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**Irreducible Elements in Compact Topological Lattices.**

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<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>vi</td>
</tr>
<tr>
<td>I PRELIMINARIES</td>
<td>1</td>
</tr>
<tr>
<td>II PERIPHERAL ELEMENTS IN LATTICES</td>
<td>13</td>
</tr>
<tr>
<td>III AN EMBEDDING THEOREM FOR COMPACT LATTICES</td>
<td>28</td>
</tr>
<tr>
<td>IV LATTICES OF BREADTH TWO</td>
<td>46</td>
</tr>
<tr>
<td>DISTRIBUTIVE LATTICES</td>
<td>46</td>
</tr>
<tr>
<td>MODULAR LATTICES</td>
<td>61</td>
</tr>
<tr>
<td>MINIMAL CONNECTED LATTICES</td>
<td>74</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>78</td>
</tr>
<tr>
<td>VITA</td>
<td>83</td>
</tr>
</tbody>
</table>
ABSTRACT

By techniques of algebraic topology, one may extend the notion of boundary in manifolds to arbitrary topological spaces. We apply these notions to the study of topological lattices. Certain points are shown always to lie in the boundary, e.g. complemented points. A variety of consequences follow. A typical result is that any compact, finite dimensional, complemented, modular topological lattice is totally disconnected.

A recent major advance in the theory of topological lattices is the result that any compact distributive topological lattice of finite breadth $n$ can be embedded topologically and algebraically in a product of $n$ compact chains. This result follows from the study of the meet irreducible elements (an element $x$ is meet irreducible if whenever $x = a \wedge b$, then $x = a$ or $x = b$). We investigate the effect of the structure of the irreducible elements on the structure of the lattice for non-distributive lattices. Pursuant to these studies certain sublattices of the two cell are classified according to the behavior of their meet irreducible elements. The results are used to study non-distributive lattices. The main result is that a compact topological lattice of finite breadth whose meet
irreducibles are the union of $n$ chains can be embedded in a product of $n$ compact chains by a join preserving homeomorphism.

Finally we consider the following problem. Given a finite sublattice $A$ of a compact, connected, modular, metric topological lattice $L$, characterize the minimal compact, connected sublattice $S \subseteq L$ such that $A \subseteq S$. In case $A$ is the sublattice generated by two unrelated elements of $L$, then $S$ is topologically isomorphic to $I \times I$. 
INTRODUCTION

Recently J. D. Lawson and B. Madison [37] gave a cohomological definition of boundary points in arbitrary topological spaces. In Chapter II we study these boundary points in topological lattices and semilattices. A. D. Wallace [48] proved that central elements of a lattice belong to the n-Rim. We show that meet complemented elements are peripheral. In [12] T. H. Choe proved that certain compact complemented modular topological lattices are totally disconnected. Using the results from our study of peripheral elements we generalize Choe's results for finite dimensional lattices.

Embeddings of compact distributive topological lattices have been given by Lawson [32] and by K. A. Baker and A. R. Stralka [10]. In Chapter III we obtain an embedding of certain compact topological lattices of finite breadth as a subsemilattice of a finite product of compact chains. The result of Baker and Stralka is a corollary. Also in this chapter we give conditions for $A \oplus B$ to be a sublattice of a modular lattice when $A$ and $B$ are chains.

In Chapter IV we investigate topological lattices of breadth two. We characterize those compact, connected topological lattices whose meet irreducible elements are
the union of two arc chains. We extend these results to those lattices whose meet irreducible elements are the union of three arc chains. Finally, these results are used to show the example of D. E. Edmondson [24] of a modular, non-distributive, compact, connected topological lattice is minimal, i.e. given the sublattice $A = \{0, a, b, c, l\}$ of a compact, connected, modular, metric topological lattice $L$, any minimal compact, connected sublattice of $L$ which contains $A$ is topologically isomorphic to Edmondson's example.
CHAPTER I
PRELIMINARIES

This chapter contains some of the basic concepts and theorems we will need in this dissertation. Those results which are not listed as theorems may be used later without reference.

I. Topological Preliminaries

The empty set is denoted by ∅. Set inclusion is denoted by ⊆. If A and B are subsets of a set X, then A\B is the set of elements of A that are not elements of B. If $\mathcal{A}$ is a collection of sets, then $\prod_{A \in \mathcal{A}} A$ denotes the Cartesian product of the sets in $\mathcal{A}$. The Cartesian product of two sets A and B is denoted by $A \times B$.

We assume that all topological spaces are Hausdorff. If A is a subset of a topological space X, then $A^\circ$ denotes the interior of A, $A^*$ denotes the closure of A, and $F(A)$ denotes the boundary of A, i.e. $F(A) = A^* \cap (X \setminus A)^\circ$. A neighborhood of a point $x \in X$ is a subset A of X such that $x \in A^\circ$.

All other set theoretic and topological notation will be consistent with that of [29]. We shall use without reference standard topological results presented therein.
In particular we shall use the theory of Moore-Smith convergence and the characterization of closed sets, compact sets, and continuous functions in terms of nets.

The following theorem is due to R. J. Koch [30, p. 399].

Theorem 1.1. Let \( A \) be a closed subset of the compact space \( X \) and suppose that \( f: Y \times X \to X \) is continuous. If \( \{y_\alpha\}_{\alpha \in D} \) clusters at \( y \in Y \) and if \( \{f(y_\alpha, A)\}_{\alpha \in D} \) is a tower such that \( f(y_\alpha, A) \subseteq f(y_\beta, A) \) whenever \( \alpha \leq \beta \), then

\[
f(y, A) = \bigcup_{\alpha \in D} f(y_\alpha, A)^*.
\]

II. Relations And Partially Ordered Sets

A relation on a set \( X \) is a subset of \( X \times X \). If \( X \) is a topological space and \( R \) is a closed subset of \( X \times X \), then \( R \) is a closed relation. A relation is reflexive if \( (x, x) \in R \) for each \( x \in X \), antisymmetric if \( (x, y) \in R \) and \( (y, x) \in R \) imply \( x = y \), and transitive if \( (x, y) \in R \) and \( (y, z) \in R \) imply \( (x, z) \in R \). A partial order is a reflexive antisymmetric, transitive relation. A partially ordered set is a pair \( (X, \leq) \), where \( X \) is a set and \( \leq \) is a partial order on \( X \). If \( (x, y) \in \leq \), we write \( x \leq y \); if \( (x, y) \in \leq \) and \( x \neq y \), we write \( x < y \).

If \( R \) is a relation on \( X \) and \( A \subseteq X \), then

\[
L(A) = \{x \in X : (x, a) \in R \text{ for some } a \in A\},
\]

\[
M(A) = \{y \in X : (a, y) \in R \text{ for some } a \in A\},
\]

\[
C(A) = L(A) \cap M(A).
\]
The lower set of \( x \) is \( L([x]) \) and is denoted by \( L(x) \);
the upper set of \( x \) is \( M([x]) \) and is denoted by \( M(x) \); if
\( a \leq b \), then the interval from \( a \) to \( b \) is \([a,b] = \{x \in X : a \leq x \leq b\}\). A set \( A \subseteq X \) is decreasing if \( L(A) \subseteq A \)
and increasing if \( M(A) \subseteq A \). The set \( A \) is increasing if and
only if \( X \setminus A \) is decreasing. The next theorem is due to L.
Nachbin [39].

Theorem 1.2. Let \( R \) be a closed relation on a
topological space \( X \). If \( A \) is a compact subset of \( X \),
then \( L(A) \) is closed.

A subset \( A \) of a partially ordered set \( X \) is convex if
\( x, z \in A \) and \( x \leq y \leq z \) imply \( y \in A \). If \( X \) is a
topological space and the partial order \( \leq \) is closed, then
\( X \) is called a partially ordered topological space. If \( X \)
has a basis of open, convex sets, \( X \) is said to be locally
convex. Nachbin [39, pp. 25-49] has shown the following.

Theorem 1.3. If \( X \) is a compact partially ordered
topological space, then \( X \) is locally convex.

Corollary 1.4. Let \( X \) be a compact partially
ordered topological space. If \( x \in X \) and \( U \) is an open
set containing \( x \), then there exists a closed, convex
neighborhood of \( X \) which is a subset of \( U \).

Proof. Since \( X \) is locally convex and regular,
there exist an open, convex set $V$ and an open set $W$ such that $x \in W$ and $W^* \subset V \subset U$. By Theorem 1.2 and its dual $M(W^*)$ and $L(W^*)$ are closed; hence $C(W^*)$ is closed. If $a, c \in C(W^*)$ and $a \leq b \leq c$, then $b \in L(W^*)$ and $b \in M(W^*)$; thus $b \in C(W^*)$; and therefore $C(W^*)$ is convex. Hence $C(W^*)$ is a closed, convex neighborhood of $x$ and $C(W^*) \subset C(V) \subset V \subset U$.

An element $x$ of a partially ordered set is maximal (minimal) if $M(x) = \{x\}$ ($L(x) = \{x\}$). The next theorem is due to A. D. Wallace [45].

**Theorem 1.5.** Let $X$ be a compact partially ordered topological space. Then every element of $X$ lies in the lower set of some maximal element and in the upper set of some minimal element.

A subset $C$ of a partially ordered set $X$ is a chain if $x, y \in C$ implies either $x \leq y$ or $y \leq x$. If $X$ is a partially ordered topological space, a subset $C$ of $X$ is an arc chain if $C$ is a compact, connected chain. For each $x \in C$, the sets $M(x) \cap C$ and $L(x) \cap C$ are closed. If these sets are taken for a subbasis for the closed sets of $C$, then the resulting topology is called the interval topology. The interval topology agrees with the relative topology since $C$ is compact.

A subset $A$ of a partially ordered set is order-dense
if \( a, c \in A \) and \( a < c \) imply there exists \( b \in A \) such that \( a < b < c \). If \( X \) is a partially ordered topological space, L. E. Ward, Jr. [51] has shown that an arc chain is order-dense and also a compact order-dense chain is connected; thus a compact order-dense chain is an arc chain. If \( X \) and \( Y \) are partially ordered sets and \( f \) is a function from \( X \) into \( Y \), then \( f \) is order-preserving if \( x, y \in X \) and \( x \leq y \) imply \( f(x) \leq f(y) \). The unit interval \( I \) is an arc chain with respect to the order inherited from the real numbers. If \( C \) is a metric arc chain, then there exists an order preserving homeomorphism from \( C \) onto \( I \). [53, p.30].

III. Topological Lattices And Semilattices

Let \( X \) be a partially ordered set. If every pair of elements of \( X \) has a greatest lower bound (least upper bound), then \( X \) is a meet (join) semilattice. If \( X \) is both a meet and a join semilattice, then \( X \) is a lattice. When we use the term semilattice we shall mean meet semilattice. If \( a, b \in X \), then \( a \land b (a \lor b) \) will denote the greatest lower bound (least upper bound) of \( a \) and \( b \); \( a \land b \) is called the meet of \( a \) and \( b \); \( a \lor b \) is called the join of \( a \) and \( b \). If a collection of elements \( \{x_{\alpha}\}_{\alpha \in A} \) has a meet (= greatest lower bound)(join = least upper bound), it will be denoted by \( \bigwedge_{\alpha \in A} x_{\alpha} (\bigvee_{\alpha \in A} x_{\alpha}) \); if \( B \) is a
subset of a semilattice S we write $AB$ for $\bigwedge_{b \in B} b$.

If we define $ab = a \wedge b$ for all $a, b$ in a semilattice $S$, then $S$ with this multiplication is a commutative idempotent semigroup. Conversely, if $S$ is a commutative, idempotent semigroup, then $S$ is a semilattice with respect to the partial order $\leq$ defined by $a \leq b$ if and only if $ab = a$. The details of this topic may be found in [16, pp. 23, 24]; for topological semigroup theory see [26].

If $S$ is a semilattice (lattice) and $A$ is a subset of $S$, the semilattice (lattice) generated by $A$ is the smallest sub-semilattice (lattice) of $S$ which contains $A$; it is denoted by $\langle A \rangle$ in either case since which one is meant will be clearly indicated where ever confusion might occur.

A semilattice $S$ is a topological semilattice if $S$ is a topological space and the function $m: S \times S \to S$ defined by $m(x, y) = x \wedge y$ is continuous. A lattice is a topological lattice if it is a topological semilattice with respect to both $\wedge$ and $\vee$.

Theorem 1.6. Let $S$ be a topological semilattice.

1) The relation $\leq$ is closed.

ii) If $S$ is compact, then $S$ has a $0$.

iii) If $U$ is an open subset of $S$, then $M(U)$ is open.
Proof. Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ be a net in $\leq$ which converges to $(x, y)$. Since $x_\alpha = x_\alpha \land y_\alpha$ for all $\alpha$ and since by continuity of $\land$ the net $\{x_\alpha \land y_\alpha\}_{\alpha \in D}$ converges to $x \land y$, we have $x = x \land y$. Thus $(x, y) \in \leq$; therefore part i) holds.

For part ii) we assume $S$ is compact. By Theorem 1.5 and part i), $S$ has a minimal element $z$. If $x \in S$, then $x \land z \leq z$, and since $z$ is minimal, $x \land z = z$. Thus $z$ is a zero for $S$. Clearly a semilattice can have at most one zero.

Let $p \in M(U)$. Then there exists $u \in U$ such that $u \leq p$. Since $p \land u = u$ and $\land$ is continuous, there exists an open set $V$ containing $p$ such that $V \land u \subseteq U$. If $y \in V$, then $y \land u \leq y$ and $y \land u \in U$. Hence $V \subseteq M(U)$. Since $p$ was arbitrary, $M(U)$ is open.

The proofs of Corollary 1.4 and Theorem 1.6 are found in [32] and are included here for convenience. The next theorem is also found there.

Theorem 1.7. Let $S$ and $T$ be topological semigroups and let $f$ be a continuous homomorphism from $S$ onto $T$. If $S$ is a topological semilattice, then $T$ is a topological semilattice and $f$ is order preserving.

Koch [31] has shown

Theorem 1.8. If $S$ is a compact, connected
topological semilattice, then every element of \( S \) lies on an arc chain containing 0.

If \( L \) is an arc chain, then \( L \) is a topological lattice. If \( f \) is a continuous meet (join) preserving function from \( L \) onto a space \( M \), then \( M \) is a continuum, i.e. a compact, connected, Hausdorff space, and so by Theorem 1.7 is totally ordered. Thus \( M \) is an arc chain or a point. In a topological semilattice the translation by \( p \) which sends \( x \) to \( x \land p \) for all \( x \) is a continuous meet preserving function; therefore the translate of an arc chain is an arc chain or a point.

In [5] L. W. Anderson has shown the following.

Theorem 1.9. A locally compact connected topological lattice is locally connected.

A lattice is distributive if \( x \land (y \lor z) = (x \land y) \lor (x \land z) \) for all \( x, y, z \). A lattice is modular if \( x \leq z \) implies \( x \lor (y \land z) = (x \lor y) \land z \) for all \( x, y, z \). A lattice may be neither modular nor distributive and it may be modular but not distributive. A complete discussion may be found in [11].

A subset \( A \) of a lattice \( L \) is meet-irredundant if \( A B > \land A \) for any proper subset \( B \subseteq A \). The breadth of \( L \) is the least upper bound of the cardinalities of all the finite meet-irredundant subsets of \( L \). Join-irredundant
is defined dually and the breadth of $L$ is the same when either meet-irredundant or join-irredundant sets are used to define it \[11\]. The width of a subset $A \subseteq L$, written $w(A)$, is the cardinality of the largest subset of $A$ each two of whose elements are unrelated. The notion of width is meaningful in any partially ordered set.

In \[23\] D. E. Edmondson has shown the following.

Theorem 1.10. If $L$ is a connected topological lattice of breadth two, then $L$ is modular.

An element $x$ of a meet (join) semilattice is meet (join) irreducible if $x = a \land b$ ($a \lor b$) implies $x = a$ or $x = b$; $x$ is meet (join) prime if $a \land b \leq x$ ($a \lor b \leq x$) implies $a \leq x$ or $b \leq x$ ($a \leq a$ or $x \leq b$). For distributive lattices the two notions coincide.

A lattice is complete if every subset has a greatest lower bound and a least upper bound. If $A$ is a subset of a compact topological lattice $L$, then the $0(1)$ of the closed sublattice generated by $A$ is a greatest lower bound (least upper bound for $A$). Thus $L$ is complete. A net $\{x_\alpha\}_{\alpha \in D}$ in a complete lattice $L$ order converges to $x$ if $\bigwedge_{\alpha \geq \beta} x_\alpha = \bigwedge_{\alpha \geq \beta} x_\beta = x$. We define a subset $A \subseteq L$ to be closed if whenever $\{x_\alpha\}$ is a net in $A$ and $\{x_\alpha\}$ order converges to $x$, then $x \in A$. The complements of the closed sets form a topology for $L$ called the order

Theorem 1.11. Suppose that \( L \) is a complete lattice with finite breadth and that the operations of \( L \) are continuous with respect to order convergence. Then every element of \( L \) is the meet of finitely many meet-irreducible elements (prime elements, if \( L \) is distributive).

IV. Algebraic Topology

The Alexander-Wallace-Spanier cohomology theory will be employed. The basic theorems and notation may be found in [42].

A topological space \( X \) is acyclic if the induced homomorphism \( f^* \) of a function \( f \) from \( X \) to a point space is an isomorphism in all dimensions. The following theorem is due to Wallace [50].

Theorem 1.12. If \( S \) is a compact, connected topological semilattice, then \( S \) is acyclic.

In [37] J. D. Lawson and B. Madison have given a cohomological definition of boundary points of a topological space. A point \( x \in X \), a topological space, is marginal if for any open set \( U \) containing \( x \), there exists an open set \( V \) containing \( x \) and contained in \( U \).
such that \( H^*(X,X\setminus V) \) is trivial; \( x \) is peripheral if for any open set \( U \) containing \( x \), there exists an open set \( V \) containing \( x \) and contained in \( U \) such that the homomorphism \( i^*:H^*(X,X\setminus V) \to H^*(X,X\setminus U) \) induced by the inclusion function \( i \) is the trivial or zero homomorphism; \( x \) is an inner point if it is not peripheral. The next three theorems are in [37]; the first two are due to Lawson and Madison and the third is due to K. H. Hofmann.

Theorem 1.13. In a regular space \( X \) the following are equivalent:

1) the point \( x \) is marginal in \( X \);
2) for any open set \( U \) containing \( x \), there exists an open set \( V \) containing \( x \) and contained in \( U \) such that the natural homomorphism \( H^*(X) \to H^*(X\setminus V) \) is an isomorphism.

Theorem 1.14. The following are equivalent in a regular space \( X \):

1) the point \( x \) is peripheral in \( X \);
2) the point \( x \) is peripheral in \( K \), a neighborhood of \( x \).

Theorem 1.15. Let \( X \) be a compact space, \( T \) a topological space with distinguished point \( 1 \), and \( F:T \times X \to X \) a continuous function with \( F(1,x) = x \) for each
x \in X$. If $p$ is an inner point of $X$, then there is an open set $U$ containing $1$ so that if $t$ is in the component of $1$ in $U$, then $p \in F(t \times X)$.

