 3-separation of \( M \) induced by \( e_i \) with \( B \subseteq B_i \). Then, by construction, \( \bigcap_{i=1}^{n} (B_i - \text{fcl}(A_i)) \) labels \( v \) in \( T'_1 \). By Theorem 1.4.5, each separation \((A_i, B_i)\) has a corresponding equivalent 3-separation \((A'_i, B'_i)\) in \( T_2 \). As \( B_i - \text{fcl}(A_i) = B'_i - \text{fcl}(A'_i) \) for all \( i \), we have that the labels of \( v \) and \( \phi(v) \) are the same if \( B \) is non-empty, or, by symmetry, if the label of \( \phi(v) \) in \( T_2 \) is non-empty. Otherwise, both labels are empty. Therefore, the labels of \( v \) and \( \phi(v) \) are equal. \( \square \)
Next, we consider properties of elements that appear more than once in the expanded 3-tree. By construction, such an element $e$ only appears in sequential bags of the expanded 3-tree. We call such elements \textit{wild}.

\textbf{Lemma 3.4.2.} Let $T'$ be the expanded 3-tree for $M$. Let $X$ and $Y$ be sequential bags labeling distinct vertices $x$ and $y$, respectively, and let $e \in X \cap Y$. Every sequential bag $Z$ labeling a vertex $z$ on the path $P$ in $T'$ from $x$ to $y$ contains $e$, unless $z$ neighbors a flower vertex $v$ and $e$ is loose in the flower displayed by $v$.

\textit{Proof.} Let $T$ be a 3-tree for $M$. If $z$ is a degree-2 vertex and both of its neighbors are labeled by non-sequential bags, then $z$ corresponds to a bag edge $f$ in $R(T)$. Let $S_f$ be represented by $(A,S_{AB},B)$. Then $e \in S_{AB} = Z$. If $z$ is a flower vertex, then $z$ came from a flower vertex of $T$ and $e$ is loose in the flower displayed by this vertex. Hence $e \in Z$. If $z$ was obtained by subdividing a non-sequential petal edge of $T$, then one neighbor of $z$ is a flower vertex $v$ that is on $P$. Thus, $e$ is loose in the flower displayed by the vertex $v$. So $e \notin Z$. It is impossible for $z$ to be a pendant vertex adjacent to a sequential petal edge unless $z$ is $x$ or $y$. The remaining case is when $z$ is degree-2 and both of its neighbors are flower vertices. If $e$ is loose in neither flower displayed by the two neighboring vertices, then $e$ is in the bag labeling $z$. \hfill \square

\textbf{Lemma 3.4.3.} Suppose $e$ and $f$ are two different wild elements in $M$ and both $e$ and $f$ are in a sequential bag $X$ labeling a vertex $x$. Then either $X$ is the only sequential bag containing both $e$ and $f$, or $x$ is a flower vertex and there is at most one other flower vertex $y$ whose label $Y$ contains $e$ and $f$.

\textit{Proof.} Suppose $X$ labels a vertex $x$ that is not a flower vertex, $Y$ labels a vertex $y$, and $\{e,f\} \subseteq X \cap Y$. Consider the path $P$ from $x$ to $y$ in $T$. If the path is a
single edge, then $y$ is a flower vertex. Since $e$ and $f$ are loose in that flower, neither $e$ nor $f$ can be in $Y$. If the path has length at least two, then by Lemma 3.4.2, $e$ and $f$ are in any sequential bag along $P$. If the path contains a vertex labeled by a non-sequential bag, then we have two inequivalent non-sequential 3-separations in $M$ in which both $e$ and $f$ are sequential. This contradicts Theorem 3.2.5. Thus every vertex on the path is labeled by a sequential bag. Since $x$ is not a flower vertex, it is adjacent to a flower vertex and we have a contradiction. Thus $x$ has degree 2 and $X$ is the only label containing $e$ and $f$. If $x$ is a flower vertex, then every other vertex $y$ that has a label containing both $e$ and $f$ is a flower vertex.

Suppose there are three flower vertices $x$, $y$, and $z$ whose labels contain $e$ and $f$. Then there is a path containing $x$, $y$, and $z$, or a vertex $w$ on the path from $x$ to $y$ such that there is a path from $w$ to $z$. Two edges on the path from $x$ to $y$ are on different sides of $z$ or $w$ and induce two inequivalent non-sequential 3-separations in $M$. Thus $z$ cannot have two wild elements, and so there are at most two flower vertices with labels containing $e$ and $f$. 

Now, we modify the expanded 3-tree $T'$ to produce the decomposition tree $T''$ for $M$. In particular, we modify $T'$ by adding wild elements of $M$ to some of the bags as follows. Note that, in this process, no bag gets more than one copy of an element. For each bag $B$ of $T'$, let $v$ be the vertex labeled by $B$ and consider $T' - v$. Suppose $x$ belongs to bags in more that one component of $T' - v$. Then $x$ is certainly wild, and we may add $x$ to $B$. Suppose $x$ belongs to a bag of just one component, say $T'_x$, of $T' - v$ and $x$ is wild. Let $X$ be the union of the bags in $T'_x$. Then $(E - \text{fcl}(X), \text{fcl}(X))$ is a 3-separation of $M$. If $x \in \text{cl}^{(*)}(E - \text{fcl}(X))$, then add $x$ to $B$. 

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As we have changed the expanded 3-tree and the expanded 3-tree depends only on \( M \), the decomposition tree depends only on \( M \). Thus, the decomposition tree for \( M \) is unique.

**Lemma 3.4.4.** Let \( M \) be a 3-connected matroid with \(|E(M)| \geq 9\). Then \( M \) has a unique decomposition tree.

Call each set that labels a vertex of the decomposition tree \( T'' \) a bag, as in previous trees. Moreover, call a vertex or bag of \( T'' \) non-sequential or sequential if its corresponding vertex or bag in the expansion tree \( T' \) is non-sequential or sequential, respectively.

### 3.5 The Pieces and the Decomposition

Let \( M \) be a 3-connected matroid with \(|E(M)| \geq 9\). The goal of this section is to extract pieces of the decomposition from the sets labeling vertices in the decomposition tree for \( M \). First, we will show how to obtain sequentially 4-connected pieces from non-sequential vertices. Then, we will obtain sequential 3-connected matroids from non-flower sequential vertices. Finally, we will obtain 3-connected pieces from flower vertices.

Let \( v \) be a degree-\( n \) vertex labeled by \( B \) in the decomposition tree \( T'' \). Delete an edge \( e \) neighboring \( v \). Then \( T'' \setminus e \) has two components \( T_1 \) and \( T_2 \). Let \( X_1 \) and \( X_2 \) be the unions of the labels of the vertices in \( T_1 \) and \( T_2 \), respectively. Now, \( X_1 \) and \( X_2 \) are not necessarily disjoint, due to the presence of wild elements. Assume \( B \subseteq X_1 \) and let \( W = X_1 \cap X_2 \). Then \((X_1, X_2 - W)\) is a 3-separation in \( M \), and we may fill the line of separation of \((X_1, X_2 - W)\). We call \((X_1, X_2 - W)\) the 3-separation induced by \( e \) and \( B \) in \( M \). Let \( \{e_1, e_2, \ldots, e_n\} \) be the edges adjacent to \( v \). For each \( i \) in \( \{1, 2, \ldots, n\} \), complete the line \( L_i \) of separation corresponding to \( e_i \) and \( B \). By Lemma 2.4.4, we can perform all such extensions one after the other and obtain a
unique extension $M'$ of $M$. Let $B' = M'|(B \cup L_1 \cup L_2 \cup \cdots \cup L_n)$. Then we say the $B'$ is obtained from $M$ and $T''$ by \textit{detaching} $B$. The following lemma proves that this operation produces a unique 3-connected matroid.

\textbf{Lemma 3.5.1.} Let $B$ be the label of a vertex in the decomposition tree $T''$ for $M$. There is a unique 3-connected matroid $B'$ obtained from $M$ and $T''$ by detaching $B$.

\textit{Proof.} Let $v$ be the degree-$n$ vertex of $T''$ labeled by $B$. By Lemma 2.4.4, there is a unique extension $M'$ of $M$ and $B'$ is a restriction of $M'$. Next we show that $B'$ is 3-connected. By Lemma 2.4.3, we may assume that any new elements were added to the side of the separation that contains $B$. Thus, by Lemma 2.4.4, we may obtain the extension $M'$ of $M$ and a set of 3-separations $\{(A'_i, B'_i) \mid i \in \{1, 2, \ldots, n\}\}$ with $B \subseteq B'_i$ and $E(M') - E(M) \subseteq B'_i$ for all $i$. By repeated applications of Theorem 2.1.1 to these separations, we obtain $B'$ and that $B'$ is 3-connected.

For each bag of the decomposition tree, we obtain a 3-connected piece. We next show that if we choose a non-sequential bag or a sequential bag that does not label a flower vertex, then we obtain a sequentially 4-connected piece.

\textbf{Lemma 3.5.2.} Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$ and let $T''$ be the decomposition tree for $M$. If $B$ is a non-sequential bag, then the matroid $B'$ obtained by detaching $B$ is sequentially 4-connected.

\textit{Proof.} Let $(X, Y)$ be a non-sequential 3-separation of $B'$. Since $|L_i| \geq 3$, $L_i$ is spanned by $X$ or $Y$, for each $i$. The 3-connected matroid $B'$ was obtained by recursive applications of Theorem 2.1.1, so we may apply Lemma 2.3.8 recursively to obtain a non-sequential 3-separation $(X', Y')$ of $M'$ with $X \subseteq X'$ and $Y \subseteq Y'$. We may apply Lemma 2.3.5 recursively to obtain a non-sequential 3-separation
(X^−, Y^−) of M. Choose a 3-tree T for M. If v is the vertex of T" that is labeled by B, then there is a corresponding bag vertex w in T. Let w be labeled by C. Now (X^−, Y^−) is equivalent to some 3-separation (X^+, Y^+) displayed by T. We may assume that either there is an edge e of T neighboring w such that e displays (X^+, Y^+), or that the flower vertex or other edge of T which displays (X^+, Y^+) is in a component of T\e different from the one containing w. Without loss of generality, we may assume C ⊆ fcl(X^+) and fcl(X^−) = fcl(X^+).

Now (X^−, Y^−) is equivalent to some 3-separation (X^+, Y^+) displayed by T. We may assume that either there is an edge e of T neighboring w such that e displays (X^+, Y^+), or that the flower vertex or other edge of T which displays (X^+, Y^+) is in a component of T\e different from the one containing w. Without loss of generality, we may assume C ⊆ fcl(X^+) and fcl(X^−) = fcl(X^+).

Now, there is a class Se of 3-separations of M that are equivalent to the one displayed by e in T, and this class can be represented by (V, S)_V Z with C ⊆ fcl(V) ⊆ fcl(X^+). Moreover, the edge corresponding to e in T" gives us a 3-separation (V', Z') induced by e and B with B ⊆ V' that is equivalent to (V \ S, V Z). Then fcl(V') = fcl(V'). Since Y ⊆ fcl(V') = fcl(V) ⊆ fcl(X^+), we have that Y ⊆ fcl(X^−). Then Y ⊆ X' and we may recursively apply the contrapositive of Lemma 2.3.8. We deduce that (X, Y) is sequential in B', a contradiction.

**Lemma 3.5.3.** Let (A, x_1, x_2, . . . , x_n, B) be a 3-sequence in M and let \(\overrightarrow{X}\) be the ordered set (x_1, x_2, . . . , x_n). Let \(L_A = \text{cl}(A) \cap \text{cl}(B \cup \overrightarrow{X})\) and \(L_B = \text{cl}(A \cup \overrightarrow{X}) \cap \text{cl}(B)\). If |L_A|, |L_B| ≥ 3, then \(M|(L_A \cup \overrightarrow{X} \cup L_B)\) is a sequential 3-connected matroid.

**Proof.** By Theorem 2.1.1, \(M|(L_A \cup \overrightarrow{X} \cup B)\) and \(M|(L_A \cup \overrightarrow{X} \cup L_B)\) are 3-connected.

As A spans \(L_A\) and |B| ≥ 2, elements of \(L_A\) in \(\overrightarrow{X}\) may be moved to the front of the sequence, and, as \(L_B\) is spanned by B and |A| ≥ 2, elements of \(L_B\) in \(\overrightarrow{X}\) may be moved to the end of the sequence. Call one such reordering \(\overrightarrow{Y}\). Now, \(M|(L_A \cup \overrightarrow{Y} \cup L_B)\) is 3-connected, since \(L_A \cup Y \cup L_B = L_A \cup X \cup L_B\). As \(2 = r(A \cup \overrightarrow{Y_i}) + r(B \cup \overrightarrow{Y_i}) - r(M) ≥ r(L_A \cup \overrightarrow{Y_i}) + r(L_B \cup \overrightarrow{Y_i}) - r(L_A \cup \overrightarrow{Y} \cup L_B) ≥ 2\), we have that \(L_A \cup \overrightarrow{Y_i}\) is 3-separating in \(M|(L_A \cup \overrightarrow{X} \cup L_B)\) for all i. Moreover, at least two elements of \(L_A\) are at the beginning of the sequence and at least

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two elements of $L_B$ are at the end. As $L_A$ and $L_B$ are segments, we can list the elements of $L_A$ and $L_B$ at the ends in any order to obtain a sequential ordering $(\overrightarrow{L_A}, \overrightarrow{Y} - (L_A \cup L_B), \overrightarrow{L_B} - L_A)$ of $L_A \cup \overrightarrow{Y} \cup L_B$.

