Analysis of Equation and Diagram Construction in Applied Calculus Problem Solving

Ahmed Ibrahim Usman
Louisiana State University and Agricultural and Mechanical College, ahmediu22@yahoo.com

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ANALYSIS OF EQUATION AND DIAGRAM CONSTRUCTION IN APPLIED CALCULUS PROBLEM SOLVING

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Educational Theory, Policy, and Practice

by
Ahmed Ibrahim Usman
B.S., Bayero University Kano, 1994
M.S., Northwest Missouri State University-Maryville, 2002
M.S., University of Missouri-Rolla, 2005
December 2008
DEDICATION

This endeavor is dedicated to my late father, Ibrahim Danbaba Mai Gyaran Agogo, my mother Hauwa Ahmadu, my wife Hafsat, my daughters Maryam, Hamida, Hanifa, and son Abdulwarith, my uncle Bala Ahmadu, and Garba Imam and his family. They are source of encouragement and inspiration for what I achieved in my life. I am grateful to them for their patience, love, support and prayers during this period of my stay in the United States of America.
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ABSTRACT

The purpose of this study was to assess algebra and geometric prerequisites skills as incorporated into the Applied Calculus Optimization Problem (ACOP) solution. The difficulties that students encounter in applying algebraic and geometric prerequisites at the early stages of the ACOP solution were identified. The study analyzes errors related to variables and equations (i.e. algebraic symbol/transformation skills), drawing of geometric diagrams (visualization skills) and those associated with application of basic differentiation concepts into ACOP solution process.

The study’s goals were addressed as seven specific research questions further subdivided into three main parts: the first four research questions investigated prerequisite algebraic and geometric skills, while question five examined the ability to use some or all of the prerequisite skills to obtain the required ACOP model. Question six is concerned with how some prerequisite (differentiation) skills are use in ACOP solution process. Finally, question seven looked into students’ ability to fully bring into play all the prerequisite skills into ACOP solution process. Furthermore, each of the seven research questions was split into quantitative and qualitative parts. The quantitative data were collected using a test instrument; and a follow up interview was conducted to collect qualitative data. These qualitative data were used to supplement, support and illuminate results from the quantitative components. The target sample is freshmen students taking calculus I in the department of mathematics, Louisiana State University, Baton Rouge.

Overall, the study has revealed that students have achieved a very low success rate on ACOP, immediately following instruction on ACOP solving in their calculus I class. In general, they failed to integrate the basic competences required in ACOP solution. Qualitative evidence from students’ test performance indicated that failure to visualize geometric diagrams from word problems tended to preclude getting the required formula. More generally, failure in at least
one competence lead to collapse in another, and hence the whole breakdown of the ACOP solution process.

The overall finding of the research was that students generally failed in integrating the independent algebraic and geometric competences; in cases where integration occurred, students face structural and procedural setbacks that ultimately led to a weakening of the ACOP solution process.
CHAPTER ONE

BACKGROUND/ORIENTATION

Introduction

Calculus is a branch of mathematics which has been developed to describe relationships between two or more things which can change continuously (Davis & Hersh, 1981). According to Young (1986) quoted by Ferrini-Mundy and Graham (1991), “calculus is our most important course . . . the future of our subjects depends upon improving it” (p. 627). Similarly, in a survey at the University of Connecticut for the requirements of all sciences and engineering major programs, thirty-two (32) key courses were found to have first-year calculus as a prerequisite (Hurley, Koehn and Ganter, 1999). Further, those thirty-two (32) courses require students to use the quantitative, analytical, and problem-solving skills conveyed by calculus. From another perspective, the importance of calculus and with particular reference to its impacts on our life, the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (1989) states that, in grades 9-12, the mathematics curriculum should include the informal exploration of calculus concepts from both a graphical and numerical perspective so that all students can determine maximum and minimum points of a graph, interpret the results in problem situations, and investigate the concepts of limit and area under a curve by examining infinite sequences and series. In addition, students intending to go to college should understand the conceptual foundations of limit, area under a curve, rate of change, and slope of a tangent line. Furthermore students should be able to analyze the graphs of polynomial, rational, radical, and transcendental functions (p. 180). Achieving these would enhance further training of competent people to conduct research and improve the life of a greater generation still unborn.
Despite the importance of calculus stated above, certain difficulties inhibit students from learning it, leading to unprecedented failure. As stated by Ferrini-Mundy and Graham (1991), annually, about 600,000 students enroll in calculus in four-year colleges and universities. Out of this number, half are in engineering, and less than half of them finish with a grade of D or higher. This clearly shows that calculus is either poorly learned or taught or both.

Some of the calculus-learning inhibition factors can be particularly related to the characteristics of first-year calculus course (Burton, 1989). According to Burton, first-year calculus course relies on some specific mathematics skills that students are presumed to have mastered in high school. This includes the language and skills of Mathematics symbols, diagrams, equations and formulas (i.e. the fundamentals). In comparison to characteristics of other first-year college-level courses, Burton further stressed that, college-level courses in some fields have no exact precursors in the high school curriculum, which requires the students to start from the beginning. On the other hand, there are some college-bound courses that only require a certain level of students’ maturity and experience. Neither of these situations holds for first-year calculus. Calculus taken during the first year in college clearly requires and depends on students’ actual mathematics skills that cut across algebra and geometry.

Further studies of calculus-learning complications were reported by Davis (1986) quoted by Ferrini-Mundy and Graham (1991). They noted that students learned calculus during their first year in notational form, full of manipulation and with total disregard of conceptual understanding. This point was further reiterated by the Calculus Reform Conference at Tulane University in conjunction with the January 1986 Joint Mathematical Meetings in New Orleans. Davis further claimed that “these students show every evidence of having suffered from their pursuit of a meaningless, ritualistic manipulation of symbols” (p. 36 quoted on p. 628).
Applied Calculus Optimization Problem (ACOP)

A good window in which to study the problems associated with calculus learning is applied calculus optimization problems (ACOP). ACOP is the set of word problems which requires the application of algebra, geometry, and basic differentiation skills from calculus to compute, interpret and analyze larger and smaller values of a model on some interval, and determine where the largest or smallest value occurs. ACOP is a good piece of calculus to study because:

- ACOP is conceptually self-contained. As part of calculus it calls upon specific conceptual elements that form a coherent set of ideas for students to master. We can use ACOP to examine students’ mastery of Calculus ideas.
- ACOP incorporates the full range of mathematics language fundamentals including symbols, diagrams, equations, and formulas. In this respect, we can gain a full picture of how these fundamentals are applied in calculus through examination of ACOP.
- ACOP problems are moderately difficult. It is worthwhile to study ACOP in detail because while difficult for many students, it is still relatively tractable as a part of calculus I. Its solution processes are consistent from problem to problem, and its components are definite. In contrast, related rates problems, another staple of the calculus curriculum, can become extremely complicated.

Because these three characteristics make ACOP extremely useful as an avenue to understanding students’ broader calculus difficulties, I have elected to do a dissertation study focused on this topic. This chapter contains the rationale of this study and its theoretical framework. It also includes discussion on the significance of the study.
Rationale

After graduation in 1994 with a Bachelor’s degree in Mathematics Education in Nigeria, I taught a range of courses from developmental mathematics to differential equations in a college of education. Upon arriving in the USA at Northwest Missouri State University, Maryville, and the University of Missouri-Rolla, I held the positions of a tutor and graduate teaching assistant for a period of three and half years. Within the period, I taught courses from College Algebra to Calculus. The idea of my study was conceived as a result of my teaching experience, and finalized after a series of discussions with my major adviser at LSU. I have witnessed many situations in which students are struggling with algebra and geometry problems in their attempt to solve calculus problems in general and Applied Calculus in particular. Specifically, some students in my calculus class displayed inability to handle an Applied Calculus Optimization Problem (ACOP) solution that involved drawing diagrams, formulating and transforming algebraic equations required to get the final model. As Peterson (2004) noted, quoting Tucker and Leitzel (1994), five main concerns associated with calculus study were articulated at the end of Calculus Reform Conference sponsored by the Sloan foundation at Tulane University in New Orleans, LA, in January 1986. The concerns are:

- Too few students successfully completed calculus.
- Students were mindlessly implementing symbolic algorithm with no understanding and little facility at using calculus in subsequent mathematics courses.
- Faculty were frustrated at the need to work so hard to help poorly prepared, poorly motivated students learn material that was a shadow of the calculus they had learned.
- Calculus was being required as an unmotivated and unnecessary filter by some disciplines that made little use of it in their own courses;
- Mathematics was lagging behind other disciplines in the use of technology (p.13).

Moreover, Bremigan (2005) stated that many problems traditionally included in studying single-variable calculus initially draw upon students’ abilities to interpret or construct a diagram representing some situation geometrically, with the application of calculus concepts and
procedures occurring much later in the problem-solving process. The skills required for constructing and interpreting a given situation from word problems form a cornerstone of Applied Calculus Optimization Problem. There is not much available research that documents students’ algebraic and geometric difficulties with respect to ACOP solution. In the two Mathematics branches (i.e. algebra and geometry), there are many mathematical skills and vocabulary to be developed so that students can express and develop their mathematical knowledge and understanding. In general, Mathematical language consists of symbols (e.g. variables), diagrams (e.g. plane & solid), notations, conventions and models (e.g. equations) that can only be interpreted by the mathematically literate. Furthermore, mathematical literacy can be enhanced if the teaching approach shifts from the traditional rote and skill learning to emphasis on conceptual understanding, as recommended by the Calculus Reform Work Conference (Tucker and Leitzel, 1994).

The goal of my study is to assess algebra and geometric prerequisite skills as incorporated into the ACOP solution. I will identify difficulties that students encounter in applying algebraic and geometric skills at the early stages of the ACOP solution, and postulate and validate misconceptions related to variables and equations usage. I will analyze errors with respect to drawing and symbol/transformation skills as well as those related to application of basic calculus concepts into the Applied Calculus Optimization Problem solution.

These goals will be accomplished using two methods of data collection: test and clinical interviews. The structure of the test instrument designed by the researcher will have four components: algebra, geometry, partial ACOP and full ACOP.

**Theoretical Framework**

“Models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using an external notation system, and that are used to
construct, describe, or explain the behaviors of other system(s) – perhaps so that the other systems can be manipulated or predicted intelligently” (Lesh & Doerr, 2003, p. 10). The proposed model to be used in this study is the conceptual model defined by Lesh, Landau, and Hamilton (1983) as an adaptive structure consisting of (a) within-concept networks of relations and operations that the student must coordinate in order to make judgments concerning the concepts; (b) between-concept systems that link and/or combine within-concept networks; (c) systems of representations (e.g. written symbols, drawings), together with coordinated systems of translations among and transformations within models; and (d) systems of modeling processes, which are dynamic mechanisms that enable the first three components to be used, or to be modified or adapted to fit real situations.

For the purpose of this study, the framework proposed here is an attempt to impose some structure and organization from component (d) of the conceptual model onto the ACOP solution process. This fourth component of the conceptual model consists of dynamic mechanisms that enable the first three components to develop and be adapted to everyday applications. The modeling process as shown in Fig. 1.1 below (i.e. the (d) component of the conceptual model) is adopted from Lesh et al. to fit into the ACOP solution processes. These processes include (1) simplifying the original problem situation by ignoring “irrelevant” characteristics in a real situation in order to focus on other characteristics; (2) establishing a mapping between the problem situation and the conceptual model(s); (3) investigating the properties of the model in order to generate information about the original situation; and (4) translating (or mapping) the predictions from the model back into the original situation and checking whether the results “fit”. The description of each component is given below.
1. Real situation.

As shown in figure 1.1, the real situations ACOP represented in word problem can be re-represented using systems of representation (e.g. written symbols, drawings). The ability to represent a problem situation using written symbols and diagrams is the first step in understanding the given problem.

2. Translate.

This is a process of translating the real situation into an appropriate model. This will involve associating geometric formulae/equations with an earlier created real situation in the diagrammatic form in 1 above.

3a. Model.

This is the end product of the modeling process. For the purpose of this study, the model is the required equation derived from geometric formulae/equations. They are the direct reflection of the geometric description found in the ACOP word problem.
b. Transformation.

This is the process of changing the model to fill gaps, (example, using symbol skills to get the required model).

4. Translate.

This is the last step of the solution process. In involves testing whether the final result fits into the original situation.

**Purpose of the Study**

Algebra and geometry are useful instruments for solving many mathematical problems. From past experiences as an instructor of college algebra, trigonometry and calculus, I realized that students experience great difficulty when solving algebraic equations as well as applying those skills to calculus problems in general and applied calculus optimization problems in particular. Specifically, students displayed inability to handle an ACOP solution that primarily uses prerequisite algebraic and geometric skills, examples, drawing diagrams, formulation and transformation of algebraic equation required to get the final model/equation.

To illustrate the ACOP solution steps, I will use the following example. Find the dimensions of a rectangular garden that would maximize the area if the fencing material has a perimeter of 100 feet. To find the model of an area to be maximized, we will represent this information in a diagram, a hypothetical rectangular garden with unknown dimensions, as shown in Fig. 1.2 below. This is called: drawing a diagram activity.

![Diagram of a rectangular garden with unknown dimensions](image)

Fig. 1.2
The next step in the solution process is labeling the diagram with variables, as shown in Fig. 1.2 above. Let $x$, $y$, and $A$ represent the length, width and area of the rectangular garden respectively.

Then area, $A = xy$, and perimeter, $P = 2x + 2y = 100$ or $y = 50 − x$. Substituting $y$ into the area formula, the required model of the area to be maximized is $A(x) = x (50 − x) = 50x − x^2$. This task is call symbol/transformation skills.

The calculus required here is a basic differentiation, i.e. finding the first derivative of the area model, which is: $\frac{dA}{dx} = 50 − 2x$. Setting the first derivative to zero will yield $50 − 2x = 0$ or $x = 25$. Thus, the maximum value of the area will occur at either $x = 0$, $x = 25$ or $x = 50$.

In this stage of applying basic differentiation skills into an ACOP solution, finding the first derivative, especially when the model involves algebraic fraction, as well as equating it to zero can be problematic.

The purpose of this study is to assess algebra and geometric prerequisite skills as incorporated into the ACOP solution. Generally, the study’s main goals are:

- Assess students’ prerequisite skills in setting up an ACOP problem.
- Develop a model of how prerequisite competences are integrated together into ACOP problem solving.

Specific research questions designed to address the two main goals of the study are further subdivided into three main parts: the first four research questions will investigate prerequisite algebra and geometric skills, while question 5 will examine the ability to use some or all of the prerequisite skills to obtain the required ACOP model. Question 6 will look at how some prerequisite skills are used in the ACOP solution process. The last set of the research
questions i.e. 7 will look into students’ ability to fully bring into play all the prerequisite skills into the ACOP solution process.

The research questions are:

1a. Are students able to construct a diagram to represent word problems?
   b. How do students construct diagrams to represent word problems?

2a. Are students able to label diagrams appropriate/inappropriately using variables?
   b. How do students label diagrams appropriate/inappropriately using variables?

3a. Are students able to associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?
   b. How do students associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

4a. Are students able to use symbol/transformation skills?
   b. What are students’ strengths/weaknesses in symbol/transformation skills?

5a. Are students able to marshal the above competences to find the model in an ACOP?
   5a*. Are there any relationships between finding the required equation/model of an ACOP and diagram construction; labeling the diagram; associating diagrams with geometric formulas/equations; and manipulating/transformation symbols skills?
   b. How do students use prerequisite skills to find the model in an ACOP?

6a. Are students able to do the calculus in an ACOP solution when the model is given?
   b. How do students use calculus in an ACOP solution when the model is given?

7a. Are students able to solve ACOP completely?
   b. How do students solve ACOP completely?
The quantitative components of the research questions are (a) parts while the qualitative components are the (b) parts; and research question (5a*) would be assessed quantitatively. The overall outcome from these research questions will be a synthesized model of how the prerequisites fit together to constitute a solution of ACOP.

**Significance of the Study**

Students experience great difficulty when manipulating algebraic equations to solve Applied Calculus Optimization Problems. Specifically, students display inability to handle an ACOP solution that involved drawing diagrams and formulating and transforming algebraic equation to get the final model/equation. One possible reason for students’ inability to handle an ACOP solution successfully may be an inadequate mathematical vocabulary, and inability to express their mathematical knowledge and understanding. Development of mathematical knowledge and vocabulary is hierarchical in nature. The hierarchical nature of concept development is very important in solving ACOP as the need arises. This study therefore, is intended to assess students’ prerequisite skills both algebraic and geometric in Applied Calculus Optimization Problem solutions. It will also assess basic differentiation skills required to solve ACOP either partially or completely.

The study will offer significant contributions to the mathematics education community in several ways. First, it will emphasize the relationship between conceptual and procedural knowledge in teaching ACOP. Since the study is done at the university level and with Freshmen Calculus I students, it will reveal the deficiencies (algebraic and geometric) that are carried along from high school. Moreover, it will analyze how these prerequisites are coordinated (or not) in a successful (and unsuccessful) ACOP solution. Finally, the study will evaluate students’ basic differentiation skills. ACOP is among the first topic that requires application of basic differentiation skills in Calculus I; hence it can be used to assess students’ basic grounding as
they advance in the Calculus curriculum. At the end, some recommendations would be made to
the teaching of algebra and geometry that are required in the Applied Calculus Optimization
Problem solution process.

**Definition of Terms/Acronyms**

**Algebra:** According to Mason (1996), algebra is “derived from the problems of *al-jabar*
(meaning, adding or multiplying both sides of an equation by the same thing in order to eliminate
negative/fractional terms), which were paralleled by problems of *al-muqabala* (subtracting the
same thing from or dividing the same thing into both sides)” (pg 73).

**Equation:** According to James and James Mathematics Dictionary (1976), equations are
mathematical statements that indicate equality between two expressions.

**Variable:** A quantity which may assume an unlimited number of values is called variable. A
quantity whose value is unchanged is called a constant. For example, in the equation of a circle
\[x^2 + y^2 = a^2\], \(x\) and \(y\) are variables, but \(a\) is constant (Osborne, 1909, 1, cited by Philipp, 1992).

**Geometry:** According to Battista (in press), geometry is a complex interconnected network of
concepts, ways of reasoning, and representation systems that is used to conceptualize and
analyze physical and imagined spatial environments.

**Visualization:** Arcavi (2003) claimed that “visualization is the ability, the process and the
product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our
minds, on paper or with technological tools, with the purpose of depicting and communicating
information, thinking about and developing previously unknown ideas and advancing
understanding” (p.217).

**Geometric diagrams:** Diagram is a graphic representation of an algebraic or geometric
relationship (dictionary.com)
**Transformation/symbol skills activities:** According to Kieren (in press), a description of transformational activities is: (referred to, by some, as the *rule-based* activities) – includes, for instance, collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on.

**Acronyms:**

*Diagcons:* Diagram construction.

*Labeldiag:* Label a diagram.

*Assodiag:* Associating a diagram with formulas.

*Mansym:* Manipulating symbol skills.

*Acopm:* ACOP model/equation.

*Diffov:* Differentiation of variable (partial ACOP).

*Optpbm:* Optimization problem (full ACOP).
CHAPTER TWO

REVIEW OF THE LITERATURE

This chapter reviews Calculus and some important concepts that are required or formed the foundation of the solution process of any calculus problem in general and Applied Calculus Optimization Problem (ACOP) in particular. These concepts are algebraic (variables, equations/formula), geometric diagrams, visualization, and symbols/transformation skills. The overall purpose of the chapter is to review literature and unveil the difficulties that students encounter in learning and understanding these concepts, as well as the application of symbol skills/transformation.

The chapter contains four parts. First part will define algebra followed by literature review on some algebraic concepts (variables, equations/formula). The second part will define geometry as well as review literature on some concepts (diagrams, visualization). The third part will review literature on symbols/transformation skills; the final section will discuss on Calculus and review its’ literature.

What is Algebra?

According to Mason (1996), algebra is “derived from the problems of al-jabar (meaning, adding or multiplying both sides of an equation by the same thing in order to eliminate negative/fractional terms), which were parallel by problems of al-muqabala (subtracting the same thing from or dividing the same thing into both sides)” (p 73). It can be inferred from above definition that algebra is confine to limited view, restricted only to the process of solving an equation. Overtime, the meaning of algebra was developed and broadened, shifting from algebra as a process to algebra as an object. Another attempt in the process of defining algebra was made by Wheeler (1996). He describes algebra “as symbolic system (its presence is recognized by symbols), a calculus (its use in computing numerical solution to problems), and
also as a representational system” (it plays major role in the mathematization of situations and experiences). Wheeler’s description has a positive implication to this research.

To some people algebra is a collection of symbols, rules and procedures, while to Mathematicians; it is much more than that. From the perspectives of Kieren (1992), algebra is conceived as a branch of Mathematics that deals with symbolizing and generalizing numerical relationships and mathematical structures, and with operating within those structures.

From other perspectives, algebra is about identifying patterns and generalizing those patterns. Generalizing involves seeing a pattern, expressing it clearly in verbal terms, and then using the symbols to express the pattern in general terms. According to Sfard cited by Bednarz et al (1996), affirms that most authors unanimously agree to the early origins of algebra because they “…spot algebra thinking wherever an attempt is made to treat computational processes in a somehow general way” (pg. 103).

Algebraic concepts to be reviewed in this part are variables and equations/formula.

**Variables**

In this subsection, the definition, classification, and researches conducted to investigate how students learn variables; difficulties and misconceptions operating with the variables will be discussed.

According to Sfard (1995), historians seem united in the opinion that the 16th century French Mathematician François Viète was the first to replace numerical givens with symbols. Viète was the inventor of parametric equations, equations with literal coefficients, i.e. variables. Students encounter the concept of a variable for the first time in introductory algebra or beginning algebra. This is followed by learning of algebraic terms and expressions, before finally getting into equations in college algebra, and general Mathematics. This arrangement is based on the fact that equations involve expressions and terms. Again, expressions and terms in turn
contain variables. Understanding the concept of a variable is fundamental to the study of algebra which in turn forms the foundation of solution of different types of problems that lead to formulating an equation or set of equations.

According to Wagner & Parker (1993), citing the research of Wagner (1983) shows that students can work with variables without fully understanding the power and flexibility of literal symbols. Because variables operate much like the numbers in Arithmetic, and because conceptually they resemble pronouns in ordinary language, most students can acquire some facility in routine algebraic manipulations. On the other hand, variables are different from numerals, example, a variable can represent many numbers simultaneously, as in \([0 \leq n \leq 20]\). Moreover, variables can be used to indicate multiplication (juxtaposition) as in \([3mn]\), in contrast to the place value interpretation that we give to numerals alone, as in \(258\). They are different from words with regards to their consistency of meaning throughout a single context. According to mathematical convention, stated by Wagner, the meaning and, in particular, the value ascribed to a literal symbol must be the same wherever that symbol appears in a given context. That is, in substituting values for \(x\) in the sentence \(3(x+2) + 5 = 17 – 2x\), the same value must be substituted wherever \(x\) occurs. In contrast to this consistency property of variables in a single context, identical words or phrases may refer to different things within a single sentence.

Moreover, according to Wagner & Parker, variables are versatile. As they have stated, variables can be used as names for numbers or other objects; as discrete unknowns in equations; as continuous unknowns in inequalities; as indeterminate in polynomials; as generalized numbers in identities; as independent and dependent variables in functions; as parameters in formulas; and so on. Understanding this classification by the students is very important and useful to the teaching of calculus and applied calculus optimization problem (ACOP) in particular. If we look at variables from the viewpoint of the roles they play in algebra, Usiskin (1988)), and the ways
that students operate with them (Kuchemann 1978), cited by Wagner & Parker (1993), it is clear that the concept of variable is in fact a multifaceted idea. Shortlists of literal symbols given by Wagner & Parker (1993) are: Unknown, Variable, Constant, Parameter, Generalized number, Name, Placeholder, Argument, and Indeterminate.

In a study conducted by Kuchemann (1981), he observed six stages through which students progress in acquiring a mental model of a variable and are identified as follows:

- **Letter evaluated:** In this case students avoid operating on a specific unknown and as such simply assign a numeral value to the unknown from the outset. The student may recall any number or recall the number fact about the expressed relationship.

- **Letter not used:** here a student may just ignore the existence of the letter, or at best acknowledge it, but does not give it meaning. For example, if \(a + b = 5\), \(a + b + 2 =?\) and the student gives & as an answer.

- **Letter used as an object:** in this case the student regards the letter as mnemonics for the objects in its own right. For example, ‘4a’ as ‘4 apples’. At this level students are able to regard expressions like \(5 + 4a\), \(p + 1\) as meaningful.

- **Letter used as a specific unknown:** the student here regards the letter as specific but unknown number, and can operate on it directly.

- **Letter as generalized number:** the letter is regarded here as representing, or at least being able to take several values rather than just one value.

- **Letter used as a variable:** this is the final stage where the student sees the letter as representing a range of unspecified values and understands that a relationship exists between two such sets of values.

Other findings from the research of Kuchemann indicates that greater number of students between the ages of 13 – 15 years treated the letters as specific unknowns than as generalized
numbers; despite the classroom experiences they had in representing number patterns as
generalized statements. Majority either treated the letters as objects or ignored them.

Another major area of research on variables was on students’ difficulty operating with
variables as well as its misconceptions. Kuchemann (1981) cited by White & Mitchelmore
(1996) suggest that very few students understand variables at his highest conceptual level. In a
related development, Eisenberg (1991) agrees and in support cites the research of Wagner
(1981), who showed that 15% of 16-year-old students treated two equations as totally different
when the only difference was the letter used to represent the variable. Such a superficial
understanding of variables is in line with Kieran’s (1989) view cited by White & Mitchelmore
(1996) that one of the main difficulties in learning algebra centers on accommodating the
meaning of letters involved, i.e. variables. Booth (1989) cited by White & Mitchelmore suggest
that the required meaning [of variable] is often neglected in the teaching and learning of algebra,
so that many students only learn manipulation rules without reference to the meaning of the
expressions being manipulated. It is a matter of some interest to find out whether students who
aspire to advanced mathematical thinking involved in calculus and [applied calculus
optimization problem in particular] have an adequate concept of a variable, White &
Mitchelmore (1996), this being one of the fundamental questions to be answered in this study.