We note that if $T$ is locally connected we may choose $U$ so that if $t \in U$, then $p \in F(t \times X)$.

We shall use the dimension function, called codimension, defined by H. Cohen [17]. If $X$ has codimension less than or equal to $n$ and $A$ is a locally compact subset of $X$, then $A$ has codimension less than or equal to $n$. The next two theorems are also from [37].

Theorem 1.16. If $X$ is a locally compact space of finite codimension, then the set of inner points of $X$ is dense in $X$.

Theorem 1.17. Let $X$ be a compact space of codimension $n$. If $A$ is a closed subset of the set of peripheral points of $L$, then the codimension of $A$ is less than $n$.

Lawson [36] has shown

Theorem 1.18. If $L$ is a locally compact, connected topological lattice, then the breadth of $L$ is equal to the codimension of $L$. 
CHAPTER II
PERIPHERAL ELEMENTS IN LATTICES

In this chapter we investigate the peripheral points in compact, connected topological lattices. The results are used to show that certain classes of compact topological lattices are totally disconnected. We also obtain some information about the codimension of the boundary of \( M(x) \) in topological lattices of finite codimension.

Definition 2.1. A topological semilattice \( S \) is said to have \textit{small semilattices} at \( x \) if \( x \) has a basis of neighborhoods which are subsemilattices of \( S \); \( S \) has \textit{small semilattices} if it has small semilattices at every point.

Following the example of Hofmann at the Second Florida Symposium on Automata and Semigroups we make the following definition.

Definition 2.2. A topological semilattice is a \textit{Lawson semilattice} if it has small semilattices.

Lawson [33] has shown

Theorem 2.3. If \( S \) is a \textit{locally compact Lawson semilattice}, then every compact subset of \( S \) is contained in a compact subsemilattice of \( S \).
Lawson and Madison [38] have proved that cutpoints of compact, connected spaces are inner points. This result and the next theorem locate all the meet irreducible elements of a semilattice.

Theorem 2.4. Let $S$ be a compact, connected, locally connected Lawson semilattice. If $p \in S$ is meet irreducible and is not a cutpoint of $S$, then $p$ is marginal in $S$.

Proof. Let $p \in U$, an open subset of $S$. Since $p$ is not a cutpoint of $S$, it is known [52] that there exists an open set $V$ containing $p$ such that $V \subseteq U$ and $S \setminus V$ is connected. Since $V$ is open in $S$, $S \setminus V$ is closed; hence compact. If $x, y \in S \setminus \{p\}$, then $x \wedge y \in S \setminus \{p\}$ because $p$ is meet irreducible. Thus $S \setminus \{p\}$ is a locally compact Lawson semilattice. By Theorem 2.3 there exists a compact subsemilattice $W \subseteq S \setminus \{p\}$ such that $S \setminus V \subseteq W$. Since $S \setminus V$ is connected, $<S \setminus V>^*$ is a compact, connected subsemilattice of $W$. By Theorem 1.12 $<S \setminus V>^*$ is acyclic; thus $H^*(S) \rightarrow H^*(<S \setminus V>^*)$ is an isomorphism. Since $S \setminus V \subseteq <S \setminus V>^* \subseteq W \subseteq S \setminus \{p\}$, we have $p \in S \setminus <S \setminus V>^* \subseteq V \subseteq U$. It follows by Theorem 1.13 that $p$ is marginal in $S$.

It is known [37] that $x$ marginal in $X$ implies $x$ is peripheral in $X$. The converse is not true however.

Corollary 2.5. Let $L$ be a compact, connected
topological lattice with small semilattices with respect to the meet operation. If \( p \in L \) is meet irreducible and is not a cutpoint of \( L \), then \( p \) is marginal in \( L \).

Proof. By Theorem 1.9 \( L \) is locally connected. Thus by Theorem 2.4 \( p \) is marginal in \( L \).

In [36] Lawson proved that any topological semilattice of finite breadth has small semilattices. The following proposition is a corollary due also to Lawson.

Proposition 2.6. A compact topological lattice \( L \) of finite breadth has small lattices, i.e. each \( x \in L \) has a neighborhood basis of sublattices of \( L \).

Proof. We actually show that the neighborhoods may be chosen to be intervals of \( L \) and thus are closed and convex. Let \( x \in U \) an open subset of \( L \). By Corollary 1.4 there exists a closed, convex set \( V \) such that \( x \in V^o \subset V \subset U \). Since \( L \) is a topological semilattice with respect to each operation, there exist a closed meet subsemilattice \( A \) and a closed join subsemilattice \( B \) such that \( x \in A^o \subset A \subset V^o \subset V \) and \( x \in B^o \subset B \subset V^o \subset V \). Since \( V \) is closed, it is compact; consequently \( A \) and \( B \) are compact. Therefore by Theorem 1.6 part ii) and its dual, \( A \) has a zero and \( B \) has a 1 , say \( a \) and \( b \) respectively. Thus \( x \in A^o \cap B^o \subset [a,b] \subset V \subset U \); hence \([a,b]\) is the required sublattice.
Corollary 2.7. Let $L$ be a compact, connected topological lattice of finite codimension. If $p \in L$ is not a cutpoint of $L$ and $p$ is either meet irreducible or $p$ is join irreducible, then $p$ is marginal in $L$.

Proof. By Theorem 1.18 $L$ has finite breadth. Hence the conclusion is immediate from Proposition 2.6 and Corollary 2.5.

Definition 2.8. An element $x$ in a lattice $L$ with $0(1)$ is meet (join) complemented if there is an element $y \in L$ such that $y \neq 0$ ($y \neq 1$) and $x \land y = 0$ ($x \lor y = 1$); $x$ is complemented if there is a $y \in L$ such that $x \land y = 0$ and $x \lor y = 1$. A lattice is meet complemented if each of its elements, except 1, is meet complemented. Similarly we define join complemented and complemented lattices.

Theorem 2.9. Let $L$ be a compact, connected topological lattice. If $a, b \in L$ and $a$ is a meet complement for $b$, then $[0, a]$ and $[0, b]$ are contained in the set of peripheral elements of $L$.

Proof. We define $F: L \times L \to L$ by $F(x, y) = x \lor y$ for all $x, y \in L$. Then $F$ is a continuous function and $F(0, y) = y$ for all $y \in L$. Let $x \in (0, a) = [0, a] \setminus \{0\}$; then $x \land b \leq a \land b = 0$ which implies $x \land b = 0$. If $x$ is not peripheral in $L$, then by Theorem 1.15 there exists an open set $U$ containing 0 such that for each $s \in U$ there
is a t.c.L for which \( s \vee t = x \). Since \([0,b] = b \Delta L\) is the continuous image of a connected space, it is connected. Thus \( U \cap (0,b) \neq \emptyset \). Let \( s \in U \cap (0,b) \) and let \( t \in L \) be such that \( s \vee t = x \). Then \( s < x \) and \( s < b \); thus \( s < x \wedge b = 0 \) which implies \( s = 0 \), contrary to \( s \in (0,b) \). Hence \( x \) is peripheral in \( L \). That \( 0 \) is actually marginal is a topological semigroup result due to Lawson and Madison [38]. This is also a consequence of the remark following Theorem 2.19 below. Thus each element of \([0,a]\) is peripheral in \( L \). The proof for \([0,b]\) is similar.

Corollary 2.10. Let \( L \) be a compact, connected topological lattice. If \( a,b \in L \) are not related, then \([a, a \vee b], [b, a \vee b], [a \wedge b, a] \) and \([a \wedge b, b]\) are contained in the set of peripheral elements of \([a \wedge b, a \vee b]\).

Proof. The lattice \([a \wedge b, a \vee b]\) is the continuous image of \([a \wedge b] \times L(a \vee b)\) which is compact and connected since \( L(a \vee b)\) is connected as was observed in the proof of Theorem 2.9. Since \( a \) and \( b \) are complements for each other in this interval, the conclusion follows from Theorem 2.9 and its dual.

Corollary 2.11. Let \( L \) be a compact, connected topological lattice. If for \( p \in L \) there is a \( q \in L \) such that \( q \) is not related to \( p \) and either \( p \in M(p \wedge q)^{\circ} \), or \( p \in L(p \vee q)^{\circ} \), then \( p \) is peripheral in \( L \).
Proof. The point $q$ is a meet complement for $p$ in $M(p \land q)$; thus by Theorem 2.9 $p$ is peripheral in $M(p \land q)$. If $p \in M(p \land q)^{\circ}$, then by Theorem 1.14 $p$ is peripheral in $L$. The case of $p \in L(p \lor q)^{\circ}$ is dual.

We do not know if there exist non-degenerate, compact, connected, meet complemented topological lattices; however we have:

Proposition 2.12. If $L$ is a non-degenerate, compact, connected, meet complemented topological lattice, then the codimension of $L$ is infinite.

Proof. Suppose the codimension of $L$ is finite. By Theorem 1.16 the inner points must be a dense subset of $L$. By Theorem 2.9 $L$ has no inner points. Thus the codimension of $L$ cannot be finite.

For convenience we state the following corollary which is immediate from the preceding.

Corollary 2.13. A compact, connected, meet complemented, topological lattice $L$ has finite codimension if and only if $L$ is a point.

Definition 2.14. A lattice is relatively meet complemented if each interval $[a,b]$ is a meet complemented lattice. Similarly we define relatively join complemented and relatively complemented.
Theorem 2.15. Let $L$ be a compact, relatively meet complemented topological lattice of finite codimension. Then $L$ is totally disconnected.

Proof. Let $p \in L$, and let $C$ be the component of $p$ in $L$. By continuity of $\land$ and $\lor$, $C \land C$ and $C \lor C$ are connected sets which contain $p$. Thus $C \land C$ and $C \lor C$ are contained in $C$ which means $C$ is a sublattice of $L$. A component is closed; therefore $C$ is compact. By Theorem 1.6 part ii) and its dual $C$ has a zero and a 1, say $a$ and $b$ respectively. We show that $C = [a,b]$. Clearly $C \subseteq [a,b]$. Let $x \in [a,b]$. Then $a = x \land a \in x \land C$ and $x \land C$ is connected; thus $x \land C \subseteq C$. Now $x = x \lor b \in x \land C$; hence $x \in C$. Thus $[a,b] \subseteq C$; consequently $C = [a,b]$. Thus $C$ is a compact, connected, meet complemented topological lattice and has finite codimension. By Corollary 2.13 $C = \{p\}$. Therefore $L$ is totally disconnected.

Definition 2.16. An orthomodular lattice is a lattice $L$ with 0 and 1 and a function $*:L \to L$ such that for all $a,b \in L$ $a \land a^* = 0$, $a \lor a^* = 1$, $(a \land b)^* = a^* \lor b^*$, $(a \lor b)^* = a^* \land b^*$, $(a^*)^* = a$, and if $a \leq b$, then $a \lor (a^* \land b) = b$.

Corollary 2.17. If $L$ is a compact topological lattice of finite codimension satisfying any one of the following conditions, i) $L$ is relatively complemented, ii) $L$ is complemented and modular,
iii) \( L \) is an orthomodular lattice, then \( L \) is totally disconnected.

Proof. Part i) is immediate from Theorem 2.15. It is known [11, pp. 16,53] that any lattice which satisfies either part ii) or part iii) is relatively complemented. Hence part ii) and part iii) follow from part i).

We now give an example of a compact, complemented topological lattice of codimension one which is not totally disconnected.

Example 2.18. Let \( L = \{(0,y) : \frac{1}{2} \leq y \leq 1\} \cup \{(0,0), (1,0), (1,1)\} \subseteq \mathbb{R} \times \mathbb{R} \), the unit square. With the lattice operations defined by the order inherited from \( \mathbb{R} \times \mathbb{R} \) and with the relative topology of the plane, \( L \) is a compact, complemented topological lattice of codimension one which is not totally disconnected.

The next theorem gives another sufficient condition for an element to be peripheral in a compact, connected, locally connected topological semilattice. We later use the results to study the codimension of \( F(M(x)) \) for \( x \) an element of certain topological lattices.

Theorem 2.19. Let \( S \) be a compact, connected, locally connected topological semilattice with \( 1 \). If \( p \in S \) and \( M(p)^o = \emptyset \), then \( M(p) \) is contained in the set
Proof. We define \( F: S \times S \to S \) by \( F(x,y) = x \wedge y \) for all \( x, y \in S \). Then \( F \) is continuous and \( F(1,x) = x \) for all \( x \in S \). If \( s \in M(p) \) and \( s \) is an inner point of \( S \), then by Theorem 1.15 there exists an open set \( U \) containing \( 1 \) such that for each \( u \in U \) there is a \( v \in S \) with \( u \wedge v = s \). This implies \( U \subseteq M(s) \subseteq M(p) \) so that \( M(p)^o \neq \Box \) contrary to hypothesis. Thus \( s \) is peripheral in \( S \), and since \( s \) was an arbitrary element of \( M(p) \), \( M(p) \) consists entirely of peripheral elements of \( S \).

A compact, connected topological lattice is locally connected by Theorem 1.9 and has a \( 1 \) by Theorem 1.6; thus if \( M(p)^o = \Box \) in such a lattice, then \( M(p) \) consists of peripheral elements of \( L \).

The set of peripheral elements of a topological space need not be closed [38]. However, we have the following.

Corollary 2.20. Let \( L \) be a compact, connected topological lattice of finite codimension. If \( A = \{ x \in L : M(x)^o = \Box \} \), then each element of \( A^* \) is peripheral in \( L \).

Proof. Let \( x \in A^* \) and suppose that \( x \) is an inner point of \( L \). Let \( \{ x_\alpha \} \) be a net in \( A \) which converges to \( x \). Then \( \{ x_\alpha \vee x \} \) also converges to \( x \), and since \( M(x_\alpha \vee x) \subseteq M(x_\alpha) \), we must have \( M(x_\alpha \vee x)^o = \Box \). Thus
By Theorem 2.19 and our assumption that \( x \) is inner, \( M(x)^\circ \neq \emptyset \). By Theorem 1.16 we may choose an inner point \( y \) of \( L \) such that \( y \in M(x)^\circ \); by Theorem 1.14 \( y \) is also an inner point of \( M(x) \).

We now show that if \( U \) is any open set of \( M(x) \) which contains \( x \), there is some \( u \in U \) such that \( u \not\leq y \). Suppose on the contrary there is an open set \( U \) of \( M(x) \) containing \( x \) for which \( u \in U \) implies \( u \leq y \). Since \( \{x_\alpha \wedge x\} \subseteq M(x) \) and converges to \( x \), there exists an \( \alpha \) such that \( x_\alpha \wedge x \in U \). Hence \( x_\alpha \wedge x \leq y \); therefore \( y \in M(x_\alpha \wedge x) \) which implies \( M(y) \subseteq M(x_\alpha \wedge x) \). But \( M(y)^\circ \subseteq M(y) \subseteq M(x_\alpha \wedge x) \) and \( M(y)^\circ \neq \emptyset \) since \( y \) is an inner point of \( L \) (Theorem 2.19); this implies \( M(x_\alpha \wedge x)^\circ \neq \emptyset \) contrary to \( x_\alpha \wedge x \in A \). Thus our claim is established.

Since \( x \) is the zero of \( M(x) \) and \( y \) is inner in \( M(x) \), it follows from the proof of Theorem 2.19 that there must be an open subset \( U \) of \( M(x) \) which contains \( x \) and such that \( u \in U \) implies \( u \leq y \). This contradiction completes the proof.

We give an example to show that Corollary 2.20 need not be true if \( L \) does not have finite codimension.

Example 2.21. Let \( L = \{ (x_i) : 0 \leq x_i \leq 1 \} \cup \{ (x_i) : -1 \leq x_i \leq 0 \} \subseteq \prod_{i=1}^{\infty} \mathbb{R}_i \), \( \mathbb{R}_i \) the set of real numbers for \( i = 1, 2, \ldots \). With the order and topology inherited from \( \prod_{i=1}^{\infty} \mathbb{R}_i \), i.e.
\((x_i) \leq (y_i)\) if and only if \(x_i \leq y_i\) for \(i = 1, 2, \ldots\), \(L\) is a compact connected topological lattice. Since \(p = (p_1)\) with \(p_i = 0\) for \(i = 1, 2, \ldots\) is a cutpoint of \(L\), by the comment preceding Theorem 2.4, \(p\) is an inner point of \(L\).

Any \((x_i) \in L\) with \(0 < x_i \leq 1\) for infinitely many \(i\) has the property that \(M((x_i))^{o}\) is empty. Thus \([(2^{-1}, 2^{-1}, \ldots); i = 1, 2, \ldots]) \subset A = [(x_i): M((x_i))^{o} = \emptyset], and\n
\([(2^{-1}, 2^{-1}, \ldots); i = 1, 2, \ldots])\) converges to \(p\).

We now investigate the codimension of \(F(M(x))\). The next lemma is immediate from Theorem 1.6 part iii).

**Lemma 2.22.** If \(S\) is a topological semilattice, \(a \in S\) and \(M(a)^{o} \neq \emptyset\), then \(M(M(a)^{o}) = M(a)^{o}\).

**Lemma 2.23.** Let \(S\) be a topological semilattice and let \(a \in S\). If \(x \in F(M(a))\), then \([a, x] \subset F(M(a))\).

**Proof.** If \(y \in [a, x]\) and \(y \not\in F(M(a))\), then \(y \in M(a)^{o}\). By Lemma 2.22 we have \(x \in M(a)^{o}\) contrary to hypothesis. Hence \(y \in F(M(a))\).

**Definition 2.24.** A net \(\{x_{\alpha}\}_{\alpha \in D}\) in a partially ordered space is **increasing** (decreasing) if \(\alpha \leq \beta\) implies \(x_{\alpha} \leq x_{\beta}\) (\(x_{\beta} \leq x_{\alpha}\)).

In [5] the following theorem was proved by Anderson.

**Theorem 2.25.** If \(L\) is a connected topological lattice and \(p \in L\), then \(p\) is a cutpoint of \(L\) if and
only if \( p \neq 0, p \neq 1, \) and \( L = L(p) \cup M(p). \)

**Theorem 2.26.** Let \( L \) be a compact, connected topological lattice of finite codimension \( n \), and let \( a \in L \setminus \{0,1\} \). Then

i) \( M(a)^0 = \emptyset \) implies the codimension of \( M(a) \) is less than \( n \),

ii) \( (M(a)^0)^* = M(a) \) implies the codimension of \( F(M(a)) \) is less than \( n \),

iii) if \( x \in F(M(a)) \), then the codimension of \( [a,x] \) is less than \( n \).

**Proof.** Part i). By Theorem 2.19 \( M(a) \) is a closed subset of the peripheral elements of \( L \); thus by Theorem 1.17 the codimension of \( M(a) \) is less than \( n \).