Lemma 3.5.4. Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$ and let $T''$ be the decomposition tree for $M$. If $B$ is a sequential bag that does not label a flower vertex, then the matroid $B'$ obtained by detaching $B$ is a sequential 3-connected matroid.

Proof. Let $v$ be the vertex of the decomposition tree labeled by $B$. Since $B$ does not label a flower vertex, the degree of $v$ is one or two. Lemma 3.5.1 gives us $B'$ and that $B'$ is 3-connected. In both cases, Lemma 3.5.1 gives us $B'$ and that $B'$ is 3-connected. Lemma 3.5.3 gives us a sequential ordering of $B'$.

Finally, we consider the pieces obtained from the flower bags of the decomposition tree. We call such a piece a flower piece. First, we consider the paddle case.

Lemma 3.5.5. Let $v$ be a vertex of the decomposition tree which displays a tight maximal paddle and is labeled by $B$. If $B'$ is obtained by detaching $B$, then $B'$ is isomorphic to $U_{2,n}$ for some $n \geq 3$.

Proof. Let $L$ be the line completing $(P_i, \bigcup_{j \neq i} P_j)$. As $\cap_M(P_i, P_j) = \cap_{M'}(P_i, P_j) = 2$ and $L \subseteq \cl_{M'}(P_i)$, we have that $L \subseteq \cl_{M'}(P_j)$ for all $j$. Thus $B' = M|L$, and the result follows.

In the case of paddles, we have obtained a special sequentially 4-connected matroid. For any other type of flower, we obtain a special 3-connected matroid that may not be sequentially 4-connected. These are the only pieces of the decomposition which we permit not to be sequentially 4-connected. Their role is similar to the role played by cycles and bonds in the decomposition of 2-connected graphs into
3-connected graphs, cycles, and bonds in that a flower piece in our decomposition inherits the crossing 3-separations that the flower displays in $M$.

Thus we have obtained sequentially 4-connected or 3-connected pieces from each vertex of the decomposition tree. Now that we have all the pieces, we can prove Theorems 3.1.1 and 3.1.2.

**Proof of Theorem 3.1.1.** Let $T''$ be the decomposition tree for $M$. For each vertex $v$ of $T''$, we may detach the piece $M_v$ at $v$ to obtain a set of pieces $\{M_v \mid v \in V(T)\}$. Since the decomposition tree is unique and the operation of detaching at a vertex produces a unique matroid, the set of pieces is unique and each piece is either sequentially 4-connected or is a flower piece that might only be 3-connected. \qed

**Proof of Theorem 3.1.2.** The chosen representation of $M$ embeds $M$ in $PG(r-1, q)$ and fixes coordinates for each point of $M$. Let $T$ be the decomposition tree for $M$. For each vertex $v$ of $T$, we may detach the piece $M_v$ at $v$ to obtain a set of pieces $\{M_v \mid v \in V(T)\}$. Since the decomposition tree is unique, the set of pieces is unique, and each piece is either sequentially 4-connected or is a flower piece that might only be 3-connected. Moreover, because $M$ is represented, each piece is represented. Also, when any new element is added to $M$ in completing a line of separation, label the point so we know that it was not originally in $M$. Thus, we may recover $M$ by identifying points with the same coordinates and then deleting those points that were labeled as not originally being in $M$. \qed
Chapter 4
The Structure of 3-Connected Graphs

4.1 Overview

The following are well-known results that relate graph connectivity and matroid connectivity:

**Lemma 4.1.1.** Let $G$ be a graph having no isolated vertices. If $|V(G)| \geq 3$, then $M(G)$ is 2-connected if and only if $G$ is 2-connected and loopless.

**Lemma 4.1.2.** Let $G$ be a graph having no isolated vertices. If $|V(G)| \geq 3$ and $G \not\cong K_3$, then $M(G)$ is 3-connected if and only if $G$ is 3-connected and simple.

We define the *vertex boundary* $\partial(A)$ of a set $A$ of edges of a graph $G$ to be $V(G[A]) \cap V(G[E - A])$. We will also refer to members of $\partial(A)$ as *vertices of attachment*.

Since graphic matroids are binary, we may apply Theorem 2.1.1 to decompose the cycle matroid $M$ of a simple 3-connected graph $G$ at a line of separation. However, the resulting matroids $M'$, $A'$, and $B'$ from Theorem 2.1.1 are not guaranteed to be graphic. For example, if we view $K_4$ as a three-petal unresolved flower and let $A$ be one of the petals, then $M'$ is the Fano matroid, which is non-graphic. But $M(K_4)$ is sequentially 4-connected, and so we do not need to decompose it. We will see that this is the only difficulty in specializing our previous results to graphs.

In Section 4.2, we will consider the properties of flowers in graphs, and then, in Section 4.4, we will specialize Theorem 3.1.2 to simple 3-connected graphs.

We also prove a characterization of sequential 3-connected graphs in Section 4.3. Consider a simple 3-connected graph $G$ whose edges can be partitioned into a
sequence of sets, each of which is a triangle or a bond, such that the sequence starts with a degree-3 vertex or triangle and then alternates between bonds or triangles, as illustrated in Figure 4.1. We call such a graph a sequential 3-path. We will prove that every sequential 3-connected graph $G$ is a minor of a sequential 3-path. Moreover, $G$ is obtained from a sequential 3-path by contracting edges in certain bonds or deleting edges in certain triangles, and then deleting all but one edge in each parallel class or contracting all but one edge in each series class.

4.2 Flowers in Graphs

Let $G$ be a simple 3-connected graph. Then we say that $\Phi$ is a flower in $G$ if $\Phi$ is a flower in $M(G)$. In this section, we will obtain properties of flowers in $G$. To use results in [23], we must restrict ourselves to graphs having at least nine edges. This restriction is not difficult to deal with, as Tutte’s Wheels and Whirls Theorem implies the following:

**Lemma 4.2.1.** Let $G$ be a simple 3-connected graph with $|E(G)| < 9$. Then $M(G)$ is sequentially 4-connected and $G$ is isomorphic to $K_4$ or $W_4$.

The main result of this section is the following:
Theorem 4.2.2. Let $G$ be a simple 3-connected graph. If $\Phi$ is a tight flower in $M(G)$ with at least three petals, then exactly one of following is true:

(i) $\Phi$ is an unresolved flower in which the three petals are the three disjoint perfect matchings in $G \cong K_4$.

(ii) $\Phi$ is a paddle, a swirl-like flower, or an unresolved flower in which each of the petals induces a connected subgraph, has at least three edges, and has a vertex boundary consisting of exactly three vertices.

In order to prove this theorem, the next lemma considers the properties of 3-separating partitions in graphs. We then apply this lemma to analyze the different types of flowers in 3-connected graphic matroids, from which Theorem 4.2.2 immediately follows.

Lemma 4.2.3. Let $(X, Y)$ be an exactly 3-separating partition of a 3-connected graphic matroid $M(G)$. Then one of the following holds:

(i) $G[X]$ and $G[Y]$ are connected and have exactly three vertices in common.

(ii) $X$ or $Y$ consists of two edges.

(iii) $X$ or $Y$ is a bond consisting of three vertex-disjoint edges.

Proof. Recall that the number of connected components of a graph $H$ is denoted by $k(H)$. As $r(X)+r(Y) = r(M)+2$ and $|V(G[X])|+|V(G[Y])|-|V(G)| = |\partial(X)|$, we have $|V(G[X])| - k(G[X]) + |V(G[Y])| - k(G[Y]) = |V(G)| - 1 + 2$, or

$$|\partial(X)| = k(G[X]) + k(G[Y]) + 1. \quad (4.1)$$

We may assume that $|X|, |Y| \geq 3$. Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_m\}$ be the sets of components of $G[X]$ and $G[Y]$, respectively. Without loss of generality,
we may assume that \( n \geq m \). Assume \( G[X] \) has \( s_x \) single-edge components and \( t_x \) other components so that \( s_x + t_x = n \). Define \( s_y \) and \( t_y \) for \( G[Y] \) similarly. Also, note that any of the components of \( G[X] \) or \( G[Y] \) other than a single edge must meet the rest of \( G \) (and hence \( G[Y] \) or \( G[X] \), respectively) in at least three vertices. Otherwise, we would have a 2-separation in \( M(G) \).

As \( k(G[X]) = n \geq k(G[Y]) \), we have

\[
1 + 2s_x + 2t_x = 1 + 2n \geq 1 + k(G[X]) + k(G[Y]) = |\partial(X)|. \tag{4.2}
\]

Since the \( X_i \)'s are vertex disjoint,

\[
|\partial(X)| = \sum_{i=1}^{n} |V(G[X_i]) \cap V(G[Y])| \geq 2s_x + 3t_x. \tag{4.3}
\]

Thus \( t_x \leq 1 \) and

\[
2s_x + (2t_x + 1) \geq |\partial(X)| \geq 2s_x + 3t_x. \tag{4.4}
\]

Now if \( Y_i \) is a single-edge component of \( G[Y] \), then the two vertices of \( Y_i \) are in different components \( X_h \) and \( X_j \) of \( G[Y] \); otherwise, \( (X \cup E(Y_i), Y - E(Y_i)) \) is a 2-separation of \( M(G) \). Furthermore, each of \( X_h \) and \( X_j \) has more than one edge; otherwise, \( G \) has a degree-2 vertex. Hence, as \( t_x \leq 1 \), we have \( s_y = 0 \).

Suppose \( t_x = 0 \). Then \( G[X] \) is composed of \( n = s_x \) single-edge components, and so \( |\partial(X)| = 2s_x \). Consider the graph \( G' \) whose vertices are the components of \( G[Y] \) and whose edges are the edges of \( X \); that is, if \( Y_i \) and \( Y_j \) are components of \( G[Y] \) and \( y_i \) and \( y_j \) are their corresponding vertices in \( G' \), then there is an edge \( y_iy_j \) in \( G' \) for each edge of \( G[X] \) from \( Y_i \) to \( Y_j \). Note that we allow loops and multiple edges in \( G' \). As \( |\partial(X)| = 2s_x \) and \( k(G[X]) = s_x \), we have \( 2s_x = s_x + k(G[Y]) + 1 \) from (4.1). Thus \( k(G[Y]) = s_x - 1 \). Now, \( |E(G')| = |X| = s_x \) and \( |V(G')| = k(G[Y]) = s_x - 1 \). By assumption, \( |X| \geq 3 \). If \( |X| = 3 \), then \( G[Y] \) has two components, and so (iii) holds.
Suppose $|X| > 3$. Then the average degree $\frac{2s_x}{s_x - 1}$ of $G'$ is less than three. Thus there is a component $Y_h$ of $G[Y]$ that meets $G[X]$ in exactly two vertices. It follows, as $G$ is connected, that $Y_h$ meets exactly two edges of $X$. Hence these two edges form an edge cut of $G$, and we obtain a contradiction to the 3-connectedness of $G$.

We may now assume that $t_x = 1$. Then equality holds throughout (4.4), and so $|\partial(X)| = 2s_x + 3$. Moreover, equality also holds in (4.2). Thus $k(G[X]) = k(G[Y])$. Hence we may apply the analysis that generated the inequalities (4.2) to $Y$. Combining this with the facts that $\partial(Y) = \partial(X)$ and $s_y = 0$, we have

$$3t_y \leq |\partial(Y)| = |\partial(X)| \leq 1 + 2t_y. \quad (4.5)$$

Hence $t_y = 1$. Since $|\partial(X)| = 2s_x + 3$, we deduce that $s_x = 0$, and so (i) holds. $\Box$

**Lemma 4.2.4.** Let $\Phi$ be a tight flower having at least three petals in a 3-connected graphic matroid $M(G)$. Then no petal of $\Phi$ is a bond in $G$ consisting of three vertex-disjoint edges.

**Proof.** Let $\Phi = (P_1, P_2, \ldots, P_n)$. Assume $P_1$ is a bond in $G$ consisting of three vertex-disjoint edges, and let $X$ and $Y$ be the edge sets of the components of $G \setminus P_1$. Suppose $n \geq 4$. Then $(P_1 \cup P_2, P_3 \cup \cdots \cup P_n)$ is a 3-separation and each side contains at least four elements. Therefore, $G[P_1 \cup P_2]$ and $G[P_3 \cup \cdots \cup P_n]$ are connected and have a common vertex boundary of size 3. Thus $P_3 \cup \cdots \cup P_n \subseteq X$ or $P_3 \cup \cdots \cup P_n \subseteq Y$. So $P_2 \supseteq X$ or $P_2 \supseteq Y$. Similarly, $G[P_1 \cup P_n]$ is connected, and so $P_n \supseteq Y$ or $P_n \supseteq X$. Thus $\{P_2, P_n\} = \{X, Y\}$, which is a contradiction since $n \geq 4$. Therefore $\Phi$ has three petals. If $G[P_2]$ and $G[P_3]$ are connected, then $\{P_2, P_3\} = \{X, Y\}$. But then $P_1$ is a loose petal since its elements are in the coclosure of $P_2$. Therefore $G[P_2]$ or $G[P_3]$ is disconnected. Now $|V(G)| \geq 6$ and each vertex has degree at least three, and so $|E(G)| \geq 9$. If $|E(G)| = 9$, then
$G[P_2 \cup P_3]$ is the vertex-disjoint union of two triangles. Let $|P_2| = 2$. Then $|P_3| = 4$. But in this case it is impossible for $P_3$ to be connected as required by Lemma 4.2.3. Then let $|P_2| = |P_3| = 3$. As $P_2$ is disconnected, by Lemma 4.2.3, it must also be a matching, which is impossible given the structure of $G$. Therefore we deduce that $|E(G)| \geq 10$. Without loss of generality, $|P_3| = 4$, and so $G[P_3]$ is connected. We may assume $P_3 \subseteq X$. Then $G[P_2]$ is disconnected and hence $|P_2| \leq 3$. But $P_2 \supseteq Y$, and $Y$ is connected and contains at least three edges. Then, as $G[P_2]$ is disconnected, $P_2 \cap X$ cannot be empty. Thus $|P_2| \geq 4$, a contradiction.

