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Absolute Z-Sets and the Z-Set Absorption Property.

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ABSTRACT

A closed subset $K$ of a separable metric space $X$ is said to be a Z-set in $X$ if for each non-empty homotopically trivial open set $U$ in $X$, $U \setminus K$ is non-empty and homotopically trivial. Z-sets have played an important role in the study of many infinite dimensional spaces, especially $l^2$, $s$, $Q$ and manifolds modeled on these spaces.

In Chapter I we discuss some of the known properties of Z-sets and state some of their more important applications. In the remainder of this paper, we consider two further properties related to Z-sets.

In Chapter II we are concerned with the Z-set absorption property. A subset $M$ of a separable metric space $X$ is said to have the Z-set absorption property in $X$ if $M = \bigcup_{i>0} M_i$ where

1. For each $i>0$, $M_i$ is a Z-set in $X$;
2. For each $i>0$, $M_i \subseteq M_{i+1}$; and
3. For each Z-set $A$ in $X$, for each closed subset $B$ of $A$ with $B \subseteq M$ and for each open (in $X$) cover $\mathcal{U}$ of $A$, there exists a homeomorphism $h$ of $X$ onto itself such that $h(A) \subseteq M$, $h|B = id$ and $h$ is limited by $\mathcal{U}$.

This property is similar to the compact absorption property,
introduced by Anderson in [5], except that the sets under consideration here are just Z-sets, which in F-manifolds need not be compact. In this chapter we present some examples of sets with the Z-set absorption property and prove theorems which are analogous to theorems proved for sets with the compact absorption property.

In Chapter III we determine which topologically complete separable metric spaces are absolute Z-sets; that is, have the property that the image of any closed embedding of the space into $l_2$ is a Z-set in $l_2$. Our main result of this chapter is the following.

**Theorem 3.1.** Let $X$ be a topologically complete separable metric space. Then $X$ is an absolute Z-set if and only if $X$ is $\sigma$-compact.
CHAPTER I

INTRODUCTION

A closed subset $K$ of a separable metric space $X$ is said to be a Z-set in $X$ (or to have Property Z in $X$) provided that for any non-empty homotopically trivial open set $U$ in $X$, $U \setminus K$ is non-empty and homotopically trivial. (A set $V$ is homotopically trivial if each map of the boundary $S^{n-1}$ of an $n$-ball $B^n$ ($n \geq 1$) into $V$ can be extended to a map of $B^n$ into $V$). Thus the removal of a Z-set does not affect the (trivial) homotopy, either global or local, of the space. We also observe that the property of being a Z-set is topological. While non-empty Z-sets do not occur at all in $n$-dimensional Euclidean space, any closed subset of the boundary of an $n$-manifold is a Z-set in the manifold. Since these sets already have the unique characteristic of being a subset of the boundary and since variously arising subsets of infinite dimensional spaces have been adequately described only in terms of Z-sets, it is proper to consider Z-sets as essentially infinite dimensional phenomena.

Let $s$ denote the countable infinite product $\prod_{i>0} I_i^o$ of the open intervals $I_i^o = (-1,1)$, and regard the Hilbert Cube, $Q$, as the countable infinite product $\prod_{i>0} I_i$
of the closed intervals $I_1 = [-1, 1]$. Note that $s$ is a subset of $Q$. Also, if $l_2$ denotes the space of square summable sequences of reals with the norm topology, then by [1], $s$ and $l_2$ are homeomorphic. In fact, it has been shown as a result of work by Bessaga and Pelczynski [9], Kadec [18] and Anderson [1] that all separable infinite dimensional Fréchet spaces are homeomorphic to $s$, where by a Fréchet space we mean a locally convex complete linear metric space. Thus, as in [8], we define a Fréchet manifold (or $F$-manifold) to be a separable metric space having an open cover, each element of which is homeomorphic to $s$. Also, as in [12], a Hilbert Cube manifold (or $Q$-manifold) is a separable metric space having an open cover where each element of the cover is homeomorphic to an open subset of $Q$. For each $i > 0$, let $\tau_i$ be the projection function of $l_2$, $s$ or $Q$ on its $i^{th}$ coordinate space; that is, $\tau_i((x_n)_{n>0}) = x_i$. A subset $K$ of $l_2$, $s$ or $Q$ is said to have infinite deficiency if for infinitely many $i$, $\tau_i(K)$ is a point.

The concept of a Z-set was introduced by Anderson in [3] as a means of giving a topological characterization of infinite deficiency. Since that time Z-sets have been widely used in the study of the topology of $l_2$, $s$, $Q$ and of manifolds modeled on these or other infinite dimensional spaces. As this paper is concerned with certain properties and uses of Z-sets, we include here a brief collection of
examples and properties of Z-sets along with some of their more important applications.

We first give some alternate definitions of Z-sets which are all equivalent to the original definition in the case that the separable metric space X is $\mathbb{R}^2$, $s$, Q or a manifold modeled on any of these.

**Definition 1.1.** K is a Z-set in X if K is closed and for each $n > 0$, each map of the n-cell $I^n$ into X can be uniformly approximated by maps of $I^n$ into $X \setminus K$.

**Definition 1.2.** K is a Z-set in X if K is closed and for each $\epsilon > 0$ there exists an $\epsilon$-small homotopy of X off K; that is, for each $\epsilon > 0$, there exists a map $H$ of $X \times I$ into X such that $H(x,0) = x$, $H(X \times \{1\}) \subset X \setminus K$ and $d(H(x,t),x) < \epsilon$.

**Definition 1.3.** K is a Z-set in X if K is closed and for each $x \in K$, there exists an open set $U_x$ in $X$, $x \in U_x$, such that $K \cap U_x$ is a Z-set in $U_x$ under the original definition of a Z-set.

The following are some useful elementary examples of Z-sets. Others may be developed by using the definition or the properties 1.1 to 1.4 which follow.

**Example 1.1.** Any compact subset of $B(Q)$ is a Z-set in Q, where $B(Q)$ is the pseudo-boundary of
Q; i.e., \( B(Q) = \{ x \in Q : \text{there exists } i > 0 \text{ with } \tau_1(x) = 1 \text{ or } -1 \} \).

**Example 1.2.** Any closed subset of infinite deficiency in \( Q, \ell_2, \) or \( s \) is a Z-set.

**Example 1.3.** Any compact subset of \( s, \ell_2 \) or an F-manifold is a Z-set.

**Example 1.4.** Let \( X \) be a Q- or F-manifold. Then any closed subset of \( X \times \{1\} \) in \( X \times (0, 1] \) is a Z-set in \( X \times (0, 1] \). (For such \( X, X \times (0, 1] \) is also a Q- or F-manifold).

Now we list several elementary properties of Z-sets, some of which are used in later proofs. In the following, let \( X = Q, s, \ell_2 \) or a Q- or F-manifold.

**Property 1.1.** If \( K \) is a Z-set in \( X \) and \( K' \) is a closed subset of \( K \), then \( K' \) is a Z-set in \( X \).

**Property 1.2.** If \( K \) is a closed subset of \( X \) which is the countable union of Z-sets in \( X \), then \( K \) is a Z-set in \( X \).

**Property 1.3.** A closed set \( K \) in \( Q \) is a Z-set in \( Q \) if and only if \( K \cap s \) is a Z-set in \( s \).

**Property 1.4.** If \( K \) is a Z-set in \( X \) and \( K' \) is a closed subset of \( X \), then \( K \setminus K' \) is a Z-set in \( X \setminus K' \).
Let $\mathcal{H}(X)$ denote the set of homeomorphisms of a space $X$ onto itself. The following two theorems were the first important results involving $Z$-sets.

**Theorem 1.1 [3]** A closed subset $K$ of $X (X = s, l_2$ or $Q)$ is a $Z$-set in $X$ if and only if there exists $h \in \mathcal{H}(X)$ such that $h(K)$ has infinite deficiency in $X$.

**Theorem 1.2 [3]** Let $K$ be a $Z$-set in $X = s, l_2$ or $Q$ and let $h$ be a homeomorphism of $K$ into $X$. Then there exists $H \in \mathcal{H}(X)$ such that $H|K = h$ if and only if $h(K)$ is a $Z$-set in $X$.

These results have been generalized to $F$-manifolds in [7] and to $Q$-manifolds by Anderson and Chapman. We cite the homeomorphism extension theorem of [7] here as it is used in later proofs. A **homotopy** $H$ of a subset $K$ of $X$ into $X$ is a continuous one-parameter family of continuous functions $H_t, 0 \leq t \leq 1$, of $K$ into $X$ such that $H_0$ is the inclusion map. If $H$ is a homotopy of $K$ into $X$ and $\mathcal{U}$ is an open cover of $X$, then $H$ is **pathwise limited** by $\mathcal{U}$ if for each $x \in K$, $\{H_t(x) : 0 \leq t \leq 1\}$ is contained in some element of $\mathcal{U}$.

**Theorem 1.3 [7]** Let $X$ be an $F$-manifold, let $K_1$ and $K_2$ be $Z$-sets in $X$, let $\mathcal{U}$ be an open cover of $X$, and let $h$ be a homeomorphism of $K_1$ onto $K_2$. If there is a homotopy $H$ of $K_1$ into $X$ such that $H_1 = h$ and $H$ is
pathwise limited by $\mathcal{U}$, then $h$ can be extended to $h^* \in \mathcal{U}(X)$ where $h^*$ is limited by $\text{st}^h(\mathcal{U})$.

The next two theorems are concerned with the concept of "negligible" sets and are also among the early important results involving Z-sets. Negligibility has played a vital role in much of infinite dimensional topology. A subset $K$ of a space $X$ is strongly negligible if for every open cover $\mathcal{U}$ of $X$, there exists a homeomorphism $h$ of $X$ onto $X\setminus K$ such that $h$ is limited by $\mathcal{U}$; i.e. for each $p \in X$, there exists $U \in \mathcal{U}$ such that both $p$ and $h(p)$ are elements of $U$.

**Theorem 1.4** [8]. A closed set $K$ in an F-manifold $X$ is strongly negligible if and only if $K$ is a Z-set in $X$.

**Theorem 1.5** [4]. A set $K$ in an F-manifold $X$ is strongly negligible if and only if $K$ is a countable union of Z-sets in $X$.

We say that $(M,N)$ is a manifold pair if $M$ and $N$ are both F-manifolds and $N$ is a Z-set in $M$. The following theorem by Henderson indicates how, under special conditions, $N$ may be thought of as the "boundary" of $M$. It is a strong form of his "open embedding theorem" since $M\setminus N$ will be open in $s$.

**Theorem 1.6.** Let $(M,N)$ be a manifold pair such that the inclusion map of $N$ into $M$ induces a homotopy
equivalence of $N$ and $M$. Then there exists an embedding $h$ of $M$ into $s$ such that the topological boundary of $h(M)$ in $s$ is $h(N)$.

In [5] Anderson and in [10] Bessaga and Pelczynski obtained useful and very similar characterizations of certain dense $\sigma$-compact subsets of the Hilbert Cube and of separable infinite dimensional Fréchet spaces. (A set is $\sigma$-compact if it is the countable union of compact sets).

Here we use the terminology of [5]. A subset $M$ of a separable metric space $X$ has the (finite-dimensional) compact absorption property, or (f-d) cap, in $X$ if (1)$M = \bigcup_{n>0} M_n$, where $M_n$ is a (finite-dimensional) compact $Z$-set in $X$ such that $M_n \subset M_{n+1}$, and (2) for each $\epsilon > 0$, each integer $m > 0$ and each (finite-dimensional) compact subset $K$ of $X$, there is an integer $n > 0$ and an embedding $h$ of $K$ into $M_n$ such that $h| (K \cap M_m) = \text{id}$ and $d(h, \text{id}) < \epsilon$. The "finite dimensionality" restriction is understood to be used throughout the definition or not at all. Thus we have two similar properties, the cap and the f-d cap.

Among the examples of such sets we have $B(Q)$ which is a cap-set in $Q$, $s_f = \{x \in s : x$ has at most finitely many non-zero coordinates\} which is an f-d cap-set in $s$, and $\ell_f = \{x \in \ell_2 : x$ has at most finitely many non-zero coordinates\} which is an f-d cap-set in $\ell_2$.

The following are some of the more important properties
of (f-d) cap-sets which are generalized in Chapter II. They are results of Anderson [5] or Bessaga and Pełczyński [10] in the case that $X = Q, l_2$ or $s$, of Chapman [12] in the case that $X$ is an F- or Q-manifold and of Torunczyk [20] for more general spaces.

A-1. If $M$ and $N$ are both cap-sets in $X$ or are both f-d cap-sets in $X$ and $\mathcal{U}$ is an open cover of $X$, then there exists $h \in \mathcal{W}(X)$ such that $h(M) = N$ and $h$ is limited by $\mathcal{U}$.

A-2. If $M$ is an (f-d) cap-set in $X$ and $K$ is a Z-set in $X$, then $M \setminus K$ is an (f-d) cap-set in $X$.

A-3. If $M$ is an (f-d) cap-set in $X$ and $N$ is a countable union of compact Z-sets in $X$ such that $M \subset N$, then $N$ is an (f-d) cap-set in $X$.

In Chapters II and III we study two further properties related to Z-sets. In Chapter II we are concerned with sets which have the so-called Z-set absorption property. This property is similar to the compact absorption property except the sets under consideration here are just Z-sets which in F-manifolds need not be compact. In the Hilbert Cube where the closed sets are compact, the two properties are the same. In this chapter we develop properties which are directly analogous to A-1, A-2 and A-3 listed above. In Chapter III we determine which spaces
are absolute Z-sets; that is, have the property that the image of any closed embedding of the space into $\ell_2$ is a Z-set in $\ell_2$. Essentially this is done by using Wong's result [22] that there exists a closed subset of $\ell_2$ which is homeomorphic to the space of irrationals but is not a Z-set in $\ell_2$, and showing that certain spaces admit closed embeddings into $\ell_2$ such that their images contain this copy of the irrationals.