Part ii). The hypothesis \( (M(a)^0)^* = M(a) \) implies \( F(M(a)) \) is nowhere dense in \( M(a) \). In \( M(a) \), \( [a,x] \) is the lower set of any \( x \in M(a) \), and by Lemma 2.23 \( [a,x] \) is contained in \( F(M(a)) \) when \( x \in F(M(a)) \). Thus the interior of \( [a,x] \) in \( M(a) \) is empty. Hence by Theorem 2.19 \( x \) is peripheral in \( M(a) \). Therefore \( F(M(a)) \) is a closed subset of the peripheral points of \( M(a) \); by Theorem 1.17 the codimension of \( F(M(a)) \) is less than the codimension of \( M(a) \) which is less than or equal to \( n \).

Part iii). Let \( x \in F(M(a)) \); let \( U(a) = L \setminus [L(a) \cup M(a)] \). If \( U(a) = \emptyset \), then by Theorem 2.25 \( a \) is a cutpoint of \( L \).
and hence \( a = x \). Thus we are done. Therefore suppose
that \( U(a) \neq \Box \) and \( x \neq a \). Since \( x \notin L(a) \) we may choose
an open set \( V \) containing \( x \) such that \( V \cap L(a) = \Box \). Now
\( x \in F(M(a)) \) implies \( V \cap L \setminus M(a) \neq \Box \); \( L = M(a) \cup L(a) \cup U(a) \);
therefore \( L \setminus M(a) = (L(a) \cup U(a)) \setminus \{a\} \). Therefore since
\( V \cap L(a) = \Box \) we must have \( V \cap U(a) \neq \Box \); also \( x \in L(x) \) and
\( L(x) \) is connected. Thus \( V \cap (L(x) \setminus \{x\}) \neq \Box \).

We claim there is a net \( \{x_{\alpha}^\prime\}_{\alpha \in D} \subseteq L(x) \cap U(a) \) which
converges to \( x \). Choose \( \{x_{\alpha}^\prime\}_{\alpha \in D} \subseteq U(a) \) such that
\( \{x_{\alpha}^\prime\}_{\alpha \in D} \) converges to \( x \). The preceding discussion concerning \( V \) shows we can choose such a net in \( U(a) \). Let
\( x_\alpha = x \wedge x_{\alpha}^\prime \) for all \( \alpha \in D \). Then \( \{x_\alpha\}_{\alpha \in D} \subseteq L(x) \) and
converges to \( x \). Suppose that \( \{x_\alpha\}_{\alpha \in K} \subseteq L(a) \) for all
\( \alpha \in K \), a cofinal subset of \( D \). Then \( \{(x_\alpha, a)\}_{\alpha \in K} \) converges
to \( (x, a) \) which implies, by Theorem 1.6 part i), that
\( x \leq a \). But \( x \leq a \) and \( x \in M(a) \) imply \( x = a \) contrary
to our choice of \( x \). Thus \( x_\alpha \notin L(a) \) for almost all
\( \alpha \in D \); therefore we may assume \( \{x_\alpha\}_{\alpha \in D} \subseteq L \setminus L(a) \). If
\( x_\alpha \in M(a) \), then \( a \leq x_\alpha = x_{\alpha}^\prime \wedge x \leq x_{\alpha}^\prime \) contrary to our
choice of \( x_{\alpha}^\prime \). Hence \( \{x_\alpha\}_{\alpha \in D} \subseteq U(a) \) and the claim is
established.

We claim now that \( \{x_\alpha\}_{\alpha \in D} \) may be assumed to be an
increasing net. Let \( y_\alpha = \bigwedge_{\beta \geq \alpha} x_\beta \) for each \( \alpha \in D \). Clearly
\( \{y_\alpha\}_{\alpha \in D} \) is an increasing net contained in \( L(x) \). Let \( U \)
be an open set containing \( x \). By Proposition 2.6 there
exists a closed sublattice $V \subseteq U$ such that $x \in V^\circ$. Since $\{x_\alpha\}_{\alpha \in D}$ converges to $x$, there exists $\beta \in D$ such that $\alpha \geq \beta$ implies $x_\alpha \in V^\circ$. Hence $y_\gamma = \bigwedge_{\alpha \geq \gamma} x_\alpha \in V$ for all $\gamma \geq \beta$. Thus $\{y_\alpha\}_{\alpha \in D}$ converges to $x$. That $\{y_\alpha\}_{\alpha \in D} \subseteq U(a)$ follows just as did the fact that $\{x_\alpha\}_{\alpha \in D} \subseteq U(a)$.

For each $\alpha \in D$, the codimension of $[a, a \wedge x_\alpha]$ is less than $n$ by Corollary 2.10 and Theorem 1.17. Thus by Theorem 1.18 the breadth of $[a, a \wedge x_\alpha]$ is less than $n$.

Since $\{x_\alpha\}_{\alpha \in D}$ is an increasing net, $[[a, a \wedge x_\alpha]]$ is a tower. Therefore $\bigcup_{\alpha \in D} [a, a \wedge x_\alpha]$ has breadth less than $n$; consequently the breadth of $\left( \bigcup_{\alpha \in D} [a, a \wedge x_\alpha] \right)^*$ is less than $n$.

The interval $[a, a \wedge x_\alpha] = (a \wedge x_\alpha) \Lambda M(a)$. Letting $\Lambda$ be the function $f$ of Theorem 1.1 we have $[a, x] = (\bigcup_{\alpha \in D} [a, a \wedge x_\alpha])^*$. By Theorem 1.18 the codimension of $[a, x]$ is less than $n$.

E. D. Shirley and Stralka [41] have proved that sublattices of finite breadth $n \geq 2$ of a connected, distributive topological lattice $L$ of breadth $n$ have non-empty interiors in $L$. We give a related result.

**Theorem 2.27.** Let $L$ be a compact, connected topological lattice of codimension $n$. If $L(x) = (L(x)^\circ)^*$, then the codimension of $[z, x] = n$ implies $([z, x])^\circ \neq \emptyset$. Also $z \in (L(x))^\circ$. 
Proof. Suppose \( z \not\in L(x)^o \). Then by the dual of Lemma 2.23 \([z,x] \subseteq F(L(x))\) which by the dual of Theorem 2.26 part ii) has codimension less than \( n \). Thus \( z \in L(x)^o \).

The upper set of \( z \) in \( L(x) \) is \([z,x]\). If the interior in \( L(x) \) of \([z,x]\) were empty, then \([z,x]\) would consist entirely of points peripheral in \( L(x) \) (by Theorem 2.19) and thus by Theorem 1.17 would have codimension less than \( n \). Thus \([z,x]\) has interior in \( L(x) \); hence there exists an open set \( U \) of \( L \) such that \( \Box \not\subseteq U \cap L(x) \subseteq [z,x] \). Since \( L(x) = (L(x)^o)^* \), \( U \cap (L(x)^o) \not= \Box \). Thus since \( U \cap L(x)^o \subseteq U \cap L(x) \subseteq [z,x] \), \(([z,x])^o \not= \Box \).

It is easy to see that in any topological lattice, if \( L(x)^o \not= \Box \), then \( L(x) = (L(x)^o)^* \) if and only if \( x \in (L(x)^o)^* \).
CHAPTER III
AN EMBEDDING THEOREM FOR COMPACT LATTICES

The main goal of this chapter is to generalize the embedding theorem of Baker and Stralka [10]. In the rest of the chapter we investigate the class of lattices introduced below. For the remainder of this dissertation we denote by \( P(L) \) the set of all meet irreducible elements of \( L \) and by \( \bar{P}(L) \) the set of all join irreducible elements of \( L \).

Lemma 3.1. Let \( S \) be a topological semilattice.

i) If \( \mathcal{C} \) is a chain in \( S \), then \( \mathcal{C}^* \) is a chain in \( S \).

ii) If \( \{x_\alpha\}_{\alpha \in D} \) is a net in \( S \) which converges to \( x \in S \) and \( x \leq x_\alpha \) for all \( \alpha \in D \), then \( x = \bigwedge_{\alpha \in D} x_\alpha \).

iii) If \( S \) is compact and \( \{x_\alpha\}_{\alpha \in D} \) is a decreasing net in \( S \), then \( \{x_\alpha\}_{\alpha \in D} \) converges to \( \bigwedge_{\alpha \in D} x_\alpha \).

Proof. Part i). Let \( \geq = \{(x,y) \in S \times S : (y,x) \in \leq\} \). It follows from Theorem 1.6 part i) that \( \geq \) is closed in \( S \times S \). Since \( \mathcal{C} \) is a chain, \( \mathcal{C} \times \mathcal{C} \subseteq \leq \cup \geq \). Thus \( \mathcal{C}^* \times \mathcal{C}^* = (\mathcal{C} \times \mathcal{C})^* \subseteq \leq \cup \geq \) which implies that \( \mathcal{C}^* \) is a chain.
in $S$.

Part ii). By hypothesis $x \leq x_\alpha$ for all $\alpha \in D$; hence

$x$ is a lower bound for $\{x_\alpha\}_{\alpha \in D}$. Suppose $y \leq x_\alpha$ for all $\alpha \in D$; then $\{(y, x_\alpha)\}_{\alpha \in D}$ is a net in $\leq$ which

converges to $(y, x)$. By Theorem 1.6 part i) $(y, x) \in \leq$.

Therefore $x$ is the greatest lower bound of $\{x_\alpha\}_{\alpha \in D}$.

Part iii) is due to D. P. Strauss [44].

Definition 3.2. If $S$ is a semilattice and $x \in S$, then $x$ is a local maximum of $S$ if $x$ is open in $M(x)$.

Lemma 3.3. Let $S$ be a compact topological semilattice such that $S = \bigcap_{i=1}^{n} C_i$, where $C_i$ is a chain

contained in $\mathcal{P}(S)$ for $i = 1, \ldots, n$. If $x, y \in C_i$, $x < y$, and for all $j \neq i$, either (i) $y \in C_j$ or (ii) $x \not\leq \land D$ for any $D \subseteq C_j \setminus \{x\}$ or (iii) $x$ is a local maximum of $C_j$ (i.e.,

$x$ is open in $M(x) \cap C_j$) then $x \in L(y)^\circ$.

Proof. For convenience we assume $x, y \in C_n$. Suppose $x \not\in L(y)^\circ$. Then each open set containing $x$ must meet $S \setminus L(y)$; hence there is a net $\{x_\alpha\}_{\alpha \in D}$ converging to $x$ such that $x_\alpha \not\leq y$ for all $\alpha \in D$. By hypothesis each $x_\alpha = x_\alpha_1 \land \ldots \land x_\alpha_n$ with $x_\alpha_i \in C_i$ for $i = 1, \ldots, n$ and $\alpha \in D$. Since each $C_i$ is a chain, $x_\alpha$ and $y$ are related whenever $y \in C_i$. If $y \in C_i$ and $x_\alpha \leq y$, then $x_\alpha \leq y$ contrary to $x_\alpha \not\leq y$. Thus $y < x_\alpha$ whenever $y \in C_i$. Choosing subnets if necessary we have
\{x_\alpha\} converging to \(x_1 \in C_1^*\) for \(i = 1, \ldots, n\). Since 
\(x_\alpha = x_\alpha_1 \land \ldots \land x_\alpha_n\) and \(\{x_\alpha\}\) converges to \(x\), 
\(x = x_1 \land \ldots \land x_n\). Assume the \(C_1\) are renumbered so that 
y \notin C_i for \(i = 1, \ldots, k-1\) with \(k-1 < n\). Since \(x\) is 
meet irreducible, \(x = x_1\) or \(x = x_2 \land \ldots \land x_n\). Suppose 
x = x_1. If \(x_{\alpha_1} \leq x\) for any \(\alpha\), then \(x_{\alpha} \leq x_{\alpha_1} \leq x < y\) 
contrary to \(x_{\alpha} \nless y\). Since \(C_1^*\) is a chain, we conclude 
x < x_{\alpha_1} for all \(\alpha\). By Lemma 3.1 \(x = x_1 = \bigwedge x_{\alpha_1}\). Also 
x is not open in \(M(x) \cap C_1\). Finally \(y \notin C_1\). Thus we 
have contradicted the hypothesis that for \(j \neq n\), either 
y \in C_j or \(x \notin \land D\) for any \(D \subseteq C_j \setminus \{x\}\) or \(x\) is a local 
maximum of \(C_j\). Therefore \(x \neq x_1\). Similarly \(x \neq x_j\) 
for \(j = 2, \ldots, k-1\). Thus \(x = x_k\) or \(x = x_{k+1} \land \ldots \land x_n\). 
As noted above, \(y\) is less than \(x_{\alpha_k}\) for all \(\alpha\) since 
y \in C_k; by Theorem 1.6 part i) \(y \leq x_k\). Hence if \(x = x_k\), 
we have \(x < y \leq x\), a contradiction. Similarly \(x \neq x_j\) for 
j = \(k+1, \ldots, n\). This contradiction completes the proof.

The type of function introduced in the next 
definition plays an important part in the rest of this 
dissertation. We shall use the symbol \(\sigma\), with or 
without subscripts, only for such functions; hence we shall 
not repeat the definition for each use of the symbol.

Definition 3.4. Let \(L\) be a lattice with \(1\) and \(C\) 
a chain in \(L\) which contains \(1\) and which is closed under 
arbitrary meets in \(L\), i.e. \(A \subseteq C\) implies \(\land A\) exists
in \( L \) and \( \wedge A \in C \). Then \( \sigma : L \to C \) is defined by
\[ \sigma(x) = \wedge \{ p \in C : x \leq p \} \]
for all \( x \in L \). A subscript, such as \( \sigma_c \), may be used to indicate the range of \( \sigma \).

The following elementary facts about \( \sigma \) are known [10].

Theorem 3.5. 1) \( \sigma \) is order preserving; \( x \leq \sigma(x) \) for all \( x \in L \); \( \sigma(p) = p \) for all \( p \in C \);

2) if \( L \) is a complete lattice, then \( \sigma \) preserves arbitrary joins, i.e.
\[ \sigma(\bigvee_{\alpha} x_{\alpha}) = \bigvee_{\alpha} \sigma(x_{\alpha}) \]
for any collection \( \{x_{\alpha}\} \subseteq L \);

3) if \( p \in C \) implies \( p \) is prime in \( L \), then \( \sigma \) is a lattice homomorphism.

We need the following results about complete chains [11, p. 112] and [25].

Theorem 3.6. Let \( L \) be a lattice with \( 1 \). If \( C \) is a chain in \( L \) which contains \( 1 \) and which is closed under arbitrary meets in \( L \), then

1) \( C \) is a complete lattice;

2) \( C \) is compact Hausdorff in its interval topology.

Lemma 3.7. Let \( L \) be a compact topological lattice of finite breadth and let \( C \) be a chain in \( L \) such that
i) \( l \in C \) and \( C \) is closed under arbitrary meets in \( L \).

ii) \( x \) a local maximum of \( C \) implies \( x \) a local maximum of \( L \).

iii) \( x, y \in C \) and \( x < y \) imply \( x \in L(y)^\circ \).

Then \( \sigma: L \to C \) is a continuous join homomorphism when \( C \) has the interval topology.

**Proof.** Let \( z = \wedge C \). Because \( C \) has the interval topology we need only show that \( \sigma^{-1}([z, p) \cap C) \) is open and \( \sigma^{-1}([z, p] \cap C) \) is closed for all \( p \in C \). Let \( W = \sigma^{-1}([z, p) \cap C) \) and let \( x \in W \). Then \( \sigma(x) < p \); thus \( \sigma(x) \in L(p)^\circ \). We consider two cases.

**Case 1)** Suppose \( \sigma(x) \) is a local maximum of \( C \).

Then \( \sigma(x) \) is a local maximum of \( L \); hence \( \sigma(x) \) is open in \( M(\sigma(x)) \). The function \( f: L \to M(\sigma(x)) \) defined by \( f(y) = \sigma(x) \lor y \) for all \( y \in L \) is continuous. Thus \( L(\sigma(x)) = f^{-1}(\sigma(x)) \) is open in \( L \). By Theorem 3.5 part 1) \( x \in L(\sigma(x)) \). Clearly \( L(\sigma(x)) \subset W \).

**Case 2)** Suppose \( \sigma(x) \) is not a local maximum of \( C \).

Then \( \sigma(x) \) is not open in \( M(\sigma(x)) \cap C \). Thus there is an \( s \in (M(\sigma(x)) \cap C) \cap ([z, p) \cap C) \) such that \( s \neq \sigma(x) \). Hence \( \sigma(x) < s < p \); therefore \( \sigma(x) \in L(s)^\circ \). By Theorem 3.5 part 1) and the dual of Lemma 2.22 \( x \in L(s)^\circ \). Clearly \( \sigma(L(s)^\circ) \subset [z, s] \cap C \). Hence \( x \in L(s)^\circ \subset W \). Since \( x \)
was arbitrary, \( W \) is open.

Let \( F = \sigma^{-1}([z,p] \cap C) \). Then \( F = L(p) \) which is closed by Theorem 1.2.

Hence we have shown that \( \sigma \) is continuous. By Theorem 3.5 part 2) \( \sigma \) is a join homomorphism.

The next definition is a generalization of a notion due to Baker and Stralka [10].

Definition 3.8. If \( L \) is a lattice and \( C \) is a chain in \( L \), then \( C \) is a coordinate chain of \( L \) if

i) \( 1 \in C \),

ii) \( C \) is closed under arbitrary meets in \( L \),

iii) \( x \in C \) implies \( x = \wedge D \) for some \( D \subseteq P(L) \cap C \).

R. P. Dilworth [19] proved the following theorem about the width of partially ordered sets:

Theorem 3.9. If \( X \) is a partially ordered set and \( w(X) = n \), then \( X \) is the union of \( n \) disjoint chains.

Lemma 3.10. Let \( L \) be a compact topological lattice such that the breadth of \( L \) is less than or equal to \( w(P(L)) = n \). Then there exist coordinate chains \( C_1, \ldots, C_n \) of \( L \) such that,

i) \( P(L) \subseteq \bigcup_{i=1}^{n} C_i \)

ii) \( x \in L \) implies \( x = \bigwedge_{i=1}^{n} \sigma_i(x) \) where \( \sigma_i : L \to C_i \)
for \( i = 1, \ldots, n, \)

iii) \( p \) a local maximum of \( C_i \) implies \( p \) a local maximum of \( L \) for \( i = 1, \ldots, n \),

iv) \( x, y \in C_i \) and \( x < y \) imply \( x \in L(y) \) for \( i = 1, \ldots, n \).

Proof. Let \( J \) be the set of all \((C_1, \ldots, C_n)\) such that for each \( i \) between 1 and \( n \), \( C_i \) is a chain in \( L \), \( 1 \in C_i \), \( x \in \mathcal{P}(L) \) and \( x = \wedge D \) for some \( D \subseteq C_i \) implies \( x \in C_i \), and \( \mathcal{P}(L) = \bigcup_{i=1}^{n} C_i \). By Theorem 3.9 we can find chains \( A_1, \ldots, A_n \) such that \( \mathcal{P}(L) = \bigcup_{i=1}^{n} A_i \). For each \( i = 1, \ldots, n \), let \( C'_i = \{1\} \cup \{x \in \mathcal{P}(L): x = \wedge D \text{ for some } D \subseteq A_i\} \).

