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Conical Representations for Direct Limits of Riemannian Symmetric Spaces.

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CONICAL REPRESENTATIONS FOR DIRECT LIMITS OF RIEMANNIAN SYMMETRIC SPACES

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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in

The Department of Mathematics

by
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Abstract

We extend the definition of conical representations for Riemannian symmetric space to a certain class of infinite-dimensional Riemannian symmetric spaces. Using an infinite-dimensional version of Weyl’s Unitary Trick, there is a correspondence between smooth representations of infinite-dimensional noncompact-type Riemannian symmetric spaces and smooth representations of infinite-dimensional compact-type symmetric spaces. We classify all smooth conical representations which are unitary on the compact-type side. Finally, a new class of non-smooth unitary conical representations appears on the compact-type side which has no analogue in the finite-dimensional case. We classify these representations and show how to decompose them into direct integrals of irreducible conical representations.
Chapter 1
Introduction

Harmonic analysis and representation theory of topological groups have been very well-studied over the past century and have produced many fruitful applications in areas such as PDEs and quantum physics. Two broad developments in the theory are brought together in this thesis: first, Helgason’s theory of horocycle spaces and conical representations for noncompact-type Riemannian symmetric spaces and second, the more recent study of representation theory and harmonic analysis on infinite-dimensional Lie groups.

In the theory of Riemannian symmetric spaces, there are two crucially important dualities. One is the duality between compact-type and noncompact-type Riemannian symmetric spaces. The other is the duality between a noncompact-type Riemannian symmetric space and its horocycle space. These dualities are intimately connected to the representation theory of their corresponding isometry groups (see [19], [21], and [22], for instance). For instance, Weyl’s unitary trick sets up a correspondence between finite-dimensional spherical representations for a compact-type symmetric space and finite-dimensional spherical representations for its corresponding noncompact-type symmetric space. In turn, the finite-dimensional spherical representations for a noncompact-type symmetric space are identical to the conical representations for its corresponding horocycle space.

More recently, researchers have turned their attention to the study of infinite-dimensional Lie groups. These are groups which are modeled by locally convex topological vector spaces in the same way that finite-dimensional Lie groups are modeled on finite-dimensional vector spaces. The simplest and “smallest” infinite-dimensional groups are the direct-limit groups, which are constructed by taking unions of increasing chains of finite-dimensional Lie groups. In a similar way, one can form an infinite-dimensional symmetric space by forming a direct limit of finite-dimensional symmetric spaces. Representation theory and even harmonic analysis questions for direct-limit groups and direct-limit symmetric spaces have been studied in some depth (e.g., see [2], [4], [41], [42], [37], [38], [52], [53], and [54] for just a few examples). A good overview of the field may be found in [43].

In particular, spherical representations for infinite-dimensional symmetric spaces are well-studied in the literature (e.g., see [7] and [42]). On the other hand, the theory of conical representations for infinite-dimensional Riemannian symmetric spaces appears to have been largely neglected up to this point. In this thesis, we begin to rectify this situation by classifying all of the smooth conical representations for direct limits of noncompact-type Riemannian symmetric spaces that satisfy certain technical conditions. Combined with the results of [7], we see that for infinite-dimensional symmetric spaces of infinite rank, none of the smooth conical representations are spherical, a situation which is in stark contrast with the
classical result of Helgason that all finite-dimensional representations are spherical if and only if they are conical. We further demonstrate the existence, in certain cases, of nonsmooth unitary conical representations for direct limits of compact-type Riemannian symmetric spaces. This is a phenomenon which has no analogue for finite-dimensional symmetric spaces. We also show how these conical representations decompose into direct integrals of irreducible representations.

The arrangement of this thesis is as follows. Chapter 2 reviews relevant theorems from elementary representation theory and harmonic analysis. Chapter 3 reviews the relevant structure theory for Riemannian symmetric spaces and their associated horocycle spaces. It also reviews the basic results about spherical representations and conical representations and their role in harmonic analysis on Riemannian symmetric spaces and horocycle spaces. Much of the theory of spherical representations are due to Harish-Chandra, and the corresponding results for conical representations are mostly due to Helgason. Chapter 4 introduces the concept of direct-limit Lie groups. It also introduces the necessary technical machinery for studying direct limits of symmetric spaces and horocycle spaces. We define what we call admissible direct limits of Riemannian symmetric spaces and show that the classical examples of direct limits of Riemannian symmetric spaces meet this definition. Chapter 5 contains several useful results about representations of direct-limit groups, including an infinite-dimensional generalization of Weyl’s Unitary Trick. Finally, Chapter 6 contains the main results of the thesis. We provide natural definitions of conical representations for infinite-dimensional Riemannian symmetric spaces. We construct and classify all unitary conical representations for direct limits of compact-type symmetric spaces. Finally, in Chapter 7 we end by describing some interesting questions which remain unanswered.

1.1 Notational Preliminaries

If $A$ is a set, then its cardinality is denoted by $\#A$. If $G$ is a group, then $e$ denotes the identity element. If $H$ and $K$ are subgroups of $G$, then $Z_H(K)$ and $N_H(K)$ denote the centralizer and normalizer, respectively, of $K$ in $H$. Similar notation is used for centralizers and normalizers of Lie algebras.

All vector spaces, except for Lie algebras, are assumed to be over the field of complex numbers unless otherwise stated. We denote by $\langle A \rangle$ the algebraic linear span of a subset $A$ of a topological vector space $V$. The closed linear span of $A$ is denoted by $\overline{\langle A \rangle}$. The space of continuous linear functionals on $V$ is denoted by $V^*$, and the space of continuous conjugate-linear functionals on $V$ is denoted by $V'$. If $\mathcal{H}$ is a Hilbert space, then the inner product of two vectors $u, v \in \mathcal{H}$ is denoted by $\langle u, v \rangle_{\mathcal{H}}$, or if the choice of Hilbert space is understood, by $\langle u, v \rangle$. We consider inner products to be linear in the first variable and conjugate-linear in the second variable. The space of bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$.

If $M$ is a manifold, then the space of smooth, compactly supported functions on $M$ is denoted by $\mathcal{D}(M)$. The usual topology given to $\mathcal{D}(M)$ gives it the structure of a lim-Fr´echet space (i.e., it is a direct limit of Fr´echet spaces). The space of distributions on $M$ is denoted by $\mathcal{D}'(M)$ and is defined to be the space of continuous
conjugate-linear functionals on $\mathcal{D}(M)$ (we choose to think of $\mathcal{D}'(M)$ as the anti-
dual of $\mathcal{D}(M)$ so that there is a continuous linear embedding $\mathcal{D}(M) \hookrightarrow \mathcal{D}'(M)$).

We give $\mathcal{D}'(M)$ the weak-* topology. We denote by $C^\infty(M)$ the space of smooth
functions on $M$. 

3
Chapter 2
A Brief Review of Harmonic Analysis and Representation Theory

Representation theory is the study of linear actions of groups on vector spaces, which are called representations. Of particular interest are the unitary representations, in which a group acts on a Hilbert space by isometries. Given a group, representation theory seeks to explicitly construct and classify the representations of that group to the broadest extent possible. It thus fits naturally into the broader theory of groups, which has been traditionally motivated primarily by the study of symmetry.

Fundamental insights in representation theory often come from relating group representations to representations of other related objects. For instance, unitary representations of locally compact groups may be integrated to yield representations of a group $C^*$-algebra, which allows the application of powerful tools from operator theory. Similarly, unitary representations of a Lie group may be differentiated to yield representations of the group’s Lie algebra, which allows the representation to be studied using basic linear algebra techniques instead of more difficult tools from differential geometry and analysis. Finally, through the beautiful and classical construction of Gelfand-Naimark-Segal, the theory of unitary representations may be connected with the theory of positive-definite functions. A distributional variant of this construction uses positive-definite distributions on Lie groups to embed unitary representations into spaces of distributions on homogeneous spaces.

The foundational task of the field of harmonic analysis, on the other hand, is to use the information provided by the action of a group to decompose a space of functions into simpler pieces. Such exploitations of symmetry, to borrow a phrase of Mackey, have many applications, particularly in the study of linear PDEs and in quantum physics. Because it is concerned with symmetries of vector spaces of functions, representation theory naturally plays a very important role, although harmonic analysis may be distinguished from the study of representation theory as an end in itself.

The material in this chapter is entirely classical and may be found in standard references on abstract harmonic analysis, such as [8], [14], and [30]. See also the survey article [29] for an excellent and concise introduction to the theory.

2.1 Unitary Representations

We begin by defining the basic terms.

**Definition 2.1.** Let $G$ be a topological group and let $V$ be a locally convex topological vector space. A representation of $G$ on $V$ is a continuous homomorphism:

$$\pi : G \to \text{GL}(V),$$
where $\operatorname{GL}(V)$ is given the strong operator topology. (We say that $\pi$ is a \textbf{norm-continuous representation} if it is continuous when $\operatorname{GL}(V)$ is given the operator norm topology.) If $V$ is a Hilbert space, then $\pi$ is said to be a \textbf{unitary representation} if $\pi(g)$ is a unitary operator for all $g \in G$.

Given two representations $(\pi, V)$ and $(\sigma, W)$ of $G$, we say that a bounded linear operator $T : V \to W$ is an \textbf{intertwining operator} if $T\pi(g) = \sigma(g)T$ for all $g \in G$. If $\pi$ and $\sigma$ possess a continuously-invertible intertwining operator between them, then we say that they are \textbf{equivalent representations}. We write $\operatorname{Hom}(\pi, \sigma)$ for the space of all intertwining operators between $\pi$ and $\sigma$.

Among more general continuous representations, unitary representations in particular possess the important property that they may be decomposed into smaller representations. In fact, suppose that $(\pi, \mathcal{H})$ is a unitary representation of a group $G$ on a Hilbert space $\mathcal{H}$ and that $V$ is a closed subspace of $\mathcal{H}$ such that $\pi(g)v \in V$ for all $v \in V$. Then we say that $V$ is an \textbf{invariant subspace} of $\mathcal{H}$. One may form a representation $\pi_V$ of $G$ on $V$ simply by restricting the action of $\pi$ on $\mathcal{H}$ to the subspace $V$. We say that $\pi_V$ is a \textbf{subrepresentation} of $\pi$.

Now consider the closed subspace

$$V^\perp = \{ w \in \mathcal{H} | \langle w, v \rangle = 0 \text{ for all } v \in V \}.$$ 

Note that if $w \in V^\perp$, then

$$\langle \pi(g)w, v \rangle = \langle w, \pi(g^{-1})v \rangle = 0$$

for all $v$ in $V$. It follows that $V^\perp$ is also an invariant subspace of $\mathcal{H}$.

In fact, we see that $\mathcal{H} = V \oplus V^\perp$ and that

$$\pi(g)(v + w) = \pi_V(g)v + \pi_{V^\perp}(g)w$$

for all $v \in V$ and $w \in V^\perp$. In this case, we write $\pi = \pi_V \oplus \pi_{V^\perp}$ and say that $\pi$ decomposes into a \textbf{direct sum of representations}.

If a representation $(\pi, \mathcal{H})$ of $G$ possesses no invariant subspaces besides $\mathcal{H}$ and $\{0\}$, then we say that $\pi$ is an \textbf{irreducible} representation. Let $\hat{G}$ denote the set of equivalence classes of irreducible representations of $G$.

\textbf{Theorem 2.2.} (\textit{Schur's Lemma}; see \cite[p. 71]{14}). Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary representation of a group $G$. Then every intertwining operator $T \in \operatorname{Hom}(\pi, \pi)$ for the representation $\pi$ may be written $T = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.

Now suppose that $\pi$ is a unitary representation of $G$ on a finite-dimensional Hilbert space $\mathcal{H}$. If $\pi$ is not irreducible, then we can repeatedly follow the process outlined above of decomposing it into sums of subrepresentations. Because $\mathcal{H}$ is finite-dimensional, the process must terminate at some point, which will occur when $\pi$ has been decomposed into a direct sum of irreducible subrepresentations. In this way, irreducible representations play a role similar to that of prime numbers in arithmetic.
More generally, we say that a representation \((\pi, \mathcal{H})\) of \(G\) is cyclic if there is a vector \(v \in \mathcal{H}\) such that the span \(<\pi(G)v>\) is dense in \(\mathcal{H}\). In that case, we say that \(v\) is a cyclic vector for \(\pi\). The following powerful and broad-ranging result may be proven using Zorn’s Lemma.

**Theorem 2.3.** ([14, p. 70]). Every unitary representation \((\pi, \mathcal{H})\) of a group \(G\) may be decomposed into an orthogonal direct sum of cyclic subrepresentations.

It is a classical result that all unitary representations of compact groups may be decomposed into a direct sum of irreducible subrepresentations. On the other hand, there are many interesting examples of representations of noncompact groups on infinite-dimensional Hilbert spaces which do not possess any irreducible subrepresentations (and thus cannot be decomposed into a direct sum of irreducible subrepresentations). However, it is possible to write such representations as a sort of “continuous” direct sum of irreducible representations in a matter which we now describe, roughly following the construction in [14, p. 219–232].

Suppose that \(\mu\) is a Borel measure on a topological space \(X\), and that for each \(x \in X\) we are given a unitary representation \((\pi_x, \mathcal{H}_x)\) of a group \(G\). Suppose we are also given a collection of maps \(s_i : X \to \bigcup_{x \in X} \mathcal{H}_x\) for \(i\) in some countable index set \(I\) such that:

1. \(s_i(x) \in \mathcal{H}_x\) for each \(x \in X\) and \(i \in I\).
2. \(<s_i(x)|i \in I>\) is dense in \(\mathcal{H}_x\) for all \(x \in X\).
3. \(x \mapsto <s_i(x), s_j(x)>_{\mathcal{H}_x}\) is a Borel-measurable function on \(X\) for all \(i, j \in I\).

The set \(\{s_i\}_{i \in I}\) is called a measurable frame. We then say that a map \(s : X \to \bigcup_{x \in X} \mathcal{H}_x\) is a measurable section if

1. \(s(x) \in \mathcal{H}_x\) for each \(x \in X\).
2. \(x \mapsto <s(x), s_i(x)>_{\mathcal{H}_x}\) is a Borel-measurable function on \(X\) for all \(i \in I\).

Finally, we define a direct-integral Hilbert space by

\[
\mathcal{H} \equiv \int_X^{\oplus} \mathcal{H}_x d\mu(x) = \left\{ \text{measurable sections } s \left| \int_X ||s(x)||_{\mathcal{H}_x}^2 d\mu(x) < \infty \right. \right\}
\]

where the inner product is given by

\[
\langle u, v \rangle = \int_X \langle u(x), v(x)\rangle_{\mathcal{H}_x} d\mu(x)
\]

for \(u, v \in \mathcal{H}\). We can also define a continuous unitary representation \(\pi \equiv \int_X^{\oplus} \pi_x d\mu(x)\) of \(G\) on \(\mathcal{H}\) by

\[
(\pi(g)s)(x) = \pi_x(g)(s(x))
\]

for all \(s \in \mathcal{H}\) and \(g \in G\). We say that \(\pi\) is a direct integral of the representations \(\mathcal{H}_x\) for \(x \in X\).
It is easy to see that orthogonal direct sums of Hilbert spaces and representations are a special case of direct integrals in which the measure is discrete. Moreover, every continuous unitary representation of a group \( G \) may be decomposed as a direct integral of irreducible representations, although there are some subtleties surrounding the uniqueness of such decompositions for certain groups.

We end this section by defining two very important classes of representations.

**Definition 2.4.** A unitary representation \((\pi, \mathcal{H})\) of a topological group \( G \) is said to be **multiplicity free** if every decomposition \( \pi = \pi_1 \oplus \pi_2 \) of \( \pi \) into a direct sum of subrepresentations has the property that no subrepresentation of \( \pi_1 \) is equivalent to a subrepresentation of \( \pi_2 \).

One can show that a unitary representation \( \pi \) is multiplicity-free if and only if its ring \( \text{Hom}(\pi, \pi) \) of intertwining operators is commutative. The term “multiplicity free” comes from the fact that a direct sum \( \pi = \bigoplus_{i \in I} \pi_i \) of irreducible representations of a group \( G \) is multiplicity free if and only if each equivalence class in \( \hat{G} \) appears at most once in the collection of \( \pi_i \)'s. This basic result is a corollary of Schur’s lemma (see [9, p. 123]).

**Definition 2.5.** A unitary representation \((\pi, \mathcal{H})\) of a topological group \( G \) is said to be **primary** if the center of its ring of intertwining operators is trivial—that is, if

\[
Z(\text{Hom}(\pi, \pi)) = \{ \lambda \text{Id} | \lambda \in \mathbb{C} \}.
\]

One can show (see [9, p. 122]) that a direct sum \( \pi = \bigoplus_{i \in I} \pi_i \) of irreducible representations of a group \( G \) is primary if and only if all the irreducible components \( \pi_i \) are equivalent to each other. However, for some groups it is possible to construct primary representations which cannot be decomposed into a direct sum of irreducible representations.

### 2.2 Invariant Measures

It is well-known that every locally-compact topological group \( G \) possesses a Radon measure \( \mu_G \) which is left-invariant under left translations of the group and such that every open subset of \( G \) has positive measure. That is,

\[
\int_G f(gx) d\mu_G(x) = \int_G f(x) d\mu_G(x) \tag{2.1}
\]

for all \( f \in C_c(G) \) and \( g \in G \). Such measures, called **Haar measures**, are unique up to multiplication by a constant. If \( G \) is a compact group, then \( \mu_G \) is a finite measure, which we will always normalize so that \( \mu_G(G) = 1 \).

The existence of Haar measures has several important and useful consequences. For example, the fact that compact groups have invariant probability measures makes it possible to construct many arguments in which one averages some object over the group:

**Theorem 2.6.** (See also [26, Proposition 4.6]). If \( G \) is a compact topological group, then every norm-continuous representation \((\pi, \mathcal{H})\) of \( G \) on a Hilbert space is equivalent to a unitary representation.
Proof. We denote the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and construct a new inner product $\langle \cdot, \cdot \rangle_\pi$ on $\mathcal{H}$ by defining:

$$\langle v, w \rangle_\pi = \int_G \langle \pi(g)v, \pi(g)w \rangle_{\mathcal{H}}d\mu_G(g)$$

for all $v, w \in \mathcal{H}$.

Now define

$$M = \sup_{g \in G} ||\pi(g)||_{\mathcal{H}}$$

and note that $M < \infty$ because $\pi$ is norm-continuous and $G$ is compact. We then have $||\pi(g)^{-1}||_{\mathcal{H}} < M$ for all $g \in G$. Thus

$$M^{-2}||v||_{\mathcal{H}}^2 \leq ||v||_\pi^2 = \int_G ||\pi(g)v||_{\mathcal{H}}^2d\mu_G(g) \leq M^2||v||_{\mathcal{H}}^2$$

for all $v \in \mathcal{H}$. Hence the identity map on $\mathcal{H}$ forms a homeomorphism between $\mathcal{H}$ under $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\mathcal{H}$ under $\langle \cdot, \cdot \rangle_\pi$.

Finally, for all $h \in G$ and $u, v \in \mathcal{H}$, we have that

$$\langle \pi(h)u, \pi(h)v \rangle_\pi = \int_G \langle \pi(gh)v, \pi(gh)w \rangle_{\mathcal{H}}d\mu_G(g)$$

$$= \int_G \langle \pi(g)v, \pi(g)w \rangle_{\mathcal{H}}d\mu_G(g)$$

$$= \langle u, v \rangle_\pi.$$

Thus, we see that $\pi$ is a unitary representation of $G$ on $\mathcal{H}$ under the inner product $\langle \cdot, \cdot \rangle_\pi$. □

For certain groups, the left-invariant Haar measure is also right invariant. That is, the Haar measure $\mu_G$ satisfies the property that

$$\int_G f(g_1xg_2)d\mu_G(x) = \int_G f(x)d\mu_G(x)$$  \hspace{1cm} (2.2)

for all $f \in C_c(G)$ and $g_1, g_2 \in G$. In this case, we say that $G$ is unimodular. Many basic results in harmonic analysis can be formulated most cleanly when the group under consideration is unimodular; fortunately several broad classes of groups are known to be unimodular, including (see [12, p. 88]) all compact groups, abelian groups, semisimple Lie groups, and connected nilpotent Lie groups (in contrast, not all solvable Lie groups are unimodular).

At any rate, with a Haar measure $\mu_G$ on $G$, we may consider the Hilbert space $L^2(G) \equiv L^2(G, \mu_G)$ of square-integrable functions on $G$. It is easy to show that the action given by

$$(g \cdot f)(x) = f(g^{-1}x)$$  \hspace{1cm} (2.3)

for $g \in G$ and $f \in L^2(G)$ gives a continuous representation of $G$ on $L^2(G)$ that is unitary by (2.1). This representation is called the left regular representation of $G$. 

8
The foundational problem of harmonic analysis is to provide, for a particular group $G$, a decomposition of the regular representation into irreducible components. A general result states that this is possible for a very broad class of locally-compact groups, called **Type I groups**.

**Definition 2.7.** A topological group $G$ is said to be of **Type I** if every primary representation of $G$ decomposes into a direct sum of copies of the same irreducible representation.

This class includes all compact groups (see [12, p. 206] and all semisimple Lie groups (see [18, p. 230]), for example.

**Theorem 2.8.** (The Abstract Plancherel Theorem; see [9, p. 368]). Let $G$ be a Type I separable, locally-compact topological group. For each $\lambda \in \hat{G}$, choose a representative irreducible representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $G$. Then there is a measure $\mu$ on $\hat{G}$ (whose measure class is uniquely determined) such that

$$L^2(G) \cong \int_{\hat{G}} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda} d\sigma(\lambda).$$

Such a decomposition is called a **Plancherel formula** for $G$.

One of the basic tasks of harmonic analysis is to make the Plancherel formula as explicit as possible for particular groups.

There are several variants of the regular representation that will be useful to us later. We can define continuous representations $L$ and $R$ of $G$ on the space $\mathcal{D}(G)$ of smooth, compactly supported functions as follows:

$$L(g)f(x) = f(g^{-1}x)$$
$$R(g)f(x) = f(xg)$$

for $g, x \in G$ and $f \in \mathcal{D}(G)$. These representations may be dualized to produce continuous representations $L$ and $R$ on the space $\mathcal{D}'(G)$ of distributions on $G$.

Similarly, one can define continuous left- and right-regular representations of $G$ on the space $C^\infty(G)$ of smooth functions on $G$. We say that a function $f \in C^\infty(G)$ is $G$-finite if the subspace $\langle L(G)f \rangle \subseteq C^\infty(G)$ generated by all $G$-translations of $f$ is finite-dimensional. We denote the space of all $G$-finite smooth functions by $C^\infty_{\text{fin}}(G)$ and note that $C^\infty_{\text{fin}}(G)$ is an invariant subspace of $C^\infty(G)$.

**2.3 Homogeneous Spaces**

More generally, we wish to study not only functions on a group $G$ but also functions on spaces on which $G$ acts. To that end, suppose that $G$ is a Lie group which acts smoothly and transitively on a manifold $X$. Let $x_o \in X$ and consider the stabilizer subgroup of $G$ given by

$$G^{x_o} = \{g \in G | g \cdot x_o = x_o\}.$$  

Note that $G^{x_o}$ is a closed subgroup of $G$. 

One can then form the space $G/G^{x_0}$ of left cosets. There is a transitive action of $G$ on $G/G^{x_0}$ given by

$$g \cdot hG^{x_0} = ghG^{x_0}.$$ 

In fact, one can show (see [5, Proposition 4.6]) that there is a $G$-equivariant diffeomorphism

$$X \rightarrow G/G^{x_0}.$$ 

In other words, we have an identification of transitive $G$-actions with quotient spaces of the form $G/H$, where $H$ is a closed subgroup of $G$. Such spaces are called **homogeneous spaces**, because the transitive group action forces them to have the same local behavior around each point. We refer to $G$ as the **translation group** of $G/H$ and to $H$ as the **isotropic subgroup** of $G$.

We would like to study harmonic analysis on homogeneous spaces. Just as for harmonic analysis on groups, the natural place to start is to construct an invariant measure. Unfortunately, not every homogeneous space $G/H$, where $G$ and $H$ are locally compact groups, possesses a Radon measure that is invariant under the action of $G$. However, as long as both $G$ and $H$ are unimodular, then such a measure always exists:

**Theorem 2.9.** (See [8, p. 41–44]). If $G$ and $H$ are locally compact unimodular topological groups, then there is a Radon measure $\mu_{G/H}$ on $G/H$, unique up to multiplication by a constant, such that

$$\int_{G/H} f(g \cdot x) d\mu_{G/H}(x) = \int_{G/H} f(x) d\mu_{G/H}(x)$$

for all $g \in G$ and $f \in C_c(G/H)$.

Furthermore, $\mu_{G/H}$ satisfies the functional equation

$$\int_G f(g) d\mu_G(g) = \int_{G/H} \int_H f(gh) d\mu_H(h) d\mu_{G/H}(g)$$

for all $f \in C_c(G)$.

As before, we construct the Hilbert space $L^2(G/H) \equiv L^2(G/H, \mu_{G/H})$ and note that a continuous unitary representation of $G$ on $L^2(G/H)$ may be constructed using the action given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for $f \in L^2(G/H)$, $g \in G$, and $x \in G/H$. This representation is also called a **regular representation** of $G$ for the homogeneous space $G/H$. Just as for $L^2(G)$, it is a basic problem of harmonic analysis to explicitly decompose $L^2(G)$ into a direct integral of irreducible representations.

In the interest of brevity, from this point forward we will use the simplified notations $dg = d\mu_G(g)$ and $dx = d\mu_{G/H}(x)$ to denote integration against a Haar measure on $G$ and against a $G$-invariant measure on $G/H$, respectively.
2.4 Gelfand-Naimark-Segal

In this section we explore the connection between unitary representations and positive-definite functions. We begin with a basic definition:

Definition 2.10. Let $G$ be a group. We say that a function $\phi: G \to \mathbb{C}$ is **positive-definite** if

$$\sum_{i,j=1}^{n} \phi(g_i^{-1}g_j)c_i \overline{c_j} > 0$$

where $g_i \in G$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$.

Positive-definite functions have several basic properties which may be proved directly from the definition (see [8, Lemma 5.1.8]):

1. $\phi(e) > 0$
2. $|\phi(g)| \leq \phi(e)$ for all $g \in G$
3. $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$

The canonical examples of positive-definite functions are provided by matrix coefficients of unitary representations. That is, if $(\pi, \mathcal{H})$ is a unitary representation of a group $G$ and $v \in \mathcal{H}\{0\}$, then the function $\phi_{\pi,v}: G \to \mathbb{C}$ given by

$$\phi_{\pi,v}(g) = \langle v, \pi(g)v \rangle$$

(2.4)

is continuous and positive-definite, as may be shown straightforwardly using the unitarity of $\pi$ and the definition of positive-definite functions.

The key insight of Gelfand-Naimark-Segal is that every continuous positive-definite function arises in this way from a unitary representation. In particular, given a continuous positive-definite function $\phi: G \to \mathbb{C}$, one can define a representation. We now show how this may be done.

For each $g \in G$, define the function $g \cdot \phi: G \to \mathbb{C}$ by

$$g \cdot \phi(x) = \phi(g^{-1}x)$$

for each $x \in G$. We can then define the vector space

$$V_\phi = \langle \{g \cdot \phi | g \in G\} \rangle,$$

which is the algebraic span of all $G$-translates of $\phi$. We define a pre-Hilbert space structure on $V_\phi$:

$$\left\langle \sum_{i=1}^{n} c_i (g_i \cdot \phi), \sum_{j=1}^{n} d_j (h_j \cdot \phi) \right\rangle = \sum_{i,j=1}^{n} \phi(g_i^{-1}h_j)c_i \overline{d_j}$$

(2.5)

where $c_i, d_j \in \mathbb{C}$ and $g_i, h_j \in G$. It can be shown that this bilinear form is well-defined on $V_\phi$ and turns it into a pre-Hilbert space.
We can then define a representation \( \pi \phi \) of \( G \) on \( V \phi \) by
\[
\pi \phi(g) v(h) = v(g^{-1}h)
\]
for all \( v \in V \) and \( g, h \in G \). It is clear from (2.5) that \( \pi \phi \) extends to a unitary representation on the Hilbert-space completion \( \mathcal{H} \phi \) of \( V \phi \). Then one has
\[
\phi(g) = \langle \phi, \pi(g) \phi \rangle_{\mathcal{H} \phi}.
\]
Thus every positive-definite function may be given the form (2.4). In fact, a stronger result may be proven:

**Theorem 2.11.** (Gelfand-Naimark-Segal; see [8, p. 54, 61].) The map
\[
(\pi, v) \mapsto \phi_{\pi,v}
\]
is a surjection from the set of all pairs \((\pi, v)\) of cyclic representations \((\pi, \mathcal{H})\) of \( G \) and cyclic vectors \( v \in \mathcal{H} \setminus \{0\} \) to the set of all continuous positive-definite functions on \( G \).

Furthermore, suppose that \((\pi, \mathcal{H})\) and \((\sigma, \mathcal{K})\) are unitary representations of \( G \) such that \( v \in \mathcal{H} \) and \( w \in \mathcal{K} \) are cyclic vectors. Then one has
\[
\phi_{\pi,v} = \phi_{\sigma,w}
\]
if and only if there is a unitary intertwining operator \( T : \mathcal{H} \to \mathcal{K} \) such that \( T(v) = w \).

Let \( G \) be a locally-compact topological group. We write \( \mathcal{P}(G) \) for the space of all positive-definite functions \( \phi \) on \( G \) such that \( \phi(e) = 1 \). One can show that \( \mathcal{P}(G) \) is a closed convex subset of the space \( L^\infty(G) \) of almost-everywhere-bounded measurable functions on \( G \). The convexity may be shown by noticing that
\[
\lambda \phi_{\pi,v} + (1 - \lambda) \phi_{\sigma,w} = \phi_{\pi \oplus \sigma, \sqrt{\lambda} v + \sqrt{1 - \lambda} w},
\]
where \((\pi, \mathcal{H})\) and \((\sigma, \mathcal{K})\) are unitary representations of \( G \) with cyclic vectors \( v \in \mathcal{H} \) and \( w \in \mathcal{K} \).

In fact, \( L^\infty(G) \) is the dual of the Banach space \( L^1(G) \) by the Riesz Representation Theorem. One can show that \( \mathcal{P}(G) \) is closed in the weak-* topology on \( L^\infty(G) \). Since \(|\phi(g)| \leq \phi(e) = 1\) for all \( \phi \) in \( \mathcal{P}(G) \) and \( g \in G \), we see that \( \mathcal{P}(G) \) is contained in the unit ball \( B_1(L^\infty(G)) \). It follows from the Banach-Alaoglu theorem that \( \mathcal{P}(G) \) is a compact convex subset of \( L^\infty(G) \) in the weak-* topology. Thus, the Krein-Milman theorem may be applied to \( \mathcal{P}(G) \):

**Theorem 2.12.** (Krein-Milman [8, Theorem 5.2.7]) If \( K \) is a compact, convex subset of a locally convex topological vector space \( V \), then
\[
K = \text{co}(\text{ex}(K)),
\]
where \( \text{co} \) denotes the convex hull and \( \text{ex}(K) \) denotes the set of extremal points of \( K \).
In other words, all normalized positive-definite functions may be formed by
taking a limit of convex combinations of normalized positive-definite functions. In
fact, by exploiting the identity in (2.6), one has the following result:

**Theorem 2.13.** Let $G$ be a locally compact topological group. Then the extremal
points of $\mathcal{P}(G)$ are given by functions of the form $\phi_{\pi,v}$, where $(\pi, \mathcal{H})$ is an irre-
ducible representation of $G$ and $v$ is a cyclic unit vector in $\mathcal{H}$.

Thus, positive-definite functions are generated in some sense by the ones coming
from irreducible representations. These are just a few examples of how powerful
theorems from functional analysis may be applied to provide insight into the de-
composition of unitary representations.

### 2.5 Smooth Vectors and Distribution Vectors

Suppose now that $G$ is a Lie group with Lie algebra $\mathfrak{g}$. In a certain sense, $\mathfrak{g}$ is a
“linearization” of $G$ that encapsulates all of the local aspects of its structure.
This is exemplified best by the famous Campbell-Baker-Hausdorff Theorem, which
shows how the group product on a Lie group may be recovered, in a neighborhood
of the identity, from the Lie bracket on its Lie algebra.

Similarly, it is desirable to recover information about a representation of a
group $G$ by first passing to a representation of $\mathfrak{g}$. If $(\pi, V)$ is a continuous finite-
dimensional representation of $G$ (not necessarily unitary), then one can show that
the map $g \mapsto \pi(g)v$ is a smooth (in fact analytic) function from $G$ to $V$. Thus, $\pi$
induces a representation $d\pi$ of $\mathfrak{g}$ on $V$ by:

$$d\pi(X)v = \frac{d}{dt} \bigg|_{t=0} (\pi(\exp(tX))v)$$

for all $X \in \mathfrak{g}$ and $v \in V$. One can show that two finite-dimensional representations
$\pi$ and $\rho$ of $G$ are equivalent if and only if $d\pi$ and $d\rho$ are equivalent.

However, the situation is more delicate for infinite-dimensional representations.
Let $(\pi, \mathcal{H})$ be a continuous representation of $G$ on a Hilbert space. We say that
$v \in \mathcal{H}$ is a smooth vector in $\mathcal{H}$ if the map $g \mapsto \pi(g)v$ is smooth. We denote the
space of all smooth vectors by $\mathcal{H}^\infty$. Similarly, we say that a vector is $G$-finite if
the $G$-invariant subspace $\langle \pi(G)v \rangle$ generated by $v$ is finite-dimensional. We denote
the space of $G$-finite vectors by $\mathcal{H}^{\text{fin}}$. It is not difficult to show that $\mathcal{H}^\infty$ and $\mathcal{H}^{\text{fin}}$
are linear subspaces of $\mathcal{H}$ and that $\mathcal{H}^{\text{fin}} \subseteq \mathcal{H}^\infty$.

Unfortunately, there are many interesting examples of infinite-dimensional rep-
resentations $(\pi, \mathcal{H})$ for which not every vector is a smooth vector. Nevertheless, a
classical result of G\r{a}rding uses the integrated representation of $\pi$ to show that $\mathcal{H}^\infty$
is a dense subspace of $\mathcal{H}$.
Theorem 2.14. (Gårding; see [8, p. 131–133]). Let \((\pi, H)\) be a continuous representation of a locally compact group \(G\) on a Hilbert space \(H\). Then \(H^\infty\) is a dense subspace of \(H\). In fact, for each \(f \in \mathcal{D}(G)\) and \(v \in H\), the vector

\[ \pi(f)v \equiv \int_G f(g)\pi(g)vdg \tag{2.7} \]

is in \(H^\infty\).

In fact, a beautiful theorem of Dixmier and Malliavin shows that the vectors Gårding constructed generate all of the smooth vectors:

**Theorem 2.15.** (The Decomposition Lemma; see [10]) If \((\pi, H)\) is a continuous representation of a locally compact group \(G\) on a Hilbert space \(H\), then every element of \(H^\infty\) may be written as a finite linear combination of vectors of the form (2.7).

Theorem 2.14 allows us to define a representation of \(g\) on the space \(H^\infty\) in the same way as before, namely

\[ d\pi(X)v = \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t} \]

for all \(X \in g\) and \(v \in H^\infty\). Furthermore,

\[ d\pi(X)d\pi(Y)v - d\pi(Y)d\pi(X)v = d\pi([X,Y])v \]

for all \(X,Y \in g\) and \(v \in H^\infty\) (see [21, p. 387]). This representation of \(g\) on \(H^\infty\) extends to a representation \(d\pi\) of the universal enveloping algebra \(U(g)\) in the natural way.

Finally, \(H^\infty\) may be given a Fréchet topology under the family of seminorms given by

\[ ||v||_D = ||d\pi(D)v||_H \]

for each \(D \in U(g)\) and \(v \in H\). Under this topology, the inclusion map

\[ H^\infty \hookrightarrow H \]

is a continuous dense embedding of a Fréchet space into a Hilbert space ([8, p. 132]).