Another milestone in investigating students’ misconception of variable is the famous
student and professor problem. In a research conducted by Rosnick (1981) on students’
 misconception concerning the concept of variable, much of the research has been based on the
student and professor problem. It was an extension of a body of research done by the Cognitive
Development Project at the University of Massachusetts. It reads as follows: [write an equation,
using the variables S and P to represent the following statement] “At this university there are six
times as many students as professor”. Use S for the number of students and P for the number of
Professors. The result indicates that 37 percent of a group of 150 entering engineering students at the University of Massachusetts were unable to write the correct equation, $S = 6P$, in any form, [i.e. not using other letters representing number of students and professors]. Moreover, students oriented towards business and social sciences did appreciably worse on these problems. According to research report, many students who write $6S = P$ believe that $S$ is a label standing for students rather than a variable standing for number of students. They will read the equation $6S = P$ as “there are six students for every one professor”, pointing to $S$ as they say students and $P$ as they say professors. Conversely, they will read $S = 6P$ as “one student for every six professors” instead of the appropriate “the number of students is equal to six times the number of professors”. The letters in equations can stand abstractly for number may sound obvious to the initiated, but it is apparently not at all obvious to the students. This result supports the hypothesis that students tend to view the use of letters in equation as labels that refer to concrete entities. Furthermore, these results also underscore the fact that students do have a great deal of difficulty with translation. Further opinion suggests that it is equally important that Mathematics educators should be aware of the distinction between writing “$P = $ professors and $P = $ number of professors”, or similar situations that are applicable, Rosnick (1981).

Acknowledged above are misconceptions and difficulties experienced by students in understanding the concepts of variable from the lens of algebra. Base on the importance variables have in ACOP solution, it would be necessary and essential to conduct research to investigate students’ difficulties and misconception of variables from ACOP perspectives.

**Equations/Formula**

In this subsection, the concepts of equations, formula, function or generalization would be used interchangeably. The definition, importance as well as three areas of researches on equations/formula i.e. difficulty formulating algebraic equations from word problems; the
significance of equality sign in equations; and preference of solution techniques of equations will be discussed.

**Definition and importance of equations**

According to James and James Mathematics Dictionary (1976), equations are mathematical statements that indicate equality between two expressions. Equations may express identities or conditional relationships between numbers and/or variables. Example, 
\[(x + 1)(x - 1) = x^2 - 1, \forall x\]  
i.e. difference of two squares is an identity, whereas an example of a conditional equation is \[(x + 1)^2 = 2x^2 + x + 1,\] where the only solutions are 0 & 1. An identity is a statement that is true for all values of the variables, except for those values of the variables for which each member of the statement of equality does not have meaning (James and James, 1976), while conditional equation is the one that is true for certain values of the variables involved. From another perspective, Hercovies & Kieren (1980) define equations from two lens, Arithmetic identities and Algebraic processes. Using the Arithmetic identities approach, equation is define as an Arithmetic identity with a hidden number, whereas the algebraic process defines an equation as any algebraic expression of equality containing a letter or (letters), i.e. [a variable or variables]. Each of these definitions have advantages depending on the context.

One of the important concepts in algebra is equation, and teaching of algebra in turn forms a significant component of the NCTM curriculum standards. It was emphasized in the curriculum that continued study of algebraic concepts such as [equations, variables] would help students represent situations that involve variable quantities with expressions, equations …, and use tables, graphs [and calculus] as tools to interpret expressions, equations, NCTM (1989). The success of any applied calculus optimization problem (ACOP) relies heavily on a good
understanding of above concepts in algebra, i.e. equation and variable. To stress the importance of variables and equations, Rosnick & Clement (1980) opined that the fundamental concepts of variables and equations should not be treated lightly in high schools and colleges, nor should we assume that our students will develop the appropriate concepts by osmosis.

**Formulating/generating equations**

Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty for algebra students, Kieren (2007). Word problem situations not only continue to be used as a means for infusing algebraic objects with meaning but have also received increased emphasis in reform programs as vehicles for introducing students to algebra. According to Kieren (2007), research in this area continues to provide evidence of students’ preferences for arithmetic reasoning and their difficulties with the use of equations to solve word problems (e.g., Bednarz & Janvier, 1996; Cortés, 1998; Swafford & Langrall, 2000). Is this possible in the solution of ACOP? Stacey and MacGregor (1999) as cited by Kieren (in press) found that, at every stage of the process of solving problems by algebra, students were deflected from the algebraic path by reverting to thinking grounded in Arithmetic problem-solving methods. Other researchers have concentrated on the difficulties associated with two major ways of translating equation from verbal data, Stacey & MacGregor (1993).

Hercovies (1989, pg 65.) cited by Stacey & MacGregor (1993) referred to these two procedures as syntactic and semantic translation. The so-called syntactic translation is accepted as a procedure frequently used by students for formulating equations from the natural language expressions, and is thought to be an important source of errors, particularly the reversal error, but conversely Kirshner et al (1991), cited again by Stacey & MacGregor (1993) affirmed that syntactic translation is a useful algebraic skill. To support the position of Kirshner, that syntactic translation is useful, Stacey & MacGregor (1993) referring the work of MacGregor, (1991);
Pimm (1981); and Schoenfeld, (1985) say that syntactic translation is valid, since textbooks teach students to form number sentences or algebraic equations by matching specific verbal cues to mathematical symbols from left to right.

According to Mestre (1988); MacGregor (1991), the sequential left to right method of translation to algebraic equations is a common procedure taught to students with little regard for meaning. In a related development, Stacey & MacGregor (1993) citing Spanos et al (1988) say that ‘from the data collected on the procedures used by 46 students working on problems in small groups found that “students often attempt to duplicate the surface word order in rendering equations into symbolic notations”.’ In the work of Laborde (1990) from literature on the interaction of language and Mathematics referring the work of Clement (1982, pg. 61), wrote that “according to Clement, [the] …variable-reversal error appeared to stem from using a left to right translation of the problem statement”. Similarly, Cocking & Chipman (1988, pg. 30) referred to “direct sequential word-for-symbol mapping” as cause of reversal, and didn’t mention any alternative causes. The description of syntactic translation stress that the translation is carried out by a word-for symbol mapping respecting word order.

The second way of formulating equations from verbal data is semantic translation. Hercovies (1989) referring to this process as requiring more than semantic knowledge alone. In order to understand a statement before representing it algebraically a combination of several processes including the application of syntactic, semantic, and pragmatic rules is involved. Stacey & MacGregor (1993) referring the work of Kirshner et al (1991) used the expression semantic/conceptual to describe this translation.

Another aspect that contributes to students’ difficulty in translating word problems into equations is language. Aspect of the natural language in which a mathematical relation is expressed may interfere with the process of translation into an algebraic representation, Stacey &
MacGregor (1993). These aspects include complex syntax, the order of items of information, and
the degree to which relations are made explicit. Stacey & MacGregor (1993) was quoting the
work of Laborde (1990) in the interaction of language and Mathematics learning, he concluded
that features of an expression in natural language can obstruct and mislead the translator.
Moreover, Kaput (1987) cited by Stacey & MacGregor, stated that the major causes of the
reversal error are the powerful and automatic use of natural-language rules of syntax and
reference.

The significance of equality sign in equations

Many students right from the elementary level up to college level have failed to interpret
the equal sign as a symbol denoting the relation between two equal quantities. They consider the
equality sign as a command to carry out a calculation Kieren (1981); Wagner & Parker (1993).
Again Kieren (1981) claimed that the procedures used by students to solve equations (i.e. using
equality sign as a separator) and to find the derivative of a function would seem to indicate that
high school and college students may also tend to interpret the equal sign in terms of an operator
symbol,… rather than as a symbol for an equivalence relation.

Another approach to writing of equations observed by Clement, Lochhead, and Monk
(1981) cited by Stacey & MacGregor (1993) is called static comparison. In this approach, the
equation is used to represent an association of related groups rather than equal numbers. Under
this interpretation, the equal sign denote correspondence or association rather than equality. In
another twist, with in the cognitive science perspective and referring the work of Davis (1984);
Stacey & MacGregor (1993) attributed the reversal error to a difficulty in selecting between two
frames:
the labels frame, dealing with label or units (e.g., 1m = 100 cm); and

the numerical-variables equation frame, dealing with relations between numbers (e.g., 

\[ x = 100y \]).

The belief that algebraic letters are abbreviated words or labels for objects (e.g., P means “professor” or symbolizes one professor) has been shown to be a common misconception Booth, 1984; Kuchemann, 1981; Malle, 1985; Mestre, 1988; cited by Stacey & MacGregor (1993). It was well documented that students misunderstand the structure and meaning of equations and the significance of the equals sign (Hercovies and Kieren, 1980; Kieren, 1988; also cited again by Stacey & MacGregor (1993). The common perception of the equal sign as a separator symbol (Kieren, 1981) is relevant to this study.

If misinterpretation of algebraic letters together with a loose interpretation of equality is an important cause of difficulty, it seems likely that the incidence of errors in formulating equations would be reduced if students were prompted to think of letters as standing for numbers and not words Stacey & MacGregor (1993); Rosnick & Clement (1980).

**Preference of solution techniques in equations**

According to Nathan and Koedinger (2000a), cited by Kieren (2007), word problems presented in verbal form are easier for students to solve than comparable questions presented in other formats, such as equations or “word-equations.” These researchers presented a set of 12 problems to a group of 76 high school students in the following formats: Two were in story-problem format (e.g., “When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?”), two were in equation format (e.g., “Solve for x: 

\[ x \cdot 6 + 66 = 81.90 \]”), and two were in word-equation format (e.g., “Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What did I start with?”). For one problem in each
format, the start value was unknown, and for the other, the result was unknown. The equation format was found to be significantly less likely to be correctly solved than either the story-problem or word-equation formats, whereas algebra story problems and algebra word-equation problems were found to be of equal difficulty. In a replication study involving 171 students (Nathan & Koedinger, 2000a), solution success rates for symbolic equations were 25% less than for story problems and nearly 20% less than for word equations. In both of these studies, the start-unknown format was more difficult than the result-unknown problems. Students’ difficulties with problems presented in equation contradict the most favored claim, according to these researchers, that story (word) problems are naturally harder than symbolic ones. The many possible ways of solution processes and the accessibility of a variety of solving approaches for story problems are judged to be factors responsible for the higher success rates with the verbal format.

In a follow-up study, Koedinger and Nathan (2004) further explored their earlier finding that students are more successful solving simple algebra story problems than solving mathematically equivalent equations. They found proof that this result is not simply a consequence of situated world knowledge helped problem-solving performance, but rather as a result of student difficulties with understanding the formal symbolic representation of quantitative relations. In fact, students translated story problems into the standard equation format only 5% of the time. According to the researchers, students even after a full year of algebra were particularly tested by the demands of understanding the letter-symbolic form of the equation: “The language of symbolic algebra presents new demands that are not common in English or in the simpler symbolic Arithmetic language of students’ past experience” (p. 149). The researchers suggested that more attention be focused on equation-solving instruction;
however, they emphasized that this be done after students have had experience with translating back and forth between English and algebra within story problem situations.

**Formula**

Formula is a symbolic statement of relationship between two or more variables, Butler, C. H. et al (1970). This definition can be characterized to reflect the function of an equation. On the basis of the features of this definition and for this study, equations and formulas are the same. Again, according to Macmillan school Dictionary, a formula is group of letters, numbers, or symbols that represent a rule in Mathematics or science. It provides an ideal medium for the transition from the earlier work to the more formal and systematic aspects of algebra and calculus. According to Butler (1970), formula involves a great many of the concepts of elementary algebra; the symbolic language of constant and variables; the concept of dependence and function; substitution and evaluation; operation with literal symbols, fractions and so on. Again, he says that a formula forms a core which has points of contact not only with the previous experiences of the students, but with many of the topics, example applied calculus optimization problem (ACOP).

Furthermore, Butler claimed that, previously in Arithmetic and informal geometry students must have contact with formulae such as those for mensuration of the simplest and most common geometric forms. It should be assumed that the student brings to his first course in calculus enough background to enable him to use as a point of departure in his work in [applied calculus optimization problem]. His or her further study of the formula in turn serves to familiarize him, and gives him experience in, the progressive mastery of the new language, concepts symbolism, and operation of algebra and calculus.

Butler did not stop there; he maintained that the main things a student should get from his more formal study of formulas are:
An understanding and appreciation of the nature and significance of the symbolism of algebra.

An appreciation of the fact that a formula is merely the translation of an English sentence into symbolic form.

A clear concept of the meaning of a constant and a variable and the distinction between the two.

A clear concept and appreciation of dependence and the meaning of relationship of independent and dependent variables.

The ability to set up simple formulae expressing relationships existing in situations within the student’s experience.

Facility and accuracy in substitution and evaluation of formulas.

The ability to represent graphically the relationships indicated by formulae involving two variables.

The ability to solve formulae, i.e., to transform an implicit relationship into an explicit relationship through application of the laws of algebraic operation.

Even though there were no direct researches investigating formula, it can be opine that it was due to sharing similar characteristics with equations; hence, equations subsume the formula.

What is Geometry?

According to Battista (2007), geometry is a complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imagined spatial environments. “Geometric reasoning consists, first and foremost, of the invention and use of formal conceptual systems to investigate shape and space” (Battista, 2001a; 2001b; & 2007).
In this subsection, geometric diagrams will be defined, as well as discussion on important research areas. These research areas are problems of geometry teaching and learning and significance of geometric diagrams, and research report on students’ performance in geometry.

**Definitions of geometric diagrams**

Diagram is a plan, sketch, drawing, or outline designed to demonstrate or explain how something works or to clarify the relationship between the parts of a whole. From mathematical point of view, diagram is a graphic representation of an algebraic or geometric relationship (dictionary.com)

According to Usiskin (1997) quoted by Pennsylvania Council of Teachers of Mathematics (PCTM, 2005) analyses of geometry concepts suggest that:

- Geometry is a branch of Mathematics that connects Mathematics with the real, physical world.
- Geometry is a branch of Mathematics that studies visual patterns.
- Geometry is a vehicle for representing phenomenon whose origin is not visual or physical.
- Geometry uses the mathematical language for describing space.

The Mathematics curriculum is being re-conceptualized to model conceptual growth from informal/intuitive understanding to generalized/formal knowledge, PCTM (2005). The focus of the new Mathematics strand cited by PCTM, (2005) as initiated by NTCM, MAA, U.S. department of Education through National Assessment of Educational Progress (NAEP) and National Voluntary Mathematics Test (NVMT) throughout the grade-levels is to develop the learners thinking abilities, use the structure and nature of the geometric concepts, to analyze and solve problems that arise in their everyday activities. These are areas that can serve as foundations upon which teaching and learning of ACOP can succeed, at least from geometrical
perspectives. Two and three dimensions spatial sense is a fundamental component of the early study and assessment of geometry, PCTM, (2005). Once students understand spatial relationships, they can use the dynamic nature of geometry to connect Mathematics to their world. Geometry is increasingly used to model and solve real-world problems, often connected to algebraic representations through a coordinate system and formulae, PCTM, (2005). Moreover, as emphasized by PCTM, (2005), students are expected to have had much experience with basic identification of shapes such as triangles, circles, and rectangles, and with identifying and measuring lines segment and angles. These skills are essential in the successful solution of any given ACOP.

Despite huge importance associated with geometry in our life both in school and the greater outside world, Usiskin, Z (1987) indicated that geometry is faced with performance and curriculum problems. This was in addition to an earlier claim made about twenty years ago by Carl Allendoerfer (1969), which he says that:

“The mathematical curriculum in our elementary and secondary schools faces a serious dilemma when it comes to geometry. It is easy to find fault with the traditional course in geometry, but sound advice on how to remedy these difficulties is hard to come by….curricular reform groups at home and abroad have tackled the problem, but with singular lack of success or agreement… We are therefore, under pressure to do something about geometry; what shall we do? (p.165; cited by PCTM, 2005).

In a related development, the result of TIMSS (1998) which the international average was 45% correct; placing US average at 32% correct, buttress the fact that the challenges identified by Allendoerfer and others have not been met (PCTM, 2005).

In an attempt to handle the recurring problem faced by teaching and learning of geometry in our schools, a developmental model of geometric thought was presented in the research of
Dina van Hiele-Geldorf and Pierre van Hiele, for which both received international recognition (PCTM, 2005). Even though both paradigms are recognized, the van Hiele model is probably the most helpful paradigm for planning the geometry strand in K-12 instruction. This model identifies the geometric thinking process in five levels: 1) visualization [recognizing and naming the figures]; 2) analysis [describing the attributes]; 3) informal deduction [classifying and generalizing by attributes]; 4) deduction [developing proofs using axioms and definitions]; 5) rigor [working in various geometrical systems], PCTM, (2005). This philosophy asserts that learners move sequentially from visualization towards the rigor level, but this being contradicted by the traditional reality in which geometry is viewed as a high schools course where most students are exposed to formal, abstract level with little or no regard for their appropriate conceptual readiness, PCTM, (2005). In accordance with NCTM (1989) standards, the geometric curriculum should develop in students the ability to deduce arguments expressed orally and in sentence or paragraph form; solve real-world applications and modeling.

The importance that a geometric diagrams can serve are multifaceted, and among them are organizing given numerals (i.e. labeling the diagram), spatial, and relational information; defining variables; and identifying physical constraints, Bremigan (2005). The diagram drawn or [constructed] represents a specific case and, when generalizing, the problem solver must discern which relationships illustrated in the single diagram remain fixed (i. e. independent), and which remain a variable, Bremigan (2005), and finally when evaluating the solution for reasonableness, the solver must take into consideration that the diagram was probably not constructed to scale.

**Geometric performance**

According to Battista (2007), the concept of measurement is woven throughout the fabric of geometric conceptualization, reasoning, and application. Measurement is critical for understanding the structure of shapes, using coordinate systems to determine locations in space,
specifying transformations, and establishing the size of objects. Geometric measurement is also embedded in the graphic representation of functions and algebraic equations.

Moreover, Battista (2007) citing the work of Kloosterman et al, (in press); Martin & Strutchens, (2000); Sowder et al, (in press), stated that, despite the importance of geometric measurement, students’ performance on measurement tasks is alarming low. According to NAEP website reported by Battista (2007), citing the study of Sowder et al, (in press), in 2000, only about 14% of grade 8 students could determine the number of square tiles it takes to cover a region of a given dimensions, and only about 25% could determine the surface area of a rectangular solid. On volume, in 1990, only 55% of 12th graders and 41% of 8th graders knew that a measurement of 48 cubic inches for a rectangular box represented volume.

The implication of these research findings stated above for majority of students is that, there is a basic disconnection between spatial and measure-based numerical reasoning, Barrett & Clements (2003); Battista (2001a); Clements et al, (1997), cited by Battista (2007). That is, many students do not properly maintain the connection between numerical measurements and the process of unit-measure iteration. For example, most students correctly use the formulas for the area of a rectangle or volume of a right rectangular prism in standard problem contexts, neither understand why the formulae work nor apply the formulae appropriately in nonstandard contexts. Another implication of the research result is that, because of an inappropriate connection between student’s spatial structuring and the numerical procedure, they did not understand that the mathematical formula applied was inappropriate. Indeed, this is a common problem for many students. To generalize the findings, Battista (2007) concluded that students working with nonstandard measurement problems (Collins Problem) must perform two critical processes: (a) they must construct a proper spatial structuring of the situation; (b) they must coordinate their spatial structuring with an appropriate numerical scheme. Too often, students skip the first
process and proceed directly to the second. Also, even when students recognize that they must perform the first process, they often have difficulty doing so. That is, because many traditional curriculum prematurely teaches numerical procedures for geometric measurement, students have little opportunity to think about the appropriateness of the numerical procedures they apply, and they have insufficient opportunities to develop skill in spatially structuring arrays of measurement units. In particular, even when students recognize that the volume formula is inappropriate for the nonstandard problem (Collin Problem), they have significant difficulty constructing an appropriate spatial structuring of the packages because they have had so few opportunities to develop their structuring skills. Battista citing the works of Chappell & Thompson (1999); Nunes, Light, & Mason (1993); Pesek & Kirshner (2000); Woodward & Byrd (1983) state that the research also shows that students commonly interchange measurement units or computational procedures.

**Visualization**

This subsection will define visualization and discuss its trends of research. The trends of research have representational and interpretational roles.

**Definition of Visualization**

According to Hershkowitz (1990), “visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information” (p.75), whereas Zimmermann & Cunningham (1991) view visualization as “the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding” (p.3). In another twist, Arcavi (2003) paraphrased the two definitions and claimed that “visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and
communicating information, thinking about and developing previously unknown ideas and advancing understanding” (p.217).

**Visualization as representational and interpretational tool**

The role of visualization as representational and interpretational tool has been the ongoing area of interest among algebra researchers (e.g., Arcavi, 2003; Dreyfus, 1991). It was reported that, understanding fundamental calculus concepts (e.g., limits, derivatives, and integral) requires the use of visual representations, and the ability to successfully solve many problems with calculus is dependent on visual images in the form of diagrams or graphs Zimmerman (1991); cited by Bremigan (2005). Prerequisites identified by Zimmerman presented in the work of Bremigan for visual thinking in calculus[(ACOP)] include the ability to extract specific information from diagrams, an understanding of algebra (variables, equations) and geometry (plane and solid) as alternative languages for the expression of mathematical ideas, and knowledge of the rules and conventions associated with mathematical graphs. More often, calculus teachers assume that their students have these prerequisites skills and that students appreciate the important role of reasoning with visual representations, Bremigan. In a contrary view, Bremigan (2005) reported the opinion of Eisenberg & Dreyfus (1991) stated that students’ reluctance to visualize is demonstrated throughout their study of Mathematics, and this behavior is particularly disturbing when displayed by students studying calculus. This finding can be validated using applied calculus optimization problem (ACOP) in this study.

According to Kieren (2007), and drawing on findings by Schoenfeld, Smith, and Arcavi (1993) and Moschkovich, Schoenfeld, and Arcavi (1993), Knuth (2000) examined 9th- to 12th-grade students’ understanding of the concept that the coordinates of any point on a line will satisfy the equation of the line, within the context of problems that require the use of this knowledge. Knuth found an overwhelming reliance on letter-symbolic representations, even on
tasks for which a graphical representation seemed more appropriate. The findings indicated to Knuth that for familiar routine problems many students have mastered the connections between the letter-symbolic and graphical representations; however, such mastery appeared to be superficial at best. Visualization in advanced problem solving by both college students (the novices of the study) and Mathematics professors (the experts) was the focus of a study reported by Stylianou and Silver (2004). The two groups judged visual representations likely to be useful with different sets of problems; furthermore, when it comes to actual usage, the experts constructed visual representations much more often than did the novices.

The connection between algebraic and geometric aspects of slope, scale, and angle was investigated, as cited by Kieren (2007). According to the research report, Zaslavsky, Sela, and Leron (2002) found evidence of much confusion regarding the connection between algebraic and geometric aspects of slope, scale, and angle. Participants, who included 11th-grade students as well as teachers, Mathematics educators, and Mathematicians, responded to a simple but nonstandard task concerning the behavior of slope under a nonhomogeneous change of scale. Results indicated two main approaches – analytic and visual – as well as a combination of the two. The researchers recommended that instruction on slope distinguishes between the erroneous conception of visual slope – the slope of a line (for which the angle is a relevant feature) – and the analytic slope – the rate of change of a function.

According to Lean and Clements (1981) cited by Fennema and Tartre (1985), students who process mathematical information by verbal-logical means outperform students who process mathematical information visually. This contradicts the idea that spatial visualization skills are highly important in the learning of Mathematics and that the development of such skills should become a major goal of Mathematics education, (Fennema and Tartre (1985). Even though
skeptics have questioned the inclusion of spatial visualization as a major mathematical goal, but it is very relevant to the teaching of applied calculus optimization problem (ACOP).

**Transformation/Symbol Skills Activities**

This subsection will describe symbol skills/transformational activities, and also discuss the trends of research in the field. The research areas are: nature of the manipulative processes used in expression simplification, equation; solution of systems of equations; and theoretical elements of transformational activities.

Kieren (2007) gives a description of transformational activities as: (referred to, by some, as the rule-based activities) – includes, for instance, collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on. A great deal of this type of activity is concerned with changing the symbolic form of an expression or equation in order to maintain equivalence. In addition to developing meaning for equivalence, this activity also includes meaning building for the use of properties and axioms in the manipulative processes themselves.

**Nature of the manipulative processes used in expression simplification and solution of equation**

Earlier research provides evidence that simplification of algebraic expressions create serious difficulties for many students Linchevski & Hercovies (1996). Students experience serious problems in grouping or combining like terms. Whereas in Arithmetic, operations yield other numbers, algebra operations may yield algebraic terms and/or expressions. Expressions encapsulate a process as instructions to calculate a numerical value, but they are also a product as objects which can be manipulated in their own right French (2002). As further claimed, “failure
to appreciate this dual nature of expressions is a major barrier to success in algebra” (p.24). Tall and Thomas cited in French (2002) and Nickson (2000), identified four obstacles frequently met by students in making sense of algebraic expressions. These are:

- The parsing obstacle;
- The expected answer obstacle;
- The lack of closure obstacle;
- The process-product obstacle.