Lemma 4.2.5. If $P$ is a petal of a tight flower in $M(G)$ having at least three petals and $G[P]$ is connected, then $G[P]$ has three vertices of attachment.

Proof. This is certainly true if $|P| \geq 3$. If $|P| = 2$, then $G[P]$ is a two-edge path. If one of the vertices of this path is not a boundary vertex, then $G$ has a vertex of degree at most two, a contradiction.

Lemma 4.2.6. Let $\Phi$ be a tight paddle or a tight copaddle in $M(G)$ having at least three petals. Then each petal contains at least three elements.

Proof. If $\Phi$ is a paddle $(P_1, P_2, \ldots, P_n)$ and $|P_1| = 2$, then $\cap(P_1, P_2) = r(P_1) + r(P_2) - r(P_1 \cup P_2) = 2$. Hence $r(P_2) \geq r(P_1 \cup P_2)$ and so $P_2$ spans $P_1$. Therefore $P_1$ is loose, a contradiction. The copaddle case follows by duality.

Lemma 4.2.7. Let $(P_1, P_2, \ldots, P_n)$ be a tight swirl-like or a tight spike-like flower $\Phi$ of $M(G)$ having at least four petals. Then each petal contains at least three elements.

Proof. Assume that $|P_1| = 2$. Then, by [23, Lemma 5.8(ii)], each element of $P_1$ is tight. Since $\Phi$ has at least four petals, $(P_1 \cup P_2, P_3 \cup \cdots \cup P_n)$ is 3-separating and each side contains at least four elements. Thus each side induces a connected
subgraph and shares three vertices in common with the other side. Now, both $G[P_1]$ and $G[P_2]$ are disconnected with size 2, or at least one of $G[P_1]$ or $G[P_2]$ is connected. Suppose $G[P_1]$ and $G[P_2]$ are both disconnected of size 2. Then, as $\cap(P_1, P_2) = 1$, we have $r(P_1 \cup P_2) = 3$. It follows, as $G[P_1 \cup P_2]$ is connected, that $P_1 \cup P_2$ is a 4-cycle. Since $P_1 \cup P_2$ meets the rest of the graph in only three vertices, there is a degree-2 vertex in $G$, a contradiction.

Now assume that $G[P_2]$ is connected. Then $P_2$ meets the rest of the graph in three vertices $\{a, b, c\}$. Since no edge of $P_1$ is loose and $G[P_1 \cup P_2]$ is connected, each edge of $P_1$ has one endpoint in $\{a, b, c\}$ and the other endpoint in $V(G) - V(G[P_2])$. Thus, without loss of generality, $G[P_1]$ is a 2-edge path $adb$ or a pair of vertex-disjoint edges $ax$ and $cy$. In the first case, the vertex boundary of $G[P_1 \cup P_2]$ is some 3-element subset of $\{a, b, c, d\}$. In particular, this vertex boundary must contain $c$. It must also contain $d$; otherwise, $d$ has degree 2. Hence we may assume this vertex boundary is $\{a, c, d\}$. Then $P_2$ cospans $bd$, a contradiction. Now suppose $G[P_1]$ is the graph containing two disjoint edges $ax$ and $cy$. Then the vertex boundary of $G[P_1 \cup P_2]$ must contain and hence equal $\{x, y, c\}$. Hence $P_2$ cospans $ax$, a contradiction.

We conclude that $G[P_2]$ is disconnected. Hence $G[P_1]$ is a 2-edge path. Moreover, the three vertices of this path must be the vertex boundary of $G[P_1 \cup P_2]$; otherwise, $G$ has a degree-2 vertex or has an edge in $P_1$ that is cospanned by $P_2$. It follows, since $G[P_2]$ is disconnected, that $G$ has a cut vertex, a contradiction.

We now analyze what types of flowers occur in graphs and describe the vertex boundaries of their petals.

**Lemma 4.2.8.** Suppose $\Phi$ is a flower in $M(G)$. Then $\Phi$ is not a Vámos-like flower.
Proof. As $M(G)$ is binary and hence representable, by [23, Corollary 6.2], $M(G)$ has no Vámos-like flowers. \qed

**Lemma 4.2.9.** Suppose $\Phi$ is a tight flower $(P_1, P_2, P_3, \ldots, P_n)$ in $M(G)$ with $n \geq 3$. Then $\Phi$ is not a copaddle.

**Proof.** Assume $\Phi$ is a copaddle. By Lemma 4.2.6, $|P_i| \geq 3$ for all $i \in \{1, 2, \ldots, n\}$. Then each $G[P_i]$ is connected by Lemmas 4.2.3 and 4.2.4, and $|\partial(G[P_i])| = 3$ by Lemma 4.2.5. Since $\Phi$ is a copaddle, $\cap(P_i, P_j) = 0$ for all distinct $i$ and $j$. This implies that each pair of petals share at most one vertex. If $n = 3$, then $P_1$ has three boundary vertices but only two other petals to meet. Hence $M(G)$ has no 3-petal tight copaddles. If $n > 3$, then, by [23, Lemma 4.9 and Lemma 5.9], $(P_1, P_2, P_3 \cup \cdots \cup P_n)$ is a tight flower. Since $\cap(P_1, P_2) = 0$, the last flower is a tight 3-petal copaddle in $M(G)$, a contradiction. \qed

**Lemma 4.2.10.** Suppose $\Phi$ is a tight flower $(P_1, P_2, P_3, \ldots, P_n)$ in $M(G)$. If $\Phi$ is a spike-like flower, then $n \leq 3$.

**Proof.** Assume $n \geq 4$. By Lemma 4.2.7, we may suppose $|P_i| \geq 3$ for all $i \in \{1, 2, \ldots, n\}$. Then each $G[P_i]$ is connected by Lemmas 4.2.3 and 4.2.4. Also, $G[P_i]$ has exactly three boundary vertices. As $\Phi$ is spike-like, $\cap(P_i, P_j) = 1$ for all distinct $i$ and $j$. This implies that each petal meets each other petal in exactly two vertices. Let $v_0$, $v_1$, and $v_2$ be the vertices of attachment of $P_1$. Without loss of generality, assume $P_2$ meets $P_1$ in $v_0$ and $v_1$. Let $v_3$ be the third vertex of attachment of $P_2$. Now $P_3$ meets $P_2$ in two vertices from $\{v_0, v_1, v_3\}$, and $P_3$ meets $P_1$ in two vertices from $\{v_0, v_1, v_2\}$. Without loss of generality, $P_3$ meets $P_2$ in $\{v_0, v_1\}$ or $\{v_0, v_3\}$.

If $P_3$ meets $P_2$ in $\{v_0, v_3\}$, then $P_3$ meets $P_1$ in $\{v_0, v_2\}$. Now $P_4$ must meet each of $P_1$, $P_2$, and $P_3$ in exactly two vertices, but this is impossible if $P_4$ meets $v_0$. \hfill 84
So $P_4$ meets $\{v_1, v_2, v_3\}$. But then $G[P_4 \cup P_1]$ has four vertices of attachment, a contradiction.

If $P_3$ meets $P_2$ in $\{v_0, v_1\}$, then $P_3$ also meets $P_1$ in $\{v_0, v_1\}$. If $v_4$ is the third vertex of attachment of $P_3$, then $v_4$ is not a vertex of $P_1$ or $P_2$, so $v_4 \notin \{v_2, v_3\}$.

Assume $n = 4$. As $P_4$ is 3-separating and meets the rest of the graph in exactly 3 vertices, $P_4$ contains $\{v_2, v_3, v_4\}$. Hence $P_4$ meets $P_1$ in a single vertex; a contradiction. Hence $M(G)$ has no spike-like flowers with exactly four petals. But if $n > 4$, then $(P_1, P_2, P_3, P_4 \cup \cdots \cup P_n)$ is such a flower; a contradiction. \hfill $\square$

**Lemma 4.2.11.** Let $\Phi$ be a tight paddle $(P_1, P_2, P_3, \ldots, P_n)$ in $M(G)$ with $n \geq 3$. Then $G$ has three vertices $\{x, y, z\}$ such that each petal of $\Phi$ induces a connected subgraph having $\{x, y, z\}$ as its vertex boundary and containing at least three edges.

**Proof.** By Lemma 4.2.6, we know that every petal $P_i$ of $\Phi$ has at least three edges. It follows by Lemmas 4.2.3 and 4.2.4 that $G[P_i]$ is connected and has a vertex boundary of size three. Now consider $P_i$ and $P_j$. Since $\cap(P_i, P_j) = 2$, these petals share exactly three vertices. The three vertices of attachment must be the boundary vertices $\{x, y, z\}$ of $P_i$. Since this is true for all $j \neq i$, the result follows. \hfill $\square$

**Lemma 4.2.12.** Let $\Phi$ be a tight swirl-like flower $(P_1, P_2, P_3, \ldots, P_n)$ in $M(G)$ with $n \geq 4$. Then $G[P_i]$ is connected for all $i$ and $V(G)$ contains a set $\{v_0, v_1, \ldots, v_n\}$ where the vertex boundary of $P_i$ is $\{v_0, v_i, v_{i+1}\}$ if $i < n$ and the vertex boundary of $P_n$ is $\{v_0, v_1, v_n\}$.

**Proof.** By Lemma 4.2.7, we may suppose that $|P_i| \geq 3$ for all $i \in \{1, 2, \ldots, n\}$. Then each $G[P_i]$ is connected, by Lemmas 4.2.3 and 4.2.4. Also, $G[P_i]$ has exactly three vertices in common with the rest of the graph. Since $\Phi$ is swirl-like, $\cap(P_i, P_{i+1}) = 1$ for all $i$. Thus $P_i$ and $P_{i+1}$ have exactly two common vertices.
Moreover, if \( j \in \{i - 1, i, i + 1\} \), then \( \cap (P_i, P_j) = 0 \), and so \( P_i \) and \( P_j \) have at most one common vertex.

Now \( G[P_1] \) and \( G[P_2] \) have two common vertices, say \( \{v_0, v_2\} \). Also, \( G[P_1] \) and \( G[P_n] \) have two common vertices. One of these must be either \( v_0 \) or \( v_2 \), so assume it is \( v_0 \); also, one of these must be neither \( v_0 \) nor \( v_2 \), so call this vertex \( v_1 \). Then \( G[P_2] \) and \( G[P_n] \) have one common vertex \( v_0 \). Then the vertex \( v_0 \) is in \( G[P_1], G[P_2], \) and \( G[P_n] \). Now \( P_3 \) has two common vertices with \( P_2 \) and at most one with \( P_1 \). Let \( v_3 \) be the third boundary vertex of \( P_2 \). Then the boundary of \( G[P_1 \cup P_2] \) includes, and so equals, \( \{v_0, v_1, v_3\} \). Hence \( v_2 \notin P_3 \), and so \( v_0 \in P_3 \). Thus \( P_2 \) meets \( P_3 \) in \( v_0 \) and \( v_3 \). Continuing in this way, we get the set of vertices \( \{v_0, v_1, v_2, \ldots, v_n\} \) such that \( v_i \) is in both \( P_{i-1} \) and \( P_i \) for \( i > 1 \), while \( v_1 \) is in both \( P_1 \) and \( P_n \), and \( v_0 \) is in \( P_j \) for all \( j \).

\[ \square \]

**Lemma 4.2.13.** Suppose \( \Phi \) is a tight unresolved flower \((P_1, P_2, P_3)\) in \( M(G) \). Then one of the following holds:

(i) Each petal is a pair of vertex-disjoint edges and \( G \cong K_4 \).

(ii) Each petal is connected with at least three edges and \( G \) has vertices \( v_0, v_1, v_2, \) and \( v_3 \) such that \( P_i \) has boundary \( \{v_0, v_i, v_{i+1}\} \) for \( i < n \) and \( P_3 \) has boundary \( \{v_0, v_1, v_3\} \).

*Proof.* Assume that \( |P_1| \geq 3 \). Then \( G[P_1] \) is connected and has vertex boundary \( \{v_0, v_1, v_2\} \). If \( G[P_2] \) and \( G[P_3] \) are both connected, then \( G[P_1] \) and \( G[P_2] \) have two common vertices and \( G[P_1] \) and \( G[P_3] \) have two common vertices. Without loss of generality, \( P_1 \) and \( P_2 \) share \( \{v_0, v_2\} \), while \( P_1 \) and \( P_3 \) share \( \{v_0, v_1\} \). We know that \( G[P_2] \) and \( G[P_3] \) have vertex boundaries of size 3. Thus there is a vertex \( v_3 \) that is common to \( P_2 \) and \( P_3 \), where \( v_3 \neq v_0, v_1, v_2 \). Thus \( v_0 \) is common to all three petals and (ii) holds.
Suppose both $P_2$ and $P_3$ are disconnected. Then $|P_2| = |P_3| = 2$. But, $G[P_2 \cup P_3]$ is connected of rank 3; therefore, $G[P_2 \cup P_3]$ is a 4-cycle. Since $\partial(G[P_2 \cup P_3])$ has three boundary vertices, $G$ has a degree-2 vertex; a contradiction. Thus either $P_2$ or $P_3$ is connected.