Now consider the anti-dual \(H^{-\infty}\) of \(H^\infty\)—that is, the space of all conjugate-linear continuous functionals on \(H^\infty\). Elements of \(H^{-\infty}\) are called distribution vectors for the representation \(\pi\). We give \(H^{-\infty}\) the weak-* topology. Then there is a continuous embedding

\[ H \hookrightarrow H^{-\infty} \]

given by mapping a vector \(v \in H\) to the conjugate-linear functional on \(H^\infty\) given by

\[ w \mapsto \langle v, w \rangle_H \]

for all \(w \in H^\infty\).
There are several ways in which the space of distribution vectors is well-behaved. Just as distributions on a manifold are infinitely differentiable in a weak sense, the derived representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ on $\mathcal{H}^\infty$ extend by dualization to representations on $\mathcal{H}^{-\infty}$. Furthermore, distribution vectors can be “smoothed out” by integration against smooth functions on $G$:

**Lemma 2.16. ([8, p. 136]).** For each $v \in \mathcal{H}^{-\infty}$ and $\phi \in \mathcal{D}(G)$, the distribution vector

$$\pi(\phi)v = \int_G \phi(g)v \, dg$$

is an element of $\mathcal{H}^\infty$.

As a corollary of this result, one has that $\mathcal{H}^\infty$ is densely contained in $\mathcal{H}^{-\infty}$. Putting everything together, we have continuous, dense embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}.$$  

### 2.6 Invariance and Harmonic Analysis on Homogeneous Spaces

Suppose that $G$ is a compact group with a closed subgroup $K$, and consider the space $G/K$ and the regular representation of $G$ on $L^2(G/K)$. The basic task of harmonic analysis on $G/K$ is to decompose $L^2(G/K)$ into a direct sum of irreducible representations. We now show how to determine which equivalence classes of unitary representations of $G$ appear in this decomposition, as well as how many times they appear.

Suppose that $(\sigma, \mathcal{H})$ is a unitary representation of $G$. We consider the space

$$\mathcal{H}^K \equiv \{ v \in V | \pi(k)v = v \text{ for all } k \in K \}$$

of $K$-invariant vectors in $\mathcal{H}$. One then has the following theorem.

**Theorem 2.17.** For each irreducible unitary representation $(\sigma, \mathcal{H})$ of $G$, we have that

$$\dim \mathcal{H}^K = \dim \text{Hom}(\sigma, L^2(G/K)).$$

That is, the multiplicity of $\sigma$ in $L^2(G/K)$ is equal to the dimension of the space of $K$-invariant vectors in $\mathcal{H}$.

This result is a special case of the Frobenius Reciprocity Theorem for unitary representations of compact groups (see [14, p. 160]).

**Corollary 2.18.** Let $G$ be a compact group. For each $(\pi, \mathcal{H}_\pi) \in \hat{G}$. Then

$$L^2(G/K) \cong_G \bigoplus_{\pi \in \hat{G}} m_\pi \mathcal{H}_\pi,$$

where $m_\pi = \dim \mathcal{H}_\pi^K$ and $m_\pi \mathcal{H}_\pi = \mathcal{H}_\pi \oplus \cdots \oplus \mathcal{H}_\pi$ refers to the direct sum of $m_\pi$ copies of $\mathcal{H}_\pi$. 

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Unfortunately, the analysis just described is not applicable to homogeneous spaces $G/H$ where either $G$ or $H$ is non-compact. If $G$ is locally compact but not compact, then $L^2(G/H)$ is no longer guaranteed to decompose into a direct sum of irreducible subrepresentations; a direct integral decomposition is necessary. Furthermore, if $H$ is non compact, then $L^2(G)$ may not possess any nontrivial $H$-invariant functions, so that $L^2(G/H)$ cannot be embedded as a subrepresentation of $L^2(G)$. The solution to this problem is to move to the theory of distributions and distribution vectors; one attempts to decompose $L^2(G/H)$ into a direct integral of irreducible representations $(\pi, \mathcal{H})$ which possess $\mathcal{H}$-invariant distribution vectors (i.e., $\mathcal{H}^{-\infty}\mathcal{H} \neq 0$).
Chapter 3
Finite-Dimensional Riemannian Symmetric Spaces

Riemannian symmetric spaces form a class of particularly well-behaved homogeneous spaces with a rich structure theory and relatively well-understood harmonic analysis. Among other important properties, they possess a Riemannian metric that is invariant under the action of the translation group. Furthermore, the isotropic subgroup is fixed under an involution on the translation group, which essentially forces the regular representations on Riemannian symmetric spaces to have multiplicity-free direct integral decompositions. We shall also see that there is a beautiful duality between compact-type and noncompact-type Riemannian symmetric spaces.

In addition, the noncompact-type Riemannian symmetric spaces possess an associated homogeneous space called a horocycle space. The relationship between a Riemannian symmetric space and its horocycle space is analogous to, for instance, the relationship between points and hyperplanes in $\mathbb{R}^n$, or the relationship between points and horocycles of hyperbolic space (it is for this reason that the terminology horocycle space was originally chosen).

In the late 1950s, Gelfand and Graev developed a “horospherical method” which relates harmonic analysis on the noncompact-type Riemannian symmetric space $\text{SL}(n,\mathbb{R})/\text{SU}(n)$ and harmonic analysis on its horocycle space (see [30, p. 283–287]). These ideas were generalized to all noncompact-type Riemannian symmetric spaces and developed quite completely in the pioneering work of Helgason (see [19], for instance). The relationship between symmetric spaces and horocycle spaces, together with its implications for representation theory, provides the primary context for this thesis.

See [20] for a comprehensive overview of the structure theory for Riemannian symmetric spaces. See also [21] and [22] for applications of representation theory to analysis on Riemannian symmetric spaces and horocycle spaces, respectively. A good concise overview of this theory from the perspective of unitary group representations may be found in [36].

3.1 Basic Definitions

Suppose that $G$ is a semisimple Lie group with finite center and that $K$ is a closed subgroup. Furthermore, we suppose that there is an involutive automorphism $\theta : G \rightarrow G$ such that

$$\left( G^\theta \right)_0 \leq K \leq G^\theta,$$

(3.1)

where $G^\theta$ is the fixed-point subgroup for $\theta$ and $(G^\theta)_0$ is the connected component of the identity for $G^\theta$. Then $G/K$ is said to be a symmetric space.

The involution $\theta$ differentiates to an involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of $G$. By (3.1), the +1-eigenspace for $\theta$ is just $\mathfrak{k}$ (i.e., the Lie algebra for $K$). We
denote the $-1$-eigenspace of $\theta$ by $p$. Just as $\mathfrak{k}$ may be naturally identified with the tangent space $T_eK$, there is a natural identification of $p$ with the tangent space $T_eR/K$ (see [20, p. 214]). We may write down the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus p.$$  

Due to the fact that $\theta$ is also a Lie algebra involution, one easily computes that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Let $G/K$ be a symmetric space with involution $\theta$, and recall that the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ provides an $\text{Ad}(G)$-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. If $B$ restricts to a positive-definite or negative-definite symmetric bilinear form on $\mathfrak{p}$, then $G/K$ is said to be a Riemannian symmetric space. This terminology comes from the fact that an $\text{Ad}(G)$-invariant positive-definite bilinear form on $\mathfrak{p}$ may be translated by the action of $g$ to produce a $G$-invariant Riemannian metric on $G/K$.

If $U/K$ is a Riemannian symmetric space with $U$ compact, then $B$ restricts to a negative-definite form on $\mathfrak{p}$ and $U/K$ is said to be a compact-type Riemannian symmetric space. On the other hand, if $G/K$ is a Riemannian symmetric space with $G$ noncompact, then $K$ is compact and $B$ restricts to a positive-definite form on $\mathfrak{p}$ and $G/K$ is said to be a noncompact-type Riemannian symmetric space.

There is a beautiful duality between compact-type and noncompact-type Riemannian symmetric spaces. Suppose that $U/K$ is a compact-type symmetric space with involution $\theta$. We make the further simplifying assumption that $G$ is simply-connected. As before, we consider the $\theta$-eigenspace decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that $\mathfrak{g}$ may be embedded in the complexified Lie algebra $\mathfrak{u}_\mathbb{C} = \mathfrak{u} \otimes \mathbb{C}$. Furthermore, $\theta$ extends to a complex Lie algebra involution on $\mathfrak{u}_\mathbb{C}$, which we also denote by $\theta$. Furthermore, the Killing form $B$ on $\mathfrak{u}$ extends to a complex bilinear form on $\mathfrak{g}_\mathbb{C}$. We can then consider the real vector space $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$ defined by

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}.$$  

It can be shown that $\mathfrak{g}$ is a real semisimple Lie algebra that is invariant under $\theta$. In fact, $\mathfrak{k}$ and $i\mathfrak{p}$ are the $+1$- and $-1$-eigenspaces for $\theta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$. Also, since $U/K$ is a compact-type Riemannian symmetric space, we see that $B(X, X) < 0$ for all $X \in \mathfrak{p}$. But then $B(iX, iX) > 0$ for all $X \in \mathfrak{p}$ and hence $B$ is positive-definite on $i\mathfrak{p}$.

We now consider the unique connected complex Lie group $U_\mathbb{C}$ with Lie algebra $\mathfrak{u}_\mathbb{C}$ such that $U$ is the analytic subgroup of $U_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$. The Lie algebra involution $\theta$ on $\mathfrak{u}_\mathbb{C}$ integrates to an involution on $U_\mathbb{C}$ by Proposition 7.5 in [26]. We then consider the analytic subgroup $G \leq U_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{g} \subset \mathfrak{u}_\mathbb{C}$. By Proposition 7.9 in [26], we see that $G$ is a closed subgroup of $U_\mathbb{C}$ and has a finite center. Putting everything together, we see that $G/K$ is a noncompact-type Riemannian symmetric space, called the $c$-dual of $U/K$.  

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3.2 The Structure of Noncompact-Type Riemannian Symmetric Spaces

In this section we review the basic structure theory for noncompact-type Riemannian symmetric spaces. All of the material is entirely classical and may be found in standard references, such as Chapters VI and VII in [26] or in Chapter VI of [20].

Let $G$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$. It can be shown (see [26, p. 355–358]) that there is an involution $\theta$ on $\mathfrak{g}$ such that the symmetric bilinear form given by

$$(X, Y) \mapsto -B(X, \theta Y)$$

is positive-definite. Such an involution is called a Cartan involution and is unique up to inner automorphisms. One shows that a Cartan involution on $\mathfrak{g}$ integrates to an involution on $G$ (see [26, p. 362]). Furthermore, if $G$ has a finite center, then $K = G^\theta$ is a maximal compact subgroup of $G$. For a subgroup $G \leq \text{GL}(n, \mathbb{C})$ which is stabilized by the taking of adjoints, then one may define Cartan involutions on $G$ and $\mathfrak{g}$ by setting $\theta(g) = (g^{-1})^*$ and $\theta(X) = -X^*$, respectively.

Now suppose that $G/K$ is a noncompact-type Riemannian symmetric space with involution $\theta$ and that $G$ has finite center. It can be shown that $\theta$ is a Cartan involution on $G$ and thus that $K$ is a maximal compact subgroup of $\mathfrak{g}$. Thus, the classification of real semisimple Lie groups may be used to provide a classification of noncompact-type Riemannian symmetric spaces.

As before, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Now let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Denote the real linear dual of $\mathfrak{a}$ by $\mathfrak{a}^*$, whose elements are called weights. For each $\alpha \in \mathfrak{a}^*$, write

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a} \}$$

Note that because $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a},$$

where $\mathfrak{m} = Z_\mathfrak{e}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The set of all $\alpha \neq 0$ in $\mathfrak{a}^*$ such that $\mathfrak{g}_\alpha \neq 0$ is denoted by $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma$. Elements of this set are called restricted roots.

Because the Killing form is positive-definite on $\mathfrak{p}$, it follows that $\mathfrak{g}$ decomposes into joint eigenspaces under the action of $\text{ad}(\mathfrak{a})$:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

(3.2)

(Note that all of the restricted roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$ are real-valued weights on $\mathfrak{a}$, in contrast with the roots of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, which are in general complex-valued.) The Jacobi identity shows that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha + \beta}$$

(3.3)

for all $\alpha, \beta \in \Sigma$. In this way the restricted root spaces provide a great deal of information about the Lie algebra structure of $\mathfrak{g}$. 

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An element $H \in \mathfrak{a}^*$ is said to be **regular** if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. The set of all regular elements of $\mathfrak{a}$ will be denoted by $\widetilde{\mathfrak{a}}$. The connected components of $\widetilde{\mathfrak{a}}$ are called **Weyl chambers**. We choose a Weyl chamber $C \subseteq \widetilde{\mathfrak{a}}$. Under this choice, a weight $\lambda \in \mathfrak{a}^*$ is said to be **positive** if $\lambda(H) > 0$ for all $H \in C$. We let $\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^+$ denote the set of all positive restricted roots. Since the negative of any restricted root is again a restricted root, one obtains a decomposition

$$\Sigma = \Sigma^+ \cup (-\Sigma^+)$$

We denote by $\Sigma^0(\mathfrak{g}, \mathfrak{a}) = \Sigma^0$ the set of nonmultiplicable restricted roots (that is, roots $\alpha \in \Sigma$ such that $c\alpha \notin \Sigma$ for all $c \neq 1$ in $\mathbb{R}$). We set $\Sigma^0_+ = \Sigma_0 \cap \Sigma^+$. Finally, it is possible to choose a set $\Psi = \{\alpha_1, \ldots, \alpha_r\} \subset \Sigma^0_+$, where $r = \dim \mathfrak{a}$, such that $\Psi$ is a basis for $\mathfrak{a}^*$. Each root $\alpha \in \Sigma^0_+$ may then be written $\alpha = n_1\alpha_1 + \cdots + n_r\alpha_r$, where $n_1, \ldots, n_r \in \mathbb{N}$. Roots in $\Psi$ are called **simple roots**.

Now consider the normalizer $M'$ of $\mathfrak{a}$ in $K$ (that is, $M'$ consists of all $k \in K$ such that $\text{Ad}(k)\mathfrak{a} = \mathfrak{a}$). Similarly, let $M = Z_K(\mathfrak{a})$ denote the centralizer of $\mathfrak{a}$ in $K$ (that is, $M$ consists of all $k \in K$ such that $\text{Ad}(k)X = X$ for all $X \in \mathfrak{a}$). Note that $M \leq M'$. The quotient group $W = M/M'$ is called the **restricted Weyl group** for $(\mathfrak{g}, \mathfrak{a})$. In fact, one may show that elements in $W$, acting by conjugation on $\mathfrak{a}$, permute the Weyl chambers. Furthermore, there is a unique element $w^* \in W$ whose action on $\mathfrak{a}$ sends the Weyl chamber $C$ to the Weyl chamber $-C$. We refer to $w^*$ as the **longest element** of the Weyl group.

As a word of caution to the reader, we note that $M = Z_K(\mathfrak{a})$ is generally not connected (even when $G$ is connected), in which case $M \neq \exp \mathfrak{m}$. We define $M_0 = \exp \mathfrak{m}$ and note that $M_0$ is the connected component of the identity for $M$. We will recall some well-known results about the structure of the component group $M/M_0$ when the need arises later.

Consider the nilpotent Lie algebras

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

$$\mathfrak{\overline{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}.$$ 

By combining (3.2) and (3.4), we then have a **triangular decomposition** of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{\overline{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$ 

There is a triangular decomposition on the level of the group $G$, as well. Consider the subgroups $N = \exp \mathfrak{n}$, $\overline{N} = \exp \mathfrak{\overline{n}}$, and $A = \exp \mathfrak{a}$ of $G$. One can show that $\overline{N}$, $M$, $A$, and $N$ are closed Lie subgroups of $G$ with Lie algebras $\mathfrak{\overline{n}}$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then the map

$$\overline{N} \times M \times A \times N \to G \quad (\mathfrak{\overline{n}}, m, a, n) \mapsto \mathfrak{\overline{n}man}$$

\footnote{It is standard in the literature to define $\Sigma_0$ to be the set of all indivisible roots, but here we follow the notation of [7].}
is a smooth embedding of the manifold $\mathcal{N} \times M \times A \times N$ into an open dense subset of $G$.

There are other decompositions of $G$ that are useful to consider. The **Iwasawa decomposition** states that

$$ g = k \oplus a \oplus n. $$

This decomposition integrates nicely to the group level; in fact, the map

$$ K \times A \times N \to G \quad (k, a, n) \mapsto kan $$

is a diffeomorphism.

Recall the choice of positive Weyl chamber $C$ in $a$. The set

$$ \tilde{A} = \exp \tilde{a} $$

is called the **regular set** of $A$. Similarly, we define

$$ A^+ = \exp C \subseteq \tilde{A}. $$

In fact, because the elements of $W$ permute the Weyl chambers of $A$, there is a natural identification $A^+ \cong \tilde{A}/W$.

One can show (see Theorem 7.39 in [26]) that $G = KAK$; that is, each $g \in G$ may be written $g = k_1ak_2$ where $a \in A$ and $k_1, k_2 \in K$. More strongly, one has the decomposition

$$ G = K\tilde{A}K. \quad (3.6) $$

In fact, if $g = k_1ak_2$ where $a \in A^+$ and $k_1, k_2 \in K$, then $a \in A^+$ is uniquely determined by $g$. In other words, there is a natural identification

$$ K\backslash G/K = A/W, \quad (3.7) $$

where $K\backslash G/K$ denotes the space of double-cosets of $G$ over $K$.

It follows easily from (3.3) that $[a, n] \subseteq n$ and $[m, n] \subseteq n$. One can in fact show that both $A$ and $M$ normalize $N$ in $G$. Since $M$ normalizes $N$, we see that $MN$ is a closed Lie subgroup of $G$ with Lie algebra $m \oplus n$. Also, since $A$ centralizes $M$ and also normalizes $N$, it follows that $MAN$ is a closed Lie subgroup of $G$ with Lie algebra $m \oplus a \oplus n$. Furthermore, $MN \trianglelefteq MAN$ and $MAN/MN \cong A$. One refers to $MAN$ as a **minimal parabolic subgroup** of $G$ (more generally, a **parabolic subgroup** of $G$ is a group $H$ such that $MAN \leq H \leq G$). Note that each choice of maximal abelian subalgebra $a$ in $p$ and Weyl chamber in $a$ produces a minimal parabolic subgroup in this way.

We move now to the final decomposition of this section. For each $w \in W$, we choose a representative $m_w \in M'$. We then consider double cosets of the form

$$ MANm_wMAN = Nm_wMAN, $$

where $K\backslash G/K$ denotes the space of double-cosets of $G$ over $K$.
called **Bruhat cells**. The **Bruhat decomposition** (see [26, Theorem 7.40]) states that $G$ decomposes into a disjoint union of Bruhat cells. That is,

$$G = \bigcup_{w \in W} N m_w MAN.$$  

(3.8)

From (3.5) we see that one of the Bruhat cells is an open and dense subset of $G$, namely the cell corresponding to the longest Weyl group element $w^*$. The Bruhat decomposition should be viewed as analogous to the decomposition in (3.6). Helgason exploited this analogy and many others to relate analysis on the symmetric space $G/K$ to analysis on the associated horocycle space, which we discuss in the next section.

### 3.3 The Horocycle Space

We continue with the same notation as in the previous section. A **horocycle** on a noncompact-type Riemannian symmetric space $G/K$ is an orbit in $G/K$ of a subgroup of $G$ that is conjugate to $N$. In other words, it takes the form

$$gN g^{-1} \cdot g_0 K \subseteq G/K,$$

where $g, g_0$ are arbitrary elements $G$. We denote the space of all Horocycles by $\Xi$.

Horocycles on Riemannian symmetric spaces were studied in detail by Helgason in the 1970s. The relationship between horocycles and points in a Riemannian symmetric space was intended to be analogous to the relationship between points and hyperplanes in $\mathbb{R}^n$ (see [22, p. 59]). Most of the results in this section may be found in either [19] and [22].

It is not difficult to see that left translations of horocycles by elements of $G$ are also horocycles. In fact,

$$h \cdot (gN g^{-1} \cdot g_0 K) = (hg)N(hg)^{-1} \cdot hg_0 K$$

where $h \in G$ and $gN g^{-1} \cdot g_0 K$ is a horocycle in $\Xi$. Thus $G$ acts on $\Xi$ by left translation. One then has the following theorem of Helgason.

**Theorem 3.1** (Theorem II.1.1 in [22]). The group $G$ acts transitively on $\Xi$, and the isotropic subgroup of $G$ which fixes the horocycle $N \cdot K$ is $MN$.

In other words, we can make the identification

$$\Xi \cong G/MN$$

**Theorem 3.2** (Proposition II.1.4 in [22]). The map

$$K/M \times A \to G/MN$$

$$(kM, a) \mapsto kaMN$$

is a diffeomorphism.
For each \( w \in W \), define a set of horocycles by \( \Xi_w = N A m_w \cdot MN \subseteq G/MN \), where \( m_w \) is a representative in \( M' \) of the Weyl group element \( w \in W = M'/M \). Using the fact that \( A \) normalizes \( N \) and that \( M' \) normalizes \( A \), we obtain the following identity for each Bruhat cell:

\[
MANm_wMAN = MANm_wMN.
\]

The Bruhat decomposition then implies (see [22, p. 63]) that \( \Xi \) decomposes disjointly as

\[
\Xi = \bigcup_{w \in W} \Xi_w,
\]

Furthermore, from the denseness of the embedding in (3.5), we see that \( \Xi_w^* \) is an open, dense subset of \( \Xi \).

**Theorem 3.3** (Proposition II.1.5 in [22]). Each element \( gMN \in G/MN \) may be written in the form

\[
gMN = mnam_wMN,
\]

where \( m \in M, n \in N, a \in A \), and \( m_w \in M' \) is a representative of \( w \in W \). Furthermore \( a \) and \( w \) are uniquely determined.

As a corollary of this result, we may make an identification

\[
MN \backslash G/MN \cong A \times W,
\]

which should be viewed in analogy with (3.7).

### 3.4 Spherical Representations

Suppose that \( G/K \) is a Riemmanian symmetric space with an involution \( \theta \). Based on the remarks in Section 2.6, one expects that representations possessing \( K \)-invariant vectors will play a crucial role in harmonic analysis on \( G/K \), which leads to the following definition:

**Definition 3.4.** We say that a Hilbert space representation \((\pi, \mathcal{H})\) of \( G \) is **spherical** if there is a nonzero cyclic vector \( v \in \mathcal{H} \) such that \( \pi(k)v = v \) for all \( k \in K \).

The following lemma allows the identification of irreducible unitary spherical representations:

**Lemma 3.5.** (See Lemma IV.3.5 in [21]) Suppose that \((\pi, \mathcal{H})\) is a unitary spherical representation of \( G \). Then \( \pi \) is irreducible if and only if \( \dim \mathcal{H}^K = 1 \).

On the other hand, the algebra \( \mathbb{D}(G/K) \) of left \( G \)-invariant differential operators on \( G/K \) is abelian (see Corollary II.5.4 in [21]). It is thus natural to look for functions which are joint eigenvectors for the operators in \( \mathbb{D}(G/K) \). In particular, we arrive at the following definition:

**Definition 3.6.** A function \( \phi \in C^\infty(G/K) \) is called a **spherical function** if

1. \( \phi \) is left-invariant under translations by elements of \( K \).
2. $\phi$ is an eigenfunction for every differential operator in $\mathcal{D}(G/K)$.

One can show that any distribution in $\mathcal{D}'(G)$ which satisfies the conditions of Definition 3.6 is automatically an analytic function (see [22, p. 105]), which is why we speak of spherical functions rather than spherical distributions. Note that spherical functions on $G/K$ may be considered as bi-$K$-invariant functions on $G$. Spherical functions are nicely characterized by a functional equation:

**Theorem 3.7** (Proposition IV.2.2 in [21]). A continuous function $\phi : G \to \mathbb{C}$ is a spherical function if and only if $\phi$ is not identically zero and

$$
\int_K \phi(xky)dk = \phi(x)\phi(y)
$$

(3.9)

for all $x, y \in G$.

In fact, (3.9) is often chosen as the definition of a spherical function, in part because it is applicable to the more general class of homogeneous spaces known as Gelfand pairs.

There is a natural correspondence between spherical functions and spherical representations, as illustrated by the following lemma.

**Lemma 3.8.** (Theorem IV.3.7 in [21]) Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary spherical representation of $G$ with a spherical vector $e$. Then the function $\phi_\pi$ on $G$ given by

$$
\phi_\pi(x) = \langle e, \pi(g)e \rangle
$$

is a positive-definite spherical function. Furthermore, every positive-definite spherical function takes the form $\phi_\pi$ for an irreducible unitary spherical representation $\pi$ that is unique up to unitary equivalence.

**Proof.** Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary spherical representation of $G$ with a spherical vector $e$. We will show that $\phi_\pi$ is spherical by demonstrating that it satisfies the condition of Lemma 3.7. Note that the orthogonal projection $P$ from $\mathcal{H}$ to $\mathcal{H}^K$ is given by:

$$
P(v) = \int_K \pi(k)v dk.
$$

Since $P(\pi(y)e) \in \mathcal{H}^K$ and $\dim \mathcal{H}^K = 1$, it follows that $P(\pi(y)e) = ce$ for some nonzero $c \in \mathbb{C}$. But then

$$
c = \langle P(\pi(y)e), e \rangle
$$

$$
= \int_K \langle \pi(ky)e, e \rangle
$$

$$
= \langle \pi(y)e, e \rangle.
$$
Hence
\[ \int_K \phi(xky) dk = \int_K \langle e, \pi(xky)e \rangle \]
\[ = \left\langle \pi(x^{-1})e, \int \pi(k)\pi(y)e \ dk \right\rangle \]
\[ = \left\langle \pi(x^{-1})e, P(\pi(y)e) \right\rangle \]
\[ = \left\langle \pi(x^{-1})e, \langle \pi(y)e, e \rangle \right\rangle \]
\[ = \langle e, \pi(x)e \rangle \langle e, \pi(y)e \rangle \]
\[ = \phi(x)\phi(y) \]

On the other hand, suppose that \( \phi \) is a positive-definite spherical function on \( G \). Because \( \phi \) is positive definite, there is a representation \((\pi, \mathcal{H})\) of \( G \) with a nonzero cyclic vector \( v \in \mathcal{H} \) such that
\[ \phi(g) = \langle v, \pi(g)v \rangle \]
It follows immediately from the bi-\( K \)-invariance of \( \phi \) that \( v \in \mathcal{H}^K \).

It remains to be shown that \( \pi \) is irreducible. To that end, we will show that \( \dim \mathcal{H}^K = 1 \). Fix \( y \in G \). From Lemma 3.7, we see that
\[ \langle \pi(x^{-1})v, P(\pi(y)v) \rangle = \int_K \phi(xky) dk \]
\[ = \phi(x)\phi(y) \]
\[ = \langle \pi(x^{-1})v, \pi(y)v \rangle \]

for all \( x \in G \). Recall that \( v \) is cyclic; that is, \( \langle \pi(G)v \rangle \) is dense in \( \mathcal{H} \), It follows that \( P(\pi(y)v) = \phi(y)v \). Using again the fact that \( v \) is cyclic, we see that
\[ \dim(\text{range } P) = 1. \]
In other words, \( \dim \mathcal{H}^K = 1 \), and thus \( \mathcal{H} \) is irreducible.

3.5 Conical Representations

In this section we assume that \( G/K \) is a noncompact-type Riemannian symmetric space. Just as was the case for a symmetric space \( G/K \), it can be shown that the algebra \( \mathbb{D}(G/MN) \) of left-\( G \)-invariant differential operators on a horocycle space \( G/MN \) is commutative (see Theorem II.2.2 in [22]), and it is natural to look for joint eigendistributions.

Definition 3.9. A distribution \( \phi \in \mathcal{D}'(G/MN) \) is called a conical distribution if it is an eigendistribution for every differential operator in \( \mathbb{D}(G/MN) \). If \( \phi \) is in fact a smooth function on \( G \), then we say that it is a conical function.

As before, we notice that a conical distribution on \( G/MN \) may be considered to be a bi-\( MN \)-invariant distribution on \( G \). In contrast to the situation for spherical
functions, a conical distribution need not be analytic and need not be a function at all.

The analogue for a horocycle space of a spherical representation is called a conical representation.

**Definition 3.10.** A Hilbert space representation \((\pi, \mathcal{H})\) of \(G\) is said to be **conical** if there is a nonzero cyclic distribution vector \(v\) in \(\mathcal{H}^{-\infty}\) such that \(\pi(MN)v = v\). In this case, \(v\) is said to be a conical distribution vector for \(\pi\).

Suppose that \((\pi, \mathcal{H})\) is a conical representation of \(G\) with a conical unit vector \(v \in \mathcal{H}^{MN}\). In this case, one obtains a conical function \(\psi_{\pi,v}\) by

\[
\psi_{\pi,v}(g) = \langle v, \pi(g)v \rangle.
\] (3.10)

Note the similarity with the way in which spherical representations give rise to spherical functions.

In general, a conical representation \((\pi, \mathcal{H})\) of \(G\) might not have a conical vector but rather may have merely a conical distribution vector. In this case, each \(v \in (\mathcal{H}^{-\infty})^{MN}\) gives rise to a conical distribution \(\psi_{\pi,v}\) on \(G\) in the following way:

Suppose that \(\pi\) is a conical representation of \(G\) with conical vector \(v \in (\mathcal{H}^{-\infty})^{MN}\). For each \(v \in \mathcal{H}^{-\infty}\) and \(f \in \mathcal{D}(G)\), consider as in Section 2.5 the vector

\[
\pi(f)v = \int_G f(g)\pi(g)v\ dg \in \mathcal{H}^{\infty}
\]

We then define a conical distribution \(\psi_{\pi,v}\) on \(G\) by

\[
\langle \psi_{\pi,v}, f \rangle = \langle v, \pi(f)v \rangle
\]

Another contrast with spherical representations is that an irreducible conical representation \((\pi, \mathcal{H})\) may have the property that \(\dim(\mathcal{H}^{-\infty})^{MN} > 1\), as we shall see later.

### 3.6 Finite-Dimensional Representations and Weyl’s Unitary Trick

The easiest representations to construct and classify are those which are finite-dimensional. For that reason, we will later be interested in studying, for infinite-dimensional Riemannian symmetric spaces, the analogues of finite-dimensional conical representations. Those representations will no longer be finite-dimensional, but they will inherit many of the features of finite-dimensional conical representations. To that end, we review the relevant material on finite-dimensional representations.

The material in this section is almost entirely classical and very well known. For a treatment of Weyl’s Unitary Trick, see Section VII.1 of [26]. The highest-weight theorem may be found in any standard reference on Lie groups, including Section V.2 of [26]. For a more algebraic treatment, see Chapter 3 of [15]. Results about finite-dimensional spherical and conical representations may be found in Section V.4 of [21] and Section II.4 of [22], respectively.
As before, we suppose that $G/K$ is a noncompact-type Riemannian symmetric space with involution $\theta$. For this section, we will further assume that $G/K$ is the c-dual of a simply-connected compact-type Riemannian symmetric space $U/K$. In other words, we have that $\mathfrak{g}_C = \mathfrak{u}_C$ and also have the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\theta$ on $\mathfrak{g}$. Furthermore, $G$ and $U$ share the same complexified group $G_C = U_C$.

**Theorem 3.11** (Weyl’s Unitary Trick). ([26, Proposition 7.15]) There is a one-to-one correspondence between the following categories of representations on a finite-dimensional vector space $V$, under which corresponding representations have the same algebra of intertwining operators:

1. representations of $G$ on $V$
2. representations of $\mathfrak{g}$ on $V$
3. complex-linear representations of $\mathfrak{g}_C$ on $V$
4. holomorphic representations of $G_C$ on $V$
5. representations of $U$ on $V$
6. representations of $\mathfrak{u}$ on $V$

**Proof.** We briefly sketch an outline of the proof. Begin with a representation $\pi$ of $G$ on $V$. We can differentiate $\pi$ to yield a representation of $\mathfrak{g}$ on $V$. Note that any two representations of $G$ with the same derived representation are equivalent, so passing from (1) to (2) is injective. We can extend the real-linear representation of $\mathfrak{g}$ on $V$ to a complex-linear representation $\mathfrak{g}_C$, and this process is bijective. We can similarly extend representations of $\mathfrak{u}$ to $\mathfrak{u}_C$. This gives the correspondences (2) $\leftrightarrow$ (3) and (6) $\leftrightarrow$ (3). Since $G_C$ is simply-connected, there is a correspondence between holomorphic representations of $G_C$ and complex-linear representations of $\mathfrak{g}_C$ given by differentiation. This gives the bijective correspondence (3) $\leftrightarrow$ (4). We can restrict the holomorphic representation of $G_C$ to the closed subgroups $U$ and $G$, giving (4) $\rightarrow$ (5) and (4) $\rightarrow$ (1). Finally, differentiating a representation of $U$ gives a representation of $\mathfrak{u}$ and this correspondence is bijective because $U$ is simply connected, yielding the correspondences (5) $\leftrightarrow$ (6).

At this point we recall Theorem 2.6, from which it follows that every finite-dimensional representation of $U$ is equivalent to a unitary representation and thus decomposes into a direct sum of irreducible representations of $U$.\(^2\) At any rate,

\(^2\)In contrast, finite-dimensional irreducible representations of the noncompact semisimple group $G$ are typically not unitary.
classifying finite-dimensional representations of \( G \) can be reduced to classifying irreducible unitary representations of \( U \). For that reason, our next step is to briefly review the highest-weight classification of irreducible representations of the compact group \( U \).

We use the notation of Section 3.2. In particular, we have a maximal abelian subalgebra \( a \) in \( p \).\(^3\) Now let \( t \) be a maximal abelian subalgebra of \( m = Z_I(a) \). It can be shown that \( h = t \oplus i a \) is a Cartan subalgebra of \( u \) and that \( \tilde{h} = t \oplus a \) is a Cartan subalgebra of \( g \). In other words, \( h_C = t_C \oplus a_C \) is a maximal abelian subalgebra of \( g_C \). For each \( \alpha \in a_C^\ast \), we define the space

\[
g_{C,\alpha} = \{ Y \in g_C | [H, Y] = \alpha(H)Y \text{ for all } H \in h_C \}.
\]

If \( g_{C,\alpha} \neq 0 \), then we say that \( \alpha \) is a root for \( (g_C, h_C) \) and denote the set of all such roots by \( \Delta = \Delta(g_C, h_C) \). Note the distinction between the restricted roots in \( \Sigma(g, a) \) and the roots in \( \Delta(g_C, h_C) \). As with the restricted roots, we choose a positive root subsystem \( \Delta^+ \subseteq \Delta \). Then \( \Delta = (\Delta^+)\hat{\cup}(-\Delta^+) \).

Now consider an irreducible unitary representation \( (\pi, V) \) of \( U \). The derived representation of \( u \) then acts on \( V \) by skew-adjoint operators, whose eigenvalues are purely imaginary. For each \( \lambda \in i h^* \), we define the weight space

\[
V_\lambda = \{ v \in V | d\pi(H)v = \lambda(H)v \text{ for all } H \in h \}.
\]

If \( V_\lambda \neq \{0\} \), then we say that \( \lambda \) is a weight for \( \pi \) and denote the set of all weights for \( \pi \) by \( \Delta(\pi) \). Because \( d\pi(h) \) is an abelian Lie algebra of skew-adjoint operators on \( V \), it follows that \( V \) decomposes into joint eigenspaces. In other words,

\[
V = \bigoplus_{\lambda \in \Delta(\pi)} V_\lambda.
\]

We say that a weight \( \lambda \in i h^* \) is dominant if \( \langle \lambda, \alpha \rangle > 0 \) for all \( \alpha \in \Delta^+ \). We can now review the famous Highest-Weight Theorem, which classifies irreducible representations of compact groups. We say that a weight \( \lambda \in i h^* \) is integral if \( \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in Z \) for each \( \alpha \in \Delta^+ \). We denote the set of all dominant, integral weights by \( \Lambda^+(u, h) \).

**Theorem 3.12. (The Highest-Weight Theorem; see Theorem 5.110 in [26])**

Let \( U \) be a simply-connected compact group.