To some students, the addition sign (+) signals an instruction to do some calculation; they are expected to produce an answer. This is what is referred to as the expected answer obstacle. The way we read from left to right is also noted to influence students to interpret for example $6 + 4x$ as saying ‘add 6 and 4 and then multiply by $x$’. This obstacle is what is termed the parsing obstacle. This obstacle also leads students into reading ‘$ab$ as $a$ and $b$ and thereby end up thinking that it is the same as $a + b$. Students show discomfort when they have to accept, say $6 + 4x$ as a final answer after some algebraic manipulations is said to be due to ‘lack of closure’ obstacle. To the students this is an incomplete answer. The process – product obstacle refers to students’ failure to appreciate the dual nature of algebraic expressions i.e. expressions can indicate an instruction and at the same time they can represent the result of the operations French (2002).

The above mention problems associated with simplification of algebraic terms and expressions lead to further complex problems that are usually found when students solve equations. Students who have insufficient conceptual knowledge about terms and expressions experience serious problems when they have to read and interpret the symbolic form of equations. Students are usually not able to make sense of the algebraic equations, as they do not really understand the structure of the relations in the equation Kieren (1992). Algebraic
manipulations include processes such as simplifying brackets, collecting like terms, factoring, etc. in solving equations these processes are performed when transforming the original equation to its simpler equivalent forms. Solutions of equations involve both procedural and structural operations Kieren (1992). Procedural refers to Arithmetic operations carried out on numbers to yield numbers. For example, if we take the algebraic expression, $3x + y$, and replace $x$ and $y$ by 4 and 5 respectively, the result is 17. The term structural, on the other hand, refers to a different set of operations that are carried out, not on numbers, but on algebraic expressions. For example, if we take the algebraic expression $3x + y + 8x$, this can be simplified to yield $11x + y$ or divide by $z$ to yield $\frac{11x + y}{z}$.

Solution of systems of equations

According to Kieren (2007), researchers have known very little about the ways in which students approach the solving of systems of equations and the manner in which they think about its underlying concepts. The recent research of Filloy, Rojano, and Solares (2003, 2004) has focused on the spontaneous approaches of 13- and 14-year-olds, who had already been introduced to the solving of one-unknown linear equations, to problems that could be solved by systems of equations. One of the aims of this research was to document the transition from one-unknown representations and manipulations to the representation and manipulation of one unknown given in terms of the other unknown. It was found that students seemed more inclined to make sense of comparison approaches than substitution methods; however, the researchers noted that manipulation difficulties contributed to making the substitution method less accessible. They also observed that the extension of the notion of transitivity of equality from the numeric to the algebraic domain, as well as the idea of substituting one expression with another, was not at all obvious to students.
Further evidence of students’ difficulties cited by Kieren (2007) with the substitution method for solving systems of equations was found by Drijvers (2003). Kieren reported that, within a Computer Algebra System (CAS) environment, 14- and 15-year-olds were asked to solve parametric equations, for example, “Solve $ax + b = 5$ for $x$.” Students experienced difficulty in accepting the expression $(5 - b)/a$ as a solution. According to Drijvers, this required that they conceptualize an expression as an object (Sfard, 1991). Other tasks such as, “Consider the equations $y = a - x$ and $x^2 + y^2 = 10$. Make one equation from these in which $y$ does not appear; you do not need to solve this new equation” (p. 260), which required substituting a variable in one equation by an expression drawn from the other equation, were equally problematic.

Another research reported by Kieren (2007) investigates the transformational activities related to quadratic equations. This is an important result with respect to solution of ACOP. According to Kieren, citing the work of Vaiyavutjamai, Ellerton, and Clements (2005), nearly 500 students from Year 9 classes in Thailand, Year 10 classes in Brunei Darussalam, and 2nd-year university students in the United States “attempted to solve the same quadratic equations, all of the form $x^2 = K$ ($K > 0$) and $(x - a)(x - b) = 0$ (where $a$ and $b$ are any real numbers)”. All students had already learned to solve such equations before participating in the study. The responses to the second type of equation, in particular, suggested the presence of serious gaps in the theoretical thinking underpinning students’ work when solving such equations. For example, in solving $(x - 3)(x - 5) = 0$, several students who correctly solved the equation checked their solutions by substituting $x = 3$ into $(x - 3)$ and $x = 5$ into $(x - 5)$ and concluded that because $0 \times 0 = 0$ their solutions were correct. Related difficulties were encountered by the U.S. students who were asked to respond True or False to the following statement: This equation
\((x - 3)(x - 5) = 0\) is equivalent to \(x^2 - 8x + 15 = 0\), which is a quadratic equation with two solutions. Thus, with \((x - 3)(x - 5) = 0\), the \(x\) in the first brackets always equals 3, and the \(x\) in the second brackets always equals 5. Fifty-five percent of respondents answered that this was indeed True. To another question, many were unsure whether the \(x\) in the “\(x^2\) term” represented the same variable as the \(x\) in the “\(x\) term” of the equation \(x^2 - 8x + 15 = 0\). The finding that most of the students in this study were confused about how the solutions of a quadratic equation related to the equation itself led these researchers to suggest that, if quadratic equations are to remain an important component of middle- and upper-secondary Mathematics curricula, then research is needed to guide teachers about how students think about quadratic equations.

**Theoretical elements of transformational activities**

Kieren (2007) claimed that, element of theoretical control that is considered basic to transformational activity is the knowledge that relates the algebra with the arithmetic. For example, numerical substitution activity within expressions and equations can help students make connections between the arithmetical and the algebraic world. Kieren reported that, Graham and Thomas (2000) conducted a teaching experiment with nearly two hundred 13- and 14-year-olds in which students used the letter stores of the graphing calculator as a model of a variable and so evaluated different expressions for a variety of inputs. In the case of equivalent expressions, students came to see that different expressions were being used to represent the same process. This activity also had an impact on students’ views of expressions and variables, which suggests that a task that is transformational in nature can simultaneously be related to generational activity if it leads to an evolution of students’ conceptions of the objects of algebra.

Also cited by Kieren (2007), the kind of errors that students can make in algebraic transformational activity (e.g., Carry et al., 1980; Lemoyne, Conne, & Brun, 1993; Matz, 1982; Sleeman, 1984) have suggested to some researchers (e.g., Kirshner, 1989) that the issue is not an
absence of theoretical control but rather a misperception of form. In a study that extended Kirshner’s earlier work on the visual syntax of algebra, Kirshner and Awtry (2004) investigated the role of visual salience in the initial learning of algebra for students in four intact classes of seventh graders (about 12 years old). They found that students did indeed engage with the visual characteristics of the symbol system in their initial learning of algebraic rules: The percentage-correct scores for recognition tasks were significantly higher for visually salient rules than for non-visually-salient rules. Similarly, Hewitt (2003), in a study of 40 teachers and a class of 11- to 12-year-olds, found that the inherent mathematical structure, and the visual impact of the notation itself, had an effect on the way in which equations were manipulated.

Calculus

In this subsection, calculus will be defined as well as discussion on its importance, followed by review of literature on the trend of research in the area. These areas include effect of working in pairs or groups of minority students in learning calculus, how technology can be helpful/detrimental with respect to students’ calculus misconception; and the implications of Calculus reform effort.

Definition of Calculus

Calculus is a branch of mathematics which has been developed to describe relationships between two or more things which can change continuously (Davis & Hersh, 1981). According to Young (1986) quoted by Ferrini-Mundy and Graham (1991), “calculus is our most important course...The future of our subjects depends upon improving it” (p.627). Similarly in a survey at the University of Connecticut for the requirements of all sciences and engineering major programs, thirty two (32) key courses were found to have first-year calculus as a prerequisite (Hurley, Koehn and Ganter, 1999). Furthermore, those thirty two (32) courses require students to use the quantitative, analytical, and problem-solving skills conveyed by calculus.
Impact of working in pairs or groups of minority students in learning Calculus

The impact of cooperative learning of Calculus on minority students has been an ongoing area of concern among calculus researchers. In a study, Moore (2005) examines the impact of cooperative learning among Emerging Ethnic Engineers (E3) at the University of Cincinnati. Moore states that “the E3 programme recognizes the significance of early academic success, particularly for freshmen students, and provides an effective academic support structure for its students through the cooperative learning calculus programme” (p.536). He concluded that, E3 students had a success rate higher than those who did not participate in the cooperative learning calculus course, fully go in for engineering degree, and extend their cooperative experience outside the walls of the University. He finally asserts that student-centered learning experience in the cooperative learning calculus programme has been established to be helpful in increasing retention rates, determination, leading to academic success for students who enrolled in the programme.

In a related study, the role of Emerging Scholars Program (ESP) on the choice of Mathematics, Sciences and Engineering (MSE) major for African Americans, Latinos, and Women at the University of Texas at Austin was examined by Moreno and Muller (1999). They analyze the influence of calculus performance on choosing a MSE major by focusing on variations by race, ethnicity, and gender and on the role of students’ participation in the ESP. They states that, ESP students earn higher calculus grades than non-ESP students and are more likely to enroll in the second semester calculus sequence. Furthermore, the consequences of earning a higher grade in calculus would lead to majoring in MSE. Finally, they asserted that, African American, Latinos, and Women excel in calculus if they receive appropriate academic challenges and support.
How technology can be helpful/detrimental with respect to students’ Calculus misconception

The calculus reform of the undergraduate curriculum has generated a lot of recommendations based on the description of its teaching and learning problem. Central to those recommendations has been the use of technology in order to put less emphasis on computation and pay more attention to learning calculus concepts.

Related to the use of technology in learning calculus, a study was conducted by Palmiter (1991) involving 78 subjects. She investigated whether there is a significant difference between students who have been taught calculus using a computer algebra system to compute limits, derivatives, and integrals and students who have used standard paper-and-pencil procedures in (a) knowledge of calculus concepts, (b) knowledge of calculus procedures. She opined that, students who were taught calculus using a computer algebra system had higher scores on a test of conceptual knowledge of calculus than the students taught by traditional methods. Furthermore, she claimed that students in the computer class also had higher scores on a calculus computational exam using the computer algebra system than students in the traditional class using paper and pencil.

In another study, Heid (1988) investigated calculus concepts learning without concurrent or previous mastery of the usual algorithmic skills of computing derivatives, computing integrals, or sketching curves. The research involves 39 college students in 13 weeks of applied calculus course, studying calculus concepts using graphical and symbolic-manipulation computer programs to perform routine manipulations. Class transcripts, student interviews, field notes, and test results were analyzed for patterns of understanding. She claimed that students showed better understanding of course concepts and perform almost as well on a final exam of routine skills when compared to a class of 100 students who had practiced the skills for the entire 15 weeks.
Implications of Calculus reform effort

One of the primary goals of calculus reform has been to make it easier for all (traditional White or Asians and male, and Female, Black, or Hispanic) students to do well in calculus (Feffer & Petechuk, n.d). While this could be good news for those who might otherwise have failed the subject entirely, better students (the traditional pool of future scientists and engineers) may pass through the calculus sequence knowing far less than they would have otherwise.

According to Feffer & Petechuk (n.d.) judging the success of calculus reform requires consideration not only of the respective curriculum, but also of the goals of calculus teaching and its position in the university curriculum. Is it for talented few or the mediocre many? This has generated a dichotomy of yes or no.

Feffer (n.d) claimed yes, and that reform-based calculus provides students with a better grasp of the real-world applications and context of mathematical principles, and it also increases the participation of student populations that have been underserved by traditional teaching methods.

Contradicting Feffer’s claimed above; Petechuk (n.d) opined that, the calculus reform project purges calculus of its mathematical rigor, resulting in a watered-down version that poorly prepares students for advanced mathematical and scientific training.

In a longitudinal study conducted by Schwingendorf, McCabe, and Kuhn (2000), performance of students from Calculus, Concepts, Computers and Cooperative Learning (C⁴L) was compared with that of traditional (TRAD) students. The fundamental questions investigated are: which program, C⁴L or TRAD, provides a student with a better understanding of the required calculus concepts? Secondly, which program better inspires students to pursue further study in calculus or, more generally in mathematics?
They reported that, C4L students earn higher grades in calculus courses than the TRAD students. It was further found that the C4L students are as prepared as the TRAD students, and also competent for mathematics courses beyond the calculus program. Finally, C4L students were better inspired more than the TRAD students to pursue further study in calculus, as well as register for non-calculus mathematics courses after the calculus sequence.

Taking into consideration many of the researches reported above that highlighted on misconceptions and difficulties associated with learning and operating with variables, equations, visualization and symbol skills (transformation activities) from the lens of algebra and geometry, as well as calculus, the propose study would aim at investigating the contribution of algebraic and geometric prerequisites skills as incorporated into students’ applied calculus optimization problem (ACOP) solution.
CHAPTER THREE

RESEARCH DESIGN OF THE STUDY

Introduction

The research design chapter contains description of procedures used in conducting the study. It also includes a definition of research design, a description of the mixed method design used in the study, and the quantitative and qualitative methods used in sequential explanatory design. Other items in the chapter are demographic description of participating students, data collection procedures and instruments, and analysis procedures.

According to de Vos (2002), a research design is “a plan, recipe or blueprint for the investigation” (p. 165). It offers a clear description of how the research is going to be conducted. Moreover, it indicates the procedures that will be followed in the sampling and data collection in order to reach the research aims and objectives. The ultimate goal for a good research design is to provide a credible answer to the research question (Macmillan & Schumacher, 2001).

This study applies a mixed methods research design. According to Creswell (2003), “a mixed methods approach is one in which the researcher tends to base knowledge claims on pragmatic grounds (e.g., consequence oriented, problem-centered, and pluralistic)” (p. 19). It employs strategies of inquiry that involve collecting data either simultaneously or sequentially to best understand research problems. The data collection also involves gathering both numeric information (e.g., on instruments) as well as text information (e.g., on interviews) so that the final database represents both quantitative and qualitative information. The sequential approach used in this study is explanatory. According to Creswell (2003) the sequential explanatory strategy is the most straightforward of the six major mixed methods approaches. It is characterized by the collection and analysis of quantitative data followed by the collection and analysis of qualitative data.
According to Creswell, Clark, Gutmann and Hanson (2003), the purpose of the sequential explanatory design is typically to use qualitative results to assist in explaining and interpreting the findings of the quantitative results. The initial quantitative phase of the study may be used to characterize individuals along certain traits of interest related to the research questions. These quantitative results can then be used to guide the purposeful sampling of participants for a qualitative study.

For the purpose of this study, the quantitative portion of the sequential explanatory mixed method research was used to assess students’ algebraic, geometric prerequisites and basic differentiation skills in ACOP solution. These skills include drawing diagrams (geometric); and formulating and transforming equations (algebraic). Moreover, the quantitative phase also assesses students’ skills in applying basic differentiation concepts from calculus into ACOP solutions. The second stage of the sequential explanatory strategy is qualitative, using clinical interviews. It was used purposefully to explore and probe a few individual students’ results from the quantitative component, in-depth.

The two processes of the sequential strategy, quantitative methods followed by the qualitative method, are used to detect errors and misconceptions with respect to the drawing of diagrams and symbols skills, the use of variables, equations, and the visualization of the solution of ACOP. Moreover, errors related to use of basic differentiation skills are noted.

Specific research questions investigated in this mixed method study are stated below:

1a. Are students able to construct a diagram to represent word problems?
   b. How do students construct diagrams to represent word problem?

2a. Are students able to label diagrams appropriate/inappropriately using variables?
   b. How do students label diagrams appropriate/inappropriately using variables?
3a. Are students able to associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

b. How do students associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

4a. Are students able to use symbol/transformation skills?

b. What are students’ strengths/weaknesses in symbol/transformation skills?

5a. Are students able to marshal the above competences to find the model in an ACOP?

5a*. Are there any relationships between finding the required equation/model of an ACOP and diagram construction; labeling diagram; associating diagrams with geometric formulas/equations; and manipulating/transformation symbols skills?

b. How do students use prerequisites skills to find the model in an ACOP?

6a. Are students able to do the calculus in an ACOP solution when the model is given?

b. How do students use calculus in an ACOP solution when the model is given?

7a. Are students able to solve ACOP completely?

b. How do students solve ACOP completely?

The quantitative components of the research questions are the (a) parts while the qualitative components are the (b) parts; the research question (5a*) was assessed quantitatively. The overall outcome from these research questions will be a synthesized model of how the prerequisites fit together to constitute a solution of ACOP.

**Quantitative Methods**

Quantitative methods may be most simply and economically defined as the techniques associated with the gathering, analysis, and interpretation of information that is presented in numerical form (Teddlie, 2006).
This portion of the quantitative method was divided into two main sections. The first section was further subdivided into seven sections which were defined by the seven research questions. The research questions were constructed to assess students’ prerequisite skills in visualization/diagram construction, labeling the diagrams constructed with appropriate variables, and associating geometric diagrams with appropriate formulae and transformation/symbols skills. An instrument was designed to conduct the above assessment. Overall, the quantitative part of the research analyzed ACOP solutions to determine the points of breakdown. Other specific points of breakdown expected from the assessment results were the application of basic differentiation concepts in the ACOP solution process.

The second part of the quantitative method examined the contribution of each individual skill stated above toward the solution process of ACOP. A model would generated to assess each point of breakdown in relation to other factors.

Qualitative Methods

According to Teddlie (2006), qualitative methods may be most simply and parsimoniously defined as the techniques associated with the gathering, analysis, and interpretation of information that is presented in a narrative form. Many qualitatively oriented researchers subscribe to a worldview known as constructivism, and its variants such as naturalism (e.g., Howe, 1988; Lincoln & Guba, 1985; Maxcy, 2003). Constructivists believe that researchers individually and collectively construct the meaning of the phenomenon under investigation.

The purpose of using qualitative methods in this study was to further investigate the results of the quantitative part. The qualitative portion was conducted using clinical interviews. According to Clement (2000), the “clinical interview is a technique pioneered by Piaget (1975) to study the form of knowledge structures and reasoning processes” (p. 547). Over the past
thirty-two years, clinical interviewing has evolved into a variety of methods, including open-ended interviews and think-aloud problem solving protocols and stimulated recall. As well, Ginsburg (1997) claimed that Piaget designed the method of clinical interview to accomplish three goals: “to depict the child’s ‘natural mental inclination’, to identify underlying thought processes, and to take into account the larger ‘mental context” (p. 48).

The strength of the clinical interview in comparison to the non-clinical method includes the ability to collect and analyze data on mental processes at the level of a student’s authentic ideas and meanings, and to expose hidden structures and processes in the student’s thinking that could not be detected by less open-ended techniques, (Clement, 2000). Due to the fact that tests (standardized) are almost written from the point of view of the teacher and designed to detect standard forms of academic knowledge, they can fail to detect key elements in students’ thinking (Clement, 2000; Ginsburg, 1997). Clinical interviews on the other hand, can be designed to elicit and document naturalistic forms of thinking. In some exploratory varieties of clinical interviewing, Clement (2000) claimed that the investigator can also react responsively to data as they are collected by asking new questions in order to clarify and extend the investigation. He stated further that, even where the detection of academic knowledge is sought, the clinical interview can give more information on the depth of conceptual understanding, because oral and graphical explanation can be collected, and classified where appropriate.

Using clinical interview in the qualitative part of this study, the researcher will ask students questions from test results they already answered in the quantitative part. The test instrument is composing of four major parts: algebra, geometry, partial ACOP, and full ACOP. The questions will be open-ended, guided by students’ solutions from the test instrument. The focus areas to be probed in-depth are the following concepts: constructing and labeling diagrams, manipulating symbol skills, and associating diagrams with equations/formulas, as well as
assessing basic differentiation skills. Their incorporation into partial and full ACOP solutions is also investigated.

For the purpose of this study, the stimulated recall research methodology were used to understand and identify underlying the thought processes of students solving problems from the test instrument. According to Henderson and Tallman (2006), “stimulated recall is an introspective method that can be used to elicit people’s thought processes and strategies when carrying out a task or activity” (p. 55). Furthermore, Gass and Mackey (2000) opined that a concrete reminder of an event will stimulate recall of the mental processes in operation during the incident itself. Stimulated recall relies on an information-processing approach whereby the use of and access to memory structure is enhanced, if not guaranteed, by a prompt that aids the recall of information (Gass & Mackey).

In most research, Henderson and Tallman stated that the concrete reminder is a video; however audiotapes (Beaufort, 2000), computer software (Henderson, 1996), the World Wide Web (Henderson, Putt, Ainge, and Coombs, 1997), and written documents (Gass and Mackey, 2000) can be used as alternative artifact prompts. In the stimulated recall interview, both participants, i.e. researcher and interviewee, are engaged the second time with the previous experience by revisiting the written documents or watching the videos, (shown in figure 3.1 below) with the sole purpose of exposing the interviewee’s thought processes at the time of the original task through verbalization of those thoughts. Figure 3.1 gives the description of the stimulated recall mechanism adapted from Henderson and Tallman (2006).
Demographics of Participating Students

The target population is freshmen students taking calculus I in the Department of Mathematics, Louisiana State University, Baton Rouge. The course is taught in the Fall, Spring and Summer semesters every year. For the purpose of this study, data were collected using students enrolled in the calculus I course in Spring semester, 2008. The university is among the elite Southern Region Universities of United States. It has an average enrollment of about 31,000 students per year at its Baton Rouge campus. Most of the students are from higher to middle socio-economic status. A significant proportion of the student population receives athletic, academic, and other scholarships.

The portion of calculus I which was used for the study was applied calculus optimization problems. As explained in the mathematics department website, students are having difficulty in understanding and applying concepts required to solve ACOP.

Students taking calculus I are split into twenty-six different sections. Each section contains not more than forty students. This represents a total enrolment of 1040 students. Six
sections of calculus I were used for the study. This represents 156 students or approximately 15% of the total enrollment in calculus I. The six sections used for the quantitative portion of the study were selected using the convenience sampling technique. Convenience sampling involves drawing elements from the group (usually most appropriately regarded as a subpopulation) that is easily accessible by the researcher (Kemper, Stringfield and Teddlie, 2003). Moreover, the sampling strategy used in selecting clinical interview participants is the mixed method sampling strategy. The qualitative data were generated using interviews with 15 students. They were selected using the sequential mixed method sampling strategy. According to Teddlie & Yu (2007), it involves selection of the unit of analysis for a mixed method study through the sequential use of probability and purpose sampling strategies. The probability sampling used in selecting the interviewee is the simple random sample, while the purposive sampling is the convenience sampling strategy. The textbook for calculus I course in mathematics department at LSU is: Calculus, Early Transcendentals, first edition by Rogawski.

Data Collection and Analysis Methods

According to Johnson & Turner (2003), “a method of data collection is simply a technique that is used to collect empirical research data. It is how researchers ‘get’ their information” (p. 298). The six major methods of data collection are questionnaires, interviews, focus group, test, observations, and secondary data. For the purpose of this study, sequential explanatory mixed method data collection procedures were used.

The data collected from this study were analyzed using the mixed method data analysis approach. According to Teddlie and Onwuegbuzie (2003), “mixed methods data analyses is defined as the use of quantitative and qualitative analytical techniques, either concurrently or sequentially, at some stage beginning with the data collection process, from which interpretations are made in either a parallel, an integrated, or an iterative manner” (p. 352–353). Furthermore,
they claimed that mixed methods data analysis allows the researcher to use the strengths of both
quantitative and qualitative analysis techniques so as to understand phenomena better. The
ability to “get more out of the data” provides the opportunity to generate more meaning, thereby
enhancing the quality of data interpretation.

The sequential mixed methods data collection procedures were conducted in two phases.
The first phase was an assessment test. The instrument for the test was designed by the
researcher to meet the requirements of the study. The second phase of the study will use clinical
interviews to explore and probe students’ conceptual understanding and skills.

Quantitative Data Collection Instrument

Test instrument

The test instrument for this study is an assessment test. The main themes to be
investigated in this study are: students’ prerequisite skills in algebra and geometry, so as to
develop a model of how prerequisite competences are integrated together into ACOP problem
solving. The test instrument structure reflects the main themes of the study, to determine if
students are able to:

- Construct a diagram to represent the given word problem
- Label a given diagram using appropriate variables
- Associate geometric diagrams with appropriate algebraic equation(s)/formula
- Manipulate symbol skills
- Use transformation/symbol skills to get the required model
- Solving a partial and full ACOP

Construction of a diagram to represent the given word problem

This part of the assessment test is primarily designed/adapted by the researcher to assess
whether students can draw a diagram to represent an ACOP and a non ACOP word problem. It is
designed to measure students’ knowledge of basic geometric shapes as well as their application to physical situations. Normally, in ACOP, a given word problem contains a description of a physical situation which students are expected to represent diagrammatically. This task takes into consideration students’ visualization ability. Basic geometric shapes such as rectangle, triangle, circle, and cylinder and so on are expected to be used fluently by the test takers.

**Label a given diagram using appropriate variables**

Labeling activity of the test assessed two themes: conceptual understanding of variables and their applications to new, independent, and appropriate situations. They were designed to assess whether students could use an appropriate variable of their choice and suitable to a given situation. An example of an appropriate situation would be to label the given cylinder using appropriate variables. In an attempt to answer the question, \( r \) for radius and \( h \) for height could be used. Another skill that was expected to be measured by the labeling task item of the test was what numerical value of the variable(s) is/are practical.

**Associating geometric diagrams with appropriate algebraic equation(s)/formula**

This item of the research instrument was designed to measure students’ ability to associate an appropriate geometric diagram with its algebraic equation(s)/formulas in relation to calculating area, perimeter or volume. Test takers were expected to match geometric formulas with their geometric diagrams from the pool of formulas and geometric diagrams. Here, the themes to be assessed were recognition of geometric diagrams and their formula, as well as application.

**Manipulating symbol skills**

This part of the test assessed students’ ability to manipulate any algebraic equation irrespective of whether or not it is related to ACOP. Particular emphasis was given to non-ACOP problems, because the skill is applicable to many situations.
Using transformation/symbol skills to get the required model

The required equation for the solution in a given ACOP can be derived straight away. In other cases, it involves transformation/symbol skills of one to two equations to get the required result. This test item would assess students’ abilities in this aspect. In essence, the main theme of this test item is transformation/symbol skill evaluation.