We now present some more notation which is used throughout this paper. Let $U$ be a subset of a metric space $(X,d)$. Then $\text{diam } U = \sup\{d(x,y): x,y \in U\}$. If $\mathcal{U}$ is a collection of subsets of $X$, then $\text{mesh } (\mathcal{U}) = \sup\{\text{diam } U: U \in \mathcal{U}\}$. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open subsets of a space $X$. Then $\mathcal{A}$ is a refinement of $\mathcal{B}$ if for each $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subset B$. If $A$ is a subset of $X$ and $\mathcal{B}$ is a collection of subsets of $X$, then $\text{st}(A,\mathcal{B}) = \bigcup\{B \in \mathcal{B}: B \cap A \neq \emptyset\}$ and $\text{st}(\mathcal{B}) = \text{st}^0(\mathcal{B}) = \{\text{st}(B,\mathcal{B}): B \in \mathcal{B}\}$. Also, for each integer $n > 0$, $\text{st}^n(\mathcal{B}) = \text{st}(\text{st}^{n-1}(\mathcal{B}))$. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open subsets of $X$. We say that $\mathcal{A}$ is a star-refinement of $\mathcal{B}$ if $\text{st}(\mathcal{A})$ refines $\mathcal{B}$. If $\mathcal{B}$ is a cover of $X$, we will require that $\mathcal{A}$ also be a cover of $X$. The set of positive integers will be denoted by $\mathbb{N}$ and the closure of a set $A$ will be denoted by $\bar{A}$ or $\text{cl}(A)$. 
CHAPTER II

SETS WITH THE Z-SET ABSORPTION PROPERTY

If $\mathcal{U}$ is a collection of open sets in a space $X$, let $\mathcal{U}^*$ denote the union of the elements of $\mathcal{U}$. We say that $h \in \mathcal{N}(X)$ is \underline{limited} by a collection $\mathcal{U}$ of open sets in $X$ if

1. For each $x \in \mathcal{U}^*$, there exists $U \in \mathcal{U}$ such that both $x$ and $h(x)$ are in $U$ and
2. $h|_{X \setminus \mathcal{U}^*} = \text{id}$.

A subset $M$ of a separable metric space $X$ is said to have the Z-set \underline{absorption property} (or to be a \underline{Zap-set}) in $X$ if $M = \bigcup_{i > 0} M_i$ where

1. For each $i > 0$, $M_i$ is a Z-set in $X$;
2. For each $i > 0$, $M_i \subseteq M_{i+1}$; and
3. For each Z-set $A$ in $X$, for each closed subset $B$ of $A$ with $B \subseteq M$, and for each open (in $X$) cover $\mathcal{U}$ of $A$, there exists $h \in \mathcal{N}(X)$ such that $h(A) \subseteq M, h|_{B} = \text{id}$ and $h$ is limited by $\mathcal{U}$.

The concept of the Z-set absorption property is suggested by the (f-d) cap property as described in Chapter I or by the similar concept of $(G, \mathcal{K})$-skeletons which H. Torunczyk discusses in [20] and which were introduced by Bessaga and Pelczynski in [10]. In [12], Chapman has studied applications of (f-d) cap-sets to problems concerning F- and Q-manifolds using techniques more specifically directed to manifold questions.

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In this Chapter we first present an example of a Zap-set in $s$ and then follow the general outline of the study of $(G,\mathcal{X})$-skeletonized sets in Torunczyk's paper [20] in studying the properties of Zap-sets. However, the $(G,\mathcal{X})$-skeletonized sets are $\sigma$-compact while for $s$, $l_2$ or $F$-manifolds, Zap-sets are not $\sigma$-compact so that much of the apparatus must be carefully redone.

Section 2.1. Preliminaries

An isotopy $\varphi$ of a subset $D$ of a space $X$ into $X$ is a continuous one-parameter family of homeomorphisms $\varphi_t$, $0 \leq t \leq 1$, of $D$ into $X$ such that $\varphi_0$ is the inclusion map. Let $D$ be a subset of a space $X$, let $\mathcal{U}$ be an open cover of $X$ and let $\varphi$ be an isotopy of $D$ into $X$. Then $\varphi$ is pathwise limited by $\mathcal{U}$ if for each $x \in D$, there exists $U \in \mathcal{U}$ such that $\{\varphi_t(x):0 \leq t \leq 1\} \subseteq U$. If $\varphi$ is an isotopy of $D \subseteq X$ into $X$ and $0 \leq a < b \leq 1$, define the isotopy $\varphi[a,b]$ by

$$
\varphi[a,b]_t = \begin{cases} 
\varphi_0 & , 0 \leq t \leq a \\
(\varphi(t-a)/(b-a)) & a \leq t \leq b \\
\varphi_1 & , b \leq t \leq 1 
\end{cases}
$$

For each $i > 0$, let $f_i$ be a homeomorphism of some subset $D_i$ of a space $X$ into $X$ such that $f_i(D_i) = D_{i+1}$. If the sequence $(f_1 \circ \ldots \circ f_i)_{i \geq 0}$ converges pointwise to a homeomorphism $f$ of $D_\perp$ into $X$, then we call $f$ the infinite
left product of \((f_i)_{i>0}\) and write \(f = \Pi_{i>0} f_i\). If \(f\) is a homeomorphism of a subset \(D\) of \(s\) into \(s\), let \(\alpha(f) = \{i \in \mathbb{N}: \text{there exists } x \in D \text{ with } \tau_i(f(x)) \neq \tau_i(x)\}\).

An open cover of a space \(X\) is said to be **star-finite** provided that the closure of each element of the cover intersects the closure of only finitely many other elements of the cover. As in [8], an open cover \(\mathcal{U}\) of an open subset \(U\) of a space \(X\) is said to be a **normal cover** provided that whenever \(f\) is a homeomorphism of \(U\) into itself which is limited by \(\mathcal{U}\), there is an extension of \(f\) to a homeomorphism of \(X\) into itself which is the identity on \(X \setminus U\). This extension, denoted \(f^*\), is said to be the **normal extension** of \(f\). A subset of \(s\) of the form \(\Pi_{i>0} J_i^o\) is called a **basic open set in** \(s\) if for each \(i>0\), \(J_i^o\) is an open subinterval of \(I_i^o = (-1,1)\) and \(J_1^o = I_1^o\) for all but finitely many \(i\).

The following known lemmas concerning refinements are employed in proofs of theorems contained in later sections. The first is non-trivial and is implied by Theorem 1 of [8].

**Lemma 2.1.** Let \(U\) be an open subset of \(s\) and let \(\mathcal{U}\) be an open cover of \(U\). Then there is an open cover \(\mathcal{V}\) of \(U\) such that:

1. \(\mathcal{V}\) is a star-refinement of \(\mathcal{U}\);
2. \(\mathcal{V}\) is a normal cover of \(U\); and
(3) each element of $\mathcal{V}$ is a basic open set in $s$.

The next lemma is found in [14, p. 167].

**Lemma 2.2.** Let $\mathcal{U}$ be an open cover of a metric space $X$. Then for each $n > 0$, there is an open cover $\mathcal{V}_n$ of $X$ such that $\text{st}^n(\mathcal{V}_n)$ is a refinement of $\mathcal{U}$.

Section 2.2. A General Example of a Zap-set

In this section we construct an example of a Zap-set in $s$, not only to show that a nice one exists, but also to provide some intuition for the proofs of several theorems which will be given later.

Let $\{a_i\}_{i>0}$ be a collection of infinite subsets of $N$ such that (1) $N\setminus a_1$ is infinite, (2) for each $i > 0$, $a_{i+1} \subseteq a_i$ and (3) $\cap_{i>0}a_i = \emptyset$. For each $i > 0$, let $M_i = \{x \in s : \tau_n(x) = 0 \text{ for each } n \in a_i\}$.

**Theorem 2.1.** If $M = \bigcup_{i>0}M_i$, then $M$ is a Zap-set in $s$.

Since for each $i > 0$, $M_i$ is a closed set of infinite deficiency in $s$, each $M_i$ is a Z-set in $s$ (Theorem 1.1) and since $a_{i+1} \subseteq a_i$, we have $M_i \subseteq M_{i+1}$. Thus the first two conditions in the definition of a Zap-set are satisfied.

To show that the third condition is also satisfied, we need several lemmas which will now be given. The first gives a condition under which the infinite left product of isotopies
will be an isotopy.

**Lemma 2.3.** For each $n > 0$, let $\varphi_t^n$ be an isotopy of a subset $D_n$ of $s$ into $s$ such that $\varphi_t^n(D_n) = D_{n+1}$ and $\delta_1 = \bigcup_{t > 0} \varphi_t^n$ exists. If for each $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $0 \leq t \leq 1$, $i \in \alpha(\varphi_t^n)$ implies $i > k$, then $\delta$ defined by $\delta_1 = \bigcup_{t > 0} \varphi_t^n[1 - 1/n, 1 - 1/(n+1)]_t$ is an isotopy.

**Proof:** Since for each $t$, $0 \leq t < 1$, $\delta_t^n$ is a finite composition of homeomorphisms and $\delta_1$ is assumed to be a homeomorphism of $D_1$ into $s$, we have that for each $t$, $\delta_t^n$ is a homeomorphism of $D_1$ into $s$. It is also clear that $\delta$ is continuous for $t < 1$. To show continuity at $t = 1$, we let $x_0 \in D_1$, let $(x_m)_{m \geq 0}$ be a sequence in $D_1$ which converges to $x_0$ and let $\epsilon > 0$. We now find $m_0 \in \mathbb{N}$ and $t_0 < 1$ such that for any $m > m_0$ and $t > t_0$, $d(\delta_t^n(x_m), \delta_t(x_0)) < \epsilon$.

Let $k \in \mathbb{N}$ such that $\sum_{i=k}^{\infty} 1/2^i < \epsilon/3$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $t, 0 \leq t \leq 1$, $i \in \alpha(\varphi_t^n)$ implies $i > k$, so that for $t, t' \geq t_0 = 1 - 1/n_0$ and $j \leq k$, we have that $\tau_j(\delta_t^n(x)) = \tau_j(\delta_{t'}(x))$ for each $x \in D_1$ and thus $d(\delta_t^n(x), \delta_{t'}(x)) < \sum_{i=k}^{\infty} 1/2^i < \epsilon/3$. Since $\delta_{t_0}^1$ is a homeomorphism, there exists $m_0 \in \mathbb{N}$ such that for every $m > m_0$, $d(\delta_{t_0}^1(x_m), \delta_{t_0}^1(x_0)) < \epsilon/3$. Now for any $m > m_0$ and
t > t_0, \ d(\dot{s}_1(x_0), \dot{s}_t(x_m)) \leq d(\dot{s}_1(x_0), \dot{s}_{t_0}(x_0)) \\
+ d(\dot{s}_{t_0}(x_0), \dot{s}_{t_0}(x_m)) + d(\dot{s}_{t_0}(x_m), \dot{s}_t(x_m)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.

Therefore, \dot{s} is continuous at \ t = 1 \ and \ thus \ is \ an \ isotopy. \ □

The next remark and lemma provide an isotopy which carries a subset of a copy of \ s onto the copy by means of rotating coordinates.

**Remark 2.1.** Let \ I^o \ be the open interval \ (-1,1). For each \ t, 0 < t < 1, and for each \ (o,x) \in \{o\} \times I^o, \ let \ y_t(o,x) = (x \cos \Pi(1-t)/2, x \sin \Pi(1-t)/2). \ Then \ \gamma = \{y_t\}_{0 < t < 1} \ is \ an \ isotopy \ of \ \{o\} \times I^o \ into \ I^o \times I^o \ such \ that \ y_1(o,x) = (x,o).

**Lemma 2.4.** Let \ n \in \mathbb{N} \ and \ let \ \mathcal{B} \ be an infinite subset of \ \mathbb{N} \ such \ that \ n \in \mathcal{B}. \ Then \ there \ exists \ an \ isotopy \ \dot{s} \ of \ \{o\} \times \Pi_{j \geq n} I^o_j \ into \ \Pi_{j \geq n} I^o_j \ such \ that \ 
\dot{s}_1([o] \times \Pi_{j \geq n} I^o_j) = \Pi_{j \geq n} I^o_j \ and \ for \ each \ t, 0 < t < 1, \alpha(\dot{s}_t) \in \mathcal{B}.

**Proof:** Let \ \{(n_i)_{i > 0}\} \ be an increasing sequence of elements of \ \mathcal{B} \ such \ that \ n_1 = n. \ For \ each \ i > 0, \ let \lambda_i = \{j \in \mathbb{N} : j \geq n \text{ and } j \neq n_i, n_{i+1}\} \ and \ let \ \phi^1_{i} \ be \ the \ isotopy \ defined \ by \ \phi^1_{j}(x,y) = (x, y_{t_j}(y)) \ where \ x \in \Pi_{j \in \lambda_i} I^o_j \ and \ y = (o,y') \in I^o \times I^o. \ Note \ that \ \phi^1_{i} \ is \ a \ homeomorphism \ of \ \Pi_{j \geq n, j \neq n_{i+1}} I^o_j \times \{o\} \ onto \ \Pi_{j \geq n, j \neq n_{i+1}} I^o_j \times \{o\}. \ Since \ \lim_{i \to \infty} \phi^1_{i} \circ \ldots \circ \phi^1_{1}(o,x_{n+1},x_{n+2}, \ldots) = (y_n,y_{n+1}, \ldots)
where $y_j = x_{n_{1}+1}$ if $j = n_1$ for some $i$, and $y_j = x_j$ if $j \neq n_1$, we have that $\xi_1 = \prod_{i>0}^{i} \varphi^i_1$ exists and in fact $\xi_1([o] \times \prod_{i>n}^{i} \varphi^i_1) = \prod_{i>n}^{i} \varphi^i_1$. It is also clear that the other condition of Lemma 2.3 is satisfied. Thus $\phi$, defined by $\phi_t = \prod_{i>0}^{i} \varphi^i_1[1-l/l,1-l/(i+1)]_t$ is an isotopy fulfilling the requirements of this lemma. □

A set $U$ is an m-basic open set in $s$ if $U = \prod_{i>0}^{i} J^o_i$ where for $0 < i \leq m$, $J^o_i$ is an open subinterval of $(-1,1)$ and for $i > m$, $J^o_i = (-1,1)$. In the following $M$ is as defined in Theorem 2.1.