1. If \( (\pi, V) \) is an irreducible representation of \( U \), then there is a unique dominant integral weight \( \lambda \in \Lambda^+(u, h) \) such that \( \lambda \in \Delta(\pi) \) and

\[
d\pi(X)v = 0
\]

for all \( v \in V_\lambda \) and \( X \in \bigoplus_{\alpha \in \Delta^+} g_{C,\alpha} \). One says that \( \lambda \) is the highest weight of \( \pi \) and that elements of \( V_\lambda \) are highest-weight vectors. Furthermore, \( \dim V_\lambda = 1 \).

3In the literature it is standard to write \( u = t \oplus p \) rather than \( u = t \oplus i p \). Thus all instances of \( a \) or \( a^* \) will be off by a factor of \( i \) from the literature on compact-type symmetric spaces.
2. If $(\pi, V)$ is an irreducible representation of $U$, then $\dim V_\lambda = \dim V_{w\lambda}$ for any $w$ in the Weyl group $W = N_U(h)/Z_U(h)$ and any $\lambda \in i\mathfrak{h}^*$. 

3. Two representations of $U$ are equivalent if and only if they possess the same highest weight.

4. Each dominant integral weight $\lambda \in \Lambda^+(\mathfrak{u}, \mathfrak{h})$ is the highest weight of some irreducible unitary representation of $U$. We denote such a representation by $(\pi_\mu, \mathcal{H}_\mu)$.

Together with Weyl’s Unitary Trick, The Highest-Weight Theorem provides a parameterization of all finite-dimensional irreducible representations of semisimple Lie groups.

### 3.7 Finite-Dimensional Conical and Spherical Representations

The problem of determining which finite-dimensional representations are spherical or conical is solved by some classical results of Helgason, which we state in this section.

**Theorem 3.13 (The Cartan-Helgason Theorem).** ([21, p. 535]) Suppose that $U/K$ is a compact-type symmetric space with c-dual $G/K$ and that $(\pi, V)$ be an irreducible representation of $U$ with highest-weight $\lambda \in i\mathfrak{h}^*$. We recall that $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{a}$. Suppose further that $U$ is simply-connected. Then the following are equivalent:

1. $\pi$ is a spherical representation of $U$

2. $\pi(M)v = v$ for each highest-weight vector $v \in V_\lambda$.

3. $\lambda(t) = 0$ and also

$$\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \text{ for all } \alpha \in \Sigma^+$$

If $\pi$ is spherical, then we say that $\lambda|_{i\mathfrak{a}}$ is the **highest restricted weight** of $\pi$. Note that there is a natural identification of purely imaginary weights on $i\mathfrak{a}$ with purely real weights on $\mathfrak{a}$. Thus, the highest restricted roots may be identified with elements of $\mathfrak{a}^*$. We write

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{g}, \mathfrak{a}) \equiv \left\{ \mu \in \mathfrak{a}^* \left| \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \text{ for all } \alpha \in \Sigma^+ \right. \right\}$$

and note that each element of $\Lambda^+$ corresponds to unique irreducible spherical representations of $U$ and $G$.

More precisely, one can show that $\Lambda^+$ has a lattice structure as follows. Define linear functionals $\xi_j \in \mathfrak{a}^*$ by

$$\frac{\langle \xi_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j} \text{ for } 1 \leq j \leq r \quad (3.11)$$
Then $\xi_1, \ldots, \xi_r \in \Lambda^+$ and

$$\Lambda^+ = \mathbb{N}\xi_1 + \cdots + \mathbb{N}\xi_r = \left\{ \sum_{j=1}^r n_j \xi_j \middle| n_j \in \mathbb{N} \right\}.$$ 

The weights $\xi_j$ are called the fundamental weights for $(g, a)$. Note that each element of $\Lambda^+$ corresponds to a unique irreducible spherical representation of $U$.

In fact, the Cartan-Helgason Theorem also gives a classification of conical representations of $G$.

**Theorem 3.14.** ([22, p. 119]) Suppose that $(\pi, V)$ is an irreducible finite-dimensional representation of $G$. Then $\pi$ is spherical if and only if it is conical, in which case $V^{MN}$ consists of the highest-weight vectors of $\pi$.

Now that the irreducible finite-dimensional spherical and conical representations have been parameterized, one may ask more generally about finite-dimensional spherical and conical representations that may not be irreducible.

To that end, suppose that $(\pi_\mu, H_\mu)$ is an irreducible $K$-spherical representation of $G$ with highest weight $\mu$ and that $(\sigma, H)$ is a unitary primary representation of $G$ consisting of representations of type $\mu$. By [16, Lemma 1.5], all cyclic primary representations of a compact group are finite-dimensional, and hence $\sigma$ extends uniquely to a holomorphic spherical representation of $G^C$. Because it is a finite-dimensional spherical representation, $\sigma$ is automatically a conical representation of $G^C$. In fact, as the following result shows, the $MN$-invariant vectors of $\sigma$ are precisely the highest-weight vectors of irreducible subrepresentations of $\sigma$.

**Lemma 3.15.** Suppose, as above, that $(\sigma, H)$ is a unitary primary representation of a compact group $G$ consisting of representations with highest weight $\mu$. If $v \in H^{MN}\{0\}$, then $v$ is a highest-weight vector that generates an irreducible spherical representation of $G$. Furthermore, if $v, w \in H^{MN}\{0\}$ and $v \perp w$, then $\langle \pi(G)v \rangle \perp \langle \pi(G)w \rangle$.

**Proof.** Consider $W = \langle \sigma(G)v \rangle$. Then we can write $W = W_1 \bigoplus \cdots \bigoplus W_n$ where each $W_i$ gives an irreducible representation of $G$ that is equivalent to $H_\mu$. It must be a finite direct sum because all cyclic primary representations of compact groups are finite-dimensional (see [16]). For each $i$, let $v_i$ be the orthogonal projection of $v$ onto $W_i$. Then $v = v_1 + \cdots + v_n$. Since each $W_i$ is a $G$-invariant subspace, it follows that each vector $v_i$ is also invariant under $MN$. Because $W_i$ is irreducible, we see that $v_i$ must be a (nonzero) highest-weight vector of weight $\mu$ (see [15, Theorem 12.3.13]). Hence $v$ is a weight vector of weight $\mu$.

Suppose that $W$ is not irreducible (that is, $n > 1$). Because $W$ is cyclic, there must be $g_1, \ldots, g_k \in G$ and $c_1, \ldots, c_k \in \mathbb{C}$ such that $\sum_{i=0}^k c_i \pi(g_i)v = v_1$ (it is sufficient to consider finite linear combinations because $W$ is finite-dimensional). It follows from the invariance of each space $W_k$ that $\sum_{i=0}^k c_i \pi(g_i)v_1 = v_1$ and

---

4The lemma is likely known by specialists, but we were not able to find a citation in the literature.
\[ \sum_{i=0}^{k} c_i \pi(g_i)v_2 = 0. \] Because \( W_1 \) and \( W_2 \) give equivalent representations of \( G \) and all highest-weight vectors of an irreducible representation are constant multiples of each other, this is a contradiction. Thus \( W \) is irreducible and \( v = v_1 \) is a highest-weight vector for \( W \).

Now suppose that \( v \) and \( w \) are nonzero \( MN \)-invariant vectors in \( \mathcal{H} \) such that \( v \perp w \). Write \( V = \langle \pi(G)v \rangle \) and \( W = \langle \pi(G)w \rangle \). We want to show that \( V \perp W \).

By the above, we know that \( V \) and \( W \) correspond to (equivalent) irreducible representations of \( G \) with highest-weight vectors \( v \) and \( w \), respectively. Hence, either \( V \cap W = \emptyset \) or \( V = W \).

Because the space of highest-weight vectors of an irreducible representation of \( G \) is one dimensional and \( v \perp w \), we cannot have \( V = W \). Thus \( V \cap W = \emptyset \).

In particular, if we write \( U \equiv \langle V \cup W \rangle \), then \( U \cong G \) \( V \oplus W \). Now consider the orthogonal complement of \( V \) in \( U \), namely \( W' \equiv U \ominus V \equiv \{ u \in U | u \perp V \} \).

Then \( U \cong G \) \( V \oplus W' \). Since \( W' \) is \( G \)-equivalent to both \( V \) and \( W \), we see that it must have an \( MN \)-invariant vector, which will be orthogonal to \( v \). Thus, since \( U^{MN} = \text{span}\{v, w\} \) and \( v \perp w \), we see that \( w \) must in fact be an \( MN \)-invariant vector in \( W' \). In particular, \( w \perp V \). It follows from the unitarity of \( \sigma \) that \( W \perp V \) as we wanted to show.

3.7.1 Applications to Harmonic Analysis

The importance of finite-dimensional spherical representations of a group \( G \) for harmonic analysis may be seen by the fact that each finite-dimensional irreducible representation \( (\pi, V) \) of \( G \) is contained in the regular representation of \( G \) on \( C^\infty_{\text{fin}}(G) \). In fact, let \( (\pi, V) \) be a finite-dimensional irreducible representation of \( G \). Fix an inner product on \( \mathcal{H} \) such that the corresponding representation of \( U \) is unitary (this inner product is unique up to multiplication by a constant). Then \( \pi(k) \) is unitary for each \( k \in K \) and \( \pi(\exp X) \) is self-adjoint for each \( X \in p \) (since \( d\pi \) acts by skew-adjoint operators on \( p \), it acts by self-adjoint operators on \( p \)). For each \( u, v \in \mathcal{H} \) the matrix coefficient function

\[ \pi_{u,v}(g) = \langle \pi(g^{-1})u, v \rangle \]

extends to a holomorphic function on \( G^C \) by Weyl’s Unitary Trick. For each \( v \in V \), the map

\[ u \mapsto \pi_{u,v} \]

is a linear intertwining operator from \( (\pi, V) \) into \( (L, C^\infty_{\text{fin}}(G)) \). It is injective because \( \pi \) is irreducible.

Furthermore, if \( (\pi, V) \) is a finite-dimensional irreducible spherical representation of \( G \) such that \( e \in V^K \) is a unit vector, then \( \pi_{u,e} \) is a right-\( K \)-invariant smooth function on \( G \) for each \( u \in V \): in fact,

\[ \pi_{u,e}(g) = \langle \pi((gk)^{-1})u, e \rangle = \langle \pi(g^{-1})u, \pi(k)e \rangle = \langle \pi(g^{-1})u, e \rangle = \pi_{u,e}(g) \]
for all \( g \in G \) and \( k \in K \). Here we have used the fact that \( \pi|_K \) is unitary. Using the identification of right-\( K \)-invariant functions on \( G \) with functions on \( G/K \), we see that
\[
\pi_{u,e} \mapsto \pi_{u_e}
\]
gives an intertwining operator from \((\pi, V)\) into \((L, C^\infty_{\text{fin}}(G/K))\). In fact, it can be shown that \( \pi_{e,e} \) is a spherical function on \( G \) (see [22, p. 106]).

Now suppose that \((\pi, V)\) is a finite-dimensional irreducible conical representation of \( G \) such that \( v \in V^{MN} \) is a unit vector. Then \( \pi_{u,v} \) is a right-\( MN \)-invariant smooth function on \( G \) for each \( u \in V \): in fact,
\[
\pi_{u,v}(gmn) = \langle \pi((gmn)^{-1})u, v \rangle = \langle \pi(n^{-1})\pi(m^{-1})\pi(g^{-1})u, v \rangle = \langle \pi(g^{-1})u, \pi(m)\pi(n^{-1})v \rangle = \langle \pi(g^{-1})u, v \rangle = \pi_{u,v}(g)
\]
for all \( g \in G \), \( m \in M \), and \( n \in N \). Here we have used the fact that \( \pi(m) \) is unitary for \( m \in M \subseteq K \) and that \( \pi(n) \) is self-adjoint for \( n \in N \subseteq \exp p \). Furthermore, it can be shown that \( \psi_{v,v} \) is a conical function (see [22, p. 113]). Using the identification of right-\( MN \)-invariant functions on \( G \) with functions on \( G/MN \), we see that
\[
\pi_{u,v} \mapsto \pi_{u,v}
\]
gives an intertwining operator from \((\pi, V)\) into \((L, C^\infty_{\text{fin}}(G/MN))\).

Because \( \dim V^K = 1 \) and \( \dim V^{MN} = 1 \), it is possible to show that \((\pi, V)\) appears in \((L, C^\infty_{\text{fin}}(G/K))\) and \((L, C^\infty_{\text{fin}}(G/MN))\), respectively, with multiplicity one. In fact, it follows from Lemma II.4.14 and Proposition II.4.15 in [22] that
\[
C^\infty_{\text{fin}}(G/MN) \cong_G \sum_{\lambda \in \Lambda^+(g,a)} \mathcal{H}_{\lambda},
\]
where \( \sum^{\oplus} \) denotes an algebraic direct sum. From Corollary 12.3.15 in [15], we know that
\[
C^\infty_{\text{fin}}(G/K) \cong_G \sum_{\lambda \in \Lambda^+(g,a)} \mathcal{H}_{\lambda}.
\]
Furthermore, one can show (see [21, Theorem V.4.3]) that
\[
C^\infty_{\text{fin}}(U/K) \cong_U \sum_{\lambda^+ \in \Lambda(g,a)} \mathcal{H}_{\lambda}.
\]

In other words, there are very natural identifications of smooth, \( G \)-finite smooth functions on \( U/K \), \( G/K \) and \( G/MN \). Consider the mapping defined by
\[
\pi_{u,e} \mapsto \pi_{u,v}
\]
for each spherical/conical representation \((\pi, \mathcal{H})\) and each \( u \in \mathcal{H} \), where \( e \in \mathcal{H}^K \) and \( v \in \mathcal{H}^{MN} \) are unit vectors. This mapping may be extended by linearity to yield a \( G \)-intertwining operator from \( C^\infty_{\text{fin}}(G/K) \) to \( C^\infty_{\text{fin}}(G/MN) \). This intertwining operator is given by a form of the celebrated Radon transform and may be defined in terms of integral operators.
3.8 Unitary Spherical and Conical Representations

We are primarily concerned in this thesis with studying the analogue of finite-dimensional conical representations for infinite-dimensional symmetric spaces, none of which are unitary. However, we briefly review the construction of unitary conical representations in order to show the important role that they play in harmonic analysis on noncompact-type Riemannian symmetric spaces, which provides an important motivation for extending the theory of conical representations to the infinite-dimensional context.

The results in this section are primarily due to Harish-Chandra (for $G/K$) and Helgason (for $G/MN$). Helgason’s exposition of Harmonic analysis on $G/K$ may be found in Chapter IV of [21] and his exposition of analysis on $G/MN$ may be found in Chapters II and VI of [22] and in the earlier paper [19]. A simpler proof of the Plancherel formula for $G/MN$ was provided by Ronald Lipsman in [28]; it is simple enough that we shall provide a brief outline here. For a good overview of these topics from a representation-theory perspective, see [36].

The standard construction of unitary spherical and conical representations for noncompact-type Riemannian symmetric spaces uses a technique known as parabolic induction. As before, let $G/K$ be a Riemannian symmetric space of noncompact type and use the notation in Section 3.2.

We begin by choosing a one-dimensional representation of $A$, which may be identified with an element $\lambda$ of $a^*_C$. Due to the fact that $MAN/MN \cong A$, there is a well-defined extension to a representation $1 \otimes \lambda \otimes 1$ of $MAN$ such that

$$(1 \otimes \lambda \otimes 1)(man) = \lambda(\log(a))$$

for all $m \in M$, $a \in A$, and $n \in N$.

We then define the spherical principal series representation $(\sigma_\lambda, K_\lambda)$ by setting

$$K_\lambda = \left\{ \psi : G \to \mathbb{C} \mid \psi(gman) = a^{-\lambda - \rho} \psi(g) \text{ and } ||\psi||^2 = \int_K |\psi(k)|^2 dk < \infty \right\}$$

and letting $\sigma_\lambda$ act on $K_\lambda$ by

$$\sigma_\lambda(g)\psi(h) = \psi(g^{-1}h).$$

In the terminology of induced representations, one writes $\sigma_\lambda = \text{Ind}_{MAN}^G (1 \otimes \lambda \otimes 1)$ and says that $\sigma_\lambda$ is the representation of $G$ induced by the representation $1 \otimes \lambda \otimes 1$ of the parabolic subgroup $MAN$.

There are several ways to interpret the space $K_\lambda$. It can be viewed as a space of square-integrable sections of a particular homogeneous line bundle over $G/MAN$. Furthermore, using the Iwasawa decomposition, one can show that there is a diffeomorphism between $G/MAN$ and $K/M$ which gives a natural identification of

\[\text{The literature typically denotes this representation by } (\pi_\lambda, \mathcal{H}_\lambda), \text{ but we need to reserve that notation for a later use.}\]
with $L^2(K/M)$, on which the action is given by:

$$\sigma_\lambda(g) f(hM) = a(g^{-1}h)^{-\lambda-\rho} f(k(g^{-1}h))$$

for all $f \in L^2(K/M)$ and $h \in K$, where

$$g^{-1}h = k(g^{-1}h)a(g^{-1}h)n(g^{-1}h)$$

is the Iwasawa decomposition of $g^{-1}h$ in $G$. This realization of $K_\lambda$ is called the compact picture. Note that $\sigma_\lambda|_K$ is just the regular representation of $K$ on $L^2(K/M)$.

One can further show that $\sigma_\lambda$ is irreducible for almost all $\lambda$. Also, for each Weyl group element $w$, we have that $\sigma_\lambda \cong \sigma_{w\lambda}$ for almost all $\lambda \in a^*_C$. Finally, $\sigma_\lambda$ is a unitary representation if $\lambda \in i a^*$.

It is easy to see from the compact picture that $\sigma_\lambda$ is a spherical representation. In particular, we note that the constant function 1 in $L^2(K/M)$ is $K$-invariant.

It is less obvious that $\sigma_\lambda$ is a conical representation. Note that there is a continuous injection $H_\lambda \hookrightarrow D'(G/MN)$. For almost all $\lambda \in i a^*$, Helgason constructs $\#W$ distinct conical distributions on $G/MN$ with eigenvalue $\lambda - \rho$ with respect to the action of $a$. It is not clear from the works of Helgason whether these conical distributions are continuous functionals on the space $(H_\lambda)^\infty$ of smooth vectors for $H_\lambda$, but this result may be seen in [28, p. 50]. In other words, one has that $\dim(H_\lambda^{\infty})_{MN} = \#W$ for almost all $\lambda \in i a^*$.

The question of whether all unitary irreducible conical representations are constructed by the unitary spherical principal series is a subtle one. In a certain moral sense, one expects the unitary spherical principal series to exhaust “almost all,” if not all, unitary irreducible conical representations [22, p. 147]. To this end, Helgason was able to classify all conical distributions with the exception of certain singular eigenvalues [22, Theorem II.5.16]. For symmetric spaces $G/K$ of rank one, the classification was completed by Hu (see [24] as well as Theorem II.6.18 and Theorem II.6.21 in [22]). However, for cases of rank higher than one it is not clear in the literature whether the answer is known.

3.8.1 Applications to Harmonic Analysis

In this section we briefly discuss the Plancherel formulas for noncompact-type Riemannian symmetric spaces and their associated horocycle spaces and note the role played by unitary spherical and conical representations.

We once again suppose that $G/K$ is a noncompact-type Riemannian symmetric space and use the terminology of Section 3.2. Because $M$ is a compact group and $N$ is a connected nilpotent group, we see that both groups are unimodular. We normalize the measure on $N$ by

$$\int_N a(n)^{-2\rho} \, dn = 1.$$ 

One can show that $MN$ is a unimodular group ([22, p. 82]). Furthermore, $G$ is a unimodular group because it is semisimple. It follows from Theorem 2.9 that
$G/MN$ possesses a $G$-invariant measure. Similarly, because $G$ and $K$ are both unimodular, the symmetric space $G/K$ possesses a $G$-invariant measure.

We can now consider the unitary regular representations of $G$ on $L^2(G/K)$ and $L^2(G/MN)$. The deep work of Harish-Chandra shows that the regular representation $(L^2_{G/K}, L^2(G/K))$ may be written as a direct integral of unitary spherical principal series representations (see, for instance, Sections 2.5 and 2.8 in [36]):

$$L^2(G/K) \cong_G \int_{ia^*/W} \mathcal{K}_\lambda |c(\lambda)|^{-2} d\lambda,$$

where the measure $|c(\lambda)|^{-2} d\lambda$ is the Lebesgue measure on $ia^*/W$ weighted by $|c(\lambda)|^{-2}$. Here $c$ is the famous Harish-Chandra $c$-function given by

$$c(\lambda) = \int_N a(n)^{-\lambda - \rho} d\bar{n}$$

for $\lambda \in ia^*$ with Re $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$. It is $W$-invariant and may be extended meromorphically to all of $ia^*$, so that (3.12) is well-defined.

On the other hand, the decomposition of $(L^2_{G/MN}, L^2(G/MN))$ may be derived using more elementary methods, and we briefly sketch the argument here. By using induction in stages, one has that

$$L^2(G/MN) \cong Ind^G_{MN}(1) \cong Ind^G_{MAN} Ind^{MAN}_{MN}(1)$$

But we also have

$$Ind^{MAN}_{MN}(1) \cong L^2(MAN/MN) \cong \int_{ia^*} 1 \otimes \lambda \otimes 1 d\lambda,$$

where the latter equality follows from the fact that $MAN/MN \cong A$. Here $d\lambda$ is Lebesgue measure on $ia^*$. Therefore, one has that

$$L^2(G/MN) \cong \int_{ia^*} Ind^G_{MAN}(1 \otimes \lambda \otimes 1)$$

where we use the fact that induction commutes with taking direct integrals. Putting everything together, we have the following result (see Section 4.2 in [36]):

$$L^2(G/MN) \cong_G \int_{ia^*/W} \mathcal{K}_\lambda d\lambda.$$

(3.13)

In fact, because $(\sigma_\lambda, \mathcal{H}_\lambda)$ is equivalent to $(\sigma_w\lambda, \mathcal{H}_w\lambda$ for almost all $\lambda \in ia^*$ and $w \in W$, one sees that

$$L^2(G/MN) \cong_G (\#W) \int_{ia^*/W} \mathcal{K}_\lambda d\lambda \cong_G (\#W) L^2(G/K).$$

That is, $L^2(G/MN)$ is equivalent to a direct sum of $\#W$ copies of $L^2(G/K)$. In fact, a variant of the Radon transform may be used to define an intertwining operator from $L^2(G/K)$ to the space $L^2_W(G/MN)$ of $W$-invariant functions on $G/MN$. See Sections 4.2 and 4.3 in [36] for more details.
Chapter 4
Direct Limits of Groups and Symmetric Spaces

Motivated in part by applications to physics, there has been an increasing amount of work done on infinite-dimensional Lie groups since the 1970s. These are topological groups which are locally modeled on locally convex topological vector spaces over $\mathbb{R}$ (in the same way that finite-dimensional Lie groups are modeled on finite-dimensional vector spaces over $\mathbb{R}$). The simplest infinite-dimensional Lie groups which may be considered are those which are formed by taking direct limits of finite-dimensional Lie groups. They occupy a sort of “middle ground” between finite-dimensional groups and other infinite-dimensional groups with finer topologies, in that they inherit many of the properties of the former but already exhibit some of the pathologies of the latter.

We refer the reader to [11] and [33] for a good overview of the basic properties of direct-limit groups. See [34] and [31] for some details about the construction of smooth manifold structures on direct-limit groups. See also [50] for an in-depth study of direct limits of abelian and nilpotent groups and for applications of direct-limit groups to physics.

4.1 Review of Direct Limits and Projective Limits

We begin in this section by very briefly reviewing several basic definitions and results about direct limits and projective limits. See, for instance, the appendices in [33] for more details.

Suppose that for each $n \in \mathbb{N}$ one has a topological space $X_n$ and continuous embeddings $p_n^{n+1}: X_n \to X_{n+1}$, which we refer to as inclusion maps. By repeated composition of these inclusion maps, we construct continuous maps $p_k^n: X_n \to X_k$ for any $n \leq k$. Note that $p_k^n \circ p_m^n = p_m^k$ for all $m \leq n \leq k$. We say that $\{X_n\}_{n \in \mathbb{N}}$ together with the inclusion maps forms a direct system.

Next, we define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{n \in \mathbb{N}} X_n$ as follows: for $x \in X_n$ and $y \in X_m$, where $n \leq m$, we write $x \sim y$ if $p_m^n(x) = y$. We then define

$$X_\infty \equiv \lim_{\rightarrow} X_n \equiv \left( \bigsqcup_{n \in \mathbb{N}} X_n \right) / \sim$$

and say that $X_\infty$ is the direct limit of $\{X_n\}_{n \in \mathbb{N}}$. Note the the inclusion map from $X_n$ to $\bigsqcup_{n \in \mathbb{N}} X_n$ factors through the quotient to give an injective map $p_n: X_n \to X_\infty$. We then give $X_\infty$ the weakest topology such that $p_n$ is continuous for each $n \in \mathbb{N}$. The direct limit possesses two important properties:

---

1We warn the reader that it is not always assumed in the literature that the inclusion maps are injective.
Lemma 4.1. Let $Y$ be a topological space. Suppose that $\{X_n, p_n^{n+1}\}$ is a direct system of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : X_n \to Y$ so that the diagram

\[
\begin{array}{c}
X_k & \xrightarrow{f_k} & Y \\
\downarrow p_n^k & & \downarrow f_n \\
X_n & \xleftarrow{p_n} & X_n
\end{array}
\]

commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : X_\infty \to Y$ such that

\[
\begin{array}{c}
X_\infty & \xrightarrow{f_\infty} & Y \\
\downarrow p_n & & \downarrow f_n \\
X_n & \xleftarrow{f_n} & X_n
\end{array}
\]

commutes for each $n \in \mathbb{N}$.

Lemma 4.2. Suppose that $\{X_n, p_n^{n+1}\}$ and $\{Y_n, q_n^{n+1}\}$ are direct systems of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : X_n \to Y_n$ so that the diagram

\[
\begin{array}{c}
X_k & \xrightarrow{f_k} & Y_k \\
\downarrow p_n^k & & \downarrow q_n^k \\
X_n & \xleftarrow{f_n} & Y_n
\end{array}
\]

commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : X_\infty \to Y_\infty$ such that

\[
\begin{array}{c}
X_\infty & \xrightarrow{f_\infty} & Y_\infty \\
\downarrow p_n & & \downarrow q_n \\
X_n & \xleftarrow{f_n} & Y_n
\end{array}
\]

commutes for each $n \in \mathbb{N}$.

In fact, these can be taken to be a sort of universal property for direct limits. Following the construction of direct limits of topological spaces, it is possible to define direct limits for the categories of topological groups, vector spaces, and Lie algebras which satisfy the previous two lemmas.

The prototypical example of a direct system is that of a collection $\{X_k\}_{k \in \mathbb{N}}$ of topological spaces such that $X_k$ is a closed subset of $X_m$ whenever $k \leq m$. Then we can identify $\varinjlim X_n$ with the set $X_\infty = \bigcup_{m \in \mathbb{N}} X_m$ given by the topology where a set $A \subset X_\infty$ is open if and only if $A \cap X_n$ is an open subset of $X_n$ for each $n \in \mathbb{N}$.

If $\{G_k\}_{k \in \mathbb{N}}$ is a collection of topological groups such that $G_k$ is a closed subgroup of $G_m$ whenever $k \leq m$, then we form the direct limit $G_\infty = \bigcup_{n \in \mathbb{N}} G_n$ in the topological category. The group product is obvious: if $a, b \in G_n$, then their product
in $G_\infty$ is equal to the group product under $G_n$. One uses Lemma 4.2 to show that the group product and inverse on $G_\infty$ are continuous.

For example, consider the groups $SU(n)$ for each $n \in \mathbb{N}$. We see that $SU(n) \leq SU(n+1)$ under the identification $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

We can form the direct-limit group $SU(\infty) = \varinjlim SU(n) = \bigcup_{n \in \mathbb{N}} SU(n)$. One can think of $SU(\infty)$ as consisting of all unitary operators on $\ell^2(\mathbb{C})$ which fix all but finitely many of the standard basis elements. Alternately, $SU(\infty)$ may be thought of as consisting of infinite complex matrices which are equal to the identity matrix outside of a finite block in the upper-left corner.

If $\{\mathcal{H}_k\}_{k \in \mathbb{N}}$ is a collection of Hilbert spaces such that such that $\mathcal{H}_k$ is a closed subgroup of $\mathcal{H}_m$ whenever $k \leq m$, then we form the direct limit $\mathcal{H}_\infty = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ in the topological category. One uses Lemma 4.2 to show that the addition and constant multiplication on $\mathcal{H}_\infty$ are continuous. Furthermore, $\mathcal{H}_\infty = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ carries a continuous inner product. However, $\mathcal{H}_\infty$ is not necessarily a Hilbert space and we must take the completion $\overline{\mathcal{H}_\infty}$ to obtain a Hilbert space.

Next, we consider the Cartesian product $\prod_{n \in \mathbb{N}} X_n$ under the product topology. We denote by $\varprojlim X_n$ the set of all sequences $(x_n)_{n \in \mathbb{N}}$ such that $p^n_m(x_n) = x_m$. We give $\varprojlim X_n$ the topology it inherits as a subspace of the Cartesian product. Note that there are projection maps $p_n : X_\infty \to X_n$ defined by $p_n((x_m)_{m \in \mathbb{N}}) = x_n$. In fact, the topology on $\varprojlim X_n$ is the weakest topology such that the $p_n$ is continuous for each $n \in \mathbb{N}$. In other words, we can form a basis for the topology on $\varprojlim X_n$ consisting of sets of the form $p^{-1}_n(A)$ where $A$ is an open subset of $X_n$ for some $n \in \mathbb{N}$. These sets are called **cylinder sets**.

Projective limits satisfy universal properties obtained by reversing the arrows for the corresponding properties of direct limits:

**Lemma 4.3.** Let $Y$ be a topological space. Suppose that $\{X_n, p^{n+1}_n\}$ is a projective system of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : Y \to X_n$ so that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f_k} & X_k \\
& \searrow & \downarrow \quad p^k_n \\
& \quad f_n & \downarrow \\
& & X_n
\end{array}
\]
commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : Y \to X_\infty$ such that

\[
\begin{array}{ccc}
Y & \xrightarrow{f_\infty} & X_\infty \\
\downarrow{f_n} & & \downarrow{p_n} \\
X_n & & 
\end{array}
\]

commutes for each $n \in \mathbb{N}$.

**Lemma 4.4.** Suppose that $\{X_n, p_{n+1}^n\}$ and $\{Y_n, q_{n+1}^n\}$ are projective systems of topological spaces and suppose that for each $n \in \mathbb{N}$ we are given a continuous map $f_n : X_n \to Y_n$ so that the diagram

\[
\begin{array}{ccc}
X_k & \xrightarrow{f_k} & Y_k \\
\downarrow{p_k^n} & & \downarrow{q_k^n} \\
X_n & \xrightarrow{f_n} & Y_n 
\end{array}
\]

commutes for each $n \leq k$. Then there is a unique continuous map $f_\infty : X_\infty \to Y_\infty$ such that

\[
\begin{array}{ccc}
X_\infty & \xrightarrow{f_\infty} & Y_\infty \\
\downarrow{p_n} & & \downarrow{q_n} \\
X_n & \xrightarrow{f_n} & Y_n 
\end{array}
\]

commutes for each $n \in \mathbb{N}$.

One may define projective limits in the category of topological groups by starting with the topological projective limit and defining the group product to be the restriction of the componentwise product of sequences in the Cartesian product. Projective limits of vector spaces and Lie algebras may be defined in similar ways.

Suppose that $(V_n, p_{n+1}^n)_{n \in \mathbb{N}}$ is a direct system of topological vector spaces. Then we can define continuous projections $q_{n+1}^n : V_{n+1}^* \to V_n^*$ by $q_{n+1}^n(\lambda)v = \lambda(p_{n+1}^n v)$ for each $v \in V_n$. This allows us to form the projective limit $\lim_{\leftarrow} (V_n^*)$. In fact, one can show that

\[
(\lim_{\leftarrow} V_n)^* \cong \lim_{\leftarrow} (V_n^*). \]

**4.2 Lie Algebras and Complexifications of Direct-Limit Groups**

Suppose that $\{G_n\}_{n \in \mathbb{N}}$ is a direct system of Lie groups with inclusion maps $p_{n+1}^n : G_n \to G_{n+1}$. Then the differentiated map $d_{p_{n+1}^n} : g_n \to g_{n+1}$ is an injective Lie algebra homomorphism for each $n \in \mathbb{N}$ because each $p_{n+1}^n$ is a smooth embedding. Thus $\{g_n\}_{n \in \mathbb{N}}$ is a direct system of Lie algebras with inclusion maps $d_{p_{n+1}^n} : g_n \to g_{n+1}$. Thus we have the direct-limit group $G_\infty = \lim_{\rightarrow} G_n$ and the direct-limit Lie algebra $g_\infty = \lim_{\rightarrow} g_n$.

It is natural to ask whether $g_\infty$ is the Lie algebra for $G_\infty$ in some sense. To that end, consider the exponential maps $\exp_n : g_n \to G_n$ for each $n \in \mathbb{N}$. One
notices that \( p_n^{n+1} \circ \exp_n = \exp_{n+1} \circ d\exp_n \) by the definition of the differentiated homomorphism \( d\exp_n \). In other words, the diagram

\[
\begin{array}{ccc}
g_{n+1} & \xrightarrow{\exp_{n+1}} & G_{n+1} \\
\downarrow{d\exp_{n+1}} & & \downarrow{\exp_{n+1}} \\
g_n & \xrightarrow{\exp_n} & G_n
\end{array}
\]

commutes for each \( n \in \mathbb{N} \). Thus we may consider the continuous map

\[ \exp_\infty : g_\infty \to G_\infty \]

defined by \( \exp_\infty(p_n(X)) = p_n(\exp(X)) \) for all \( X \in g_n \). Under certain technical conditions which include all of the classical direct-limit groups, it has been shown that \( \exp_\infty \) is a local homeomorphism (see Proposition 7.1 in [31]).\(^2\) However, we will not need to use this result for our purposes.

Now suppose that \( \{U_n\}_{n \in \mathbb{N}} \) is a direct system of connected compact Lie groups with inclusion maps \( p_n^{n+1} : U_n \to U_{n+1} \). Following the process described in Proposition 3.6 of [33], we construct complexifications of \( u_\infty = \lim_{\to} u_n \) and \( U_\infty = \lim_{\to} U_n \). As before, we consider the direct-limit group \( U_\infty = \lim_{\to} U_n \) and its Lie algebra \( u_\infty = \lim_{\to} u_n \). For each \( n \in \mathbb{N} \), we consider the complexified Lie algebra \( (u_n)_\mathbb{C} = u_n \otimes_{\mathbb{R}} \mathbb{C} \). Then the inclusion maps \( d\exp_n^{n+1} : u_n \to u_{n+1} \) may be complexified to yield complex-linear injective Lie algebra homomorphisms \( (d\exp_n^{n+1})_\mathbb{C} : (u_n)_\mathbb{C} \to (u_{n+1})_\mathbb{C} \).

We may thus consider the complex Lie algebra

\[ (u_\infty)_\mathbb{C} = \lim_{\to} (u_n)_\mathbb{C}. \]

Furthermore, because \( (d\exp_n^{n+1})_\mathbb{C} \) is the complexification of the linear map \( d\exp_n^{n+1} \), we see that the inclusions \( i_n : u_n \to (u_n)_\mathbb{C} \) satisfy the following commutative diagram:

\[
\begin{array}{ccc}
u_{n+1} & \xrightarrow{i_{n+1}} & (u_{n+1})_\mathbb{C} \\
\downarrow{d\exp_n^{n+1}} & & \downarrow{(d\exp_n^{n+1})_\mathbb{C}} \\
u_n & \xrightarrow{i_n} & (u_n)_\mathbb{C}
\end{array}
\]

We thus obtain an injective homomorphism \( i_\infty : u_\infty \to (u_\infty)_\mathbb{C} \). One can show that \( (u_\infty)_\mathbb{C} \) is the complexification of the Lie algebra \( u_\infty \).