Solving partial ACOP

As part of the research instrument, this item will measure students’ ability (skill) to solve a partial ACOP. A partial ACOP is a problem which does not require some prerequisites, especially geometric prerequisites. The main focus of this test item is to assess whether students can solve an ACOP in which the required geometric parts of the solution are already given. This will show the extent to which geometric skills or their absences play a role in ACOP solution. Moreover, it will clearly assess students’ algebraic and basic differentiation skills in the ACOP solution.

Solving full ACOP

This part of the test instrument will assess student’s overall algebraic, geometric and basic differentiation skills in ACOP solutions. The task measured students’ skills or their absence into the ACOP solution process. The complete ACOP solution will show the exact point of breakdown: either algebraic, geometric, or basic differentiation skills.

Quantitative Data Analysis

To quantify the test result, a rubric was developed assigning scores for each item on the test instrument. A score of 0 to 9 points was distributed, 3 points for each question in the section depending on the level of performance shown by the students. Example, a correct response will received 9 points for the whole section, while a totally wrong or no response received zero points. The generated scores were then analyzed using descriptive statistics and regression
analysis. The data collected from the test instrument which has seven components were interpreted using frequencies, correlation, descriptive statistics (mean, standard deviation, minimum, and maximum) to compare students’ performance.

The second portion of the quantitative method used multiple regression analysis to interpret the data collected from test instrument. Multiple regression analysis has basic assumptions of linearity, homoscedasticity (equal variance), and normality which were tested using a scatter plot, Q – Q plot, and histogram respectively from Statistical Package for Social Sciences (SPSS) software output.

A p-value of $\alpha = 0.05$ were used for interpreting the quantitative data. A p-value is a measure of how much evidence we have against the null hypothesis Simon (2007). The null hypothesis, traditionally represented by the symbol $H_0$, represents the hypothesis of no change or no effect. The smaller the p-value, the more evidence we have against $H_0$. It is also a measure of how likely we are to get a certain sample result or a result “more extreme,” assuming $H_0$ is true.

The Multiple Regression Model is given as:

$$ACOPM = DIAGCONSb_1 + LABELDIAGb_2 + ASSODIAGb_3 + MANSYMb_4,$$

where the dependent-variable is ACOP model/equation (ACOPM), and independent-variables are constructing diagrams (DIAGCONS), label a diagram (LABELDIAG), associating diagram with formula (ASSODIAG) and manipulating symbol skills (MANSYM, while $b_1$, $b_2$, $b_3$, $b_4$ are coefficients of the variables in the models.

**Test interpretation**

The test analysis has two main purposes: i) to determine the quantitative results from the quantitative part (described above); ii) to describe the solution processes used by students. The description and analyses of the test solution processes were conducted with interviewees selected using the sequential mixed method sampling strategy, and created profiles of students’
performance. The sequential mixed method sampling strategy uses probability and purpose sampling strategies. A total of 15 interviews were conducted: with three students selected from the S-section; two students each from the L-section; T-section; V-section; W-section; and four students from J-section.

Profiles of students’ performance were developed based on the test results. The profiles characterized the combinations of prerequisite competences which students displayed in the first four sections of the test. An “A” represents above-average performance in each section, whereas “B” is below-average performance. The four prerequisite competences are: constructing diagrams, labeling a diagram, associating a diagram with a formula, and manipulating symbol. AAAA represents above-average performance in all the four prerequisites competences; AAAB represent above-average performance in three competences, and one below-average competence. AABB represent above and below average performances in two competences each. ABBB represent one above and three below average performances. BBBB represents below-average performance in all the four competences.

Within each profile, performances of students on other sections were examined, using the same criteria of above and below average. It examined whether students were able to obtain above or below average-performance in tasks that require application of prerequisite skills or competences. These tasks are ACOP model/equation (ACOPM), solving partial ACOP (DIFFOV), and solving complete ACOP (OPTPBM).

**Qualitative Sampling Strategy**

The qualitative part of the study uses sequential mixed method sampling strategy to generate data. According to Teddlie & Yu (2007), sequential mixed method sampling strategy involves selection of unit of analysis for a mixed method study through the sequential use of probability and purpose sampling strategies. The probability sampling used in selecting the
interviewee is simple random sample, while the purposive sampling is convenience sampling strategy.

**Qualitative Data Collection Methods**

**Clinical interviews**

In this second phase of data collection of the mixed method design, clinical interviews were used. Clinical interviews are proposed to explore in-depth students’ conceptual understanding using the stimulated recall technique related to: constructing diagrams and labeling, manipulating symbol skills and associating diagrams with equations/formulas, partial or full ACOP. The Stimulated recall interviews were conducted after tests were taken from the quantitative part.

Specifically, the purpose of using stimulated recall interviews was to collect data on students’ thought processes accompanying solution of the test items; assess the quality and character of their prerequisite competencies regarding algebra and geometry; and finally, develop a model of how student integrate prerequisite competencies in the solution of ACOP problems.

The data collected using stimulated recall methods were carefully analyzed, and the testing results highlighted apparent strengths and weaknesses in prerequisites competences. Moreover, the data had described thought processes and were follow up with directed questioning intended to explore the apparent strength and weaknesses based on theories of algebraic and geometric competencies found in the literature.

**Qualitative Data Analysis**

The qualitative data analyses were approached from two themes: Test interpretation (described above) and interview analysis. The qualitative test interpretations described above were used to generate hypotheses about the student’s competencies, which were then followed up with interviews for further qualitative analysis.
Stimulated recall interview analysis

For the purpose of selecting participants for the stimulated recall interviews, a simple random sample and convenient sampling were used (sequential mixed method approach). For each interviewee, an interview protocol was prepared before the interview based on the participants’ test script. Each protocol contained hypotheses and probes on the student’s performance from the test. After the interviews, data generated were first transcribed. The text data were then described, guided by the specific research questions related to the following areas: constructing diagrams and labeling, manipulating symbol skills and associating diagrams with equations/formulas, partial or full ACOP. These were followed by a thick description of students’ thought processes as related by interviewees. Finally, the data were analyzed with an evolution of hypotheses concerning strengths and weaknesses of students as developed in the interview process.
CHAPTER FOUR

ANALYSIS OF THE DATA

The purpose of this study was to assess algebraic and geometric prerequisite skills as incorporated into the ACOP solution. Generally, the study’s main themes are:

- Assess students’ prerequisite skills in setting up an ACOP problem.
- Develop a model of how prerequisites competences are integrated together into ACOP problem solving.

The study contains both quantitative and qualitative data in an effort to investigate the role played by algebraic and geometric skills in ACOP solution process. Chapter four contains the quantitative and qualitative data analysis in two different sections.

Quantitative Data Analysis

This section of chapter 4 contains the descriptive statistics and regression analysis results from the quantitative data collected. The quantitative results were analyzed based on the seven sections of the research questions. The research questions are:

1a. Are students able to construct a diagram to represent word problems?

2a. Are students able to label diagrams appropriate/inappropriately using variables?

3a. Are students able to associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

4a. Are students able to use symbol/transformation skills?

5a. Are students able to marshal the above competences to find the model in an ACOP?

5a*. Are there any relationships among finding the required equation/model of an ACOP and diagram construction; labeling a diagram; associating diagrams with geometric formulas/equations; and manipulating/transformation symbols skills?

6a. Are students able to do the calculus in an ACOP solution when the model is given?
7a. Are students able to solve ACOP problems completely?

The total number of students involved in the study was 155. They were enrolled in Math 1550 calculus I course for Spring 2008 Semester at Louisiana State University, Baton Rouge. The sample size consisted of six out of 25 sections that were initially enrolled with population size of approximately 900 students during the Spring Semester. The reliability test for the data collected is Cronbach’s Alpha of 0.767 ($\alpha = 0.767$), which is acceptable. According to George & Mallery (2005), “the Cronbach’s alpha is designed as a measure of internal consistency; that is, do all items within the instrument measure the same thing?”

Further details of the 155 students sample size consist of J-section (39, 25.2%); T-section (24, 15.5%); W-section (12.2%); V-section (23, 14.8%); L-section (12.9%); and S-section (30, 19.4%). The demographic characteristics of the sample is contain in Table 4.1 below.

Table 4.1: Frequencies and Percentages for Gender/Race

<table>
<thead>
<tr>
<th>Gender/Race</th>
<th>Frequencies</th>
<th>Percentages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>92</td>
<td>59.40%</td>
</tr>
<tr>
<td>Female</td>
<td>62</td>
<td>40.60%</td>
</tr>
<tr>
<td>White</td>
<td>114</td>
<td>73.50%</td>
</tr>
<tr>
<td>African-American</td>
<td>22</td>
<td>14.20%</td>
</tr>
<tr>
<td>Other</td>
<td>15</td>
<td>9.70%</td>
</tr>
</tbody>
</table>

Table 4.2: Mean & Standard Deviation of ACT scores for Gender/Race

<table>
<thead>
<tr>
<th>Gender/Race</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>25.21</td>
<td>2.8</td>
</tr>
<tr>
<td>Female</td>
<td>25.22</td>
<td>3.19</td>
</tr>
<tr>
<td>White</td>
<td>25.58</td>
<td>2.56</td>
</tr>
<tr>
<td>African-American</td>
<td>23.65</td>
<td>3.5</td>
</tr>
<tr>
<td>Other</td>
<td>24.42</td>
<td>3.83</td>
</tr>
</tbody>
</table>

The mean ACT scores for the males and females indicate little distinction between the genders, but females have higher standard deviation than the males. This showed there is more
variability around the mean ACT scores among females than males. Moreover, there is wide
disparity between Whites and African–Americans as indicated by the mean ACT scores.
Similarly, the higher standard deviation of the African–Americans signifies that there is more
variability among them than the Whites. Others also have a higher variability than Whites, but
there is little difference between Others and African–Americans.

Table 4.3: Descriptive Statistics of Variables for African-Americans

<table>
<thead>
<tr>
<th>Variables</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagram construction</td>
<td>22</td>
<td>2</td>
<td>9</td>
<td>5.77</td>
<td>2.045</td>
</tr>
<tr>
<td>Label a diagram</td>
<td>22</td>
<td>3</td>
<td>7</td>
<td>5.77</td>
<td>1.232</td>
</tr>
<tr>
<td>Associating diagram with formula</td>
<td>22</td>
<td>1</td>
<td>6</td>
<td>4.045</td>
<td>1.5729</td>
</tr>
<tr>
<td>Manipulating symbol skills</td>
<td>22</td>
<td>1</td>
<td>6</td>
<td>2.86</td>
<td>1.246</td>
</tr>
<tr>
<td>ACOP equation</td>
<td>22</td>
<td>0</td>
<td>5</td>
<td>2.56</td>
<td>1.681</td>
</tr>
<tr>
<td>Differentiating/optimizing a variable</td>
<td>22</td>
<td>1</td>
<td>5</td>
<td>2.86</td>
<td>1.246</td>
</tr>
<tr>
<td>Optimization problem</td>
<td>22</td>
<td>0</td>
<td>3</td>
<td>1.18</td>
<td>1.14</td>
</tr>
</tbody>
</table>

Table 4.4: Descriptive Statistics of Variables for Whites

<table>
<thead>
<tr>
<th>Variables</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagram construction</td>
<td>114</td>
<td>1</td>
<td>9</td>
<td>6.53</td>
<td>2.910</td>
</tr>
<tr>
<td>Label a diagram</td>
<td>114</td>
<td>1</td>
<td>9</td>
<td>6.50</td>
<td>1.453</td>
</tr>
<tr>
<td>Associating diagram with formula</td>
<td>114</td>
<td>1</td>
<td>6</td>
<td>4.965</td>
<td>1.212</td>
</tr>
<tr>
<td>Manipulating symbol skills</td>
<td>114</td>
<td>0</td>
<td>8</td>
<td>3.98</td>
<td>1.709</td>
</tr>
<tr>
<td>ACOP equation</td>
<td>114</td>
<td>0</td>
<td>9</td>
<td>3.31</td>
<td>1.873</td>
</tr>
<tr>
<td>Differentiating/optimizing a variable</td>
<td>114</td>
<td>0</td>
<td>6</td>
<td>3.30</td>
<td>1.698</td>
</tr>
<tr>
<td>Optimization problem</td>
<td>114</td>
<td>0</td>
<td>6</td>
<td>1.96</td>
<td>1.420</td>
</tr>
</tbody>
</table>
Table 4.5: Descriptive Statistics of Variables for Others

<table>
<thead>
<tr>
<th>Variables</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagram construction</td>
<td>15</td>
<td>0</td>
<td>9</td>
<td>5.87</td>
<td>2.475</td>
</tr>
<tr>
<td>Label a diagram</td>
<td>15</td>
<td>3</td>
<td>8</td>
<td>5.87</td>
<td>1.727</td>
</tr>
<tr>
<td>Associating diagram with formula</td>
<td>15</td>
<td>1</td>
<td>6</td>
<td>3.70</td>
<td>1.771</td>
</tr>
<tr>
<td>Manipulating symbol skills</td>
<td>15</td>
<td>1</td>
<td>6</td>
<td>3.60</td>
<td>1.595</td>
</tr>
<tr>
<td>ACOP equation</td>
<td>15</td>
<td>1</td>
<td>8</td>
<td>3.47</td>
<td>1.846</td>
</tr>
<tr>
<td>Differentiating/optimizing a variable</td>
<td>15</td>
<td>0</td>
<td>5</td>
<td>2.60</td>
<td>1.639</td>
</tr>
<tr>
<td>Optimization problem</td>
<td>15</td>
<td>0</td>
<td>4</td>
<td>1.80</td>
<td>1.207</td>
</tr>
</tbody>
</table>

The performance of the overall sample was compared based on the Race. The mean scores from Tables 4.4, 4.5 & 4.6 indicates that there are wide differences between Whites and African–Americans on individual variables, such as Diagram construction, Label a diagram, Associating diagram with formula, Manipulating symbol skills, ACOP equation, differentiating/optimizing a variable (partial ACOP), and optimizing problem (complete ACOP). However, the standard deviation shows that the amount of variability among Whites students is higher than African–Americans with the exception of two variables: diagram construction and associating diagram with formulas. On a similar note, a broad variation also exist between Others and African – Americans, because Others have higher mean score. Comparing the standard deviations of these two groups again, there is more variability among Others in all the variables except one (Optimizing problem) than African–Americans. On comparing the Whites with Others, Tables 4.4, 4.5, & 4.6 indicates higher mean scores among Whites in all but one variable (ACOP equation) than Others. On the contrast, Others have higher variability in three out of seven variables when put side by side with Whites.
Table 4.6: Descriptive Statistics of All Variables and Subjects

<table>
<thead>
<tr>
<th>Variables</th>
<th>Mean</th>
<th>Mode</th>
<th>Std. Deviation</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Maximum possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagcons*</td>
<td>6.23</td>
<td>7</td>
<td>2.04</td>
<td>0</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Labeldiag*</td>
<td>6.34</td>
<td>7</td>
<td>1.5</td>
<td>1</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Assdiag*</td>
<td>4.7</td>
<td>6</td>
<td>1.41</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Mansym*</td>
<td>3.78</td>
<td>4</td>
<td>1.69</td>
<td>0</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Acopm*</td>
<td>3.21</td>
<td>3</td>
<td>1.86</td>
<td>0</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Diffov*</td>
<td>3.16</td>
<td>2</td>
<td>1.66</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Optpbm*</td>
<td>1.83</td>
<td>2</td>
<td>1.41</td>
<td>0</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

*Meaning of variables ACRONYMS was given in Chapter one.

Research question 1a: Are students able to construct a diagram to represent word problems?

The first research question investigated whether students were able to construct diagram from word problem. The descriptive statistics given in table 6 shows an average score of 6.23 out of maximum of 9 points (maximum possible is 9), with most scored point of 7. The frequency shows that 49 students (32.3%) scored between 0 to 5 points; 21 (13.5%) scored 6 points; 45 (29.0%) scored 7 points; 6 (3.9%) achieved a total of 8 points; and 33 (21.3%) achieved 9 points. Overall, more than 65% of the total sample scored at least 6 points. Based on this statistical fact it can be determined that the students had reasonable success in constructing diagrams from word problems.

Research question 2a: Are students able to label diagrams appropriate/inappropriately using variables?

This research question examines whether students could label a constructed diagram using one variable explicitly, or calculated dimensions labeling or non-canonical variables (dimensions).

The mean mark attained in this task was 6.34 points, with most occurring point of 7. The frequency shows that 46 students (29.7%) scored between 0 to 5 points; 17 (11%) scored 6 points; 67 (43.2%) achieved a total of 7 points; 18 (11.6%) scored 8 points, and 7 (4.5%)
achieved 9 points. Splitting the sample into two ranges i.e. from 0 to 5 points and 6 to 9 points indicate that first range of the total population have ended up with a cumulative percentages of 29.7 %, whereas majority of the sample (70.1 %) fall within the second range. This statistical information indicates that the task of labeling a geometric diagram either explicitly, calculated or from non-canonical geometric diagram was well accomplished by most students.

**Research question 3a:** Are students able to associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

The focus of this task was to assess whether students were able to associate formulas with appropriate geometric diagrams. The mean score in this task was 4.7 out of 6 possible, with a mode of 6 and a standard deviation of 1.41. The occurrence of scores shows that 34 students (21.9%) scored between 0 to 3.5 points; 27 (17.4%) scored between 4 and 4.5; 43 (27.7%) achieved a score of range 5 to 5.5, and 51 (32.9%) scored 6 points. Taken as a whole, more than 78 % of the sample scored at least 4 points in the task. It can be inferred that the students are able to associate geometric diagrams with their appropriate formulas successfully.

**Research question 4a:** Are students able to use symbol/transformation skills?

The focal point of this task was assessing students’ competencies in algebraic skills. This capability reflects an accumulation of skills over a period of time and can indicate the trajectory that student has passed through.

The mean score for this task was 3.78 out of 9 possible points, with a standard deviation of 1.69. The minimum score earned was 0 while the maximum earned was 8. Frequency of scores group in the range of 0 to 4 points and 5 to 9 are 108 (69.7 %) and 47 (30.3 %), respectively. In general, more than half of the sample was unable to successfully do the tasks.
Research question 5a: Are students able to marshal the above competences to find the model in an ACOP?

The focus of this research question was to assess whether students could use their algebra and geometric competencies measured above to come up with a model in ACOP solution. The statistical outcome of the tasks shows mean score of 3.21 out of 9 possible points, with a standard deviation of 1.87, with the most frequently occurring point of 3. The frequencies obtained from the statistical output were grouped into two main categories: 0 to 4 and 5 to 9 points. There were 120 students (77.4 %) for the first category, and 20 (22.6 %) for the second.

It can be inferred from the statistical data that the greater percentage of the sample fell below expectation and were unable to successfully derive the required model from the given word problems. In essence, they found it difficult to use the expected algebra and geometric competences with ease in deriving the expected model.

Research question 6a: Are students able to do the calculus in an ACOP solution when the model is given?

The focus of this task was to assess whether students could solve ACOP in which the required geometric part was either given or suppressed. Basically, it would appraise students’ algebraic and basic differentiation skills in the ACOP solution process. Data from SPSS statistical output for this task indicates an average score of 3.16 out of maximum points of 6, with a standard deviation of 1.66, and most occurring score of 2. The sample was grouped base on valid scores ranging from 0 to 3 and 4 to 6 points. The frequencies found from the output indicate that the first group includes 92 students (59.3 %), whereas the second include 63 (40.7 %). It can be concluded from these statistical information that the sample was partially divided between those that could effectively manipulate algebraic and basic differentiation skills in an ACOP solution and those that couldn’t.
**Research question 7a:** Are students able to solve ACOP problems completely?

The center point of this task was to measure students’ skills or lack of skills in ACOP solution process, as well as finds a point of weaknesses that were either algebraic, geometric or basic differentiation skills.

Evidence from the statistical output shows a mean score of 1.83 out of 9 possible points, with a standard deviation of 1.41, and a mode of 2. The minimum point score is 0, while the maximum is 6.

Overall sample performance on this tasks base on this output shows that 106 students (69.4%) score between 0 to 2 points, while 49 (31.6%) earn between 3 to 6 points. Drawing conclusions based on these facts indicates that a significant segment of the sample is unable to solve a complete ACOP successfully. When compared with other tasks in the test, it emerges with the least performance.

**Regression Analysis**

Research question 5a*: Are there any relationships between finding the required equation/model of an ACOP and diagram construction, labeling diagram, associating diagrams with geometric formulas/equations, and manipulating/transformation symbols skills?

This research question assessed whether there was a relationship between getting the required equation/model of an ACOP and some algebra and geometric competences. A multiple regression analysis, ANOVA, and correlation were used to get the best predictors among the independent variables.

The multiple regressions assumptions are normality, linearity, homoscedasticity and co-linearity. The normality was checked using histogram and normal Q-Q plot of regression which shows that the assumption was met. The homoscedasticity is good, except for some outliers, and linearity was also met. Similarly, linearity, normality, and homoscedasticity for the predictors look
good from the Q-Q plot graphs. The co-linearity assumption was satisfied as was indicated in Table 4.3: there was either no correlation among the independent variables or it was low negative correlation. We can conclude from these results that the regressions assumptions were met.

A total of five variables are used in the multiple regression analysis for the study in which the dependent variable is ACOP Equation, whereas the independent variables are Diagram Construction, Label a diagram, Associating diagrams with formula, and Manipulating symbol skill.

Table 4.7: Pearson Correlation between Dependent and Independent Variables

<table>
<thead>
<tr>
<th></th>
<th>Diagram construction</th>
<th>Label a diagram</th>
<th>Associating diagram with formulae</th>
<th>Manipulating symbol skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACOP equation</td>
<td>0.375(0.001)*</td>
<td>0.431(0.001)*</td>
<td>0.239(0.003)*</td>
<td>0.235(0.003)*</td>
</tr>
</tbody>
</table>

* = Correlation is significant at the 0.05 level (2-tailed).
Dependent variable: ACOP Equation.
Independent variables: Diagram construction, Label a diagram, Associating diagram with formula, Manipulating symbol skill.

Table 4.8: Coefficient Correlations between Independent Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>Manipulating symbol skill</th>
<th>Label a diagram</th>
<th>Associating diagram with formulae</th>
<th>Diagram construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manipulating symbol skills</td>
<td>1</td>
<td>-0.091</td>
<td>-0.173</td>
<td>-0.71</td>
</tr>
<tr>
<td>Label a diagram</td>
<td>1</td>
<td>0.008</td>
<td>-0.409</td>
<td></td>
</tr>
<tr>
<td>Associating diagram with formulae</td>
<td>1</td>
<td></td>
<td>1</td>
<td>-0.271</td>
</tr>
<tr>
<td>Diagram construction</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

The null hypothesis is there is no correlation between the dependent variable (ACOP Equation) and the four predictors (Diagram Construction, Label a diagram, Associating diagrams with formula, and Manipulating symbol skill).
As shown in Table 4.7 and 4.8, there is low positive correlation between the dependent variable and the four independent variables, and independent variables are not well correlated among themselves which shows that they are not co-linear.

Table 4.9: ANOVA of Four Predictor Variables

<table>
<thead>
<tr>
<th>Model</th>
<th>Sum of squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>139.057</td>
<td>4</td>
<td>34.764</td>
<td>13.091</td>
<td>0.001</td>
</tr>
<tr>
<td>Residual</td>
<td>398.337</td>
<td>150</td>
<td>2.656</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>537.394</td>
<td>154</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.10: ANOVA of Three Predictor Variables

<table>
<thead>
<tr>
<th>Model</th>
<th>Sum of squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>133.750</td>
<td>3</td>
<td>44.583</td>
<td>16.676</td>
<td>0.001</td>
</tr>
<tr>
<td>Residual</td>
<td>403.643</td>
<td>151</td>
<td>2.673</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>537.394</td>
<td>154</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The null hypothesis is that there is no difference between the mean scores of the four and three predictors. As shown in Tables 4.9 and 4.10, the ANOVA results for both the four variables and three variables are \( F (4, 150) = 34.764, p = 0.00 < 0.05 \), and \( F (3, 151) = 16.676, p = 0.000 < 0.05 \), respectively. Since the F-values are significant for both situations, we can reject the null hypothesis and conclude that there are differences between the means. This can allow us to investigate the contribution of each variable as regards to predicting ACOP equation scores in the study.
Table 4.11: Stepwise Model Summary of Regression for Three Predictors

<table>
<thead>
<tr>
<th>Model</th>
<th>R</th>
<th>R Square</th>
<th>Adjusted R Square</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.431</td>
<td>0.185</td>
<td>0.185</td>
<td>1.692</td>
</tr>
<tr>
<td>2</td>
<td>0.478</td>
<td>0.228</td>
<td>0.218</td>
<td>1.652</td>
</tr>
<tr>
<td>3</td>
<td>0.499</td>
<td>0.249</td>
<td>0.234</td>
<td>1.635</td>
</tr>
</tbody>
</table>

1 Predictors: (Constant), Label a diagram.
2 Predictors: (Constant), Label a diagram, Diagram construction.
3 Predictors: (Constant), Label a diagram, Diagram construction, Manipulating symbol skill.
Dependent variable: ACOP Equation.

Table 4.12: Model Summary of Regression for the Four Predictors

<table>
<thead>
<tr>
<th>Model</th>
<th>R</th>
<th>R Square</th>
<th>Adjusted R Square</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.509</td>
<td>0.259</td>
<td>0.239</td>
<td>1.630</td>
</tr>
</tbody>
</table>

1 Predictors: (Constant), Manipulating symbol skill, Label a diagram, Associating diagram with formula, Diagram construction.
Dependent variable: ACOP Equation.