We may now assume that $P_3$ is a pair of vertex-disjoint edges and $G[P_2]$ is connected. Then $G[P_2]$ has two vertices $\{v_0, v_1\}$ in common with $G[P_1]$. The third boundary vertex of $P_2$, say $v_3$, which is not $v_2$, must be in $P_3$. Since $r(P_i \cup P_3) = r(P_i) + 1$ for $i = 1, 2$, it follows that $v_2v_3$ is an edge of $P_3$ and both $P_1$ and $P_2$ span an edge in $P_3$, which must be $v_0v_1$. Hence $P_3$ is loose, a contradiction.

We conclude that no petal of $\Phi$ has more than two edges. Hence, each petal consists of a pair of vertex-disjoint edges, and the union of any two petals is a 4-cycle. Thus $G \cong K_4$, and this flower is tight since each of these 2-element petals is tight. Thus (i) holds. \qed

4.3 3-Sequences in 3-Connected Graphs

In this section we will examine the properties of 3-sequences in 3-connected graphic matroids. First, we examine some of the types of structures as defined in [12] that appear graphic matroids. Second, we prove that every sequential 3-connected graph can be obtained from a special class of graphs we call sequential 3-paths. The results on sequential 3-connected graphs is a special case of a more general result whose proof will appear in a paper currently in preparation; the work on the general matroid case is joint with James Oxley. The next result relies on Lemma 4.2.3:

**Lemma 4.3.1.** Let $(A, \overline{X}, B)$ be a 3-sequence in a 3-connected graphic matroid $M(G)$. Then one of the following holds:

1. $G[A]$ and $G[B]$ are connected and $|\partial(A)| = |\partial(B)| = 3$. 

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(ii) \(|A| = 2, X = \{x\}, |B| \geq 6, and G[A \cup x] is a bond consisting of three disjoint edges.\)

Proof. If \(G[A]\) and \(G[B]\) are connected, then we have (i) by Lemma 4.2.3. Suppose \(|A| = 3\) and \(G[A]\) is disconnected. Then, \(A\) is a bond of \(G\) consisting of three disjoint edges. Consider \(A \cup x_1\) and \((A \cup x_1, B \cup (\overrightarrow{X} - x_1))\). If \(|E(G)| < 9\), then \(G \cong K_4\) or \(W_4\), which is impossible. Thus \(|B \cup (\overrightarrow{X} - x_1)| \geq 5\) and \(G[A \cup x_1]\) is connected, which is impossible. We now suppose that \(|A| = 2\) and \(G[A]\) is not connected. If \(G[A \cup x_1]\) is not connected, then \(A \cup x_1\) is a bond consisting of three disjoint edges. Thus \(x_1\) is the only member of \(\overrightarrow{X}\) and \(|B| \geq 6\), so (ii) holds. \(\square\)

A 3-sequence \((A, \overrightarrow{X}, B)\) is a graphic matroid \(M(G)\) is a graphic 3-sequence if \(G[A]\) and \(G[B]\) are connected. A 3-sequence of the type identified in Lemma 4.3.1(ii) is trivial and will not be considered further. The following lemma is a consequence of previous lemmas and the properties of closure and coclosure.

Lemma 4.3.2. Let \((A, \overrightarrow{X}, B)\) be a graphic 3-sequence with \(\overrightarrow{X} = (x_1, x_2, \ldots, x_n)\).

(i) If \(x_i\) is a guts element then \(\partial(A \cup \overrightarrow{X}_{i-1}) = \partial(A \cup \overrightarrow{X}_i).\)

(ii) If \(x_i = cd\) is a coguts element with endpoints \(c\) and \(d\), then, after possibly interchanging \(c\) and \(d\), \(c \in \partial(A \cup \overrightarrow{X}_{i-1})\) and \(\partial(A \cup \overrightarrow{X}_i) = (\partial(A \cup \overrightarrow{X}_{i-1}) - c) \cup d.\) \(\square\)

Next, we analyze the properties of some of the different types of sequential structures that appear in 3-connected graphs.

Lemma 4.3.3. In a graphic matroid \(M(G)\), if \(S\) is a maximal segment or maximal cosegment in a graphic 3-sequence \((A, \overrightarrow{X}, B)\), then \(2 \leq |X| \leq 3\). Moreover,
(i) If $S$ is a maximal segment and $\overrightarrow{X} = (\overrightarrow{L}, S, \overrightarrow{R})$, then $\partial(S) = \partial(A \cup \overrightarrow{L}) = \partial(\overrightarrow{R} \cup B)$, and $\partial(S)$ is the vertex boundary of the 3-separation $(A \cup \overrightarrow{L} \cup S, \overrightarrow{R} \cup B)$ of $G$;

(ii) If $S$ is a maximal cosegment of size three, then $S$ is a bond in $G$;

(iii) If $S$ is a maximal cosegment of size two, then $G \setminus S$ has a cut vertex $c$, and the end vertices of the edges in $S$ can be partitioned into sets $S_1$ and $S_2$ such that $S_1 \cup c$ and $S_2 \cup c$ are vertex boundaries of 3-separations of $G$.

Proof. Either $M(G)|S$ or $M^*(G)|S$ is uniform of rank 2. Since $M(G)$ is binary, it follows that $|S| \leq 3$. By Theorem 6.8 and Theorem 6.9 in [12], there is a graphic 3-sequence $(A, \overrightarrow{Y}, B)$ of the form $(A, L, S, R, B)$, where $L \cup S \cup R = X = Y$. Suppose $S$ is a maximal segment. Let $\partial(A \cup L) = \{a, b, c\}$ and $e \in S$. Then, as $e$ is a guts element, we know that $e$ is $ab$, $ac$, or $bc$. Moreover, as $|S| \geq 2$, we have $E(G) \geq 6$ and $(A \cup L \cup e, E(G) - (A \cup L \cup e))$ is a 3-separation. Suppose $S$ is a maximal cosegment. If $|S| = 3$, then $S$ is clearly a bond. If $|S| = 2$, then one boundary vertex of $A \cup L$ is not an end vertex of a member of $S$, and so is a cut-vertex. 

Lemma 4.3.4. Suppose $S$ is a fan in $G$. Then $G$ has a 3-sequence of the form $(A, L, S, R, B)$, and $S$ has three boundary vertices. Moreover, $|\partial(A \cup L) \cap \partial(B \cup R)| = 2$ and there is a special vertex $v$ such that each guts element of $S$ has $v$ as an end vertex and no coguts element of $S$ has $v$ as an end vertex.

Proof. Let $(A, \overrightarrow{X}, B)$ be a 3-sequence of $M(G)$ containing a fan. Let $\partial(A \cup L) = \{a, b, c\}$ and $\partial(B \cup R) = \{d, e, f\}$. If the first element $s_1$ in an ordering of $S$ is a guts element, then $\partial(A \cup L \cup s_1) = \{a, b, c\}$ and we may assume that $s_1 = ab$. Then $s_2$ is a coguts element. By [12, Theorem 6.9], $s_2$ can never come before $s_1$ in a 3-sequence, and so $s_2$ must meet exactly one vertex of $s_1$, say $a$, and we may
assume $s_2 = ad$ by Lemma 4.3.2. In fact, by [12, Theorem 6.9], in any 3-sequence obtained by reordering $X$, the order of the elements in $S$ is the same as the order induced by $X$. Moreover, there is a reordering of $X$ in which the elements of $S$ are consecutive, and the elements of $S$ are alternately guts and coguts elements in the induced order.

If $|S| = 2$, then, without loss of generality, $b = e$ and $c = f$. Thus $\partial(A \cup L) \cap \partial(B \cup R) = \{b, c\}$ and $b$ is the special vertex. If $s_1$ is a coguts element and $s_2$ is a guts element, then we may assume that $s_1 = ad, s_2 = db, b = e, and c = f$. Thus $\partial(A \cup L) \cap \partial(B \cup R) = \{b, c\}$ and $b$ is the special vertex.

If $|S| = 3$ and $s_1$ is a guts element, then $S$ is a triangle. Moreover, we may assume that $s_1 = ab, s_2 = ad, s_3 = db, b = e, and c = f$. Thus the lemma holds in this case, and $b$ is the special vertex. If $|S| = 3$ and $s_1$ is a coguts element, then we may assume that there is a vertex $g$ of $M$ that is not $a, b, c, d, e, or f$ such that $s_1 = ag, s_2 = gb, s_3 = gd, b = e, and c = f$. Thus the lemma holds in this case, and $b$ is the special vertex.

If $|S| \geq 4$, then by the same arguments as above, we may assume that $b = e$ and $c = f$. Moreover, there is a path from $a$ to $d$. Each guts element in $S$ is of the form $vb$ for some element $v$ on the path from $a$ to $d$, possibly including $a$ or $d$. Each coguts element in $S$ is of the form $v_1v_2$ for two elements on the path from $a$ to $d$, possibly including $a$ or $d$. Thus the lemma holds with $b$ being the special vertex.

We say that the special vertex in the previous lemma is the hub vertex of the fan $S$. In [12], Hall, Oxley, and Semple defined four other structures in 3-sequences. While these structures do appear in graphic 3-sequences, their properties are not as crucial to describing the 3-sequences as they are in matroids in general. The
key idea in describing the structure of graphic 3-sequences is the characterization of vertex boundaries in Lemma 4.3.2.

Let $M(G)$ be a 3-connected sequential matroid and let $(A, \vec{X}, B)$ be a graphic 3-sequence of $M(G)$. As $G[A]$ is connected, it has three boundary vertices $a, b,$ and $c$. If the first element $x_1$ in $X$ is guts, then $x_1$ is $ab$, $ac$, or $bc$, and the vertex boundary of $G[A \cup x_1]$ is also $\{a, b, c\}$. If $x_1$ is a coguts element, then $x_1$ has one end vertex in $\{a, b, c\}$, say $a$, and one end vertex outside $\{a, b, c\}$, say $d$. Then the vertex boundary of $G[A \cup x_1]$ is $\{d, b, c\}$. In other words, we can view each pair of separations $((A \cup \vec{X}_i, \vec{X}_i \cup B), (A \cup \vec{X}_{i+1}, \vec{X}_{i+1} \cup B))$, in terms of the corresponding change to the vertex boundary. Moving a guts element does not change the boundary and moving a coguts element changes one vertex of the boundary. This idea of a 3-sequence as being a list of vertex boundaries is what we need to characterize 3-sequences in graphs and sequential 3-connected graphs.

Now we will introduce a special class of graphs. Take three disjoint paths $P_1$, $P_2$, and $P_3$ of the same length $n$ and let these paths have the following ordered sets of vertices: $P_1 = (x_0, x_1, x_2, \ldots, x_n)$, $P_2 = (y_0, y_1, y_2, \ldots, y_n)$, and $P_3 = (z_0, z_1, z_2, \ldots, z_n)$. Obtain the graph $P_\Delta^n$ from these three paths of length $n$ by adding the edges $\{x_iy_i, x_iz_i, y_iz_i\}$ for all $i \in \{0, 1, 2, \ldots, n\}$. We say that $P_\Delta^n$ is a \textit{sequential 3-path with triangle ends}. Observe that $P_\Delta^n$ is the triangle having vertices $\{x_0, y_0, z_0\}$. We say that the graph $P_\gamma^n$ is a \textit{sequential 3-path with triad ends} if $P_\gamma^n$ is obtained from $P_\Delta^{n-2}$ by adding two degree-3 vertices $p_0$, adjacent to $x_0$, $y_0$, and $z_0$, and $p_{n-2}$, adjacent to $x_{n-2}$, $y_{n-2}$, and $z_{n-2}$. The graph $P_\gamma^2$ is isomorphic to the graph obtained by deleting an edge of $K_5$. We say that the graph $P_\mu^n$ is a \textit{sequential 3-path with mixed ends} if $P_\mu^n$ is obtained from $P_\gamma^{n-1}$ by adding one degree-3 vertex $p_0$ adjacent to $x_0$, $y_0$, and $z_0$. The graph $P_\mu^n$ is the graph isomorphic to $K_4$ with one vertex designated as $p_0$ and the others as $x_0$, $y_0$, and $z_0$. Finally, we say that a graph is a
sequential 3-path if it is isomorphic to any of the three types above. Moreover, we say that each of $P^n_\Delta$, $P^n_Y$, and $P^n_M$ has length $n$. Thus the edge set of a sequential 3-path can be partitioned into a sequence of 3-element sets that alternate between triangles and triads. We say that the triangle or triad at the beginning or end of the sequence is an end triangle or an end triad, respectively, and more generally, an end of the sequential 3-path. All other triangles and triads in a sequential 3-path are called internal. We will show that sequential 3-paths are 3-connected and that any sequential 3-connected graph is a special minor of a sequential 3-path.

First, recall that a triangle-sum of two graphs is obtained by identifying a triangle common to both graphs and then deleting the edges of the triangle. We wish to use Theorem 2.1.1 for graphs. In this case, the generalized parallel connection of two graphic matroids identifies a common triangle, but the edges of the triangle are not deleted. Thus, we define the modified triangle-sum of two simple 3-connected graphs $G_1$ and $G_2$ to be the 3-connected graph $G$ obtained by identifying a triangle common to $G_1$ and $G_2$ but not deleting the identified edges. The following lemma is a straightforward consequence of the construction of sequential 3-paths.