For each \( n \in \mathbb{N} \), we consider the complexification \( (U_n)_\mathbb{C} \) of the compact Lie group \( U_n \). We recall that \( (U_n)_\mathbb{C} \) has Lie algebra \( (u_n)_\mathbb{C} \) and that \( U_n \) is the closed analytic subgroup of \( (U_n)_\mathbb{C} \) corresponding to the Lie algebra \( u_n \). By [26, Proposition 7.5], each homomorphism \( p_n^{n+1} \) induces a holomorphic homomorphism \( (p_n^{n+1})_\mathbb{C} : (U_n)_\mathbb{C} \to (U_{n+1})_\mathbb{C} \) whose differential is \( (d\exp_n^{n+1})_\mathbb{C} \). We may thus consider the direct-limit group

\[ (U_\infty)_\mathbb{C} = \lim_{\to} (U_n)_\mathbb{C}. \]

\(^2\)In fact, once the proper definitions for infinite-dimensional manifolds have been made, it can be shown under these technical conditions that \( \exp_\infty \) is a local diffeomorphism (see Theorem 8.2).
Furthermore, by [26, Proposition 7.5] it follows that the inclusion maps \( i_n : U_n \to (U_n)_\mathbb{C} \) satisfy the following commutative diagram:

\[
\begin{array}{ccc}
U_{n+1} & \xrightarrow{i_{n+1}} & (U_{n+1})_\mathbb{C} \\
\downarrow{p_{n+1}^0} & & \downarrow{(p_{n+1})_\mathbb{C}} \\
U_n & \xrightarrow{i_n} & (U_n)_\mathbb{C}
\end{array}
\]

We thus obtain a continuous injective homomorphism \( i_\infty : U_\infty \to (U_\infty)_\mathbb{C} \). Because the image of \( U_n \) under \( i_n \) is closed in \( (U_n)_\mathbb{C} \) for each \( n \in \mathbb{N} \), we see that the image of \( U_\infty \) under \( i_\infty \) is a closed subgroup of \( (U_\infty)_\mathbb{C} \). For these reasons, we say that \( (U_\infty)_\mathbb{C} \) is the \textbf{complexification} of \( U_\infty \).

\section{4.3 Direct Systems of Riemannian Symmetric Spaces}

Suppose that \( \{G_n\}_{n \in \mathbb{N}} \) is a direct system of semisimple Lie groups and that for each \( n \in \mathbb{N} \) we have an involution \( \theta_n : G_n \to G_n \) such that \( G_n/(G_n)\theta \) is a Riemannian symmetric space and the diagram

\[
\begin{array}{ccc}
G_{n+1} & \xrightarrow{\theta_{n+1}} & G_{n+1} \\
\downarrow{p_{n+1}^0} & & \downarrow{(p_{n+1})_\mathbb{C}} \\
G_n & \xrightarrow{\theta_n} & G_n
\end{array}
\]

commutes. We thus have a continuous involution \( \theta_\infty : G_\infty \to G_\infty \). Write \( K_n = (G_n)\theta \) for each \( n \in \mathbb{N} \). We see that \( \{K_n\}_{n \in \mathbb{N}} \) forms a direct system with inclusion maps given by \( p_{n+1}^0|_{K_n} \). Furthermore, (4.1) implies that \( p_{n+1}^0(K_n) = p_{n+1}^0(G_n) \cap K_{n+1} \) for each \( n \in \mathbb{N} \), so there are well-defined inclusion maps from the quotient space \( G_n/K_n \) to \( G_{n+1}/K_{n+1} \). We thus obtain a direct system of homogeneous spaces \( \{G_n/K_n\}_{n \in \mathbb{N}} \). Now construct the direct limits \( G_\infty = \lim\downarrow G_n, K_\infty = \lim\downarrow K_n, \) and \( G_\infty/K_\infty = \lim\downarrow G_n/K_n \). Finally, one can show that \( K_\infty = (G_\infty)\theta_\infty \).

We say that \( G_\infty/K_\infty \) is a \textbf{lim-Riemannian symmetric space}. If \( G_n/K_n \) is a compact-type symmetric space for each \( n \in \mathbb{N} \), then \( G_\infty/K_\infty \) is said to be a \textbf{lim-compact Riemannian symmetric space}. Similarly, if \( G_n/K_n \) is a noncompact-type Riemannian symmetric space for all \( n \in \mathbb{N} \), then \( G_\infty/K_\infty \) is said to be a \textbf{lim-noncompact Riemannian symmetric space}.

For each \( m \in \mathbb{N} \), denote the Killing form on \( \mathfrak{g}_k \) by \( B_k \). Note that for each \( k \leq m \), the Killing form \( B_m : \mathfrak{g}_m \times \mathfrak{g}_m \to \mathbb{C} \) restricts to an \( \text{ad}(\mathfrak{g}_k) \)-invariant bilinear form on \( \mathfrak{g}_k \). If \( \mathfrak{g}_k \) is a simple Lie algebra for all \( k \in \mathbb{N} \), then all such \( \text{ad}(\mathfrak{g}_k) \)-invariant bilinear forms are constant multiples of each other, and hence \( B_m|_{\mathfrak{g}_k \times \mathfrak{g}_k} = cB_k \) for some constant \( c \in \mathbb{C} \). In the interest of consistency, we replace each Killing form \( B_k \) in this case with a constant multiple in such a way that \( B_m|_{\mathfrak{g}_k \times \mathfrak{g}_k} = B_k \) for all \( k \leq m \). In other words, we will shall normalize the Killing forms of the \( \mathfrak{g}_k \)'s so that they are consistent with each other.

Similarly, if \( \{\mathfrak{g}_k\}_{k \in \mathbb{N}} \) is a direct system of simple Lie algebras, then one constructs a direct system \( \{\mathfrak{g}_k \times \mathfrak{g}_k\}_{k \in \mathbb{N}} \) of semisimple Lie groups. The same construction as before allows us to consistently normalize Killing forms on \( \mathfrak{g}_k \times \mathfrak{g}_k \) for each \( k \in \mathbb{N} \).
We now see how the notion of c-duals may be extended to lim-Riemannian symmetric spaces. Suppose that \( \{ U_n/K_n \}_{n \in \mathbb{N}} \) is a direct system of Riemannian symmetric spaces with involutions \( \theta_n : U_n \to U_n \) and inclusion maps \( p_n^{n+1} : U_n \to U_{n+1} \). We follow the constructions in Section 4.2 to produce a complexification \( (U_\infty)_C = \lim \to (U_n)_C \) for the lim-compact group \( U_\infty = \lim U_n \). To simplify notation we assume that \( (U_n)_C \subseteq (U_{n+1})_C \) and therefore \( U_n \subseteq U_{n+1} \) for each \( n \in \mathbb{N} \).

We recall that the involutions \( \theta_n : U_n \to U_n \) extend to holomorphic involutions \( \theta_n : (U_n)_C \to (U_n)_C \). The fact that the diagram
\[
\begin{array}{ccc}
U_{n+1} & \xrightarrow{\theta_{n+1}} & U_{n+1} \\
\uparrow & & \uparrow \\
U_n & \xrightarrow{\theta_n} & U_n
\end{array}
\]
commutes implies that
\[
\begin{array}{ccc}
(U_{n+1})_C & \xrightarrow{\theta_{n+1}} & (U_{n+1})_C \\
\uparrow & & \uparrow \\
(U_n)_C & \xrightarrow{\theta_n} & (U_n)_C
\end{array}
\] (4.2)
commutes by [26, Proposition 7.5].

As before, we write \( u_n = \mathfrak{f}_n \oplus \mathfrak{p}_n \)
for each \( n \in \mathbb{N} \), where \( \mathfrak{f}_n \) and \( \mathfrak{p}_n \) are the +1- and -1-eigenspaces of \( \theta_n \). From (4.1) it follows that
\( \mathfrak{f}_n = \mathfrak{f}_{n+1} \cap u_n \) and \( \mathfrak{p}_n = \mathfrak{p}_{n+1} \cap u_n \)
and hence that \( \mathfrak{f}_n \subseteq \mathfrak{f}_{n+1} \) and \( \mathfrak{p}_n \subseteq \mathfrak{p}_{n+1} \). For each \( n \), we construct the c-dual Lie algebra
\[ g_n = \mathfrak{f}_n \oplus i \mathfrak{p}_n \subseteq (u_n)_C \]
and note that \( g_n \subseteq g_{n+1} \). Finally, we construct the analytic subgroup \( G_n \) of \( (U_n)_C \) which corresponds to the Lie algebra \( g_n \) and recall that \( G_n \) is closed in \( (U_n)_C \). Thus \( G_n \) is a closed subgroup of \( G_{n+1} \) for each \( n \). It follows that the direct-limit group \( G_\infty = \lim \to G_n \) is a closed subgroup of \( (U_\infty)_C \) and possesses the direct-limit Lie algebra \( g_\infty = \lim \to g_n \).

Reviewing the construction of finite-dimensional c-dual spaces, we see that the complexified involution \( \theta_n : (u_n)_C \to (u_n)_C \) restricts to an involution \( \theta_n : g_n \to g_n \) and that \( \mathfrak{f}_n \) and \( i \mathfrak{p}_n \) are the +1- and -1-eigenspaces of \( \theta_n \) in \( g_n \). Furthermore, because \( g_n \) is \( \theta_n \)-stable, the holomorphic involution \( \theta_n : (U_n)_C \to (U_n)_C \) restricts to an involution \( \theta_n : G_n \to G_n \) such that \( (G_n)_{\theta_n} = K_n \). Finally, the restriction of (4.2) implies that the diagram
\[
\begin{array}{ccc}
G_{n+1} & \xrightarrow{\theta_{n+1}} & G_{n+1} \\
\uparrow & & \uparrow \\
G_n & \xrightarrow{\theta_n} & G_n
\end{array}
\]
commutes. Thus \( \{G_n/K_n\}_{n \in \mathbb{N}} \) is a direct system of noncompact-type Riemannian symmetric spaces. We say that \( G_\infty/K_\infty = \lim G_n/K_n \) is the \textbf{c-dual} of \( U_\infty/K_\infty \).

In order to align our notation with that of Chapter 3, we set \( p_n = \tilde{\imath}p_n \) for each \( n \), so that

\[
\begin{align*}
g_n &= \mathfrak{k}_n \oplus p_n, \\
u_n &= \mathfrak{k}_n \oplus \imath p_n.
\end{align*}
\]

Finally, we notice that

\[
\begin{align*}
g_\infty &= \mathfrak{k}_\infty \oplus p_\infty, \\
u_\infty &= \mathfrak{k}_\infty \oplus \imath p_\infty,
\end{align*}
\]

where \( \mathfrak{k}_\infty = \lim \mathfrak{k}_n \) and \( p_\infty = \lim p_n \) are the +1-and −1-eigenspaces of \( \theta_\infty \) in \( g_\infty \).

### 4.4 Propagated Direct Limits

As before, we assume that \( G_\infty/K_\infty \) is a lim-noncompact Riemannian symmetric spaces which is the c-dual of a direct limit \( U_\infty/K_\infty \) of simply-connected compact Riemannian symmetric space. We need to put some further technical conditions on \( G_\infty/K_\infty \) in order to prove our results about conical representations. The first condition is that of \textit{propagation}, which was introduced by Ólafsson and Wolf. See [40], [52], and [54] for more details on this construction.

We begin this section by examining the restricted root data of \( G_\infty/K_\infty \), using the notation of Section 4.3. We recursively choose maximal commutative subspaces \( a_k \subset p_k \) such that \( a_n \subseteq a_k \) for \( n \leq k \) and define \( a_\infty = \lim a_n \). We then obtain the restricted root system \( \Sigma_n = \Sigma(g_n, a_n) \) for each \( n \in \mathbb{N} \). Note that

\[
\Sigma_n \subseteq \Sigma_k |_{a_n} \setminus \{0\}
\]

whenever \( n \leq k \).

Next, we recursively choose positive subsystems \( \Sigma_n^+ \subseteq \Sigma_n \) in such a way that

\[
\Sigma_n^+ \subseteq \Sigma_k^+ |_{a_n} \setminus \{0\}.
\]

The projective limit \( \Sigma_\infty^+ = \lim \Sigma_n^+ \) plays the role of the positive root subsystem for \( (g_\infty, a_\infty) \).

For each \( n \in \mathbb{N} \), we let \( (\Sigma_n)_0 \) denote the set of nonmultipliable roots in \( \Sigma_n \) and set \( (\Sigma_n)_0^+ = (\Sigma_n)_0 \cap \Sigma_n^+ \). Denote the set of simple roots in \( (\Sigma_n)_0^+ \) by \( \Psi_n = \{\alpha_1, \ldots, \alpha_{r_n}\} \), where \( r_n = \dim a_n \). Since we will be dealing with direct limits we may assume that \( \Sigma \), and hence \( \Sigma_0 \), is one of the classical root systems. We number the simple roots in the following way:
We are now ready to introduce the definition of propagated direct-limits of symmetric spaces.

**Definition 4.5.** We say that a lim-noncompact symmetric space $G_\infty / K_\infty$ is propagated if

1. For each simple root $\alpha \in \Psi_k$ there is a unique simple root $\tilde{\alpha} \in \Psi_n$ such that $\tilde{\alpha}|_{\Phi_k} = \alpha$, whenever $k \leq n$.

2. There is a choice of ordering on the roots in $\Psi_k$ for each $k \in \mathbb{N}$ such that either $\Phi_n = \Phi_k$ or else $\Psi_k$ extends $\Psi_n$ for $n \leq k$ only by adding simple roots at the left end. (In particular, each $\Psi_k$ has the same Dynkin diagram type.)

We also introduce an analogous notion of propagation for lim-compact groups. Let $U_\infty = \lim_{\rightarrow} U_n$ be a direct limit of compact Lie groups. Choose a Cartan subalgebra $h_n \subseteq g_n$ for each $n$ in such a way that $h_n \subseteq h_k$ whenever $n \leq k$. One then obtains a root system $\Delta_n = \Delta(g_n, h_n)$ for each $n$. After recursively choosing positive subsystems $\Delta_n^+ \subseteq \Delta_n$ such that $\Delta_n^+ \subseteq \Delta_k^+ |_{h_n \setminus \{0\}}$, for $n \leq k$, we arrive at a set $\Xi_n$ of simple roots in $\Delta_n^+$. We order these simple roots the same way as in Table 4.3.

**Definition 4.6.** We say that the lim-compact group $U_\infty$ is propagated if

1. For each simple root $\alpha \in \Xi_k$ there is a unique simple root $\tilde{\alpha} \in \Xi_n$ such that $\tilde{\alpha}|_{h_k} = \alpha$, whenever $k \leq n$.

2. There is a choice of ordering on the roots in $\Xi_k$ for each $k \in \mathbb{N}$ such that either $h_n = h_k$ or else $\Xi_k$ extends $\Xi_n$ for $n \leq k$ only by adding simple roots at the left end.

Suppose that $U_\infty$ is a propagated direct limit of compact, simply-connected semisimple Lie groups. Then each $U_k$ may be decomposed into a product of compact simple Lie groups, say $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^{d_k}$. We can recursively choose Cartan subalgebras $h_k = h_k^1 \oplus h_k^2 \oplus \cdots \oplus h_k^{d_k}$ where each $h_k^i$ is a Cartan subalgebra of $u_k^i$. 44
The definition of propagation then implies that \( d_n = d_m \equiv d \) for each \( n, m \in \mathbb{N} \) and that the indices may be ordered in such a way that \( \{U^i_k\}_{n \in \mathbb{N}} \) is a propagated direct system of compact simple Lie groups for each \( 1 \leq i \leq d \).

Following the exposition in [7], we make note of the details of each root system for later use. We identify \( a \) with \( \mathbb{R}^r \) so that, as usual, \( a = \{(x_{r+1}, \ldots, x_1) \mid x_1 + \ldots + x_{r+1} = 0\} \) if \( \Psi = A_r \) and otherwise \( a = \mathbb{R}^r \). Set \( e_1 = (0, \ldots, 0, 1, 0), \ldots, e_n = (1, 0, \ldots, 0) \) where \( n = r + 1 \) for \( A_r \) and otherwise \( n = r \).

We view the vectors \( e_j \) also as elements in \( a^\ast \) via the standard inner product in \( \mathbb{R}^{r+1} \) in the case \( \Psi = A_r \) and otherwise \( \mathbb{R}^r \). Note that in the case \( \Psi = A_r \) this gives a map \( \mathbb{R}^{r+1} \to a^\ast \) which is not injective.

For \( \Psi = A_r \), we have \( \Sigma_0^+ = \{e_j - e_i \mid 1 \leq i < j \leq n\} \) and \( \alpha_j = e_{j+1} - e_j \), \( j = 1, \ldots, r \). The Weyl group \( W \) consists of linear maps given by

\[
 w_\sigma(e_i) = e_\sigma(i)
\]

for permutations in the symmetric group \( S_{r+1} \). One can show that the fundamental weights are

\[
 \xi_j = 2 \sum_{i=j+1}^{r+1} e_i .
\]

If \( \Psi \) is of type \( B_r \) then we have \( \Sigma_0^+ = \{e_j \mid j = 1, \ldots, r\} \cup \{e_j \pm f_i \mid 1 \leq i < j \leq r\} \) and \( \Psi = \{\alpha_1 = f_1\} \cup \{\alpha_i = e_i - e_{i-1} \mid i = 2, \ldots, r\} \). The Weyl group consists of linear maps generated by the involutions

\[
 w_i(e_i) = -e_i \text{ and } w_i(e_j) = e_j \text{ for } j \neq i
\]

for \( i \leq r \) and the maps

\[
 w_\sigma(e_i) = e_\sigma(i)
\]

for permutations in the symmetric group \( S_r \). Furthermore, one shows that the fundamental weights are

\[
 \xi_1 = \sum_{j=1}^{r} e_j \text{ and } \xi_j = 2 \sum_{i=j}^{r} e_i , \quad j > 1 .
\]

If \( \Psi \) is of type \( C_r \) then we have \( \Sigma_0^+ = \{2e_j \mid j = 1, \ldots, r\} \cup \{e_j \pm e_i \mid 1 \leq i < j \leq r\} \) and \( \Psi = \{\alpha_1 = 2e_1\} \cup \{\alpha_j = e_j - e_{j-1} \mid j = 2, \ldots, r\} \). The Weyl group consists of linear maps generated by the involutions

\[
 w_i(e_i) = -e_i \text{ and } w_i(e_j) = e_j \text{ for } j \neq i
\]

for \( i \leq r \) and the maps

\[
 w_\sigma(e_i) = e_\sigma(i)
\]

for permutations in the symmetric group \( S_r \). Furthermore, one shows that the fundamental weights are

\[
 \xi_j = 2 \sum_{i=j}^{r} f_i
\]
If $\Psi$ is of type $D_r$ then $\alpha_1 = e_1 + e_2$ and $\alpha_j = e_j - e_{j-1}$ for $j \geq 2$. The Weyl group consists of linear maps generated by the involutions

$$w_i(e_i) = -e_i \quad \text{and} \quad w_i(e_j) = e_j \quad \text{for} \quad j \neq i$$

for $2 \leq i \leq r$ and the maps

$$w_\sigma(e_i) = e_\sigma(i)$$

for permutations in the symmetric group $S_r$. One shows that the fundamental weights are

$$\xi_1 = \sum_{i=1}^{r} e_i, \quad \xi_2 = -e_1 + \sum_{j=2}^{r} e_j, \quad \text{and} \quad \xi_j = 2 \sum_{i=j}^{r} e_i \quad \text{for} \quad j \geq 3.$$

Thus if we take a propagated symmetric space $G_\infty/K_\infty$ or a propagated direct-limit group $U_\infty$, then one uses the above formulations to construct countable bases $\{e_1, e_2, \ldots\}$ for $a_\infty$ and $h_\infty$, respectively.

### 4.5 Admissible Direct Limits

We continue to examine the root data for lim-noncompact symmetric spaces $G_\infty/K_\infty$ by analogy with Section 3.2. For each $k \in \mathbb{N}$ and each restricted root $\alpha \in \Sigma_k$, we define as before the root space

$$g_{k,\alpha} = \{Y \in g_k \mid [H, Y] = \alpha(H)Y \text{ for all } H \in a_k\}.$$

Next we define the subalgebras

$$n_k = \bigoplus_{\alpha \in \Sigma_k^+} g_{k,\alpha}$$

and

$$m_k = Z_{g_k}(a_k)$$

of $g_k$. Similarly, we define the subgroups $N_k = \exp(n_k)$ and $M_k = Z_{g_k}(a_k)$ of $G_k$.

For each $k \in K$, the conical representations of $G_k$ are the representations which possess a nonzero vector (or, more generally, distribution vector) which is invariant under the action of the group $M_kN_k$. Hence, in order to define conical representations of $G_\infty$, one would like to define a subgroup $M_\infty N_\infty = \lim M_nN_n$. In order for such a group to be well-defined, we need to make a technical assumption.

**Definition 4.7.** A lim-noncompact symmetric space $G_\infty/K_\infty$ is said to be admissible if $M_kN_k \leq M_mN_m$ whenever $k \leq m$.

As a consequence of the following lemmas, it is sufficient to assume that $m_k \subseteq m_m$ for $k \leq m$:

**Lemma 4.8.** If $G_\infty/K_\infty$ is a lim-noncompact symmetric space, then $N_k \leq N_m$ for $k \leq m$. 

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\textit{Proof.} We will show that \( n_k \subseteq n_m \). The result will then follow from the fact that \( N_k = \exp n_k \) and \( N_m = \exp n_m \).

In fact, it suffices to show that \( g_{k,\alpha} \subseteq n_m \) for all \( \alpha \in \Sigma_k^+ \). Suppose that \( X \in g_{k,\alpha} \). Consider the decomposition of \( X \) into \( a_m \)-root vectors:

\[
X = \sum_{\beta \in \Sigma_m} X_{\beta},
\]

where \( X_{\beta} \in g_{m,\beta} \) for each \((g_m, a_m)\)-root \( \beta \). Because this decomposition is unique and \( X \) is a root vector for \( a_k \subseteq a_m \), it follows that \( \beta|_{a_k} = \alpha \) for all \( \beta \in \Sigma_m \) such that \( X_{\beta} \neq 0 \).

Now recall that we have made a consistent choice of positive root subsystems \( \Sigma_k^+ \) of \( \Sigma_k \) and \( \Sigma_m^+ \) of \( \Sigma_m \). In other words, \( \beta \in \Sigma_m \) is positive if \( \beta|_{a_k} \) is positive. Since \( \alpha \in \Sigma_k^+ \), it follows that \( X \) is a sum of \( \Sigma_m^+ \)-root vectors. Hence, \( X \in N_m \). \hfill \Box

Due to the fact that \( M_k \) is typically a disconnected subgroup of \( G_n \), it is not clear \textit{a priori} that requiring \( m_k \subseteq m_m \) for \( k \leq m \) is sufficient to imply that \( M_k \leq M_m \). However, the following lemma shows that this Lie algebra condition is, in fact, sufficient:

\textbf{Lemma 4.9.} Suppose that \( G_\infty / K_\infty \) is a propagated lim-noncompact symmetric space such that \( m_k \subseteq m_m \) for all \( k \leq m \). Then \( M_k \leq M_m \) for \( k \leq m \).

\textit{Proof.} By Theorem 7.53 in [26] we see that for each \( k \in \mathbb{N} \) there is a finite discrete subgroup \( F_k \subseteq M_k \) such that \( M_k = F_k(M_k)_0 \), where \((M_k)_0 = \exp m_k \) is the connected component of the identity in \( M_k \). Because \( m_k \subseteq m_m \) for all \( k \leq m \), we see that \((M_k)_0 \leq (M_m)_0 \). We must show that \( F_k \leq M_m \) for \( k \leq m \).

In fact, [26, Theorem 7.53] shows that \( F_k \subseteq \exp (ia_k) \cap K_k \) for each \( k \in \mathbb{N} \). Since \( ia_k \subseteq ia_m \) when \( k \leq m \), it follows that \( \exp (ia_k) \subseteq \exp (ia_m) \). Thus \( F_k \subseteq \exp (ia_m) \). Since \( a_k \oplus ia_k \) is a commutative subalgebra of \( u_C \), it is clear that every element of \( \exp (ia_m) \) commutes with \( \exp (a_m) \) and thus that \( \exp (ia_m) \) centralizes \( a_m \). It follows that \( F_k \leq M_m \) and thus \( M_k \leq M_m \). \hfill \Box

At this point we do not know whether every propagated direct limit of noncompact-type Riemannian symmetric spaces is admissible, but in any case this assumption is not a restrictive one, as it is satisfied by each of the classical direct limits, as we demonstrate in the next section.

\textbf{4.6 Admissibility of Classical Direct Limits}

The classical propagated direct systems of Riemannian symmetric spaces may be found in Table 4.4, where each row gives a noncompact-type symmetric space \( G_n / K_n \) and its simply-connected compact dual space \( U_n / K_n \), and where the restricted roots exhibit the Dynkin diagram \( \Psi_n \). For each row, the limit \( G_\infty / K_\infty = \lim G_n / K_n \) is propagated and also that it is possible to choose Cartan subalgebras of \( U_n \) for each \( n \in \mathbb{N} \) so that \( U_\infty = \lim U_n \) is a propagated direct-limit group (see, for instance, [37, Section 2] or [52, Section 3]).
Note that in each row of Table 4.4, the symmetric space $U_n/K_n$ is simply-connected. However, in certain rows the group $U_n$ is not simply-connected. We may remove this obstruction simply by passing to the universal cover $\widetilde{U}_n$ of $U_n$. In fact, that the involution $\theta_n$ on $u_n$ integrates to an involution $\tilde{\theta}_n$ on $\widetilde{U}_n$. Denote the fixed-point subgroup for $\tilde{\theta}_n$ in $\widetilde{U}_n$ by $\tilde{K}_n$. By simply-connectedness all of the inclusions on the Lie algebra level integrate to inclusions on the group level, so that $\widetilde{U}_n/\tilde{K}_n$ forms a propagated direct system of compact-type symmetric spaces. Furthermore, one sees that if $p : \widetilde{U}_n \to U_n$ is the covering map, then $p \left( \tilde{K}_n \right) \subseteq K_n$. Hence $p$ factors to a covering map from $\widetilde{U}_n/\tilde{K}_n$ to $U_n/K_n$ (see [20, p. 213]). Since $U_n/K_n$ is already simply-connected, we see that $\widetilde{U}_n/\tilde{K}_n$ is diffeomorphic to $U_n/K_n$.

While we do not know whether it is possible to show that all propagated direct systems of Riemannian symmetric spaces are admissible in the sense of 4.7, the aim of this section is to show that each classical example is admissible. For the explicit matrix realizations of the compact-type Riemannian symmetric spaces, see [20, p. 446, 451–455].

<table>
<thead>
<tr>
<th>Classical direct systems of irreducible Riemannian symmetric spaces</th>
</tr>
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<tbody>
<tr>
<td>$G_n$</td>
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<td>13</td>
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</table>

4.6.1 A General Strategy for Proving Admissibility

The embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$ A \mapsto \begin{pmatrix} I & A \\ A & I \end{pmatrix} $$

for the systems in rows 5, 6, and 7. In all other cases in Table 4.4, the embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$ A \mapsto \begin{pmatrix} A & I \\ I \end{pmatrix}, $$

where $I$ is a $1 \times 1$, $2 \times 2$, or $4 \times 4$ identity matrix.
Suppose we can choose \( a_n \) for each \( n \) in such a way that

\[
\begin{pmatrix}
  * & 0 & *
  \\
  0 & a_n & 0
  \\
  * & 0 & *
\end{pmatrix}
\] (4.7)

or

\[
\begin{pmatrix}
  a_n & 0 \\
  0 & *
\end{pmatrix}
\] (4.8)

(depending on the type of embedding \( G_n \hookrightarrow G_{n+1} \)). In this case, since \( a_n \) commutes with \( M_n = Z_K(a_n) \) by definition, it follows from (4.7) and (4.8) that \( a_{n+1} \) commutes with

\[
\begin{pmatrix}
  I & 0 & 0 \\
  0 & M_n & 0 \\
  0 & 0 & I
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  M_n & 0 \\
  0 & I
\end{pmatrix},
\]

respectively, depending on the type of embedding \( G_n \hookrightarrow G_{n+1} \). In other words, \( M_n \leq Z_{K_{n+1}}(a_{n+1}) = M_{n+1} \).

Hence, in order to prove that a propagated direct limit is admissible, it is sufficient to show that either (4.7) or (4.8) holds. In most cases, our proof of admissibility will take this form.

4.6.2 \( U_n = L_n \times L_n \) and \( K_n = \text{diag} \ L_n \)

This case corresponds to the first four rows in Table 4.4. In this case, one sees that

\[
u_n = t_n \times t_n \\
\frak t_n = \{(X, X) \in u_n | X \in t_n\} \\
\frak i\frak p_n = \{(X, -X) \in u_n | X \in t_n\}.
\]

Furthermore, if we fix a Cartan subalgebra \( \frak h_n \subseteq t_n \) for each \( n \), then we can choose

\[
i\frak a_n = \{(X, -X) \in u_n | X \in \frak h_n\}.
\]

Now suppose that \( g \in L_n \) and that \((g, g) \in M_n = Z_{K_n}(a_n)\). Then \( g \in Z_{L_n}(\frak h_n)\); that is, \( g \) centralizes the Cartan subalgebra \( \frak h_n \) of \( t_n \). Since \( K_n \) is connected, it follows that \( g \in H_n \equiv \text{exp}(\frak h_n) \). Thus \( M_n = \text{diag} \ H_n \) for each \( n \). It follows that \( M_k \leq M_n \) for \( k \leq n \).

4.6.3 \( \text{Rank}(G_\infty/K_\infty) \equiv \dim a_\infty < \infty \)

This case corresponds to rows 5_1, 6_1, and 7_1 in Table 4.4. If \( \dim a_\infty < \infty \), then for \( k \) large enough, one has \( a_k = a_\infty \). Suppose \( k \leq n \) and \( g \in M_k \). That is, \( g \in K_k \) and \( g \) centralizes \( a_k \). But \( a_k = a_n = a_\infty \) and \( K_k \leq K_n \). Thus \( g \in M_n \).
4.6.4 $\text{Rank}(G_n/K_n) = \text{Rank}(G_n)$ for all $n \in \mathbb{N}$

This case corresponds to rows 8 and 11 in Table 4.4. One has that $a_n$ is a Cartan subalgebra for $g_n$. In particular, $Z_{\mathfrak{g}_n}(a_n) = a_n$. Since $a_n \cap \mathfrak{k}_n = \{0\}$, one has that $m_n \equiv Z_{\mathfrak{g}_n}(a_n) = \{0\}$ for all $n \in \mathbb{N}$.

For example, if we let $G_n = \text{SL}(n, \mathbb{R})$ and $K_n = \text{SO}(n)$ and make the standard choice of $a_n = \{\text{diag}(a_1, \ldots, a_n) | a_i \in \mathbb{R}\}$, then one has $M_n = \{\text{diag}(\pm1, \ldots, \pm1)\}$. Thus $M_k \leq M_n$ for $k \leq n$.

4.6.5 $U_n/K_n = \text{SU}(2n)/S(\text{SU}(n) \times \text{SU}(n))$

This case corresponds to row 52 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{su}(n, n)$, $\mathfrak{u}_n = \mathfrak{su}(2n)$, and $\mathfrak{k}_n = \mathfrak{s}(\mathfrak{su}(n) \oplus \mathfrak{su}(n))$. The involution is given by $\theta_n : A \mapsto J_nAJ_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in M(2n, \mathbb{C}) \left| \begin{array}{c} A^* = -A, D^* = -D, \\
\text{and } \text{Tr}(A) + \text{Tr}(D) = 0 \end{array} \right. \right\}$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in M(2n, \mathbb{C}) \left| \begin{array}{c} A^* = -A, D^* = -D, \\
\text{and } \text{Tr}(A) + \text{Tr}(D) = 0 \end{array} \right. \right\}$$

$$\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in M(2n, \mathbb{C}) \right\}.$$

We choose

$$i\mathfrak{a}_n = \left\{ \begin{pmatrix} \cdots & a_n \\ -a_n & \cdots \\ -a_1 \\ \cdots \\
& \cdots \end{pmatrix} \left| a_i \in \mathbb{R} \right. \right\}$$

Thus condition (4.7) is satisfied and so $G_\infty/K_\infty$ is admissible.

4.6.6 $U_n/K_n = \text{SO}(2n)/\text{SO}(n) \times \text{SO}(n)$

This case corresponds to row 62 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{so}(n, n)$, $\mathfrak{u}_n = \mathfrak{so}(2n)$, and $\mathfrak{k}_n = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$. The involution is given by $\theta_n : A \mapsto J_nAJ_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix}.$$
More explicitly, one has

\[ u_n = \left\{ \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \in M(2n, \mathbb{R}) \ \middle| \ A^T = -A \text{ and } D^T = -D \right\} \]

\[ t_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in M(2n, \mathbb{R}) \ \middle| \ A^T = -A \text{ and } D^T = -D \right\} \]

\[ i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \in M(2n, \mathbb{R}) \right\} . \]

We choose

\[ i\mathfrak{a}_n = \left\{ \begin{pmatrix} \begin{pmatrix} \cdots & a_n \\ -a_1 \\ \cdots \\ -a_n \end{pmatrix} \end{pmatrix} \ \middle| \ a_i \in \mathbb{R} \right\} . \]

Thus condition (4.8) is satisfied and so \( G_\infty/K_\infty \) is admissible.

4.6.7 \( U_n/K_n = \text{Sp}(2n)/(\text{Sp}(n) \times \text{Sp}(n)) \)

This case corresponds to row 7 in Table 4.4. One has \( g_n = \text{sp}(n, n), u_n = \text{sp}(2n), \) and \( t_n = \text{sp}(n) \oplus \text{sp}(n) \). The involution is given by \( \theta_n : A \mapsto J_n A J_n^{-1} \), where

\[ J_n = \begin{pmatrix} I_n \\ -I_n \end{pmatrix} . \]

More explicitly, one has

\[ u_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in M(2n, \mathbb{H}) \ \middle| \ A^* = -A \text{ and } D^* = -D \right\} \]

\[ t_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in M(2n, \mathbb{H}) \ \middle| \ A^* = -A \text{ and } D^* = -D \right\} \]

\[ i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in M(2n, \mathbb{H}) \right\} . \]

We choose

\[ i\mathfrak{a}_n = \left\{ \begin{pmatrix} \begin{pmatrix} \cdots & a_n \\ -a_1 \\ \cdots \\ -a_n \end{pmatrix} \end{pmatrix} \ \middle| \ a_i \in \mathbb{R} \right\} . \]

Thus condition (4.7) is satisfied and so \( G_\infty/K_\infty \) is admissible.
4.6.8 $U_n/K_n = SU(2n)/Sp(n)$

This case corresponds to row 9 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{u}_n = \mathfrak{su}(2n)$ and $\mathfrak{k}_n = \mathfrak{sp}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where $J_n$ is given by

$$J_n = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
& \ddots \\
& & 0 & -1 \\
& & & 1 & 0
\end{pmatrix}.$$

(4.9)

One can also obtain the same symmetric space by using the involution $\tilde{\theta}_n : A \mapsto \tilde{J}_n A \tilde{J}_n^{-1}$, where

$$\tilde{J}_n = \begin{pmatrix}
& & & & -1 \\
1 & & & \ddots & \\
& & \ddots & & \\
& & & 1 & \\
& & & & 1
\end{pmatrix}.$$

(4.10)

The calculations will be easier if we use $\tilde{\theta}_n$ instead of $\theta_n$. However, we must use $\theta_n$ in order for the inclusions $U_n \to U_{n+1}$ to take the form of (4.6). We can move freely between these pictures, however, because $J_n = E_{\sigma_n} J_n E_{\sigma_n}^{-1}$, where $E_{\sigma_n} \in M(2n, \mathbb{C})$ is the permutation matrix corresponding to the permutation

$$\sigma = (1 \ n)(2 \ (n+1)) \cdots ((n-1) \ 2n) \in S_{2n}.$$

In other words, the rows and columns are interwoven, so that the first $n$ basis elements of $\mathbb{C}^{2n}$ are mapped to odd-numbered basis elements and the final $n$ basis elements of $\mathbb{C}^{2n}$ are sent to even-numbered basis elements.