A stepwise multiple regressions, as indicated in Table 4.11, showed that multiple correlations $R$ increased from 0.431 with one predictor to 0.499 with three predictors. One predictor (associating diagrams with formula) was taken out of the model because the $t$-value was not significant at $\alpha = 0.05$. Similarly, $R$ square increased from 0.185 with one predictor to 0.249 with three predictors. This indicates that a variation in the dependent variable changed from 18.5% (one predictor) to 24.9% (three predictors). Again, when all the predictors are included in the model, it gives a multiple correlation, $R$ of 0.509, an increase of 0.010 from 0.499 when the model contained only three predictors. In a related development, $R$ square increased from 0.249 (three predictors) to 0.259 (four predictors). The change of 0.010 (1%) signifies an increase in the contribution of independent variables towards a variation in the dependent variable.
Table 4.13: Coefficients of the Regression Model for Three Predictors

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-1.116</td>
<td>0.63</td>
<td>-1.772</td>
<td>0.078</td>
</tr>
<tr>
<td>Label a Diag</td>
<td>0.393</td>
<td>0.098</td>
<td>0.315</td>
<td>3.995</td>
</tr>
<tr>
<td>Diag. construction</td>
<td>0.193</td>
<td>0.073</td>
<td>0.210</td>
<td>2.653</td>
</tr>
<tr>
<td>Manipulating symbol skill</td>
<td>0.162</td>
<td>0.079</td>
<td>0.147</td>
<td>2.044</td>
</tr>
</tbody>
</table>

Dependent Variable: ACOP Equation.

Table 4.14: Coefficients of the Regression Model for Four Predictors

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-1.530</td>
<td>0.693</td>
<td>-2.209</td>
<td>0.029</td>
</tr>
<tr>
<td>Label a Diag</td>
<td>0.394</td>
<td>0.098</td>
<td>0.316</td>
<td>4.020</td>
</tr>
<tr>
<td>Diag. construction</td>
<td>0.164</td>
<td>0.075</td>
<td>0.179</td>
<td>2.179</td>
</tr>
<tr>
<td>Manipulating symbol skill</td>
<td>0.142</td>
<td>0.080</td>
<td>0.129</td>
<td>1.776</td>
</tr>
<tr>
<td>Assoc. diag. with formula</td>
<td>0.141</td>
<td>0.100</td>
<td>0.107</td>
<td>1.414</td>
</tr>
</tbody>
</table>

Dependent Equation: ACOP Equation.

The regression model for the three predictors is given as:

\[
ACOPM = \text{DIAGCONS (0.193)} + \text{LABELDIAG (0.393)} + \text{MANSYM (0.162)} - 1.116
\]

As indicated in Table 4.10, the t-values for the three predictor variables are all significant at 0.05 values. The Standardized Beta values for the predictor variables indicate the contribution of the individual predictors to the dependent variable in the model. As contained in Table 4.13, Label a diagram contributed more than the other two predictors, followed by Diagram construction, and the least input to the dependent variable by the predictor variable was Manipulating symbol skills. To test the significance of including the three predictors in the model, a sampling
distribution F distribution for the test statistic was used. The F critical is $F(3, 150) @ 0.05$ level of significance is 2.68. Since the computed value (25.03) exceeds the critical of 2.68, the null hypothesis is rejected and concluded that the inclusion of the three predictor variables is statistically significant.

The regression model when all the four predictor variables are included is given below:

$$ACOPM = \text{DIAGCONS} (0.164) + \text{LABELDIAG} (0.394) + \text{MANSYM} (0.162) + \text{ASSODIAG} (0.141) - 1.530$$

The t-values for the four predictor variables (see Table 4.14) suggest that two are statistically significant while the other two are not at $\alpha = 0.05$. The table also supports this claim because the standardized Beta values point out that they are the least values of 0.107 and 0.129. This is a sign that they added less to the variation of the dependent variable.

As was stated earlier, the R square has changed from 0.499 to 0.509, (see Tables 4.11 & 4.12) a change of 0.010 (1 %). This indicates that the inclusion of the predictor (ASSODIAG) in the model accounts for an additional 1 % of the variation in the dependent variable, i.e. ACOP Equation.

To test the statistical significance of this variation we have used the following formulae, and:

$$F = \frac{(R^2_1 - R^2_2)/(k_1 - k_2)}{(1 - R^2_1)/(n - k_1 - 1)}$$

$$= \frac{1/1}{(1-0.509)/(150)} = 0.00667$$

The F critical, $F(1, 150) @ 0.05$ is given as 3.92. Since the computed value is less than the critical value, then we do not reject the null hypothesis and conclude that the addition of the predictor ASSOCIATING DIAGRAM WITH FORMULA resulted in a statistically not significant increase in the multiple R. Other evidence (see Table 4.14) indicates that, the t-value
for the variable was not significant and the standardized Beta value was also the least among the four variables. It can be concluded that the best model is the one with three predictors (Table 4.13).

Based on the evidence and facts, it can be concluded that there are relationship between equation/model of an ACOP and diagram construction, labeling diagram, associating diagrams with geometric formulas/equations, and manipulating/transformation symbols skills.

**Frequency Analysis of Students’ Performance Profile**

<table>
<thead>
<tr>
<th>Profiles</th>
<th>Above Average Frequency</th>
<th>ACOPM</th>
<th>DIFFOV</th>
<th>OPTPBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAAA</td>
<td>35</td>
<td>21</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>AAAB</td>
<td>37</td>
<td>13</td>
<td>13</td>
<td>24</td>
</tr>
<tr>
<td>AABB</td>
<td>38</td>
<td>13</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>ABBB</td>
<td>30</td>
<td>3</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>BBBB</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

The quantitative frequency results from Table 4.15 indicated that 35 students (22.6%) obtained an above-average score in all the four sections that assessed their algebraic and geometric prerequisite competences. Within the above-average stratum for the four competences, 21 students (60%) obtained above-average scores in derivation of ACOP model/equation, 20 students (57%) in solving partial ACOP, and 25 students (71.4%) or (16.1% of the whole sample) in solving the complete ACOP (optimization). The low frequencies of students achieving above-average for ACOPM and DIFFOV within this profile relative to ACOP may seems mysterious, in that the competencies needed to do ACOP includes the competencies needed to do ACOPM and DIFFOV. As explained, below, a possible explanation of this seeming anomaly may be the overall difficulty of the ACOP items, causing a basement (floor) effect.

According to Colman (2001), floor effect is an artificial lower limit on the value that a variable can attain, causing the distribution of scores to be skewed. The distribution of scores on
an ability test will be skewed by a floor effect if the test is much difficult for many of the
respondents and many of them got zero scores. Looking further into the frequency data indicated
that, there are 36 (23.2%) students who score zero, 28 (18.1%) scoring one point in complete
ACOP section of the test. Overall, 41.3% of the sample has score either one or zero. These are
sufficient reasons to cause the distribution to skew.

The inherent difficulties associated with this section of the test were inferred from
comments made by students on their test performance within the 35 students who obtained
above-average scores in all four competences. Later in the section, other general comments
reveling sources of difficulties will be elaborated. For example, after making an attempt to
answer question 7c, a student within the first group wrote “I am not sure how to work out this
problem.” What can be inferred from this statement was that, she/he could not set up the
equation after partial visualization of the word problem. Setting the equation up was an
important component in the solution process of complete ACOP. Still on 7c, a different student
asserted “I have no idea where to start.” This was coming on the heels of lack of visualization
ability. The student’s statement indicated lack of familiarity or experience with rain gutter. As
claimed by Hershkowitz (1990), the two factors, familiarity and experience have influence on the
students’ visualization ability. Another student on question 7b claimed, “needed a calculator to
find the dimensions” of the poster with minimum sides. This had happen after completely
visualizing and labeling sides of the required diagram. The approached adopted here was
arithmetic, trying different numbers that are likely factors of 384. Clearly, the students couldn’t
set up the equation to solve the problem despite knowing what the question requires. In the same
trend, on question 7a, a student stated that “I don’t know how to find the length.” This statement
was made after the student had successfully visualized the rectangular box. It is worth noting
here that setting up the required equation base on the constraints given in the word problem was
difficult, hence the compelling revelation. Base on these exposures, it could be inferred that lack of visualization ability, failure to set-up required equation base on given constraints thwarted students’ efforts in this subgroup to successfully accomplish what they are expected to achieve.

Further examination of the results indicated that 127 (81.9%) have finished or attempted all the questions (section seven inclusive) in the test, against 28 (18.1%) who did not finished or failed to attempted all the questions. There are also only 6 (3.9%) students who had written on their test performance script that “they ran out of time.” Additional close inspection of the quantitative outcome showed that all the 25 students who obtained above-average performance had also finished the test. Within the context of those who finished or attempted all the questions, some students make statements that exposed their point of weaknesses. These includes “I do not understand optimization”, “just learned optimization, so I am doubtful that I will produce a successful answer”, “had trouble setting it up”, “I can’t answer because it involves visualization and trigonometry”, “not good with trigonometry and understanding the concepts”. It can be inferred base on these statements that students have difficulties integrating algebraic or geometric competences at one point or another into complete ACOP solution processes, hence causing lower performances in this section.

Another look at the vast majority who were unable to obtain above-average scores in all the four competences was further subdivided into four categories. The first group (see Table 4.15) consists of 37 students (23.9%) performed below-average in one of the four competences. Within this group, 13 students (35.1%) performed above-average in ACOP model/equation derivation tasks, 13 students (35.1%) in solving partial ACOP, and 24 students (64.9%) or (15.5% of the entire sample) in complete ACOP. The second group consists of 38 students (24.1%) performed above-average in only two competences. Inside this stratum, 13 students (34.2%) performed above-average in ACOP model/equation tasks, 15 students (39.5%) in partial
ACOP, and 16 students (42.1%) in solving complete ACOP. The third group had 30 students (19.4%) who performed above-average in only one competence. Within this level, 3 students (10%) performed above-average in ACOP model/equation derivation tasks, 10 students (33.3%) in partial ACOP solution, and 16 students (53.3%) in complete ACOP solution. The last group consisted of 15 students (9.7%) who were unable to score an above-average performance in any of the four competences. Within this level, no student was able to obtained above-average score in either ACOPM (ACOP model/equation) or DIFFOV (partial ACOP), but surprisingly, 5 (33.3%) were able to secure an above-average performance in complete ACOP solution. This can be attributing to the basement effect for scores in the section.

Table 4.16: Pearson Correlation between ACT and All Variables

<table>
<thead>
<tr>
<th></th>
<th>Diagram const.</th>
<th>Label diag.</th>
<th>Associating diagram with formula</th>
<th>Manipulating symbol skills</th>
<th>ACOP equation</th>
<th>Differentiating/ opt a variable</th>
<th>Optimization problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACT Scores</td>
<td>0.440</td>
<td>0.405</td>
<td>0.219</td>
<td>0.168</td>
<td>0.385</td>
<td>0.232</td>
<td>0.312</td>
</tr>
<tr>
<td>( )* = significant at $\alpha = 0.05$.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.16 showed a correlation between ACT scores and all variables used in the study. There were low positive correlation between ACT and four out of seven variables (diagram construction, label a diagram, ACOP equation, and optimization). This indicated a lower positive linear relationship. On the other side, there were little if any correlation between ACT scores and three of the remaining variables used in the study (associating symbol skills, manipulating symbol skills, and differentiating /optimization a variable. This indicated somewhat little linear relationship.

Summary of the Quantitative Data Analysis

All seven variables used in the study could be classified as either isolated and non-isolated tasks. Isolated cases are tasks that require students to represent geometric word problems
diagrammatically, solve algebraic problems with connections to others areas such as trigonometry. The non-isolated tasks are where geometric, algebraic and trigonometric knowledge come to play in the solution processes of complete word problems. The isolated tasks are constructing diagrams, label a diagram, associating a diagram with formula and manipulating symbol skills. On the other hand, the non–isolated tasks include deriving ACOP equation, differentiation of variable, and lastly optimization problem. Overall, based on the statistical information (see Table 4.6), it can be inferred that majority of students in the sample shows some strength of performance in the isolated tasks, with relative exception of manipulating symbol skills. Further evidence of the high-quality performance was manifested in the lower variability among the isolated cases with the exception of one variable (diagram construction). They were able to develop understanding, follow procedures, and make connections where necessary (in explicit and implicit cases) in order to successfully complete the tasks as required. Still within the isolated tasks, there are some that tend to follow the procedures mindlessly, without making connections where desirable.

On the other hand, i.e. the non–isolated tasks, statistical evidences (mean scores and standard deviation) indicate that, generally, students’ performance was lower than that of the isolated tasks. Comparing the mean scores for the three non–isolated tasks to that of the isolated, it comes out they are by far less. Similarly, the standard deviation are also higher, pointing out that there are more variability among them than the isolated tasks. The majority of students were unable to explore, understand the nature of the mathematical concepts, processes and establish relationships as required by the tasks. Some found it hard to regulate their thoughts in connection to the tasks. The end product of all these is that they failed to incorporate algebraic and geometric competences into complete ACOP solution processes.
**Qualitative Data Analysis**

The qualitative data were collected to support, illuminate, reinforce, enhance and supplement the outcome found from the quantitative part of the study. The qualitative data were generated using interviews with 15 students. They were selected using sequential mixed method sampling strategy. According to Teddlie & Yu (2007), this sampling strategy involves selection of the unit of analysis for a mixed method study through the sequential use of probability and purpose sampling strategies. The probability sampling used in selecting the interviewee is simple random sample, while the purposive sampling is convenience sampling strategy. This portion of Chapter four will be concluded with a summary of the qualitative results.

The focal point in this study was to measure students’ skills or absence of skills in complete ACOP solution process. Are these students in calculus I who have just been taught ACOP solving able to solve typical ACOP problems? This is the context in which the intentions of the further qualitative analysis were based. The following is the detailed data analyzed based on individual research questions.

**Research question 1b**: How do students construct diagrams to represent word problems?

The focus of this research question is to measure students’ knowledge of basic geometric shapes as well as their application to physical situations. It also takes into account students’ visualization ability and how it is incorporated into basic geometric shapes such as rectangle, triangle, circle, and cylinder, and so on. The visualization capability is expected to be used fluently.

Three test items were designed/adapted from calculus textbooks by the researcher to assess these attributes of the first research question. The concepts assessed by the test items in this section include distinguishing a square from a rectangle as well as locating the squares that are supposed to be cut upon the rectangular diagram; visualizing a cylinder inside a hemisphere;
and understanding the concepts of cylinder, hemisphere, and inscription. Other anticipated conceptual difficulties comprise visualizing a movement from ordinary plane paper (two-dimensional) to solid (three-dimensional).

Results from the quantitative portion of the test (see the first section of Chapter 4) indicate that the average score is 6.23 out of maximum of 9 possible points. The results also show that 70 students (45.8%) scored between 0 & 6; whereas 84 students (54.2%) scored between 7 & 9 points.

From the indicated quantitative outcome, the sample can be divided into two categories: above and below expectations. The above expectations accounted for more than fifty per cent of the sample in this isolated task, because they performed minimal mistakes in their solution process. The isolated task is a task that requires students to represent geometric word problems diagrammatically. A qualitative illustration of a representative sample from each group, above and below expectations, will be given below.

Subject W-2 has demonstrated an ability to visualize the geometric word problem fluently as well as in writing. (W–represent a section of the calculus I selected to take the test, and 2 represent a second student selected randomly and conveniently agree to do the interview). Below gives an account of our interaction:

RCH: I want to know how you interpret the question.

W-2: You have cardboard box and in making it (demonstrating with hand how to fold it).

RCH: So you visualize it before you write?

W-2: Yeah

RCH: How about the second one?
W-2: For this one I wasn’t quite sure how to do the top…kind of a cut off. The visual representation doesn’t need to be perfect. But again, you just need to visualize, because it is purely conceptual.

W-2’s test script and verbal demonstration have indicated an understanding of the relationship between concept definition and concept image (Hershkowitz, 1990) of rectangular box. The interviewee clearly describes a rectangular box when the four squares are cut out before it was bent up. Similarly, the concepts of hemisphere, cylinder and inscription were perfectly explained according to what the task requires. This portrays what is in his mind and was able to put it in writing, however, the interviewee had some little difficulty reaching a consensus between what a sector looks like and the actual concept of a sector (concept and concept image). The only set back shown by the interviewee in this task is the inability to draw with correct properties, the diagram of a sector, as shown in figure 4.1. Below is the reproduced diagram of what W-2 draw to represent a sector:

![Fig. 4.1](image)

On the other hand, among those who scored below average on this task, as was revealed from the interview results, the knowledge of basic concepts of geometric diagrams as well as misunderstanding the question poses some problems. To illustrate this deficiency, T-20 claimed that “I didn’t understand what the question says. All that I knew was circular cylinder and the rest, I didn’t know…such as hemisphere, inscription”. What provoked this claim was her insufficient knowledge of basic geometric shapes. In a similar vein, V-2 demonstrated the
absence of basic geometric concepts in the course of our interaction during the interview. The excerpt from the interview is given below:

RCH: Let us start with question 1b, even though you got 1a right, tell me how you solve it?

V-2: Uhm…right circular cylinder…I don’t know what that is…I have never heard of that. I don’t know if it is like a cylinder or it has a right angle in it. But I know it is inscribe in a hemisphere.

RCH: And you think this is a hemisphere?

V-2: Yeah…probably wrong (laughter).

RCH: Can you give an example of a hemisphere in real life?

V-2: Just like the earth is circle…I don’t know…I mean…

RCH: You are not sure of the concept of a hemisphere?

V-2: Yes.

RCH: How about question 1c.

V-2: I have probably didn’t know how to start.

RCH: Okay, but you have a cone right?

V-2: Uhm…I got it from here (pointing at diagram from question three).

RCH: What is a sector? Did you know what a sector is?

V-2: No.

RCH: Can you recall a situation where you learn about a sector?

V-2: I may probably do…I don’t remember that.

V-2 clearly shows some deficiency or absence of facts on basic geometric diagrams. For example, the concept of hemisphere was illusive even though there was an attempt to make a connection with real life, which itself was hazy as illustrated in Figure 4.2. In the interviewee’s
test script (see Figure 4.2), a cylinder was inscribed in a circle (assuming circle to be a sphere), but the task required a hemisphere. So according the interviewee’s understanding as indicated by the test script, a sphere is considered to be a hemisphere. In the course of interview, the student was not sure whether the solution given, i.e. Fig. 4.2, was correct. On a related issue, the concept of a sector again poses a setback to the interviewee. Evidently, from the student’s written work (Fig. 4.3), it was revealed that the student was unable to cut a sector from a circle as required by the task.

Fig. 4.2                                                                  Fig. 4.3

In summarizing the qualitative results for the first research question, most students showed a good understanding of the basic geometric shapes that are represented in word problems, even though there are some that illustrated lack of this knowledge. There were also indications that visualization ability was a problem among the students, most especially moving from 2-dimensional to 3-dimensional diagrams. Misunderstanding or unfamiliarity with the concepts of hemisphere and inscription were a hurdle among below average performing students in this task. Other revelations from the qualitative results are that few claim misunderstanding of the question, which was clearly grounded on their insufficient facts on basic geometric concepts. Overall, establishing a relation between concept definition and concept image as it is reflected in the minds of individuals (Hershkowitz); what (Vinner, 1983) called the product of concept formation process in the mind proved difficult.
**Research question 2b:** How do students label diagrams appropriately/inappropriately using variables?

The focus of this research question is to assess students’ conceptual understanding of variables and their applications to new, independent and appropriate situations.

There were three test questions designed to assess these targeted qualities. The concepts evaluated include expressing dimensions using a single variable via the terms “twice at” or “a third of”; the concept of dimension itself; and establishing relationship between $L$, length of the rectangular piece of metal and $2\pi r$, circumference of the top of the resultant cylinder. Another conceptual hurdle includes labeling the sides of non-routine and non-canonical geometric diagram using only four variables.

Information derived from the quantitative portion of the study indicated that the mean score for this test item is 6.34 points, with 7 points being the most frequent score. The quantitative data from this research question can be divided into two categories, using the average score as a relative cut off point. This indicated that 63 students (40.7%) scored between 0 to 6 points; whereas 92 (59.3%) achieved a score between 7 to 9. Using this statistical platform and mean score as benchmark, the two categories are those above and below expectations.

A general inference drawn from the examination of the qualitative data was that the interviewees vary on their capability and the majority of the below average students found it difficult to use geometric (mathematical) language, for example the dimensions of a cylinder. Moreover, a establishing relationship between the length of a rectangle and its symbolic representation in the new transformed situation (cylinder), as well as picking four variables (two each horizontally and vertically) for the labeling of non-canonical geometric figure remained a very difficult task.
V-15 was among the few who was able to fully come to terms with the concept of the dimensions of a cylinder, and in fact set up a relationship between length $L$, of rectangular piece of metal and the perimeter (circumference of the top of the cylinder), before expressing the required dimension, i.e. $r$ in terms of $L$. Below is the excerpt reproduced:

$$P = L$$
$$P = \pi 2 \ r$$
$$L = 2 \pi r$$
$$\therefore r = \frac{L}{2 \pi}$$

Among the below-average students a majority of them including L-7, J-23, and S-3 found it difficult to interpret the question. L-7 clearly had no idea what the dimensions of a cylinder were at the beginning, but later picked it up when the concept was associated with some familiar situation. On the other hand, J-23 and S-3 had a similar struggle even when a familiar situation was cited; but still they could not get it. Below was how the conversation went with J-23:

RCH: When you transform the rectangle into a cylinder, you generate a circumference here.
J-23: Right.
RCH: And then you generate a height so, what are the dimensions of a cylinder in your new object?
RCH: Lets look at here, what are the dimensions of a rectangle?
J-23: Length times width.
RCH: Then what are the dimensions of the cylinder?
J-23: Aaa…are going to take into account volume or not necessarily? Is that what you are asking?
RCH: No…no. The dimensions….you know what the dimensions means?
J-23: Yeah….are the ….would the circumference, plus the width…
The concept of dimension as mathematical (geometric) language caused some hardship for J-23 in doing calculating labeling. Calculating labeling tasks require setting up a relationship between two expressions and a variable of interest which is expressed in terms of the other(s).
For example, using the relation (equation) \( L = 2\pi r \) to express \( r \) in terms of \( L \). The student showed misunderstanding of the meaning of dimensions itself as well as its application to the specified situation. The student tried as much as possible to guess what the meaning of dimension was by making a statement and asking if the statement referred to dimension, as well as by thinking of dimension in the sense of 1-dimensional, 2-dimensional, 3-dimensional geometric figures. This is an innovation to cover up for the difficulty experienced as well as an effort to fill-in the gaps. Two separate situations like “. . . take into account volume or not necessarily?” and “. . . would the circumference” show these dual novelty of “cover-up and fill-in” effort.

In the task that involves the non-canonical geometric diagram, V-2 demonstrated a good understanding of the relationship between horizontal and vertical sides of the figure and was able to use it successfully. My interaction with V-2 follows:

RCH: Okay…lets look at last part of 2c…can you explain how you arrived at your solution?
V-2: Because this is L and if you subtract that, you got that small part.
RCH: So you are able to relate the longer distance with these smaller ones?
V-2: Yes.

In conclusion, students’ inability to fully grasp the concept of dimension from geometric perspectives with particular reference to a cylinder leads to misunderstanding what the task requires and consequently to making mistakes in the solution. Similarly, non-familiarity (lack of connection) with the concepts again poses some hindrance to the success of this task. The overall adverse effect of this shortcoming is that students are unable to express \( r \) in terms of other variables. To a certain extent, some cannot even correctly interpret the statement “express \( r \) in terms of \( L \)”. This is a clear deficiency in students’ mathematical language. The non-canonical task was done relatively well with a majority of the students establishing a good relationship
between the sides (longer and shorter horizontal and vertical) of the geometric figure. It can be inferred from this activity that a good understanding and usage of the concepts of variables was well accomplished. The explicit labeling activity involving the use of variables was also fully achieved as a significant portion of the students was successful.

**Research question 3b**: How do students associate geometric diagrams with appropriate algebraic equation(s)/formula(s)?

The focal point of this research question is to assess how students recognize and associate geometric diagrams with their formulas. They are expected to match geometric formulas with geometric diagrams from the pool of formulas and geometric diagrams.

The test instrument designed by the researcher to appraise these characteristics contains six geometric diagrams (2 and 3–dimensional), while there are twelve formulas in the pool. Both formulas and diagrams are arranged randomly.

Although some formulas may be more or less familiar than others, the knowledge of formulas and their relationship with geometric diagrams is factual information. The items explored whether students experienced some difficulty finding an appropriate match as well as whether they could distinguish between 2-dimensional and 3-dimensional diagrams. Moreover, the items examined students’ capability in differentiating the concept of area and surface area with particular reference to 2-and 3–dimensions.

Statistical results from quantitative data showed an average score of 4.7 points out of maximum 6 points possible, with incidentally, the most frequent score being 6. The samples were grouped into two using the mean as a benchmark. The first group has a range of 0 to 4.5 points with 61 students (39.3 %); while the second group had a range of 5 to 6 points with 94 students (60.6 %). The two groups can be classified as below and above expectations. Further qualities of these groups will be explored below.
Generally, students used process of elimination to associate appropriate geometric diagrams with their formulas. This can be called use of strategic knowledge in a negative way. Strategic knowledge refers to knowing when to use a rule. But process of elimination can be used even if the rules are not well known. The below expectations category of students used non-conceptual processes, leading to circumstances where 2-dimensional figures ended up with a volume formula or a 3-dimensional diagram got an area formula instead of surface area formula. A good example of the use of non-conceptual process of elimination emerges from my interaction with T-20 and our conversation went as follows:

RCH: Okay, almost everything here is perfect (question 3) except amm…what is this a rectangle right? The first diagram here is a rectangle right, so here you say volume, does a rectangle has a volume?
T-20: (Laughter)…..no.
RCH: So what happens?
T-20: I have no idea.
RCH: You just probably write.
T-20: Because I was just doing the process of elimination. I don’t know why I am doing this.
RCH: Here you have a box and left it blank, probably you want to write it here but you didn’t. Since you follow the process of elimination, can you describe how get the volume of a cylinder?
T-20: Amm…Okay…I didn’t know …but I knew this is a circle…

Despite the fact that T-20 succeeded in associating the given formulas with their appropriate diagram, the procedures followed were non-strategic. Similar characteristics were shown by L-7 and V-2.