**Lemma 4.3.5.** A sequential 3-path $G$ of length $n$ has an internal triangle $T$ if and only if $G$ is the modified triangle-sum of two sequential 3-paths $G_1$ and $G_2$ of lengths $n_1$ and $n_2$, respectively, such that (i) $G_1$ and $G_2$ have a common end triangle $T$; and (ii) $n_1, n_2 \geq 1$ and $n_1 + n_2 = n$.

**Proposition 4.3.6.** Let $G$ be a sequential 3-path with $|E(G)| = n$. Then, $G$ is a planar graph and its cycle matroid $M(G)$ is 3-connected and sequential. The dual $M^*(G)$ is also the cycle matroid of a sequential 3-path. Moreover, if $|E(G)| \geq 6$, then there is a sequential ordering $(e_1, e_2, e_3, \ldots, e_{n-2}, e_{n-1}, e_n)$ for $M(G)$ that has the following properties:
(i) For all \(i \equiv 1 \pmod{3}\), the set \(\{e_i, e_{i+1}, e_{i+2}\}\) is a triangle or a bond in \(G\).

(ii) For all \(i \equiv 1 \pmod{3}\), either \(\{e_i, e_{i+1}, e_{i+2}\}\) is a triangle and \(\{e_{i+3}, e_{i+4}, e_{i+5}\}\) is a bond in \(G\), or \(\{e_i, e_{i+1}, e_{i+2}\}\) is a bond and \(\{e_{i+3}, e_{i+4}, e_{i+5}\}\) is a triangle in \(G\).

(iii) The two ends of the sequential 3-path are \(\{e_1, e_2, e_3\}\) and \(\{e_{n-2}, e_{n-1}, e_n\}\).

Proof. Let \(G\) be a sequential 3-path. The graph \(G\) is clearly planar and 3-connected. Moreover, it is straightforward to see that the graphs \(P^n_\Delta\) and \(P^n_Y\) are planar duals of each other, while the planar dual of \(P^n_M\) is isomorphic to \(P^n_M\). To complete the proof of the proposition, we argue, by induction on the length \(k\) of the 3-path, that \(M(G)\) has a sequential ordering with the desired properties. If \(G \cong P^n_\Delta\), then \(M(G)\) is isomorphic to \(M(K_3)\). If \(G \cong P^n_M\), then \(M(G)\) is isomorphic to \(M(K_4)\), which has a sequential ordering with the desired properties. If \(G \cong P^n_Y\), then \(M(G)\) is isomorphic to a single-element deletion of \(M(K_5)\), which has a sequential ordering with the desired properties.

Suppose the result is true for all sequential 3-paths of length \(j\) with \(j < k\). Consider a sequential 3-path of length \(k \geq 1\). Either \(G \cong P^n_\Delta\) or \(G\) has an internal triangle. The cycle matroid of \(P^n_\Delta\) has a sequential ordering with the desired properties. If \(G\) has an internal triangle, then, by Lemma 4.3.5, \(G\) is the modified triangle-sum of two sequential 3-paths \(G_1\) and \(G_2\). By the induction hypothesis, each of these has a 3-sequence of the desired type. Moreover, we obtain \(G\) by identifying a triangle \(T\) that is an end triangle of both \(G_1\) and \(G_2\). Let \(|E(G_1)| = k\) and \(|E(G_2)| = l\). We may assume that \(T\) is \(\{e_{k-2}, e_{k-1}, e_k\}\) in a 3-sequence \((e_1, e_2, \ldots, e_k)\) for \(M(G_1)\) and \(T\) is \(\{f_1, f_2, f_3\}\) in a 3-sequence \((f_1, f_2, \ldots, f_l)\) for \(M(G_2)\). Thus \((e_1, e_2, \ldots, e_k, f_4, f_5, \ldots, f_l)\) is a sequential order-
ing of $M(G)$ that has the desired properties. Therefore, by induction, $M(G)$ has a sequential ordering with the desired properties.

We say that a sequential ordering that has the properties listed in Lemma 4.3.6 is a \emph{3-path ordering}.

For the next lemma, we go outside of graphs to the class of binary matroids. We do this because it is straightforward to extend our main results to sequential 3-connected binary matroids.

**Lemma 4.3.7.** If $M$ is a sequential 3-connected binary matroid, then $M$ is the cycle matroid of a planar graph.

**Proof.** By [13, Lemma 4.2] and the discussion preceding it, every 3-connected minor of a sequential 3-connected matroid is sequential. As $F_7$, $F_7^*$, $M(K_5)$, $M^*(K_5)$, $M(K_{3,3})$ and $M^*(K_{3,3})$ are not sequential, $M$ has no minor isomorphic to one of these six matroids. Thus, by [19, Corollary 6.6.6], $M$ is the cycle matroid of a planar graph.

**Theorem 4.3.8.** Let $M(G)$ be a sequential 3-connected graphic matroid. Then $G$ can be obtained from a sequential 3-path by contracting members of internal triads or deleting members of internal triangles and then deleting all but one element from each non-trivial parallel class and contracting all but one element from each non-trivial series class.

**Proof.** Consider a 3-sequence $(e_1, e_2, \ldots, e_m)$ for $M(G)$. If $m < 9$, then $G$ is isomorphic to $K_3$, $K_4$, or $W_4$. The graphs $K_3$ and $K_4$ are sequential 3-paths, and $W_4$ can be obtained from $P_2^2$ by deleting one edge from the triangle adjacent to the two triads. Now, we may assume $m \geq 9$. We will show how to obtain a sequential 3-path $P$ from which we may obtain $G$ by the operations listed in the statement.
of the theorem. Let \( n \) be the number of coguts elements in the sequence outside of \( \{ e_1, e_2, e_3, e_{m-2}, e_{m-1}, e_m \} \). The set \( T_1 = \{ e_1, e_2, e_3 \} \) is a triangle or triad in \( M(G) \). We will complete the proof by induction on \( n \). If \( n = 0 \), then since we have at least nine edges, the edges outside of \( \{ e_1, e_2, e_3, e_{m-2}, e_{m-1}, e_m \} \) are obtained by closure. This is impossible if \( T_1 \) or \( \{ e_{m-2}, e_{m-1}, e_m \} \) is a triangle. Thus \( T_1 \) and \( \{ e_{m-2}, e_{m-1}, e_m \} \) are triads and \( \{ e_4, e_5, e_6 \} \) is a triangle. Thus, \( G \) is a sequential 3-path.

Suppose that the theorem is true for \( k < n \). Suppose \( \{ e_1, e_2, e_3 \} \) is a triangle. Then \( e_4 \) is a coguts element. By Lemma 4.3.2, we may assume that the end vertices of \( \{ e_1, e_2, e_3 \} \) are \( \{ a, b, c \} \), that the vertex boundary of \( \{ e_1, e_2, e_3, e_4 \} \) is \( \{ a, b, c \} \), and \( e_4 = cd \). There is a 3-connected graph \( G' \) obtained from \( G \) by adding the edges \( ad \) and \( bd \), if they were not already present in \( G \). Then there is a graph \( G_1 \) isomorphic to \( K_4 \) with edges \( \{ ac, ab, bc, cd, ad, bd \} \). Moreover, \( \{ ac, bc, cd, ab, ad, bd \} \) is a 3-path ordering of \( G_1 \). There is a simple 3-connected graph \( G_2 \) which is the subgraph of \( G' \) obtained by deleting \( \{ ac, bc, cd \} \). Moreover, \( G_2 \) is sequential, has a sequential ordering in which the first three elements are \( \{ ab, ad, bd \} \), and, in this sequential ordering, there are \( n - 1 \) coguts elements. By the induction hypothesis, there is a sequential 3-path \( H_2 \) having a 3-path ordering starting with \( \{ ab, ad, bd \} \).

Consider the graph \( H_1 \) that is isomorphic to \( P_1 \) as shown in Figure 4.2. Let \( H \) be the sequential 3-path obtained by the modified triangle-sum of \( H_1 \) and \( H_2 \). We can recover \( G' \) from \( H \) by applying the operations that convert \( H_2 \) to \( G_2 \), contracting \( a_0a \), contracting \( b_0b \), and deleting one element from the resulting non-trivial parallel class. Then we may obtain \( G \) from \( G' \) by deleting \( ad \) and \( bd \), if necessary.

The remaining case is when \( \{ e_1, e_2, e_3 \} \) is a triad. Let \( e \) be the first coguts element in the sequential ordering after \( \{ e_1, e_2, e_3 \} \). There is a degree-3 vertex \( a \) of \( G \) that is an end vertex of \( e_1, e_2, \) and \( e_3 \), and the other ends of these edges are
FIGURE 4.2. The graphs $G_1$ and $H_1$

the vertex boundary $\{b, c, d\}$. Thus, after $\{e_1, e_2, e_3\}$ in the sequence is a set $Z$ of at most three guts elements whose end vertices are contained in $\{b, c, d\}$. After these elements of $Z$ comes $e$. There is a graph $G'$ obtained from $G$ by adding the edges of $\{bc, bd, cd\} - Z$ to $G$. Let $G_2 = G'\{e_1, e_2, e_3\}$. Then $G_2$ is 3-connected by Theorem 2.1.1 and has a sequential ordering starting with $bc$, $bd$, $cd$, and $e$.

We may apply the argument in the previous case to obtain a sequential 3-path $H_2$ whose sequential ordering starts with $bc$, $bd$, $cd$, and $e$. Let $H_1$ be the graph isomorphic to $K_4$ having 3-path ordering $(ab, ac, ad, bc, bd, cd)$. Then we may take the modified triangle-sum of $H_1$ and $H_2$ to obtain a sequential 3-path $H$. We can recover $G'$ by applying the operations to the $H_2$ part of $H$ to recover $G_2$. Then we may delete members of $\{bc, bd, cd\}$, if necessary, to obtain $G$.

\[\square\]

\textbf{Theorem 4.3.9.} Let $G$ be a graph obtained from a sequential 3-path by contracting members of internal triads or deleting members of internal triangles, and then deleting all but one element from each non-trivial parallel class or contracting all but one element each non-trivial series class. Then $G$ is a sequential 3-connected graph.
Proof. Let $G$ be obtained from the sequential 3-path $H$ by the permitted operations, and consider a 3-path ordering of $H$. If $e$ is an element in an internal triangle, then $e$ is a guts element in the 3-path. By Lemma 4.3.2, $G/e$ has a non-minimal 2-separation. Thus, by [19, Proposition 8.4.6], the matroid obtained by contracting all but one element in each non-trivial series class of $M(G)\setminus e$ is 3-connected. Similarly, if $e$ is an element in an internal triad, then $e$ is a coguts element in the 3-path ordering. Then the matroid obtained by deleting all but one element in each non-trivial parallel class of $M(G)/e$ is 3-connected. Thus $G$ is 3-connected. Finally, by [13, Lemma 4.2] and the discussion preceding it, any 3-connected minor of a sequential 3-connected matroid is sequential. Therefore, $G$ is a sequential 3-connected graph.

Finally, we state a corollary for 3-connected binary matroids that follows easily from Lemma 4.3.7, Theorem 4.3.8, and Theorem 4.3.9.

**Corollary 4.3.10.** Let $M$ be a sequential 3-connected binary matroid. Then $M$ can be obtained from the cycle matroid of a sequential 3-path by contracting members of internal triads or deleting members of internal triangles and then deleting all but one element from each non-trivial parallel class and contracting all but one element from each non-trivial series class. Moreover, any binary matroid $M$ that is obtained in such a way from the cycle matroid of a sequential 3-path is a sequential 3-connected binary matroid.

### 4.4 A Decomposition of 3-Connected Graphs

In this section, we will specialize the decomposition in Chapter 3 to simple 3-connected graphs. Let $G$ be a simple 3-connected graph. To use those results, we must assume that $|E(G)| \geq 9$. If $|E(G)| < 9$, then, by Lemma 4.2.1, $M(G)$ is sequentially 4-connected.
In Section 4.2, we saw that there are only two types of flowers in $G$. Moreover, we know the vertex boundaries of the petals of these flowers. A swirl-like flower has a configuration of petal boundaries that is similar to those found in wheels, as shown in Figure 4.3. A set of three vertices whose removal leaves at least three components indicates the presence of a paddle. Clearly, the special flower piece we get from the a flower vertex that displays a paddle is a triangle. The following lemma proves that we get wheels from swirl-like flowers:

**Lemma 4.4.1.** Let $M(G)$ be a 3-connected graphic matroid and let $T''$ be the decomposition tree for $M(G)$. If $v$ is a flower vertex that displays a swirl-like flower $\Phi = (P_1, P_2, \ldots, P_n)$, then the piece obtained by detaching $v$ is a wheel.

*Proof.* By Lemma 4.2.12, $G$ has a special vertex $v_0$ that is on the boundary of every petal of $\Phi$. By Theorem 3.2.4, the loose elements of $\Phi$ are fans. The set of tight elements $T_i$ in a petal $P_i$ are 3-separating, and so the vertex boundary of $T_i$ has size 3. By Lemma 4.3.4, there is a special vertex that is an end vertex of each guts element in the fans that are contained in $\text{fcl}(P_i)$, for each $i$. This vertex must
be $v_0$. The vertex boundary of $T_i$ is in $V(B')$ and, by Lemma 4.3.4, contains $v_0$. Therefore, $B'$ is a wheel.