We proceed by using $\tilde{\theta}_n$. We have

$$\mathfrak{su}(2n) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in M(2n, \mathbb{C}) \mid \begin{array}{c}
A^* = -A, D^* = -D, \text{ and } \\
\text{Tr}(A) + \text{Tr}(D) = 0
\end{array} \right\}$$

$$\mathfrak{sp}(n) \cong \mathfrak{k}_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M(2n, \mathbb{C}) \mid \begin{array}{c}
A^* = -A, \text{ and } B^T = B
\end{array} \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M(2n, \mathbb{C}) \mid \begin{array}{c}
A^* = -A, B^T = -B, \text{ and } \\
\text{Tr}(A) = 0
\end{array} \right\}.$$
for $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, and we can choose

$$i\widetilde{a}_n = \left\{ \begin{pmatrix} ia_1 \\ \vdots \\ ia_n \\ ia_1 \\ \vdots \\ ia_n \end{pmatrix} \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right. \right\} \left| \begin{array}{c} a_i \in \mathbb{R} \text{ and } \sum_{i=1}^n a_i = 0 \end{array} \right\}.$$  

We now proceed to the $\theta_n$ picture. Conjugation of $\widetilde{h}_n$ by $E_{\sigma_n}$ (followed by renumbering the indices) yields the $\theta_n$-stable Cartan subalgebra

$$\mathfrak{h}_n = \mathfrak{h}_n = \left\{ \begin{pmatrix} ia_1 \\ \vdots \\ ia_{2n} \end{pmatrix} \left| \begin{array}{c} a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{2n} a_i = 0 \end{array} \right\}.$$  

Finally, conjugation of $\widetilde{a}_n$ by $E_{\sigma_n}$ yields

$$i\mathfrak{a}_n = \left\{ \begin{pmatrix} ia_1 \\ \vdots \\ ia_n \\ ia_1 \\ \vdots \\ ia_n \end{pmatrix} \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right. \right\} \left| \begin{array}{c} a_i \in \mathbb{R} \text{ and } \sum_{i=1}^n a_i = 0 \end{array} \right\}.$$  

While condition (4.8) is not quite satisfied, we do have that

$$\mathfrak{a}_{n+1} \subseteq \left( \begin{pmatrix} \mathfrak{a}_n + \mathbb{C}\text{Id} \\ 0 \end{pmatrix} \right).$$  

(4.11)

Since $\mathfrak{m}_n$ centralizes $\mathfrak{a}_n$, it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_n + \mathbb{C}\text{Id}$. Thus by (4.11), it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_{n+1}$. Thus $\mathfrak{m}_m \subseteq \mathfrak{m}_n$ for $m \leq n$, and it follows that $G/\mathbb{K}$ is admissible.

4.6.9 $U_n/\mathbb{K} = \text{SO}(4n)/\text{U}(2n)$

This case corresponds to row 10 in Table 4.4. One has $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, $\mathfrak{u}_n = \mathfrak{so}(4n)$ and $\mathfrak{t}_n = \mathfrak{u}(2n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where $J_n$ is given by (4.9). As in the previous example, one can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where $\widetilde{J}_n$ is given by (4.10).
We work first on the $\bar{\theta}_n$-side. We have

$$\mathfrak{so}(4n) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \in M(4n, \mathbb{R}) \right\}$$

$$A^T = -A \text{ and } D^T = -D$$

$$\mathfrak{u}(2n) \cong \mathfrak{t}_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M(4n, \mathbb{R}) \right\}$$

$$A^T = -A \text{ and } B^T = B$$

$$\mathfrak{i} \mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M(4n, \mathbb{R}) \right\}$$

$$A^T = -A \text{ and } B^T = -B$$

There is a $\bar{\theta}_n$-stable Cartan subalgebra

$$\bar{\mathfrak{h}}_n = \left\{ \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \\ & 0 & a_2 \\ & -a_2 & 0 \\ & & & \ddots \\ & & & 0 & a_{2n} \\ & & & -a_{2n} & 0 \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$

and we can choose

$$\bar{\mathfrak{a}}_n = \left\{ \begin{pmatrix} 0 & a_1 & \cdots & 0 \\ -a_1 & 0 & \cdots & 0 \\ & 0 & a_n & \cdots \\ & & -a_n & 0 \\ & & & \ddots \\ & & & 0 & -a_n \\ & & & & a_n & 0 \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$

Moving to the $\theta_n$-picture, we conjugate everything by $E_{\sigma_n}$ and renumber the indices to arrive at the $\theta_n$-stable Cartan algebra

$$\mathfrak{h}_n = \left\{ \begin{pmatrix} 0 & 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2 & \cdots & 0 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_4 & \cdots & 0 \\ -a_3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{2n-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_{2n} & \cdots & 0 \\ -a_{2n-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_{2n} & 0 & 0 & \cdots & 0 \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$
\[ i\mathfrak{a}_n = \begin{cases} 
\begin{pmatrix} 
0 & 0 & a_1 & 0 \\
0 & 0 & 0 & -a_1 \\
-a_1 & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 
\end{pmatrix} & a_i \in \mathbb{R} \\
\end{cases} \]

Hence \( \mathfrak{a}_n \) is block-diagonal, and moving from \( \mathfrak{a}_n \) to \( \mathfrak{a}_{n+1} \) is simply a matter of adding another \( 4 \times 4 \) block. Thus we see that condition (4.8) is satisfied and hence \( G_\infty/K_\infty \) is admissible.

4.6.10 \( U_n/K_n = \text{SO}(2(2n + 1))/U(2n + 1) \)

This case corresponds to row 102 in Table 4.4. One has \( \mathfrak{g}_n = \mathfrak{so}^\ast(2(2n + 1)), \mathfrak{u}_n = \mathfrak{so}(4n) \) and \( \mathfrak{t}_n = \mathfrak{u}(2n) \). As in the previous example, one can also obtain the same symmetric space by using the involution \( \tilde{\theta}_n : A \mapsto \tilde{J}_n A \tilde{J}_n^{-1} \), where \( \tilde{J}_n \) is given by (4.10).

We first work on the \( \tilde{\theta}_n \) side. We then have

\[ \mathfrak{so}(2(2n + 1)) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\
-B^T & D \end{pmatrix} \in M(2(2n + 1), \mathbb{R}) \bigg| A^T = -A, D^T = -D \right. \]

\[ \mathfrak{u}(2n + 1) \cong \mathfrak{t}_n = \left\{ \begin{pmatrix} A & B \\
-B & A \end{pmatrix} \in M(2(2n + 1), \mathbb{R}) \bigg| A^T = -A, B^T = B \right. \]

\[ i\mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\
B & -A \end{pmatrix} \in M(2(2n + 1), \mathbb{R}) \bigg| A^T = -A, B^T = -B \right. \]

There is a \( \tilde{\theta}_n \)-stable Cartan subalgebra

\[ \tilde{\mathfrak{h}}_n = \begin{cases} 
\begin{pmatrix} 
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & -a_2 \\
-a_2 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 
\end{pmatrix} & a_i \in \mathbb{R} \\
\end{cases} \]
and we can choose
\[ i\tilde{a}_n = \begin{cases} 
0 & a_1 \\
-1 & 0 \\
& \ddots \\
& 0 & a_n \\
& -a_n & 0 \end{cases} 
\] 
and finally
\[ ia_n = \begin{cases} 
0 & 0 \\
& \ddots \\
& 0 & a_n \\
& 0 & -a_n \end{cases}. \]

Moving to the \( \theta_n \)-picture, we conjugate everything by \( E_{\sigma_n} \) and renumber the indices to arrive at the \( \theta_n \)-stable Cartan algebra
\[ h_n = \] 
and hence \( a_n \) is block-diagonal, and moving from \( a_n \) to \( a_{n+1} \) is simply a matter of adding another \( 4 \times 4 \) block. Thus we see that condition (4.8) is satisfied and hence \( G_\infty/K_\infty \) is admissible.
Chapter 5
Representations of Direct-Limit Groups

In this chapter we review some important results about representations for direct-limit groups and lim-Riemannian symmetric spaces. See [43] for a quite comprehensive overview of representation theory for classical direct limits of symmetric spaces. See also [11] and [33] for many basic results on representations of direct-limit groups.

We begin this chapter by reviewing how one can construct representations of a direct-limit group by forming a direct limit of representations of finite-dimensional Lie groups. This construction provides the simplest way to construct unitary or even irreducible unitary representations for direct-limit groups.

Next we begin to tackle the issue of smoothness for representations of direct-limit groups. We review several useful results from the literature (especially from [31] and [11]) which provide equivalent conditions for smoothness.

Next we discuss a generalization of Weyl’s unitary trick which identifies smooth representations of a lim-compact symmetric space $U_{\infty}/K_{\infty}$ with smooth representations of its c-dual $G_{\infty}/K_{\infty}$. This brings up the question of unitarizability of representations of the lim-compact group $U_{\infty}$, which is unfortunately rather subtle.

Making things more concrete, we follow earlier constructions in [37],[52], and [54] to define highest-weight representations for $U_{\infty}$. We end the chapter by recalling the main result of [7] on spherical representations.

5.1 Direct Limits of Representations

Suppose that $G_{\infty} = \varinjlim G_n$ is a direct-limit group with inclusion maps $p_{n+1}^n : G_n \rightarrow G_{n+1}$ and that for each $n \in \mathbb{N}$ we are given a continuous Hilbert representation $(\pi_n, \mathcal{H}_n)$ and partial isometries $j_{n+1}^n : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ such that the diagram

$$
\begin{array}{ccc}
G_{n+1} \times \mathcal{H}_{n+1} & \xrightarrow{\pi_{n+1}} & \mathcal{H}_{n+1} \\
\downarrow p_{n+1}^n \times j_{n+1}^n & & \downarrow j_{n+1}^n \\
G_n \times \mathcal{H}_n & \xrightarrow{\pi_n} & \mathcal{H}_n
\end{array}
$$

commutes (see Section 2 in [31]). A continuous map $\pi_\infty : G_\infty \times \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is induced, where $\mathcal{H}_\infty = \varinjlim \mathcal{H}_n$. It may be readily shown that $\pi_\infty$ is in fact a continuous representation of $G_\infty$ on $\mathcal{H}_\infty$. In fact, $\pi_\infty$ extends by continuity to a continuous representation on the Hilbert space completion $\overline{\mathcal{H}_\infty}$ (see, for instance, Proposition B.10 in [33]). One can also show that if $\pi_n$ is unitary for each $n \in \mathbb{N}$, then $\pi_\infty$ is a unitary representation of $G_\infty$ on $\overline{\mathcal{H}_\infty}$.

For a more intuitive perspective on this situation, suppose that $\{G_n\}_{n \in \mathbb{N}}$ is an increasing sequence of Lie groups (i.e., $G_n$ is a closed subgroup of $G_m$ for $n \leq m$) and that for each $n$ we are provided with a continuous Hilbert representation $(\pi_n, \mathcal{H}_n)$.
such that \((\pi_n, \mathcal{H})\) is equivalent (by a unitary intertwining operator) to a subrepresentation of \((\pi_{n+1}|_{G_n}, \mathcal{H}_{n+1})\). Then one has a direct system of representations and may form a direct-limit representation \((\pi_\infty, \mathcal{H}_\infty)\) of \(G_\infty\).

One of the key tools in representation theory is the study of intertwining operators for representations. It is clear that an operator \(T \in \mathcal{B}(\mathcal{H})\) is an intertwining operator for a Hilbert representation \((\pi, \mathcal{H})\) of a direct-limit group \(G_\infty = \lim \rightarrow G_n\) if and only if it is an intertwining operator for \(\pi|_{G_n}\) for each \(n\). If \(\pi\) is a direct-limit representation, then we can say more:

**Lemma 5.1.** ([27]) If \((\pi, \mathcal{H}) = (\lim \rightarrow \pi_n, \lim \rightarrow \mathcal{H}_n)\) is a direct limit of Hilbert representations, then a bounded operator \(T \in \mathcal{B}(\mathcal{H})\) is an intertwining operator for \(\pi\) if and only if \(T|_{\mathcal{H}_n}\) is an intertwining operator for \(\pi|_{G_n}\) for each \(n \in \mathbb{N}\).

**Proof.** One direction is obvious. To prove the other direction, we suppose that \(T|_{\mathcal{H}_n}\) is an intertwining operator for \(\pi|_{G_n}\) for each \(n \in \mathbb{N}\). It is thus clear that \(T\pi(g)v = \pi(g)Tv\) for any \(g \in G_\infty\) and any \(v\) in the algebraic direct limit space \(\mathcal{H}_\infty = \lim \rightarrow \mathcal{H}_n\). The lemma follows since \(\mathcal{H}_\infty\) is a dense subspace of \(\mathcal{H}\) and since \(\pi(g)\) is continuous for each \(g \in G_\infty\). □

Direct-limit representations are the easiest representations to construct for \(G_\infty\). The following theorem shows that they may be in fact be used to construct a large class of irreducible unitary representations:

**Theorem 5.2.** ([27]) Suppose that \(\{G_n\}_{n \in \mathbb{N}}\) is a direct system of locally compact groups and that \(\{(\pi_n, \mathcal{H}_n)\}_{n \in \mathbb{N}}\) is a compatible direct system of irreducible unitary representations of \(G_n\) for each \(n \in \mathbb{N}\). Then \((\pi, \mathcal{H}) \equiv (\lim \rightarrow \pi_n, \lim \rightarrow \mathcal{H}_n)\) is an irreducible unitary representation of \(G_\infty\).

**Proof.** Suppose that \(T \in \mathcal{B}(\mathcal{H})\) is an intertwining operator for \(\pi\). Then \(T|_{\mathcal{H}_n}\) is a \(G_n\)-intertwining operator for \(\pi_n\). Since \(\pi_n\) is irreducible, it follows from Schur's Lemma that \(T|_{\mathcal{H}_n} = c \text{Id}\) for some constant \(c \in \mathbb{C}\). Because \(\mathcal{H}_n \subseteq \mathcal{H}_k\) for \(n \leq k\), we see that the constant is independent of \(n\). Thus, \(T|_{\mathcal{H}_\infty} = c \text{Id}\), where \(\mathcal{H}_\infty = \lim \rightarrow \mathcal{H}_n\) is the algebraic direct limit space. By continuity we then have that \(T = c \text{Id}\) since \(\mathcal{H}_\infty\) is a dense subspace of \(\mathcal{H}\). Because the intertwining operator \(T \in \mathcal{B}(\mathcal{H})\) was arbitrary, it follows immediately that \(\mathcal{H}\) is an irreducible representation. □

We caution the reader that there are many examples of irreducible representations of direct-limit groups which are not given by direct limits of irreducible representations (see [11, p. 971]).

### 5.2 Smoothness and Local Finiteness

Just as for finite-dimensional Lie groups, it is natural to try to gather information about a representation of a direct-limit group by differentiating it to obtain a representation of its Lie algebra. We begin with some natural definitions.

**Definition 5.3.** Suppose that \((\pi, \mathcal{H})\) is a continuous Hilbert representation of a direct-limit group \(G_\infty = \lim \rightarrow G_n\) and that \(v \in \mathcal{H}\). We say that \(v\) is a **smooth**
vector for \( \pi \) if it is a smooth vector for the restricted representation \((\pi|_{G_n}, \mathcal{H})\) of \( G_n \) for each \( n \in \mathbb{N} \). We denote by \( \mathcal{H}^\infty \) the space of all smooth vectors for \( \pi \).

Similarly, we say that \( v \) is a **locally finite vector** for \( \pi \) if it is a smooth vector for the restricted representation \((\pi|_{G_n}, \mathcal{H})\) of \( G_n \) for each \( n \in \mathbb{N} \). We denote by \( \mathcal{H}^{\text{fin}} \) the space of locally finite vectors for \( \pi \). Note that \( \mathcal{H}^{\text{fin}} \subseteq \mathcal{H}^\infty \).

Given a Hilbert representation \((\pi, \mathcal{H})\) of \( G_\infty \), we may construct a representation of \( g_\infty \) on \( \mathcal{H}^\infty \) as follows. For each \( n \in \mathbb{N} \), we have the differentiated representation \( \mathcal{D}(\pi|_{G_n}) \) of \( g_n \) on \( \mathcal{H}_n \) with

\[
d(\pi|_{G_n}) = \lim_{t \to 0} \frac{\pi|_{G_n}(\exp tX)v - v}{t},
\]

for each \( X \in g_n \) and \( v \in \mathcal{H}^\infty \). We see that

\[
d(\pi|_{G_{n+1}})(X)v = \lim_{t \to 0} \frac{\pi|_{G_{n+1}}(\exp(tX))v - v}{t}
= \lim_{t \to 0} \frac{\pi|_{G_n}(\exp tX)v - v}{t}
= d(\pi|_{G_n})(X)v,
\]

and thus there is a well-defined map \( d\pi(X) : \mathcal{H}^\infty \to \mathcal{H}^\infty \) for each \( X \in g_\infty = \varinjlim g_n \), given by \( d\pi(X)v = d(\pi|_{G_n})v \) for each \( X \in g_n \). It is a straightforward argument to show that

\[
d\pi(X + Y)v = \pi(X)v + \pi(Y)v
\]

and

\[
d\pi([X,Y])v = \pi(X)\pi(Y)v - \pi(X)\pi(Y)v
\]

for all \( v \in \mathcal{H}^\infty \) and \( X,Y \in g_\infty \).

It is not at all clear from the definitions that a representation of \( G_\infty \) is guaranteed to possess any smooth vectors or locally-finite vectors. In fact, the existence of smooth vectors is far more subtle for representations of infinite-dimensional Lie groups than for finite-dimensional Lie groups, where every continuous representation on a Frechet space admits a dense subspace of smooth vectors. There are examples of unitary representations of Banach-Lie groups which do not possess any \( C^1 \) vectors, much less any smooth vectors (see [3]). For direct-limit groups, however, a beautiful theorem of Danilenko shows that unitary representations always admit smooth vectors.

**Theorem 5.4.** ([6]; see also [34, Theorem 11.3]) Suppose that \((\pi, \mathcal{H})\) is a unitary representation of a countable direct limit of locally compact topological groups. Then \( \mathcal{H}^\infty \) is a dense subspace of \( \mathcal{H} \).

We may thus consider the space \( \mathcal{H}^{-\infty} = (\mathcal{H}^\infty)' \) of **distribution vectors** for a unitary representation \((\pi, \mathcal{H})\) of \( G_\infty \) and obtain dense embeddings

\[
\mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}
\]

as we saw for representations of finite-dimensional Lie groups.

Some representations may consist entirely of smooth vectors:
Definition 5.5. Suppose that $G_\infty$ is a direct-limit Lie group. We say that a continuous Hilbert representation $(\pi, H)$ of $G_\infty$ is smooth if $H_\infty = H$.

If $G_\infty$ is a direct limit of complex Lie groups, then a continuous Hilbert representation $(\pi, H)$ of $G_\infty$ is holomorphic if $\pi|_{G_n}$ is holomorphic for each $n \in \mathbb{N}$.

In fact, we will be primarily concerned with smooth representations in this thesis. They play a role for direct-limit groups that is similar to the role played by finite-dimensional representations for finite-dimensional Lie groups. There are several conditions which are equivalent to smoothness:

Theorem 5.6. Let $(\pi, H)$ be a continuous Hilbert representation of a Lie group $G$. Then the following are equivalent:

1. $\pi$ is smooth

2. There is a Lie algebra representation $d\pi : g \rightarrow \mathcal{B}(H)$ (for which $g$ acts by bounded operators) such that

$$\pi(\exp X) = \exp(d\pi(X))$$

for each $X \in g$.

3. $\pi$ is norm-continuous.

Proof. First we prove (1) $\rightarrow$ (2). Suppose that $\pi$ is smooth. Then $H_\infty = H$ and it follows that for each $X \in g$, we have a strongly-continuous one-parameter group $\{Q(t)\}_{t \in \mathbb{R}}$ of bounded operators on $H$ given by

$$Q(t) = \pi(\exp tX).$$

Since $H_\infty = H$, we see that the limit

$$d\pi(X)v = \lim_{t \to 0} \frac{\pi(\exp tX)v - v}{t}$$

exists in $H$ for all $v \in H$. Following the terminology of [49, p. 375], we have that the domain of $d\pi(X)$ is all of $H$ (i.e., $D(d\pi(X)) = H$). By [49, Theorem 13.36], this implies that $d\pi(X) \in B(H)$ and that

$$\pi(\exp tX) = \exp(td\pi(X))$$

for all $X \in g$ and $t \in \mathbb{R}$. This establishes (5.1).

Next we demonstrate that (2) $\rightarrow$ (3). Suppose that (5.1) holds. Then

$$||\pi(\exp(X)) - \Id|| = ||\exp(d\pi(X)) - \Id||$$

$$= \left|\left| \sum_{n=1}^{\infty} \frac{d\pi(X)^n}{n!} \right|\right|$$

$$\leq \sum_{n=1}^{\infty} \frac{||d\pi(X)||^n}{n!}$$

$$= \exp(||d\pi(X)||) - 1$$
for all \( X \in \mathfrak{g} \).

Let \( X_1, \ldots, X_d \) be a basis for \( \mathfrak{g} \), where \( d = \dim \mathfrak{g} \), and set

\[
M = \max_{1 \leq i \leq d} ||d\pi(X_i)||.
\]

It follows that

\[
||d\pi \left( \sum_{i=1}^{d} c_i X_i \right)|| \leq \left( \sum_{i=1}^{d} c_i \right) M
\]

whenever \( c_i \in \mathbb{R} \) for \( 1 \leq i \leq d \). Hence, it follows that if \( X = \sum_{i=1}^{d} c_i X_i \) with \( \sum_{i=1}^{d} c_i < \epsilon \), then \( ||\pi(\exp(X)) - \Id|| \leq \exp(\epsilon M) - 1 \). Thus, we see that \( X \mapsto \pi(\exp X) \) is norm-continuous. The result then follows that \( \pi \) is norm-continuous from the fact that \( \exp : \mathfrak{g} \to G \) is a local diffeomorphism.

Finally, \((3) \to (1)\) is a straightforward application of [49, Theorem-13.36], which says that if \( \lim_{t \to 0} ||\pi(\exp(tX)) - \Id|| = 0 \) for all \( X \in \mathfrak{g} \), then the infinitesimal generator is a bounded operator (that is, the differential exists everywhere). \( \square \)

It is certainly possible to construct continuous unitary representations of direct-limit groups which possess no locally finite vectors. This behavior is already present for finite-dimensional Lie groups, however: an irreducible infinite-dimensional representation of a noncompact Lie group \( G \) does not possess any \( G \)-finite vectors. More surprisingly, it is possible to construct an irreducible unitary representation of a lim-compact group which has no locally finite vectors ([35]). However, as a corollary of the following theorem, smooth representations of direct-limit groups always consist entirely of locally finite vectors.

**Theorem 5.7.** Suppose that \((\pi, \mathcal{H})\) is a continuous Hilbert representation of a Lie group \( G \). Then \( \mathcal{H}^{\infty} = \mathcal{H} \) if and only if \( \mathcal{H}^{\text{fin}} = \mathcal{H} \).

**Proof.** Because \( \mathcal{H}^{\text{fin}} \subseteq \mathcal{H}^{\infty} \), one direction is obvious. To prove the other direction, suppose that \( \mathcal{H}^{\infty} = \mathcal{H} \).

Theorem 5.6 ensures that \( d\pi \) acts by bounded operators on \( \mathcal{H} \). Fix \( v \in \mathcal{H} \) and consider the subspace

\[
V \equiv \{ \{d\pi(X)v|X \in \mathfrak{g}\} \subseteq \mathcal{H} \}.
\]

Since \( \mathfrak{g} \) is a finite-dimensional Lie algebra, we see that \( V \) must be finite-dimensional (if \( \{X_1, \ldots, X_n\} \) is a basis for \( \mathfrak{g} \), then \( V \) is generated by \( \{\pi(X_1)v, \ldots, \pi(X_n)v\} \)). Note that \( V \) is a closed subspace of \( \mathcal{H} \) because it is finite-dimensional.

Next we show that

\[
V = \langle \pi(G)v \rangle,
\]

from which the lemma will follow. Let \( X \in \mathfrak{g} \). Because \( d\pi(X) \in B(\mathcal{H}) \), we have that

\[
\pi(\exp X)v = \exp(d\pi(X))v = \sum_{n=0}^{\infty} \frac{d\pi(X)^n}{n!}v.
\]

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It is thus clear that $\pi(\exp X)v \in \langle d\pi(g)v \rangle = V$. Finally, if $g \in G$, then there are $X_1, \ldots, X_i \in \mathfrak{g}$ such that $g = \exp(X_1) \cdots \exp(X_i)$. Hence $\pi(g)v \in V$ for all $g \in G$ and we are done.

**Corollary 5.8.** Suppose that $(\pi, \mathcal{H})$ is a continuous Hilbert representation of a direct-limit group $G_\infty$. Then $\mathcal{H}_\infty = \mathcal{H}$ if and only if $\mathcal{H}^\text{fin} = \mathcal{H}$.

It is well known that every continuous, finite-dimensional representation of a Lie group is smooth. However, it is also possible to construct infinite-dimensional Hilbert representations which are smooth. Suppose that $U$ is a compact Lie group and that $(\pi, V)$ is a finite-dimensional representation of $U$. Without loss of generality, we may assume that $\pi$ is unitary. Now consider the representation

$$(N\pi, NV) \equiv \left( \bigoplus_{n \in \mathbb{N}} \pi \bigoplus_{n \in \mathbb{N}} V \right)$$

constructed by taking a Hilbert space direct sum of countably many copies of $(\pi, V)$. For each $v \in NV$, we consider the closed invariant subspace $W = \langle N\pi(U)v \rangle$ generated by $v$. Then $W$ gives a cyclic primary representation of $U$ and decomposes into a direct sum of representations equivalent to $(\pi, V)$. From [16] we see that every cyclic primary representation of the compact group $U$ is finite-dimensional. Thus $\dim W < \infty$ and so $v$ is a $U$-finite vector.

In fact, the next theorem shows that in a certain sense, primary representations (or more precisely, finite direct sums of them) provide the only way to obtain infinite-dimensional smooth representations of $U$:

**Theorem 5.9.** Let $(\pi, \mathcal{H})$ be a unitary representation of a compact Lie group $U$. Then $\pi$ is smooth if and only if $\pi$ decomposes into a finite direct sum of primary representations of $U$.

Before we prove this theorem, we need to introduce the following useful lemma, which we will also make use of several times in the next chapter:

**Lemma 5.10.** Let $G$ be a topological group and let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Let $\mathcal{A}$ be a finite or countably infinite index set, and suppose that

$$v = \sum_{i \in \mathcal{A}} v_i,$$

where $v_i \in \mathcal{H}$ for each $i \in \mathcal{A}$ and where $\langle \pi(G)v_i \rangle$ and $\langle \pi(G)v_j \rangle$ give mutually distinct irreducible representations of $G$ for $i \neq j$. Then

$$\langle \pi(G)v \rangle = \bigoplus_{i \in \mathcal{A}} \langle \pi(G)v_i \rangle.$$

**Proof (of Lemma 5.10).** Write $V = \langle \pi(G)v \rangle$. The fact that $V_i = \langle \pi(G)v_i \rangle$ and $V_j = \langle \pi(G)v_j \rangle$ give disjoint representations of $G$ for $i \neq j$ implies that $V_i \perp V_j$. It is obvious that

$$\langle \pi(G)v \rangle \subseteq \bigoplus_{i \in \mathcal{A}} \langle \pi(G)v_i \rangle.$$
so we prove the opposite containment. It suffices to show that \( v_i \in V \) for all \( i \in A \).

Suppose that \( v_i \not\in V \) for some \( i \in A \). Define
\[
w = \sum_{j \neq i} v_j \quad \text{and} \quad W = \langle \pi(G)w \rangle \subseteq \bigoplus_{j \neq i} V_j.
\]
Then \( V_i \perp W \) and \( v = v_i + w \). Furthermore, \( V_i \) and \( W \) give disjoint representations of \( G \).

Now let \( c_1, \ldots, c_k \in \mathbb{C} \) and \( g_1, \ldots, g_k \in G \). Then
\[
\sum_{j=1}^{k} c_j \pi(g_j) v = \left( \sum_{j=1}^{k} c_j \pi(g_j) v_i \right) + \left( \sum_{j=1}^{k} c_j \pi(g_j) w \right).
\]
Because \( v_i \not\in V \) and \( V_i \) is irreducible, we see that \( V \cap V_i = \emptyset \). It follows that
\[
\sum_{j=1}^{k} c_j \pi(g_j) v_i = 0 \quad \text{if and only if} \quad \sum_{j=1}^{k} c_j \pi(g_j) w = 0.
\]

Hence there is a well-defined, nonzero intertwining operator \( L : V_i \to W \) such that \( L(v_i) = w \), which contradicts the fact that \( V_i \) and \( W \) give disjoint representations of \( G \).

**Proof (of Theorem 5.9).** Let \((\pi, \mathcal{H})\) be a unitary representation of \( U \). Then we can write
\[
\mathcal{H} \cong_G \bigoplus_{\delta \in \hat{G}} \mathcal{H}_\delta,
\]
where \( \mathcal{H}_\delta \) is the space of \( \delta \)-isotypic vectors for each \( \delta \in \hat{G} \) (that is, vectors in \( \mathcal{H}_\delta \) generate primary representations that are direct sums of copies of \( \delta \)). Then \( \pi \) is a finite direct sum of primary representations if and only if \( \mathcal{H}_\delta = \{0\} \) for all but finitely many \( \delta \in \hat{G} \).

Suppose that
\[
\mathcal{H} \cong_G \bigoplus_{i=1}^{n} \mathcal{H}_{\delta_i},
\]
where \( \delta_i \in \hat{G} \) for each \( i \). We will show that \( \pi \) is smooth. For each \( v \in \mathcal{H} \), we can write \( v = v_1 + \cdots + v_n \), where \( v_i \in \mathcal{H}_{\delta_i} \). Then
\[
\langle \pi(G)v \rangle \subseteq \bigoplus_{i=1}^{n} \langle \pi(G)v_i \rangle.
\]
However, because each space \( \langle \pi(G)v_i \rangle \) gives a cyclic primary representation of \( U \), we see that it is finite-dimensional (see [16]). Thus \( v \) is \( G \)-finite. Because \( v \in \mathcal{H} \) was arbitrary, it follows from Theorem 5.7 that \( \pi \) is smooth.
To prove the other direction, suppose that
\[ \mathcal{H} \cong_G \bigoplus_{i=1}^{\infty} \mathcal{H}_{\delta_i}, \]
where \( \delta_i \in \hat{G} \) and \( \mathcal{H}_{\delta_i} \neq \{0\} \) for each \( i \). We will show that \( \pi \) is not smooth.

For each \( i \in \mathbb{N} \), choose a nonzero unit vector \( v_i \in \mathcal{H}_{\delta_i} \) such that \( \langle \pi(G)v_i \rangle \) is irreducible. Note that \( v_i \perp v_j \) for \( i \neq j \). Furthermore, \( \langle \pi(G)v_j \rangle \) give primary representations of type \( \delta_i \) and \( \delta_j \), respectively, and are therefore disjoint. Consider the vector
\[
v \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} v_i \in \bigoplus_{i=1}^{\infty} \mathcal{H}_{\delta_i}.
\]

For each \( i \in \mathbb{N} \), we define
\[
w_j \equiv \sum_{i \neq j} \frac{1}{2^i} v_i \in \bigoplus_{i \neq j} \mathcal{H}_{\delta_i}.
\]

It is clear that the representation of \( U \) on \( \langle \pi(G)v_i \rangle \) is disjoint from the representation on \( \langle \pi(G)v_j \rangle \). Since \( v = v_i + w_i \), Lemma 5.10 implies that \( v_i \in \langle \pi(G)v \rangle \). Because this is true for each \( i \in \mathbb{N} \) and \( v_i \perp v_j \) for \( i \neq j \), it follows that \( \langle \pi(G)v \rangle \) is infinite-dimensional. Therefore, \( \pi \) is not smooth by Theorem 5.7.

The following corollaries restate the conclusion of the previous theorem in terms of weights. A slightly different proof may be found in Lemma 3.5 and Proposition 3.6 of [33].

**Corollary 5.11.** Fix a Cartan subalgebra \( \mathfrak{h} \) in \( \mathfrak{u} \), and suppose that \( (\pi, \mathcal{H}) \) is a unitary representation of \( U \). Then \( \pi \) is smooth if and only if \( \# \Delta(\pi) < \infty \) (that is, \( \pi \) has only finitely many weights).

**Proof.** Let \( H \leq U \) be the maximal torus corresponding to \( \mathfrak{h} \). If \( \pi \) is smooth, then in particular \( \pi|_H \) is smooth and thus there are only finitely many equivalence classes of irreducible (i.e., one-dimensional) representations of \( H_n \) which appear in \( (\pi|_H, \mathcal{H}) \). Thus \( \mathcal{H} \) decomposes under \( d\pi|_h \) into finitely many weight spaces and we are done.

Now suppose that \( \pi \) is not smooth. By Theorem 5.9, there are infinitely many inequivalent equivalence classes of irreducible representations of \( G_n \) which appear in \( (\pi|_H, \mathcal{H}) \). Because they are mutually inequivalent, these irreducible representations have mutually distinct highest weights and hence \( \Delta(\pi) \) is an infinite set.

**Corollary 5.12.** Suppose that \( U_\infty \) is a lim-compact group. As before, we fix a subalgebra \( \mathfrak{h}_\infty = \varinjlim \mathfrak{h}_n \) in \( \mathfrak{u}_\infty \), where each \( \mathfrak{h}_n \) is a Cartan subalgebra of \( \mathfrak{u}_n \). Suppose that \( (\pi, \mathcal{H}) \) is a unitary representation of \( U \). Then \( \pi \) is smooth if and only if \( \# \Delta(\pi|_{U_n}) < \infty \) (that is, \( \pi \) has only finitely many weights) for each \( n \in \mathbb{N} \).
Proof. This result follows immediately from Corollary 5.11 and the definition of smoothness for direct-limit groups.

Suppose now that $U_\infty$ is a propagated lim-compact group. We recursively choose a countable orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ for $\mathfrak{h}_\infty$ as in Section 4.4. Consider the supremum norm of a weight $\lambda \in i\mathfrak{h}_n^*$, given by

$$||\lambda||_\infty = \max_{1 \leq i \leq r_n} |\lambda(e_i)|$$

We then obtain the following useful theorem, which is a modification of Proposition 3.14 in [33].

**Theorem 5.13.** A unitary representation $(\pi, \mathcal{H})$ of a propagated direct limit $U_\infty$ of simply-connected compact semisimple Lie groups is smooth if and only if there is $M > 0$ such that for all $n$ one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$.

**Proof.** First we prove the theorem in the case that $U_\infty$ is a direct limit of compact simple Lie groups.

Let $(\pi, \mathcal{H})$ be a unitary representation of $U_\infty$. Suppose there is $M \in \mathbb{N}$ such that for all $n$ one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$. If $\lambda \in i\mathfrak{h}_{k}^*$ is a highest weight which appears in $\pi|_{U_n}$, then it has the form

$$\lambda = \sum_{i=1}^{r_n} a_i e_i,$$

where $a_i \in \mathbb{Z}$ and $-M \leq a_i \leq M$.

Thus, there are only $(2M)^{r_n}$ possible values for $\lambda$. In other words, $\pi|_{U_n}$ may be written as a direct sum of finitely many primary representations and is thus smooth by Theorem 5.9. Because $n \in \mathbb{N}$ was arbitrary, we have that $\pi$ is smooth.

To prove the other direction, suppose that for each $M > 0$ there is $n \in \mathbb{N}$ and a highest weight $\lambda \in i\mathfrak{h}_n^*$ of an irreducible subrepresentation of $\pi|_{U_n}$ such that $||\lambda||_\infty > M$. Fix $M > 0$ and pick $n \in \mathbb{N}$ and $\lambda \in i\mathfrak{h}_n^*$ satisfying those conditions. Then $\lambda = \sum_{i=1}^{r_n} c_i e_i$, where $c_i \in \mathbb{Z}$ for each $i$. Because $||\lambda||_\infty > M$, we see that there is some index $j$ such that $|c_j| > M$.