The strategic use of process of elimination included a good knowledge of and/or familiarity with basic geometric diagrams and their formulas. J-23 and W-2 used this approach and were able to successfully complete the task. Specifically, J-23 separated or grouped 2-dimensional geometric diagrams from the 3-dimensional ones before associating them with their appropriate formulas. Below gives an account of how our interaction went:
RCH: Talk to me about your understanding/misunderstanding of associating geometric diagrams with appropriate formula?

J-23: So following each formula with it figure...for area...perimeter...okay. This...this...this and this (circling the 3 – D) have surface areas...they are the only 3 – D shapes.

RCH: Okay.

J-23: ...and like I cross them out, and I use ...for some of them I use process of elimination. But for the circle I knew...the area and the perimeter...and for the rectangle I knew the area and perimeter. Then, from there...the cube was the only one you can use L, because the cube all the sides do the same. The only way you can do length times width times height, all of them be the same. So that the way you do the cube...and then for it surface area, there is six sides and each side is L times L to get the area of it and then times six, because there are six of them. For the rectangle (pointing at rectangular box), I knew that the area (volume, added mine) is length times width times height. For the surface area is the same thing, two of the length times height (2lh), two of the width times height (2wh), and two of length times width (2lw). The cone...cone and cylinder ...am...I actually I don’t remember how I got those, only because.....let me think (silently). I won’t say I knew this was the volume because we just done that...in class we just went over about the volume of the cylinder.

RCH: Okay.

J-23: So I knew the volume was this (pointing at the formula). This has to be the volume by default because there are no other volumes. (RCH interject with laughter). And so...

RCH: So you mean by default....since there are no any other options, this has to be the volume?

J-23: Right,...this has to be the volume.

RCH: You have already eliminated....

J-23: Three out of four....and then amm...let see ...this is the surface area that seems to go with... (long silence) this I guess ....because...I think about that, but this...if actually this work together ...then surface area would be smaller than the volume .....that is correct. Am so...I choose ..... RCH: \( \pi rL \)....

J-23: Yes....so the cone and the cylinder had a little bit of trouble and it was more of a kind of process of elimination ...actually like figuring it out. And because this was two and two, this would be one and the bottom would be one, this side of it and the back side of it would be one.

RCH: \( \pi rL \)....

J-23: Yes....so the cone and the cylinder had a little bit of trouble and it was more of a kind of process of elimination ...actually like figuring it out. And because this was two and two, this would be one and the bottom would be one, this side of it and the back side of it would be one.

The evidence indicates that the student has the ability to create ways of handling tasks that require matching of concepts, in addition to using previous experiences that are relevant. This shows J-23 is resourceful; but certainly, J-23 lacked knowledge of the formulas for the cone and cylinder. The formulas have to be found by elimination, since all other options are gone.
In summing up the qualitative results, students experienced some setbacks finding suitable matches between formulas and diagrams, and distinguishing between 2-dimensional and 3-dimensional diagrams, whereas others efficiently did the task. The hurdles encountered are grounded on the students’ unfamiliarity with geometric formulas and difficulty recognizing some shapes by name. Overall, process of elimination was used by many students as strategic knowledge, but it was use in a negative way. Use of strategic knowledge is to know when to use the rules. Despite the fact that some knew the formulas as well as the geometric diagrams, finding an appropriate match without the use of process of elimination was hard. Another important finding from the study was students were unable to establish proper relationships between concept definition and concept image with respect to area and surface area of 2- and 3-dimensional geometric diagrams.

**Research question 4b:** What are students’ strengths/weaknesses in symbol/transformation skills?

This research question assesses students’ ability to manipulate any algebraic or non-algebraic equation, irrespective of whether they are related to ACOP.

Three test items were conceived with the intent of assessing those characteristics stated above. Conceptual areas probed included expressing algebraic fractions as single; negative/positive sign errors in computation; and identifying and understanding the structure of quadratic equations and their solutions in two variables. Others consist of recognizing the quadratic structure in trigonometric equations and their solutions and how they are connected to algebraic quadratic equations; use of appropriate trigonometric identities; and detecting erroneous solutions and eliminating them from the solution set.

From the quantitative portion of the study, results obtained indicate a mean score of 3.78 out of 9 possible points. The frequencies of scores were grouped in two and classified as below
and above expectations. 108 students (69.7 %) were below expectations, and 47 students (30.3 %) were above expectations. The below expectations category have made many mistakes that can’t be ignored ranging from structural to procedural, whereas the above expectations mostly commit procedural mistakes. Qualitative results will be used to elaborate on these categories.

W-2 approached this task in question 4b from arithmetic frame, which is using trial and error and substitutions. The consequences of this attempt led him to lose one set of solutions for the quadratic equation without detecting realizing it. It can be argued that W-2 couldn’t relate the quadratic solution to its structure. The quadratic equation’s structural unfamiliarity led to an inability to recognize quadratic equations in two variables. The combined consequences of these deficiencies (operating from arithmetic frame and failure to recognize quadratic equation structure) led to a breakdown in making connections to the trigonometric equation solution process.

The probe “can you explain to me what the question is asking?”, S-3 responded by giving an explicit explanation about the procedure required to solve the equation (4b), but putting this procedures into practice seems difficult due to some algebraic error that are either non-intentional or careless. Overall, S-3 can visualize the solution conceptually, and below shows some erroneous solutions, even though the procedures are explained conceptually.

The two systems of equations given to students to solve are:

\begin{align*}
2x^2 - y &= 0 \\
3x - y &= -2
\end{align*}

S–3 attempted the solution by first expressing \( x \) in terms of \( y \) from the second equation, i.e.

\( x = \frac{-2 + y}{3} \) and then, substituting it in the first equation.

The solution process after substituting for \( x \) in second equation was given below:
Structurally, S-3 has lost the 3 by not squaring the denominator and then begins to eliminate the same by multiplying by 3. Again, the second term in the equation needs to be multiplied by the square of 3, but S-3 missed that important step. Another algebraic error is failure to multiply 2 with all the three terms in the bracket i.e. \(2(y^2 - 4y + 4)\) ≠ \(2y^2 - 8y + 4\). Similarly, S-3 was unable to add \(-8y\) and \(-y\) and would not factor the assumed correct equation successfully: \(2y^2 - 7y + 4 \neq (2y - 8)(y - 1)\). These chains of algebraic mistakes prevent S-3 from getting the correct solution. Even though, S-3 has demonstrated the strategic knowledge of the solution process verbally, from our interaction given below, that didn’t come up well procedurally (Mathematically):

RCH: Tell me about your solution in 4b?
S-3: (shook her head in disapproval or lack of confidence).
RCH: Can you explain to me what the question is asking?
S-3: What I did was . . . I was trying to solve for y here. . . one of the variables and plug the variable in the second equation and solve for the other variable. I didn’t work it out the math way.
RCH: That means you really understand the procedures but using it create some kind of problems?
S-3: Yeah.

Lack of a define strategy in the solution process of equations in question 4b and 4c as well as making connection to the trigonometric equation had been manifested by T-20.
Specifically, T-20 found it difficult to adapt a solution process for equation 4b, and evidence was in our interaction given below:

RCH: Tell me about question 4b?
T-20: Oh I didn’t do it right…I solved for y and it wasn’t coming up easy as I thought. I changed and solved for x and put in there.
RCH: If you solve for y from here it could have been easier.
T-20: I think the first time I did it, I must have done something wrong.

In an attempt to solve question 4c, which is a trigonometric equation, V-2, L-7, W-2, and S-3 can only go as far as clearing the fraction, but other algebraic processes required are elusive. For example these 4 students all successfully calculated:

\[
\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \sqrt{3}
\]

\[
\frac{1 + \cos \theta}{\sin \theta} = \sqrt{3}
\]

They were all able to make connection from adding algebraic fraction to adding trigonometric fractions. But this is where most of the students stop, although V-2 attempts three more steps by clearing the fraction as well as trying to solve for theta in futility. Clearly, V-2 was using non-strategic knowledge to solve the equation after clearing the fraction. For example, V-2 was unable to square both sides of the equation get to rid of the radical. Moreover, the product generated if both sides of the equation were squared, together with the use of appropriate trigonometric identities can lead to successful derivation of a trigonometric equation (quadratic in structure). The solution process has to be connected to an algebraic quadratic equation, which is another hurdle most students were unable to cross. Finally, erroneous solutions have to be eliminated from the solution set. Below is the reproduced work of V-2:
Exploring further among the unsuccessful ones is V-15 on the solution of 4c. V-15’s approach was trial and error using substitution for the value of theta but without success. Below was the reproduce work of V-15:

$$1 + \cos \theta = \sin \theta \sqrt{3}$$

$$\cos \theta = \sin \theta \sqrt{3} - 1$$

$$\cot \theta \left( \frac{1}{\sqrt{3}} \right) + 1 = 0$$

Finally, the value of theta calculated was within the given range, but didn’t satisfy the equation; hence theta value was erroneous. Despite this erroneous property that the solution has, the student couldn’t realize it, because of the non-conceptual approach to the whole solution process.

On the other hand, among the partial success story is J-37’s, whose solution process approach was algebraic. J-37 successfully cleared the trigonometric fraction as well as getting rid of the radical. In order to obtain the solution, J-37 strategically used appropriate trigonometric identity but the simplification processes were met with algebraic errors. Specifically, the errors were related to collecting $\cos^2 \theta$ like terms. Below gives the reproduce solution of the problem:
\[ 1 + 2 \cos \theta + \cos^2 \theta = 3 + 3 \cos^2 \theta \]
\[ 1 + 2 \cos \theta = 3 \]
\[ 2 \cos \theta = 3 - 1 \]
\[ \cos \theta = 1 \]
\[ \therefore \theta = 0 \]

The second step led to failure in getting the correct value(s) of theta as a solution.

In summarizing the outcome from this research question, the conceptual difficulties that students experienced and exhibited include an inability to express algebraic fractions into single fractions; occurrence of negative/positive sign errors in computation (non-intentional or otherwise); failure to recognize quadratic solutions from its structure; as well as identifying a quadratic equation in trigonometric form (structural). Other difficulties included lack of ability to appropriately use trigonometric identities in the solution process.

**Research question 5b:** How do students use prerequisites skills to find the model in an ACOP?

The required equation for the solution in a given ACOP can be derived straight away. In other cases, it involves transformation/symbol skills of one to two equations to get the required result, as well as the use of other geometric concepts.

There are three test items designed to evaluate these competences mentioned above. The degree of difficulty varied among the different test questions in this section. It is hypothesized that the conceptual difficulties associated with this test question include representing the dimensions using the concept of “twice the”; finding relationship between actual volume (numerical, given) and algebraic volume (formula). Others are difficulty equating the numerical perimeter (given) and the perimeter of the expected figures using labeled variables (algebraic); finding areas of the figures (i.e. square and circle) in terms of a single variable; as well as coordinating the concept of a similar triangle to deduce the required area formula.
Quantitative test results show the mean score of 3.21, out of 9 possible points. Two groups were generated and classified as below and above expectations. 120 students (77.4%) were below expectations, and 35 students (22.6%) were above expectations. It can be inferred from these statistical data that a greater percentage of the sample fell below expectation and are unable to successfully derived the required model from the given word problem. This conclusion will be buttressed further with qualitative results below.

Overgeneralization of the surface area formula have led J-23 not to detect what was irrelevant (i.e. using closed rectangular surface area formula instead of the open rectangular as required by the task in question 5a), despite establishing needed relationship between volume and area base on information given. J-23 was able to set up the required model after expressing the dimensions in a single independent variable. Below is the model and the individual dimensions expressed in a single variable:

\[
\begin{align*}
  h &= \frac{10}{2w^2} \\
  l &= 2w \\
  w &= w
\end{align*}
\]

\[
SA = \frac{30}{w} + 4w^2
\]

The dimensions expressed were correct, but the total surface area wasn’t correct, because the surface area of the top was included in the calculation.

W-2 was able to visualize and obtain the correct formula, but with a few mistakes due to oversimplification of the given formula (sharing the same attribute with J-23). Moreover, W-2 succeeded in relating perimeter (given in the task question 5b) with the perimeters of the required geometric diagrams, and was successful in expressing the variables in terms of the given perimeter.

Below shows W-2’s reproduced work:
\[4x + 2\pi r = 10(\text{perimeter})\]
\[x^2 + \pi r^2 = \text{area}\]
\[\frac{10 - 4x}{2\pi} = r\]

Sum of areas = \[x^2 + \left(\frac{10 - 4x}{2\pi}\right)^2\pi\]

This is a clear demonstration of understanding the relationship between area and perimeter. On a similar note, W-2 misinterpreted the task in 5c as trigonometric undertaking, despite labeling the sides of the inscribed rectangle, and claimed “we go back to trigonometry . . . that is where I have some trouble”. On the other hand, V-2 and J-37 were able to visualized the required diagram presented in word problem for question 5c, but labeling the unknown sides of the complex geometric diagram had generate a setback that had snowballing effect. For example, both understood the need to use similar triangle concepts but failure in correct labeling prevented J-37 from picking the correct dimensions. Below is reproduce work of J-37:

![Fig.4.5](image)

\[
\frac{3}{y} = \frac{4}{x} \quad A = (4 - x)(3 - y)
\]
\[4y = 3x \quad A = (4 - x)\left(3 - \frac{3x}{4}\right)\]

The labeling difficulty experienced here was related with double labeling of the inscribed rectangle. The student labeled the sides as \(x\) and 4 (horizontal), and \(y\) and 3 (vertical). The
students didn’t take into account the distance from the end of the inscribe rectangle to the tips of a triangle, which is supposed to be $4 - x$ and $3 - y$ horizontally and vertically, respectively. Moreover, this has complicated the choice of appropriate dimensions that are supposed to be use in the similar triangle relation. Despite conceptually understanding what to do, i.e. use a similar triangle concept to solve the task, failure in the labeling activity intertwine with picking correct length complicated the whole solution process.

On a similar note, algebraic statements such as “twice the” were successfully translated, but mathematical language such as “express in a single independent variable” became difficult for S-3. However the student succeeded in visualizing 5c, but still faced the challenge of labeling it using appropriate variables.

T-20 got the right model from question 5a using visualization and proper algebraic processes, but was operating from arithmetic frame when it came to establishing a relation between perimeter and area in 5b. T-20 states orally, the individual area formulas of the required geometric diagrams, but was looking at the possibility to substitute numerical values of the variables to answer the question.

To sum it up, some student exhibited a capability to visualize, label and used appropriate formulas for a 3-dimensional geometric diagram to derived the required equation, but the majority are met with conceptual intricacies on how to represent the dimensions using the concept such as “twice the”. Other evidences of difficulties found include how to express surface area in terms of the given dimensions as well as finding a relation between given (numerical) and algebraic (formula) volume. The biggest hurdle in the surface area task was the inability to detect the unwanted items in the formula, which normally reflect the nature of the word problem. Moreover, it was manifested many times that equating the numerical perimeter and the perimeters of the expected figures using labeled variables are conceptually difficult.
Consequently, this results in finding areas of the figures (i.e. square and circle) in terms of a single variable extremely hard. Other findings emerging from the qualitative data revealed that coordinating the concept of a similar triangle to deduce the final required equation as well as distinguishing the two legs from the hypotenuse in a right triangle, identifying and drawing right triangle itself and the concept of inscription appeared to be elusive. Moreover, labeling the remaining part of the length from the end of the inscribed rectangle to the tips of the triangle was equally difficult.

**Research question 6b:** How do students use calculus in an ACOP solution when the model is given?

This item will measure students’ ability (skill) to solve a partial ACOP. A partial ACOP is a problem which does not require the use of some prerequisite skills, especially geometric. The main focus of this test item is to assess whether students can solve an ACOP in which the required geometric parts of the solution are already given or not provided. This will show the extent to which geometric skills or their absences play a role in ACOP solution. Moreover, it will clearly assess students’ algebraic and basic differentiation skills in the ACOP solution process.

Two test items are designed to measure the expectations of this research question. The first contains application of differentiation skills and the second combines the use of algebraic and basic differentiation abilities. The skills include identifying and applying the basic rules of differentiation (power, quotient or product rule). Concepts include setting the first derivative to zero to get the stationary point, its nature and the principle behind that; recognizing the quadratic structure and its solution in the first derivative.

A test was administered to measure these skills quantitatively, and results indicate that the mean score is 3.16 out of 6 possible points. The sample was grouped based on valid scores ranging from 0 to 3 and 4 to 6 points as below and above expectations, respectively. 92 students
(59.3%) were below expectations, and 63 students (40.7%) were above expectations. Qualitative results from the interview with each subsample will broaden the description of the two groups.

J-23 was among the successful group and was able to transform the polynomial into index form for easy calculation of the derivatives, but was confused in applying the power rule because of anti-derivative (integration) learning currently taking place in class. Moreover the use of the second derivative test to find the nature of a stationary point was a success, but the reason for setting up the first derivative to zero to obtain the stationary point was conceptually obscured. In the same trend, L-7 perfectly came up with the derivative of the first equation using the quotient rule on the middle term and the power rule on the remaining terms. The test script didn’t show the full techniques of quotient rule; rather L-7 did it mentally and wrote the correct answer.

On the other hand, among the less successful ones was W-2, who fully understood and uses the procedures of finding the derivative of question 6b successfully, but committed a serious algebraic mistake leading to a wrong solution. Below is the solution process:

\[
V' = 300 - \frac{3b^2}{4} = 0
\]

\[
\frac{3 \cdot b^2 \cdot 4}{16} = 0
\]

\[
3b^2 = 75
\]

\[
b = 5
\]

The algebraic mistake was dividing 300 by 4 instead of multiplying it to get the correct outcome. On a similar pattern, L-7 solved 6b, which requires straight application of the power rule to get the first derivative, but instead, L-7 resorted to use the quotient rule again. Other algebraic procedures required to get the stationary point were effectively used, but the conceptual reasoning associated with setting the first derivative to zero in order to obtain the critical point remained uncertain. Despite this success recorded, T-20 has also done well in
getting the first derivative of the equation in 6a, but unfortunately can’t apply that skill to find
the maximum base of the box whose volume equation was given. Rather, T-20 chose to apply
algebraic procedures right away before getting the first derivative. Below is what T-20 did:

$$V = 300b - \frac{b^3}{4}$$
$$0 = 300b - \frac{b^3}{4}$$
$$\frac{b^3}{4} = 300b$$
$$b^3 = 1200b$$

Ironically, T-20 uses all the strategic knowledge required to obtain the critical value, but
without taking the first derivative.

Overall, slightly less than majority of the students were able to differentiate the equation
using the term-by-term technique since it is a regular polynomial (power rule), but experienced
some difficulty applying the quotient rule on the middle term despite the fact that the quotient
rule was not really necessary. The power rule can easily be applied to obtain the derivative of the
middle term, but misunderstanding the structure of the polynomial leads students to faced
structural difficulty in transforming the middle term into index form before taking the derivative.
Moreover, there are also intricacies associated with applying the basic differentiation rule (power
rule) to obtain the first derivative, as well as theoretical underpinning of setting it (the first
derivative) to zero. Similarly, finding the nature of the stationary point was equally problematic.
Consequently, these observable facts led some students to break down in their solution
processes.

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Research question 7b: How do students solve ACOP completely?

This part of the research question assessed student’s overall algebraic, geometric and basic differentiation skills in ACOP solutions. It measured students’ skills or their absence in the ACOP solution process. The complete ACOP solution will show the exact point of breakdown either algebraic, geometric or basic differentiation skills.

There are three tasks designed to address this research question. The tasks are different from each other with a certain degree of conceptual variability. Overall, the conceptual difficulties include recognizing the structure of a negative polynomial and calculating its derivative; finding the dimensions expressed in single variable, area formula, the derivative, the stationary point(s) and their nature of the poster problem. Others consist of identifying a trapezoidal structure from a sketch of the word problem, labeling with new variables, and deriving the required equation in terms of variable of interest. Structural inabilities might be apparent in calculating the derivatives of the derived trigonometric equation, use of appropriate identities and finding the stationary points by avoiding lost solution errors.

Results from the quantitative portion indicate that the mean score of this task is 1.83 with 9 possible points. Overall sample performance on the task based on this output were generated into two groups and classified as below and above expectations, with range of 0 to 2, and 3 to 6 points, respectively. 106 students (86.4%) were below expectations, and 49 students (31.6%) were above expectations. Drawing a remark base on these facts indicates that a significant percentage of the sample fell below expectations in the complete ACOP tasks. Qualitative results will be used to support this claim.

L-7 succeeded in visualizing, drawing and labeling the sides of the required geometric diagram in question 7a, but setting up relationship between numerical and algebraic volume and deriving appropriate formula were obscured in the first part of question seven. In the second part
of question seven (7b), only visualization and drawing were accomplished; and in last section, nothing came up. A similar success story was recorded with W-2 and J-37, where first section of the question was completely solved, but getting the derivative after setting up the equation in the second part of the question was not successful. Below is the reproduced work of W-2:

\[ a = (y + 12)(x + 8) \]
\[ a' = \frac{384}{y} + 8 + y + 12 \]
\[ xy = 384 \text{cm}^2 \]
\[ -20 = \frac{384}{y} + y \]
\[ x = \frac{384}{y} \]
\[ y^2 + 20y + 384 \]
\[ y^2 + 20y = 384 \]

Clearly, there is a structural failure in applying the product rule to differentiate the derived equation. Moreover, W-2 was unable to recognize the quadratic structure in the assumed correct first derivative, hence leading to a un-strategic solution approach. Finally, W-2 can’t visualize the rain gutter from the next question (7c). In the same trend, S-3 had difficulty in visualizing and drawing the poster and rain gutter from the last two parts of question seven, but succeeded in expressing the relationship between numerical and algebraic volume from the first part of the question. In a related issue, V-2 couldn’t visualize and draw the poster and rain gutter as well as labeling the diagrams. In fact, interpretation of mathematical language of “material required to construct” were hard. The student was undecided whether to use between area and volume formulas.

Among the relative success stories in this section was J-37. The student clearly visualizes the rain gutter, label it using appropriate variables, derived the required equation and calculate its
derive. However, the used of less simple variables define in trigonometric functions lead to incorrect results. Despite this shortcoming from the student, it can be considered as a success.

In summarizing the qualitative data of the seventh research question, structural difficulties experienced by the students were finding the derivative of a negative polynomial, accompanied critical/stationary point(s) and its nature. Other areas of complexity encountered during the solution process of the full ACOP are visualization, drawing, and in some instances labeling (when it was not apparent), as well as application of appropriate differentiation rule (power, product or quotient rule). Moreover, interpreting mathematical language such as “least material”, making connection between algebra and trigonometry (structural and procedural) as well as fluent used of trigonometric identities appeared to be causing some nuisance.

Summary of the Qualitative Data Analysis

The qualitative data generated can be categorized into two, based on what the solution processes requires by each task. Seven questions are used, each broken into either one or two or three parts, designed to measure the seven research questions. The two main categories are isolated and non-isolated tasks. The isolated tasks are those that require algebraic expertise (symbol manipulation skills), and geometric proficiency (visualization, labeling) as well as differentiation skills alone. While the non-isolated tasks requires the incorporation of those competences stated above to derived required ACOP equation, solve partial and complete optimization problems.

Generally, students had performed well in the isolated tasks. Most students showed a good understanding of the basic geometric shapes that were represented in word problems, but others illustrated lack of this ability, which was clearly grounded on their insufficient knowledge of the fundamental geometric concepts.
The deficiency in students’ mathematical language have hindered labeling task, but the non-canonical task was relatively accomplished with the majority of the students establishing a good relationship between the sides (longer and shorter horizontal and vertical) sides of the geometric figure. In a similar trend, students had demonstrated familiarity with factual knowledge of geometric formulas, but blinded use of the process of elimination and overgeneralizations as well as misinterpreting question’s instruction prevented a good achievement.

The qualitative data from another isolated task indicate that students have experienced some hardship with algebraic processes, for example adding algebraic fractions and connections to non-algebraic situation for example trigonometry; negative/positive signs errors in computation that are non-intentional or otherwise; structural, for example recognition of quadratic solutions, identifying quadratic equation in trigonometric form, and connection that is used of appropriate trigonometric identities in the solution process and detecting some erroneous solution of theta and eliminating it from the solution set.

Data from research question 5 generated can be categorized as non-isolated task, because the solution processes requires other information that were either geometric or algebra or both. It was found from this result that students had exhibited a capability to visualize, label and use appropriate formula for 3- and 2-dimensional geometric diagrams to derive the required equation, but some are met with conceptual hardship related to the interpretation and use of mathematical languages.

Overall conclusion from research question on partial ACOP revealed that students were skillful in differentiating the equations using power (term by term) and quotient rule, but experienced some sporadic difficulty in applying same to obtain the maximum variable. In fact, only one student among those interviewed was able to use the second derivative test to find the
nature of the stationary point(s). The second derivative test is weak because it does not work always. Similarly, theoretical reasoning associated with setting up the first derivative to zero was hazy among the students.