**Lemma 2.5.** Let $U$ be an m-basic open set in $s$ and let $n > m$. Then there exists an isotopy $h$ of $s$ into $s$ such that:

1. for each $t$, $0 \leq t \leq 1$, $h_t|s \backslash U = id$;
2. for each $p \in U$, $h_1(p) \in M$;
3. for each $i \leq n$, each $p \in s$ and each $t$, $\tau_i(h_t(p)) = \tau_i(p)$.

**Proof:** Since in the definition of $M$ we have that for each $i > o$, $\alpha_{i+1} \subset \alpha_i$ and that $\cap_{i>0} \alpha_i = \emptyset$, then there exists $k \in N$ such that $j \in \alpha_k$ implies $j > n$. Let $\alpha_k = \{m_i\}_{i>0}$ where $m_i < m_j$ if and only if $i < j$. As $N \backslash \alpha_k$ is infinite, let $[\beta_i]_{i>0}$ be a collection of pairwise disjoint infinite subsets of $N$ such that $j \in \beta_i$ implies $j \geq m_i$ and $\beta_i \cap \alpha_k = \{m_i\}$. By Lemma 2.4 let $\varphi^1_1$ be an
isotopy of \((0) \times \Pi_{j>m_1} I_j^0\) into \(\Pi_{j \geq m_1} I_j^0\) such that
\[
\varphi_1((0) \times \Pi_{j>m_1} I_j^0) = \Pi_{j \geq m_1} I_j^0
\]
and for each \(t\), \(\alpha(\varphi_t^1) \subset \beta_1\).

Now, applying Lemma 2.4 again, for each \(i > o\) let \(\varphi_i^1\) be
an isotopy of \(\Pi_{j=m_1}^{m_1-1} I_j^0 \times (0) \times \Pi_{j>m_1} I_j^0\) into \(\Pi_{j \geq m_1} I_j^0\) such
that \(\varphi_1^1(\Pi_{j=m_1}^{m_1-1} I_j^0 \times (0) \times \Pi_{j=m_1} I_j^0) = \Pi_{j \geq m_1} I_j^0\) and for each
\(t\), \(\alpha(\varphi_t^i) \subset \beta_1\). Also, for each \(i > o\), let
\[D_i = \{x \in \Pi_{j \geq m_1} I_j^0: \text{for each } k \geq i, \tau_{m_k}(x) = o\}.\]
If for each \(i > o\) and each \(t, 0 \leq t \leq 1\), we let \(\psi_t^i = \varphi_t^i|D_1\) then
\(\psi_1(D_1) = D_{i+1}\). By observations similar to those in the
proof of Lemma 2.4, it is easy to see that \(\psi_1 = \Pi_{i>0} \psi_t^i\)
e
exists and in fact \(\psi_1(D_1) = \Pi_{j \geq m_1} I_j^0\). Also the other
condition of Lemma 2.3 is satisfied since \(\alpha(\psi_t^i) \subset \beta_1\) and
\(j \in \beta_1\) implies \(j \geq m_1\). Thus \(\psi\) defined by
\(\psi_t = \Pi_{i>0} \psi^i[\frac{1}{1-i/1,1-1/(i+1)}]_t\) is an isotopy.

For each \(t, 0 \leq t \leq 1\), define \(g_t\) of \(\Pi_{j \geq m_1} I_j^0\) into
itself by \(g_t = \psi_{1-t} \circ \psi_t^{-1}\). Then \(g_0 = id\) and
\(g_1(\Pi_{j \geq m_1} I_j^0) = D_1\). In fact for each \(t \neq o\), there exists
\(i > o\) such that \(g_t(\Pi_{j \geq m_1} I_j^0) \subset \{x \in \Pi_{j \geq m_1} I_j^0: \tau_n(x) = o \text{ for every } n \in \alpha_1\}\).

Let \(u\) be a map of \(s\) into \([0,1]\) such that
\(\{x \in s: u(x) = o\} = S \setminus U\). For each \(t, 0 \leq t \leq 1\), define \(h_t\)
of \(s\) into \(s\) by \(h_t(x, y) = (x, g_u(x, y), t(y))\) where
\(x \in \Pi_{j < m_1} I_j^0\) and \(y \in \Pi_{j \geq m_1} I_j^0\). Then \(h\) is an isotopy of
\(s\) into \(s\) such that:
(1) $h_0 = \text{id}$ since $h_0(x,y) = (x,g_0(y)) = (x,y)$;
(2) $h_t|s\setminus U = \text{id}$ since $(x,y) \in s\setminus U$ implies $u(x,y) = o$ and thus $h_t(x,y) = (x,g_0(y)) = (x,y)$;
(3) for each $(x,y) \in U$, $h_1(x,y) = (x,g_u(x,y)(y)) \in M$ since $u(x,y) \neq o$; and
(4) for each $i \leq n$, each $p \in s$ and each $t$,
$\tau_i(h_t(p)) = \tau_i(p)$ since $i \leq n$ implies $i < m_1$.

Hence, $h$ is the desired isotopy. □

The following convergence procedure which is Theorem 2 of [8] is used in the proof of Theorem 2.1.

**Convergence Procedure A:** Let $\mathcal{U}$ be a countable star-finite open cover of a space $X$. Then there exists an ordering $\{U_i\}_{i>0}$ of the members of $\mathcal{U}$ such that for any sequence $(f_i)_{i>0}$ of homeomorphisms of $X$ into itself for which $f_i$ is the identity on $X\setminus U_i$, we have that $f = \lim_{i>0} f_i$ exists.

**Proof of Theorem 2.1:** Now we are in a position to show that $M$ satisfies the third condition in the definition of a Zap-set. Let $A$ be a $Z$-set in $s$, let $B$ be a closed subset of $A$ such that $B \subset M$, and let $\mathcal{U}$ be an open (in $s$) cover of $A$. (Note that $A$ is a $Z$-set in $\mathcal{U}^*$ and that in the following we use only this fact and not that $A$ is a $Z$-set in $s$). By Lemma 2.1, let $\mathcal{U}'$ be a normal open cover of $\mathcal{U}^*$ such that $\mathcal{U}'$ is a refinement of $\mathcal{U}$.
By Lemma 2.2, let \( \mathcal{U} \) be an open cover of \( \mathcal{U}^* \) such that \( \text{st}^4(\mathcal{U}) \) is a refinement of \( \mathcal{U}' \). Applying Lemma 2.1 again, let \( \mathcal{V} = \{ V_i \}_{i>0} \) be a normal star-finite open cover of \( \mathcal{U}^* \setminus \mathcal{B} \) such that for each \( i > 0 \), \( V_i \) is a basic open set in \( s \), \( V_i \cap \mathcal{B} = \emptyset \) and \( \mathcal{V} \) is a refinement of \( \mathcal{U} \). Assume that \( \mathcal{V} \) has been ordered in accordance with Convergence Procedure A.

For each \( i > 0 \), let \( n_i = \max \{ n \in \mathbb{N} : V_j \in \mathcal{V} \text{ is an } n \text{-basic open set and } V_j \cap V_1 \neq \emptyset \} \). Then by Lemma 2.5, for each \( i > 0 \), there exists an isotopy \( h_1 \) of \( s \) into \( s \) such that:

1. for each \( p \in V_1 \), \( h_1^i(p) \in M \);
2. for each \( t, 0 \leq t \leq 1 \), \( h_1^i|_{s \setminus V_1} = \text{id} \); and
3. for each \( n \leq n_i \), each \( p \in s \) and each \( t, \tau_n(h_1^i(p)) = \tau_n(p) \).

Now for each \( t, 0 \leq t \leq 1 \), let \( h_t = \text{lim}_{i \to \infty} h_1^i(\mathcal{U}^* \setminus \mathcal{B}) \) (which exists by Convergence Procedure A). Then \( h = \{ h_t \}_{0 \leq t \leq 1} \) is an isotopy of \( \mathcal{U}^* \setminus \mathcal{B} \) into itself as is seen by observing that \( H_1 \) defined on \( s \times [0,1] \) by \( H_1(x,t) = (h_t^1(x),t) \) has the property that \( H_1|_{(s \times [0,1]) \setminus (V_1 \times [0,1])} = \text{id} \) so that \( H = \text{lim}_{i \to \infty} H_1 \) exists. We wish to prove that \( h \) is pathwise limited by \( \mathcal{V} \). To do this we let \( x \in V_1 \in \mathcal{V} \) and show that for each \( t, 0 \leq t \leq 1 \), \( h_t(x) \in V_1 \). Assume that \( V_1 \) is an \( m \)-basic open set in \( s \) and let \( V_j \) be any
element of \( \mathcal{V} \). If \( V_j \cap V_i \neq \emptyset \), then \( h^J_j \) has the property that for \( n \leq n_j \) and \( 0 \leq t \leq 1 \), 
\[ \tau_n(h^J_t(p)) = \tau_n(p) \]
for any \( p \in s \). However, for \( k \leq m \), 
\[ k \leq n_j \]
since \( V_1 \cap V_j \neq \emptyset \), so that \( \tau_k(h^J_t(p)) = \tau_k(p) \) and 
\[ h^J_t(p) \in V_1 \]
if \( p \in V_i \). If \( V_j \cap V_i = \emptyset \), then for 
\[ p \in V_1, h^J_t(p) = p. \]
Thus by the above and the fact that 
\( \mathcal{V} \) is star-finite, for each \( t, 0 \leq t \leq 1 \), 
\[ h^J_t(x) \in V_1 \]
if \( x \in V_i \) and hence \( h \) is pathwise limited by \( \mathcal{V} \). Since for each \( t \), \( h^J_t \) is limited by the normal cover \( \mathcal{V} \), we may think of \( h^J_t \) as having been defined so as to be the identity on \( B \) and therefore we may regard \( h \) as an isotopy of \( \mathcal{U}^* \) into itself which is pathwise limited by \( \mathcal{U} \) (\( \mathcal{V} \) is a refinement of \( \mathcal{U} \)). Hence \( g = h^J_1|A \) is a homeomorphism of \( A \) into \( M \) which is isotopic to the identity by an isotopy, namely \( h \), which is pathwise limited by \( \mathcal{U} \). Also \( g(A) \) is a Z-set in \( \mathcal{U}^* \) since each \( g(A) \cap M_1 \) is. Using the Homeomorphism Extension Theorem 1.3, there exists \( f' \in \mathcal{K}(\mathcal{U}^*) \) such that \( f'|A = g \) and \( f' \) is limited by \( \text{st}^4(\mathcal{U}) \) and thus \( \mathcal{U}' \). Let \( f \) be the normal extension of \( f' \) to the identity on \( s \setminus \mathcal{U}^* \). Then \( f \in \mathcal{K}(s), f(A) \subset M, f|B = \text{id} \) and \( f \) is limited by \( \mathcal{U} \). Therefore, \( M \) is a Zap-set in \( s \). \( \square \)

**Corollary 2.1.** If \( U \) is an open set in \( s \), then \( U \cap M \) is a Zap-set in \( U \).

**Proof:** For each \( i > o, M_i \cap U \) is a Z-set in \( U \).
and \( M \cap U = \bigcup_{i>0} (M_i \cap U) \). Let \( A \) be a Z-set in \( U \), let \( B \) be a closed subset of \( A \) with \( B \subset M \cap U \) and let \( \mathcal{U} \) be an open cover of \( A \). Then the proof of Theorem 2.1 applies and \( M \cap U \) is a Zap-set in \( U \). □

Example 2.1. Define \( f \) of \( s_f \times s \) into \( s \) by
\[
 f(x,y) = z \text{ where } \tau_k(z) = \tau_{(k+1)/2}(x) \text{ if } k \text{ is odd and } \tau_k(z) = \tau_{k/2}(y) \text{ if } k \text{ is even. Clearly, } f \text{ is an embedding of } s_f \times s \text{ into } s. \]
For each \( i > 0 \), let \( a_i = \{2n-1\}_{n \geq i} \) and let \( M_i = \{x \in s: f(x) = 0 \text{ if } j \in a_i \} \).
Letting \( M = \bigcup_{i>0} M_i \), we have that \( M = f(s_f \times s) \) and by Theorem 2.1, \( M \) is a Zap-set in \( s \).

Section 2.3. Some Basic Results Concerning Zap-sets

The theorems in this section show that Zap-sets have properties which are directly analogous to the Properties A-1, A-2 and A-3 given in Chapter I. Since many of the proofs, as presented here, require results of Chapman on Z-sets and these results are not known in general, these properties are shown only for Zap-sets in \( F \)- and \( Q \)-manifolds. As we also prove that the Z-set absorption property and the compact absorption property are equivalent in \( Q \)-manifolds, many of the theorems in this section are stated for Zap-sets in \( F \)-manifolds only.

The next remark is immediate since the concept of a Zap-set is topological.
Remark 2.2. If $M$ is a Zap-set in a separable metric space $X$ and $h$ is a homeomorphism of $X$ onto $Y$, then $h(M)$ is a Zap-set in $Y$.

Example 2.2. By a result of [5] or [10], there is a homeomorphism $h$ of $s$ onto $\ell_2$ such that $h(s_f) = \ell_f$. Let $f$ be the homeomorphism of $s \times s$ onto $\ell_2 \times \ell_2$ defined by $f(x,y) = (h(x), h(y))$. Then $f(s_f \times s) = \ell_f \times \ell_2$ and thus by Remark 2.2 and Example 2.1, $\ell_f \times \ell_2$ is a Zap-set in $\ell_2 \times \ell_2$.

For the remainder of this chapter, if $f_1, \ldots, f_\lambda$ are homeomorphisms, we let $F_1 = f_1 \circ \ldots \circ f_\lambda$. Also, if $\mathcal{U}$ is a cover of a space $X$ and $h \in \mathcal{W}(X)$, let $h(\mathcal{U}) = \{h(U) : U \in \mathcal{U}\}$.

The following convergence procedure is Lemma 1 of [21] and is a modification of the inductive convergence criterion in [6]. It, together with the next lemma, provides a method for obtaining a sequence of homeomorphisms whose infinite left product exists and is limited by a given cover.