A graph is \textit{sequentially 4-connected} if $M(G)$ is sequentially 4-connected. Theorem 3.1.2 gives us a decomposition of graphs into sequentially 4-connected graphs:

\textbf{Theorem 4.4.2.} Let $G$ be a simple 3-connected graph. Then, $G$ has a decomposition into a unique set of labeled sequentially 4-connected graphs. Moreover, $M$ can be recovered by identifying identically-labeled edges, followed by deletions of certain special labeled edges.

\textit{Proof.} By Theorem 3.1.2, $G$ has a decomposition into a unique set of sequentially 4-connected graphs and flower pieces. The flower pieces from paddles are graphs isomorphic to $K_3$, and, by Lemma 4.4.1, the pieces from swirl-like flowers are wheels. When we complete the line of separation, we add up to three edges to certain sets of three vertices in $G$. Uniquely label each of the edges in a line of separation and note those edges that were not originally in $G$. Then $G$ can be recovered by identifying identically-labeled edges, and then deleting the noted edges. \hfill $\square$
Chapter 5

Pancyclic and Hamiltonian Matroids

5.1 Overview

Bondy [3] proved that an \( n \)-vertex Hamiltonian graph with sufficiently many edges has cycles of all sizes. In Section 5.2, we prove that a rank-\( r \) GF\((q)\)-representable matroid with sufficiently many elements has circuits of all sizes. Section 5.2 is a corrected version of a paper, joint with James Oxley, that has appeared in *Discrete Mathematics* [2]. In Section 5.3, we prove that every element in a simple rank-\( r \) binary matroid \( M \) on at least \( 2^{r-1} \) elements is in a Hamiltonian circuit unless \( M \) is isomorphic to one of six exceptional matroids.

5.2 Pancyclic Representable Matroids

A simple graph \( G \) with vertex set \( V(G) \) is *pancyclic* if it contains cycles of all lengths \( l \), for \( 3 \leq l \leq |V(G)| \). Bondy [3] proved the following:

**Theorem 5.2.1.** Let \( G \) be a simple Hamiltonian graph with \( |V(G)| = n \). If \( |E(G)| \geq n^2/4 \), then \( G \) is pancyclic unless \( G \) is isomorphic to \( K_{n/2,n/2} \).

The exceptional graph \( K_{n/2,n/2} \) is special in that it has many edges and many even cycles, but no odd cycles. A similar role is played in binary matroids by affine geometries, which also have many elements and many even circuits, but no odd circuits. It is natural to ask whether Bondy’s theorem has an analog for binary or even for GF\((q)\)-representable matroids. Toward this end, we define a simple rank-\( r \) matroid \( M \) to be *Hamiltonian* if it has a circuit of size \( r + 1 \) and to be *pancyclic* if it has circuits of all sizes \( s \), for \( 3 \leq s \leq r + 1 \). We will prove the following:

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Theorem 5.2.2. Let $M$ be a simple rank-$r$ binary matroid. If $|E(M)| \geq 2^{r-1}$, then $M$ is pancyclic unless $M$ is isomorphic to $AG(r-1, 2)$ or $PG(r-2, 2) \oplus U_{1,1}$.

Note that if we add the condition that $M$ is Hamiltonian, then $M$ must be pancyclic unless it is an affine geometry of odd rank. The main result of this section is a theorem on the existence of circuits of every size in matroids with no $U_{2,q+2}$-minor. This will imply the above result for binary matroids and the following result for $GF(q)$-representable matroids.

Theorem 5.2.3. Let $M$ be a simple rank-$r$ $GF(q)$-representable matroid.

(i) If $|E(M)| \geq \frac{q^{r-1}-1}{q-1} + q + 1$, then, for all $s$ in $\{3, 4, \ldots, r+1\}$ and all $e$ in $E(M)$, there is an $s$-circuit containing $e$.

(ii) If $|E(M)| \geq \frac{q^{r-1}-1}{q-1} + 2$, then, for all $s$ in $\{3, 4, \ldots, r+1\}$ and all but at most one $e$ in $E(M)$, there is an $s$-circuit containing $e$.

(iii) If $|E(M)| = \frac{q^{r-1}-1}{q-1} + 1$, then $M$ is pancyclic unless $M$ is isomorphic to one of the following matroids:

(a) $U_{3,q+2}$ for $q$ a power of 2,

(b) $PG(r-2, q) \oplus U_{1,1}$ if $r \geq 3$, or

(c) $AG(r-1, 2)$.

Matroid terminology used here follows Oxley [19] with the following exceptions: the simple matroid associated with a matroid $M$ is denoted by $si(M)$; and if $x$ and $y$ are elements of a simple matroid $M$, then $xy$ denotes the line of $M$ spanned by $\{x, y\}$. 

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The next theorem is the main result of this section. Note that $S(3, 6, 22)$ is the rank-4 paving matroid of the unique Steiner system $S(3, 6, 22)$. The blocks of the Steiner system are the hyperplanes of the matroid.

**Theorem 5.2.4.** Let $M$ be a simple rank-$r$ matroid with no $U_{2,q+2}$-minor, for some integer $q$ greater than one.

(i) If $|E(M)| \geq \frac{q^{r-1}-1}{q-1} + q + 1$, then, for all $s$ in $\{3, 4, \ldots, r+1\}$ and all $e$ in $E(M)$, there is an $s$-circuit containing $e$.

(ii) If $|E(M)| \geq \frac{q^{r-1}-1}{q-1} + 2$, then, for all $s$ in $\{3, 4, \ldots, r+1\}$ and all but at most one $e$ in $E(M)$, there is an $s$-circuit containing $e$.

(iii) If $|E(M)| = \frac{q^{r-1}-1}{q-1} + 1$, then $M$ is pancyclic unless $M$ is isomorphic to one of the following matroids:

(a) $U_{3,q+2}$,

(b) $U_{2,q+1} \oplus U_{1,1}$,

(c) $N_q \oplus U_{1,1}$, where $N_q$ is projective plane of order $q$,

(d) $PG(r-2, q) \oplus U_{1,1}$ if $r > 4$,

(e) $AG(r-1, 2)$, or

(f) $S(3, 6, 22)$.

The proof of this theorem uses the following results. The first and second are due to Kung [15] and Murty [18], respectively. The third is a straightforward consequence of the second, while the fourth and fifth use standard techniques. The sixth follows from results of Doyen and Hubaut [8] (see Welsh [29, pp.214-5]) and Lam, Thiel, and Swiercz [16].

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Theorem 5.2.5. Let $q$ be an integer exceeding one. If $M$ is a rank-$r$ matroid with no $U_{2,q+2}$-minor, then $|E(M)| \leq \frac{q^r-1}{q-1}$. For $r \geq 4$, equality holds in this bound if and only if $M \cong PG(r-1,q)$. When $r = 3$, equality holds if and only if $M$ is a projective plane of order $q$.

Lemma 5.2.6. Let $C_1$ and $C_2$ be circuits of a matroid $M$ with $C_2 = \{e, f, g\}$ and $C_1 \cap C_2 = \{g\}$. If $(C_1 - g) \cup e$ is independent in $M$, then $(C_1 \cup C_2) - g$ is a circuit.

Lemma 5.2.7. Let $\{e, f, g\}$ be a circuit of $M$, and let $C_g$ be a circuit of $si(M/e)$ containing $g$. Then either $C_g \cup e$ or $(C_g - g) \cup \{e, f\}$ is a circuit of $M$.

Proof. As $C_g$ is a circuit of $M/e$, either $C_g \cup e$ or $C_g$ is a circuit of $M$. We may assume the latter. Noting that $r_M((C_g - g) \cup e) = r_{M/e}(C_g - g) + r_M(e) = |C_g - g| + 1 = |(C_g - g) \cup e|$, we have that $(C_g - g) \cup e$ is independent. By Lemma 5.2.6, $(C_g - g) \cup \{e, f\} \in C(M)$. □

Lemma 5.2.8. Let $M$ be a simple rank-$r$ matroid having no $U_{2,q+2}$-minor where $q \geq 2$. If $|E(M)| \geq \frac{q^r-1}{q-1} + a$ where $a \geq 1$ and $e \in E(M)$, then $|E(si(M/e))| \geq \frac{q^{r-2}-1}{q-1} + \left\lceil \frac{a}{q} \right\rceil$.

Proof. As every line of $M$ through $e$ has at most $q$ other points, $|E(si(M/e))| \geq \left\lceil \frac{1}{q} \left(\frac{q^{r-2}-1}{q-1} + a - 1\right)\right\rceil = \left\lceil \frac{1}{q} \left((q^{r-2} + q^{r-3} + \ldots + q + 1) + a - 1\right)\right\rceil = \left[q^{r-3} + q^{r-4} + \ldots + q + 1 + \frac{a}{q}\right] = \frac{q^{r-2}-1}{q-1} + \left\lceil \frac{a}{q} \right\rceil$. □

Lemma 5.2.9. Let $M$ be a simple rank-$r$ matroid having no $U_{2,q+2}$-minor where $q \geq 2$. Suppose $|E(M)| = \frac{q^r-1}{q-1} + a$ and $|E(si(M/e))| = \frac{q^{r-2}-1}{q-1} + b$. If $M/e$ has exactly $c$ elements in trivial parallel classes, then $c \leq b + \frac{b-a}{q-1}$. Moreover, if equality holds, then each nontrivial parallel class of $M/e$ has exactly $q$ elements.
Proof. The following inequalities are equivalent:

\[
|E(M)| \leq q(|E(si(M/e))| - c) + c + 1
\]

\[
\frac{q^{r-1}-1}{q-1} + a \leq q\left(\frac{q^{r-2}-1}{q-1} + b - c\right) + c + 1
\]

\[
q^{r-2} + q^{r-3} + \ldots + q + 1 + a \leq (q^{r-2} + q^{r-3} + \ldots + q) + q(b - c) + c + 1
\]

\[
a \leq qb - qc + c
\]

\[
(q - 1)c \leq qb - a
\]

\[
(q - 1)c \leq (q - 1)b + b - a
\]

\[
c \leq b + \frac{b-a}{q-1}.
\]

If equality holds in the last line, then equality must hold in the first line, and so 
every nontrivial parallel class of \(M/e\) has exactly \(q\) members.

\[\square\]

**Lemma 5.2.10.** Let \(M\) be a simple matroid with rank \(r \geq 4\). If \(M\) has no triangles and if every single-element contraction of \(M\) is a projective space, then \(M \cong AG(r-1, 2)\) or \(M \cong S(3, 6, 22)\).

**Proof.** By Doyen and Hubaut [8], if \(r > 4\), then \(M \cong PG(r-1, q)\) or \(M \cong AG(r-1, q)\); and if \(r = 4\), then (i) \(M \cong PG(3, q)\), (ii) \(M \cong AG(3, q)\), (iii) every single-element contraction of \(M\) is a projective plane of order 4, or (iv) every single-element contraction of \(M\) is a projective plane of order 10. Because \(M\) has no triangles, \(M/e\) is simple for all \(e \in E(M)\). Now \(PG(r-1, q)\) has triangles for all \(q\) and \(AG(r-1, q)\) has triangles if \(q \neq 2\). The remaining possibility is \(AG(r-1, 2)\), all of whose single-element contractions are isomorphic to \(PG(r-2, 2)\).

By Lam, Thiel, and Swiercz [16], there are no projective planes of order 10, and, by Doyen and Hubaut [8], \(S(3, 6, 22)\) is the unique matroid all of whose single-element contractions are projective planes of order 4.

\[\square\]
Proof of Theorem 5.2.4. We argue by induction on \( r \) to prove all three parts simultaneously. The result is easily checked if \( r = 2 \). Assume \( r = 3 \). If \( |E(M)| = q + 2 \), then either \( M \cong U_{3,q+2} \), or \( M \) has a nontrivial line and at least one other point not on this line. If there is exactly one point not on the line, then \( M \cong U_{2,q+1} \oplus U_{1,1} \).

If there are at least two points not on the line, then there is a 4-circuit containing these two points. Thus \( M \) has a 3-circuit and a 4-circuit and (iii) holds.

Now let \( |E(M)| \geq q + 3 \). Suppose \( e \in E(M) \) and \( |\mathrm{si}(M/e)| > 2 \). Then \( 3 \leq |\mathrm{si}(M/e)| \leq q + 1 \) and there is at least one 2-circuit \( \{f, g\} \) in \( M/e \). As \( \mathrm{si}(M/e) \) is a nontrivial line, it has a triangle \( C \) through \( g \). Since \( \{e, f, g\} \) is a triangle of \( M \), Lemma 5.2.7 implies that \( C \cup e \) or \( (C - g) \cup \{e, f\} \) is a 4-circuit of \( M \) containing \( e \). Thus if \( |\mathrm{si}(M/e)| > 2 \), then \( e \) is in both a 3-circuit and a 4-circuit of \( M \). We deduce that (i) and (ii) hold unless \( M \) has an element \( e \) such that \( |\mathrm{si}(M/e)| = 2 \). Consider the exceptional case. Then \( |E(M)| < 2q + 2 \) and \( M \) consists of two lines meeting in \( e \). Thus \( M \) has 3- and 4-circuits through every point except \( e \). Hence, in the exceptional case, (ii) holds and (i) holds vacuously since \( |E(M)| < \frac{q^{k-1} - 1}{q-1} + q + 1 \).

We conclude that the theorem holds when \( r = 3 \).