Lastly, the qualitative data up-and-coming from last research question i.e. seven, indicates that students had structural troubles finding the derivative of negative polynomial, accompanied by critical/stationary point(s) calculation as well as visualization, drawing, labeling of the full ACOP. It was opined that they found it difficult to accessed appropriate knowledge (algebraic, trigonometric, geometric and basic differentiation); use their previous understanding; and make connections in an interwoven way towards the solution of the tasks. Furthermore, it was clearly revealed that some students couldn’t control or regulate their cognitive processes during the solution procedures of the tasks.
CHAPTER 5
CONCLUSIONS, DISCUSSION AND RECOMMENDATIONS

Review of Study’s Goal and Research Design.

The purpose of this study was to assess algebra and geometric prerequisites skills as incorporated into the Applied Calculus Optimization Problem (ACOP) solution. The difficulties that students encounter in applying algebraic and geometric prerequisites at the early stages of the ACOP solution were identified. The study analyzed errors related to variables and equations (i.e. algebraic symbol/transformation skills), drawing of geometric diagrams (visualization skills) and those associated with application of basic differentiation concepts into a ACOP solution process.

The study’s goals was addressed as seven specific research questions further subdivided into three main parts: the first four research question investigated prerequisite algebra and geometric skills, while questions five examined the ability to used some or all of the prerequisites skills to obtain the required ACOP model. Question six was concerned with how some prerequisite (differentiation) skills are used in the ACOP solution process. Research question seven looked into students’ ability to fully bring into play all the prerequisite skills into the ACOP solution process. Furthermore, each of the seven research questions was split into quantitative and qualitative parts. The quantitative components of the research questions are labeled with an “a”, while the qualitative components with a “b”; and research question (5a*) was assessed quantitatively using regressions analysis. The overall outcome from these research questions was synthesized model of how the prerequisites fit together to constitute a solution of the ACOP.

The first research question assessed whether students could draw a diagram to represent the ACOP or non ACOP word problems quantitatively (a) and qualitatively (b). It was designed
to measure students’ knowledge of basic geometric shapes as well as their application to physical situations. The second research question was concerned with labeling activity and was designed to measure two themes, conceptual understanding of variables and their applications to new, independent, and appropriate situations. It also assessed whether students could use an appropriate variable of their choice and suitable to a given circumstances.

The third research question measured students’ ability in associating an appropriate geometric diagram with its algebraic equation(s)/formula in relation to calculating area, perimeter or volume, while fourth research question measured students’ ability to manipulate any algebraic equation irrespective of whether or not it was related to the ACOP. The fifth research question appraised how students were able to derive the required equation for the solution of a given ACOP. In most cases, it involves transformation/symbol skills of one to two equations to get the required one. As part of the research question, the sixth question measured students’ ability (skill) to solve a partial ACOP. A partial ACOP was a problem which does not require some prerequisites, especially geometric prerequisites. The main focus assessed whether students could solve an ACOP in which the required geometric parts of the solution are already given or not suppressed. The last set of the research question measured student’s overall algebraic, geometric and basic differentiation skills in the complete ACOP solutions. The complete ACOP solution process will show the exact point of breakdown either algebraic, geometric or basic differentiation skills.

After the quantitative data was collected using test instrument, a follow up interview was conducted to collect qualitative data. These qualitative data were use to supplement, support and illuminate results from the quantitative components.

The target sample was freshmen students taking calculus I in the department of Mathematics, Louisiana State University, Baton Rouge. The course was taught in Fall, Spring
and Summer semesters every year. For the purpose of this study, data were collected using students who enrolled in calculus I course in Spring semester, 2008. The university is among the elite Southern Region University of United States. It had an average enrollment of about 31,000 students per year at its Baton Rouge campus. Most of the students are from higher to middle socio-economic status. A significant proportion of students receive athletics, academic and other scholarships.

The part of calculus I used for the study was applied calculus optimization problem. As explained in the mathematics department website, students are having difficulty in understanding and applying concepts required to solve the ACOP.

Students taking calculus I was split into twenty six (26) different sections. Each section contained not more than forty (40) students. This represented an approximate enrolment of 1040 students. Six sections of calculus I were used for the study, with size of 155 students or approximately 15% of the total enrollment in calculus I. The six sections used for the quantitative portion of the study were selected using convenience sampling technique, in that the instructors of these sections were ready to participate in the study. Moreover, a sequential mixed method sampling strategy (simple random sampling strategy combines with convenience sampling) was used in selecting clinical interviews participants.

**Conclusions**

The focal point in this study was to measure students’ skills or their absence in the complete ACOP solution process. Evidence from the statistical output for the complete ACOP task showed a mean score of 1.83 out of 9 possible points, with a standard deviation of 1.41, and a mode of 2. The minimum point scored was 0, while the maximum was 6.

Overall sample performance on this task based on this output shows that 106 students (69.4%) scored between 0 to 2 points, while 49 students (31.6%) achieved a score between 3 and
6. Drawing conclusion based on these facts indicated that a significant portion of the sample was unable to solve a complete ACOP successfully. Why are these students in calculus I who have just been taught ACOP solving unable to solve typical ACOP problems? This is the context in which the intentions of the conclusions were base.

The outcome of the study was revealed in the conclusions section and followed by discussion and recommendations.

The analysis of both quantitative and qualitative results revealed that, generally, students had demonstrated aptitude in isolated tasks that requires geometric proficiency (visualization, labeling) as well as differentiation skills, and relatively fairly good performance in algebraic expertise (symbol manipulation skills), and but was unable to incorporate these competences in some non-isolated situations (model/equation derivation, partial and full optimization problems). The themes that come into sight as a result of analyzing the qualitative data were:

- Isolated knowledge of geometric proficiency (visualization and labeling). Students were able to visualized and labeled geometric diagrams that was either presented in word problems or were given as blank figures, but in some occasions, these capabilities were not applied properly.

- Lack of algebraic manipulative/symbol skills. Students were struggling with the structural and procedural knowledge required for the solution of algebraic and non-algebraic (trigonometric) equations.

- Detached proficiency in basic differentiation skills. At this level, students were able to found the derivatives of some terms in a polynomial but do not always succeeded as a result of structural failures.

- Students’ knowledge of algebra, geometry and basic differentiation and breakdown of application to optimization problem. Students rarely apply the geometric, algebraic and
basic differentiation expertise to derive the required equation/model or find the solutions of a partial or the complete ACOP.

**Isolated knowledge of geometric proficiency (visualization and labeling).**

Generally, students performed well in the isolated tasks of constructing and labeling a diagram. Results from the quantitative data indicated high mean score and this was equally supported by the qualitative outcome. Most students showed a good understanding of the basic geometric shapes that are represented in word problems, although a few illustrated lack of this ability. For example, the unsuccessful ones in this activity were hindered by their low level ability to transform 2–dimensional to 3–dimensional objects, and vise versa, a result that was consistent with Mitchelmore (1980) cited by Hershkowitz (1990). According to Hershkowitz, three factors influenced the description and interpretation of 3–dimensional drawings and these are culture, experience and familiarity. It was evident from the qualitative results that lack of familiarity and in-experience plays a major role in hindering students in this activities. A good example is the tasks of transforming a sector of a circle into a cone. Students interviewed acknowledged that, either they didn’t know what a sector was (in-experience) or they couldn’t fully describe what a sector looks like (familiarity). Cultural factor may not be significant factor in obstructing success in this situation because there are so many cultural attendances for basic geometric shapes. Overall, the lingering difficulties that thwarted these few students’ success in these tasks were clearly grounded on their insufficient knowledge of fundamental geometric concepts.

The deficiencies in students’ mathematical language had hindered calculating labeling task, but non-standard shapes created for the test in which sides were explicitly labeled, students were relatively successful. One major constraint face by students in the non–standard labeling task was their inability to recognize the functional relations among related elements of the
diagram. Establishing this relationship was a serious problem among the unsuccessful students. The calculating labeling activity was hindered by geometric language deficiency. Specifically, students had demonstrated lack of familiarity and understanding with the concept of dimensions of a cylinder. This language deficit had totally hindered a majority of these students who failed on these items from accomplishing what the task requires. Other findings revealed that using expressions like “in terms of these variables” seems confusing to some. All these were rooted as a result of the shortage of students’ mathematical (geometric) language. In a dissimilar trend, students had demonstrated familiarity with factual knowledge of geometric formulas, but blinded use of the process of elimination and overgeneralizations as well as misinterpreting the question’s instruction prevented some students from achieving what was required by tasks.

**Lack of algebraic manipulative/symbol skills.**

This is another isolated task that was measured in the quantitative part of the study. Results from the quantitative data showed that more than two-third of the sample was unable to successfully do the tasks. Most of the students had committed what Donaldson (1963) cited by Orton (1983) called structural and executive (procedural) errors. According to Donaldson, structural errors are those “which arose from some failure to appreciate the relationships involved in the problem solution” (p. 4). Executive (procedural) errors were those which involved failure to carry out manipulations, though the principles involved may have been understood.

Most students exhibit structural errors in this task. For example, a student solving a pair of equations that clearly should require application of quadratic equation ended up using substitution and trial and error method. This worked to limited extent, because the numbers tried coming up to be true, but the completed solution set goes beyond what was found. In essence, the student failed to saw from the structure of the equation, how many solutions were possible. This
approach of substitution as documented by Kieren (1985) supported the fact that students who used it to solve an equation do understood the equivalence role of the equal sign.

It was equally found in the study that students were unsuccessful in this task because they performed executive (procedural) errors. Some of these errors included incorrect multiplying out terms in a bracket with either a constant or a variable. This was what Ayers (2000) called pre-multiplier error. Other operational errors include failure to correctly factor a quadratic equation and negative/positive sign occurrence in the solution due to misunderstanding equivalence role of equal sign in the equation. One other error commonly found was misinterpretation of the concept of variables. For example, students added $3x$ and $2x^2$ to got $5x$ or $5x^2$. Clearly, this demonstrated absence of understanding of the distinction between linear and quadratic terms (structural). Moreover, misinterpretation of some mathematical language such as “express $p$ in terms $a$” have led to incorrect solution in many instances.

Overall, an accumulation of all these structural and procedural errors help in worsening the solution processes of non-algebraic equations (trigonometric). Most common among operational breakdown was inability to found the sum of two trigonometric fractions as well as clearing the resultant fraction. Moreover, students found it hard to identify, recall, and use appropriate trigonometric identity, in addition to recognizing quadratic structures express in trigonometric terms in order to get its solution.

**Detached proficiency in basic differentiation skills.**

The focus here was to assess students’ competences in basic differentiation skills. Quantitative results showed that students’ performances were moderately well accomplished.

The use of power rule to differentiate a polynomial was understood and applied to terms with positive exponents, but difficulties emerged when the variable of interest was found in the denominator of the algebraic fraction term. In most cases, students unnecessarily attempted to
apply quotient rule, but end up getting unsuccessful results. The major obstacle encountered in
the application of quotient rule is misunderstanding how the rule was derived and subsequent
inability to detect mistakes whenever they occurred. An alternative method of differentiating the
polynomial is by transforming the term $\left( \frac{\sqrt{10}}{t^6} \right)$ into index form with negative exponent and
subsequently applying the power rule. Since the power rule was well understood by the students,
its application might not cause some havoc. On a similar note, the application of power rule was
met with another structural difficulty when differentiating a term in the polynomial with
exponent of one. The lingering difficulty was associated with the value of $t^{1-1} = t^0$, but
expressing the value of $t^0$ as 1 was not obvious to many students. This was pure algebraic
structural problem. In the same way, subtracting one from negative two-fifth i.e. $-\frac{2}{5} - 1$ was
challenging to some students. It led to variation of answer such as $-\frac{4}{5}$, $-\frac{3}{5}$ and so on. This was
another structural as well as procedural error. For those who succeed in finding the correct
derivative and setting it to zero, it became problematic for them to explain or interpret why $Y'$
or $\frac{dy}{dx} = 0$. Another conceptual error was inability to obtain and explained the nature of the
stationary point (i.e. maximum or minimum). In some instances, procedural errors could add up
to the conceptual difficulty and make solution unreachable.
Students’ knowledge of algebra, geometry, basic differentiation and breakdown of application to optimization.

One of the hallmarks of this study was to investigate how these competences could aid or abate successful solution of optimization problems. Quantitative results indicated that there was reasonable success in the isolated tasks that requires application of visualization of geometric figures alone with high mean score, moderately well accomplished performance in basic differentiation skills and fairly successful success in algebraic symbol skills. These success stories didn’t translate well with a lot of students when they became an integral part of optimization problem solution process.

Optimizations solution processes started with visualization of geometric diagram described in word problems. It was clear that students were unable to visualize geometrically. For example, one word problem described a rain gutter and required students to relate it to a familiar geometric shape. The failure to put down the visual image of the rain gutter, followed by inability to construct additional geometric modification to complete the diagram, had led to chain reaction collapse (geometric, algebraic or basic differentiation) in the solution process. The first source of breakdown was inability to came up with appropriate geometric formula, as a result of visualization deficiency and transforming it to suit the situation. What Donaldson (1963) cite by Orton (1983) called failure to incorporate the constraints laid down in what was given. Another source of breakdown is labeling a complex diagram in terms of information given. The concepts of variable were well understood, but non-strategic solution process led to derivation of complicated equation. Such equations, though hard to solve, are still solvable. One of the steps in the solution process of optimization problem requires setting up the first derivative to zero and solve for the variable of interest algebraically. Finding meaning to the statement $\frac{dy}{dx} = 0$ was already shown to be out of reach, i.e. students couldn’t explain why it was done. Subsequently,
procedural errors that were algebraic may sometimes hinder the possibilities of finding the correct critical point.

In the solution process of optimization problem that involves trigonometric functions, certain constraints often came into play, because it is a periodic function. Students tend to ignore this constraint as noted earlier by Donaldson.

Basic differentiation rules i.e. power, product and quotient were most commonly used in the solution processes of optimization. Improper uses of these rules usher in another source of error and complicate the solution process. Specifically, students found it hard to apply the quotient rule correctly, similarly the product rule. For example, the derivative of

\[ a = \left( \frac{384}{y} + 8 \right)(y + 12) \]

was given as

\[ a' = \frac{384}{y} + 8 + y + 12 \]

using product rule by one student. What could be inferred here was that the student simply changes the product of two equations by removing the brackets to addition. This was typical situation which shows collapsed of an understanding of the concept of product rule and how it was applied.

**Discussion and Recommendations**

There are a lot of educational implications of this study. The discussion of results was centered on three major components (geometry (visualization), algebra (manipulation/symbol skills) and basic differentiation skills) and how each influences the solution of optimization problem.

According to Hershkowitz (1990), “visualization generally refers to the ability to represent, transform, generate, communicate, and reflect on visual information” (p.75). She further cited Bishop (1989), who claimed that, “visualization is important not only for its own sake but also because the type of mental processes involved are necessary for, and can transfer to, other areas of mathematics” (p.76), example geometry.
Students participating in this study had an average ACT score, high enough to expect a good performance in the geometric component of the study that required basic geometric knowledge (visualization in particular). Indeed, the results from both quantitative and qualitative portion revealed that prospect to a certain degree, but integrating those capabilities proved difficult in the solution of optimization problem. The results exposed that students had faced some difficulty creating concept image from concept definition Vinner (1983) cited by Hershkowitz (1990), that is they were unable to interpret and show some understanding of the concept presented in word problems and translated them into concept images. According to Hershkowitz, concept is derived from its mathematical definition and has attributes. Those attributes were used to separate examples from non-examples. The educational implications of this finding for the school curriculum and college teaching were to adapt geometry teaching methods that emphasize establishing a good relationship between concepts definitions and concept images. It was revealed that visualization skills involving interpreting figural information is trainable, Bishop (1989). Based on this, it is imperative for teachers of geometry to teach geometry (visualization) and other aspects of it from conceptual perspectives and avoid rote learning which is highly likely forgettable.

Another important area of concern is algebra. The average ACT score noted during the quantitative segment of the study as claimed earlier was promising. The general impression created by that was students were ready to build their advanced mathematical knowledge since they had the prerequisites. Contrary to that, both qualitative and quantitative results from the study indicated a serious short fall in the expected algebra/symbol skills of the students. There were certainly a lot of algebraic difficulties that obscure success in themselves (isolated tasks) or related to the solution of optimization problem (non-isolated tasks). Some of the confusions were caused by structural or executive (procedural) errors. There were sporadic conceptual
understandings, but executing them or adopting a pathway, or using defined strategies or procedures in optimization solution process led to emergence of errors. The most vulnerable un-strategic procedure adopted was the use of substitution to solve optimization problem. This was clearly grounded on the influence that Arithmetic had over Algebra which students carried from high school. Most common occurrence of such was students’ inability to establish equivalence relationship between numerical given value (area, perimeter or volume) with its algebraic form. Consider for example, the equations $l = \frac{4}{h^2}$, $w = 2h$, and $SA = 2lw + 2lh + 2wh$. Make one equation from these in which $l$ and $w$ didn’t not appear. This procedure of substituting the variables $l$ and $w$ in $SA$, or expressing $SA$ in terms of one single variable $h$, were equally problematic. This, naturally, prevented expressing a variable of interest in terms of others and the given quantity which may ultimately be used in further computational process. The consequence was that, sequence of optimization problem solution process is stall. The educational implication of this was that, the teaching and learning of algebra from structural perspectives example, Kieren (1989), Wagner et.al (1984), Kirshner (2001) should be given attention.

Researches have shown that beginning college and university calculus students are face with less knowledge, skill and understanding than assume, Dreyfus (1990). But in this study, there was relative high assumption on students’ basic knowledge, skill and understanding preparatory to taking calculus class. The bases for making this conclusion was that students who participate in this study had average ACT score, high enough to build confidence that they could succeeded in their calculus class. This study results indicated that the rule for differentiation is used as an algebraic algorithm, without much meaning and understanding attached to it. Similarly, investigating the nature of stationary point was also trailed algorithmically. The outcome was that students had relegated themselves to following the rules mindlessly, without
engaging the conceptual thoughts that lie beneath the procedure in order to successfully apply them in optimization problem solution process. The educational implication was that teachers should work with students to learn to derive differentiation rules, examined the nature of stationary points and its conceptual underpinnings as a summary of investigatory work that involves graphical and numerical activities.

**Limitations**

This subsection discussed various limitations of the study and associated implications for further research.

**Validity**

Validity is defined as the best available approximation to the truth of a given proposition, inference or conclusion (Burns, 1997). In other words, when we make some claim, does the evidence support our conclusions? From another perspective, validity is concerned with the study’s success at measuring what the researcher set out to measure.

For any research, there are threats to its internal and external validity. In this section, discussions on two major threats to internal validity which are instrumentation (Burns, 1997) for the quantitative portion and triangulation (Creswell, 2003; Patton, 2002) for the qualitative component were given.

The study was designed base on mixed methods research design. It used data collection and analysis procedures from quantitative and qualitative perspectives. According to Creswell, it is characterize by the collection and analysis of data quantitatively as well as qualitatively.

Instrumentation as a threat to internal validity was caused by inconsistencies with the testing instrument, i.e. grader or the test itself (Burns, 1997). It is sometimes called Experimenter bias. This becomes a problem when the generated data were subjective. To avoid this problem, substantial amount of time and effort was invested in the construction of the test instrument, as
well as preparing interview protocol for each interviewee selected for the interview. Since this was somehow a guided interview, directed by the nature of the interviewee’s test results, however, the process was not followed strictly. There were some elements of adaptability allowed to accommodate some unexpected development from the side of the interviewee. Appendix B contains documentation of the interview protocols and transcripts to support the attempt made by the researcher to handle this major validity constrains.

From the qualitative perspectives, validity is seen as the strength of the research, and is used to determining whether the findings were accurate from the standpoint of the researcher, the participant, or the readers of an account (Creswell & Miller, 2000 cited by Creswell, 2003). Eight ways of checking the accuracy of the findings are available depending on the user and among them is triangulation. For this study, triangulation strategy was used to check truthfulness of the findings. According to Patton (2002), “the logic of triangulation is based on the premise that no single method ever adequately solves the problem of rival explanations. Because each method reveals different aspects of empirical reality, multiple methods of data collection and analysis provide more grist for the research mill” (p.555 – 556). Furthermore, to make the research findings more accurate and valid, combinations of interviewing, observation, and document analysis are expected in much fieldwork. In this study, interviewing based on protocols developed for each interviewee were conducted as well as thorough examination of individual’s test script.

Other limitations of the study

The strength of any research is its setback. These limitations can be associated with the status of the instrument use. Clinical interviews, in particular stimulated recall interviews and all other methods of verbal reporting can be used for data collection. However, stimulated recall cannot provide a complete means of capturing all of a participant’s thoughts and strategies,
because they can only report what is in their consciousness (Henderson & Tallman, 2006). Furthermore, one major problem associated with verbal descriptions of cognitive processes and experience was that such reports do not relate clearly to any specific observable behavior (Ericsson & Simon, 1993). Moreover, data from verbal reports are time consuming and costly to interpret. However, this was not enough set back to disqualify the method. Not all test results requires generalization since it was not a large scale study. It was a small study that aims at finding solution to some specifics problems.

On the other hand, researchers (e.g. Bloom, 1954; Ericsson and Simon, 1980; and Lieberman, 1979) have revealed that stimulated recall and think-aloud reports “are reliable measures and that results obtained using verbal reports do corresponds with the actual behavior” (Gass & Mackey, 2000, p. 17).

The instrument use in this study was developed by the researcher (experimenter bias); this is another threat to validity. Moreover, the researcher also serves as the person who conducted the interview as well as preparing the interview protocols. It was possible that his anticipation, belief and expectations may likely influence the data collection and analysis, as well as the outcome of the study.

External validity is another threat to research. It is referred as the drawing conclusions from the sample data to other samples, different location and past or future location (Creswell, 2003). In other words it means generalizing the findings farther than given situation. This study was a small research, hence it couldn’t be generalized. It was the intention of the researcher not to cover a larger sample. It was also intentional not to cover other aspects of calculus. It was a bit difficult getting students for the interview after they took the test, going by the fact that they were not anticipating any compensation in form of extra credit.
Implications for further research

It is important that the investigation be conducted with a lager sample. It should focus on algebra and geometry competences as incorporated into ACOP solution. Moreover, this type of research should be replicated to others areas of applied calculus that became “nightmare” for first-year students such as related-rate and Riemann sum.

Another area that deserves attention is algebraic and geometric (mathematical) language particularly used in areas of applied calculus such as ACOP, related-rate and Riemann sum.

Final Summary

Overall, the study revealed that students had failed to integrate the basic competences required in ACOP solution. These competences were assessed as isolated tasks and quantitative results, Table 4.15, showed that 72 students (46.5%) had obtained an above average score in at least three of the four prerequisite skills, but only 41 (26.5%) were able to score an above average performance in the complete optimization tasks. In a similar trend, the link between the competences also contributes towards poor performance in ACOP solution. Qualitative evidences from students’ test performance indicated that, failure to visualize geometric diagrams from word problems rarely allows getting the required formula. In the same way, those that were unable to calculate correct first derivative couldn’t obtain stationary point(s) or critical value. It was clear that failure in at least one competence led to collapsed in another, hence a whole breakdown in ACOP solution process. The hallmark of the research indicated that students failed in integrating individual algebraic and geometric competences, and where such were successful, they were met with structural and procedural setbacks that ultimately led to weaken ACOP solution processes.
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APPENDIX A

TEST INSTRUMENT AND SCHEDULE

This appendix contains letter to Math 1550 instructors asking for their cooperation to allow their students take part in the study; the test instrument and grading rubric and test schedule for each section. Students in the six sections have taken the test, at different time and date, but not all from each section. The time allow for the test for each section is fifty minutes.
Letter to Section’s Instructors

Date: 25 January 2008.

To: MATH 1550 Instructors

Copy to: Calculus Coordinator, Britt Paul

Study with Math 1550 Students: A Reminder

Hello, I am Ahmed Ibrahim Usman, a doctoral candidate in the College of Education, department of Educational theory, policy and Practice, LSU and my area of specialization is Mathematics education. My prospectus was approved in Fall 2007, and now at the stage of planning how to collect data. My area of interest is algebra and geometry skills related to the solution of applied calculus optimization problem. I have talk to the calculus coordinator Paul Britt, who said I should contact you guys for help, hence I write to solicit your permission and support to use the students in your section for study. The data collection procedures will be in two phases. The first phase is the quantitative part which requires students to take a test, while the second phase involves interviews with some test-takers selected randomly. The test will be given immediately after the optimization topic has been covered, somewhere around the third or fourth week of February. I would like you to kindly accommodate me in your plans. The test would last 50 minutes. If you agree that your section would be used for the study, kindly indicate your acceptance by email, and we will continue to plan.

Thank and appreciate your support.

Ahmed Ibrahim Usman.

Email: ausman1@lsu.edu
Test Instrument

Print email/phone number____________________________________

Gender:__________  Race:___________        ACT Scores ______________
First time taking Math 1550_____ Second time taking Math 1550__________

The test instrument is designed to measure algebraic, geometric and basic differentiation skills as they are incorporated into applied calculus optimization problem (ACOP). Algebra and geometry skills are considered important and required for any student entering into Calculus I as well as further Calculus courses. Students normally shown their algebraic and geometric skills in isolated situations with little consideration to application in a assume practical setting.

Answer all questions in the seven sections. THE BEGINNING QUESTIONS SHOULD REQUIRE LITTLE TIME TO COMPLETE, LEAVING YOU MORE TIME FOR THE LATER QUESTIONS. TRY TO LEAVE YOURSELF ENOUGH TIME TO answer all questions. Calculators ARE NOT PERMITTED. IF YOU CANNOT ANSWER A QUESTION, PLEASE DESCRIBE THE DIFFICULTIES YOU ENCOUNTERED.

Construction of a diagram to represent the given word problem.
Question 1a.
An open box is to be made from a rectangular piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Make a diagram that shows the cardboard after the corners have been cut, but before the sides have been bended up.
Question 1b.
Draw a diagram to represent a right circular cylinder inscribed in a hemisphere.
Question 1c.
A conical drinking cup is made from a circular piece of paper by cutting out a sector and joining the edges. Sketch the diagrams after the sector is removed from the circle, and then again when the cup is made.