**Convergence Procedure B.** Let $(f_n)_{n>0}$ be a sequence of homeomorphisms of a complete metric space $X$ onto itself, and let $\mathcal{U}$ be any open cover of $X$. If $[\mathcal{U}_n]_{n>0}$ is a collection of open covers of $X$ such that (1) $\text{st}_2(\mathcal{U}_0)$ is a refinement of $\mathcal{U}$, (2) for each $n>0$, ...
mesh $u_n < 1/2^n$ and $u_n$ is a star-refinement of $u_{n-1}$, and (3) for each $n > 0$, $f_{n+1}$ is limited by $u_n$ and mesh $(F^{-1}_n(u_n)) < 1/2^n$, then $f = \lim_{n \to \infty} f_n \in \mathcal{K}(X)$ and $f$ is limited by $u$.

Lemma 2.6. Let $\mathcal{U}$ be an open cover of a metric space $X$, let $\varepsilon > 0$ and let $f_n, \ldots, f_1$ be in $\mathcal{K}(X)$. Then there exists an open cover $\mathcal{V}$ of $X$ such that $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, mesh ($\mathcal{V}$) < $\varepsilon$ and mesh ($F^{-1}_n(\mathcal{V})$) < $\varepsilon$.

Proof: Since $f_n, \ldots, f_1$ are homeomorphisms, so is $F_n^{-1}$ and hence $F_n^{-1}$ is continuous. Thus for each $p \in X$, let $V_p$ be an open set containing $p$ such that $\text{diam}(F_n^{-1}(V_p)) < \varepsilon/2$ and $V_p$ is contained in a member of $\mathcal{U}$. By Lemma 2.2, let $\mathcal{V}$ be a star-refinement of $(V_p : p \in X)$ such that mesh ($\mathcal{V}$) < $\varepsilon$. □

The following theorem corresponds to Property A-1.

Theorem 2.2. Let $M$ and $N$ be Zap-sets in a topologically complete separable metric space $X$, let $\mathcal{U}$ be any open cover of $X$ and let $A$ be any $Z$-set in $X$ such that $A \subseteq M \cap N$. Then there exists $h \in \mathcal{K}(X)$ such that $h(M) = N$, $h|A = \text{id}$ and $h$ is limited by $\mathcal{U}$.

Proof: Let $M = U_{1>0}M_i$ and $N = U_{1>0}N_i$ as in the definition of a Zap-set. Using condition (3) in this definition, we shall inductively construct a sequence $(K_i)_{i>0}$
of subsets of $\mathcal{M}$, a sequence $(\mathcal{V}_i)_{i \geq 0}$ of open covers of $X$, and a sequence $(f_i)_{i > 0}$ in $\mathcal{N}(X)$ such that:

1. for each $i > 0$, $K_i$ is a $Z$-set in $X$, $A \subset K_i \subset K_{i+1}$ and $M_i \subset K_i \subset M$;

2. $\text{st}^2(\mathcal{V}_0)$ refines $\mathcal{U}$, and for each $i > 0$, $\mathcal{V}_i$ is a star-refinement of $\mathcal{V}_{i-1}$, $\text{mesh}(\mathcal{V}_i) < 1/2^i$ and $\text{mesh}(F_i^{-1}(\mathcal{V}_i)) < 1/2^i$; and

3. $f_i|A = \text{id}$, and for each $i > 0$, $f_{i+1}|F_i(K_i) = \text{id}$, $N_{i+1} \subset F_{i+1}(K_{i+1}) \subset N$ and $f_{i+1}$ is limited by $\mathcal{V}_i$.

Suppose we have done this. Then by Convergence Procedure B, $f = \lim_{i \to \infty} f_i \in \mathcal{N}(X)$ and $f$ is limited by $\mathcal{U}$.

Since $M = \bigcup_{i > 0} K_i$ and for each $i > 0$, $K_i \subset K_{i+1}, F_i(K_i) \subset N$ and $f_{i+1}|F_i(K_i) = \text{id}$, we have that $f(M) \subset N$. Likewise since $N = \bigcup_{i > 0} N_i$ and for each $i > 0$, $N_i \subset F_i(K_i)$ with $f_{i+1}|F_i(K_i) = \text{id}$, $N \subset f(M)$ and thus $f(M) = N$. Also, $f_i|A = \text{id}$ and $A \subset K_i$ implies that $f|A = \text{id}$.

We now begin the inductive process. Let $\mathcal{V}_0$ be an open cover of $X$ such that $\text{st}^2(\mathcal{V}_0)$ refines $\mathcal{U}$ and $\text{mesh}(\mathcal{V}_0) < 1$. Let $\mathcal{V}_0'$ be a star-refinement of $\mathcal{V}_0$. Since $M$ is a Zap-set in $X$ and since $N_1 \cup A$ is a $Z$-set in $X$ with $A \subset M$, there exists $h_1 \in \mathcal{N}(X)$ such that $h_1(N_1) \subset M, h_1|A = \text{id}$ and $h_1$ is limited by $\mathcal{V}_0'$. Let $K_1 = (h_1(N_1) \cup M_1 \cup A) \subset M$. Then $h_1^{-1}(K_1) = N_1 \cup h_1^{-1}(M_1) \cup A$.

Since $N$ is a Zap-set, $h_1^{-1}(K_1)$ is a $Z$-set and $N_1 \cup A \subset N$, there exists $g_1 \in \mathcal{N}(X)$ such that $g_1(h_1^{-1}(K_1)) \subset N, g_1|\left(N_1 \cup A\right) = \text{id}$.
and $g_1$ is limited by $\nu'$. Let $f_1 = g_1 \circ h_1^{-1}$. Then $N_1 \subseteq f_1(K_1) \subset N$, $f_1|A = \text{id}$ and since $h_1^{-1}$ and $g_1$ are limited by $\nu'_o$, $f_1$ is limited by $\text{st}(\nu'_o)$ and thus $\nu'_o$.

Now assume that $K_1, \ldots, K_n, f_1, \ldots, f_n$, and $\nu'_o, \ldots, \nu'_{n-1}$ have been defined so that conditions (1), (2) and (3) are satisfied for each $i$, $1 \leq i \leq n$. By Lemma 2.6, let $\nu_n$ be a star-refinement of $\nu_{n-1}$ such that $\text{mesh} (\nu_n) < 1/2^n$, $\text{mesh} (F_n^{-1}(\nu_n)) < 1/2^n$, and let $\nu'_n$ be a star-refinement of $\nu'_n$. Since $F_n \in \mathcal{V}(X)$, $F_n(M)$ is a Zap-set in $X$ and $N_{n+1} \cup F_n(K_n)$ is a Z-set in $X$. Thus there exists $h_{n+1} \in \mathcal{V}(X)$ such that $h_{n+1}(N_{n+1} \cup F_n(K_n)) \subseteq F_n(M)$, $h_{n+1}|F_n(K_n) = \text{id}$ and $h_{n+1}$ is limited by $\nu'_n$. Let $K_{n+1} = F_n^{-1}[h_{n+1}(F_n(K_n) \cup N_{n+1})] \cup M_{n+1}$. Since $h_{n+1}^{-1} \circ F_n(K_{n+1}) = F_n(K_n) \cup N_{n+1} \cup h_{n+1}^{-1}(F_n(M_{n+1}))$ is a Z-set, there exists $g_{n+1} \in \mathcal{V}(X)$ such that $g_{n+1}(h_{n+1}^{-1} \circ F_n(K_{n+1})) \subseteq N$, $g_{n+1}|[F_n(K_n) \cup N_{n+1}] = \text{id}$ and $g_{n+1}$ is limited by $\nu'_n$. Let $f_{n+1} = g_{n+1} \circ h_{n+1}^{-1}$. Then $K_1, \ldots, K_{n+1}, f_1, \ldots, f_{n+1}$, and $\nu'_o, \ldots, \nu'_{n}$ satisfy conditions (1), (2) and (3) for each $i$, $1 \leq i \leq n+1$. □

Now we are in a position to show the equivalence of the Z-set absorption property and the compact absorption property in Q-manifolds.

**Remark 2.3.** Let $X$ be a Q-manifold. Then $M$ is a Zap-set in $X$ if and only if $M$ is a cap-set in $X$. 
Proof. Let $M$ be a cap-set in $X$. It is clear that $M$ satisfies the first two conditions in the definition of a Zap-set. To show $M$ also satisfies the third, let $A$ be a Z-set in $X$, let $B$ be a closed subset of $A$ with $B \subset M$ and let $\mathcal{U}$ be an open (in $X$) cover of $A$. Since $B$ is closed in $X$, $\mathcal{U}^* \setminus B$ is open and by Lemma 5.4 of [12], $M \cap \mathcal{U}^* \setminus B$ is a cap-set in $\mathcal{U}^* \setminus B$. Also, since $A$ is a Z-set in $X$, $A \setminus B$ is the countable union of compact $Z$-sets and hence by Theorem 6.6 of [12], $M \cup A \setminus B$ is a cap-set in $X$ so that $(M \cup A \setminus B) \cap \mathcal{U}^* \setminus B$ is a cap-set in $\mathcal{U}^* \setminus B$. Let $\mathcal{V}$ be a normal open cover of $\mathcal{U}^* \setminus B$ such that $\mathcal{V}$ refines $\mathcal{U}$. Then by Property A-1, there exists $h \in \mathcal{V}(\mathcal{U}^* \setminus B)$ such that $h((M \cup A \setminus B) \cap \mathcal{U}^* \setminus B) = M \cap \mathcal{U}^* \setminus B$ and $h$ is limited by $\mathcal{V}$. Let $h^* \in \mathcal{V}(X)$ be the normal extension of $h$. Then $h^*(A) \subset M$, $h^* \setminus B = id$ and $h$ is limited by $\mathcal{U}$. Thus $M$ is a Zap-set in $X$.

Now let $M$ be a Zap-set in $X$. By Lemma 5.6 of [12], there exists a cap-set $N$ in $X$ and by the preceding part of this proof, $N$ is a Zap-set in $X$. Also, by Theorem 2.2, there exists $h \in \mathcal{V}(X)$ such that $h(M) = N$. Therefore, $M$ is a cap-set in $X$ since the compact absorption property is topological. □

In the following we are primarily concerned with Fréchet manifolds. The next remark is an immediate consequence of Henderson's open embedding theorem [15], Corollary 2.1
and Remark 2.2.

**Remark 2.4.** Every F-manifold contains a Zap-set.

The following result is used in several later proofs.

**Remark 2.5.** If \( N \) is a Zap-set in an F-manifold \( X \) and \( U \) is an open subset of \( X \), then \( U \cap N \) is a Zap-set in \( U \).

**Proof:** Let \( f \) be an open embedding of \( X \) into \( S \) and let \( M \) be as in Theorem 2.1. Then by Corollary 2.1, \( f(X) \cap M \) is a Zap-set in \( f(X) \) and by Theorem 2.2, there exists \( h \in \mathcal{U}(f(X)) \) such that \( h(f(N)) = f(X) \cap M \). Also \( h(f(U \cap N)) = h(f(U)) \cap h(f(N)) = h(f(U)) \cap M \) which again by Corollary 2.1 is a Zap-set in \( h(f(U)) \). Thus \( U \cap N \) is a Zap-set in \( U \). \( \square \)

Let \( \mathcal{U} \) be an open cover of a space \( X \) and let \( f, g \in \mathcal{U}(X) \). Then we say \( f \) is \( \mathcal{U} \)-close to \( g \) if for each \( x \in X \), there exists \( U \in \mathcal{U} \) such that \( f(x), g(x) \in U \).

**Corollary 2.2.** Let \( M \) and \( N \) be Zap-sets in an F-manifold \( X \) and let \( A \) be a Z-set in \( X \). Then for each \( g \in \mathcal{U}(X) \) such that \( g(A \cap M) = g(A) \cap N \) and for each open cover \( \mathcal{U} \) of \( X \), there exists \( h \in \mathcal{U}(X) \) such that \( h(M) = N \), \( h|A = g|A \) and \( h \) is \( \mathcal{U} \)-close to \( g \).

**Proof:** Let \( N' = g^{-1}(N) \) and let \( \mathcal{V} \) be a normal
open cover of $X \setminus A$ such that $\gamma$ is a refinement of $g^{-1}(\mathcal{U})$. By Remark 2.5, $M \setminus A$ and $N \setminus A$ are Zap-sets in $X \setminus A$ so that by Theorem 2.2, there exists $f \in \mathcal{U}(X \setminus A)$ such that $f(M \setminus A) = N \setminus A$ and $f$ is limited by $\gamma$. Let $f^* \in \mathcal{U}(X)$ be the normal extension of $f$ and let $h = g \circ f^*$. Then $f^*$ is limited by $g^{-1}(\mathcal{U})$ and thus $h$ is $\mathcal{U}$-close to $g$. Also, $h(M) = h((M \setminus A) \cup (M \cap A)) = g(f^*(M \setminus A))$
$\cup g(f^*(M \cap A)) = g(N \setminus A) \cup g(M \cap A) = (N \setminus g(A)) \cup (g(A) \cap N) = N$
and $h|A = g \circ f|A = g|A$. □

**Theorem 2.3.** If $M$ is a Zap-set in an F-manifold $X$ and $K$ is a Z-set in $X$, then $M \setminus K$ is a Zap-set in $X$.