From the details in Section 4.4, there is a Weyl group element $w \in W(\mathfrak{g}_n, \mathfrak{a}_n)$ such that $w(e_1) = e_i$ and $w(e_i) = e_1$. Then $|w\lambda(e_1)| = |c_j| > M$. By the Highest-Weight Theorem, we see that $w\lambda \in \Delta(\pi|_{U_n})$; that is, $w\lambda$ is a $\mathfrak{h}_n$-weight for $\pi|_{U_n}$. It is then clear that $(w\lambda)|_{\mathfrak{h}_k}$ is an $\mathfrak{h}_k$-weight for $\pi|_{U_k}$ whenever $k \leq n$ (since every $w\lambda$-weight vector in $\mathcal{H}$ is automatically a $(w\lambda)|_{\mathfrak{h}_k}$-weight vector). Furthermore, since $|(w\lambda)|_{\mathfrak{e}_n}(e_1)|| = |c_j| > M$, we see that $||w\lambda||_{\mathfrak{e}_n}|_\infty > M$.

Thus, if $k \in \mathbb{N}$ is fixed, then for each $M \in \mathbb{N}$ there is a weight $\lambda \in \Delta(\pi|_{U_k})$ such that $||\lambda||_\infty > M$. Hence $\Delta(\pi|_{U_k})$ is not a finite set and thus by Corollary 5.12 it follows that $\pi$ is not smooth.

Suppose more generally that $U_\infty$ is a propagated direct limit of semisimple Lie groups. Then we can write $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^d$ for all $k \in \mathbb{N}$ in such a way that
\(\{U_n^i\}_{n \in \mathbb{N}}\) is a propagated direct system of compact simple Lie groups for each \(1 \leq i \leq d\). We can then recursively choose Cartan subalgebras \(\mathfrak{h}_n^i = \mathfrak{h}_n^1 \oplus \mathfrak{h}_n^2 \oplus \cdots \oplus \mathfrak{h}_n^d\), where \(\mathfrak{h}_n^i\) is a Cartan subalgebra of \(u_n^i\) for each \(i\) and \(n\). A weight in \(\lambda \in \mathfrak{h}_n^*\) is dominant integral if and only if \(\lambda|_{\mathfrak{h}_n^i}\) is dominant integral for each \(1 \leq i \leq d\). Since \(U_n^i\) is a propagated direct limit of compact simple Lie groups, it follows that there is \(M_i > 0\) such that for all \(n \in \mathbb{N}\) one has that ||\(\lambda||_{\infty} < M_i\) for each highest weight \(\lambda \in \mathfrak{h}_n^*\) appearing in \(\pi|_{U_n^i}\). Since \(\max_{1 \leq i \leq d} M_i < \infty\), we are done.

We end the section with the following remarkable result, which implies that the smoothness of a representation of a direct limit of simple groups is controlled by the smoothness of the restriction to any nontrivial one-dimensional analytic subgroup.

**Theorem 5.14.** Let \(U\) be a compact simple Lie group. Then a unitary representation \((\pi, \mathcal{H})\) of \(U\) is smooth if and only if there is \(X \in u\{0\}\) such that \(d\pi(X)\) is a bounded operator on \(\mathcal{H}\).

**Proof.** One direction is obvious. To show the other direction, suppose that \((\pi, \mathcal{H})\) is a non-smooth unitary representation of \(U\). We will show that \(d\pi(X)\) has an unbounded spectrum. Let \(\mathfrak{h}\) be any Cartan subalgebra for \(U\).

Because \(\pi\) is not smooth, it follows that there is for each \(M > 0\) weight \(\lambda \in \Delta(\pi)\) with \(||\lambda||_{\infty} > M\). As in the proof of Theorem 5.13, we see that for each Weyl-group element \(w \in W(u, \mathfrak{h})\), the weight \(w\lambda\) is in \(\Delta(\pi)\). If we write \(\lambda = \sum_{i=1}^{r} a_i e_i\), then there is some \(j\) such that \(|a_j| > M\). We can use the Weyl group to permute the basis elements so that \(a_j\) appears as the \(i^{th}\) coefficient of a weight in \(\Delta(\pi|_{U})\). Thus we have that the set

\[\{(\lambda, e_i)|\lambda \in \Delta(\pi)\}\]

of \(i^{th}\) coefficients of weights of \(\pi\) is unbounded for all \(i \leq r\).

In other words, one has for each \(n \in \mathbb{N}\) that the set of weights in \(\Delta(\pi)\) is unbounded in every direction on \(\mathfrak{h}\). It follows that \(d\pi(X)\) has an unbounded spectrum for all \(X \in \mathfrak{h}\). Because every element of \(u\) is contained in some Cartan subalgebra, the result follows.

**Corollary 5.15.** Let \(U_\infty\) be a direct limit of compact simple Lie groups. Then a unitary representation \((\pi, \mathcal{H})\) of \(U_\infty\) is smooth if and only if there is \(X \in u\{0\}\) such that \(d\pi(X)\) is a bounded operator on \(\mathcal{H}\).

**Proof.** This corollary follows immediately by applying Lemma 5.14 to \(U_n\) for each \(n\) in \(\mathbb{N}\).

Note that this result is false for non-simple compact groups: suppose that \(J\) and \(T\) are compact Lie groups, that \((\pi, \mathcal{H})\) is a smooth unitary representation of \(J\), and that \((\sigma, \mathcal{K})\) is a non-smooth unitary representation of \(T\). Then the outer tensor product representation \((\pi \boxtimes \sigma, \mathcal{H} \otimes \mathcal{K})\) of \(J \times T\) has the property that \(\pi|_J\) is smooth but \(\pi|_T\) is non-smooth.
5.3 Generalizing Weyl’s Unitary Trick

Weyl’s Unitary Trick plays a crucial role in understanding finite-dimensional representations of finite-dimensional Lie groups. There is a natural extension of Weyl’s Unitary Trick to smooth representations of direct-limit groups. The first step is to extend Weyl’s unitary trick to smooth representations of finite-dimensional groups. We begin with a well-known lemma on intertwining operators of smooth representations.

**Lemma 5.16.** Suppose that \((\pi, \mathcal{H})\) is a smooth Hilbert representation of a Lie group \(G\). Then the derived representation \(d\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})\) possesses the same algebra of intertwining operators as \(\pi\).

**Proof.** Suppose that \(T\) is an intertwining operator for \(d\pi\). That is, \(d\pi(X)T = Td\pi(X)\) for all \(X \in \mathfrak{g}\). It immediately follows that \(\exp(X)T = T\exp(X)\) for all \(X \in \mathfrak{g}\) and thus \(T\) is an intertwining operator for \(\pi\) by Theorem 5.6. Next suppose that \(T\) is an intertwining operator for \(\pi\). Then \(T\pi(\exp tX) = \pi(\exp tX)T\) for all \(X \in \mathfrak{g}\) and \(t \in \mathbb{R}\). It follows by differentiation at \(t = 0\) that \(Td\pi(X) = d\pi(X)T\) for all \(X \in \mathfrak{g}\). \(\square\)

Now we are ready to extend Weyl’s Trick to smooth representations of finite-dimensional groups.

**Theorem 5.17.** Suppose that \(G/K\) is a noncompact-type Riemannian symmetric space which is the c-dual of a simply-connected compact symmetric space \(U/K\) where \(U\) is simply-connected. Finally, let \(\mathcal{H}\) be a Hilbert space. There are one-to-one correspondences between the following categories of representations on \(\mathcal{H}\) which preserve the algebras of intertwining operators:

1. Smooth representations of \(G\) on \(\mathcal{H}\)
2. Holomorphic representations of \(U_{\mathbb{C}}\) on \(\mathcal{H}\)
3. Smooth representations of \(U\) on \(\mathcal{H}\)

**Proof.** We will construct the correspondences \((1) \to (2)\) and \((2) \to (1)\). The proofs for \((2) \to (3)\) and \((3) \to (2)\) are identical.

One passes from \((2)\) to \((1)\) quite easily: if \((\pi, \mathcal{H})\) is a holomorphic representation of \(U_{\mathbb{C}}\), then it is clear that \(\pi|_G\) is a smooth representation of \(G\).

To construct \((1) \to (2)\), we suppose that \((\pi, \mathcal{H})\) is a smooth representation of \(G\). We wish to construct a holomorphic representation \(\pi_{\mathbb{C}}\) of \(U_{\mathbb{C}}\) on \(\mathcal{H}\) such that \(\pi_{\mathbb{C}}|_G = \pi\). First we notice that each vector \(v \in \mathcal{H}\) is contained in a finite-dimensional \(G\)-invariant subspace \(W\). Write \(\pi_W^W\) for the subrepresentation of \(\pi\) corresponding to \(W\). By the finite-dimensional Weyl Trick, we see that \(\pi_W^W\) uniquely extends to a holomorphic representation \(\pi_{\mathbb{C}}^W\) of \(U_{\mathbb{C}}\) on \(\mathcal{H}\). We define \(\pi_{\mathbb{C}}(g)v = \pi_{\mathbb{C}}^W(g)v\) for each \(v \in W\) and \(g \in U_{\mathbb{C}}\). If \(V\) and \(W\) are finite-dimensional invariant subspaces of \(\mathcal{H}\) and \(v \in V \cap W\), then the uniqueness of the holomorphic extension shows that \(\pi_{\mathbb{C}}^W(g)v = \pi_{\mathbb{C}}^V(g)v\) and thus \(\pi_{\mathbb{C}}\) is well-defined.
It is clear that $\pi_C$ is a vector space representation of $U_C$ on $\mathcal{H}$, but we must still show that $\pi_C$ acts by bounded operators and that it acts holomorphically. Since $\pi$ is smooth, Theorem 5.6 implies the existence of a Lie algebra representation $d\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ such that

$$\pi(\exp X) = \exp(d\pi(X))$$

for all $X \in \mathcal{B}(\mathcal{H})$. Notice that $d\pi$ uniquely extends to a complex-linear Lie algebra representation $d\pi_C : \mathfrak{u}_C \to \mathcal{B}(\mathcal{H})$ by setting

$$d\pi_C(X + iY) = d\pi(X) + id\pi(Y)$$

for all $X, Y \in \mathfrak{g}$.

By restricting to finite-dimensional invariant subspaces of $\mathcal{H}$ and applying the finite-dimensional Unitary Trick, we verify that

$$\pi_C(\exp X)v = \exp(d\pi_C(X))v$$

(5.2)

for all $X \in \mathfrak{u}_C$ and $v \in \mathcal{H}$. In particular, we see that $\pi_C(g) \in \mathcal{B}(\mathcal{H})$ for all $g \in U_C$ and also that $\pi_C$ is smooth.

Next, we note that $\pi^C$ gives a holomorphic representation on $W$ for every finite-dimensional $U$-invariant subspace of $\mathcal{H}$. Since every vector in $\mathcal{H}$ is contained in such a finite-dimensional invariant subspace, we see that the map

$$U_C \mapsto \mathcal{H}$$

$$g \mapsto \pi_C(g)v$$

is holomorphic for each $v \in V$. Thus $\pi_C$ is holomorphic.

It is clear that the real Lie algebra representation $d\pi$ and the complex Lie algebra representation $d\pi_C$ possess the same algebra of intertwining operators. Thus $\pi$ and $\pi_C$ possess the same algebra of intertwining operators by Lemma 5.16. Furthermore, the uniqueness of the complexification $\pi_C$ follows from its uniqueness on every finite-dimensional invariant subspace of $\mathcal{H}$.

Our infinite-dimensional version of Weyl’s Trick is then an immediate corollary (see [33, Proposition 3.6] for a partial version of this result and a different proof):

**Corollary 5.18.** Suppose that $G_\infty/K_\infty$ is a lim-noncompact Riemannian symmetric spaces which is the c-dual of a lim-compact symmetric space $U_\infty/K_\infty$ where $U_n/K_n$ and $U_n$ are simply-connected for each $n$. Finally, let $\mathcal{H}$ be a Hilbert space. There are one-to-one correspondences between the following categories of representations on $\mathcal{H}$ which preserve the algebras of intertwining operators:

1. Smooth representations of $G_\infty$ on $\mathcal{H}$
2. Holomorphic representations of $(U_\infty)_C$ on $\mathcal{H}$
3. Smooth representations of $U_\infty$ on $\mathcal{H}$

Proof. This corollary follows immediately by applying Theorem 5.17 to representations of $G_n$, $(U_n)_\mathbb{C}$, and $U_n$ on $\mathcal{H}$ for each $n \in \mathbb{N}$.

There is one crucial aspect of the finite-dimensional version of Weyl’s Unitary Trick which we have as yet failed to mention: every smooth (i.e., norm-continuous) Hilbert representation of a compact Lie group is unitarizable. This key property is what gives Weyl’s Trick much of its power, since it allows us to treat finite-dimensional representations of noncompact semisimple Lie groups as if they were unitary. We take a moment, therefore, to explore what can be said about unitarizability of representations of $U_\infty$.

The first thing we note is that the representation $(\pi|_{U_n}, \mathcal{H})$ may be unitarized for each $n \in \mathbb{N}$, because $U_n$ is a compact group. Furthermore, a unitarization of $\pi|_{U_\infty}$ automatically unitarizes the restrictions $\pi|_{U_j}$ for $j \leq n$. However, it is not clear a priori whether or not it is possible to simultaneously unitarize $\pi|_{U_n}$ for all $n \in \mathbb{N}$, which is what would be required in order to unitarize $\pi$.

Recall that the trick we used to show that representations of compact groups are unitarizable was to integrate an inner product over the group using Haar measure. While $U_\infty$ is not locally compact, and thus does not possess a Haar measure, one can show that it possesses the next-best thing:

**Theorem 5.19.** ([47, Proposition 13.6]). Let $\text{UCB}(U_\infty)$ denote the Banach space of uniformly-continuous, bounded functions on $G$, then there is a continuous functional $\mu \in \text{UCB}(U_\infty)^*$ such that

1. $\mu(1) = 1$, where $1$ is the constant-one function
2. $\mu(f) \geq 0$ whenever $f \geq 0$
3. $|\mu(f)| \leq ||f||_\infty$ for all $f \in \text{UCB}(U_\infty)$
4. $\mu(R_g f) = \mu(L_g f) = \mu(f)$ for all $g \in U_\infty$ and $f \in \text{UCB}(U_\infty)$

We say that $\mu$ is an **invariant mean** for $U_\infty$.

Proof. For each $n \in \mathbb{N}$, we define a functional $\mu_n \in \text{UCB}(U_\infty)^*$ by

$$\mu_n(f) = \int_{U_n} f|_{U_n}(g) dg$$

for each $f \in \text{UCB}(U_\infty)$. It is clear that each $\mu_n$ satisfies the first three conditions of an invariant mean. Furthermore, we see that $\mu_n(R_g f) = \mu_n(L_g f) = f$ whenever $g \in U_n \leq U_\infty$. Thus, any weak-* cluster point of the set $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \text{UCB}(U_\infty)^*$ will possess property (4). But by the Banach Alaoglu theorem, the unit ball in $\text{UCB}(U_\infty)^*$ is weak-* compact and thus our sequence must possess a cluster point (property (3) shows that the sequence is contained in the unit ball).
Because $\text{UCB}(U_\infty)$ is not separable, the unit ball in $\text{UCB}(U_\infty)^*$ is not guaranteed to be weak-* sequentially compact. Thus there is no reason to expect that $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \text{UCB}(U_\infty)^*$ will possess a convergent sequence. In fact, an application of the Axiom of Choice is required to construct an invariant mean on $U_\infty$. There are also an uncountable number of distinct invariant means on $U_\infty$, so we are far from the uniqueness properties of Haar measures.

Invariant means in some ways behave as finitely-additive invariant integrals on $U_\infty$. For that reason, we often use the notation

$$\mu(f) = \int_{U_\infty} f(g) d\mu(g),$$

although we must be careful to note that $\mu$ is not in any sense a countably-additive measure on $U_\infty$.

Nevertheless, once a group $G$ possesses an invariant mean, it is possible to use the “integration” trick to show that all uniformly bounded representations of $G$ are unitarizable:

**Theorem 5.20. ([47, Proposition 17.5]).** Suppose that $G$ is an amenable group and that $\pi$ is a uniformly bounded continuous representation of $G$ on a separable Hilbert space $\mathcal{H}$ (that is, $\sup_{g \in U_\infty} ||\pi(g)|| < \infty$). Then $\pi$ is equivalent to a unitary representation.

**Proof.** Let $M = \sup_{g \in U_\infty} ||\pi(g)||$. Clearly, $M = \sup_{g \in U_\infty} ||\pi(g)^{-1}||$; it follows that

$$M^{-1}||u|| \leq ||\pi(g)u|| \leq M||u||$$

for all $g \in U_\infty$.

Now let $\mu$ be a bi-invariant mean on $G$. We denote the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and define a new inner product $\langle \cdot, \cdot \rangle_\mu$ on $\mathcal{H}$ by

$$\langle u, v \rangle_\mu = \int_G \langle \pi(g)u, \pi(g)v \rangle_{\mathcal{H}} d\mu(g)$$

for all $u, v \in \mathcal{H}$. We use the fact that $g \mapsto \langle \pi(g)u, \pi(g)v \rangle_{\mathcal{H}}$ is a uniformly continuous, bounded function on $G$ (since $\pi$ is continuous and uniformly bounded). It is clear that $\langle \cdot, \cdot \rangle_\mu$ provides a positive semi-definite Hermitian form on $\mathcal{H}$.

Note that for $u \in \mathcal{H}$ one has that

$$0 < M^{-2}||u||^2_{\mathcal{H}} \leq ||u||^2_\mu = \int_G ||\pi(g)u||^2_{\mathcal{H}} d\mu(g) < M^2||u||^2_{\mathcal{H}}.$$

Thus $\langle \cdot, \cdot \rangle_\mu$ is strictly positive-definite and continuous with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. 

Taking stock again of our situation, we see that all uniformly-bounded Hilbert representations of $U_\infty$ are unitarizable. Furthermore, if a continuous Hilbert representation $(\pi, \mathcal{H})$ is unitarizable, then $\pi$ is uniformly bounded. In fact, if an invertible bounded intertwining operator $T \in \text{GL}(\mathcal{H})$ unitarizes $\pi$, then we see that $T\pi(g)T^{-1}$ is unitary and thus $||\pi(g)|| < ||T|| ||T^{-1}||$ for all $g \in U_\infty$. 

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Unfortunately, it is not possible to say much more, because it is possible to construct a smooth Hilbert representation of $U_\infty$ which is not unitarizable, as we now show.

Consider the group $U_\infty = SU(\infty) = \lim_{n \to \infty} SU(2n)$. For each $n \in \mathbb{N}$, consider the standard representation $\pi_n$ of $SU(2n)$ on $\mathcal{H}_n = \mathbb{C}^{2n}$ (that is, $\pi_n(g) v = g \cdot v$ for all $g \in SU(2n)$). By taking the direct limit, we may form a unitary representation $\pi = \lim_{n \to \infty} \pi_n$ of $SU(\infty)$ on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{C}) = \lim_{n \to \infty} \mathbb{C}^{2n}$ of square-summable sequences of complex numbers. Note that $SU(2n)$ acts trivially on the orthogonal complement of $\mathcal{H}_n$. It follows that $\pi|_{SU(2n)}$ decomposes into a direct sum of the standard representation $\pi_n$ and infinitely many copies of the trivial irreducible representation. That is, $\pi|_{SU(2n)} = \pi_n \oplus \mathbb{N} \text{Id}_{SU(2n)}$, where $\text{Id}_{SU(2n)}$ denotes the trivial irreducible representation of $SU(2n)$ on $\mathbb{C}$. Thus, by Theorem 5.9, it follows that $\pi|_{SU(2n)}$ is smooth for each $n \in \mathbb{N}$ and hence that $\pi$ is smooth.

Now let $V_1 = \mathcal{H}_1$ and define $V_n = \mathcal{H}_n \ominus \mathcal{H}_{n-1}$ for each $n > 1$. Note that $\dim V_n = 2$ for each $n \in \mathbb{N}$. We now completely discard unitarity and choose some new inner product $\langle , \rangle_{V_n}$ on $V_n$ under which $\|\pi(g)|_{V_n}\| \geq n$ for some $g \in SU(2n)$. For instance, if $\pi(g) v = w$, where $v, w \in V_n$ are linearly independent, then we can choose any inner product $\langle , \rangle_{V_n}$ on $V_n$ such that $\|v\|_{V_n} = 1$ and $\|w\|_{V_n} = n$.

Next we define for each $n \in \mathbb{N}$ the finite-dimensional Hilbert space

$$
\mathcal{K}_n = \bigoplus_{i=1}^{n} V_i,
$$

where each $V_i$ is given the new inner product we just defined. As vector spaces, $\mathcal{K}_n = \mathcal{H}_n$, but they possess different inner products. Now $\{(\pi_n, \mathcal{K}_n)\}_{n \in \mathbb{N}}$ forms a direct system of continuous Hilbert representations. We consider the representation $(\widetilde{\pi}_\infty, \mathcal{K}_\infty) = (\lim_{n \to \infty} \pi_n, \lim_{n \to \infty} \mathcal{K}_n)$. Note that $\pi|_{SU(2n)}$ and $\widetilde{\pi}|_{SU(2n)}$ possess the same irreducible subrepresentations for each $n \in \mathbb{N}$. In particular, $\widetilde{\pi}$ is smooth. Finally, it is clear that $\widetilde{\pi}$ is not uniformly bounded (since $\sup_{g \in SU(2n)} \|\pi(g)\| \geq n$ for each $n \in \mathbb{N}$), and is therefore not unitarizable.

Heuristically, it seems that the smooth Hilbert representations of $U_\infty$ which are not unitarizable have in some sense been given an unnatural or “incorrect” topology. For that reason, we will for the rest of the thesis work only with unitary representations of $U_\infty$ and with smooth representations of $G_\infty$ which correspond to smooth unitary representations of $U_\infty$ under Weyl’s Trick.

### 5.4 Highest-Weight Representations

Now suppose that $G_\infty/K_\infty$ is an admissible lim-noncompact symmetric space which is the c-dual of a lim-compact symmetric space $U_\infty/K_\infty$. We wish to construct irreducible spherical and conical representations for $G_\infty/K_\infty$ and $U_\infty/K_\infty$. 
The most natural way to do this would be to construct a direct limit of spherical/conical representations. The following lemma provides the foundation for this construction and is a generalization of a result proved by Olafsson and Wolf in Lemma 5.8 of [40].

**Theorem 5.21.** Let $U_n/K_n$ be a propagated lim-compact symmetric space such that $U_n/K_n$ is simply connected for each $n \in \mathbb{N}$. Fix indices $n < m$ and dominant weights $\lambda \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ and $\mu \in \Lambda^+(\mathfrak{g}_m, \mathfrak{a}_m)$ such that $\mu|_{\mathfrak{a}_n} = \lambda$. Consider the irreducible spherical representations $(\pi_\mu, \mathcal{H}_\mu)$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U_m$ and $G_n$, respectively, with respective highest weights $\mu$ and $\lambda$. Let $w$ be a highest-weight vector for $\pi_\mu$.

Then the representation of $U_n$ on $W = \langle \pi_\mu(U_n)w \rangle$ is equivalent to $\pi_\lambda$.

**Proof.** For each dominant weight $\nu$ in $\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$, let $w_\nu$ be the orthogonal projection of $w$ onto the space of $\pi_\nu$-isotypic vectors in $W$. Then $w = \sum_\nu w_\nu$ (note that $w_\nu = 0$ for all but finitely many choices of $\nu$).

Write $W_\nu = \langle \pi_\mu(U_n)w_\nu \rangle$ for each $\nu$. Because $W_\nu$ consists of $U_n$-isotypic vectors of type $\nu$, we see that the action of $U_n$ on $W_\nu$ is $U_n$-isomorphic to a direct sum of copies of the irreducible representation $(\pi_\nu, \mathcal{H}_\nu)$ with highest-weight $\nu$.

Since $w$ is a $U_m$-highest-weight vector for $\pi_\mu$, $\pi(M_nN_n)w = w$. In particular, $\pi(M_nN_n)w = w$. Since the space of isotypic vectors in $W$ of type $\pi_\nu$ is invariant under $G_n$, it follows that $w_\nu$ is fixed under $M_nN_n$ for each $\nu \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$. Thus Lemma 3.15 shows that if $w_\nu \neq 0$, then $W_\nu$ is a $U_n$-irreducible subspace of $W$ that is $U_n$-isomorphic to $\mathcal{H}_\nu$ and that $w_\nu$ is a highest-weight vector for $W_\nu$. In particular, $w_\nu$ is a weight vector of weight $\nu$.

On the other hand, since $w$ is a $U_m$-weight vector of weight $\mu$, it follows that it is a $U_n$-weight vector of weight $\lambda = \mu|_{\mathfrak{a}_n}$. But we also have that $w = \sum_\nu w_\nu$, where each $w_\nu$ is a weight vector of weight $\nu$. Hence $w = w_\lambda$ and $W = W_\lambda$, and so we are done. \qed

We follow the construction in [52, p. 464–466], and more details may be found at that source. For each $n$, we denote the set of fundamental weights by $\xi_{n,1}, \ldots, \xi_{n,r_n}$, where $r_n = \dim \mathfrak{a}_n$ and where we have numbered the fundamental weights according to the roots as in Section 4.4. Suppose $k \leq n$. One can show that

$$\xi_{n,i}|_{\mathfrak{a}_k} = \xi_{k,i}$$  \hspace{1cm} (5.3)

for all $n \in \mathbb{N}$ and $i \leq r_k$. Furthermore, one can check that $\xi_{n,i}|_{\mathfrak{a}_k} = 0$ for $r_k < i \leq r_n$. Thus

$$\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n) = \mathbb{N}\xi_{n,1} + \cdots + \mathbb{N}\xi_{n,r_n} = \left\{ \sum_{j=1}^{r_n} c_j \xi_{n,j} \mid c_j \in \mathbb{N} \right\}$$  \hspace{1cm} (5.4)

and

$$\left( \sum_{j=1}^{r_n} c_j \xi_{n,j} \right)_{\mathfrak{a}_k} = \left( \sum_{j=1}^{r_k} c_j \xi_{k,j} \right) \in \Lambda^+(\mathfrak{g}_k, \mathfrak{a}_k)$$  \hspace{1cm} (5.5)

whenever $k \leq n$. 

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We can thus form a projective limit

$$\Lambda^+ \equiv \Lambda^+(g_\infty, a_\infty) = \lim_{\longleftarrow} \Lambda^+(g_n, a_n).$$

We say that $\Lambda^+(g_\infty, a_\infty)$ is the set of dominant integral weights for the restricted root system $\Sigma(g_\infty, a_\infty)$. That is, $\Lambda^+$ consists of the elements $\lambda$ of $a_\infty^*$ such that $\lambda_{|a_n}$ is dominant and integral for every $n$. Notice that (5.3) implies that for each $i \in \mathbb{N}$ there is a weight $\xi_i \in a^*_\infty$ such that $\xi_i|a_n = \xi_{n,i}$ for each $n \in \mathbb{N}$.

If $\dim a_\infty = \infty$, then (5.4) and (5.4) imply that $\Lambda^+(g_\infty, a_\infty)$ is equal to the set of formal sums $\sum_{i \in \mathbb{N}} c_i \xi_i$ where $(c_i) \in \mathbb{N}$ is any sequence in $\mathbb{N}$. On the other hand, if $a_\infty$ is finite-dimensional, say with dimension $r$, then $\Lambda^+(g_\infty, a_\infty)$ is equal to the set of sums $\sum_{i=1}^r c_i \xi_i$ where $c_1, \ldots, c_r \in \mathbb{N}$.

Just as in the finite-dimensional case, weights in $\Lambda^+$ can be used to create highest-weight representations of $U_\infty$. To see this, fix $\mu \in \Lambda^+$. For $n \in \mathbb{N}$, let $(\pi_{\mu_n}, \mathcal{H}_{\mu_n})$ be the irreducible representation of $U_n$ with highest weight $\mu_n \equiv \mu|_{a_n}$, and let $v_n \in \mathcal{H}_{\mu_n}$ be a nonzero highest-weight vector. By Theorem 5.21, we see that $\pi_{\mu_n}$ may be embedded unitarily into $\pi_{\mu_{n+1}}$ by identifying the respective highest-weight vectors $v_n$ with $v_{n+1}$. The corresponding unitary representation of $U_\infty$ constructed by the direct limit of $\pi_{\mu_n}, n \in \mathbb{N}$ is denoted by

$$(\pi_{\mu}, \mathcal{H}_{\mu}) = \left( \lim_{\longleftarrow} \pi_{\mu_n}, \lim_{\longleftarrow} \mathcal{H}_{\mu_n} \right),$$

where $\mathcal{H}_{\mu} = \lim_{\longleftarrow} \mathcal{H}_{\mu_n}$ is the Hilbert completion of the algebraic direct limit $\lim_{\longleftarrow} \mathcal{H}_{\mu_n}$ of Hilbert spaces. We refer to $\pi_{\mu}$ as the **highest-weight representation with highest weight** $\mu$. Note that a direct limit of irreducible representations of $U_n$ is an irreducible representation of $U_\infty$ by 5.2.

If $\dim a_\infty = \infty$, then we can write elements of $a^*$ as sequences $(a_i) \in \mathbb{Z}$ of integers, so that a sequence $(a_i) \in \mathbb{Z}$ corresponds to the formal sum $\sum_{i \in \mathbb{N}} a_i e_i \in a^*_\infty$. We now use this notation to write down the fundamental weights for $\Sigma(g_\infty, a_\infty)$ for some infinite Dynkin-diagram types.

If $\Sigma(g_\infty, a_\infty)$ has type $A_\infty$, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots)$$

where the first $i$ entries in $\xi_i$ are zeros.

If $\Sigma(g_\infty, a_\infty)$ has type $B_\infty$, then

$$\xi_1 = (1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, \ldots) \text{ for } i > 1,$$

where the first $i - 1$ entries in $\xi_i$ are zero for $i > 1$.

If $\Sigma(g_\infty, a_\infty)$ has type $C_\infty$, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots),$$

where the first $i - 1$ entries in $\xi_i$ are zero.
If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type $D_\infty$, then

$$\xi_1 = (1, 1, 1, \ldots), \xi_2 = (-1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, 2, \ldots) \text{ for } i \geq 3,$$

where the first $i-1$ entries in $\xi_i$ are zero for $i \geq 3$.

By examining the fundamental weights in each case and extending them to weights on $\mathfrak{h}_\infty$, it follows from the boundedness condition in Theorem 5.13 that a highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ will be smooth if and only if we can write $\lambda$ as a finite linear combination

$$\lambda = \sum_{i=1}^{n} c_i \xi_i,$$

where $c_i \in \mathbb{N}$ for each $n$. In particular, if $\dim \mathfrak{a}_\infty < \infty$, then every highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is smooth.

### 5.5 Spherical Representations for Lim-Compact Symmetric Spaces

In preparation for our study of conical representations, we end this chapter by reviewing the main result of our earlier paper [7], which concerned spherical representations for propagated lim-compact symmetric spaces.

Suppose that $U_\infty/K_\infty$ is a lim-compact symmetric space (as usual, we assume that $U_n/K_n$ is simply-connected for each $n \in \mathbb{N}$ for the sake of clarity). The definitions of spherical representations and spherical functions are entirely analogous to the definitions for finite-dimensional symmetric spaces.

**Definition 5.22.** A continuous unitary representation $(\pi, \mathcal{H})$ of $U_\infty$ is said to be $(K_\infty)$-spherical if there is a nonzero cyclic vector $v \in \mathcal{H}$ such that $\pi(K_n)v = v$ for each $n \in \mathbb{N}$.

**Definition 5.23.** (See [13]) A continuous, bi-$K_\infty$-invariant function $\phi : U_\infty \to \mathbb{C}$ is said to be a spherical function if

$$\phi(x)\phi(y) = \lim_{n \to \infty} \int_{K_n} \phi(xky)dk$$

for all $x, y \in U_\infty$.

It is natural to ask whether one may form an irreducible $K_\infty$-spherical representation of $U_\infty$ merely by taking a direct limit of irreducible unitary spherical representations of the $K_n$'s. The most appealing candidates would be the unitary highest-weight representations constructed in the previous section. In [7] we showed that this scheme only works for certain symmetric spaces:

**Theorem 5.24.** ([7, Theorem 4.5]) Let $\mu \in \Lambda^+(\mathfrak{u}_\infty, \mathfrak{t}_\infty)$ and consider the corresponding unitary highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ of $U_\infty$. (Recall that $\pi_\mu$ was constructed as a direct limit of spherical representations.) Then $\pi_\mu$ is a spherical representation if and only if

$$\text{Rank } U_\infty/K_\infty = \dim \mathfrak{a}_\infty < \infty,$$
that is, if $U_\infty/K_\infty$ is a symmetric space with a finite rank.

In the case that $U_\infty/K_\infty$ has finite rank, the function $\phi_\mu : U_\infty \to \mathbb{C}$ defined by

$$\phi_\mu(g) = \langle e, \pi(g)e \rangle,$$

where $e \in H^K$ is a unit vector, is a positive-definite spherical function.

As a side note, the only classical finite-rank lim-compact symmetric spaces are the finite-rank Grassmannian spaces $\text{SO}(p + \infty)/\text{SO}(p) \times \text{SO}(\infty)$, $\text{SU}(p + \infty)/\text{SU}(p) \times \text{SU}(\infty)$, and $\text{Sp}(p + \infty)/\text{Sp}(p) \times \text{Sp}(\infty)$, which correspond to the space of $p$-dimensional subspaces of $\mathbb{R}^\infty$, $\mathbb{C}^\infty$, and $\mathbb{H}^\infty$, respectively. The other classical lim-compact symmetric spaces in Table 4.4 all have infinite rank.

Theorem 5.24 demonstrates that there is a striking difference in behavior between finite-rank lim-Riemannian symmetric spaces and infinite-rank lim-Riemannian symmetric spaces, and we shall note this divergence of behavior again in the next chapter.

Finally, we note that for the case of a finite-rank lim-compact symmetric space $U_\infty/K_\infty$, the classification of spherical functions in [48] implies that the highest-weight representations $(\pi_\mu, \mathcal{H}_\mu)$ with highest-weight $\mu \in \Lambda^+(g_\infty, \mathfrak{k}_\infty)$ exhaust all irreducible spherical representations of $U_\infty$. 
Chapter 6
Conical Representations for Admissible Direct Limits

This chapter contains the main results of the thesis. In the first section, we give a natural definition for conical representations of admissible lim-noncompact symmetric spaces $G_\infty/K_\infty$. As before, we assume that $G_\infty/K_\infty$ is the c-dual of a propagated lim-compact symmetric space $U_\infty/K_\infty$. By using the generalization of Weyl’s Unitary Trick from the previous chapter, each smooth cyclic representation of $U_\infty$ gives rise to a smooth cyclic representation of $G_\infty$, and it is natural to say that a smooth cyclic representation of $U_\infty$ is conical if the corresponding representation of $G_\infty$ is conical.

In fact, we will see that in some cases it is possible to define nonsmooth unitary representations of $U_\infty$ which are conical but do not correspond to continuous Hilbert representations of $G_\infty$. This is a strange situation which does not occur in the finite-dimensional case.

With these definitions, we classify all of the irreducible cyclic unitary representations of $U_\infty$ which are conical. Next we see that smooth conical unitary representations of $U_\infty$ decompose into a discrete direct sum of highest-weight representations.

Combining our results with Theorem 5.24, we will show that, if $\text{Rank } U_\infty/K_\infty = \infty$, then there are no smooth unitary representations of $U_\infty$ which are both spherical and conical. On the other hand, if $\text{Rank } U_\infty/K_\infty < \infty$, then we will see that a smooth irreducible unitary representation of $U_\infty$ is spherical if and only if it is conical. This situation is also in stark contrast to the situation for finite-dimensional symmetric spaces, for which finite-dimensional representations are spherical if and only if they are conical.

In the final section, we show how to disintegrate (possibly nonsmooth) conical representations into direct integrals of irreducible representations by integrating over a set of paths in a tree of highest weights. We also show that cyclic conical representations are always multiplicity-free representations (and hence are Type I representations).

6.1 Definition of Conical Representations

Let $G_\infty/K_\infty$ be the c-dual of a propagated lim-compact symmetric space $U_\infty/K_\infty$ such that $U_n/K_n$ and $U_n$ are simply-connected for each $n$ and assume that $G_\infty/K_\infty$ is admissible. We begin by defining unitary conical representations of $G_\infty$:

Definition 6.1. Suppose that $(\pi, \mathcal{H})$ is a unitary representation of $G_\infty$. We say that $\pi$ is conical if there is a cyclic distribution vector $v \in \mathcal{H}^{-\infty}$ such that $\pi(mn)v = v$ for all $m \in M_\infty$ and $n \in \mathbb{N}_\infty$.