Assume the theorem holds for \( r < k \) and let \( r = k > 3 \). First, we consider (i). Suppose that \( |E(M)| \geq \frac{q^{k-1} - 1}{q-1} + q + 1 \) and let \( e \in E(M) \). Then \( M \) has at least two nontrivial lines through \( e \) since \( |E(\mathrm{si}(M/e))| \leq \frac{q^{k-1} - 1}{q-1} \). By Lemma 5.2.8, \( |E(\mathrm{si}(M/e))| \geq \frac{q^{k-2} - 1}{q-1} + 2 \). Then, by the induction hypothesis, every element but at most one of \( \mathrm{si}(M/e) \) is in circuits of all sizes from 3 to \( k \). By choosing a triangle containing \( e \) and an element of \( \mathrm{si}(M/e) \) that is in circuits of all sizes from 3 to \( k \), we apply Lemma 5.2.7 to get circuits in \( M \) of all sizes from 4 to \( k + 1 \) through \( e \). Since \( e \) is also in a triangle, (i) holds.

Next we consider (ii). Assume \( |E(M)| = \frac{q^{k-1} - 1}{q-1} + a \) with \( 2 \leq a \leq q \) and let \( e \in E(M) \). Then, as \( a \geq 2 \), it follows that \( e \) is in a triangle of \( M \). Moreover,
\[ |E(\text{si}(M/e))| \geq \frac{q^{r-2} - 1}{q-1} + 1 \] by Lemma 5.2.8. If \[ |E(\text{si}(M/e))| \geq \frac{q^{r-2} - 1}{q-1} + q + 1, \] then every element of \( \text{si}(M/e) \) is in circuits of every size from 3 to \( k \). Choose an element \( g \) of \( \text{si}(M/e) \) that is in a triangle of \( M \) with \( e \). By Lemma 5.2.7, the triangle containing both \( e \) and \( g \) and the circuits of every size from 3 to \( k \) containing \( g \) yield circuits of \( M \) containing \( e \) of all sizes from 3 to \( k + 1 \).

Suppose that \( c \) elements of \( M/e \) are in trivial parallel classes. Assume that \[ |E(\text{si}(M/e))| = \frac{q^{r-2} - 1}{q-1} + b \] with \( 2 \leq b \leq q \). Then, by Lemma 5.2.9, \( c \leq b + \frac{b-a}{q-1} \). Since \( b \geq 2 \) and \( a \leq q \), we assert that \( c \leq q \). To see this, suppose that \( c \geq q + 1 \). Then \( b + \frac{b-a}{q-1} \geq q + 1 \), and so \((q-1)b+b-a \geq (q+1)(q-1)\). Thus \(qb-a \geq q^2-1\), and hence we obtain the contradiction that \(-1 \geq 1 - a \geq q^2 - qb = q(q-b) \geq 0\).

We conclude that \( c \leq q \). Let \( U \) be the set of elements of \( M/e \) that are in trivial parallel classes. By the induction hypothesis, all but at most one element, say \( p \), of \( \text{si}(M/e) \) is in circuits of all sizes from 3 to \( k \) in \( \text{si}(M/e) \). Assume \( p \) is not in a trivial parallel class of \( M/e \). Adjoin to \( U \) all points on the line \( ep \) of \( M \). Thus \( U \) has at most \( 2q + 1 \) elements. As \( |E(M)| = \frac{q^{r-1} - 1}{q-1} + a \) and \( r \geq 4 \), it follows that \( |E(M)| \geq q^2 + q + 1 + a \). Thus \( |E(M) - U| \geq (q^2 + q + 1 + a) - (2q + 1) = q^2 - q + a > 0 \).

Hence \( M \) has at least \( q^2 - q + a \) elements that are in nontrivial parallel classes of \( M/e \) and avoid \( U \). Take \( g \) to be one such element that is also in \( \text{si}(M/e) \). As \( g \) is not \( p \), there are circuits of all sizes from 3 to \( k \) containing \( g \) in \( \text{si}(M/e) \), and \( \{e, g\} \) is contained in a triangle of \( M \). Thus, by Lemma 5.2.7, \( M \) has circuits of all sizes from 3 to \( k + 1 \) containing \( e \).

Now assume \[ |E(\text{si}(M/e))| = \frac{q^{r-2} - 1}{q-1} + 1 \]. By Lemma 5.2.9, \( c \leq 1 + \frac{1-a}{q-1} < 1 \). Then every element of \( M/e \) is in a nontrivial parallel class. Moreover, by the induction hypothesis, \( \text{si}(M/e) \) has circuits of all sizes from 3 to \( k \) unless \( \text{si}(M/e) \) is one of the exceptions (a)–(f). By Lemma 5.2.7, we deduce that \( M \) has circuits containing \( e \) of all sizes from 3 to \( k + 1 \) unless \( \text{si}(M/e) \) is one of (a)–(f). Now part (ii) holds...
unless there are at least two elements $f$ and $g$ of $M$ such that each of $\text{si}(M/f)$ and $\text{si}(M/g)$ is one of (a)–(f). We may assume that $g \in \text{si}(M/f)$. Because every element of $M/g$ is in a nontrivial parallel class, $g$ is in a triangle with every other element of $\text{si}(M/f)$. This is not possible in any of (a)–(f), so (ii) holds.

Finally, we consider (iii). Assume that $|E(M)| = \frac{q^r-1}{q-1} + 1$. Suppose first that $M$ has no triangles. Then, for all $e$ in $E(M)$, we have $|E(\text{si}(M/e))| = \frac{q^r-1}{q-1}$, and so, by Theorem 5.2.5, every single-element contraction of $M$ is a projective space.

By Lemma 5.2.10, $M \cong AG(r-1, 2)$ or $M \cong S(3, 6, 22)$.

We may now assume that $M$ has a triangle and that this triangle contains $e$. If $|E(\text{si}(M/e))| \geq \frac{q^r-2}{q-1} + q + 1$, then every element of $\text{si}(M/e)$ is in circuits of all sizes from 3 to $k$. So $M$ has circuits of all sizes from 3 to $k + 1$ by Lemma 5.2.7.

If $|E(\text{si}(M/e))| = \frac{q^r-2}{q-1} + b$ for $2 \leq b \leq q$, then all but at most one element, say $p$, of $\text{si}(M/e)$ is in circuits of all sizes from 3 to $k$. By Lemma 5.2.9, $c \leq b + \frac{b-1}{q-1} \leq b + 1 \leq q + 1$. Let $U$ be the set consisting of those elements of $M/e$ that are in trivial parallel classes. Assume $p$ is in a nontrivial parallel class and adjoin to $U$ all points on the line $ep$. Thus $|U| \leq 2q + 2$. Since $r \geq 4$, we have $|E(M)| \geq q^2 + q + 2$. Hence $|E(M) - U| \geq (q^2 + q + 2) - (2q + 2) = q^2 - q > 0$. So we may choose $g$ from $E(M) - U$ in $\text{si}(M/e)$ such that $\{e, g\}$ is in a triangle of $M$ and $e$, $g$, and $p$ are not collinear. Then, since $\text{si}(M/e)$ has circuits of all sizes from 3 to $k$ containing $g$, Lemma 5.2.7 imples that $M$ has circuits of all sizes from 3 to $k + 1$ containing $e$.

If $|E(\text{si}(M/e))| = \frac{q^r-2}{q-1} + 1$, then by Lemma 5.2.9, $c \leq 1 + \frac{1-1}{q-1} = 1$, that is, at most one element of $M/e$ is in a trivial parallel class. Hence $M$ has a 3-circuit. Moreover, we get a 4-circuit in $M$ by taking two elements from each of two nontrivial parallel classes of $M/e$. If $\text{si}(M/e)$ has circuits of all sizes from 3 to $k$, then $M$ has circuits of all sizes from 3 to $k + 1$ by Lemma 5.2.7. Thus we may assume that $\text{si}(M/e)$ is one of the exceptions (a)–(f), and next we consider each of these,
noting that we have already shown that $M$ has both 3- and 4-circuits. Suppose first that $\text{si}(M/e)$ is $U_{3,q+2}$. Then we use Lemma 5.2.7 to get a circuit of size 5. Suppose next that $\text{si}(M/e)$ is $S(3, 6, 22)$. Then $M$ has 5- and 6-circuits by Lemma 5.2.7.

Next suppose that $\text{si}(M/e)$ is the direct sum of a coloop $g$ and a projective space of rank at least two. Either $g$ is the unique element of $M/e$ in a trivial parallel class or not. In the first case, $g$ is also a coloop of $M$. By Lemma 5.2.9, each nontrivial parallel class of $M/e$ has $q$ elements. Thus $|E(M\setminus g)| = \frac{q^r - 1}{q-1}$ and, by Theorem 5.2.5, $M\setminus g$ is a projective space, and $M$ is (c) or (d). Now suppose $g$ is in a nontrivial parallel class. We now have that $M$ is the parallel connection, with basepoint $e$, of the line $eg$ and a matroid of rank $r - 1$, and that the line $eg$ has at least one other point $f$. We may use circuits of sizes from 3 to $r - 1$ of $\text{si}(M/e)$ to obtain circuits of $M$ of sizes 4 to $r$ that contain $e$ and avoid all other points on the line $eg$. Then, we take an $r$-circuit $C$ of $M$ containing $e$ and apply Lemma 5.2.6 to get that $(C - e) \cup \{f, g\}$ is an $(r + 1)$-circuit of $M$.

Finally, we consider the case when $\text{si}(M/e)$ is a binary affine geometry. Then $q = 2$ and so $M$ is binary, as $M$ has no $U_{2,4}$-minor. In $M$, there is exactly one trivial line through $e$. We can obtain a binary representation for a single-element extension $M'$ of $M$ as follows. If $AG(r - 2, 2)$ is represented by the matrix $A$, then

$$
\begin{bmatrix}
1 & 1^T & 0^T \\
0 & A & A
\end{bmatrix}
$$

represents $M'$, where the first column of this matrix corresponds to $e$, and $0$ and $1$ are vectors of all zeros and all ones, respectively, of appropriate size. Since $A$ can be chosen so that its columns are all vectors of $V(r - 1, 2)$ with first coordinate 1, it follows that $M'\setminus e \cong AG(r - 1, 2)$. Thus $M'$ is the unique simple rank-$r$ binary single-element extension of $AG(r - 1, 2)$ and hence $M$ is pancyclic.

\[\square\]
The next two lemmas were proved by Kantor [14] (see Welsh [29, p.215]) and Bose [4] (see Oxley [19, p.206]), respectively.

**Lemma 5.2.11.** The matroid $S(3, 6, 22)$ is not representable over any field.

**Lemma 5.2.12.** The matroid $U_{3,q+2}$ is representable over $GF(q)$ if and only if $q$ is even.

On combining these lemmas with Theorem 5.2.4, we immediately obtain Theorems 5.2.2 and 5.2.3.

### 5.3 Binary Hamiltonian Matroids

Let $M$ be a simple rank-$r$ binary matroid $M$ on at least $2^{r-1}$ elements. We will now prove that every element of $E(M)$ is in a Hamiltonian circuit unless $M$ is isomorphic to one of six exceptional matroids. We have seen two of these in Section 5.2: $AG(r - 1, 2)$, for $r$ even, and $PG(r - 2, 2) \oplus U_{1,1}$. The other four exceptions are related to these two.

The matroid $P_e(PG(r - 2, 2), U_{2,3})$ that is the parallel connection with basepoint $e$ of $PG(r - 2, 2)$ and $U_{2,3}$ has no Hamiltonian circuits that contain $e$. However, every other point is in circuits of all sizes. Note that by deleting a point $h$ other than $e$ from the $U_{2,3}$ part of $M$, we obtain a matroid isomorphic to $PG(r - 2, 2) \oplus U_{1,1}$. If we delete a point $h$ other than $e$ from the $PG(r - 2, 2)$ part of $M$, then the resulting matroid $P_e(PG(r - 2, 2) \setminus h, U_{2,3})$ is easily seen to be unique up to isomorphism. In it, $e$ is still in no Hamiltonian circuits, but every other point is in circuits of all sizes.

Up to isomorphism, the matroid $AG(r - 1, 2)$ has a unique extension $M$. We shall write this matroid as $AG(r - 1, 2) + e$. When $r$ is odd, $M$ has many Hamiltonian circuits. Moreover, any point besides $e$ is in circuits of all odd sizes with $e$ and circuits of all even sizes avoiding $e$. However, $e$ is in no even-sized circuits and hence
no Hamiltonian circuits. Moreover, if we delete any point from the $AG(r-1, 2)$ part of $M$, we obtain a matroid that is unique up to isomorphism. This matroid still has no Hamiltonian circuits containing $e$ but has every other point in a Hamiltonian circuit.

**Theorem 5.3.1.** Let $M$ be a simple rank-$r$ binary matroid with $r \geq 3$ and $|E(M)| \geq 2^{r-1}$, and let $e \in E(M)$. Then every member of $E(M)$ is in a Hamiltonian circuit of $M$ unless $M$ is isomorphic to one of the following matroids:

(a) $PG(r - 2, 2) \oplus U_{1,1}$;

(b) $P_e(PG(r - 2, 2), U_{2,3})$;

(c) $P_e(PG(r - 2, 2) \setminus h, U_{2,3})$, for some $h \neq e$;

(d) $AG(r - 1, 2)$, for $r$ even;

(e) $AG(r - 1, 2) + e$, for $r$ odd; or

(f) $(AG(r - 1, 2) \setminus h) + e$, for $r$ odd and for some element $h \neq e$.

**Proof.** We proceed by induction on $r$. Let $r = 3$. Then $M$ is isomorphic to a restriction of the Fano matroid $F_7$, and $|E(M)| \geq 4$. If $M$ has six or seven elements, then $M \cong M(K_4)$ or $M \cong F_7$, respectively, in which each element is in a Hamiltonian circuit. If $M$ has five elements, then $M \cong P_e(PG(1, 2), U_{2,3}) \cong AG(2, 2) + e$.