Since \( C'_i \subseteq A_i^* \cup \{1\} \), Lemma 3.1 part i) implies \( C'_i \) is a chain; \( 1 \in C'_i \); \( \mathcal{P}(L) = \bigcup_{i=1}^{n} C'_i \). Thus to conclude that \((C'_1, \ldots, C'_n) \in J \) we need to show that \( D \subseteq C'_i \) and \( \wedge D \in \mathcal{P}(L) \) imply \( \wedge D \in C'_i \) for \( i = 1, \ldots, n \). Let \( D \subseteq C'_i \) and suppose \( \wedge D \in \mathcal{P}(L) \). Each \( x \in D \) belongs to \( A_i \) or is equal to \( \wedge D_x \) for some \( D_x \subseteq A_i \). Let \( E = (\bigcup \{D_x: x \in D \setminus A_i\}) \cup (D \cap A_i) \).

Then \( E \subseteq A_i \) and \( \wedge E = \wedge D \); hence \( \wedge D \in C'_i \). Thus \( J \neq \emptyset \).

We partially order \( J \) by coordinatewise set inclusion. Let \( \{((C_{1a}, \ldots, C_{na})\}_{a \in D} \) be a chain in \( J \). Let \( C''_i = \bigcap_{a \in D} C_i \) for \( i = 1, \ldots, n \). Clearly each \( C''_i \) is a chain, contains 1, and contains \( \wedge E \) whenever \( E \subseteq C''_i \) and \( \wedge E \in \mathcal{P}(L) \). We show that \( \mathcal{P}(L) = \bigcup_{i=1}^{n} C''_i \). Suppose there exists \( x \in \mathcal{P}(L) \) such that \( x \not\in \bigcup_{i=1}^{n} C''_i \). Then for each \( i \) there is an \( a(i) \) such that \( x \not\in C_{i a(i)} \) whenever
\((C_{1\beta} , \ldots, C_{n\beta}) \subseteq (C_{1\alpha(1)} , \ldots, C_{n\alpha(1)}) \). If \(\gamma\) is chosen such that \(\alpha(i) < \gamma\) for \(i = 1, \ldots, n\), then \((C_{1\gamma} , \ldots, C_{n\gamma}) \subseteq (C_{1\alpha(i)} , \ldots, C_{n\alpha(i)})\) for \(i = 1, \ldots, n\). Hence \(x \notin \bigcap_{i=1}^{n} C_{i\gamma}\) contrary to \((C_{1\gamma} , \ldots, C_{n\gamma}) \in \mathcal{J}\). Therefore \((C_{1}'' , \ldots, C_{n}'' \) \(\in \mathcal{J}\). Thus any chain in \(\mathcal{J}\) has a lower bound in \(\mathcal{J}\); therefore by Zorn's Lemma \(\mathcal{J}\) must contain a minimal element, say \((M_{1} , \ldots, M_{n})\).

Let \(C_{i}\) be the meet closure of \(M_{i}\) for \(i = 1, \ldots, n\). It follows that \(C_{1}, \ldots, C_{n}\) are coordinate chains of \(L\) which satisfy part i). By Theorem 1.11 part ii) is satisfied.

To prove part iii), suppose \(p\) is a local maximum of \(C_{i}\); then \(p \in M_{i}\) for otherwise \(p = AD\) for some \(D \subseteq M_{i}\) and \(D\) is a decreasing net since \(M_{i}\) is a chain; hence \(D\) converges to \(\wedge D\) by Lemma 3.1 part iii). Now suppose that \(p\) is not a local maximum of \(L\). There must exist a net \(\{x_{\alpha}\}_{\alpha \in D}\) in \(L\) such that \(p < x_{\alpha}\) and \(\{x_{\alpha}\}_{\alpha \in D}\) converges to \(p\). By part ii) \(x_{\alpha} = \bigwedge_{i=1}^{n} \sigma_{i}(x_{\alpha})\) for all \(\alpha \in D\). Choosing subnets if necessary \(\{\sigma_{i}(x_{\alpha})\}_{\alpha \in D}\) converges to \(p_{i}\) for \(i = 1, \ldots, n\). Since \(p \in M_{i}\), \(p\) is meet irreducible. Therefore \(p = p_{j}\) for some \(j\). By Lemma 3.1 part ii) \(p_{j} = \bigwedge_{\alpha \in D} \sigma_{j}(x_{\alpha})\). Since \(p\) is a local maximum of \(C_{i}\), \(j \neq i\). But now \((M_{1}, \ldots, M_{i}\setminus \{p\}, \ldots, M_{n}) \in \mathcal{J}\) and is strictly less than \((M_{1}, \ldots, M_{n})\) contrary to the minimality of \((M_{1}, \ldots, M_{n})\). Hence \(p\) must be a local maximum of \(L\).
For part iv) let \( x, y \in C_i \) and \( x < y \). By the way \( C_i \) is defined, we may choose \( p, q \in M_i \) such that \( x \leq p < q \leq y \). If \( p \in L(q)^\circ \), then \( x \in L(q)^\circ \subset L(y) \) (dual of Lemma 2.22). Thus \( x \in L(y)^\circ \). Therefore we may assume that \( x, y \in M_i \). Suppose that \( x \notin L(y)^\circ \). By Lemma 3.3 there exists \( j \neq i \) and \( D \subseteq M_j \) such that \( x = \wedge D \). Thus \( x \in M_j \). We consider two cases.

Case 1) Suppose \( x \) is a local maximum of \( M_i \).

Since \( x \in M_j \), \( (M_1, \ldots, M_i \setminus \{x\}, \ldots, M_n) \in \mathcal{J} \) contrary to the minimality of \( (M_1, \ldots, M_n) \).

Case 2) Suppose \( x \) is not a local maximum of \( M_i \).

Then for each \( s \in M_i \) such that \( x < s < y \), \( s \notin L(y)^\circ \) since \( s \in L(y)^\circ \) implies \( x \in L(y)^\circ \) (dual of Lemma 2.22). Thus by Lemma 3.3 for each \( s \in M_i \) such that \( x < s < y \) there exists \( k \neq i \) such that \( s \in M_k \). Therefore \( (M_1, \ldots, (M_i \cap [y, 1]) \cup (M_i \cap L(x)), \ldots, M_n) \in \mathcal{J} \) contrary to the minimality of \( (M_1, \ldots, M_n) \). Since each case leads to a contradiction, \( x \in L(y)^\circ \).

With these preliminaries finished we are now able to obtain the main theorem of this chapter.

Theorem 3.11. Let \( L \) be a compact topological lattice such that the breadth of \( L \) is less than or equal to \( w(P(L)) = n \). Then \( L \) can be embedded in a product of \( n \) compact chains by a join preserving homeomorphism.
Proof. Let $C_1, \ldots, C_n$ be the $n$ coordinate chains constructed in Lemma 3.10. By Theorem 3.6 each of these chains is a compact topological lattice when given the interval topology. By Lemma 3.7 $\sigma_i: L \rightarrow C_i$ is a continuous join homomorphism for $i = 1, \ldots, n$. Thus $f: L \rightarrow \prod_{i=1}^{n} C_i$ defined by $f(x) = (\sigma_1(x), \ldots, \sigma_n(x))$ is a continuous join homomorphism. To see that $f$ is one-to-one suppose that for some $x, y \in L$, $f(x) = f(y)$. Then $(\sigma_1(x), \ldots, \sigma_n(x)) = (\sigma_1(y), \ldots, \sigma_n(y))$ which implies $\sigma_i(x) = \sigma_i(y)$ for $i = 1, \ldots, n$. Thus $x = \prod_{i=1}^{n} \sigma_i(x) = \prod_{i=1}^{n} \sigma_i(y) = y$; hence $f$ is one-to-one. Since $L$ is compact and $\prod_{i=1}^{n} C_i$ is Hausdorff, $f$ is a homeomorphism.

If $L$ is distributive, then our coordinate chains and the function $f$ are precisely the coordinate chains and the embedding function obtained by Baker and Stralka [10]. Thus their result is a corollary of Theorem 3.11. Stralka [43] has obtained a different generalization for distributive lattices by replacing compact by locally convex.

The following proposition about subsemilattices of finite products of chains will allow us to conclude that $n$ is the least number of chains for which we can obtain an embedding of the type in Theorem 3.11. Stralka is also aware that $n$ is the least number although his proof is quite different.
Proposition 3.12. Let $L$ be a compact topological lattice that is a join subsemilattice of a product of $n$ chains. Then $w(P(L)) \leq n$.

Proof. Suppose not. Then we may find $n+1$ pairwise unrelated elements of $P(L)$, say $x_0, \ldots, x_n$, where $x_i = (x_{i1}, \ldots, x_{in})$ for $i = 0, \ldots, n$. Since there are only $n$ coordinates for each $x_i$, there must be one point, say $x_0$, such that for each $j = 1, \ldots, n$, there is some $i$ for which $x_{0j} \neq x_{ij}$. Let $y_i = x_0 \lor x_i$, $i = 1, \ldots, n$. Note that for each $j$, there exists an $i$ such that $x_{0j} = y_{ij}$. Then $y_i \neq x_0$, since $x_i$ and $x_0$ are unrelated, but $y_1 \land \ldots \land y_n = x_0$. This contradicts the meet irreducibility of $x_0$. Thus $w(P(L)) \leq n$.

Suppose $L$ is compact and has finite breadth $n$. By Theorem 1.11 each $x \in L$ can be written as the meet of at most $n$ meet irreducibles. If $w(P(L))$ is less than $n$, then each $x \in L$ can actually be written as a meet of less than $n$ meet irreducibles. It follows that any finite meet-irredundant set contains less than $n$ elements contrary to the assumption that $L$ has breadth $n$. We have proved:

Proposition 3.13. If $L$ is a compact topological lattice of finite breadth $n$, then $w(P(L)) \geq n$.

As an immediate consequence we may state:
Corollary 3.14. Let \( L \) be a compact topological lattice of finite breadth. Then \( w(P(L)) = n \) if and only if \( n \) is the least number of chains in which \( L \) can be embedded as a join subsemilattice.

The next lemma is an oral communication of Stralka. The proof is by this author.

Lemma 3.15. Let \( L \) be a modular lattice such that the breadth of \( L \) is \( n \geq 2 \). If \( a \) and \( b \) are unrelated elements of \( L \), then the breadth of \([a \wedge b, a]\) is less than \( n \).

Proof. Let \( x_1, \ldots, x_n \) be \( n \) elements of \([a \wedge b, a]\). Since the breadth of \( L \) is \( n \) and \([b, x_1, \ldots, x_n]\) has \( n+1 \) elements, there is a proper subset \( A \) of \([b, x_1, \ldots, x_n]\) such that \( \forall A = \forall [b, x_1, \ldots, x_n] \). Thus there is some \( y \in [b, x_1, \ldots, x_n] \) such that \( y \leq \forall ([b, x_1, \ldots, x_n] \setminus [y]) \). If \( y = b \), then \( b \leq x_1 \lor \ldots \lor x_n \); \( x_1 \lor \ldots \lor x_n \leq a \) since \( x_i \leq a \) for \( i = 1, \ldots, n \). This implies \( b \leq a \) contrary to \( a \) and \( b \) not related. Therefore \( y = x_j \) for some \( j \) between 1 and \( n \). Let \( y = \lor_{i \neq j} x_i \). We have \( x_j \leq b \lor y \) which implies \( b \lor x_j \leq b \lor (b \lor y) = b \lor y \); hence \( a \wedge (b \lor x_j) \leq a \wedge (b \lor y) \).

Since \( L \) is modular, \( a \wedge (b \lor x_j) = (a \wedge b) \lor x_j = x_j \) and \( a \wedge (b \lor y) = (a \wedge b) \lor y = y \). Therefore \( x_j \leq y = \lor_{i \neq j} x_i \). This is equivalent to \([a \wedge b, a]\) having breadth less than \( n \), for no set of \( n \) elements can be join-irredundant.

**Theorem 3.16.** If \( L \) is a connected topological lattice with 0 and 1, then \( L \) is a chain if and only if \( L \) is irreducibly connected about 0 and 1.

**Theorem 3.17.** Let \( L \) be a compact, modular topological lattice of breadth two. If \( A \) and \( B \) are compact, connected chains in \( L \), each containing 1, then \( A \wedge B \) is a compact, connected, distributive sublattice of \( L \).

Proof. Clearly \( A \wedge B \) is a compact, connected meet subsemilattice of \( L \). Suppose that \( x = x_1 \wedge y_1 \) and \( y = x_2 \wedge y_2 \) are elements of \( A \wedge B \) with \( x_1, x_2 \in A \) and \( y_1, y_2 \in B \). If \( x \leq y \) or \( y \leq x \), then \( x \vee y \in A \wedge B \). If \( x \) is not related to \( y \), then since \( A \) and \( B \) are chains, either \( x_1 \leq x_2 \) and \( y_2 \leq y_1 \) or \( x_2 \leq x_1 \) and \( y_1 \leq y_2 \). We assume \( x_1 \leq x_2 \) and \( y_2 \leq y_1 \). Since \( L \) is modular, the following equalities hold: \( x \vee y = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) = x_2 \wedge ((x_1 \wedge y_1) \vee y_2) = x_2 \wedge (x_1 \vee y_2) \wedge y_1 \). It follows then that \( x \vee y \in A \wedge B \) if \( x_1 \vee y_2 \in A \wedge B \); thus we need only show that \( a \vee b \in A \wedge B \) whenever \( a \in A \) and \( b \in B \). To that end suppose that \( a \in A \), \( b \in B \), and \( a \) is not related to \( b \). Let \( a' = \sigma_A(b) \). Note that \( a < a' \).

Since \( A \) is connected we may choose an increasing net \( \{a_\alpha\}_{\alpha \in D} \) in \( A \) which converges to \( a' \). By the dual of Lemma 3.1 part iii) \( a' = \bigvee a_\alpha \); hence \( a_\alpha < a' \) for all \( \alpha \in D \).
which implies $a_\alpha$ is not related to $b$. Let $b_\alpha = b \lor a_\alpha$ for all $\alpha \in D$. Then $[b_\alpha]$ converges to $a'$. By Lemma 3.15 the breadth of $[b, b_\alpha]$ is one for each $\alpha \in D$. As in the proof of Theorem 2.26 $[b, a'] = (\bigcup_{\alpha \in D} [b, b_\alpha])^*$; thus $[b, a']$ has breadth one. Hence $[b, a']$ is a chain. As noted above $a < a'$; hence $a \lor b \in [b, a']$. Since $B \cap [b, 1]$ is just the subchain of $B$ from $b$ to $1$, it is connected. Thus $a' \wedge (B \cap [b, 1])$ is a connected subset of the chain $[b, a']$, which contains both $b$ and $a'$. By Theorem 3.16 $[b, a'] = a' \wedge (B \cap [b, 1])$. Since $a' \wedge (B \cap [b, 1]) \subseteq a' \wedge B$ and $a \lor b \in [b, a']$, we have $a \lor b \in a' \wedge B \subseteq A \wedge B$. Hence $A \wedge B$ is a sublattice of $L$. It is well known [11] or [28] that in a modular lattice any two chains generate a distributive sublattice. Since $A, B \subseteq A \wedge B \subseteq < A \cup B >$, $A \wedge B = < A \cup B >$; hence $A \wedge B$ is distributive.

We use Theorem 3.17 to give a complete proof of the next theorem which was contained in the previously mentioned communication; Stralka's original proof apparently contains a gap.

**Theorem 3.18.** Let $L$ be a compact, connected topological lattice of breadth two and with $w(P(L)) = n$. Then $L$ is the union of a finite number of compact, connected, distributive sublattices of $L$.

**Proof.** By Theorem 3.9 $P(L) = \bigcup_{i=1}^{n} C_i$, $C_i$ a chain for $i = 1, \ldots, n$; by Theorem 1.8 each $C_i$ may be extended to
an arc chain $C'_i$ containing 0 and 1 ($i = 1, \ldots, n$). From Theorem 1.10 $L$ is modular; thus by Theorem 3.17 $C'_i \land C'_j$ is a compact, connected, distributive sublattice of $L$ for $i \neq j$, $i, j = 1, \ldots, n$. We claim that $L = \bigcup_{i \neq j} (C'_i \land C'_j)$. If $x \in P(L)$, then $x \in C'_i$ for some $i$; hence $x \in C'_i \land C'_j$ for all $j \neq i$. If $x \not\in P(L)$, then by Theorem 1.11 $x = y \land z$ for some $y, z \in P(L)$. There exist $i \neq j$ such that $y \in C'_i$ and $z \in C'_j$; thus $x \in C'_i \land C'_j$. Note that each $C'_i \land C'_j$ contains both 0 and 1.

The word "connected" may be replaced in the hypothesis of Theorem 3.18 by "modular" provided "connected" is deleted from the conclusion. In this case we do not in general know that the sublattice generated by two chains $A, B$ is $A \land B$. The next theorem gives a sufficient condition for $A \land B$ to be the sublattice generated by the not necessarily connected chains $A$ and $B$.

Theorem 3.19. Let $L$ be a compact, modular topological lattice of breadth less than or equal to $w(P(L)) = n$. Suppose that $C_1, \ldots, C_n$ are chains in $L$ satisfying

1) $P(L) = \bigcup_{i=1}^{n} C_i$,
2) $C_i \cap C_j = \{l\}$ whenever $i \neq j$, $i, j = 1, \ldots, n$,
3) $x \leq y$ implies $y = 1$ and $y \leq x$ implies $x = 1$ whenever $x \in C_1 \cup C_2$ and $y \in C_j$, $j = 3, \ldots, n$.

Then $C_1 \land C_2$ is a distributive sublattice of $L$. 
Proof. As noted in the proof of Theorem 3.17 we only need show that $x \lor y \in C_1 \lor C_2$ whenever $x \in C_1$ and $y \in C_2$. By Theorem 1.11 $x \lor y = x_1 \land \ldots \land x_m$ where $\{x_1, \ldots, x_m\}$ is meet-irredundant and $m \leq n$. Suppose that $x_i \not\in C_1 \lor C_2$ for some $i$ between 1 and $m$. Then $x \leq x \lor y \leq x_i$; hence $x_i = 1$ which contradicts the fact that $\{x_1, \ldots, x_m\}$ is meet irredundant. Thus $x_i \in C_1 \lor C_2$ for all $i = 1, \ldots, m$; hence $m \leq 2$ and $x \lor y \in C_1 \lor C_2$.

Dilworth [20] has shown that the number of meet irreducible elements is equal to the number of join irreducible elements in any finite modular lattice. A similar result for lattices of breadth two is the following.

**Theorem 3.20.** Let $L$ be a compact connected topological lattice of breadth two for which $\omega(\mathcal{P}(L)) = n$. Then $\omega(\mathcal{P}(L)) = n$.