However, we are primarily concerned in this thesis with smooth conical representations (though unitary conical representations of $G_\infty$ are an area of interest for further study):
Definition 6.2. Suppose that \((\pi, \mathcal{H})\) is a smooth Hilbert representation of \(G_\infty\). We say that \(\pi\) is \textbf{conical} if there is a cyclic vector \(v \in \mathcal{H}\) such that \(\pi(mn)v = v\) for all \(m \in M_\infty\) and \(n \in N_\infty\). In that case, we say that \(v\) is a \textbf{conical vector} for \(\pi\).

For finite-dimensional symmetric spaces, it is possible to consider a finite-dimensional conical representation to be a representation of either \(G\) or \(U\) (where \(G/K\) is the c-dual of the compact symmetric space \(U/K\)). On the one hand, the harmonic analysis applications of conical representations appear on the horocycle space \(G/MN\), so in a certain sense it is most natural to speak of conical representations of \(G\). On the other hand, these representations are only unitary if we move to the compact group \(U\). Similarly, in order to study smooth conical representations of \(G_\infty\), we shall take the roundabout approach of instead studying unitary conical representations of \(U_\infty\).

The generalization of Weyl’s unitary trick from the previous chapter shows that smooth cyclic representations of \(U_\infty\) correspond to smooth cyclic representations of \(G_\infty\). It is therefore natural to consider smooth cyclic representations of \(U_\infty\) which correspond to conical representations of \(G_\infty\). In that sense it is natural to speak of \textbf{conical representations of \(U_\infty\)}:

Definition 6.3. Let \((\pi, \mathcal{H})\) be a smooth Hilbert representation of \(U_\infty\), and note that it extends to a smooth representation of \(G_\infty\), which we denote also by \(\pi\). We say that \(\pi\) is \textbf{conical} if there is a cyclic vector \(v \in \mathcal{H}\) such that \(\pi(mn)v = v\) for all \(m \in M_\infty\) and \(n \in N_\infty\).

However, most of our machinery is only useful for \textit{unitary} representations of \(U_\infty\). We therefore need a definition of conical representations of \(U_\infty\).

Definition 6.4. A unitary representation \((\pi, \mathcal{H})\) of \(U_\infty\) is said to be \textbf{conical} if there is a nonzero cyclic vector \(v \in \mathcal{H}^{\text{fin}}\) such that \(\pi(M_nN_n)v = v\) for all \(n \in \mathbb{N}\). In that case, we say that \(v\) is a \textbf{conical vector} for \(\pi\).

Just as was the case for \(G_\infty\), notice that we do not require that unitary conical representations of \(U_\infty\) be smooth. This opens the door to the possibility of constructing conical representations of \(U_\infty\) which do not correspond to conical representations of \(G_\infty\). However, we will eventually determine which unitary conical representations are smooth (and therefore do correspond to smooth conical representations of \(G_\infty\)).

6.2 Classification of Conical Representations

In this section we begin to classify the unitary conical representations of \(U_\infty\). We determine which representations are irreducible and show how conical representations decompose into subrepresentations.

Theorem 6.5. Suppose that \(U_\infty/K_\infty\) is a propagated lim-compact symmetric space with \(U_n\) and \(U_n/K_n\) simply-connected for each \(n\) and such that the c-dual \(G_\infty/K_\infty\) is admissible. Suppose further that \((\pi, \mathcal{H})\) is a conical representation with a conical vector \(v\). For each \(n\), write \(\Gamma_n(\pi, v)\) for the set of highest weights \(\mu\) in \(\Lambda^+(u_n, a_n)\)
such that the projection $v_\mu = \text{pr}_{H_\mu} v$ of $v$ onto the space of $U_n$-isotypic vectors of type $\mu$ is nonzero. Then

1. For each $n \in \mathbb{N}$ and $\mu \in \Gamma_n(\pi, v)$, the action of $U_\infty$ on $\langle \pi(U_\infty)v_\mu \rangle$ gives a conical representation of $U_\infty$ with conical vector $v_\mu$.

2. $\pi$ decomposes into an orthogonal direct sum of disjoint conical representations as follows:

$$\mathcal{H} = \langle \pi(U_\infty)v \rangle = \bigoplus_{\mu \in \Gamma_n(\pi, v)} \langle \pi(U_\infty)v_\mu \rangle$$

3. If $\pi$ is irreducible, then $\pi$ is equivalent to a highest-weight representation $\pi_\mu$ for some $\mu \in \Lambda^+(\mathfrak{g}_\infty, a_\infty)$.

4. If $\pi$ is irreducible, then $\dim \mathcal{H}^{M_\infty N_\infty} = 1$.

**Proof.** For each $n \in \mathbb{N}$, the set $\Gamma_n(\pi, v)$ is finite because $v$ is $U_n$-finite for all $n$. Then the decomposition of $v$ into $U_n$-isotypic vectors may be written

$$v = \sum_{\mu \in \Gamma_n(\pi, v)} v_\mu,$$

where $v_\mu = \text{pr}_{H_\mu} v$. Since each isotypic subspace is $U_n$-invariant, it follows that $v_\mu \in H^{M_n N_n}$ for each $\mu \in \Gamma_n(\pi, v)$. Note that $\langle \pi(U_n)v_\mu \rangle$ gives a primary representation of $U_n$ of type $\mu$. Hence, by Lemma 3.15, it is an irreducible representation with highest-weight vector $v_\mu$.

We repeat the same process for $U_{n+1}$, writing the decomposition of $v$ into $U_{n+1}$-isotypic vectors as

$$v = \sum_{\lambda \in \Gamma_{n+1}(\pi, v)} v_\lambda \quad (6.1)$$

By Theorem 5.21 it follows for each $\lambda \in \Gamma_{n+1}(\pi, v)$ that $\langle \pi(U_n)v_\lambda \rangle$ is a $U_n$-irreducible subspace for which $v_\lambda$ is a highest-weight vector of weight $\lambda|_{h_n}$. In other words, $v_\lambda$ is also a $U_n$-isotypic vector, so $\lambda|_{h_n} \in \Gamma_n(\pi, v)$. Furthermore, since (6.1) is a decomposition of $v$ into $U_n$- and $U_{n+1}$-isotypic vectors, we see that for each $\mu \in \Gamma_n(\pi, v)$ there is $\lambda \in \Gamma_{n+1}(\pi, v)$ such that $\lambda|_{h_n} = \mu$.

In other words, if we consider all the highest weights of irreducible subrepresentations $\pi(U_n)$ and allow $n \in \mathbb{N}$ to vary, then the highest weights may be naturally arranged into a tree, as in Figure 6.1.

Next we prove (1). First note that $V_\lambda = \langle \pi(U_\infty)v_\lambda \rangle$ is a $U_\infty$-invariant subspace of $\mathcal{H}$ for each $\lambda \in \Gamma_n(\pi, v)$. Suppose $m > n$, and write

$$u_\lambda = \sum_{\nu \in \Gamma_m(\pi, v) \text{ s.t. } \nu|_{a_n} = \lambda} v_\nu$$

for each $\lambda \in \Gamma_n(\pi, v)$. Then $u_\lambda$ is a $U_n$-isotypic vector of type $\lambda$. Because $v = \sum_{\nu \in \Gamma_m(\pi, v)} v_\nu$, we see that $v = \sum_{\lambda \in \Gamma_n(\pi, v)} u_\lambda$ since every $U_m$-highest-weight vector
\[ v_\nu \text{ appears as a summand in exactly one } u_\lambda. \] Since \( v = \sum_{\lambda \in \Gamma_n(\pi, v)} v_\lambda \) is also a decomposition of \( v \) into \( U_n \)-isotypic vectors, it follows that \( v_\lambda = u_\lambda \) for each \( \lambda \in \Gamma_n(\pi, v) \). In particular, \( v_\lambda \) is \( M_m N_m \)-invariant for all \( m \geq n \). It follows that \( V_\lambda = \langle \pi(U_\infty) v_\lambda \rangle \) gives a conical representation of \( U_\infty \), proving (1).

To prove (2), we need to show that \( V_{\mu_1} \perp V_{\mu_2} \) for all \( \mu_1 \neq \mu_2 \) in \( \Gamma_n(\pi, v) \). It is sufficient to show that \( V_{\mu_1}^m = \langle \pi(U_m) v_{\mu_1} \rangle \) and \( V_{\mu_2}^m = \langle \pi(U_m) v_{\mu_2} \rangle \) are orthogonal for all \( m \). We apply Lemma 5.10 to see that

\[ \langle \pi(U_m) v_\lambda \rangle = \bigoplus_{\nu \in \Gamma_m(\pi, v) \text{ s.t. } \nu|_{\Gamma_n} = \lambda} \langle \pi(U_m) v_\nu \rangle. \]

It follows that \( \langle \pi(U_m) v_{\mu_1} \rangle \) and \( \langle \pi(U_m) v_{\mu_2} \rangle \) are orthogonal for all \( m \) and hence that \( V = \bigcup_m \langle \pi(U_m) v_{\mu_1} \rangle \) and \( W = \bigcup_m \langle \pi(U_m) v_{\mu_2} \rangle \) are orthogonal \( G \)-invariant subspaces of \( \mathcal{H} \), proving (2). Figure 6.2 demonstrates how the decomposition of \( U_m \)-representations matches the tree structure of the highest weights that was exhibited in Figure 6.1.

To prove (3), we assume that \( \pi \) is irreducible. Suppose that there is \( n \) such that \( \# \Gamma_n(\pi, v) > 1 \) (that is, there is more than one \( U_m \)-highest weight in \( \pi|_{U_m} \)). Then (2) produces orthogonal, nonzero invariant subspaces of \( \mathcal{H} \), which contradicts the assumption that \( \pi \) is irreducible. Hence \( \# \Gamma_n(\pi, v) = 1 \) for all \( m \).

For each \( n \), let \( \mu_n \) refer to the single element of \( \Gamma_n(\pi, v) \). From this it follows that \( v \) is a \( U_m \)-highest-weight vector of weight \( \mu_m \) for each \( m \) with the property that \( \mu_m|_{\Gamma_n} = \mu_n \) for \( m \geq n \). Furthermore, \( V_n = \langle \pi(U_n) v \rangle \) is a \( U_n \)-irreducible subspace of \( \mathcal{H} \) for each \( n \), and we can write \( \pi = \lim_{n \to \infty} \pi_n \), where \( \pi_n \) is the representation of \( U_n \) on \( V_n \) induced by \( \pi \). Thus \( \pi \) is a highest-weight representation and (3) is proved.

To prove that \( \dim \mathcal{H}^{M_m N_m} = 1 \), suppose that \( v \) and \( w \) are nonzero conical vectors for \( \pi \) such that \( v \perp w \). Write \( V_n = \langle \pi(U_n) v \rangle \) and \( W_n = \langle \pi(U_n) w \rangle \) for each \( n \). We
Figure 6.2. Example of a decomposition of $\langle \pi(U_n)v \rangle$ into $U_n$-isotypic subspaces (direct sums are taken vertically)

$$
\begin{align*}
\langle \pi(U_1)v \rangle & \quad \langle \pi(U_2)v \rangle & \quad \langle \pi(U_3)v \rangle & \quad \ldots \\
\oplus & & \oplus & \\
\langle \pi(U_1)v^1_1 \rangle & \quad \langle \pi(U_2)v^2_1 \rangle & \quad \langle \pi(U_3)v^3_1 \rangle & \quad \ldots \\
\oplus & & \oplus & \\
\langle \pi(U_1)v^1_2 \rangle & \quad \langle \pi(U_2)v^2_2 \rangle & \quad \langle \pi(U_3)v^3_2 \rangle & \quad \ldots \\
\oplus & & \oplus & \\
\langle \pi(U_2)v^2_3 \rangle & \quad \langle \pi(U_3)v^3_3 \rangle & \quad \ldots \\
\oplus & & \oplus & \\
\langle \pi(U_3)v^3_4 \rangle & \quad \ldots \\
\end{align*}
$$

see that $V_n$ and $W_n$ are both equivalent to $\pi_{\mu_n}$ and have $v$ and $w$ as respective highest-weight vectors. By Lemma 3.15, it follows that $V_n \perp W_n$ for each $n$. Hence $v$ and $w$ generate nonzero, orthogonal invariant subspaces of $\mathcal{H}$, contradicting the irreducibility of $\pi$.

Notice that the maps $p^{n+1}_n : \Gamma_{n+1}(\pi, v) \rightarrow \Gamma_n(\pi, v)$ defined by $p_n(\lambda) = \lambda|_{a_n}$ define a projective system. We refer to the set $\Gamma(\pi, v) = \lim \Gamma_n(\pi, v) \subseteq \Lambda^+(u_{\infty}, a_{\infty})$ as the highest-weight support of $\pi$. If we arrange the highest weights in a tree as in Figure 6.1, then we see that elements of $\Gamma(\pi, v)$ correspond to infinite paths.

We now examine the connection between conical and spherical representations of $G$. Recall that for a finite-dimensional Riemannian symmetric space the irreducible finite-dimensional conical and spherical representations are identical. The situation is much different for infinite-dimensional symmetric spaces, as the following corollary shows.

**Corollary 6.6.** If $\text{Rank}(U_{\infty}/K_{\infty}) < \infty$, then a unitary irreducible representation is spherical if and only if it is conical. If $\text{Rank}(U_{\infty}/K_{\infty}) = \infty$, then no unitary irreducible representation is both spherical and conical.

**Proof.** By part (3) of Theorem 6.5, we see that the irreducible conical representations are precisely the highest-weight representations of $U_{\infty}$ with highest weight $\mu \in \Lambda^+(U_{\infty}, K_{\infty})$. By Theorem 5.24, it follows that these highest-weight representations of $U_{\infty}$ are spherical if and only if $\text{Rank}(U_{\infty}/K_{\infty}) < \infty$. Furthermore, if $\text{Rank}(U_{\infty}/K_{\infty}) < \infty$, then the spherical representations of $U_{\infty}$ are exhausted by the irreducible highest-weight representations. \qed
6.3 Highest-Weight Supports of Conical Representations

In this section we explore some of the properties of the highest-weight trees associated with conical representations. These trees form an invariant for conical representations, but as we shall see it is possible for two distinct conical representations to possess the same highest-weight tree.

First we show that the tree set of a conical representation is independent of the choice of conical vector:

**Theorem 6.7.** Let \((\pi, \mathcal{H})\) be a unitary conical representation of \(U_\infty\). Then \(\Gamma_n(\pi, v) = \Gamma_n(\pi, w)\) for any conical vectors \(v, w\) in \(\mathcal{H}\).

**Proof.** Suppose that both \(v\) and \(w\) are conical vectors in \(\mathcal{H}\) and that \(\mu \in \Gamma_n(\pi, w)\) but \(\mu \notin \Gamma_n(\pi, v)\). Write \(w_\mu\) for the projection of \(w\) onto the \(\mu\)-isotypic vectors in \(\mathcal{H}\). Since \(\mu \in \Gamma_n(\pi, w)\), it follows that \(w_\mu \neq 0\). Define \(W = \langle \pi(U_\infty)w_\mu \rangle\) and \(V = \langle \pi(U_\infty)v \rangle\). We claim that \(W \perp V\), which will be a contradiction since \(V\) is dense in \(\mathcal{H}\).

Note that \(W = \bigcup_{m \geq n} \langle \pi(U_m)w_\mu \rangle\) and \(V = \bigcup_{m \geq n} \langle \pi(U_m)v \rangle\). It is sufficient to show that \(\langle \pi(U_m)w_\mu \rangle \perp \langle \pi(U_m)v \rangle\) for \(m \geq n\). As before, we see from Lemma 3.15 and Theorem 5.21 that

\[
\langle \pi(U_m)v \rangle = \bigoplus_{\lambda \in \Gamma_m(\pi, v)} \langle \pi(U_m)v_\lambda \rangle \cong_{U_m} \bigoplus_{\lambda \in \Gamma_m(\pi, v)} \mathcal{H}_\lambda
\]

and

\[
\langle \pi(U_m)w_\mu \rangle = \bigoplus_{\nu \in \Gamma_m(\pi, w)} \langle \pi(U_m)w_\nu \rangle \cong_{U_m} \bigoplus_{\nu \in \Gamma_m(\pi, w)} \mathcal{H}_\nu,
\]

where \(\Gamma_m(\pi, w) = \{\nu \in \Gamma_m(\pi, w) \text{ s.t. } \nu|_{\mathfrak{a}_n} = \mu\}\).

Fix \(m \geq n\). Since \(\mu \notin \Gamma_n(\pi, v)\), it follows that \(\lambda|_{\mathfrak{a}_n} \neq \mu\) for all \(\lambda \in \Gamma(\pi, v)\). Thus \(\Gamma_m(\pi, v)\) and \(\Gamma_m(\pi, w)\) are disjoint. This means that \(\langle \pi(U_m)v_\lambda \rangle \perp \langle \pi(U_m)w_\nu \rangle\) for each \(\lambda \in \Gamma_m(\pi, v)\) and \(\nu \in \Gamma_m(\pi, w)\). Hence \(\langle \pi(U_m)v \rangle \perp \langle \pi(U_m)w_\mu \rangle\) for all \(m\), as we wanted to show. \(\square\)

From now on, we write \(\Gamma_n(\pi) = \Gamma_n(\pi, v)\) and \(\Gamma(\pi) = \lim_{\downarrow \infty} \Gamma_n(\pi, v)\), where \(v\) is any conical vector of a conical representation \(\pi\) of \(U_\infty\).

**Corollary 6.8.** Let \((\pi, \mathcal{H})\) and \((\rho, \mathcal{K})\) be unitary conical representations of \((U_\infty, K_\infty)\). If there is \(n \in \mathbb{N}\) such that \(\Gamma_n(\pi) \neq \Gamma_n(\rho)\), then \(\pi \not\cong \rho\).

In particular, we have shown that having the same highest-weight tree is a necessary condition for two conical representations to be equivalent. Later we will provide examples of inequivalent conical representations with the same highest-weight trees. However, two conical representations with the same highest-weight trees are nonetheless almost equivalent in a certain sense, as the following theorem shows.

**Theorem 6.9.** Let \((\pi, \mathcal{H})\) and \((\rho, \mathcal{K})\) be conical representations of \((U_\infty, K_\infty)\) with respective conical vectors \(v\) and \(w\) such that \(\Gamma_n(\pi) = \Gamma_n(\rho)\) for each \(n\). Consider
\[ V = \langle \pi(U_\infty)v \rangle \text{ and } W = \langle \rho(U_\infty)w \rangle. \] Write \( \pi_V \) and \( \rho_W \) for the representations of \( U_\infty \) given by restricting \( \pi \) and \( \rho \) to the dense invariant subspaces \( V \) and \( W \) of \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Then

1. \( \pi_V \cong \rho_W \)
2. \( \pi|_{U_n} \cong \rho|_{U_n} \) for each \( n \).

Proof. We begin by proving (1). We claim that the map \( L : V \to W \) induced by \( \pi(g)v \mapsto \rho(g)w \) is a well-defined invertible \( U_\infty \)-intertwining operator.

As before, write \( V_m = \langle \pi(U_m)v \rangle \) and \( W_m = \langle \pi(U_m)w \rangle \), so that \( V = \bigcup_{m \geq n} V_m \) and \( W = \bigcup_{m \geq n} W_m \). Then

\[
V_m = \bigoplus_{\lambda \in \Gamma_m} \langle \pi(U_m)v_\lambda \rangle \cong U_m \bigoplus_{\lambda \in \Gamma_m} \mathcal{H}_\lambda
\]

and

\[
W_m = \bigoplus_{\lambda \in \Gamma_m} \langle \rho(U_m)w_\lambda \rangle \cong U_m \bigoplus_{\lambda \in \Gamma_m} \mathcal{H}_\lambda,
\]

where \( \Gamma_m = \Gamma_m(\pi) = \Gamma_m(\rho) \). Thus \( V_m \) and \( W_m \) are \( U_m \)-isomorphic. We must show that there is an invertible \( U_m \)-intertwining operator \( L^m : V_m \to W_m \) that maps \( v \) to \( w \).

In fact, we note that for each \( \lambda \in \Gamma_m \) there is a (not necessarily unitary) \( U_m \)-intertwining operator \( L_\lambda : \langle \pi(U_m)v_\lambda \rangle \to \langle \rho(U_m)w_\lambda \rangle \) given by \( \pi(g)v_\lambda \mapsto \rho(g)w_\lambda \).

We can then define

\[
L^m = \bigoplus_{\lambda \in \Gamma_m} L_\lambda : V_m = \bigoplus_{\lambda \in \Gamma_m} \langle \pi(U_m)v_\lambda \rangle \to \bigoplus_{\lambda \in \Gamma_m} \langle \rho(U_m)w_\lambda \rangle = W_m.
\]

Hence \( L^m v = L^m(\sum_{\lambda \in \Gamma_m} v_\lambda) = \sum_{\lambda \in \Gamma_m} w_\lambda = w \).

Since \( v \) and \( w \) are cyclic vectors in \( V_m \) and \( W_m \), respectively, \( L^m \) is in fact uniquely determined as an intertwining operator by the fact that it maps \( v \) to \( w \). In particular, \( L^n|_{V_n} = L^n \) for all \( n \leq m \). Thus the family \( \{ L^m \}_{m \in \mathbb{N}} \) is a direct system of intertwining operators that induces a continuous \( U_\infty \)-intertwining operator

\[
L : V = \lim_{\to} V_m \to \lim_{\to} W_m = W
\]
such that \( L v = w \).

Next we prove (2). Fix \( n \in \mathbb{N} \). Define \( \tilde{V}_n = V_n \) and \( \tilde{V}_m = V_m \ominus V_{m-1} \) for \( m > n \), where the orthogonal complement is taken with respect to the Hilbert space structure inherited by \( V_n \) as a closed subspace of \( \mathcal{H} \). Notice that \( \tilde{V}_m \) is a finite-dimensional \( U_n \)-invariant subspace of \( \mathcal{H} \) for each \( m \geq n \). We define \( U_n \)-invariant spaces \( \tilde{W}_m \subseteq \mathcal{K} \) for each \( m \geq n \) in exactly the same way.

Recall that \( V_m \) and \( W_m \) give equivalent representations of \( U_n \) for each \( m \geq n \) under the intertwining operator \( L^m \). It follows that \( \tilde{V}_m = V_m \ominus V_{m-1} \) and \( \tilde{W}_m = W_m \ominus W_{m-1} \) are \( U_n \)-isomorphic for all \( m > n \). Note that

\[
\mathcal{H} = \bigoplus_{m \geq n} \tilde{V}_m \text{ and } \mathcal{K} = \bigoplus_{m \geq n} \tilde{W}_m.
\]
where the direct sums are orthogonal. Since there is a unitary $U_n$ intertwining operator between $\tilde{V}_m$ and $\tilde{W}_m$ for all $m \geq n$, it follows that there is a unitary $U_n$-intertwining operator between $\mathcal{H}$ and $\mathcal{K}$.

6.4 Smooth Conical Representations

Next we consider smooth conical representations of $U_\infty$. These are of interest because they are precisely the conical representations which extend to smooth conical representations of the c-dual $G_\infty$. Our next theorem classifies the smooth representations.

**Theorem 6.10.** Suppose that $(\pi, \mathcal{H})$ is a smooth conical representation of $U_\infty$. Then $\pi$ decomposes into a direct sum of irreducible smooth highest-weight representations.

**Proof.** Let $v$ be a conical vector for $\pi$. For each $U_n$, write

$$v = \sum_{\lambda \in \Gamma_n(\pi)} v_\lambda$$

as before. As in Section 4.4, we recursively construct a countable basis $\{e_i\}_{n \in \mathbb{N}}$ for $a_\infty$ such that $\{e_1, \ldots, e_{r_n}\}$ is a basis for $a_n$ for each $n$. For each $\lambda \in a_n^*$, write

$$||\lambda||_\infty = \max_{1 \leq i \leq n} |\lambda(e_i)|.$$

In fact, if $\lambda \in A^+(g_n, a_n)$ and $\lambda = \sum_{i=1}^{r_n} a_i e_i$, then we see from the data in Section 4.4 that $a_i \leq a_j$ when $i \leq j$; thus $||\lambda||_\infty = a_{r_n}$.

For each $\mu \in \Gamma_n(\pi)$, let $\Gamma_{n+1}(\pi) = \{\lambda \in \Gamma_{n+1}(\pi) : \lambda|_{a_n} = \mu\}$. Hence we have $||\lambda||_\infty \geq ||\mu||_\infty$ for each $\lambda \in \Gamma_{n+1}$.

Now suppose that $\mu \in \Gamma_n(\pi)$ and that there are distinct weights $\lambda_1, \lambda_2 \in \Gamma_{n+1}(\pi)$. In this case we say that $\mu$ splits with respect to $\pi$. Because $\lambda_1$ and $\lambda_2$ in $A^+(g_n, a_n)$ are by assumption distinct and agree on the first $r_n$ coordinates, we see that they must differ on a coordinate $i$ with $r_n < i \leq r_{n+1}$. Since the coefficients of dominant weights form an increasing sequence, we see that either $||\lambda_1||_\infty > ||\lambda_2||_\infty \geq ||\mu||_\infty$ or $||\lambda_2||_\infty > ||\lambda_1||_\infty \geq ||\mu||_\infty$.

In other words, if a highest weight $\mu \in \Gamma_n(\pi)$ splits, then there is a $U_{n+1}$-highest weight in $\Gamma_{n+1}(\pi)$ with a coefficient which is strictly greater than all the coefficients in $\mu$. It follows that unless there is a weight $\mu_n \in \Gamma_n(\pi)$ for some $n$ which does not split and such that each $\lambda \in \Gamma_m(\pi)$ for any $m \geq n$ does not split, then we can repeat this process to obtain arbitrarily large coefficients of highest weights of representations appearing in $\pi$, contradicting Lemma 5.13. Hence, there is some highest weight $\mu \in \Gamma_n(\pi)$ such that, for each $m \geq n$, the vector $v_\mu$ is a $U_m$-highest-weight vector. Thus $\langle \pi(U_\infty)v_\mu \rangle$ gives a highest-weight representation of $U_\infty$.

Furthermore, we see that

$$v - v_\mu = \sum_{\lambda \in \Gamma_n(\pi) \backslash \mu} v_\lambda.$$
generates a conical representation by Theorem 6.5 and that
\[ \mathcal{H} = \langle \pi(U_\infty)v_\mu \rangle \oplus \langle \pi(U_\infty)(v - v_\mu) \rangle. \]

We have shown that every smooth unitary conical representation possesses an irreducible subrepresentation and that the orthogonal complement is also a smooth unitary conical representation. A standard Zorn’s Lemma argument then shows that \( \mathcal{H} \) decomposes into an orthogonal direct sum of irreducible smooth conical representations.

It follows from Theorems 5.13 and 6.10 that every smooth unitary conical representation \((\pi, \mathcal{H})\) of \( U_\infty \) is an orthogonal direct sum of smooth highest-weight representations:
\[ \pi \cong \bigoplus_{i \in \mathcal{A}} \pi_{\mu_i}, \]
where \( \mu_i \in \Lambda^+ \) for each \( i \in \mathcal{A} \). Write each highest weight \( \mu_i \) in terms of fundamental weights as in Section 5.4:
\[ \mu_i = \sum_{n=1}^{k_i} a_n^i \xi_i, \]
where \( a_n^i \in \mathbb{N} \) for each \( i \) and \( n \) (each \( \mu_i \) is a finite sum over the fundamental weights is finite because \( \pi_{\mu_i} \) is a smooth highest-weight representation). By Theorem 5.13, the smoothness of \( \pi \) is equivalent to the existence of a bound \( M > 0 \) such that \( \sum_{n=1}^{k_i} a_n^i < M \) for all \( i \in \mathcal{A} \).

### 6.5 Disintegration of Conical Representations

If we remove the assumption in Theorem 6.10 that the conical representation \((\pi, \mathcal{H})\) is smooth, then we can no longer be assured that \( \pi \) has an irreducible subrepresentation. However, we would still like to describe general conical representations in terms of the irreducible ones. This sort of description is possible with a direct-integral decomposition.

Recall that
\[ \Lambda^+ \equiv \Lambda^+(u_\infty, a_\infty) \equiv \varprojlim \Lambda^+(u_n, a_n) \subseteq a_\infty^*, \]
denotes the set of dominant integral weights for the root system \( \Sigma(u_\infty, a_\infty) \). We start by putting a topology on \( \Lambda^+ \). Each lattice \( \Lambda^+(u_n, a_n) \) carries the discrete topology. We then consider the projective limit topology on \( \Lambda^+ \), which we shall refer to as the **tree topology**. This topology is defined by a basis consisting of the cylinder sets \( B_\lambda = \{ \mu \in \Lambda^+ | [\mu]_{a_n} = \lambda \} \), where \( \lambda \) is a dominant integral weight on \( a_n \). We refer to these cylinder sets as **node sets** for reasons that will become apparent later. Note that any two node sets are disjoint or else one contains the other, so that our basis is closed under intersections. Furthermore, \( \Lambda^+ \) is second-countable under this topology, since there are only countably many dominant integral weights on \( i a_n \), for each fixed \( n \in \mathbb{N} \), so that our basis is a countable union of countable sets.
Because it is second-countable, this topology is described entirely by sequences. Note that a sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) in \( \Lambda^+ \) converges to \( \mu \) exactly when for each \( m \in \mathbb{N} \) there is \( N \) such that \( \mu_n|_{a_m} = \mu|_{a_m} \) for all \( n \geq N \).

This topology is also Hausdorff; if \( \mu \) and \( \lambda \) are distinct elements of \( \Lambda^+ \), then there is \( m \) such that \( \mu|_{a_m} \neq \lambda|_{a_m} \). Hence \( B_{\mu|_{a_m}} \) and \( B_{\lambda|_{a_m}} \) are disjoint open sets containing \( \mu \) and \( \lambda \), respectively.

In fact, \( \Lambda^+ \) is highly disconnected; every node set is both open and closed. To see this, if we consider \( B_\lambda \) for some \( \lambda \in \Lambda^+_n \), then we note that

\[
\Lambda^+ \setminus B_\lambda = \{ \mu \in \Lambda^+ | \mu|_{a_n} \neq \lambda \} = \bigcup_{\mu \in \Lambda^+_n \setminus \{\lambda\}} B_\mu,
\]

and hence \( \Lambda^+ \setminus B_\lambda \) is open.

Next consider closed subsets \( \Gamma \) of \( \Lambda^+ \) with the property that, for each \( n \in \mathbb{N} \), we have \( \Gamma \cap B_\lambda = \emptyset \) for all but finitely many \( \lambda \) in \( \Lambda^+_n \). We will refer to such sets as tree sets because, as we shall soon see, they are in one-to-one correspondence with trees of a certain type. We give each tree set \( \Gamma \) the subspace topology, so that it inherits the second-countability and Hausdorff properties from \( \Lambda^+ \). Write \( \Gamma^\lambda = B_\lambda \cap \Gamma = \{ \mu \in \Gamma | \mu|_{a_n} = \lambda \} \) for each \( n \) and each \( \lambda \in \Lambda^+_n \). We refer to these sets as node sets for \( \Gamma \). If \( \lambda \in \Lambda^+_n \) and \( \Gamma^\lambda \neq \emptyset \) (that is, there is \( \mu \in \Gamma \) such that \( \mu|_{a_n} = \lambda \)), then we say that \( \lambda \) is a node of the tree set \( \Gamma \). We write \( \Gamma_n = \{ \mu|_{a_n} | \mu \in \Gamma \} \) for the set of all nodes of \( \Gamma \) that lie in \( \Lambda^+_n \).

Now we spend a few moments explaining our tree-centric choice of terminology. For each tree set \( \Gamma \), we can construct a tree as follows. Each element of \( \Gamma_n \) for each \( n \in \mathbb{N} \) forms a node of the tree. Draw an edge from a node \( \lambda \) in \( \Gamma_n \) to a node \( \mu \) in \( \Gamma_{n+1} \) if \( \mu|_{a_n} = \lambda \). There is a correspondence between infinite paths in this tree and elements of \( \Gamma \). Each infinite path \( \{\lambda_n \in \Gamma_n \}_{n \in \mathbb{N}} \) of nodes of the tree defines a dominant weight \( \lambda \in \Lambda^+ \), since \( \lambda_m|_{a_n} = \lambda_n \) for \( m > n \). Because \( \Gamma \) is closed in the projective limit topology on \( \Lambda^+ \), it follows that \( \lambda \in \Gamma \). Similarly, each dominant weight \( \lambda \) in \( \Gamma \) defines a path \( \{\lambda|_{a_n} \in \Gamma_n \}_{n \in \mathbb{N}} \) in the tree. Hence, if \( \lambda \) is a node of \( \Gamma \), then the node set \( \Gamma^\lambda \) corresponds to the set of all infinite paths in the tree which pass through the node \( \lambda \).

It may also be readily seen that if \( \pi \) is a conical representation of \( U_{\infty} \), then the highest-weight tree \( \Gamma(\pi) \subseteq \Lambda^+ \) is a tree set.

Every tree set \( \Gamma \) is sequentially compact (and hence compact, since \( \Lambda^+ \) is second-countable). In fact, suppose that \( \{\mu_n\}_{n \in \mathbb{N}} \) is a sequence in \( \Gamma \). Now \( \Gamma_n = \{ \mu|_{a_n} | \mu \in \Gamma \} \) is finite for each \( n \). In particular, there is a subsequence \( \mu_{k_m} \) such that \( \mu_{k_m}|_{a_1} = \mu_{k_m}|_{a_1} \) for each \( m \) and \( n \). Repeating the process on this subsequence, we form a nested family of subsequences \( \{\mu_{k_{m_n}}\}_{n \in \mathbb{N}} \) such that \( \mu_{k_{m_n}}|_{a_m} = \mu_{k_{m_n}}|_{a_m} \) for each \( m \) and \( n \). Then \( \{\mu_{k_n}\}_{n \in \mathbb{N}} \) is a subsequence that converges in the tree topology on \( \Gamma \). Similarly, every node set in \( \Gamma \) is compact.

The complement of a node set in \( \Gamma \) is a finite union of node sets since \( \Gamma_n \) is finite for each \( n \). The collection \( \mathcal{F} \) of finite unions of node sets for \( \Gamma \) thus forms an algebra of sets which generates the Borel \( \sigma \)-algebra \( \mathcal{B} \) for the tree topology on \( \Gamma \).
We can use $\Gamma$ to define a measurable family of Hilbert spaces $\lambda \mapsto H_\lambda$ over $\lambda \in \Gamma$. For each $\lambda \in \Gamma$, consider the representation $(\pi_\lambda, H_\lambda)$ of $U_\infty$ with highest-weight $\lambda$. For each such representation, pick out a unit highest-weight vector $v_\lambda \in H_\lambda$.

To tie these Hilbert spaces together in a measurable way, we consider the family $\{s_g|g \in U_\infty\}$ of maps $s_g : \Gamma \to \bigcup_{\lambda \in \Gamma} H_\lambda$ given by $s_g(\lambda) = \pi_\lambda(g)v_\lambda$. Now choose a countable dense subset $E \subseteq U_\infty$ (recall that $U_\infty = \lim_\to U_n$ is separable) and consider the countable family $\{s_g|g \in E\}$ of sections. We shall use this family as a measurable frame for our family of Hilbert spaces. Hence, we need to show that $\lambda \mapsto \langle s_g(\lambda), s_h(\lambda) \rangle = \langle \pi_\lambda(g)v_\lambda, \pi_\lambda(h)v_\lambda \rangle$ (6.2)
is $\mathcal{B}$-measurable for each $g, h \in E$. Suppose that $g, h \in U_n$ for some $n \in \mathbb{N}$. Then the representation of $U_n$ on $\langle \pi_\lambda(U_n)v_\lambda \rangle$ is equivalent to $\pi_{\lambda\lambda_n}$ for each $\lambda$. Thus the map in (6.2) is constant on each node set $\Gamma^{\lambda\lambda_n}$ where $\lambda \in \Gamma$ and is hence $\mathcal{B}$-measurable. Finally, note that $\{s_g(\lambda) = \pi_\lambda(g)v_\lambda|g \in E\}$ is dense in $H_\lambda$ since $\pi_\lambda$ is irreducible and $E$ is dense in $U_\infty$. Thus, $\lambda \mapsto H_\lambda$ is a measurable field of Hilbert spaces.