If $M$ has four elements, then either $M \cong AG(2, 2)$, in which case each element is in a Hamiltonian circuit, or $M \cong PG(1, 2) \oplus U_{1,1} \cong P_e(PG(1, 2) \setminus h, U_{2,3}) \cong (AG(2, 2) \setminus h) + e$.

We may now assume that $r \geq 4$ and that the theorem is true for all $k < r$. Suppose that $M$ has an element $e$ that is not in a Hamiltonian circuit. Assume first that $e$ is in no triangles. Then $M/e \cong PG(r - 2, 2)$. Choose a Hamiltonian
circuit $C$ of $M/e$. Now either $C \cup e$ or $C$ is a circuit of $M$. The former would be a Hamiltonian circuit of $M$ containing $e$. Thus $C$ is a circuit of $M$. Hence $C$ spans a hyperplane of $M$ that does not contain $e$.

Consider an $r \times 2^{r-1}$ matrix $A$ with the property that the first column has a one in the first row and all zeroes below, and when the first row is deleted the remaining submatrix consists of the $2^{r-1}$ vectors of $V(r-1, 2)$. The matroid $M$ can be represented by this matrix $A$ with the first column representing $e$. Only the first entry of the first row of $A$ has so far been determined. Since $C$ spans a hyperplane of $M$ avoiding $e$, we can use row and column operations in $A$ to ensure that the first $r$ columns form an $r \times r$ identity matrix and columns 2 through $r$, which we label as $e_2, e_3, \ldots, e_r$, represent $r-1$ elements of $C$. Since $C \cup e$ cannot be a circuit of $M$, the remaining element of $C$ corresponds to the column $(0, 1, 1, \ldots, 1)^T$.

Let $x$ be a column vector of $A$ having exactly one zero among the last $r-1$ rows and the value $x_{r-2}$ in the first row. Let $y$ be a column vector with exactly 2 ones in the lower $r-1$ rows such that one of these ones is in the same row of $A$ as the zero in $x$. Let $y$ have the value $x_2$ in the first row. Take $e, x, y$, and the appropriate $r-2$ basis vectors to obtain a submatrix of $A$ where all rows below the first sum to zero. The matrix in Figure 5.1(a) is an example when $r = 6$.

\[
\begin{bmatrix}
1 & x_{r-2} & x_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & x_3 & x_2 & x_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(a) (b)

FIGURE 5.1. Example matrices
If \( x_{r-2} \neq x_2 \), then there is a Hamiltonian circuit through \( e \). Thus \( x_{r-2} = x_2 \). Similarly, we can analyze each column vector with \( r - 1 \) or 2 ones below the first row to obtain that its first-row entry must also be equal to \( x_2 \).

Let \( l \) be an integer such that \( 2 < l < r - 2 \) and let \( x \) be a column vector of \( A \) with exactly \( l \) ones after the first entry. Let \( x_l \) be the entry of \( x \) in the first row. Consider the submatrix of \( A \) with the following columns: \( e; x; \) the \( r - 1 - l \) vectors \( y_1, y_2, \ldots, y_{r-1-l} \) with two ones below the first row which doubly cover the zero entries of \( x \) except for one, cover one of the ones of \( x \), and single cover the remaining zero; \( l - 1 \) basis vectors covering the ones of \( x \) not covered in the previous subset of vectors; and one basis vector covering the undoubled one left remaining among the lower \( r - 1 \) rows. The matrix in Figure 5.1(b) is an example when \( r = 6 \).

This set of \( r+1 \) vectors is a Hamiltonian circuit containing \( e \) if \( x_l + (r-1-l)x_2 \equiv 1 \) (mod 2). Thus \( x_l \equiv (r-1-l)x_2 \) (mod 2). If \( x_2 = 0 \), then \( x_l = 0 \). Since \( l \) was arbitrary, then the only one in the first row is the one in the column for \( e \). Thus \( e \) is a coloop and \( M \cong PG(r - 2, 2) \oplus U_{1,1} \). Assume \( x_2 = 1 \). Then \( x_3 \equiv r - 4 \), \( x_4 \equiv r - 5 \), and so on, mod 2. That is, the \( x_i \)'s alternate between zero and one. Then \( r - 2 \) is even and hence \( r \) is even. Consider \( x \) with first coordinate \( x_l \). If \( l \) is even, then \( x_l = 1 \), and there are an odd number of ones in \( x \). If \( l \) is odd, then \( x_l = 0 \) and there are an odd number of ones in \( x \). Thus every vector in \( A \) has coordinates whose sum is 1 (mod 2). Thus \( M \cong AG(r - 1, 2) \) for \( r \) even.

We may now assume that \( e \) is in a triangle \( T = \{e, f, g\} \) of \( M \). If there is a Hamiltonian circuit \( C_g \) of \( si(M/e) \) through \( g \), then, by Lemma 5.2.7, there is a Hamiltonian circuit of \( M \) through \( e \). Assume \( si(M/e) \) has no Hamiltonian circuits through \( g \). Since \( |E(M)| \geq 2^{r-1} \), we know that \( |E(si(M/e))| \geq 2^{r-2} \), by Lemma 5.2.8, and we may apply the induction hypothesis. Thus \( si(M/e) \) is iso-
morphic to one of the exceptions, and \( g \) is an element of \( E(si(M/e)) \) not in any Hamiltonian circuits of \( si(M/e) \).

If \( si(M/e) \cong PG(r-3,2) \oplus U_{1,1} \), then the only point not in a Hamiltonian circuit is the coloop. Also, \(|E(M)| = 2^{r-1} + 1\) or \(|E(M)| = 2^{r-1}\). If the former, then all points are doubled in \( M/e \). Thus \( M \cong P_e(PG(r-2,2),U_{2,3}) \). In the latter case, \( M \) is isomorphic to a single-element deletion of \( P_e(PG(r-2,2),U_{2,3}) \), so that there is an element \( h' \) of \( M \) that is not doubled in \( M/e \). Let \( h \) be third point of the triangle containing \( h' \) and \( e \) in \( P_e(PG(r-2,2),U_{2,3}) \). Since \( h \neq f \), the undoubled point \( h' \) is in the \( PG(r-3,2) \) part of \( si(M/e) \). Thus \( M \cong P_e(PG(r-2,2) \setminus h, U_{2,3}) \).

If \( si(M/e) \cong AG(r-2,2) \) for \( r-1 \) even, then no member of \( si(M/e) \) is in a Hamiltonian circuit. Also, \(|E(M)| = 2^{r-1} + 1\) or \(|E(M)| = 2^{r-1}\). Suppose the former. Then, every point of \( M/e \) is doubled. Consider a matrix \( A \) representing \( AG(r-2,2) \) whose first row consists of all ones. Consider the \( r \times (2^{r-1} + 1) \) matrix \( A' = \begin{bmatrix} 1 & 1^T & 0^T \\ 0 & A & A \end{bmatrix} \). Now \( A' \) represents \( M \), where the first column of this matrix corresponds to \( e \), and \( 0 \) and \( 1 \) are vectors of all zeros and all ones, respectively, of appropriate size. The second row is all ones in columns \( 2 \) through \( 2^{r-1} + 1 \), and so deleting the first column from \( A' \) leaves a matrix that represents \( AG(r-1,2) \). Thus \( M \cong AG(r-1,2) + e \) with \( r \) odd. If \(|E(M)| = 2^{r-1}\), then we delete one element \( h \) distinct from \( e \) from the matroid obtained in the first case. This gives \( (AG(r-1,2) \setminus h) + e \).

We consider the remaining exceptions in pairs. If \( si(M/e) \) is isomorphic to \( P_g(PG(r-3,2),U_{2,3}) \) or \( P_g(PG(r-3,2) \setminus h, U_{2,3}) \), then \( M \) is a restriction of a matroid \( M' \) that is isomorphic to \( P_T(PG(r-2,2),F_7) \), the generalized parallel connection across the triangle \( T \) of \( PG(r-2,2) \) and \( F_7 \). Note that \( M' \) has \( 2^{r-1} + 3 \) elements, and so, we must consider every 0-, 1-, 2-, or 3-element dele-
tion of $M'$ that still contains $T$. If we delete 3 elements from the $F_7$ part, then $M \cong PG(r - 2, 2) \oplus U_{1,1}$. If we delete 2 elements from the $F_7$ part, then we have $M \cong P_x(PG(r - 2, 2), U_{2,3})$ or $M \cong P_x(PG(r - 2, 2) \backslash h, U_{2,3})$, where $h$ is the other deleted point and $x \in \{e, f, g\}$. So either $e$ is in a Hamiltonian circuit, or $x = e$ and we have one of the exceptions.

If $r = 4$, then we are looking at restrictions of $P_T(F_7, F_7)$. Because $F_7 \cong PG(3, 2)$, these were all considered in the previous paragraph. Assume that $r \geq 5$ and that we delete at least two points from the $PG(r - 2, 2)$ part avoiding $f$.

Since $r > 4$, the points deleted from the $PG(r - 2, 2)$ part of $M'$ span at most a hyperplane of $PG(r - 2, 2)$. We can choose a hyperplane $H$ of $PG(r - 2, 2)$ which spans the deleted points. Either $T$ is contained in $H$ or at most one member of $T$ is contained in $H$.

If $T \subseteq H$, then choose a basis $B_H$ of $H$ containing $e$ and $g$ but avoiding the deleted points. This is possible since $r - 2 \geq 3$. Extend $B_H$ to a basis $B_P$ of $PG(r - 2, 2)$. The element we need to add to $B_P$ to obtain an $r$-circuit avoids $H$. By Lemma 5.2.7, we can use a triangle in the $F_7$ part containing $g$ but not $e$ to get an $r$-circuit of $M$ containing $e$.

Now suppose $x$ is the only element of $T$ in $H$. We can choose a basis $B_H$ of $H$ containing $x$ but avoiding the deleted points. Extend $B_H$ to a basis $B_P$ of $PG(r - 2, 2)$ by adding $e$ if $x = f$ or $x = g$, or by adding $g$ if $x = e$. In each case, the element we need to add to $B_P$ to complete an $r$-circuit avoids $H$. By Lemma 5.2.7, we can use a triangle in the $F_7$ part containing $g$ or $f$ but not $e$ to get an $(r + 1)$-circuit of $M$ containing $e$.

Now, suppose $si(M/e)$ is $AG(r - 2, 2) + g$ or $(AG(r - 2, 2) \backslash h) + g$ for $r - 1$ odd. The matroid $AG(r - 1, 2) + T$ is unique, up to isomorphism; this can be seen by using complements. Then $M$ is a 0-, 1-, 2-, or 3-element deletion of $M' \cong AG(r - 1, 2) + T$
for $r$ even. First, if $r = 4$, we have $M' \cong P_T(F_7, F_7)$, and this has been previously considered. We assume $r \geq 6$.

Consider a matrix $A$ representing $AG(r - 2, 2)$ whose first row consists of all ones. Consider the $r \times (2^r - 1 + 1)$ matrix $A' = \begin{bmatrix} 1 & 1^T & 0^T \\ 0 & A & A \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Now $A'$ represents $M' \setminus \{f, g\}$, where the first column of this matrix corresponds to $e$, and $0$ and $1$ are vectors of all zeros and all ones, respectively, of appropriate size. The second row is all ones in columns 2 through $2^r - 1 + 1$, and so deleting the first column from $A'$ leaves a matrix that represents $AG(r - 1, 2)$. If we delete three points of $AG(r - 1, 2)$, then, in $M' \setminus \{f, g\}/e$, there is at least one trivial parallel class. Further, either there are two other trivial parallel classes or $si(M' \setminus \{f, g\}/e)$ is isomorphic to a single-element deletion of $AG(r - 2, 2)$. In either case, we can construct a representation $A''$ for $M$ such that the deleted points are represented by vectors having at most two ones.

Consider the submatrix $A_0$ of $A''$ with $r + 1$ column vectors consisting of $e$, the vector of all ones, the vector of all ones except in the first row, the $r - 3$ vectors having all ones except for zeros in rows $i$ and $i + 1$ for $3 \leq i \leq r - 1$, and the vector having all ones except in rows 3 and $r$. This submatrix consists of columns representing elements not deleted from $M'$. The case when $r = 6$ is shown in Figure 5.2. Thus there is a Hamiltonian circuit containing $e$. \hfill \square

We obtain the following corollary from Theorem 5.2.4 or Theorem 5.3.1.
Corollary 5.3.2. Let $M$ be a simple rank-$r$ binary matroid with $r \geq 3$ and $|E(M)| \geq 2^{r-1}$. Then $M$ has a Hamiltonian circuit unless either $M \cong PG(r - 2, 1) \oplus U_{1,1}$, or $r$ is even and $M \cong AG(r - 1, 2)$. 
References


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Vita

Brian Daniel Beavers was raised in Anacoco, Louisiana, and graduated from the Louisiana School for Math, Science, and the Arts in Natchitoches, Louisiana. He completed his undergraduate studies in mathematics at Louisiana Tech University in May 2000. He earned a Master of Science degree at Louisiana State University in May 2002. He studied under Professor James Oxley and is currently a candidate for the degree of Doctor of Philosophy in mathematics, to be awarded in August 2006. He has accepted a position as Assistant Professor of Mathematics and Statistics at Stephen F. Austin State University in Nacogdoches, Texas, beginning August 2006.