Proof. By Theorem 3.9 $\mathcal{P}(L) = \bigcup_{i=1}^{n} C_i$, $C_i$ a chain for $i = 1, \ldots, n$. Let $C'_1, \ldots, C'_n$ be the chains constructed in Theorem 3.18. By the duals of Theorem 3.17 and Theorem 3.18, $C'_i \lor C'_j$ is a sublattice of $L$ for $i \neq j$, and $L = \bigcup_{i \neq j} (C'_i \lor C'_j)$. Clearly we have $\mathcal{P}(L) \subseteq \bigcup_{i=1}^{n} C'_i$; thus $\omega(\mathcal{P}(L)) \leq n$. The dual argument shows that $\omega(\mathcal{P}(L)) \leq \omega(\mathcal{P}(L))$. Hence $\omega(\mathcal{P}(L)) = n$.

We close this chapter with some examples.

**Example 3.21.** Let $L = (I \times I) \setminus (I \times \{0\})$, where $I$ is
the unit interval. Then \( L \) is a locally compact, connected, metric, distributive topological lattice of breadth two.

However, \( \mathcal{P}(L) = (I \times \{0\}) \cup (\{1\} \times (0,1]) \) and \( \mathcal{P}(L) = \{0\} \times (0,1]. \) Hence \( w(\mathcal{P}(L)) = 1 < 2 = w(\mathcal{P}(L)). \)

Example 3.22. Let \( L = \{(0,0),(0,\frac{1}{2}),(0,\frac{3}{4}),(\frac{1}{2},0),(1,0), (\frac{1}{2},\frac{1}{2}),(1,1)\} \subseteq I \times I. \) Define \( (0,\frac{3}{4}) \lor (\frac{1}{2},\frac{1}{2}) = (1,1) = (0,\frac{3}{4}) \lor (\frac{1}{2},0) = (0,\frac{3}{4}) \lor (1,0) = (\frac{1}{2},\frac{1}{2}) \lor (1,0) = (1,0) \land (0,\frac{1}{2}). \) Define all other joins and all meets of the seven elements to be the usual joins and meets in \( I \times I. \) Then the breadth of \( L \) is two and \( w(\mathcal{P}(L)) = 2 < 3 = w(\mathcal{P}(L)). \)

Example 3.23. Let \( L' = L \cup \{(\frac{2^n-1}{2^n},\frac{1}{2^n}): n = 2,3,\ldots\} \cup P \) where \( L \) is the lattice of Example 3.22 and \( P = L \land \{(\frac{2^n-1}{2^n},\frac{1}{2^n}): n = 2,3,\ldots\}. \) The meets are the usual meets in \( I \times I; \) joins are defined in the same manner as in Example 3.22. Then \( L' \) is compact and has breadth two, \( w(\mathcal{P}(L')) = 2, \) and \( w(\mathcal{P}(L')) = \infty. \)

We need the theorem of D. R. Brown found in [35].

**Theorem 3.24.** Let \( S \) be a compact topological semilattice. Then the space \( S' \) of all closed ideals \((A \subseteq S \text{ is an ideal} \iff L(A) = A), \) ordered by inclusion, is a compact, distributive topological lattice. The mapping sending \( s \) into \( L(s) \) is a topological isomorphism from \( S \) into \( S'. \) If \( S \) is connected (metric), then \( S' \) is connected (metric).
Example 3.25. Let $S$ be the usual meet topological semilattice on $I \times I$, and let $L$ be the space of all closed ideals of $S$. By Theorem 3.24 $L$ is a compact, connected, metric, distributive topological lattice. It is easily seen that the breadth of $L$ is not finite while $w(\mathcal{P}(L)) = 2$. 
CHAPTER IV
LATTICES OF BREADTH TWO

In Chapter III we showed that certain compact topological lattices of breadth two are the union of a finite number of compact, distributive sublattices. We will now investigate such lattices further by studying these distributive sublattices. The irreducible elements, both meet and join, play a key role.

I. Distributive Lattices

In this section we characterize the compact, connected sublattices of $I \times I$ which contain 0 and 1. Theorem 4.1 characterizes $I \times I$; Theorem 4.10 characterizes those sublattices for which $P(L) = \bar{P}(L)$, the so-called "banana". Theorem 4.11 gives the characterization of the triangle sublattice $\{(x,y) \in I \times I : y \leq x\}$.

Theorem 4.1. Let $L$ be a compact, connected topological lattice of breadth two and $w(P(L)) = 2$. If there exist $z_1$ and $z_2$ in $L \setminus \{0,1\}$ such that $z_1 \wedge z_2 = 0$ and $z_1 \vee z_2 = 1$, then $L$ is topologically isomorphic to a product of two arc chains.

Proof. Since $z_1, z_2 \in L \setminus \{0,1\}$ and $z_1 \wedge z_2 = 0$, they
are not related. By Theorem 1.10 L is modular. By Lemma 3.15 \([z_1,1]\) has breadth one; thus \([z_1,1]\) is an arc chain for \(i = 1,2\). By Theorem 3.18 \(L = C_1 \wedge C_2\), where \(C_1\) and \(C_2\) are chains, and \(C_1 \wedge C_2\) is distributive. Since \(L\) is distributive each meet irreducible is prime. Hence \(z_1 \wedge z_2 = 0 \leq x\) implies \(z_1 \leq x\) or \(z_2 \leq x\) whenever \(x \in \mathcal{P}(L)\); consequently \(\mathcal{P}(L) \subset [z_1,1] \cup [z_2,1]\). Let \(x \in [z_1,1]\), and suppose that \(x = a \wedge b\). Then \(a, b \in [z_1,1]\); thus either \(a \leq b\) or \(b \leq a\). This implies \(x = a\) or \(x = b\). Hence \([z_1,1] \subset \mathcal{P}(L)\) for \(i = 1,2\). We have shown \(\mathcal{P}(L) = [z_1,1] \cup [z_2,1]\). Let \(C_1' = [z_1,1]\) and \(C_2' = [z_2,1]\). By Theorem 1.11 \(L = C_1' \wedge C_2'\).

If \(x \in C_1' \cap C_2'\), then \(l = z_1 \vee z_2 \leq x\); hence \(C_1' \cap C_2' = \{l\}\). Define \(f: L \rightarrow C_1' \times C_2'\) by \(f(x) = (\sigma_1(x), \sigma_2(x))\) for all \(x \in L\). By Theorem 1.11 \(x \in L\) implies \(x = \sigma_1(x) \wedge \sigma_2(x)\). Since \(C_1' \cap C_2' = \{l\}\), \(x, y \in C_1'\) and \(x < y\) imply \(x \notin C_2'^* = C_2'\). Similarly for \(x, y \in C_2'\) and \(x < y\). Therefore from Lemma 3.3 \(x, y \in C_1'\) and \(x < y\) imply \(x \in \mathcal{L}(y)\); by Lemma 3.7 \(\sigma_i\) is continuous for \(i = 1,2\). As in Theorem 3.11 \(f\) is a homeomorphism. That \(f\) is an isomorphism follows from Theorem 3.5 part 3).

To see that \(f\) is onto \(C_1' \times C_2'\), let \((x, y) \in C_1' \times C_2'\). We consider three cases.

Case 1) If \(x \leq y\), then \(y = l\); hence \(f(x) = (x, y)\).
Case 2) If $y \leq x$, then $f(y) = (x,y)$.

Case 3) If $x$ is not related to $y$, then since $L$ is distributive, by Theorem 3.5 $\sigma_i$ is a homomorphism; hence $\sigma_1(x \land y) = \sigma_1(x) \land \sigma_1(y) = \sigma_1(x) = x$ and $\sigma_2(x \land y) = \sigma_2(x) \land \sigma_2(y) = \sigma_2(y) = y$. Thus $f(x \land y) = (x,y)$.

The following lemmas will be useful in the sequel.

**Lemma 4.2.** Let $L$ be a topological lattice with $0$ and $1$, and let $A$ and $B$ be arc chains from $0$ to $1$. If

i) $f = \sigma_B|A$,

ii) $f(x) = 0$ if and only if $x = 0$,

iii) $y \in M(x)^0$ whenever $x, y \in A$ and $x < y$,

then $f$ is one-to-one.

**Proof.** Suppose that $f$ is not one-to-one. Then for some $x, y \in A, 0 < x < y$ and $f(x) = f(y)$. By hypothesis $y \in M(x)^0$; by Theorem 3.5 $y \leq f(y) = f(x)$; therefore by Lemma 2.22 $f(x)$ and $f(y) \in M(x)^0$. If $p \in B$ and $x \leq p$, then $f(x) \leq f(p) = p$; thus $p \in M(x)^0$. If $p \in B$ and $x \not\leq p$, then $p \in L \setminus M(x)$. In particular $0 \in L \setminus M(x)$. Thus $B = (B \cap M(x)^0) \cup (B \cap L \setminus M(x))$, a union of two disjoint, non-empty open sets. But $B$ is connected. Thus $f$ is one-to-one.

**Remark 4.3.** If $L$ is a topological lattice with
0 and 1, and C is an arc chain in L containing 0 and 1, then C is a maximal chain in L. Otherwise there would exist a chain M properly containing C. Then for any \( x \in M \setminus C \), \((C \cap L(x)) \cup (C \cap M(x))\) is a separation of C contrary to C connected. This fact was first observed by Anderson [1].

Lemma 4.4. Let L be a compact topological lattice with the breadth of L and \( w(P(L)) \) both equal to two. Let A and B be arc chains from 0 to 1 such that
i) \( A \subseteq P(L) \),
ii) \( B \cap P(L) = C \) and C is a non-degenerate arc chain from \( p \neq 0 \) to 1,
iii) \( A \cap B = \{0,1\} \),
iv) \( P(L) = \text{All} \cup C \).

If \( x \in A \), then \( \sigma_c(x) = p \) if and only if \( x = 0 \).

Proof. Clearly \( x = 0 \) implies \( \sigma_c(x) = p \). For the converse, suppose there is an element \( x_1 \in A \) such that \( x_1 \neq 0 \) and \( \sigma_c(x_1) = p \). By Theorem 3.5 \( x_1 < p \); thus by the maximality (Remark 4.3) of B we may choose \( x_2 \in B \) such that \( 0 < x_2 < p \) and \( x_2 \) is not related to \( x_1 \). By Theorem 1.11 \( x_2 = x' \wedge x'' \) for some \( x' \in A \) and \( x'' \in C \); since \( p \leq x'' \) and \( x_2 \leq p \), we have \( x_2 = x' \wedge p \). If \( x_1 \leq x' \), then \( x_1 = x_1 \wedge p \leq x' \wedge p = x_2 \) contrary to our choice of \( x_2 \). If \( x' \leq x_1 \), then \( x_2 = x' \wedge p \leq x_1 \wedge p = x_1 \) again contrary to our choice of \( x_2 \). Therefore there is
no such $x_1 \in A$.

We note that conditions i) and iv) may be replaced by $P(L) \subseteq A \cup C$ and $C \subseteq P(L)$.

Corollary 4.5. Let $L$, $A$, $B$, and $C$ be as in Lemma 4.4. Then $\sigma_{C}\mid A$ is one-to-one.

Proof. Lemma 4.4 together with the proof of Lemma 4.2.

We remark that the dual of Lemma 4.4 implies that the only element of $A$ that is above $\vee(P(L) \cap B) = p$ is 1.

Throughout the remainder of this section we assume that $L$ is a compact, connected, metric topological lattice of breadth two; note that this implies by Theorem 1.10 that $L$ is modular.

The following lemma is found in [11].

Lemma 4.6. Any order preserving bijection with order preserving inverse is a lattice isomorphism.

Lemma 4.7. Suppose that $P(L) = \bar{P}(L) = C_1 \cup C_2$, where $C_1$ and $C_2$ are arc chains from 0 to 1, and $C_1 \cap C_2 = \{0,1\}$. Let $f_{11}: C_i \to C_j$ be defined by $f_{11}(x) = \vee\{y \in C_j : y \leq x\}$ and $f_{21}: C_i \to C_j$ by $f_{21}(x) = \wedge\{y \in C_j : x \leq y\}$ for $i \neq j$, $i,j = 1,2$. Then these functions are topological isomorphisms with $l_{c_1} = f_{22}f_{11} = f_{12}f_{21}$ and $l_{c_2} = f_{11}f_{22} = f_{21}f_{12}$.
Proof. By Lemma 3.3 and its dual $x, y \in C_1$ and $x < y$ imply $x \in L(y)^\circ$ and $y \in M(x)^\circ$; thus by Lemma 4.2 and its dual all the functions are one-to-one. By Theorem 3.5 and its dual they are order preserving.

Let $x \in C_2$. Then $x \leq f_{22}(x)$; hence $x \leq f_{11}f_{22}(x) \leq f_{22}(x)$. Thus $f_{22}(x) \leq f_{22}(f_{11}f_{22}(x)) \leq f_{22}(x)$; therefore $f_{22}(x) = f_{22}(f_{11}f_{22}(x))$. Since $f_{22}$ is one-to-one, $x = f_{11}f_{22}(x)$. The other cases are similar. Therefore by Lemma 4.6 the functions are lattice isomorphisms. The functions are continuous by Lemma 3.7 and its dual.

We remark that $L$ is distributive since $P(L) = C_1 \cup C_2$, as observed in the proof of Theorem 3.17. We continue with the notation, hypotheses, and hence conclusions of Lemma 4.7.

Lemma 4.8. Let $q_0 \in C_2 \setminus \{0, 1\}$. We define:

- $p_0 = f_{12}(q_0)$,
- $p_{2n-1} = f_{12}(f_{11}f_{12})^n(q_0), n = 1, 2, \ldots$,
- $p_{2n} = f_{22}(f_{21}f_{22})^{n-1}(q_0), n = 1, 2, \ldots$,
- $q_{2n-1} = (f_{11}f_{12})^n(q_0), n = 1, 2, \ldots$,
- $q_{2n} = (f_{21}f_{22})^n(q_0), n = 1, 2, \ldots$.

Then \( \{p_{2n-1}\} \) and \( \{q_{2n-1}\} \) converge to 0, and \( \{p_{2n}\} \) and \( \{q_{2n}\} \) converge to 1; $x \in C_1$ implies $p_1 \leq x \leq p_0$, $p_0 \leq x \leq p_2$ or there is an $n \geq 1$ such that $p_{2n} \leq x \leq p_{2n+2}$ or $p_{2n+1} \leq x \leq p_{2n-1}$; similarly for $x \in C_2$. 
\[ q_0 \leq x \leq q_2, \quad q_1 \leq x \leq q_0, \quad \text{or there is an } n \geq 1 \text{ such that } q_{2n} \leq x \leq q_{2n+2} \quad \text{or } q_{2n+1} \leq x \leq q_{2n-1}. \]

Proof. Since \( 0 < q_0 < 1 \) and the functions are all one-to-one, \( p_0, p_{2n-1}, p_{2n}, q_{2n-1}, \) and \( q_{2n} \) are all different from 0 and 1. To show that \( \{p_{2n-1}\} \) converges to 0, suppose not; i.e. suppose \( \{p_{2n-1}\} \) converges to \( p > 0; \) it converges since it is decreasing (Lemma 3.1).

Since \( f_{11} \) is continuous, \( \{f_{11}(p_{2n-1})\} \) converges to \( f_{11}(p) \) and \( \{f_{12}(f_{11}(p_{2n-1}))\} \) converges to \( f_{12}f_{11}(p). \) By definition \( f_{12}(f_{11}(p_{2n-1})) = p_{2n-3} \) for all \( n; \) \( f_{12}f_{11}(p) < p \) for \( f_{12}f_{11}(p) = p \) implies \( f_{11}(p) = f_{21}(f_{12}f_{11}(p)) = f_{21}(p). \) This implies \( f_{11}(p) \in C_1 \cap C_2 \) contrary to \( p \neq 0,1. \) Thus \( \{p_{2n-1}\} \) has two distinct points of convergence which is impossible. Hence \( \{p_{2n-1}\} \) converges to 0. The other claims of convergence are proved in a similar manner. The remaining claims follow easily.

The next proposition appears in [11].

**Proposition 4.9.** If \( L \) is a distributive lattice and \( \{x_1, \ldots, x_n\} \) and \( \{y_1, \ldots, y_m\} \) are meet-irredundant subsets of \( P(L) \) such that \( x_1 \wedge \ldots \wedge x_n = y_1 \wedge \ldots \wedge y_m, \) then \( \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m\}. \)

Suppose now that \( L \) and \( L' \) are two lattices satisfying the hypotheses of Lemma 4.7 and that \( C_1, C_2, f_{11}, f_{21}, q_0, \) etc., are as above for \( L \) and \( C_1', C_2', f_{11}', \)
Theorem 4.10. $L$ is topologically isomorphic to $L'$.

Proof. We define: $P_{01} = C_1 \cap [p_1, p_0]$, $P_{02} = C_1 \cap [p_0, p_2]$, $P_{2n-1,2n+1} = C_1 \cap [p_{2n+1}, p_{2n-1}]$, $n = 1, 2, \ldots$, $P_{2n,2n+2} = C_1 \cap [p_{2n}, p_{2n+2}]$, $n = 1, 2, \ldots$.

We define $Q_{01}, Q_{02}, Q_{2n-1,2n+1}$, and $Q_{2n,2n+2}$ similarly using $q_0, q_1$, etc. Primes are used to denote the corresponding sets in $L'$.

Let $\varphi: P_{02} \to P'_{02}$ be any order preserving homeomorphism from $P_{02}$ onto $P'_{02}$. By Lemma 4.6 $\varphi$ is a lattice isomorphism. We define $\alpha: L \to L'$ by

1) $\alpha|_{P_{02}} = \varphi$;
2) $\alpha(0) = 0, \alpha(1) = 1, \alpha(q_o) = q'_o$;
3) if $x \in Q_{01}$, then $\alpha(x) = f'_{11} \varphi f'_{22}(x)$;
4) if $x \in Q_{02}$, then $\alpha(x) = f'_{21} \varphi f'_{12}(x)$;
5) if $x \in P_{01}$, then $\alpha(x) = f'_{12} f'_{11} \varphi f'_{22} f'_{21}(x)$;
6) if $x \in Q_{2n-1,2n+1}$, then $\alpha(x) = f'_{11} (f'_{12} f'_{11})^n \varphi f'_{22} (f_{21} f_{22})^n(x)$, $n \geq 1$, note that $\alpha(x) = f'_{11} \alpha f_{22}(x)$;
7) if $x \in Q_{2n,2n+2}$, then $\alpha(x) = f'_{21} (f'_{22} f'_{21})^n \varphi f_{12} (f_{11} f_{12})^n(x)$, $n \geq 1$, note that $\alpha(x) = f'_{21} \alpha f_{12}(x)$;
viii) if \( x \in P_{2n-1,2n+1} \), then \( \alpha(x) = (f'_{12} f_{11})^{n+1} \varphi(f_{22} f_{21})^{n+1}(x) \), \( n \geq 1 \), note that 
\[ \alpha(x) = f'_{12} \alpha f_{21}(x); \]

ix) if \( x \in P_{2n,2n+2} \), then \( \alpha(x) = (f'_{22} f'_{21})^{n} \varphi(f'_{12} f_{11})^{n}(x) \), \( n \geq 1 \), note that 
\[ \alpha(x) = f'_{22} \alpha f_{11}(x); \]

x) if \( x \in L \setminus (C_1 \cup C_2) \), then \( \alpha(x) = \alpha(x_1) \land \alpha(x_2) \), where 
\( \{x_1,x_2\} \) is the unique meet-irredundant set such that \( x_i \in C_i, i = 1, 2 \), and \( x = x_1 \land x_2 \) (Theorem 1.11 and Proposition 4.9).