**Proof:** Let $M = \cup_{i>0} M_i$ as in the definition of a Zap-set and for each $i>0$, let $M_i' = \{x \in M_i : d(x, K) \geq 1/i\}$. Then $M_i'$ is a Z-set in $X$, $M_i' \subset M_{i+1}'$ and $M \setminus K = \cup_{i>0} M_i'$.
Thus to show $M \setminus K$ is a Zap-set, let $A$ and $B$ be Z-sets in $X$ with $B \subset A \cap (M \setminus K)$ and let $\mathcal{U}$ be an open (in $X$) cover of $A$. By Lemmas 2.1 and 2.2, let $\mathcal{V}$ be a normal open cover of $\mathcal{U} \setminus B$ such that st($\mathcal{V}$) refines $\mathcal{U}$. Since $A \setminus B$ is a countable union of Z-sets and since $\mathcal{U} \setminus B$ is an F-manifold, by Lemma 7.2 of [12], there exists $f_1 \in \mathcal{U}(\mathcal{U} \setminus B)$ such that $f_1(A \setminus B) \cap K = \emptyset$ and $f_1$ is limited by $\mathcal{V}$. Let $f_1^* \in \mathcal{U}(X)$ be the normal extension of $f_1$. Let $\gamma$ be an open refinement of $\mathcal{V}$ such that $f_1^*(a) \subset \gamma^*$ and $\gamma^* \cap K = \emptyset$.
Since $M$ is a Zap-set in $X$, there exists $f_2 \in \mathcal{U}(X)$ such
that \( f_2(f_1^*(A)) \subseteq M \), \( f_2|f_1^*(B) = \text{id} \) and \( f_2 \) is limited by \( \mathcal{U} \). Note that \( f_1^*|B = \text{id} \) and \( f_2(f_1^*(A)) \cap K = \emptyset \) so that if \( f = f_2 \circ f_1^* \), then \( f(A) \subseteq M \setminus K \), \( f|B = \text{id} \) and \( f \) is limited by \( \mathcal{U} \). Thus \( f \) is the desired homeomorphism and \( M \setminus K \) is a Zap-set in \( X \). \( \Box \)

The following theorem shows that Zap-sets in \( F \)-manifolds are "maximal".

**Theorem 2.4.** Let \( M \) be a Zap-set in an \( F \)-manifold \( X \) and let \( N = \bigcup_{i > 0} N_i \) where \( M \subseteq N \) and for each \( i > 0 \), \( N_i \) is a \( Z \)-set in \( X \). Then \( N \) is a Zap-set in \( X \).

**Proof:** We first consider the case \( N = M \cup K \) where \( K \) is a \( Z \)-set in \( X \). Since \( M \) is a Zap-set in \( X \), there exists \( f_1 \in \mathcal{N}(X) \) such that \( f_1(K) \subseteq M \). Let \( \mathcal{U} \) be a normal open cover of \( X \setminus f_1(K) \). By Remark 2.5, \( M \setminus f_1(K) \) and \( f_1(M) \setminus f_1(K) \) are Zap-sets in \( X \setminus f_1(K) \) so that by Theorem 2.2, there exists \( f_2 \in \mathcal{N}(X \setminus f_1(K)) \) such that \( f_2(f_1(M) \setminus f_1(K)) = M \setminus f_1(K) \) and \( f_2 \) is limited by \( \mathcal{U} \). Let \( f_2^* \) be the normal extension of \( f_2 \) and let \( f = f_2^* \circ f_1 \). Then \( f \in \mathcal{N}(X) \) and \( f(M \cup K) = f(M \setminus K) \cup f(K) = f^*(f_1(M) \setminus f_1(K)) \cup f_2^*(f_1(K)) = (M \setminus f_1(K)) \cup f_1(K) = M \). Thus \( N = M \cup K \) is a Zap-set in \( X \).

Now let \( N = \bigcup_{i > 0} N_i \) where \( M \subseteq N \) and \( N_i \) is a \( Z \)-set in \( X \). Also assume that \( N_i \subseteq N_{i+1} \). Using the previous case we inductively construct a sequence \( (N_i)_{i > 0} \) of open
covers of \( X \) and a sequence \((f_i)_{i>0}\) in \( \mathcal{V}(X) \) such that:

1. For each \( i \geq 0 \), \( F_{i+1}(M) = M \cup N_{i+1} \) \( (f_i = f_{i} \circ \ldots \circ f_1) \);
2. For each \( i \geq 0 \), \( f_{i+1} \mid (F_i(M) \cup N_i) = id \); and
3. For each \( i > 0 \), mesh \((\mathcal{V}_i) < 1/2^i \), mesh \((f_{i}^{-1}(\mathcal{V}_i)) < 1/2^i \) and \( f_{i+1} \) is limited by \( \mathcal{V}_i \).

Assume that we have done this. Then by Convergence Procedure B, \( f = L \Pi_{i>0} f_i \in \mathcal{V}(X) \). Also for each \( p \in M \) there exists \( i > 0 \) such that \( p \in M_i \) so that \( F_i(p) \in N \) and \( f(p) = F_i(p) \) since \( M_i \subset M_{i+1} \) and \( f_{i+1} \mid F_i(M_i) = id \). Thus \( f(M) \subset N \). Now let \( p \in N \) so that \( p \in N_i \) for some \( i \).

Since \( F_i(M) = M \cup N_i \), there exists \( x \in M \) such that \( F_i(x) = p \). As \( N_i \subset N_{i+1} \) and \( f_{i+1} \mid N_i = id \), we have \( f(x) = p \). Therefore \( N \subset f(M) \), \( N = f(M) \) and hence by Remark 2.2, \( N \) is a Zap-set in \( X \).

Beginning the inductive construction we have, by the first case, that \( M \cup N_1 \) is a Zap-set in \( X \) and thus by Theorem 2.2 there exists \( f_1 \in \mathcal{V}(X) \) such that \( f_1(M) = M \cup N_1 \).

By Lemma 2.6 let \( \mathcal{V}_1 \) be an open cover of \( X \) such that mesh \((\mathcal{V}_1) < 1/2 \) and mesh \((f_1^{-1}(\mathcal{V}_1)) < 1/2 \).

Assume \( f_1, \ldots, f_n \) and \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) have been defined so as to satisfy conditions (1), (2) and (3). To obtain \( f_{n+1} \), we observe that \( F_n(M) \) is a Zap-set in \( X \) and \( M \cup N_{n+1} \) is a Zap-set in \( X \). Also \( F_n(M_n) \cup N_n \subset F_n(M) \cap (M \cup N_{n+1}) \). Thus by Theorem 2.2, there exists \( f_{n+1} \in \mathcal{V}(X) \).
such that $f_{n+1}(F_n(M)) = M \cup N_{n+1}$, $f_{n+1}|(F_n(M) \cup N_n) = \text{id}$ and $f_{n+1}$ is limited by $\mathcal{V}_n$. By Lemma 2.6, let $\mathcal{V}_{n+1}$ be an open cover of $X$ such that $\text{mesh}(\mathcal{V}_{n+1}) < 1/2^{n+1}$ and $\text{mesh}(F_{n+1}^{-1}(\mathcal{V}_{n+1})) < 1/2^{n+1}$. □

Section 2.4 Zap-sets in F-manifolds

The following two remarks provide a relationship between (f-d) cap-sets and Zap-sets in F-manifolds.

**Remark 2.6.** Let $N$ be an f-d cap-set in an F-manifold $X$. Then $N \times s$ is a Zap-set in $X \times s$.

**Proof:** Let $f$ be an open embedding of $X$ into $s$. By Lemma 5.4 of [12], $f(N)$ and $s_f \cap f(X)$ are f-d cap-sets in $f(X)$ so that by Theorem 6.1 of [12] there exists $g \in \mathcal{V}(f(X))$ such that $g(f(N)) = s_f \cap f(X)$. Let $h$ be the open embedding of $X \times s$ into $s \times s$ defined by $h(x,y) = (g \circ f(x), y)$. Then $h(N \times s) = (s_f \cap f(X)) \times s = (s_f \times s) \cap h(X \times s)$ and thus by Example 2.1 and Corollary 2.1, $h(N \times s)$ is a Zap-set in $h(X \times s)$ so that $N \times s$ is a Zap-set in $X \times s$. □

**Remark 2.7.** If $M$ is a cap-set in an F-manifold $X$, then $M \times s$ is a Zap-set in $X \times s$.

**Proof:** Let $N$ be an f-d cap-set in $X$. Then by Theorem 8.1 of [12], $N \times Q$ is a cap-set in $X \times Q$ and thus there exists a homeomorphism $g$ of $X$ onto $X \times Q$ such
that \( g(M) = N \times Q \). Let \( g' \) be the homeomorphism of \( X \times s \) onto \( (X \times Q) \times s \) defined by \( g'(x,y) = (g(x),y) \). Let \( h \) be any homeomorphism of \( Q \times s \) onto \( s \) \([2]\), and let \( h' \) of \( X \times (Q \times s) \) onto \( X \times s \) be defined by \( h'(x,y) = (x,h(y)) \). Then \( h' \circ g' \) is a homeomorphism of \( X \times s \) onto itself such that \( h' \circ g'(M \times s) = N \times s \). Thus by Remark 2.6, \( M \times s \) is a Zap-set in \( X \times s \). □

Using arguments similar to the above and the indicated results of Chapman, we have the next three results.

**Remark 2.8.** If \( M \) is a Zap-set in an F-manifold \( X \), then \( M \times s \) is a Zap-set in \( X \times s \). (Remark 2.6)

**Remark 2.9.** If \( M \) and \( N \) are Zap-sets in the F-manifolds \( X \) and \( Y \) respectively, then \( M \times N \) is a Zap-set in \( X \times Y \). (Theorem 6.8 of \([12]\))

**Remark 2.10.** If \( K \) is a countable locally-finite simplicial complex, then \( |K| \times (s_{f} \times s) \) is a Zap-set in \( |K| \times s \). (Theorem 8.2 of \([12]\))

Section 2.5 Manifolds Modeled on Zap-sets

In accordance with \([16]\), if \( F \) is a topological vector space, define \( F^{W} \) to be the countable infinite product and let \( F^{W}_{f} = \{(x_{i})_{i>0} \in F^{W}: \text{for at most finitely many } i, x_{i} \neq 0\} \). In the following "\( \simeq \)" means "is homeomorphic to". This paper \([16]\) by Henderson and West
contains several theorems concerning spaces $F$ which have the property that $F \simeq F_f^W$ and others about manifolds modeled on such spaces. Here we show that $s_f \times s \simeq (s_f \times s)_f^W$, thus obtaining the applicable results of [16] for Zap-sets in $s$ or $l_2$. We list several of the most significant of these results at the end of this section.

**Theorem 2.5.** $(s_f \times s)_f^W \simeq s_f \times s$

**Proof:** We establish the following sequence of homeomorphisms:

$$(s_f \times s)_f^W \overset{h_1}{\sim} (s_f)_f^W \times s_f \overset{h_2}{\sim} s_f \times s_f \overset{h_3}{\sim} s_f \times s \overset{h_4}{\sim} s_f \times s.$$

The first homeomorphism $h_1$ is given by the next lemma.

**Lemma 2.7.** Let $X$ and $Y$ be topological vector spaces. Then $(X \times Y)_f^W \simeq X_f^W \times Y_f^W$.

**Proof:** Let $h$ be the homeomorphism of $(X \times Y)_f^W$ onto $X_f^W \times Y_f^W$ defined by $h((x_1, y_1)_{i>0}) = ((x_i)_{i>0}, (y_i)_{i>0})$. Since $(x_1, y_1)_{i>0} \in (X \times Y)_f^W$ implies $((x_i)_{i>0}, (y_i)_{i>0}) \in X_f^W \times Y_f^W$, and vice versa, we have that $h((X \times Y)_f^W) = X_f^W \times Y_f^W$. □

The second homeomorphism $h_2$ is obtained by using the easily verified fact that $(s_f)_f^W \simeq s_f$ [21].

The third homeomorphism $h_3$ is a result of the following lemma.
Lemma 2.8. $s^w_f \simeq s_f \times s$.

Proof: We show that there is a homeomorphism $f$ of $s^w$ onto $s$ such that $f(s^w_f)$ is a Zap-set in $s$. This fact, together with Example 2.1 and Theorem 2.2, implies that $s^w_f$ and $s_f \times s$ are homeomorphic.

To define the homeomorphism $f$, for each $i > 0$, we let $\beta_i = \{2^n + 2^{i-1}\}_{n \geq 0}$ and then define $f$ of $s^w$ onto $s$ by

$$f((x^1_k)_{k \geq 0}, (x^2_k)_{k \geq 0}, \ldots, (x^i_k)_{k \geq 0}, \ldots) = (y^1_k)_{k \geq 0}$$

where

$$y^i_k = x^{i-1}_{(k+2^{i-1}-1)/2}$$

if $k \in \beta_i$. Also, letting

$$N_j = \{(x^i_k)_{k \geq 0} \in s^w: \text{for every } i > j, x^i_0 = 0\}$$

we see that

$$s^w_f = \bigcup_{j > 0} N_j.$$

Now let $\alpha_1 = \{2^n\}_{n \geq 0}$ and for each $i > 1$, let

$$\alpha_i = \alpha_{i-1} \setminus \beta_i.$$

Note that $\alpha_{i+1} \subset \alpha_i$, each $\alpha_i$ is infinite and since $\bigcup_{i > 1} \beta_i = \alpha_1, \bigcap_{i > 0} \alpha_i = \emptyset$. For each $i > 0$, let $M_i = \{x \in s: \tau_j(x) = 0 \text{ for every } j \in \alpha_i\}$. Then by Theorem 2.1, $M = \bigcup_{i > 0} M_i$ is a Zap-set in $s$. Also, from the definition of $f$ we have that for each $j > 0$, $f(N_j) = M_j$ so that $f(s^w_f) = M$ and hence $f(s^w_f)$ is a Zap-set in $s$.

The last homeomorphism $h_4$ is obtained by observing that, directly from the definition of $s_f$, we have

$$s_f \times s_f \simeq s_f.$$

By an $(s_f \times s)$-manifold we mean a separable metric space having an open cover by sets each homeomorphic to an open subset of $s_f \times s$. The next theorems follow from
Theorem 2.5 above and the indicated results in [16].

**Theorem 2.6.** If \( X \) is an \((s_f \times s)\)-manifold, then 
\( X \times (s_f \times s) \simeq X \). (Theorem 5 of [16])

**Theorem 2.7.** If \( X \) is a connected \((s_f \times s)\)-manifold, then \( X \) can be embedded as an open subset of \( s_f \times s \).
(Theorem 7 of [16])

**Theorem 2.8.** If \( X \) and \( Y \) are \((s_f \times s)\)-manifolds of the same homotopy type, then \( X \simeq Y \). (Theorem 11 of [16])

**Theorem 2.9.** If \( X \) is an \((s_f \times s)\)-manifold and if \( K \) is a \( Z \)-set in \( X \), then \( K \) is negligible. (Theorem 12 of [16]).
CHAPTER III

ABSOLUTE Z-SETS

Let $f$ and $g$ be embeddings of a space $X$ into a space $Y$. Then $f$ and $g$ are said to be weakly equivalent if there exists $h \in \mathcal{H}(X)$ such that $h(f(X)) = g(X)$, and are said to be equivalent if there exists $h \in \mathcal{H}(X)$ such that $h \circ f = g$. Clearly, equivalence implies weak equivalence.