Next, we note that $s_g$ is a measurable section for all $g \in U_\infty$. In fact, every $g \in U_\infty$ is a limit of a sequence $\{g_i\}_{i \in \mathbb{N}} \subseteq E$. Hence, we have that $\lambda \mapsto \langle s_g(\lambda), s_h(\lambda) \rangle = \lim_{i \to \infty} \langle s_{g_i}(\lambda), s_h(\lambda) \rangle$ is a measurable function for all $h \in E$, so that $s_g$ is a measurable section.

In order to construct a direct integral of representations $(\pi_\lambda, H_\lambda)$ over $\lambda \in \Gamma$, we still need a suitable choice of measure on $(\Gamma, \mathcal{B})$. In particular, we need to choose a finite measure whose support is all of $\Gamma$ (we will refer to such measures as having full support). The compactness of the node sets makes this easy because any finitely additive measure on $(\Gamma, \mathcal{F})$ extends uniquely to a countably additive measure on $(\Gamma, \mathcal{B})$.

This last claim follows from the E. Hopf Extension Theorem from measure theory, which states that a finitely additive measure $\mu$ on an algebra $\mathcal{F}$ of subsets of $X$ extends to a countably additive measure on the $\sigma$-algebra $\mathcal{B}$ generated by $\mathcal{F}$ if the measure is countably additive on $\mathcal{F}$. That is, we must show that if $A = \bigcup_{n \in \mathbb{N}} A_n$, where $A \in \mathcal{F}$ and $A_n \in \mathcal{F}$ for each $n$, then

$$\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n).$$

However, in our case, the algebra $\mathcal{F}$ consists of finite disjoint unions of node sets, and since every set in $\mathcal{F}$ is compact, it follows that there is no decomposition of a set in $\mathcal{F}$ into an infinite disjoint union of node sets.

Hence, all that we need to do is specify a (finitely additive) measure on the algebra of finite disjoint unions of node sets. We can do this rather easily. Start
with the “top-level” node sets; that is, the node sets $\Gamma^{\nu}$ for $\nu \in \Gamma_1$. We can assign a measure $\mu(\Gamma^{\nu})$ to each set in any way such that $\mu(\Gamma^{\nu}) > 0$ for each $\nu \in \Gamma_1$ and $\sum_{\nu \in \Gamma_1} \mu(\Gamma^{\nu}) = 1$. Next, for each $\lambda \in \Gamma_1$, consider

$$\Gamma_2^\lambda = \{ \nu \in \Gamma_2 | \nu|_{a_1} = \lambda \}.$$ 

We can then assign $\mu(\Gamma^{\nu})$ for each $\nu \in \Gamma_2^\lambda$ in any way such that $\mu(\Gamma^{\nu}) > 0$ and $\sum_{\nu \in \Gamma_2^\lambda} \mu(\Gamma^{\nu}) = \mu(\Gamma^\lambda)$. We can repeat this process, defining

$$\Gamma_{n+1}^\lambda = \{ \nu \in \Gamma_{n+1} | \nu|_{a_n} = \lambda \}$$

for each $\lambda \in \Gamma_n$. Then we can assign $\mu(\Gamma^{\nu})$ for all $\nu \in \Gamma_{n+1}^\lambda$ in such a way that $\mu(\Gamma^{\nu}) > 0$ for each $\nu$ and $\sum_{\nu \in \Gamma_{n+1}^\lambda} \mu(\Gamma^{\nu}) = \mu(\Gamma^\lambda)$. Doing this for all $\lambda \in \Gamma_n$ defines the measures of all node sets for weights in $\Gamma_{n+1}$. This procedure always produces a Borel measure on $\Gamma$, and every finite Borel measure of full support on $\Gamma$ can be constructed this way.

For instance, we can assign $\mu(\Gamma^{\nu}) = \frac{1}{\# \Gamma_1}$ for each $\nu \in \Gamma_1$. Then, for $\nu \in \Gamma_{n+1}$, recursively define $\mu(\Gamma^{\nu}) = \frac{1}{\# \Gamma_{n+1}} \mu(\Gamma^\lambda)$ if $\lambda \in \Gamma_n$ and $\nu \in \Gamma^\lambda$. We have now defined the measures of all node sets from weights in $\Gamma_2$. This same method can be repeated recursively to define the measures of every node set in $\Gamma$. We will refer to this example method of assignment as giving the **recursively uniform measure**.

Given a finite Borel measure $\mu$ on $\Gamma$ of full support, we may consider the direct integral $\mathcal{H} = \int_{\Gamma}^{\oplus} \mathcal{H}_\mu d\mu(\lambda)$. Elements of this direct integral consist of measurable sections $x : \lambda \mapsto x(\lambda)$ of the field $\lambda \mapsto \mathcal{H}_\lambda$ such that the norm given by $||x||^2 = \int_{\Gamma} ||x(\lambda)||^2_{\mathcal{H}_\lambda} d\mu(\lambda)$ is finite.

Our next task is to show that $\lambda \mapsto \pi_{\lambda}$ is a $\mu$-measurable family of representations. Let $x \in \mathcal{H}$, and fix $g$ in $U_\infty$. We need to show that $\lambda \mapsto \pi_{\lambda}(g)x(\lambda)$ is in $\mathcal{H}$. Now

$$\lambda \mapsto \langle \pi_{\lambda}(g)x(\lambda), s_h(\lambda) \rangle = \langle \pi_{\lambda}(g)x(\lambda), \pi_{\lambda}(h)\psi_{\lambda} \rangle$$

$$= \langle x(\lambda), \pi_{\lambda}(g^{-1}h)\psi_{\lambda} \rangle$$

$$= \langle x(\lambda), s_{g^{-1}h}(\lambda) \rangle$$

is measurable for all $h$ in $U_\infty$ since $x$ is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Thus $\lambda \mapsto \pi_{\lambda}(g)x(\lambda)$ is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Furthermore, since each $\pi_{\lambda}$ is unitary, it follows that $||\pi(g)x||_{\mathcal{H}} = ||x||_{\mathcal{H}} < \infty$. Hence $\pi = \int_{\Gamma}^{\oplus} \pi_{\lambda} d\mu(\lambda)$ is a unitary representation of $U_\infty$. Our next task is to show that $\pi$ is conical and classify all of its conical vectors.

The **essential support** of a function $f : \Gamma \to \mathbb{C}$ is defined to be the complement in $\Gamma$ of the union of all open sets on which $f$ vanishes $\mu$-almost everywhere. That is, $\text{ess supp } f = \Gamma \setminus \bigcup \{ A \subseteq \Gamma | A \text{ is open and } f|A = 0 \text{ a.e.} \}$.

**Theorem 6.11.** Let $\Gamma$ be a tree set and let $\mu$ be a finite Borel measure of full support on $\Gamma$. Consider the representation

$$\langle \pi, \mathcal{H} \rangle \equiv \left( \int_{\Gamma} \pi_{\lambda} d\mu(\lambda), \int_{\Gamma} \mathcal{H}_{\lambda} d\mu(\lambda) \right)$$

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and suppose that \( w \) is any nonzero vector in \( \mathcal{H} \). Then \( w \) generates a unitary conical representation of \( U_\infty \) if and only if there is \( f \in L^2(\Gamma, \mu) \) such that \( w = \int_\Gamma f(\lambda)v_\lambda d\mu(\lambda) \). Furthermore, in that case \( w \) generates a conical representation with highest-weight support \( \text{ess supp } f \) and

\[
\langle \pi(U_\infty)w \rangle = \int_{\Gamma \setminus f^{-1}(0)} \mathcal{H}_\lambda d\mu(\lambda)
\]

In particular, \( \pi \) is a conical representation with conical vector \( v = \int_\Gamma v_\lambda d\mu(\lambda) \).

**Proof.** \((\Rightarrow)\) Suppose that \( w \) is a conical vector for a subrepresentation of \( \pi \) and fix \( n \) in \( \mathbb{N} \). Then \( V_n \equiv \langle \pi(U_n)w \rangle \) is finite-dimensional, say with dimension \( d \). We must show that \( w(\lambda) = w(\lambda) \) is a conical vector in \( \mathcal{H}_\lambda \) for almost all \( \lambda \in \Gamma \). Our first task is to show that \( V_n(\lambda) = \langle \pi(U_n)w \rangle \) is finite-dimensional for almost all \( \lambda \in \Gamma \). It is intuitively obvious that \( \dim V_n(\lambda) \leq \dim V_n \) for almost all \( \lambda \). The next three paragraphs contain the technical details necessary to prove this statement.

Write \( d = \dim V_n \). Fix an orthonormal basis \( w_1, \ldots, w_d \) for \( V_n \) and write \( W(\lambda) = \langle w_1(\lambda), \ldots, w_d(\lambda) \rangle \). We will show that \( W(\lambda) = V_n(\lambda) \) (and hence \( \dim V_n(\lambda) \leq d \)) for almost all \( \lambda \). Apply a Gram-Schmidt orthonormalization process to the collection \( w_1(\lambda), \ldots, w_d(\lambda) \) for each \( \lambda \). We then obtain a collection \( \tilde{w}_1(\lambda), \ldots, \tilde{w}_d(\lambda) \) with the property that \( \langle \tilde{w}_i(\lambda), \tilde{w}_j(\lambda) \rangle = 0 \) for \( i \neq j \) and \( \langle \tilde{w}_i(\lambda), \tilde{w}_i(\lambda) \rangle \in \{0, 1\} \). One can show that \( \lambda \mapsto \tilde{w}_i(\lambda) \) is measurable and thus that \( \tilde{w}_i \in \mathcal{H} \) for each \( i \).

Now \( W(\lambda) = V_n(\lambda) \) if and only if \( \pi(g)w(\lambda) \in W(\lambda) \) for all \( g \) in \( U_\infty \). Choose a countable dense subset \( \{g_n\}_{n \in \mathbb{N}} \) in \( U_\infty \) (one notes that \( U_\infty \) is separable because it is a countable direct union of separable spaces). By the strong continuity of \( \pi \), we see that \( W(\lambda) = V_n(\lambda) \) if and only if \( \pi(g_m)w(\lambda) \in W(\lambda) \) for all \( m \) in \( \mathbb{N} \) (recall that \( W(\lambda) \) is closed because it is finite-dimensional). In turn, this happens exactly when \( \pi(g_m)w(\lambda) \) is equal to its orthogonal projection onto \( W(\lambda) \). In other words, \( W(\lambda) = V_n(\lambda) \) if and only if \( F_m(\lambda) = 0 \) for all \( m \in \mathbb{N} \), where \( F_m \) is the non-negative measurable function on \( \Gamma \) defined by

\[
F_m : \lambda \mapsto ||\pi(g_m)w(\lambda)||^2 - \sum_{i=1}^d |\langle \pi(g_m)w(\lambda), \tilde{w}_i(\lambda) \rangle|^2.
\]

for all \( m \in \mathbb{N} \).

Write \( A = \{\lambda \in \Gamma | W(\lambda) \neq V_n(\lambda) \} \) and \( A_m = \{\lambda \in \Gamma | \pi(g_m)w(\lambda) \notin W(\lambda) \} \). Then \( A = \bigcup_{m \in \mathbb{N}} A_m \). Furthermore, \( A_m \) is measurable for each \( m \) since \( A_m = F_m^{-1}(0) \) and \( F_m \) is a measurable function.

Suppose that it is not true that \( W(\lambda) = V_n(\lambda) \) for almost all \( \lambda \) in \( \Gamma \). Then \( \mu(A) > 0 \). Since \( A = \bigcup_{m \in \mathbb{N}} A_m \), it follows that \( \mu(A_m) > 0 \) for some \( m \). Since \( \pi(g_m)w(\lambda) \notin W(\lambda) \) for all \( \lambda \in A_m \), we see that \( \pi(g_m)w \notin \langle w_1, \ldots, w_d \rangle \), which contradicts the assumption that \( w_1, \ldots, w_d \) is a basis for \( V_n = \langle \pi(g_m)w \rangle \). Therefore, \( W(\lambda) = V_n(\lambda) \) (and, in particular, \( \dim V_n(\lambda) \leq d \)) for almost all \( \lambda \). In particular, \( w(\lambda) \) is \( U_n \)-finite for almost all \( \lambda \in \Gamma \).
Fix $n \in \mathbb{N}$. Since $\pi(M_n)w = w$, it follows that $\pi(M_n)w(\lambda) = w(\lambda)$ for almost all $\lambda$. Next, $\pi(n_n)w = w$ because $\pi(N_n)w = w$. In fact, $\pi(X)w = \int_{\Gamma} \pi(X)w(\lambda) d\mu(\lambda)$ for $X \in U^c_n$ by [1]. Thus $\pi(n_n)w(\lambda) = w(\lambda)$ for almost all $\lambda$, from which it follows that $\pi(N_n)w(\lambda) = w(\lambda)$ for almost all $\lambda$.

Since $\pi(M_nN_n)w(\lambda) = w(\lambda)$ for all $n$ and almost all $\lambda \in \Gamma$, it follows from part (4) of Theorem 6.5 that for almost all $\lambda$ there is $f(\lambda) \in \mathbb{C}$ such that $w(\lambda) = f(\lambda)v_\lambda$.

Since $\lambda \mapsto f(\lambda) = \langle w(\lambda), v_\lambda \rangle$ is measurable and

$$
||f||^2 = \int_{\Gamma} |f(\lambda)|^2 d\mu(\lambda) = \int_{\Gamma} ||w(\lambda)||^2 d\mu(\lambda) = ||w||^2, 
$$

we see that $f \in L^2(\Gamma, \mu)$, as was to be shown.

$(\Leftarrow)$ Now suppose that $w = \int_{\Gamma} f(\lambda)v_\lambda d\mu(\lambda)$, where $f \in L^2(\Gamma, \mu)$. We show that $w$ generates a conical representation of $U_\infty$ with highest-weight support $\text{ess \ supp} f$.

Consider $V_n = \langle \pi(U_n)w \rangle$. We will show that $V_n$ is finite-dimensional. As before,

$$
\pi \cong \bigoplus_{\mu \in \Gamma_n} \left( \bigoplus_{\lambda} \pi_\lambda d\mu(\lambda) \right).
$$

Write $w = \sum_{\mu \in \Gamma_n} w_\mu$, where $w_\mu = 1_{N_n}w \in \int_{\Gamma_n} \pi_\lambda d\mu(\lambda) \subseteq \mathcal{H}_n$ for each $\mu$.

Of course, if $f|_{\Gamma_n} = 0$, then $w_\mu = 0$. On the other hand, we claim that if $f|_{\Gamma_n} \neq 0$, then $\langle \pi(U_n)w_\mu \rangle \cong_{U_n} \pi_\mu$. In fact,

$$
\sum_{i=1}^{k} c_i \pi(g_i)w_\mu = \int_{\Gamma_n} \sum_{i=1}^{k} c_i \pi(g_i)f(\lambda)v_\lambda d\mu(\lambda).
$$

where $c_i \in \mathbb{C}$ and $g_i \in U_n$. Fix $\lambda \in \Gamma_n$ such that $f(\lambda) \neq 0$. Since $\lambda|_{\Gamma_n} = \mu$, we see that $\langle \pi(U_n)f(\lambda)v_\lambda \rangle$ is $U_n$-isomorphic to $\pi_\mu$.

Now $\sum_{i=1}^{k} c_i \pi(g_i)w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^{k} c_i \pi(g_i)f(\lambda)v_\lambda = 0$ in $\mathcal{H}_\lambda$ for $\mu$-almost all $\lambda$ in $\Gamma_n$. For any $\lambda$ in $\Gamma_n$ such that $f(\lambda) = 0$, it follows automatically that $\sum_{i=1}^{k} c_i \pi(g_i)f(\lambda)v_\lambda = 0$. But for any fixed $\lambda$ in $\Gamma_n$ such that $f(\lambda) \neq 0$, we see that $\sum_{i=1}^{k} c_i \pi(g_i)f(\lambda)v_\lambda = 0$ in $\mathcal{H}_\lambda$ if and only if $\sum_{i=1}^{k} c_i \pi(g_i)v_\lambda = 0$ in $\mathcal{H}_\mu$.

Since $f$ is not almost-everywhere zero on $\Gamma_n$, we see that $\sum_{i=1}^{k} c_i \pi(g_i)w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^{k} c_i \pi(g_i)v_\mu = 0$ in $\mathcal{H}_\mu$. Hence there is an injective $U_n$-intertwining operator $L : \langle \pi(U_n)w_\mu \rangle \to \mathcal{H}_\mu$ with the property that $Lw_\mu = v_\mu$.

Since $\pi_\mu$ is irreducible, it follows that $\langle \pi(U_n)w_\mu \rangle \cong_{U_n} \pi_\mu$, as we wanted to show.

It follows from Lemma 5.10 that

$$
\langle \pi(U_n)w \rangle \cong_{U_n} \bigoplus_{\mu \in \Gamma_n \text{ s.t. } w_\mu \neq 0} \langle \pi(U_n)w_\mu \rangle.
$$

Furthermore, since $w = \sum_{\mu \in \Gamma_n} w_\mu$ and each $w_\mu$ is $M_nN_n$-invariant, we see that $w$ is $M_nN_n$-invariant. Since this holds for all $n$, it follows that $w$ generates a conical
subrepresentation of $\pi$. The fact that this subrepresentation has highest-weight support \( \mu \) follows from the fact that \( w_\mu = 0 \) if and only if \( f|\Gamma_\mu = 0 \) (recall that \( w_\mu \) is the projection of \( w \) onto the \( \mu \)-isotypic vectors in \( \mathcal{H} \)).

Our final task is to prove the statement about the subrepresentations generated by conical vectors. Next suppose that \( f \in L^2(\Gamma, \mu) \) such that \( w = fu : \lambda \to f(\lambda)v_\lambda \) is a conical vector in \( \mathcal{H}_\Gamma \). We need to show that

\[
\langle \pi(U_\infty)w \rangle = \bigoplus_{\Gamma \setminus \Gamma^{-1}(0)} \mathcal{H}_\lambda d\mu(\lambda).
\]

It suffices to show that

\[
\langle \pi(U_\infty)w \rangle^\perp = \bigoplus_{f^{-1}(0)} \mathcal{H}_\lambda d\mu(\lambda).
\]

One direction of containment is clear: for any \( x \in \langle \pi(U_\infty)w \rangle \), we see that \( x(\lambda) = 0 \) for almost all \( \lambda \) such that \( f(\lambda) = 0 \) (since \( w(\lambda) = 0 \) if and only if \( f(\lambda) = 0 \)). Hence, if \( y \in \mathcal{H} \) such that \( y|\Gamma \setminus f^{-1}(0) = 0 \), then \( (x, y) = \int_{\Gamma} \langle x(\lambda), y(\lambda) \rangle d\mu(\lambda) = 0 \). In other words, \( \int_{f^{-1}(0)} \mathcal{H}_\lambda d\mu(\lambda) \subseteq \langle \pi(U_\infty)w \rangle^\perp \).

To prove the other containment, we first show that \( hw \in \langle \pi(U_\infty)w \rangle \) for all \( h \in L^\infty(\Gamma, \mu) \). We begin by showing that \( 1_{\Gamma_\mu}w \in \langle \pi(U_\infty)w \rangle \) for every node set \( \Gamma^\mu \). As before, we choose \( c_1, \ldots, c_d \in \mathbb{C} \) and \( g_1, \ldots, g_d \in U_\infty \) such that \( \sum_{i=1}^k c_i\pi_{\mu}(g_i)v_\mu = v_\mu \) and \( \sum_{i=1}^k c_i\pi_{\nu}(g_i)v_\nu = 0 \) for all \( \nu \neq \mu \) in \( \Gamma_n \). We claim that \( 1_{\Gamma_\mu}w = \sum_{i=1}^k c_i\pi_{\mu}(g_i)w \). If \( f(\lambda) = 0 \), then \( w(\lambda) = 0 \) and hence equality holds automatically. On the other hand, if \( f(\lambda) \neq 0 \), then recall that \( \langle \pi(U_\infty)w \rangle \) is equivalent to \( \pi_{\lambda|_n} \) by identifying \( w(\lambda) = f(\lambda)v_\lambda \) with \( v_{\lambda|_n} \). Hence \( \sum_{i=1}^k c_i\pi_{\mu}(g_i)v_\lambda = v_\mu \) if \( \lambda|_n = \mu \) (i.e., if \( \lambda \in \Gamma^\mu \)) and \( \sum_{i=1}^k c_i\pi_{\mu}(g_i)v_\lambda = 0 \) otherwise. Thus \( 1_{\Gamma_\mu}w = \sum_{i=1}^k c_i\pi_{\mu}(g_i)w \) and so \( 1_{\Gamma_\mu}w \in \langle \pi(U_\infty)w \rangle \).

Next we see that \( 1_Aw \in \langle \pi(U_\infty)w \rangle \) for all open sets \( A \) in \( \Gamma \). Every open set \( A \) can be written as a disjoint union \( A = \bigcup_{i=1}^\infty N_i \) of node sets. Write \( A_n = \bigcup_{i=1}^n N_i \) for each \( n \) and note that \( 1_{A_n} = \sum_{i=1}^k 1_{N_i} \) is in \( \langle \pi(U_\infty)v \rangle \) by the previous paragraph. One then sees that

\[
\int_{\Gamma} 1_{A_n}(\lambda)f(\lambda)v_\lambda d\mu(\lambda) = 1_{A_n}w \rightarrow 1_Aw = \int_{\Gamma} 1_A(\lambda)f(\lambda)v_\lambda d\mu(\lambda)
\]

in \( \mathcal{H} \) since \( 1_{A_n}f \to 1_Af \) in \( L^2(\Gamma, \mu) \). Thus \( 1_Aw \in \langle \pi(U_\infty)v \rangle \).

Next we show that \( 1_Bv \in \langle \pi(U_\infty)v \rangle \) for every Borel set \( B \) in \( \Gamma \). This follows since

\[
\mu(B) = \inf \left\{ \mu \left( \bigcup_{i=1}^\infty F_i \right) \left| B \subseteq \bigcup_{i=1}^\infty F_i \right. \text{ and } F_i \in \mathcal{F} \right\}
\]

\[
= \inf \{ \mu(A) | B \subseteq A \text{ and } A \text{ open} \}.
\]
Thus $1_B f$ can be approximated in $L^2(\Gamma, \mu)$ by a sequence $1_{A_n} f$ given by open sets $A_n$, so that $1_{A_n} w \to 1_B w$ in $H$. Hence $1_B w \in \pi(U_\infty)$.

Finally, note that if $h_n \to h$ in $L^\infty(\Gamma, \mu)$, then $h_n f \to h f$ in $L^2(\Gamma, \mu)$ and hence $h_n w \to hw$ in $H_\Gamma$. Because the measurable simple functions are dense in $L^\infty(\Gamma, \mu)$ (recall that $\mu$ is a finite measure), we see that $hw \in \pi(U_\infty)$ for all $h \in L^\infty(\Gamma, \mu)$.

Now suppose that $x \perp \pi(U_\infty)$. Define $h \in L^\infty(\Gamma, \mu)$ by

$$h(\lambda) = \frac{\langle x(\lambda), \pi(\lambda)(g) v(\lambda) \rangle}{|\langle x(\lambda), \pi(\lambda)(g) v(\lambda) \rangle|}.$$  

Then

$$0 = \langle x, \pi(g) hw \rangle = \int_\Gamma |\langle x(\lambda), \pi(\lambda)(g) f(\lambda) v(\lambda) \rangle| d\mu(\lambda).$$

for all $g$. Hence, for almost all $\lambda$, $\langle x(\lambda), \pi(\lambda)(g) f(\lambda) v(\lambda) \rangle = 0$ for all $g \in U_\infty$. It follows that, for almost all $\lambda$, either $x(\lambda) = 0$ or $f(\lambda) = 0$. Hence, $x(\lambda) = 0$ for almost all $\lambda$ such that $f(\lambda) \neq 0$. In other words, $x \in \int_{\Gamma - 1(0)} H_\lambda d\mu(\lambda)$, and we are therefore done.

\[\Box\]

**Corollary 6.12.** Every unitary conical representation of $U_\infty$ is multiplicity-free and hence of Type I.

**Proof.** Let $(\pi, H) \equiv \left(\int_\Gamma \pi_\lambda d\mu(\lambda), \int_\Gamma \mathcal{H}_\lambda d\mu(\lambda)\right)$ be a conical representation and suppose that $L : H \to \mathcal{H}$ is a $U_\infty$-intertwining operator. Consider the conical vector $v = \int_\Gamma v_\lambda d\mu(\lambda)$. Then $L v$ is a conical vector for a subrepresentation of $\pi$ and can thus be written $L v = f v$ for some $f \in L^2(\Gamma, \mu)$. It follows that

$$L(\pi(g)v) = \pi(g)(f v) = \int_\Gamma \pi(g)f(\lambda) v_\lambda d\mu(\lambda) = f \pi(g)v$$

for all $g \in U_\infty$ and hence $L y = f y$ for all $y \in H$. In other words, intertwining operators for $\pi$ may be identified with multiplier operators, and thus the ring of intertwining operators for $\pi$ is commutative. Hence $\pi$ is multiplicity-free. \[\Box\]

We now show that every unitary conical representation of $U_\infty$ disintegrates into highest-weight representations as in the last theorem.

**Theorem 6.13.** Suppose that $(\pi, H)$ is a unitary conical representation of $U_\infty$ and $w \in H \setminus \{0\}$ is a conical vector. Then there is a unique Borel measure $\mu$ on its highest-weight support $\Gamma(\pi)$ such that there is a unitary intertwining operator

$$U : H \to \int_{\Gamma(\pi)} \mathcal{H}_\lambda d\mu(\lambda)$$

such that $U w = \int_{\Gamma(\pi)} v_\lambda d\mu(\lambda)$.  

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Proof. Without loss of generality, suppose that \( ||w|| = 1 \). We begin by constructing a suitable measure \( \mu \). For each \( \lambda \) in \( \Gamma_n(\pi) \), define \( \mu(\lambda) = ||w_\lambda||^2 \). Observe that

\[
\mu(\Gamma) = ||w||^2 = \sum_{\lambda \in \Gamma_n(\pi)} ||w_\lambda||^2 = \sum_{\lambda \in \Gamma_n(\pi)} \mu(\lambda).
\]

Similarly,

\[
\sum_{\lambda \in \Gamma_n(\pi)} \mu(\lambda) = \sum_{\lambda \in \Gamma_n(\pi)} ||w_\lambda||^2 = ||w||^2 = 1
\]

Thus \( \mu \) extends uniquely to a Borel measure on \( \Gamma(\pi) \).

Consider the representation \((\tilde{\pi}, \tilde{\mathcal{H}}) \equiv \left( \int_{\lambda \in \Gamma(\pi)} \pi_\lambda d\mu(\lambda), \int_{\lambda \in \mathcal{H}} \mathcal{H}_\lambda d\mu(\lambda) \right) \) and let \( \tilde{w} \equiv \int_{\Gamma(\pi)} v_\lambda d\mu(\lambda) \). Then \( \tilde{\pi} \) is conical with conical vector \( \tilde{w} \) and highest-weight support \( \Gamma(\pi) \). We construct a unitary intertwining operator \( U : \mathcal{H} \to \tilde{\mathcal{H}} \) such that \( Uw = \tilde{w} \).

By Theorem 6.9 (i), there is a \( U_\infty \)-intertwining operator \( L : \langle \pi(U_\infty)w \rangle \to \langle \tilde{\pi}(U_\infty)\tilde{w} \rangle \) given by \( Lw = \tilde{w} \). For each \( n \) and each \( \nu \in \Gamma_n(\pi) \), \( L \) restricts to an intertwining operator between \( \langle \pi(U_n)w_\nu \rangle \) and \( \langle \tilde{\pi}(U_n)\tilde{w}_\nu \rangle \) such that \( L(w_\nu) = \tilde{w}_\nu \).

Furthermore,

\[
||\tilde{w}_\nu||^2 = \int_{\Gamma' \nu} ||\tilde{w}_\lambda||^2 d\mu(\lambda) = \int_{\Gamma' \nu} 1 d\mu(\lambda) = \mu(\Gamma') = ||w_\nu||^2.
\]

Hence, \( L \) restricts to a unitary operator on \( \langle \pi(U_n)w_\nu \rangle \) for every \( n \) and every \( \nu \in \Gamma_n(\pi) \). Because \( \langle \pi(U_\infty)w_\nu \rangle \) and \( \langle \pi(U_\infty)\tilde{w}_\nu \rangle \) are dense in \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \), respectively, \( L \) extends to a unitary intertwining operator from \( \mathcal{H} \) to \( \tilde{\mathcal{H}} \).

Now suppose that \( \mu' \) is any Borel measure on \( \Gamma(\pi) \) such that the representation \((\pi', \mathcal{H}') \equiv \left( \int_{\lambda \in \Gamma(\pi)} \pi_\lambda d\mu'(\lambda), \int_{\lambda \in \mathcal{H}} \mathcal{H}_\lambda d\mu'(\lambda) \right) \) is equivalent to \((\pi, \mathcal{H}) \) via a unitary intertwining operator \( U : \mathcal{H} \to \mathcal{H}' \) such that \( Uw = w' \), where \( w' = \int_{\Gamma(\pi)} v_\lambda d\mu'(\lambda) \). Then \( Uw_\nu = w'_\nu \) for all \( \nu \in \Gamma_n(\pi) \) and all \( n \in \mathbb{N} \) by Theorem 6.9. In particular,

\[
||w_\nu|| = ||w'_\nu||
\]

and so we have that

\[
\mu'(\Gamma') = \int_{\Gamma' \nu} ||w_\nu||^2 d\mu'(\lambda) = ||w'_\nu||^2 = ||w_\nu||^2 = \mu(\Gamma').
\]

Since \( \mu \) and \( \mu' \) agree on all node sets, it follows that \( \mu = \mu' \).

As promised before, we now show that there are typically a very large number of inequivalent conical representations of \( U_\infty \) with a given highest-weight support \( \Gamma \). By Theorem 6.11, this problem is equivalent to finding a large number of Borel measures with full support on \( \Gamma \) that are absolutely discontinuous with respect to each other.

We have already discussed the recursively-uniform measure \( \mu_{\text{rec}} \) on \( \Gamma \). One can see quite easily that the atoms of \( \mu_{\text{rec}} \) are precisely the isolated points of the
topological space $\Gamma$. All other singleton sets have measure zero under $\mu_{rec}$. We now show that for any point $x$ in $\Gamma$ we can construct a Borel measure $\mu_x$ of full support on $\Gamma$ whose atoms are precisely the isolated points of $\Gamma$ and $x$. Thus, if $x \neq y$ are non-isolated points in $\Gamma$, then $\mu_x$, $\mu_y$, and $\mu_{rec}$ lie in distinct measure classes since their null sets do not agree:

$$
\mu_x(\{x\}) > 0, \quad \mu_x(\{y\}) = 0 \\
\mu_y(\{x\}) = 0, \quad \mu_y(\{y\}) > 0 \\
\mu_{rec}(\{x\}) = 0, \quad \mu_{rec}(\{y\}) = 0
$$

There are many ways to construct $\mu_x$ given $x \in \Gamma$, but we shall use the following method, which involves a simple modification to the recursively uniform measure. For $\lambda \in \Gamma$, define $\mu_x(\Gamma^\lambda) = \frac{2}{3}$ if $x|_{a_n} = \lambda$ and $\mu_x(\Gamma^\lambda) = \left(\frac{1}{\#\Gamma_{n-1}}\right)\frac{1}{4}$ otherwise. Next suppose that $\mu_x(\Gamma^n)$ has been defined for all $\nu \in \Gamma_n$. For $\lambda \in \Gamma_{n+1}$, we define

$$
\mu_x(\Gamma^\lambda) = \begin{cases} 
\frac{1}{2} + \frac{1}{2^{n+1}} & \text{if } x \in \Gamma^\lambda \\
\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) \frac{1}{\#\Gamma^\lambda_{a_n} - 1} & \text{if } x \notin \Gamma^\lambda \text{ and } x \in \Gamma^\lambda|_{a_n} \\
\frac{1}{\#\Gamma^\lambda_{a_n}} \mu_x(\Gamma^\lambda|_{a_n}) & \text{otherwise},
\end{cases}
$$

where, as before, $\Gamma^\nu_n = \{\gamma \in \Gamma_n | \gamma|_{a_n} = \nu\}$. We have thus recursively defined a countably additive Borel measure $\mu_x$ on $\Gamma$. Note that $\mu_x$ has full support on $\Gamma$ because $\mu_x(\Gamma^\lambda) > 0$ for every open basis set $\Gamma^\lambda \subseteq \Gamma$. Furthermore, one can easily check that $\mu_x(\{x\}) = \frac{1}{2}$ and that $\mu_x(\{y\}) = 0$ if $y \neq x$ and $y$ is not an isolated point of $\Gamma$. 

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Chapter 7
Closing Remarks and Further Research

We have managed to prove several results for the unitary conical representations of \( U_\infty \), including the classification of unitary smooth conical representations, which generalize the finite-dimensional conical representations of finite-dimensional symmetric spaces. However, the question remains of whether it is possible to construct unitary conical representations of \( G_\infty \). The most likely approach would be to construct a sort of unitary spherical principal series representation, perhaps by a direct limit of unitary principal series representations. See also [55] for one approach to constructing an analogue of the principal series for direct-limit groups.

Several questions about harmonic analysis on the symmetric space \( G_\infty/K_\infty \) and \( G_\infty/M_\infty N_\infty \) remain. While neither of these infinite-dimensional spaces possess \( G_\infty \)-invariant measures, there is a possibility of constructing \( G_\infty \)-invariant measures on larger spaces. We briefly overview this construction now.

Consider a direct system \( \{ G_n \}_{n \in \mathbb{N}} \) of Lie groups and suppose that there are measurable (not necessarily continuous) projections \( p_n : G_{n+1} \to G_n \) such that \( p_n \) is \( G_n \)-equivariant and \( p_n(g) = g \) for \( g \in G_n \). In other words, one has a projective system of \( \sigma \)-algebras dual to the direct system of groups. The resulting projective-limit space \( G_\infty = \lim_{\leftarrow} G_n \) is acted on by the direct-limit group \( G_\infty = \lim_{\to} G_n \). Each group \( G_n \) possesses a \( G_n \)-quasi-invariant probability measure \( \mu_n \).

It is then possible to define a projective-limit probability measure \( \mu_\infty = \lim_{\leftarrow} \mu_n \) on \( G_\infty \) using Kolmogorov’s theorem. If this measure is quasi-invariant under the action of \( G_\infty \) on \( G_\infty \) then it is possible to define a unitary “regular representation” of \( G_\infty \) on \( L^2(G_\infty, \mu_\infty) \). This “regular representation” can then be decomposed into irreducible representations.

In fact, precisely this scheme was used by Doug Pickrell in [44] to study analysis on an infinite-dimensional Grassmannian space and later by Olshanski and Borodin in [4] to develop a theory of harmonic analysis on the infinite-dimensional unitary group \( U(\infty) \). The role played by probability theory in the latter context was crucial. In fact, the problem was shown to be related to the study of infinite point processes. Most intriguingly, probabilistic models from statistical mechanics appeared.

It would be interesting to consider a similar analysis on the infinite-dimensional symmetric space \( G_\infty/K_\infty \) and the horocycle space \( G_\infty/M_\infty N_\infty \). That is, one would construct projective-limit spaces \( \overline{G_\infty/K_\infty} \) and \( \overline{G_\infty/M_\infty N_\infty} \) which possess \( G_\infty \)-quasi-invariant measures. The problem, then, would be to decompose the corresponding unitary representations of \( G_\infty \) on \( L^2(\overline{G_\infty/K_\infty}) \) and \( L^2(\overline{G_\infty/M_\infty N_\infty}) \) into irreducible subrepresentations. One interesting question is whether those representations decompose into direct integrals of unitary spherical and conical representations of \( G_\infty \), respectively.
Also of interest is whether a sort of Radon transform may be constructed between functions on $G_\infty/K_\infty$ and functions on $G_\infty/M_\infty N_\infty$. In fact, for spaces of regular functions this has been done in the recent paper [23]. However, it would be interesting if it were possible to develop a Hilbert space analogue of the Radon transform, perhaps mapping between functions in $L^2(G_\infty/K_\infty)$ and functions in $L^2(G_\infty/M_\infty N_\infty)$. 
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Vita

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