We define \( \beta : L' \to L \) similarly letting \( \beta|_{P_{02}} = \varphi^{-1} \). We show \( \alpha \), hence \( \beta \), is order preserving and \( \beta \) is the inverse of \( \alpha \).

1) \( \alpha(q_k) = q'_k \) and \( \alpha(p_k) = p'_k \) for all \( k = 0,1,2,... \).

By definition \( \alpha(q_0) = q'_0 \), \( \varphi(p_0) = p'_0 \), and \( \varphi(p_2) = p'_2 \). Since \( q_0 \in Q_{01} \), \( \alpha(q_0) = f'_{11} \varphi f_{22}(q_0) = f'_{11} \varphi(p_2) = f'_{11}(p'_2) = f'_{11}(f'_{22}(q'_0)) = q'_0 \); since \( q_0 \in Q_{02} \),
\[ \alpha(q_0) = f'_{21} \varphi f_{12}(q_0) = f'_{21} \varphi(p_0) = f'_{21}(p'_0) = f'_{21}(f'_{12}(q'_0)) = q'_0. \]

Since \( p_0 \in P_{02} \), \( \alpha(p_0) = \varphi(p_0) = p'_0 \); also \( p_0 \in P_{01} \); therefore \( \alpha(p_0) = f'_{12} f'_{11} \varphi f_{22} f_{21}(p_0) = f'_{12} f'_{11} \varphi f_{22} f_{21}(f_{12}(q_0)) = f'_{12} f'_{11} \varphi f_{22}(q_0) = f'_{12}(q'_0) = p'_0 \); the next to last equality comes from the preceding argument involving \( q_0 \). Thus for all definitions, \( \alpha(q_0) = q'_0 \) and \( \alpha(p_0) = p'_0 \).

Suppose that \( n = 2m, m \geq 1 \). Then \( q_n = (f_{21} f_{22})^m(q_0) \).
and \( q_n \in Q_{2m-2,2m} \cap Q_{2m,2m+2} \). For \( q_n \in Q_{2m-2,2m} \),

\[
\alpha(q_n) = f_2\left(f_2 f_1\right)^{m-1} \varphi f_1\left(f_1 f_2\right)^{m-1}(q_n) = f_2(f_2 f_1)^{m-1} \varphi f_1\left(f_1 f_2\right)^{m-1}(q_o) = f_2(f_2 f_1)^{m-1} \varphi f_2(q_o) = f_2(f_2 f_1)^{m-1}(p_2) = f_2(f_2 f_1)^{m-1}(f_2(q_o)) = (f_2 f_1)^{m}(q_o) = q_n \quad \text{for} \quad q_n \in Q_{2m,2m+2},
\]

\[
\alpha(q_n) = f_2\left(f_2 f_1\right)^{m-1} \varphi f_1\left(f_1 f_2\right)^{m-1}(q_n) = f_2(f_2 f_1)^{m-1} \varphi f_1\left(f_1 f_2\right)^{m-1}(q_o) = f_2(f_2 f_1)^{m-1} \varphi f_2(q_o) = f_2(f_2 f_1)^{m-1}(p_2) = f_2(f_2 f_1)^{m-1}(f_2(q_o)) = (f_2 f_1)^{m}(q_o) = q_n'. \quad \text{The proofs for} \quad n = 2m-1 \quad \text{and for} \quad p_n \quad \text{are similar.}
\]

Let \( x, y \in \mathbb{L} \) and \( y \leq x \). We show that \( \alpha(y) \leq \alpha(x) \).

2) Suppose \( x, y \in C_1 \).

The proof is by induction on the number \( k \) of \( p_n \)'s between \( y \) and \( x \). If \( k = 0 \), then \( y \) and \( x \) belong to the same subinterval \( P_n \) of \( C_1 \), and since all the functions in the composition which define \( \alpha|_{P_n} \) are order preserving, \( \alpha(y) \leq \alpha(x) \). Suppose that \( \alpha(y) \leq \alpha(x) \) whenever there are \( k' < k \) \( p_n \)'s between \( y \) and \( x \).

Suppose now that \( y < p_n(1) < \ldots < p_n(k) < x \). By the inductive hypothesis \( \alpha(y) \leq \alpha(p_n(1)) \) and \( \alpha(p_n(1)) \leq \alpha(x) \); hence \( \alpha(y) \leq \alpha(x) \).

3) Suppose \( x, y \in C_2 \).

The proof is similar to that in part 2).

4) Suppose \( x \in C_1 \) and \( y \in C_2 \).
If $x \in P_{12}^n$, then $y \leq f_{11}(x)$; thus $a(y) \leq a f_{11}(x) = f_{11} \circ f_{22}(f_{11}(x)) = f_{11} \circ f\phi(x) \leq f\phi(x) = a(x)$. If $x \in P_{01}$ or $P_{2n-1,2n+1}$ for some $n \geq 1$, then $f_{22}(y) \leq x$. Hence $a f_{22}(y) \leq a(x)$ by part 2); $a(y) = f_{11} \circ a f_{22}(y) \leq a f_{22}(y)$. Therefore $a(y) \leq a(x)$.

Finally, suppose $x \in P_{2n,2n+2}$. Then $y \leq f_{11}(x)$; thus $a(y) \leq a f_{11}(x) \leq a(x)$. The last inequality holds because $a(x) = f_{22} \circ a f_{11}(x)$ and $a f_{11}(x) \leq f_{22}(a f_{11}(x))$.

5) Suppose $x \in C_2$ and $y \in C_1$.

The proof is similar to that in part 4).

6) Suppose $x \in C_1$ and $y = y_1 \land y_2 \in L \setminus (C_1 \cup C_2)$.

Since $L$ is distributive, $x$ is prime; thus either $y_1 \leq x$ or $y_2 \leq x$. Therefore $a(y) = a(y_1) \land a(y_2) \leq a(y_1)$, $i = 1, 2$, and $a(y_1) \leq a(x)$ or $a(y_2) \leq a(x)$. Hence $a(y) \leq a(x)$.

7) Suppose $x = x_1 \land x_2 \in L \setminus (C_1 \cup C_2)$ and $y \in C_2$.

Since $y \leq x = x_1 \land x_2 \leq x_i$, $i = 1, 2$, $a(y) \leq a(x_1)$ and $a(y) \leq a(x_2)$. Thus $a(y) \leq a(x_1) \land a(x_2) = a(x)$.

8) Suppose $x = x_1 \land x_2$ and $y = y_1 \land y_2 \in L \setminus (C_1 \cup C_2)$.

Then $y_1 \leq x_1$ or $y_2 \leq x_1$ and $y_1 \leq x_2$ or $y_2 \leq x_2$, since $x_1$ and $x_2$ are prime. Thus $a(y_1) \leq a(x_1)$ or $a(y_2) \leq a(x_1)$ and $a(y_1) \leq a(x_2)$ or $a(y_2) \leq a(x_2)$. In any case $a(y) = a(y_1) \land a(y_2) \leq a(x_1) \land a(x_2) = a(x)$. We conclude that $a(\beta)$ is order preserving on $L(L')$. 
It is clear from the definitions of $\alpha$ and $\beta$ that if $\alpha$ is restricted to $C_1 \cup C_2$, then $\beta$ is an inverse for $\alpha$.

9) $\alpha(L \setminus (C_1 \cup C_2)) \subset L' \setminus (C'_1 \cup C'_2)$.

Let $x = x_1 \land x_2 \in L \setminus (C_1 \cup C_2)$, where \{x_1, x_2\} is meet-irredundant, $x_1 \in C_1$ and $x_2 \in C_2$. If $\alpha(x) = \alpha(x_1) \land \alpha(x_2) \in C'_1 \cup C'_2$, then $\alpha(x_1) \leq \alpha(x_2)$ or $\alpha(x_2) \leq \alpha(x_1)$. Suppose $\alpha(x_1) \leq \alpha(x_2)$; then $x_1 = \beta \alpha(x_1) \leq \beta \alpha(x_2) = x_2$ contrary to \{x_1, x_2\} being meet-irredundant. Similarly $\alpha(x_2) \leq \alpha(x_1)$ is not possible. Therefore $\alpha(x) \in L' \setminus (C'_1 \cup C'_2)$.

10) Let $x = x_1 \land x_2 \in L \setminus (C_1 \cup C_2)$. By part 9) $\alpha(x_1) \land \alpha(x_2)$ is the unique meet-irredundant representation for $\alpha(x)$; hence $\beta(\alpha(x)) = \beta(\alpha(x_1)) \land \beta(\alpha(x_2)) = x_1 \land x_2 = x$. Hence $\beta$ is the inverse of $\alpha$. By Lemma 4.6 $\alpha$ is a lattice isomorphism.

11) $\alpha$ is continuous.

By Proposition 2.6 $L$ and $L'$ both have small lattices; thus they have the interval topology [44], i.e. the intervals $[a, b]$ form a subbasis for the closed sets. Therefore to show that $\alpha$ is continuous we show $\alpha^{-1}([a, b])$ is closed for each $[a, b] \subset L'$. If $[a, b] \subset L'$, then $[a, b] = M(a) \cap L(b)$. Since $\alpha$ is a lattice isomorphism, $\alpha^{-1}(M(a)) = M(\alpha^{-1}(a))$. Thus $\alpha^{-1}(M(a))$ is closed. Dually $\alpha^{-1}(L(b))$ is closed; hence $\alpha^{-1}([a, b])$ is closed. Therefore
\( \alpha \) is continuous. Since \( L \) is compact and \( L' \) is Hausdorff, \( \alpha \) is a homeomorphism.

Suppose now that \( \mathcal{P}(L) = C_1 \cup C_2 \) and \( \mathcal{P}(L) = C_1 \cup B \), where \( C_1 \) is an arc chain from 0 to 1, \( C_2 \cup B \) is an arc chain from 0 to 1, \( C_2 \cap B = \{ p \} \subset L \setminus \{ 0, 1 \} \), and \( C_1 \cap (C_2 \cup B) = \{ 0, 1 \} \). If \( L' \) is another such lattice with \( C_1', C_2', B' \) and \( p' \) the corresponding chains and point respectively, then we have:

**Theorem 4.11.** \( L \) is topologically isomorphic to \( L' \).

**Proof.** We define \( f'_{21}:C_1 \rightarrow C_2', f_{12}:C_2 \rightarrow C_1', f_{21}:C_1' \rightarrow C_2' \), and \( f'_{12}:C_2' \rightarrow C_1' \) as in Lemma 4.7. By Corollary 4.5 \( f_{21} \) and \( f'_{21} \) are one-to-one; as in Lemma 4.7 \( f_{12}(f'_{12}) \) is the inverse of \( f_{21}(f'_{21}) \). Since \( C_1 \) and \( C_2 \cup B \) are closed and \( C_1 \cap (C_2 \cup B) = \{ 0, 1 \} \), by Lemma 3.7 \( f_{21} \) and \( f'_{21} \) are continuous. Dually \( f_{12} \) and \( f'_{12} \) are continuous.

Let \( \varphi:C_1 \rightarrow C_1' \) be any order preserving homeomorphism. As noted in Theorem 4.10, \( \varphi \) is a lattice isomorphism. We define \( \alpha:L \rightarrow L' \) by

i) \( \alpha|_{C_1} = \varphi \),

ii) if \( x \in C_2 \), then \( \alpha(x) = f'_{21} \varphi f_{12}(x) \),

iii) if \( x \in L \setminus (C_1 \cup C_2) \), then \( \alpha(x) = \alpha(x_1) \wedge \alpha(x_2) \),

where \( x_1 \wedge x_2 \) is the unique representation of \( x \) as a meet of meet irreducible elements. We define \( \beta:L' \rightarrow L \) similarly with \( \beta|_{C_1'} = \varphi^{-1} \). To see that \( \alpha \), hence \( \beta \), is
order preserving, let \( x, y \in L \) with \( x \leq y \). Clearly \( \alpha(x) \leq \alpha(y) \) if \( x, y \in C_i \) for \( i = 1,2 \).

1) Suppose \( x \in C_1 \) and \( y \in C_2 \).

In this case, \( x \leq \varphi_{12}(y) \); thus \( \alpha(x) = \varphi(x) \leq \varphi_{12}(y) \leq \varphi_{21}(\varphi_{12}(y)) = \alpha(y) \).

2) Suppose \( x \in C_2 \) and \( y \in C_1 \).

From the note following Corollary 4.5 \( y = 1 \); hence \( \alpha(x) \leq \alpha(y) \).

3) All other cases follow precisely as part 6), part 7), and part 8) of Theorem 4.10.

Part 9), part 10), and part 11) of Theorem 4.10 also hold here without modification. Thus \( \alpha \) is a topological isomorphism from \( L \) onto \( L' \).

The methods developed thus far may be used to complete the characterization of those lattices for which \( P(L) \) and \( \bar{P}(L) \) are each the union of two arc chains and \( P(L) \cup \bar{P}(L) \) is the union of two arc chains \( A \) and \( B \) with \( A \cap B = \{0,1\} \)

For the arc chain \( A \) we have five possibilities:

i) \( A \subseteq P(L) \cap \bar{P}(L) \),

ii) \( A \cap \bar{P}(L) \cap P(L) = \{p\} \) and \( p \neq 0,1 \),

iii) \( A \subseteq P(L) \) and \( A \cap \bar{P}(L) \) is an arc chain from \( 0 \) to \( p, 0 < p < 1 \),

iv) \( A \subseteq \bar{P}(L) \) and \( A \cap P(L) \) is an arc chain from \( p \) to \( 1, 0 < p < 1 \),
v) $A \cap P(L) \cap \overline{P}(L)$ is an arc chain from $p$ to $q$, $0 < p < q < 1$.

Since $B$ also may satisfy any of these five conditions, there are twenty-five possible combinations for $A$ and $B$. Theorem 4.1 shows that there is only one lattice (up to isomorphism) for which both $A$ and $B$ satisfy ii). By Theorem 4.10 there is only one lattice for which both $A$ and $B$ satisfy i). From Theorem 4.11 there is only one lattice for which $A$ satisfies i) and $B$ satisfies ii).

By using combinations of the techniques of Theorem 4.10 and Theorem 4.11 we can prove there is only one lattice for each of the following combinations:

1) $A$ satisfies i) and $B$ satisfies iii),
2) $A$ satisfies ii) and $B$ satisfies v),
3) $A$ satisfies ii) and $B$ satisfies iii),
4) $A$ and $B$ both satisfy iii).

The duals of 1), 3), and 4) are of course also unique.

Combinations for which we may have non-isomorphic lattices are:

5) $A$ satisfies i) and $B$ satisfies v),
6) $A$ and $B$ both satisfy v),
7) $A$ satisfies iii) and $B$ satisfies v),
8) $A$ satisfies iii) and $B$ satisfies iv).

Suppose that $L$ satisfies 5). Then $\sigma_A(p)$ and
A point may be the same point of \( L \), or they may be distinct points of \( L \). Clearly a lattice in which they are equal cannot be isomorphic to one in which they are distinct.

II. Modular Lattices

Edmondson [24] has given an example of a modular non-distributive topological lattice. In this section we characterize this lattice.

Let \( L \) be a compact, connected topological lattice of breadth two such that \( w(P(L)) = 3 \). Then \( L \) cannot be distributive, for Baker and Stralka have proved [10] that \( L \) can be embedded as a sublattice of a product of two chains; thus by Proposition 3.12 \( w(P(L)) \leq 2 \) contrary to \( w(P(L)) = 3 \). However, by Theorem 1.10 \( L \) is modular. In this section we continue the investigations of Section II with the new hypothesis that \( w(P(L)) = 3 \). Throughout this section \( L \) will be a compact, connected, metric topological lattice of breadth two such that \( w(P(L)) = 3 \).

By Theorem 3.18 \( L = \bigcup_{i \neq j} (C_i \land C_j) \) where \( C_i \) is a chain in \( L \). The next two lemmas improve this result in certain cases.

Lemma 4.12. Suppose \( P(L) = \bigcup_{i \neq 1} C_i \) and

1) \( C_i \) is an arc chain, \( i = 1, 2, 3 \),
ii) \( \tilde{P}(L) = \frac{3}{2} \sum_{i=1}^{3} B_i \), \( B_i \) is an arc chain, \( i = 1, 2, 3 \)

iii) \( A_i = B_i \cup C_i \) is an arc chain from 0 to 1,
\( A_i \cap A_j = [0, 1] \) if \( i \neq j \), \( i, j = 1, 2, 3 \)

iv) \( B_i \cap C_i = \{ z_i \} \neq \{0\} \), \( i = 1, 2, 3 \)

v) \( z_i \land z_j = 0 \) if \( i \neq j \), \( i, j = 1, 2, 3 \)

Then \( B_i = [0, z_i] \) and \( B_i \subset C_i \land C_j \) if \( i \neq j \), \( i, j = 1, 2, 3 \)

Proof. Since \( z_i \neq 0 \) and \( z_i \land z_j = 0 \), \( z_i \) and \( z_j \) are unrelated. Thus by Lemma 3.15 \( [0, z_i] \) has breadth 1; hence \( [0, z_i] \) is an arc chain from 0 to \( z_i \). Since \( B_i \) is a connected chain from 0 to \( z_i \), \( B_i \subset [0, z_i] \). Hence by Theorem 3.16 \( B_i = [0, z_i] \). Clearly \( z_i \land C_j \) is an arc chain from 0 to \( z_i \) if \( i \neq j \); thus \( B_i = z_i \land C_j \subset C_i \land C_j \)

Continuing with the hypotheses of Lemma 4.12 we state:

Lemma 4.13. If \( p \in L \), then \( p \) belongs to at least two of \( C_1 \land C_2 \), \( C_1 \land C_3 \), \( C_2 \land C_3 \).

Proof. Since \( L = \bigcup_{i \neq j} (C_i \land C_j) \), \( p \) must belong to one of them, say \( p \in C_1 \land C_2 \). Then by Theorem 1.11
\( p = x_1 \land x_2 \land x_3 = x_1 \land x_2 \), where \( x_i = \sigma_i(p) \) for \( i = 1, 2, 3 \)

Suppose that \( p \) belongs only to \( C_1 \land C_2 \). Then \( p < x_1 \land x_3 \) and \( p < x_2 \land x_3 \).