It is known that any two embeddings of the Cantor set into the plane are equivalent. However, Antoine's Necklace, a topological Cantor set in three-dimensional Euclidean space, $E^3$, whose complement is not simply connected [17], implies the existence of two embeddings of the Cantor set into $E^3$ which are not weakly equivalent. Blankinship, using generalizations of Antoine's methods, shows that for each $n \geq 3$, there are embeddings of the Cantor set into $E^n$ which are not weakly equivalent [11]. On the other hand, Klee proves that any two embeddings of the Cantor set into $\ell_2$ are equivalent [19].

In this chapter we determine which spaces $X$ have the property that any two closed embeddings of $X$ into $\ell_2$ (or $Q$) are equivalent. For there to exist a closed embedding of
a space $X$ into $l_2$ (or $Q$), $X$ must be a topologically complete (or compact) separable metric space. It is known that any topologically complete (or compact) separable metric space can be embedded in $l_2$ (or $Q$) as a Z-set. Also, Theorem 1.2, restated in terms of equivalent embeddings, says that any two embeddings of a space into $l_2$ (or $Q$) as Z-sets are equivalent. Thus, a space $X$ has the property that any two closed embeddings of $X$ into $l_2$ (or $Q$) are equivalent if and only if $X$ has the property that the image of each closed embedding of $X$ into $l_2$ (or $Q$) is a Z-set in $l_2$ (or $Q$).

We define an **absolute Z-set** to be a topologically complete separable metric space $X$ such that for each closed embedding $f$ of $X$ into $l_2$, $f(X)$ is a Z-set in $l_2$. Also, an **absolute compact Z-set** is a compact separable metric space $X$ such that for each embedding $f$ of $X$ into $Q$, $f(X)$ is a Z-set in $Q$.

The main result in this chapter is the following characterization of absolute Z-sets.

**Theorem 3.1.** Let $X$ be a topologically complete separable metric space. Then $X$ is an absolute Z-set if and only if $X$ is $c$-compact.

The next theorem provides an alternate characterization of absolute Z-sets.
Theorem 3.2. Let $X$ be a topologically complete separable metric space. Then $X$ is an absolute $Z$-set if and only if $X$ does not contain a closed set which is homeomorphic to the space of irrationals.

The equivalence of these two characterizations is shown by the following theorem.

Theorem 3.3. Let $X$ be a topologically complete separable metric space. Then $X$ is $\sigma$-compact if and only if $X$ does not contain a closed set which is homeomorphic to the space of irrationals.

It is clear that if a space contains a closed topological copy of the space of irrationals, then it is not $\sigma$-compact. The next two lemmas show that a topologically complete separable metric space which is not $\sigma$-compact contains a closed set which is homeomorphic to the space of irrationals, and thus complete the proof of Theorem 3.3.

Lemma 3.1. If $X$ is a separable metric space which is not $\sigma$-compact, then $X$ contains a non-empty closed nowhere locally compact subset.

Proof: Let $A = \{p \in X : \text{for each open set } U \text{ in } X \text{ containing } p, U \text{ is not } \sigma\text{-compact}\}$. $A$ is not empty since otherwise, for each $p \in X$, there would exist an open set $U_p$ such that $p \in U_p$ and $U_p$ is $\sigma$-compact. As $X$ is
separable and metric, we would then have that $X = \bigcup_{i>0} U_i$ where $U_i$ is a $\sigma$-compact. Thus $X$ would be $\sigma$-compact, which would be a contradiction. To show $A$ is closed, let $p \in \bar{A}$ and let $U$ be an open set containing $p$. Then there exists $q \in U \cap A$, but $q \in A$ implies that $U$ is not $\sigma$-compact so that $p \in A$ and thus $A$ is closed.

Suppose that $A$ is locally compact at some point $p$ in $A$. Then there exists an open set $U$ in $A$ such that $p \in U$ and $\bar{U} \cap A$ is compact. However, $\bar{U} \cap A$ is $\sigma$-compact since $X \setminus A$ is, and as $\bar{U} \cap A$ is compact, we have that $\bar{U}$ is $\sigma$-compact and thus $U$ is $\sigma$-compact. But, this contradicts the fact that $p \in A$. Therefore, $A$ is nowhere locally compact and indeed, $A$ is a non-empty, closed, nowhere locally compact subset of $X$. □

A collection $\mathcal{B} = \{B_i\}_{i>0}$ of sets in a space $X$ is said to be discrete if (1) $i \neq j$ implies $\overline{B_i} \cap \overline{B_j} = \emptyset$ and (2) for each subset $\alpha$ of $\mathbb{N}$, $\bigcup_{i \in \alpha} \overline{B_i}$ is closed.

Lemma 3.2. Every nowhere locally compact topologically complete separable metric space contains a closed topological copy of the space of irrationals.

Proof: Let $X$ be a nowhere locally compact topologically complete separable metric space. Then $X$ is paracompact and non-compact. Thus, let $\mathcal{U} = \{U_i\}_{i>0}$ be a locally finite open cover of $X$ with no finite subcover of
Also, assume that \( \mathcal{U} \) has the property that for each \( i, V_i \setminus ( \bigcup_{j \neq i} V_j ) \neq \emptyset \). Let \( U_1 \) be an open subset of \( X \) such that \( U_1 \subset V_1 \) and \( \text{diam } U_1 < 1/2 \). Assume that \( U_1, \ldots, U_n \) have been defined so that for each \( i, 1 \leq i \leq n \),

1. \( U_i \) is an open subset of \( V_i \),
2. \( \text{diam } U_i < 1/2 \), and
3. \( \bar{U}_i \cap \bar{U}_j = \emptyset \) if \( i \neq j \). Now let \( U_{n+1} \) be an open subset of \( V_{n+1} \) such that \( \text{diam } U_{n+1} < 1/2 \) and \( \bar{U}_{n+1} \cap \bigcup_{i=1}^n \bar{U}_i = \emptyset \). As \( \mathcal{U} \) is locally finite, the infinite collection \( \mathcal{U}_1 = \{U_1\}_{i>0} \), obtained in this manner, is discrete.

If \( \mathcal{U} \) is any discrete collection of open sets in \( X \), then for each \( U \in \mathcal{U} \), we may let \( W \) be an open set in \( X \) such that \( \bar{W} \subset U \), and then repeat the above process for the nowhere locally compact set \( \bar{W} \). Thus, by induction, we assert the existence of a sequence, \( (\mathcal{U}_n)_{n>0} \) of discrete collections of open sets in \( X \) such that for each \( n > 0 \):

1. \( \text{mesh } (\mathcal{U}_n) < 1/2^n \);
2. \( \{U : U \in \mathcal{U}_n\} \) is a refinement of \( \mathcal{U}_{n-1} \); and
3. each element of \( \mathcal{U}_n \) contains infinitely many elements of \( \mathcal{U}_{n+1} \).

For each \( n > 0 \), let \( \mathcal{U}_n = \{U_i^n\}_{i>0} \) and let \( F_n = U_{i>0} \bar{U}_i^n \). Then \( F_n \) is closed since \( \mathcal{U}_n \) is a discrete collection. Let \( F = \bigcap_{n>0} F_n \). Then \( F \) is a closed zero-dimensional subset of \( X \) since for each \( n > 0 \), \( \mathcal{U}_n \) is discrete and \( \text{mesh } (\mathcal{U}_n) < 1/2^n \). Also, for each \( p \in F \)
and open set $V$ in $X$ such that $p \in V$, there exists $n > 0$ such that $V$ contains infinitely many elements of $U_n$. As $X$ is topologically complete, this implies that $F$ is nowhere locally compact. Since $F$ is closed in $X$, $F$ is also topologically complete. Therefore, $F$ is a closed subset of $X$ which is homeomorphic to the space of irrational.

Since each closed $\sigma$-compact subset of $l_2$ is a $Z$-set in $l_2$ [13] and since the image of a closed embedding of a $\sigma$-compact space is closed and $\sigma$-compact, it is clear that every $\sigma$-compact topological complete separable metric space is an absolute $Z$-set. The following result and theorem complete the characterization of absolute $Z$-sets.

Result 3.1 [22]. There exists a closed topological copy of the space of irrationals in $l_2$ which is not a $Z$-set in $l_2$.

Theorem 3.4. If $F$ is a closed topological copy of the space of irrationals in $l_2$ and if $X$ is a topologically complete separable metric space containing a closed set homeomorphic to the space of irrationals, then there exists a closed embedding $h$ of $X$ into $l_2$ such that $F \subset h(X)$.

To see that this actually completes the characterization, we need only let $F$ be the set asserted to exist by Remark 3.1 and observe that by Property 1.1, if $h(X)$ were
a Z-set, then \( F \) would also be a Z-set. The results in Section 3.2 and Section 3.3 are obtained primarily to provide the proof of Theorem 3.4 which is contained in Section 3.3.

The corresponding characterizations of absolute compact Z-sets, given by the following theorems, are more easily obtained.

**Theorem 3.5.** Let \( X \) be a compact separable metric space. Then \( X \) is an absolute compact Z-set if and only if \( X \) is countable.

Since a compact separable metric space is countable if and only if it does not contain a topological Cantor set, the next theorem is equivalent to Theorem 3.5.

**Theorem 3.6.** Let \( X \) be a compact separable metric space. Then \( X \) is an absolute compact Z-set if and only if \( X \) does not contain a topological Cantor set.

As each compact countable subset of \( \mathbb{Q} \) is a Z-set in \( \mathbb{Q} \), a compact countable separable metric space is an absolute compact Z-set. The remainder of the characterization is given by the next result and theorem.

**Result 3.2 [22].** There exists a topological Cantor set in \( \mathbb{Q} \) which is not a Z-set in \( \mathbb{Q} \).

**Theorem 3.7.** If \( C \) is a topological Cantor set in
$Q$ and if $X$ contains a topological Cantor set, then there exists an embedding $h$ of $X$ into $Q$ such that $C \subseteq h(X)$.

The method used in the proofs of Theorem 3.4 and Theorem 3.7 is suggested by the proof of the theorem, known at least in the folklore, which states that if $C$ is any topological Cantor set in $E^n$, then there is a closed embedding $h$ of $E^{n-1}$ into $E^n$ such that $C \subseteq h(E^{n-1})$. This embedding is obtained in much the same way that Alexander's Horned Sphere [17] is obtained, with the exception that in this construction, the "horns" approach a given topological Cantor set in $E^n$. The details of the proof of Theorem 3.7 are not given. However, with suitable simplifications, such as using finite instead of infinite collections, the lemmas applicable to Theorem 3.4; indeed, the proof of the theorem itself, can be easily modified to yield the proof of Theorem 3.7.

Section 3.1. Preliminary Definitions and Lemmas

In this section we give some of the definitions and lemmas which will be used in the proofs of the main theorems in this chapter. By $e_1$ we will always mean the point in $E^2$ with $i^{th}$ coordinate 1 and all the other coordinates 0, and $Q_1$ will denote $\{te_1: 0 \leq t \leq 1\}$. The distance function, $d$, on $E^2$ used here is defined by

$$d((x_1)_{i>0}, (y_1)_{i>0}) = (\Sigma_{i>0} (x_i - y_i)^2)^{1/2}.$$
we let \( M = \{ (x_i)_{i>0} \in \ell^2: x_1 = 0 \} \) and \( \ell^+_2 = \{ (x_i)_{i>0} \in \ell^2: x_1 > 0 \} \).

The following lemmas are used several times in later proofs.

**Lemma 3.3.** Let \( K \) be a closed subset of a separable metric space \( X \) and let \( \{A_i\}_{i>0} \) be a discrete collection of closed subsets of \( X \) such that for each \( i>0 \), \( \text{diam } A_i < \varepsilon_1 \) and \( A_i \cap K = \emptyset \). Then there exists a discrete collection \( \{U_i\}_{i>0} \) of open sets in \( X \) such that for each \( i>0 \), \( A_i \subseteq U_i \), \( U_i \cap K = \emptyset \) and \( \text{diam } U_i < 2\varepsilon_1 \).

The proof is elementary, first using the normality of \( X \) to obtain a pairwise disjoint collection \( \{W_i\}_{i>0} \) of open sets in \( X \) such that for each \( i>0 \), \( A_i \subseteq W_i \) and \( W_i \cap K = \emptyset \), and then letting \( U_i \subseteq W_i \) be an \( \varepsilon_1/i \)-neighborhood of \( A_i \). □

The proof of the next lemma is routine and is not given explicitly; the proof uses the previous lemma and the fact that if \( F \) is a copy of the irrationals, then there is an infinite discrete cover of \( F \) by open and closed sets in \( F \).

**Lemma 3.4.** Let \( F \) and \( K \) be closed subsets of a separable metric space \( X \) such that \( F \) is a copy of the irrationals and \( F \cap K = \emptyset \). Then for each \( \varepsilon>0 \), there exists an infinite discrete collection \( \mathcal{V} = \{V_i\}_{i>0} \) of open sets in \( X \) such that:
(1) for each \( i > o \), \( V_i \cap F \neq \emptyset \);

(2) \( F \subset U_{i>0}V_i \), \( K \cap U_{i>0}V_i = \emptyset \); and

(3) mesh \( \nu < \epsilon \).

A repeated use of Lemma 3.4 provides a straightforward inductive proof of the following lemma.

**Lemma 3.5.** Let \( F \) be a copy of the irrationals which is closed in \( \mathbb{R}^+ \). Then there exists a sequence \( \nu = (\nu_n)_{n>0} \) of discrete covers of \( F \) such that:

1. for each \( n > o \), \( V \in \nu_n \) implies that \( V \) is open in \( \mathbb{R}^+ \), \( \bar{V} \subset \mathbb{R}^+ \) and \( V \cap F \neq \emptyset \);

2. for each \( n > o \), \( \nu_n \) is countably infinite and mesh \( (\nu_n) < 1/2^n \);

3. for each \( n > o \), \( \nu_{n+1} \) is a refinement of \( \nu_n \); and

4. each element of \( \nu_n \) contains infinitely many elements of \( \nu_{n+1} \).

We observe, by Lemma 3.2, that such a set \( F \) exists.

**Section 3.2 Construction of an "w-stage starset system"**

In this section we construct what we call an \( \omega \)-stage starset system. This system, which is a specific union of arcs in \( \mathbb{R}^+ \) converging to a closed copy of the irrationals, is used in obtaining the embedding of the next section.
We define a subset $S$ of $\ell_2$ to be a star set at $x$ if there exists an infinite subset $\alpha$ of $\mathbb{N}$ and $h \in \mathcal{H}(\ell_2)$ such that $h(\theta) = x$ and $h(\bigcup_{i \in \alpha} Q_i) = S$, where $\theta$ is the origin of $\ell_2$. We call $x$ the base point of $S$, and for each $i \in \alpha$, we call $h(Q_i)$ a basic arc of $S$ and $h(e_i)$ a tip of $S$. If, in a union of star sets, a tip of one star set is not the base point of another, it is said to be a free tip.

A 1-stage star set system, $S^1$, in $\ell_2$ is the union of a discrete collection, $\{S^1_i\}_{i > 0}$, of star sets in $\ell_2$.

Inductively, an $n$-stage star set system, $S^n$, in $\ell_2$ is the union of an $(n-1)$-stage star set system, $S^{n-1}$, and a set $T$, where $T$ is the union of a discrete collection $\{S^n_i\}_{i > 0}$ of star sets in $\ell_2$ such that:

1. for each $i > 0$, $S^n_i \cap S^{n-1} = \{x\}$, where $x$ is the base point of $S^n_i$ and a free tip of a star set in $S^{n-1}$; and

2. for each free tip, $x$, in $S^{n-1}$, there exists $i > 0$ such that $x$ is the base point of $S^n_i$.

In this case we say that $S^n$ is derived from $S^{n-1}$. Let $(S^n)_{n > 0}$ be a sequence of $n$-stage star set systems such that for each $n > 1$, $S^n$ is derived from $S^{n-1}$. Then $S^* = \bigcup_{n > 0} S^n$ is said to be the $\omega$-stage star set system derived from $(S^n)_{n > 0}$. If, in addition, $\{A_i\}_{i > 0}$ is a discrete collection of arcs in $\ell_2$ such that for each
\( i > o, A_1 \cap S^* = \{x_1\}, \) where \( x_1 \) is the base point of the starset \( S_1^{1} \) in \( S^* \) and is an endpoint of \( A_1 \), then \( S^* \) is said to be based on \( A = U_{1>0} A_1 \) at \( \{x_1\}_{1>0} \). The following lemma provides a means for constructing an \( n \)-stage starset system from a given \((n-1)\)-stage starset system.

**Lemma 3.6.** Let \( x \in \mathbb{L}_2 \) and let \( U \) be an open set in \( \mathbb{L}_2 \) containing \( x \). If \( K \) is a \( Z \)-set in \( \mathbb{L}_2 \) and \( x \in K \), then there is a starset, \( S \), at \( x \) such that \( S \subseteq U \) and \( S \cap K = \{x\} \).

**Proof:** Let \( \alpha \subset \mathbb{N} \) such that both \( \alpha \) and \( \mathbb{N} \setminus \alpha \) are infinite. Since \( K \) is a \( Z \)-set in \( \mathbb{L}_2 \), there exists \( g \in \mathcal{U}(\mathbb{L}_2) \) such that \( g(x) = \emptyset \) and for each \( i \in \alpha \), \( \tau_1(g(K)) = 0 \) (Theorem 1.1). Let \( \epsilon > 0 \) be such that \( B_\epsilon(\emptyset) \subseteq g(U) \) and define \( f \in \mathcal{U}(\mathbb{L}_2) \) by \( f((x_1)_1>0) = (\epsilon x_1)_1>0 \). Then \( S = g^{-1} \circ f(U_{i \in \alpha} Q_i) \) is a starset at \( x \) such that \( S \subseteq U \) and \( S \cap K = \{x\} \). □

Let \( S^* \) be an \( \omega \)-stage starset system derived from \((S^n)_n>0\). Then \( J = U_{n>0} J_n \) is a basic union of arcs in \( S^* \) if for each \( n > 0 \), \( J_n \) is a basic arc in \( \text{cl}(S^n \setminus S^{n-1}) \) and \( J_m \cap J_n \neq \emptyset \) if and only if \( |m-n| = 1 \). The next lemma provides the needed \( \omega \)-stage starset system.

**Lemma 3.7.** Let \( F \) be a copy of the irrationals in \( \mathbb{L}_2^+ \) with \( F \) closed in \( \mathbb{L}_2 \) and let \((r_n)_n>0\) be a sequence of covers of \( F \) as in Lemma 3.5. If \( A \) is the union of
a discrete collection \( \{A_i\}_{i>0} \) of arcs in \( M \cup F^+ \) such that for each \( i>0 \), \( A_i \) has an endpoint \( p_i \in V_1^1 \), where \( V_1 = \{V_i^1\}_{i>0} \), and \( A \cap F = \emptyset \), then there exists an \( \omega \)-stage starset system \( S^* = U_{n>0}S^n \) based on \( A \) at \( \{p_i\}_{i>0} \) and having the following properties:

1. If \( J = U_{n>0}J_n \) is a basic union of arcs in \( S^* \), then \( \overline{J \setminus J} \) consists of a single element of \( F \) and \( \lim_{n \to \infty} \text{diam}(U_{m \geq n}J_m) = 0 \);
2. If \( J \) and \( J' \) are distinct basic unions of arcs in \( S^* \), then \( \overline{J \setminus J} \neq \overline{J' \setminus J'} \);
3. \( S^* = S^* \cup F \), \( S^* \cap F = \emptyset \) and \( S^* \cap M = \emptyset \); and
4. For each \( \epsilon > 0 \), there exists \( n>0 \) such that the free tips of \( S^n \) are contained in the \( \epsilon \)-neighborhood of \( F \).

**Proof:** We will inductively construct a sequence, \( (S^n)_{n>0} \) of \( n \)-stage starset systems such that the \( \omega \)-stage starset system, \( S^* \), has the desired properties.

By Lemma 3.6, for each \( i>0 \) let \( T_1^i = U_{j>0}J_{ij}^1 \) be a starset at \( p_i \) such that \( T_1^i \subset V_1^1 \setminus F \) and \( T_1^i \cap A = \{p_i\} \).

Let \( s_{ij}^1 \) denote the free tip of \( J_{ij}^1 \) in \( T_1^i \). By Lemma 3.5, each \( V_1^1 \) contains infinitely many elements of \( V_2 \).

Denote this set of elements by \( \{V_{ij}^2\}_{j>0} \) and let \( t_{ij}^1 \in V_{ij}^2 \setminus (F \cup T_1^i \cup A) \). Then both \( s_{ij}^1 \) and \( t_{ij}^1 \) are in \( V_1^1 \) and thus \( d(s_{ij}^1, t_{ij}^1) < 1/2 \) - since \( \text{diam} V_1^1 < 1/2 \). Since \( A \cap F = \emptyset \), since each arc \( A_i \) is compact and since
A = \bigcup_{i>0} A_{i1} is closed, then by Example 1.3 and Property 1.2, A is a Z-set in the F-manifold \((M \cup \mathcal{L}_2^+)\setminus F\). Similarly, \(U_{i>0, j>0} \{s_{ij}^1\}\) and \(U_{i>0, j>0} \{t_{ij}^1\}\) are Z-sets. By Example 1.4, \(M\) is also a Z-set in \(M \cup \mathcal{L}_2^+\). As \(\{s_{ij}^1\}\) and \(\{t_{ij}^1\}\) are discrete collections, we may define a homeomorphism \(f_1'\) by \(f_1'(s_{ij}^1) = t_{ij}^1\) and \(f_1'|_{(A \cup M)} = \text{id}\). Let 
\[ a = \{B_{1/2}(x) : x \in (M \cup \mathcal{L}_2^+) \setminus F\}, \text{ where } B_{1/2}(x) = \{y \in (M \cup \mathcal{L}_2^+) \setminus F : d(x, y) < 1/2\}. \]
Since for each \(i > 0\) and \(j > 0\), \(t_{ij}^1 \in B_{1/2}(s_{ij}^1)\) and \(B_{1/2}(s_{ij}^1)\) is arcwise connected, \(f_1'\) is clearly homotopic to the identity by a homotopy which is pathwise limited by \(a\). By the Homeomorphism Extension Theorem 1.3, there exists \(f_1 \in \mathcal{N}(M \cup \mathcal{L}_2^+) \setminus F\) such that \(f_1\) extends \(f_1'\) and \(f_1\) is limited by \(st^h(a)\). Note that \(f_1\) limited by \(st^h(a)\) implies that if \(y, z \in B_{1/2}(x)\) for some \(x\), then 
\(d(f_1(y), f_1(z)) < 17\). Thus, diam \(f_1(J_{ij}^1) < 17\) since \(J_{ij}^1 \subset B_{1/2}(s_{ij}^1)\). For each \(i > 0\), let \(S_{i1}^1 = \bigcup_{j>0} f_1(J_{ij}^1)\) and let \(S_1^1 = \bigcup_{i>0} S_{i1}^1\). Then \(S_1^1\) is a starset at \(p_1\) and \(S_1^1\) is a 1-stage starset system.

Now assume an \((n-1)\)-stage starset system, \(S_{n-1}^n\), has been defined so that the following conditions are satisfied:

1. \(S_1^1 \subset S_{n-1}^n, S_{n-1}^n \cap (M \cup F) = \emptyset\) and \(S_{n-1}^n \cap A = \bigcup_{i>0} \{p_i\}\);
2. if \(\{s_i^{n-1}\}_{i>0}\) denotes the set of free tips in \(S_{n-1}^n\) and if \(V_n = \{V_{i1}^n\}_{i>0}\), then \(s_i^{n-1} \in V_{i1}^n\); and
(3) if \( s^{n-1}_1 \in V^n_1 \) is a free tip of a basic arc \( J \)
of a starset whose base point is in \( V \in V_{n-1} \),
then \( V^n_1 \subset V \) and \( \text{diam } J < 3^4/2^{n-1} \).

Since \( S^{n-1} U A \) is a closed \( \sigma \)-compact subset of
\( M U \ell_2^+ \), \( S^{n-1} U A \) is a Z-set in \( M U \ell_2^+ \). By Lemma 3.6, for
each \( i > 0 \), let \( T^n_1 = U_{j>i}^{n+1} \) be a starset at \( s^{n-1}_1 \) such
that \( T^n_1 \subset V^n_1 \setminus F \) and \( T^n_1 \cap (S^{n-1} U A) = \{s^{n-1}_1\} \). For each
\( i > 0 \), let \( \{V^n_{i+1}\}_{j>0} \) denote the infinite set of elements of
\( V_{n+1} \) which are contained in \( V^n_1 \) and let
\( t^n_{ij} \in V^n_{ij} \setminus (F U S^{n-1} U A U M) \). Denote the free tips of
\( T^n_1 \) by \( \{s^n_{ij}\}_{j>0} \) and observe that, as in the case for \( n = 1 \),
\( s^{n-1}, U_{i>0,j>0}^{n} \{s^n_{ij}\}, U_{i>0,j>0}^{n} \{t^n_{ij}\}, M \) and \( A \) are all
Z-sets in \( (M U \ell_2^+) \setminus F \). Let \( \mathfrak{A}_n = \{B_{1/2^n}(x): x \in (M U \ell_2^+) \setminus F\} \),
where \( B_{1/2^n}(x) = \{y \in (M U \ell_2^+) \setminus F: d(x,y) < 1/2^n\} \). Note that
\( t^n_{ij} \in B_{1/2^n}(s^n_{ij}) \), \( J^n_{ij} \subset B_{1/2^n}(s^n_{ij}) \), and \( B_{1/2^n}(s^n_{ij}) \) is
arcwise connected. Thus let \( f_n \in \chi((M U \ell_2^+) \setminus F) \) such that
\( f_n(s^n_{ij}) = t^n_{ij} \), \( f_n|(M U A U S^{n-1}) = \text{id} \) and \( f_n \) is limited
by \( \text{st}_{+}(\mathfrak{A}_n) \). Observe that \( \text{diam } f_n(J^n_{ij}) < 3^4/2^n \). As
before, let \( S^n_i = U_{j>0} f_n(J^n_{ij}) \) and let \( S^n = U_{i>0} S^n_i \). Then
it is easily verified that the derived \( w \)-stage starset system,
\( S^* \), satisfies the conditions given in the statement of the
lemma. \( \square \)

If \( S^* \) is an \( w \)-stage starset system which fulfills
the requirements of Lemma 3.7, then we say that \( S^* \)
converges to $F$.

Section 3.3  An Embedding Theorem

The first lemma of this section is essentially the inductive step in the proof of Theorem 3.8 which follows. A halfspace in $\ell_2$ is a pair $[Y, K]$ of subsets of $\ell_2$ for which there exists $h \in \mathcal{U}(\ell_2)$ such that $h(Y) = M \cup \ell_2^+$ and $h(K) = M$, where as before $M = \{x_1\}_{i>0} \in \ell_2: x_1 = 0\}$ and $\ell_2^+ = \{(x_1)_{i>0} \in \ell_2: x_1 > 0\}$.

Lemma 3.8. Let $[Y,K]$ be a halfspace in $\ell_2$, let $A$ be an arc in $Y$ with endpoints $p$ and $q$, where $A \cap K = \{p\}$, and let $S = \cup_{i>0} J_1$ be a starset at $q$ such that $S \cap (A \cup K) = \{q\}$. If $W$ is an open set in $\ell_2$ such that $A \subset W$, then for any open set $V$ in $\ell_2$ such that $q \in V$, there exist $(1)$ a relative basic open set $U$ in $K$ with $p \in U \subset W$ and $(2)$ a $G \in \mathcal{U}(\ell_2)$ such that $G|_{\ell_2 \setminus W} = \text{id}$, $G(U) \subset (V \cap W)$ and for each $i > 0$, $G(U) \cap J_1$ is a point different from $q$.