For a first case we suppose there exists an \( x \in C_3 \) such that \( x < x_3 \); then \( x_1 \land x_2 \land x < p \). Since \( L \) has breadth two, \( x_1 \land x_2 \land x = x_1 \land x \) or \( x_1 \land x_2 \land x = x_2 \land x \).
Since $C_3$ is order dense, there exists a net $\{x_\alpha\}_{\alpha \in D}$ converging to $x_3$ such that $x_\alpha < x_3$ and $x_1 \wedge x_2 \wedge x_\alpha = x_1 \wedge x_\alpha$ or $x_1 \wedge x_2 \wedge x_\alpha = x_2 \wedge x_\alpha$. Thus for some cofinal subset $\Delta \subseteq D$, $x_1 \wedge x_2 \wedge x_\alpha = x_1 \wedge x_\alpha$ for all $\alpha \in \Delta$ or $x_1 \wedge x_2 \wedge x_\alpha = x_2 \wedge x_\alpha$ for all $\alpha \in \Delta$. Suppose the former holds for all $\alpha \in \Delta$. Then for each $\alpha \in \Delta$, $(x_1 \wedge x_\alpha, p) \in \leq$, and $\{(x_1 \wedge x_\alpha, p)\}_{\alpha \in \Delta}$ converges to $(x_1 \wedge x_3, p)$. By Theorem 1.6 part i) $(x_1 \wedge x_3, p) \in \leq$; therefore $x_1 \wedge x_3 \leq p$ contrary to $p < x_1 \wedge x_3$. Thus there is no $x \in C_3$ such that $x < x_3$; hence $x_3 = z_3$.

Consequently $p < x_1 \wedge x_3 = x_1 \wedge z_3 \in [0, z_3] = B_3$ (by Lemma 4.12). Thus $p \in B_3$. By Lemma 4.12 $B_3 \subseteq C_1 \wedge C_3$; hence $p$ belongs to $C_1 \wedge C_3$ contrary to the assumption that $p \in C_1 \wedge C_2$ only.

We note that Lemma 4.12 and Lemma 4.13 hold without the restriction $A_1 \cap A_j = [0,1]$.

The lack of a unique representation in terms of meet irreducible elements in non-distributive lattices leads to difficulties when we attempt to use the methods of Section I to define lattice homomorphisms. Our next goal is to show the existence of an arc chain from 0 to 1 which we may use to overcome these difficulties.

Lemma 4.14. Let $A = \{y \in L: y = \wedge[(x \wedge C_1) \cap (C_2 \wedge C_3)]$ for some $x \in C_3\}$. 

1) $A$ is an arc chain from 0 to 1,
2) $x, y \in A$ and $x < y$ imply $y \in M(x)^\circ$. 
Proof. 1) If \( x, y \in C_3 \) and \( x \neq y \), then
\[
(x \land C_1) \cap (y \land C_1) = \emptyset.
\]
Suppose not. Then there are elements \( x', y' \in C_1 \) such that \( x \land x' = y \land y' \). Since \( C_1 \land C_3 \) is distributive (Theorem 3.17), \( x = y \) and \( x' = y' \) contrary to \( x \neq y \). Hence
\[
(x \land C_1) \cap (y \land C_1) = \emptyset.
\]
2) \( A \) is a chain.

Let \( p, q \in A \); then \( p = x \land x' \) and \( q = y \land y' \) with \( x, y \in C_3 \) and \( x', y' \in C_1 \). Suppose \( x \leq y \). Then
\[
x \land q = y \land q = q,
\]
and since \( x \land y' = x \land q \in (x \land C_1) \cap (C_2 \land C_3), p \leq x \land q \).
Thus \( p \leq q \). Similarly \( q \leq p \) if \( y \leq x \). Therefore \( A \) is a chain.

3) \( A \) is order dense.

Suppose \( p, q \in A \) and \( p < q \). Let \( x, y \in C_3 \) and \( x', y' \in C_1 \) such that \( p = x \land x' \) and \( q = y \land y' \). By part 1) and part 2) \( x < y \). Since \( C_3 \) is order dense, we choose \( z \in C_3 \) such that \( x < z < y \). Then
\[
v = z \land (z \land C_1) \cap (C_2 \land C_3)
\]
is in \( A \), and by part 1) and part 2) \( p < v < q \). Thus \( A \) is order dense.

4) If \( p \in A \), then \([p, x'] \subseteq C_2 \land C_3 \) for \( x' = \sigma_3(p) \).

Let \( x \in [p, x'] \). Then \( z_3 \land x \in B_3 \subseteq C_2 \land C_3 \), the containment from Lemma 4.12. Since \( C_2 \land C_3 \) is a sublattice (Theorem 3.17) and \( p, z_3 \land x \in C_2 \land C_3 \), \( p \lor (z_3 \land x) \in C_2 \land C_3 \). We
claim \( pv(z_3 \wedge x) = (p \vee z_3) \wedge x = x' \wedge x = x \). The first equality is just the modularity of \( L \) and the last equality holds because \( x \in [p,x'] \). We show the middle equality. It is clear that \( pvz_3 \leq x' \); by the dual of Lemma 4.12 \( pvz_3 \in C_3 \). If there is an \( x'' \in C_3 \) such that \( pvz_3 \leq x'' \leq x' \), then \( p = x' \wedge ay' = x'' \wedge ay' \) contrary to part 1). Thus \( pvz_3 = x' \); hence the middle equality holds.

5) \( A \subseteq \bigcap_{i \neq j} (C_i \wedge C_j) \).

Let \( p \in A \). Then \( p = x \wedge x' \) for some \( x \in C_3 \) and \( x' \in C_1 \). Let \( q = v[(x \wedge C_1) \cap (C_1 \wedge C_2)] \). By the dual of part 4) \( [z_1 \wedge x,q] \subseteq C_1 \wedge C_2 \). Since \( p,q \) both belong to the chain \( x \wedge C_1, p \leq q \) or \( q \leq p \). If \( p \leq q \), then \( p \in [z_1 \wedge x,q] \); thus \( p \in C_1 \wedge C_2 \). In this case, since \( p \in (C_1 \wedge C_3) \cap (C_2 \wedge C_3) \) by definition, we are done. Suppose \( q < p ; x \wedge C_1 \) is order dense; consequently there exists \( v \in x \wedge C_1 \) such that \( q < v < p \). From Lemma 4.13 \( v \in C_1 \wedge C_2 \) or \( v \in C_2 \wedge C_3 \). If \( v \in C_1 \wedge C_2 \), then \( q \neq v[(x \wedge C_1) \cap (C_1 \wedge C_2)] \); if \( v \in C_2 \wedge C_3 \), then \( p \neq \wedge[(x \wedge C_1) \cap (C_2 \wedge C_3)] \). In either case we reach a contradiction; therefore \( p \in C_1 \wedge C_2 \). Since \( p \) was arbitrary, \( A \subseteq \bigcap_{i \neq j} (C_i \wedge C_j) \).

6) Let \( A' = \{ y \in L; y = \wedge[(x \wedge C_1) \cap (C_2 \wedge C_3)] \} \) for some \( x \in C_2 \).

If \( x = x_1 \wedge x_2 \wedge x_3 \in A(x_1 = a_1(x) \) for \( i = 1,2,3 \), and if \( y = \wedge[(x_2 \wedge C_1) \cap (C_2 \wedge C_3)] \in A' \), then \( x = y \).

Since \( x = x_1 \wedge x_2 = x_1 \wedge x_3 \), \( x \wedge C_1 = (x_1 \wedge x_2) \wedge C_1 = (x_1 \wedge x_3) \wedge C_1 \).
$z_1$ and $x_2$ are not related; hence $[z_1 \wedge x_2, x_2]$ has
breadth one (Lemma 3.15). Since $[z_1 \wedge x_2, x] \subset [z_1 \wedge x_2, x_2]$, the
breadth of $[z_1 \wedge x_2, x]$ is one. Thus $[z_1 \wedge x_2, x]$ is an arc chain
from $z_1 \wedge x_2$ to $x$. Since $(x_1 \wedge x_2) \wedge C_1 \subset [z_1 \wedge x_2, x]$ and is
also an arc chain from $z_1 \wedge x_2$ to $x$, by Theorem 3.16
$x \wedge C_1 = [z_1 \wedge x_2, x]$. Similarly $x \wedge C_1 = [z_1 \wedge x_3, x]$. Thus
$x \in (x_2 \wedge C_1) \cap (C_2 \wedge C_3)$; therefore $y \leq x$. Also
$y \in (x_3 \wedge C_1) \cap (C_2 \wedge C_3)$; hence $x \leq y$. Consequently $x = y$.

7) $\bigcap_{i \neq j} (C_i \wedge C_j) \subset A$.

Let $x \in \bigcap_{i \neq j} (C_i \wedge C_j)$. Then $x = x_1 \wedge x_2 \wedge x_3$, where
$x_i = \sigma_i(x)$ for $i = 1, 2, 3$, implies $x = x_1 \wedge x_2 = x_1 \wedge x_3 = x_2 \wedge x_3$.
Let $p = p_1 \wedge p_2 \wedge x_3 \in A$; thus $p = p_1 \wedge x_2 = p_2 \wedge x_3$.
Finally let $q = q_1 \wedge x_2 \wedge q_3 \in A'$. By definition of $A'$,
$q = \wedge ((x_2 \wedge C_1) \cap (C_2 \wedge C_3))$; therefore $x \in (x_2 \wedge C_1) \cap (C_2 \wedge C_3)$ implies
$q \leq x$. Hence $q_3 = \sigma_3(q) \leq \sigma_3(x) = x_3$. Similarly $p_2 \leq x_2$.
By part 6) $p = \wedge ((p_2 \wedge C_1) \cap (C_2 \wedge C_3))$; thus by part 2) $p \leq q$.
We have then $x_3 = \sigma_3(p) \leq \sigma_3(q) = q_3$. Consequently
$x_3 = q_3$. By part 6) $p = q$; $q = x_2 \wedge q_3 = x_2 \wedge x_3 = x$. Therefore
$x \in A$. Since $x$ was arbitrary, $\bigcap_{i \neq j} (C_i \wedge C_j) \subset A$.

8) $A = \bigcap_{i \neq j} (C_i \wedge C_j)$.

Since each $C_i \wedge C_j$ is compact, $A$ is compact. Thus
we have proved $A$ is an arc chain.

9) If $x, y \in A$ and $\sigma_i(x) = \sigma_i(y) = x_i$ for at least one
$i$ between 1 and 3, then $x = y$. 


If \( i = 3 \), then the claim follows from Part 1). Since \( A = \cap_{j \neq i} (C_i \land C_j) \), we may permute \( C_1, C_2 \), and \( C_3 \) in the definition of \( A \) and still obtain \( A \). Hence the claim for \( i = 1 \) and \( i = 2 \) follows by duality.

We now prove part ii). If \( x, y \in A \) and \( x < y \), then by part 9) \( \sigma_i(x) < \sigma_i(y) \) for \( i = 1, 2, 3 \). By Theorem 4.1 \( C_i \land C_j \) is isomorphic to \( C_i \times C_j \) with \( (\sigma_i(x), \sigma_j(x)) \) the image of \( \sigma_i(x) \land \sigma_j(x) = x \) under the isomorphism. Similarly \( (\sigma_i(y), \sigma_j(y)) \) is the image of \( y \). Since \( \sigma_i(x) < \sigma_i(y) \) and \( \sigma_j(x) < \sigma_j(y) \), \( (\sigma_i(y), \sigma_j(y)) \epsilon M((\sigma_i(x), \sigma_j(x)))^\circ \). This result is true for each \( i, j \) between 1 and 3 with \( i \neq j \). Consequently \( y \in M(x)^\circ \) in \( C_i \land C_j \) for all \( i \neq j \), and since \( L = \bigcup_{i \neq j} (C_i \land C_j), y \in M(x)^\circ \) in \( L \).

Let \( f: L \rightarrow A \) be defined by \( f(x) = \bigvee \{y \in A: y \leq x\} \). Since \( x, y \in A \) and \( x < y \) imply \( y \in M(x)^\circ \), the duals of Corollary 4.5 and Lemma 3.7 imply \( f|_{C_1} \) is a topological isomorphism with inverse \( \sigma_i|A \) for \( i = 1, 2, 3 \). We remark that for each \( x \in L \), \( f(\sigma_i(x)) \) is related to \( x \) for \( i = 1, 2, 3 \). This follows easily since \( \sigma_i(f(\sigma_i(x)) = \sigma_i(x) \) and \( \sigma_j(x) \) is related to \( \sigma_j(f(\sigma_i(x)) \).

Lemma 4.15. If \( x = x_1 \land x_3 = x_2 \land x_3 \neq x_1 \land x_2 \), where \( x_i = \sigma_i(x) \) for \( i = 1, 2, 3 \), then \( x = f(x_1) \land x_3 = f(x_2) \land x_3 \) and \( f(x_1) = f(x_2) \). Furthermore \( f(x_1) \land x_3 = f(y_1) \land y_3 \) if and only if \( x_1 = y_1 \) and \( x_3 = y_3 \).
Proof. It follows from part 8) of Lemma 4.14 that $A = \{y \in L: y = \wedge[(x \land C_3) \cap (C_1 \land C_2)] \text{ for some } x \in C_1\}$. Thus if $f(x_1) \leq x$, then by part 4) of Lemma 4.14 $x \in [f(x_1), x_1] \subseteq C_1 \land C_2$ contrary to $x \notin C_1 \land C_2$. Hence $x < f(x_1)$. This implies $x = x_1 \land x_3 \leq f(x_1) \land x_3$. Since $f(x_1) \leq x_1$, $f(x_1) \land x_3 \leq x_1 \land x_3 = x$. Therefore $x = f(x_1) \land x_3$.

Similarly $x = f(x_2) \land x_3$. Since $C_1 \land C_3$ is distributive (Theorem 3.17), by Theorem 3.5 part 3) $\sigma_1$ is a lattice homomorphism on $C_1 \land C_3$. By Lemma 4.14 part 8) $f(x_1)$ and $f(x_2) \in C_1 \land C_3; x_3 \in C_3$; thus $\sigma_1(f(x_1)) \land \sigma_1(x_3) = \sigma_1(f(x_1) \land x_3) = \sigma_1(f(x_2)) \land \sigma_1(x_3)$. Since $1 = z_1 \lor z_2 \leq \sigma_1(x_3)$, $\sigma_1(x_3) = 1$. Hence $\sigma_1(f(x_1)) = \sigma_1(f(x_2))$; thus by part 9) of Lemma 4.14 $f(x_1) = f(x_2)$.

If $x \in A$, then $x = f(x_1) \land x_j$ for $i \neq j, i, j = 1, 2, 3$. If $f(x_1) \land x_3 = f(y_1) \land y_3$, then $x_1 \land x_3 = f(x_1) \land x_3 = f(y_1) \land y_3 = y_1 \land y_3$. Since $C_1 \land C_3$ is distributive, this implies $x_1 = y_1$ and $x_3 = y_3$. Thus $f(x_1) = f(y_1)$.

We are finally ready to state the main theorem of this section.

Theorem 4.16. If $L$ and $L'$ are compact, connected, metric topological lattices with breadth two and $w(\mathcal{P}(L)) = w(\mathcal{P}(L')) = 3$, and which both satisfy the hypotheses of Lemma 4.12, then $L$ is topologically isomorphic to $L'$.

Proof. Let $A$, $C_i$, $B_i$, $f$, and $\sigma_i$ be the sets and
functions of the preceding discussion for $L$ and $A'$, $C'_i$, $B'_i$, $f'$, and $\sigma'_1$, the corresponding sets and functions for $L'$. Let $\phi:A \rightarrow A'$ be any order preserving homeomorphism.

We define $\alpha:L \rightarrow L'$ by

i) $\alpha|A = \phi$,

ii) if $x \in C'_i$, then $\alpha(x) = \sigma'_1 \phi f(x)$,

iii) if $x \in L \setminus \bigcup_{i=1}^3 (C'_i \cup A)$, then

$$\alpha(x) = \phi f(x_1) \land \sigma'_j \phi f(x_j),$$

where $x = f(x_1) \land x_j = x_1 \land x_j$, $x_i = \sigma_i(x)$, $x_j = \sigma_j(x)$.

By Lemma 4.15 if $x = f(x_k) \land x_j$, then $f(x_1) = f(x_k)$; hence the definition of $\alpha$ is independent of the representation chosen for $x$.

We show that $\alpha$ is an isomorphism on each $C_i \land C_j$ for $i \neq j$. For convenience suppose $i = 1$ and $j = 2$.

1) If $x = x_1 \land x_2 \notin A$, then $\alpha(x) \notin A'$.

Suppose $\alpha(x) \in A'$. By definition

$$\alpha(x) = \phi f(x_1) \land \sigma'_2 \phi f(x_2)$$

or $\alpha(x) = \phi f(x_2) \land \sigma'_1 \phi f(x_1)$; thus either $\phi f(x_1) \land \sigma'_2 \phi f(x_2) = \phi f(x')$ or $\phi f(x_2) \land \sigma'_1 \phi f(x_1) = \phi f(x')$ for some $x' \in C'_1$. Suppose $\alpha(x) = \phi f(x_1) \land \sigma'_2 \phi f(x_2)$; then

$$\phi f(x_1) \land \sigma'_2 \phi f(x_2) \in C'_1 \land C'_2$$

which is distributive; as noted in the proof of Lemma 4.15 $\sigma'_1$ is a lattice homomorphism on $C'_1 \land C'_2$. Hence $\sigma'_1 \phi f(x_1) = \sigma'_1 \phi f(x_1) \land 1 = \sigma'_1 \phi f(x_1) \land \sigma'_1 (\sigma'_2 \phi f(x_2))$

$$= \sigma'_1 (\phi f(x_1) \land \sigma'_2 \phi f(x_2)) = \sigma'_1 \phi f(x').$$

This implies $f(x_1) = f(x')$ since $\sigma'_1$ and $\phi$ are one-to-one. Thus

$$\phi f(x_1) \land \sigma'_2 \phi f(x_2) = \phi f(x_1)$$

which implies $\phi f(x_1) \leq \sigma'_2 \phi f(x_2)$;
hence \( \sigma_2' \varphi(x_1) \leq \sigma_2' \varphi(x_2) \). This implies \( f(x_1) \leq f(x_2) \)
since both \( \sigma_2' \) and \( \varphi \) are order preserving. Since
we assumed \( x = f(x_1) \wedge x_2 \), \( x = (f(x_1) \wedge x_2) \wedge f(x_2) = f(x_1) \in A \).
But \( x \not\in A \). We obtain a similar contradiction if
\( x = f(x_2) \wedge x_1 \). Thus \( \alpha(x) \notin A' \).

2) If \( x = x_1 \wedge x_2 \notin A \), then \( \alpha(x) = \alpha(x_1) \wedge \alpha(x_2) \).

If \( x = x_1 \wedge x_2 \notin A \), then \( x = f(x_1) \wedge x_2 \) or
\( x = f(x_2) \wedge x_1 \). From part 1) it follows that \( f(x_2) < f(x_1) \)
or \( f(x_1) < f(x_2) \); also \( \alpha(x) \notin A' \). Suppose \( x = f(x_1) \wedge x_2 \);
thus \( f(x_2) < f(x_1) \). In \( L' \), \( \alpha(x_1) \wedge \alpha(x_2) = \sigma_1' \varphi(x_1) \wedge \sigma_2' \varphi(x_2) \)
and \( \alpha(x_1) \wedge \alpha(x_2) = \sigma_1' \varphi(x_1) \wedge \varphi f(x_2) \) or \( \alpha(x_1) \wedge \alpha(x_2) = \varphi f(x_1) \wedge \sigma_2' \varphi(x_2) \). The latter is \( \alpha(x) \); hence suppose
\( \sigma_1' \varphi f(x_1) \wedge \sigma_2' \varphi f(x_2) = \sigma_1' \varphi f(x_1) \wedge \varphi f(x_2) \). Using the homomorphism
property of \( \sigma_1' \) on \( C_1 \wedge C_2 \), \( \sigma_1' \varphi f(x_1) = \sigma_1' \varphi f(x_1) \wedge \alpha_1 = \sigma_1' \varphi f(x_1) \wedge \sigma_1' \varphi f(x_2) \) = \( \sigma_1' \varphi f(x_1) \wedge \sigma_1' \varphi f(x_2) \leq \sigma_1' \varphi f(x_2) \). Again
since \( \sigma_1' \) and \( \varphi \) are order preserving \( f(x_1) \leq f(x_2) \). This
contradicts \( f(x_2) < f(x_1) \). A similar contradiction is
reached if \( x = f(x_2) \wedge x_1 \). Thus \( \alpha(x) = \alpha(x_1) \wedge \alpha(x_2) \).

We show \( \alpha \) is order preserving on \( C_1 \wedge C_2 \). Let
\( x, y \in C_1 \wedge C_2 \) and \( x \leq y \). We consider six cases.

a) Suppose \( x, y \in A \).

Then \( \alpha(x) = \varphi(x) \leq \varphi(y) = \alpha(y) \).

b) Suppose \( x, y \in C_i, i = 1, 2 \).

Then \( f(x) \leq f(y) \); thus by part 1) \( \varphi f(x) \leq \varphi f(y) \);
hence \( \alpha(x) = \sigma' \varphi f(x) \leq \sigma' \varphi f(y) = \alpha(y) \).

c) Suppose \( x \in C_1 \) and \( y \in C_2 \) (or \( x \in C_2 \) and \( y \in C_1 \)).
Then \( 1 = z_1 \vee z_2 \leq y \). Thus \( \alpha(x) \leq \alpha(y) \).

d) Suppose \( x \in C_1, y \in (C_1 \wedge C_2) \setminus (C_1 \cup C_2) \).
This is not possible since \( M(x) = [x,1] \subseteq C_1 \).

e) Suppose \( x \in (C_1 \wedge C_2) \setminus (C_1 \cup C_2) \) and \( y \in C_i, i = 1, 2 \).
Then \( x = x_1 \wedge x_2 = f(x_1) \wedge x_2 \) or \( x = f(x_2) \wedge x_1 \); since
\( C_1 \wedge C_2 \) is distributive, \( y \) is prime; thus \( x_1 \wedge x_2 = x \leq y \)
implies \( x_1 \leq y \) or \( x_2 \leq y \). Hence \( f(x_1) \leq f(y) \) or
\( f(x_2) \leq f(y) \). Then \( \varphi f(x_1) \leq \sigma' \varphi f(x_1) \leq \sigma' \varphi f(y) = \alpha(y) \).
It is easily verified that for any combination such as \( y \in C_1 \),
\( x = f(x_1) \wedge x_2 \), and \( x_1 \leq y \), \( \alpha(x) \leq \alpha(y) \).

f) Suppose \( x, y \in (C_1 \wedge C_2) \setminus (C_1 \cup C_2) \).

We consider three possibilities.

i) If \( x \notin A \) and \( y \in A \), then \( \alpha(x) \leq \varphi f(x_1) \leq \varphi f(y) = \alpha(y) \) since \( x_1 = \sigma(x) \leq \sigma(y) = y_1 \) for \( i = 1, 2 \).

ii) If \( x \in A \) and \( y \notin A \), then \( \alpha(x) = \varphi f(x_1) \leq \varphi f(y) \).

iii) If \( x, y \notin A \), then \( x_1 \leq y_1 \) implies
\( \alpha(x_1) \leq \alpha(y_1) \) for \( i = 1, 2 \). Thus by part 2) \( \alpha(x) = \alpha(x_1) \wedge \alpha(x_2) \leq \alpha(y_1) \wedge \alpha(y_2) = \alpha(y) \).
Thus \( \alpha \) is order preserving on \( C_1 \wedge C_2 \). Similarly \( \alpha \) is order preserving on
\( C_1 \wedge C_3 \) and \( C_2 \wedge C_3 \). If \( \beta : L' \to L \) is defined as \( \alpha \), but
with \( \beta|A' = \varphi^{-1} \), then \( \beta \) is also order preserving on
\( C_i \wedge C_j \), \( i \neq j \) and \( i, j = 1, 2, 3 \). Clearly \( \beta \) is the inverse
of \( \alpha|(A \cup C_1 \cup C_2) \); hence the proof that \( \beta \) is the inverse
of \( \alpha \) on \( C_1 \wedge C_2 \) is the same as that given in part 10) of Theorem 4.10.

Now suppose \( x, y \in L \). By Lemma 4.13 there exist \( i \) and \( j \) such that \( x, y \in C_i \wedge C_j \). By Lemma 4.15 \( \alpha(x) \) and \( \alpha(y) \) are independent of the representations chosen for \( x \) and \( y \). Thus since \( \alpha \) is an isomorphism on \( C_i \wedge C_j \), by Lemma 4.6 \( \alpha(x \lor y) = \alpha(x) \lor \alpha(y) \) and \( \alpha(x \land y) = \alpha(x) \land \alpha(y) \).

Hence \( \alpha \) is an isomorphism on \( L \). As in Theorem 4.10 \( \alpha \) is continuous on each of \( C_1 \wedge C_2, C_1 \wedge C_3, \) and \( C_2 \wedge C_3 \); hence \( \alpha \) is continuous on \( L \). Therefore \( \alpha \) is a homeomorphism.

We do not attempt to completely catalogue the lattices for which \( w(\mathcal{P}(L)) = 3 \). The methods used in Theorem 4.16 do not readily extend to all cases. Some of these lattices are not isomorphic even when the three distributive sub-lattices are unique in the sense of Section I. An example will be discussed at the end of this section.

We give one further example which is unique.

Let \( L \) be a compact, connected, metric topological lattice of breadth two such that

1) \( \mathcal{P}(L) = \bigcup_{i=1}^{3} C_i \), \( C_i \) an arc chain from \( z_i \) to \( 1 \), \( 0 < z_i < 1 \) for \( i = 1, 2, 3 \),

2) \( C_1 \cap C_2 = \{1\} \) for \( i = 2, 3 \),

3) \( C_2 \cap C_3 = [u, 1], z_i < u < 1, i = 2, 3 \),

4) \( \mathcal{P}(L) = \bigcup_{i=1}^{3} B_i \), \( B_i \) an arc chain from \( 0 \) to
z_1, i = 1, 2, 3,

v) \( B_1 \cap B_1 = \{0\} \) for \( i = 2, 3 \),

vi) \( B_2 \cap B_3 = [0, z], 0 < z < z_1, i = 2, 3 \),  

vii) \( z_2 \wedge z_3 = z, z_2 \vee z_3 = u, z_1 \wedge z_1 = 0, z_1 \vee z_1 = 1 \) for \( i = 2, 3 \).

We show that \( S = [z, u] \) satisfies the hypotheses of Theorem 4.16. Using the methods of Section I together with Theorem 4.16 it follows readily that \( L \) is unique.

Lemma 4.17. The sublattice \( S \) satisfies the hypotheses of Theorem 4.15.

Proof. Let \( z_0 = x_0 \wedge u \), where \( x_0 = \sigma_1(z) \). Then \( z_0 \wedge z_2 = x_0 \wedge u \wedge z_2 = x_0 \wedge z_2 = z \). The last equality holds since \( z \in B_2 \). Similarly \( z_0 \wedge z_3 = x_0 \wedge z_3 = z \). Now \( z_0 \vee z_2 = (x_0 \wedge u) \vee z_2 = (x_0 \vee z_2) \wedge u = 1 \wedge u = u \). Similarly \( z_0 \vee z_3 = u \).

Since \( u \) and \( x_0 \) are not related, by Lemma 3.15 \( [z_0, u] \) has breadth one. Thus \( [z_0, u] \) is an arc chain. Hence any element of \( [z_0, u] \) is meet irreducible in \( S \). Let \( C_i = [z_0, u] \) and \( C_i = [z_i, u] \) for \( i = 2, 3 \). We show that \( P(S) = \bigcup_{i=1}^{3} C_i \). Clearly \( C_i \subseteq P(S) \) for \( i = 1, 2, 3 \). Suppose \( x \in S \setminus \bigcup_{i=1}^{3} C_i \). Then \( x \notin P(L) \). By Theorem 1.11 \( x = x' \wedge x_j \) for some \( i, j \) between 1 and 3. If \( x = x_1 \wedge x_2 \), then \( x_0 \leq x_1 \) and \( x_2 \leq u \); thus \( x = (x_1 \wedge u) \wedge x_2 \in C_1 \cap C_2 \); \( x \neq x_1 \wedge u \) and \( x \neq x_2 \). If \( x = x_1 \wedge x_3 \), then \( x = (x_1 \wedge u) \wedge x_3 \in C_1 \cap C_3 \) and \( x \neq x_1 \wedge u \) and \( x \neq x_3 \). Finally if \( x = x_2 \wedge x_3 \), then
\[ x \in C_2 \setminus C_3 \quad \text{and} \quad x \not\in x_2, x \not\in x_3. \] Thus if \( x \not\in \bigcup_1^3 C_1 \), then \( x \not\in P(S) \). Dually \( \tilde{P}(S) = \bigcup_1^3 B'_i \) where \( B'_1 = [z, z_0] \) and \( B'_i = [z, z_i] \) for \( i = 2, 3 \). It is easily verified that \( S \) together with the chains \( C'_i \) and \( B'_i, i = 1, 2, 3 \), satisfy the hypotheses of Theorem 4.16.

If we change condition i) and iv) to \( C_1 = B_1 \) and \( C_1 \) is an arc chain from 0 to 1, then \( z_0 \) may belong to \( C_1 \) or \( z_0 \) may belong to \( L \setminus C_1 \). Clearly a lattice where \( z_0 \in C_1 \) cannot be isomorphic to a lattice where \( z_0 \in L \setminus C_1 \).

III. Minimal Connected Lattices

In this section we let \( L \) be a compact, connected, modular, metric topological lattice. We do not assume that \( L \) has finite breadth. If \( A \) is a finite sublattice of \( L \) and \( A \) is a chain, then clearly a minimal compact, connected sublattice of \( L \) which contains \( A \) is an arc chain from \( \sup A \) to \( \inf A \). If \( a \) and \( b \) are not related, then \( A = [a \wedge b, a \vee b] \) is a distributive sublattice of \( L \). Let \( S \) be a minimal compact, connected sublattice of \( L \) which contains \( A \).

Theorem 4.18. \( S \) is topologically isomorphic to the product of two arc chains.

Proof. Since \([a \wedge b, a \vee b]\) is a compact, connected
sublattice of \( L \) which contains \( A, [a\wedge b, a\vee b] \) contains \( S \). Thus we may assume \( a\wedge b = 0 \) and \( a\vee b = 1 \). By the dual of Theorem 1.8 there exist arc chains \( T \) from \( a \) to \( 1 \) and \( T' \) from \( b \) to \( 1 \) in \( S \). We show \( S = T \wedge T' \).

Since \( \langle T \cup T' \rangle \) is a compact, connected sublattice of \( L \) which contains \( A \), \( S \subset \langle T \cup T' \rangle \); hence \( S \) is distributive since \( \langle T \cup T' \rangle \) is \([11]\). As noted in the proof of Theorem 3.17, we only need to show that \( tvt' \in T \wedge T' \) whenever \( t \in T \) and \( t' \in T' \). Now \( t \in T \) and \( t' \in T' \) imply \( a \leq t \) and \( b \leq t' \); thus \( 1 = a\vee b \leq tvt' \). Therefore \( tvt' = 1 \in T \wedge T' \). Thus \( T \wedge T' \) is the sublattice \( \langle T \cup T' \rangle \) and \( T \wedge T' \subset S \); consequently \( S = T \wedge T' \). Clearly \( S \) has breadth two. We need to show that \( \overline{P(S)} = T \cup T' \). Since \( S = T \wedge T' \), \( \overline{P(L)} \subset T \cup T' \). Let \( t \in T \) and suppose \( t = p \wedge q \) with \( p \in T \) and \( q \in T' \). Then \( t \leq q \); hence \( 1 = a\vee b \leq q \). Therefore \( q = 1 \) and \( t = p \) which implies \( t \in \overline{P(S)} \). Similarly \( T' \subset \overline{P(S)} \). By Theorem 4.1 \( S \) is topologically isomorphic to \( T \times T' \).

Suppose now that \( A = \{0,a,b,c,1\} \) where \( a\vee b = a\vee c = b\vee c = 1 \) and \( a\wedge b = a\wedge c = b\wedge c = 0 \). Again we let \( S \) be a minimal compact, connected sublattice of \( L \) which contains \( A \). We wish to show that \( S \) is the lattice of Theorem 4.16. We assume there exists an arc chain \( T \) from \( a \) to \( 1 \) in \( S \) such that for each \( t \in T \), \( t = a\vee (c\wedge (b\vee (a\wedge (c\vee (b\wedge t)))))) \). The proof that such an arc chain exists is due to Lawson;
however, we do not include the proof here.

Let \( T' = bv(a \wedge cv(b \wedge aT)) \) and \( T'' = cv(b \wedge aT) \); then 
\( T' \) is an arc chain from \( b \) to \( l \) in \( S \), and \( T'' \) is an arc chain from \( c \) to \( l \) in \( S \).

**Lemma 4.19.** For each \( t' \in T' \), 
\[ t' = bv(a \wedge (cv(b \wedge (av(c \wedge (bv(cA(cA(bA(aV(cAt^))))))))) \) ; for each \( t'' \in T'' \), 
\[ t'' = cv(b \wedge (av(c \wedge (bv(a \wedge (cv(bAt^)))))))) \).

**Proof.** Let \( t' \in T' \); then for some \( t^* \in T \), 
\[ t' = bv(a \wedge (cv(b \wedge aT^))) \). Thus \[ bv(a \wedge (cv(b \wedge (av(c \wedge (bv(cA(cA(bA(aV(cAt^))))))))) \) = 
\[ bv(a \wedge (cv(b \wedge (av(c \wedge (bv(a \wedge (cv(bAt^)))))))))) \) = 
\[ bv(a \wedge (cv(bAt^))) = t' \) since \( av(c \wedge (bv(a \wedge (cv(bAt^))))) = t^* \) by hypothesis.

Let \( t'' \in T'' \); then for some \( t^* \in T \), \( t'' = cv(b \wedge aT^) \). 
Thus \[ cv(b \wedge (av(c \wedge (bv(aT^)))))) = cv(b \wedge (av(c \wedge (bv(a \wedge (cv(bAt^))))))) \) = 
\[ cv(b \wedge aT^) = t'' \), since \( av(c \wedge (bv(a \wedge (cv(bAt^))))) = t^* \) .

Since \(<TUT'UT''> \) is a compact, connected sub-lattice of \( L \) which contains \( A, S \subset <TUT'UT''> \). Also 
\( TUT'UT'' \subset S \); hence \( <TUT'UT''> \subset S \). Thus \( S \) is generated by \( TUT'UT'' \). We show \( S \) has breadth two.

Let \( t_1 \in T, bv(a \wedge (cv(b \wedge aT))) \in T' \), and 
\( cv(b \wedge aT^) \in T'' \), where \( t_2 \) and \( t_3 \in T \). Suppose that \( t_1 \) and \( t_2 \) are less than or equal to \( t_3 \); then
\[ t_1 \land (c \lor (b \land t_2)) \land (b \lor (a \land (c \lor (b \land t_3)))) = \]
\[ \lnot \frac{t_1 \land \left[ \left( c \land \left( b \lor (a \land (c \lor (b \land t_3))) \right) \right) \land \left( b \lor t_2 \right) \right]}{b \lor \left( a \land (c \lor (b \land t_3))) \right) \land \left( b \lor t_2 \right) \right]} \text{ by modularity and } \]
\[ b \land t_2 \leq b. \text{ Since } t_3 = a \land (c \lor (b \lor (a \land (c \lor (b \land t_3)))), c \land t_3 = \]
\[ a \land (c \land (b \lor (a \land (c \lor (b \land t_3))))) = (c \land a) \lor (c \lor (b \lor (a \land (c \lor (b \land t_3))))) = c \lor (b \lor (a \land (c \lor (b \land t_3)))) \text{; the second equality by modularity, } \]
\[ \text{and the last since } c \land a = 0. \text{ Thus we have } \]
\[ t_1 \land \left[ \left( c \land \left( b \lor (a \land (c \lor (b \land t_3))) \right) \right) \land \left( b \lor t_2 \right) \right] = \]
\[ t_1 \land \left[ \left( c \land t_3 \right) \lor (b \land t_2) \right] = t_1 \land \left[ \left( c \lor (b \land t_2)) \land \left( a \land (c \lor (b \land t_3))) \right) \right] \text{; the second equality by modularity and } b \land t_2 \leq t_2 \leq t_3, \text{ and } \]
\[ \text{the last by } t_1 \leq t_3. \text{ By Lemma 4.19 the preceding argument } \]
\[ t = t_1 \land (c \lor (b \land t_2)) \land (b \lor (a \land (c \lor (b \land t_3)))) = t_1 \land (b \lor (a \land (c \lor (b \land t_3)))) \text{ when } t_1, t_3 \leq t_2, \text{ and } \]
\[ t = (c \lor (b \land t_2)) \land (b \lor (a \land (c \lor (b \land t_3)))) \text{ when } t_2, t_3 \leq t_1. \text{ We } \]
\[ \text{have proved:} \]

Lemma 4.20. The breadth of \( S \) is two.

By Theorem 3.19 \( T \land T' \), \( T \land T'' \), and \( T' \land T'' \) are distributive sublattices of \( S \). By the dual of Theorem 3.17 \( b \land T) \lor (a \land T'' \) is a sublattice of \( S \); in fact \( b \land T) \lor (a \land T'' \) = \( T \land T' \). Similarly \( T \land T'' \) = \( a \land T'' \) \lor (c \land T') and \( T' \land T'' \) = \( b \land T) \lor (c \land T'). \text{ It follows that } \]
\[ S = (T \land T') \lor (T \land T'') \lor (T' \land T''). \text{ Thus by arguments similar } \]
\[ \text{to those in Theorem 4.18, } \varphi(S) = T \lor T' \lor T''. \text{ It is now } \]
\[ \text{easily verified that } S \text{ satisfies the hypotheses of Lemma } 4.12. \]
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