Proof: Since $[Y,K]$ is a halfspace in $\ell_2$, there exists $f \in \mathcal{U}(\ell_2)$ such that $f(Y) = M \cup \ell_2^+$ and $f(K) = M$. Let $A' = \{te_1: 0 \leq t \leq 1\}$ and for each $i > 0$, let $J'_i = \{(1+t)e_1 + te_{i+1}: 0 \leq t \leq 1\}$ and let $S' = \cup_{i>0} J'_i$. Let $g'$ be a homeomorphism of $M \cup f(A) \cup f(S)$ onto $M \cup A' \cup S'$ such that $g'|M = \text{id}$, $g'(f(A)) = A'$ and $g'(f(S)) = S'$. As
\[ M \cup f(A) \cup f(S) \text{ and } M \cup A' \cup S' \text{ are } Z\text{-sets in } M \cup \mathcal{L}^+_2, \]

let \( g \in \mathcal{U}(\mathcal{L}_2) \) be an extension of \( g' \) and let \( h = g \circ f \).

Considering \( \mathcal{L}_2 \) as \( R \times M \), let \( r \) be such that if

\[ B_\varepsilon(h(p)) = \{ x \in M : d(x, h(p)) < \varepsilon \}, \text{ then } [-r, l+r] \times B_{2r}(h(p)) \subset h(W) \text{ and } (l+r/2) \times B_r(h(p)) \subset h(W \cap V). \]

Note that for each \( i > 0 \), \((l+r/2) \times B_r(h(p))) \cap J^i = \{(i+r/2)e_1 + (r/2)e_{i+1}\} \).

Let \( \varphi \) be a piecewise linear homeomorphism of \([-r, l+r]\)

onto itself such that \( \varphi(-r) = -r, \varphi(0) = l+r/2 \) and

\( \varphi(l+r) = l+r \). For each \( t, 0 \leq t \leq 2r \), let \( \varphi_t = \varphi \) if \( t \leq r \) and \( \varphi_t = ((2r-t)/r)\varphi + ((t-r)/r)\text{id} \) if \( t \geq r \).

Now for each \( x \in B_{2r}(h(p)) \) and each \( u \in [-r, l+r] \), let

\[ H(x, u) = (x, \|x\|(u)) \text{ and let } H \text{ be the identity on } \]

\( \mathcal{L}_2 \setminus ((l+r) \times B_{2r}(h(p))) \). Let \( G \in \mathcal{U}(\mathcal{L}_2) \) be defined by

\[ G = h^{-1} \circ H \circ h \text{ and let } U = h^{-1}(B_r(h(p))). \]

We say that a subset \( K \) of a space \( X \) is locally bicollared at \( p \in K \) if there exists a relative open set \( U \) in \( K \) containing \( p \) and an open embedding \( h \) of \( U \times (-1, 1) \) into \( X \) such that for each \( x \in U \), \( h(x, 0) = x \).

**Theorem 3.8.** Let \( Y = \{(x_i)_{i \geq 0} \in \mathcal{L}_2 : x_1 \geq 2\} \) and let \( F \) be a closed copy of the space of irrationals in \( Y \).

Then there exists an embedding \( H \) of \( M \) into \( \mathcal{L}_2 \) such that \( F \subset H(M) \) and \( H^{-1}(F) \) is a \( Z \)-set in \( M \). Moreover, we can require that \( H(M) \) be locally bicollared at each point in \( H(M) \setminus F \).
Proof: Let \((\mathcal{V}_n)_{n \geq 0}\) be a sequence of covers of \(F\) as described in Lemma 3.5. Let \(\alpha \in \mathbb{N}\) such that both \(\alpha\) and \(\mathbb{N}\setminus \alpha\) are infinite and let \(D\) be a closed topological copy of the space of irrationals in \(M\) such that for each \(i \in \alpha\), \(\tau_i(D)\) is a point (Lemma 3.2). By Lemma 3.4, let \(\mathcal{J} = \{G_i\}_{i \geq 0}\) be a discrete collection of relative open sets in \(M\) such that:

1. \(D \subset \bigcup_{i \geq 0} G_i\);
2. for each \(i \geq 0\), \(G_i \cap D \neq \emptyset\); and
3. \(\text{mesh}(\mathcal{J}) < 1/2\)

For each \(i \geq 0\), let \(p_i \in G_i \cap D\) and let \(A_i = \{te_1 + p_i : 0 \leq t \leq 1\}\).

Also, let \(q_i \in V_1 F\), where \(V_1 = \{V_1^n\}_{n \geq 0}\), and define \(f'\) by \(f'(e_1 + p_i) = q_i\) and \(f'|M = \text{id}\). Since \((M \cup \ell_2^+)\setminus F\) is arcwise connected, \(f'\) is homotopic to the identity map on \((M \cup \ell_2^+)\setminus F\).

Also since \(U_{i > 0}(e_1 + p_i), U_{i > 0}(q_i)\) and \(M\) are \(Z\)-sets in the \(F\)-manifold \((M \cup \ell_2^+)\setminus F\), by Theorem 1.3, let \(f \in \mathcal{K}(\ell_2^+)\setminus F\) such that \(f\) extends \(f'\). Note that for each \(i \geq 0\), \(f(A_i) \cap F = \emptyset\). By Lemma 3.7, let \(S^*\) be an \(\omega\)-stage starset system in \(\ell_2^+\) which is based on \(A = \bigcup_{i \geq 0} f(A_i)\) at \(\{q_i\}_{i \geq 0}\) and which converges to \(F\). In the rest of this proof, we use the notation developed in the proof of Lemma 3.7.

Now by Lemma 3.3, let \(\{W_{i_1}^1\}_{i_1 \geq 0}\) be a discrete collection of open sets in \(\ell_2\) such that for each \(i \geq 0\), \(f(A_i) \subset W_{i_1}^1\), \(W_{i_1}^1 \cap F = \emptyset\) and \(W_{i_1}^1 \cap M \subset G_i\). Since for each \(i \geq 0\), \(S_i^1 = \bigcup_{j > 0} W_{i_1}^1\) is a starset in \(S^*\) at \(q_i\), by Lemma 3.8, there exist (1) a relative basic open set \(U_{i_1}^1\) in \(M\)
where $p_1 \in U_1^1 \subset (W_1^1 \cap M)$ and (2) $h_1^1 \in \mathcal{H}(\mathcal{E}_2)$ such that $h_1^1 \mid (\mathcal{E}_2 \setminus W_1^1) = \text{id}$, $h_1^1(U_1^1) \subset W_1^1 \cap V_1^1$ and for each $j > 0$, $h_1^1(U_1^1) \cap J_{1j}^1$ is a point different from $q_1$. Define $h_1$ by $h_1^1 \mid W_1^1 = h_1^1 \mid W_1^1$ and $h_1 \mid (\mathcal{E}_2 \setminus U_1^1 \cap W_1^1) = \text{id}$. Then $h_1 \in \mathcal{H}(\mathcal{E}_2)$ since $\{W_1^1\}_{i>0}$ is a discrete collection.

In the remainder of this proof we let $H_n = h_n \circ \ldots \circ h_1$. Assume $h_n \in \mathcal{H}(\mathcal{E}_2)$ has been defined so that it has the following properties:

1. for each star set $S_1^n = U_{j>0}^n J_{1j}^n$ in $cl(S^n \setminus S^{n-1})$, there exists a relative basic open set $U_1^n$ in $M$ such that $\text{diam } U_1^n < 1/2^n$, $U_1^n \cap \mathcal{D} \neq \emptyset$, $H_n(U_1^n) \subset V_1^n \setminus F$ where $V_1^n = \{V_1^1\}_{i>0}$ and for each $j > 0$, $H_n(U_1^n) \cap J_{1j}^n$ is a point, $p_{ij}$, different from the base point of $S_1^n$;

2. $h_n \mid H_{n-1}(M \setminus U_{i>0} U_1^n) = \text{id}$;

3. $d(h_n, \text{id}) < 3 \text{ mesh } ([J_{1j}^n]_{i>0, j>0})$; and

4. $H_n(M) \cap F = \emptyset$.

For each $i > 0$ and $j > 0$, let $A_{ij}^n = J_{ij}^n \cap H_n(M \cup \mathcal{E}_2^+)$. By Lemma 3.3, let $\{W_{ij}^{n+1}\}_{i>0, j>0}$ be a discrete collection of open sets in $\mathcal{E}_2$ such that $A_{ij}^n \subset W_{ij}^{n+1}$, $W_{ij}^{n+1} \cap F = \emptyset$, $(W_{ij}^{n+1} \cap H_n(M)) \subset H_n(U_{ij}^n)$ and $\text{diam } W_{ij}^{n+1} < 2 \text{ diam } J_{1j}^n$. Since for each $i > 0$ and $j > 0$, we have a star set, $S_{ij}^{n+1}$, in $S^*$ at the fee tip of $J_{ij}^n$, we apply Lemma 3.8 again to obtain
(1) a relative basic open set $U_{i,j}^{n+1}$ in $M$ such that
\[ p_{i,j}^n \in H_n(U_{i,j}^{n+1}) \subset W_{i,j}^{n+1} \text{ and } U_{i,j}^{n+1} \cap D \neq \emptyset, \text{ and (2)} \]
\[ h_{i,j}^{n+1} \in \mathcal{K}(\ell_2) \text{ such that } h_{i,j}^{n+1} |_{\ell_2 \backslash W_{i,j}^{n+1}} = \text{id}, \]
\[ h_{i,j}^{n+1}(H_n(U_{i,j}^{n+1})) \subset W_{i,j}^{n+1} \cap V_{i,j}^{n+1} \text{ and the intersection of } \]
\[ h_{i,j}^{n+1}(H_n(U_{i,j}^{n+1})) \text{ with each basic arc in } S_{i,j}^{n+1} \text{ is a point } \]
other than $p_{i,j}$. Define $h_{n+1}$ by $h_{n+1} |_{W_{i,j}^{n+1}} = h_{i,j}^{n+1} |_{W_{i,j}^{n+1}}$
and $h_{n+1} |_{\ell_2 \backslash U_{i>0,j>0} W_{i,j}^{n+1}} = \text{id}$. Then $h_{n+1} \in \mathcal{K}(\ell_2)$ since
\[ \{W_{i,j}^{n+1} \}_{i>0,j>0} \text{ is a discrete collection. Note that } \]
distance($h_{n+1}, \text{id}$) $\leq$ mesh $\{W_{i,j}^{n+1}\} < 3$ mesh $\{J_{i,j}^{n+1}\}$.

Since for each $n > 0$, distance($h_n, \text{id}$) $< 3$ mesh $\{J_{i,j}^{n} \}_{i>0,j>0}$ and since mesh $\{J_{i,j}^{n} \}_{i>0,j>0} < 35/2^n$, we have that for each $x \in M, (H_n(x))_{n>0}$ is a Cauchy sequence and thus converges to a point in $\ell_2$. If $H$ of $M$ into $\ell_2$ is defined by $H(x) = \lim_{n \to \infty} H_n(x)$, then, by the above and the fact that each $H_n$ is continuous, it is easily seen that $H$ is a continuous function. To show $H$ is one-to-one, let $x, y \in M$ with $x \neq y$. If there exists $n > 0$ such that $x, y \in M \backslash (U_{i>0,j>0} U_{i,j}^n)$, then $H_n(x) = H(x)$ and $H_n(y) = H(y)$ so that $H(x) \neq H(y)$. If for some $n, x \in U_{i,j}^n$ and $y \in U_{k}^n$ with $j \neq k$, then $H_n(U_{i,j}^n) \subset V_{i,j}^n$ and $H_n(U_{k}^n) \subset V_{k}^n$ so that $H(U_{i,j}^n) \subset V_{i,j}^n$ and $H(U_{k}^n) \subset V_{k}^n$ and therefore, $H(x) \neq H(y)$.

Finally, if $x \in \bigcup_{i>0} U_{i,j}^n$ for each $n > 0$ and there exists $m > 0$ such that $y \in M \backslash \bigcup_{i>0} U_{i,j}^m$, then $H(x) \in F, H(y) \in H_m(M)$ and $H_m(M) \cap F = \emptyset$ so that $H(x) \neq H(y)$ and hence $H$ is
one-to-one. As \( h_{n+1}|H_n(M \cup U_{i>0} U^n) = \text{id} \) and for each \( i > 0 \), \( H_n(U^n_i) \subseteq V^n_i \), we have that \( H^{-1} \) is continuous. Thus \( H \) is an embedding. Also, \( F \subseteq H(M) \) since for each \( n > 0 \), \( \gamma_n = \{V^n_i\}_{i>0} \) is a cover of \( F \) and \( H_n(U^n_i) \subseteq V^n_i \). Since \( H^{-1}(F) \subseteq D \), we have that \( H^{-1}(F) \) is a Z-set in \( M \). Moreover, \( h_{n+1}|H_n(M \cup U_{i>0} U^n) = \text{id} \) implies that \( H(M) \) is locally bicollared at each point in \( H(M) \setminus F \). □

**Proof of Theorem 3.4.** Let \( F \) be a closed set in \( \ell^2 \) which is homeomorphic to the space of irrationals and let \( X \) be a topologically complete separable metric space which contains a closed copy, \( E \), of the space of irrationals. Let \( \varphi \) be a homeomorphism of \( \ell^2 \) onto \( Y = \{(x_1)_{i>0} \in \ell^2 : x_1 \geq 2\} \). Then \( \varphi(F) \) is a closed copy of the space of irrationals in \( Y \) and thus, by Theorem 3.8, there exists a closed embedding \( H \) of \( M \) into \( \ell^2 \) such that \( \varphi(F) \subseteq H(M) \) and \( H^{-1}(\varphi(F)) \) is a Z-set in \( M \). Let \( f \) be a closed embedding of \( X \) into \( M \) such that \( f(X) \) is a Z-set in \( M \). Then \( f(E) \) is a Z-set in \( M \), so let \( g \in \mathcal{U}(M) \) such that \( g(f(E)) = H^{-1}(\varphi(F)) \). Define \( h \) of \( X \) into \( \ell^2 \) by \( h = \varphi^{-1} \circ H \circ g \circ f \). Then \( h \) is the desired embedding. □
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