Labeling a diagram using appropriate variables.
Question 2a.
A rectangular box has its length, width and height of different dimension, such that the length is twice the width, and the height is a third of the width as given below. Use a single variable to label the diagram.
Question 2b.
A water tank has been made in the shape of a right circular cylinder, as given below, from a rectangular piece of metal. If the length of the original rectangle is $L$ and its width is $W$, then label the dimensions of the cylinder in terms of these variables.

Question 2c.
Select four (4) variables and use them to represent all six (6) sides of the diagram below.
Associating geometric diagram with appropriate algebraic equation(s)/formula.

Question 3.
Consider the following geometric diagrams. Name and associate each figure with its formula for area, perimeter, volume, and surface area as the case may apply. Use the list of formulas given.

\[
\begin{align*}
S &= 6l^2, \quad A = l \times w, \quad S = 2\pi r^2 + 2\pi rh, \quad V = \pi r^2h, \quad A = \pi r^2, \quad P = 2l + 2w, \\
V &= l \times w \times h, \quad V = l^3, \quad S = 2lh + 2wh + 2lw, \quad V = \frac{1}{3} \pi r^2 h, \quad S = \pi r l, \quad C = 2\pi r
\end{align*}
\]

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<tr>
<th>Name of Geometric diagram</th>
<th>Area formula, ( A )</th>
<th>Perimeter formula, ( P )</th>
<th>Volume formula, ( V )</th>
<th>Surface area formula, ( S )</th>
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Algebra symbol skills (show all your work).

Question 4a.
Consider the equation below. Simplify, and express $p$ in terms of $a$.
\[
\frac{1}{2}(p + a) - 4 = \frac{1}{3}(2p - 1)
\]

Question 4b.
Solve the following system of equations
\[
\begin{align*}
3x - y &= -2 \\
2x^2 - y &= 0
\end{align*}
\]

Question 4c.
Find exact solution(s) of the equation.
\[
\cos \theta \sin \theta + \frac{1}{\sin \theta} = \sqrt{3} \quad \text{for } 0^0 \leq \theta < \pi
\]

Finding the equation.

Question 5a.
A rectangular storage container with an open top is to have a volume of 10 m³. The length of its base is twice the width. Find the formula for calculating the total surface area in terms of a single independent variable.

Question 5b.
A piece of wire 10 cm long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. Find a formula for the sum of the areas for the two figures expressed in terms of a single independent variable. (Do not simplify your answer)

Question 5c.
Find the formula for area of a rectangle inscribed in a right triangle with legs of length 3 cm and 4 cm. The two sides of the rectangle lie along the legs. Express your solution in terms of a single independent variable.

Differentiating and optimizing a variable.

Question 6a.
The derivative of:
\[
y = t^{\frac{2}{5}} + \frac{\sqrt{10}}{t^6} - t
\]

Question 6b.
The formula for the volume of a certain box in terms of the base is $V(b) = 300b - \frac{b^3}{4}$. Find the largest possible volume of the box.

Optimization problems.

Question 7a.
A closed rectangular box that has its width twice the height and a different length. The volume of the box is 8 cm³, find the dimensions of the box that uses the least material in construction.

Question 7b.
The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed materials on the poster is fixed at 384 cm², find the dimensions of the poster with the smallest area.
Question 7c.
A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle of \( \theta \left( 0 \leq \theta \leq \frac{\pi}{2} \right) \). How should \( \theta \) be chosen so that the gutter will carry the maximum amount of water?

**Grading Rubric**

The test instrument is designed to measure algebraic, geometric and basic differentiation skills as they are incorporated into applied calculus optimization problem (ACOP). Algebra and geometry skills are considered important and required for any student entering into Calculus I as well as further Calculus courses. Students normally shown their algebraic and geometric skills in isolated situations with little consideration to application in assume practical setting. For the scores of this test, they range from 0 to 9 points for each question 1, 2, 4, 5, 6 and 7 and 0 to 6 for question 3.

**Question 1.**

a. – 3 – sketch for rectangular shape with corners cut out.
   1 – Sketch for rectangular shape without cutting the corners.
   0 – sketch which is not rectangular/no attempt.
b. – 3- sketch of a cylinder inside a hemisphere.
   1- Two independently sketch figures, not put together.
   0 – no attempt/blank space.
c. – 3 – sketch of a sector and cone separately.
   2 – Sketch of a cone alone.
   1 – Sketch of a sector alone.
   0 – no attempt/blank space.

**Question 2.**

a. – 3 – length, \( l = 2w, w = w, \) and \( h = \frac{1}{3}w; \) \( 2w, w, \) and \( \frac{1}{3}w. \)
   2 – for using only two variables, either \( l \) and \( w, \) or \( h \) and \( w. \)
   1 – \( l, w, h. \)
   0 – no attempt/blank space.
b. – 3 – equating \( l \) to \( 2\pi r \) and \( w \) to \( h \quad \Rightarrow \quad r = \frac{l}{2\pi}, \quad h = w \)
   2 – equating \( l \) to \( 2\pi r, \) and \( h \) to \( w \)
   1 – \( r \) and \( h. \)
   0 – no attempt/blank.
c. – 3 – any combination of 4 variables i.e. \( a, b, c, \) and \( d \) or \( w, x, y, \) and \( z \) to label the given diagram or anything else that is acceptable.
2 – using either vertical/horizontal labeling first leading to some errors.
1 – using six different variables.
0 – no attempt/blank space.

Question 3.
A – 1 – rectangle; area = lb; perimeter = 2(l+b) or 2(a+b) or any acceptable solution.
B – 1 – cylinder; volume = \( \pi r^2 h \); surface area = \( 2\pi r^2 + 2\pi rh \).
C – 1 – cube; volume = \( l^3 \); surface area = \( 6l^2 \).
D – 1 – rectangular box; volume = \( l\times b \times h \); surface area = \( 2(lb + lh + bh) \).
E – 1 – cone; volume = \( \frac{1}{3} \pi r^2 h \); surface area = \( \pi rl \).
F – 1 – circle; area = \( \pi r^2 \); perimeter = \( 2\pi r \).

Question 4.
a. – 3 – complete correct solution, involving multiplying each term in the equation with the LCM or expressing the RHS and LHS in terms of algebraic fractions and the cross multiplying to clear the fraction.
2 – Expressing LHS and RHS in algebraic fraction and collecting correct like terms.
1 – Wrong solution due to using positive/negative errors.
0 – no attempt/blank space.
b. – 3 – complete correct solution, involving eliminating y variable by either substitution or subtraction/addition of the two equations. Solving a quadratic equation in x.
2 – Correct quadratic factoring with only x – solution (values).
1 – Correct factoring without x and y solution (values).
0 – no attempt/blank space.
c. – 3 – complete correct solution, involving cleared the trigonometric fraction, squaring both sides and using appropriate trigonometric identity.
2 – Substituting for trigonometric identity and factoring the resulting equation in terms of cosine.
1 – Simplifying the LHS and cross multiplying to clear the fraction.
0 – no attempt/blank space.

Question 5.
a. 3 – Complete correct solution with drawn diagram rectangular box, label sides, using surface area formula and algebraic skills, and getting required model/equation.
2 – Correct model/equation without the diagram shown.
1 – Correct diagram with no correct required equation/model.
0 – no attempt/blank space.
b. 3- Complete correct solution with drawn diagram assume wire length, square and circle; label sides of given length; resulting figures of square and circle. Using area formulas for the square and circle to get the required model/equation.
2 – Correct model/equation without the diagram shown.
1 – Correct diagram(s) moving from 1-dimension to two dimensions without the
model/equation.
0 – no attempt/blank space.

c. 3 – complete correct solution with drawn diagram of triangle with inscribe rectangle;
label sides, use of similar triangle concept, and getting the required model/equation.
2 – Using similar triangle concept with some simplification errors.
1 – Correct diagram with no further algebraic work shown.
0 – no attempt/ blank space.

Question 6.

a. 3 – Complete solution with transformation to algebraic form and correct derivatives.
2 – Correct transformation to algebraic form with derivatives of some terms.
1 - Correct transformation to algebraic form without correct derivative.

b. 3 – Complete correct solution with derivatives; maximum dimensions e.i. base, and
volume.
2 – Correct derivative with maximum base only.
1 – Correct derivatives with no critical value, i.e. base.
0 – no attempt/blank space.

Question 7.

a. 3 – Complete correct solution with drawn rectangular box, label sides, use of appropriate
formula (including algebraic symbol skills), derivative and critical values, and minimum
value of dimension.
2 – Correct surface area formula, derivative and at least one minimum value.
1 – Correct derivative of the volume of the rectangular box with no critical value or
dimensions.
0 – no attempt/blank space.

b. 3 – complete correct solution with drawn rectangles one inscribe into another, label sides,
use of appropriate formula (algebra symbol skills), derivative and critical values,
minimum value of dimension.
2 – Correct derivative, area formula with one minimum value.
1 – Correct derivative with no critical value or dimensions.
0 – no attempt/blank space.

c. 3 – Complete correct solution with drawn trapezoid, label sides, use of appropriate
formula (including trigonometric identities/algebra symbol skills), derivative and critical
values, and maximum value of the required angle.
2 – Correct area formula, sides of triangle in terms of cosine and sine functions.
1 – Correct derivative of the required area to be maximize with no critical value.
0 – no attempt/blank space.
## Test Schedule for Each Section

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>March 10, 9.30 am and 1.30pm</td>
<td>S-section and L-section</td>
</tr>
<tr>
<td>March 13, 2.30 pm</td>
<td>W-section</td>
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<tr>
<td>March 14, 2.30 pm</td>
<td>V-section</td>
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<tr>
<td>March 28, 8.30 am</td>
<td>T-section</td>
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<tr>
<td>March 31, 10.30 am</td>
<td>J-section</td>
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APPENDIX B

INTERVIEW PROTOCOLS

This section of the appendix contains interview protocols information such as introductory letter before each interview and interview schedule with all interviewees.

Introduction to the Interview

You have taken a MATH 1550 test that covers some algebra, geometry and optimization. These are topics you covered either in MATH 1550 or in earlier high school or college classes. I am going to ask you some questions stemming from what you wrote in the test. As part of my research, I am trying to understand how you reasoned and solved problems on the test. I will be asking you questions like "How did you solve that problem?" "What does this solution mean to you?" I am probing certain aspects of your work not because what you did was right or wrong, but because my research project is to analyze how students think about the mathematics they are using. I will videotape our discussion. The tape will allow me to listen to you and extract additional meaning from your response later. I want to thank you in advance for your participation in this interview, which is entirely voluntary. This interview is not a requirement of your MATH 1550 course, nor will the quality of your responses influence your grade in any. The questions that follow are either verbal or written, based on your test scripts. Did you have any question before we start?
### First Week Interview Schedule: Monday, March 24 to Friday, March 28th, 2008.

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### Second Week Interview Schedule: Monday, March 31 to Friday, April 4, 2008.

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### Third Week Interview Schedule: Monday, April 7 to Thursday April 10, 2008.

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<td>J-23</td>
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APPENDIX C

CONSENT SCRIPT FORM

This part of appendix C contains consent script used for both quantitative and qualitative data collection.

Consent Script

1). Study Title: Analysis of Equation and Diagram Construction in Applied Calculus Optimization Problem (ACOP).

2). Performance Site: Louisiana State University and Agricultural and Mechanical College.

3). Investigators: The following investigators are available for questions about this study, Monday to Friday, 8:00 A.M.  To 4:00 P.M.

4). Purpose of the Study: The purpose of this study is to assess algebra and geometric prerequisites skills as incorporated into the ACOP solution. Generally, the study’s main themes are:

- Assess students’ prerequisite skills in setting up an ACOP problem.
- Develop a model of how prerequisites competences are integrated together into ACOP problem solving.

5). Subject Inclusion: Individual between the ages of 18 and 65 who register for Calculus I (Math 1550) in some sections for the Spring Semester, 2008.

6). Study Procedures: The study will be performed in two parts. In the first part, subjects will take a test on prerequisites algebraic, geometric and calculus (differentiation) skills as corporate into applied calculus optimization problem. The second part will conduct interviews with some subject. The interview will last between 30 to 45 minutes with each subject selected.

7). Benefits: The study would offer significant contributions to the mathematics education community in several ways. First, it will emphasize relationship between conceptual and procedural knowledge in teaching ACOP. It will reveal the deficiencies (algebraic and geometric) that are carried along from high school. Moreover, it will analyze how these prerequisites are coordinate (or not) in successful (and unsuccessful) ACOP solution. Finally, the study will evaluate students’ basic differentiation skills

8). Risks: The only likely study risk is the unintentional release of the sensitive information found in both test results and interview transcripts.

9). Right to Refuse: Subjects may decide not to participate or to withdraw from the study at any time without penalty.
10). **Privacy**: Results of the study may be published, but no names or identifying information will be included in the publication. Subject identity will remain confidential unless disclosure is required by law.

11). **IRB Contact information**: If I have questions about subjects' rights or other concerns, I can contact Robert C. Mathews, Institutional Review Board,(225) 578-8692.
APPENDIX D

INDIVIDUAL INTERVIEW PROTOCOL/TRANSCRIPTION

This section of the appendix contains some individual interview transcripts and their corresponding protocols for S-3, V-2, W-2, J-23, L-7, and T-20.

S-3 Protocol

Question 1.
Hypotheses: Concepts of hemisphere, sector and inscription are vague.
Probe: Describe/draw a hemisphere, sector.

Question 2.
Hypotheses: Couldn’t establish the relationship between L from rectangle and circumference of a cylinder.
Does not know the circumference in terms of a formula, i.e. \(2\pi r\).
Does not know the concept of dimension of a cylinder.
Could not figure the relationship between \(y\) and \(h\) as well as \(w\) and \(x\)
Probes: State/mention the dimension of a cylinder.
State the formula of a circumference of a circle.
What would the sum of \(h\) and \(y\) represent?
Does \(x - w\) equivalent to \(w - x\)?

Question 3.
Hypotheses: Did not associate figures with appropriate formulas.

Probes: Can this formula or that represent the same figure?

Question 4.
Hypotheses: Detachment of a term from the indicated operation.
Unable able to find the LCM of a non-fractional term.
Can’t distinguish between \(1 + \cos \theta\) and \(1 - \cos^2 \theta\) from the identity
\[1 = \cos^2 \theta + \sin^2 \theta\]

Probe: How do you collect like terms and change their signs if the need arise?
Can you clear the fraction of this equation?
Express \(\cos^2 \theta\) in terms of \(1\) and \(\sin^2 \theta\) from the identity.

Question 5.
Hypotheses: Can’t represent the diagram from word problem.
Can’t write the surface area formula in terms of given dimensions.
Can’t express perimeter in terms of dimensions of the new figures.
Difficulty labeling sides of the rectangle inscribe in a triangle.

Probes: Can you kindly read again and represent the information a diagram?
What is the formula for the surface area of a close rectangular box you have drawn?
Use any variable and split the given perimeter into parts.
What are the sides of the inscribe rectangle?
Question 6.
Hypotheses: Difficulty transforming the middle term into index form. Unable to find the LCM of a non-fractional term (as one).

Probes: Can you write $\frac{\sqrt{10}}{t^6}$ in index form?
Can you clear the fraction in this equation?

Question 7.
Hypotheses: Likely can’t represent the word problem into a diagram. Can’t figure out the surface area formula, and use the given dimensions. Can’t visualize the information completely and set up the equation.

Probes: Can you draw a diagram to represent the information in word problem? Write the surface area formula of the rectangle. Re-represent the information from 7b in a diagram.

S-3 Transcription

RCH: You got the first question correct, but in the second one what can you say about the concept of a hemisphere?

S-3: (long silence)…I don’t know…I just draw what it look like…I am bad in math.

RCH: Read the question (1b) for S-3.

S-3: I don’t know what that is, I just got it from here (from the previous question).

RCH: So you got the cylinder, but you have difficulty getting the hemisphere?

S-3: Yes.

RCH: How about the concept of inscription? Does it sound strange or you are familiar with it?

S-3: I know what inscribe is but in this concept (context) I don’t know what it is.

RCH: Okay. Let’s look at the second one, you have cone but you have to make a cone from a sector. What does the concept of a sector mean to you? Have you seen or use it before?

S-3: I have used it before, but …

RCH: Okay…how does…
S-3: I can picture the conical drinking cup.

RCH: Okay, how does the sector look like?

S-3: I don’t know.

RCH: You can’t not recall?

S-3: Uhm . . . uhm.

RCH: Just try and let me see.

S-3: A sector…

RCH: Yeah…

S-3: Like this folded it …(folded the paper to form a cone).

RCH: Okay…

S-3: No (laughter).

RCH: You got this one right…but the second question asks about the dimensions. What are the dimensions of a cylinder?

S-3: I don’t know ( shook her head).

RCH: Okay.

S-3: See, I know that of a rectangle…I don’t know that of cylinder.

RCH: So the dimensions of cylinder you don’t know.

S-3: Uhm . . . uhm.

RCH: Okay. You got these ones right..h and w, is it suppose to be h + w, or h – w?

S-3: It is mistake, it is h + w.

RCH: You got it right in question 3 except for that…did you refer D as a rectangle? But you wrote a rectangle here, what does that mean?

S-3: The rectangle, cube…it is like a block.

RCH: Did you call it a rectangular block or cube?
S-3: Yes.

RCH: What is the surface area of a rectangular block?

S-3: Amm...(little silence).

RCH: You have some formulas you can choose.

S-3: Is it in here?

RCH: Yeah. Then what happens in question 4. You clear the fraction but you move 4 to other side and you didn’t change the sign, what happen?

S-3: I must have forgotten, it is a careless mistake.

RCH: (Laughter).

S-3: I know and did that every day.

RCH: Tell me about your solution in 4b?

S-3: (shock her head in disapproval or lack of confidence).

RCH: Can you explain to me what the question is asking?

S-3: What I did was …I was trying to solve for y here…one of the variables and plug the variable in the second equation and solve for the other variable. I didn’t work it out the math way.

RCH: That means you really understand the procedures but using it creates some kind of problems?

S-3: Yeah.

RCH: What happens in 4c, you simplify the fraction here but….

S-3: I was trying to do the trigonometric identity but I couldn’t remember.

RCH: Can you tell me the trigonometric identity you are trying to thinking about?

S-3: \( \cos^2 \theta + \sin^2 \theta = 1 \).

RCH: Right, then how did you clear this fraction?
S-3: By multiplying with sinθ.

RCH: So what is the next thing to do?

S-3: (Long silence). You can bring the sine over (no specific strategy).

RCH: Lets move to question 5, tell me about it? You got the volume, which is equal to

\[ lwh \ldots \] which is the correct formula, then you set the volume in terms of what

was given \( 2w \). \ldots \ What is the question mark?

S-3: I didn’t know.

RCH: So can you complete the diagram here?

S-3: (long silence)…no response.

RCH: Normally when you solve optimization, how did you start it?

S-3: I started by drawing a picture.

RCH: Is this a rectangular box?

S-3: No because it says it has an open top?

RCH: Oh…okay…so you are referring this to represent an open box?

S-3: Yeah.

RCH: Oh…

S-3: Am I wrong?

RCH: No you didn’t do wrong actually. The issue is not either wrong or right, but the

way you represent your understanding of the question.

S-3: The problem is that it says using a single independent variable, I don’t know how

make this length, the width and the height all in a single variable.

RCH: Tell me about 5b?

S-3: (Long silence)…when you add them together, my problem again it says one

single independent variable. I didn’t know how to do it.
RCH: You major problem is how to express it in a single independent variable?
S-3: (shook her head)...yes.
RCH: But you have the main ideas, but changing it into a single independent variable is you have some trouble?
S-3: Yes.
RCH: What about 5c? What is the dimension of the rectangle inside the triangle?
S-3: I don’t know?
RCH: What did you mean that you don’t know?
S-3: Surface area...oh...length times width.
RCH: Right. If the length is x, what is width?
S-3: y.
RCH: What are the sizes of the unknown sides?
S-3: x – 4 and y – 3.
RCH: This is an interesting question, i.e. question 6 (a,b). tell me what happens in 6a?
S-3: Aaaa...looking for the derivative, it is addition so this one is down and subtract one.
RCH: Yeah, you did this one right, what happens with the middle term?
S-3: Technically you can use quotient rule, but (long silence)...I don’t know.
RCH: Okay...can you rewrite one all over t squared in index form?
S-3: t to the negative 2.
RCH: This is a constant right, so if you write it ....
S-3: Negative six t, the negative seven.
RCH: That is the derivative?
S-3: Yeah.
RCH: But, would it affect the value you have obtain?
S-3: I guess you use that.
RCH: You got this one right, but I don’t know how you loss this from…you got the
derivative right…but in the simplification, you missed the form. You clear the
fraction, right?
S-3: Yes, I got it multiply over here…clear less mistakes (laughter).
RCH: Yeah (laughter). That is consequences of a exam.
S-3: Yeah…I made a lot of mistakes.
RCH: In the first part of the last question…(reread the question). Is this a representation
of the box?
S-3: No I guess not.
RCH: How would you transform this into a box?
S-3: Like a cube box?
RCH: Rectangular box, which is what the question says.
S-3: (Drawing the box) drawing whatever….
RCH: Okay…what happens in the test you didn’t take your time to …are you in a hurry
or what?
S-3: Yes…I had a quiz to work on after the calculus class, which is why I rush to do
this (laughter).
RCH: Then how about this one (7b) what kind of difficulty did you face in an attempt to
solve this?
S-3: Well I guess I didn’t draw it right.
RCH: What happens in last part of question 7, i.e. 7c?
S-3: Light bulb did go off.
V-2 Interview Protocol/Transcription

V-2 Protocol

Question 1.
Hypotheses: The concepts of hemisphere and sector of circle are vague.
Probes: Can you think of an example of a hemisphere in outside world and what does a sector of a circle means?

Question 2.
Hypotheses: Could establish relationship between L and circumference, but expressing radius in terms of the length, as well as relating width and height was not clear.
Probes: Explain to me the relationship between L in the rectangle and C, circumference of the cylinder. Can they be represented symbolically?
Tell more about the dimensions of a cylinder?

Question 3.
Hypotheses: Misinterpretation of the differences between 2-D and 3-D and general geometric weakness.
Probes: Make a shape and ask what it is. Ask whether 2-D has a volume or ask its associated formula.

Question 4.
Hypotheses: Skills not connected with conceptual understanding.
Probes: How do you solve this set of equations?
Explain your solution processes for this trigonometric equation?

Question 5.
Hypotheses: Misconception and interpretation of the concept of variables as it is connected to geometric diagrams.
Difficulty interpreting relationship between perimeter and area.
Probes: Can you explain how you got the surface area equation?
Tell me more about your area formula of circle and square?
What happens next in your solution process?

Question 6.
Hypotheses: Transformation difficulty and non-reflective/conceptual limitation of algebra skills.
Probes: Can you write \(\frac{\sqrt{10}}{t}\) in index form?
What are the likely value(s) for b and which one make sense to you in the context of the problem?
Question 7.

Hypotheses: Struggling with representing the word problem in a diagram (visualization), getting the required equation.

Probes: Can you represent this problem in a diagrammatic form and label its sides using variables? Explain your understanding of this question. What does it require you to do?

V-2 Transcription

RCH: Let us start with question 1b, even though you got 1a right, tell me how you solve it?

V-2: Uhm…right circular cylinder…I don’t know what that is…I have never heard of that. I don’t know if it is like a cylinder or it has a right angle in it. But I know it is inscribe in a hemisphere.

RCH: And you think this is a hemisphere?

V-2: Yeah…probably wrong. (laughter).

RCH: Can you give an example of a hemisphere in real life?

V-2: Just like the earth is circle…I don’t know…I mean…

RCH: You are not sure of the concept of a hemisphere?

V-2: Yes.

RCH: How about question 1c.

V-2: I have probably didn’t know how to start.

RCH: Okay, but you have a cone right?

V-2: Uhm…I got from here (pointing at diagram).

RCH: What is a sector? Did you know what a sector is?

V-2: No.

RCH: Can you recall a situation where you learn about a sector?

V-2: I may probably did…I don’t remember that.
RCH: You got 2a right, actually everything went right, then what happens in 2b? you establish the relationship that the circumference here is $2\pi r$, right but you wrote $r$ equal to one-half $L$, so I don’t where the pi goes?

V-2: (laughter)…I don’t know either. May be because I had $\pi L$ and…

RCH: Okay what does the $\pi L$ represent?

V-2: I don’t know, (short silence)…I think I was thinking about the diameter…I don’t know.

RCH: Did you know about the dimensions of a cylinder?

V-2: I don’t really understand the amm…like cylinder, rectangles, stuff. This is kind of confuses me.

RCH: You mean you didn’t learn it in school?

V-2: I learnt it in school, but I didn’t necessarily understand it.

RCH: What makes you not necessarily understand it?

V-2: I probably didn’t remember it.

RCH: Okay…lets look at last part of 2c…can you explain how you arrived at your solution?

V-2: Because this is $L$ and if you subtract that, you got that small part.

RCH: So you are able to relate the longer distance with these smaller ones?

V-2: Yes.

RCH: This takes us to question 3.

V-2: That was so confusing (laughter).

RCH: What makes you a little bit confusing about the question?

V-2: I don’t know…I think that they are just so many,…can’t remember those…

RCH: So many formulas…and then so many geometric shapes?
V-2: Yes.

RCH: What did you mean by so confusing?

V-2: I guess…I don’t know…I just get confuse. I just couldn’t apply the formulas with the concepts…or whatever.

RCH: You mean that you couldn’t get the concepts of….

V-2: How to apply this (formulas) to these shapes… I guess.

RCH: So you have difficulty relating individual formulas with individual geometric figures?

V-2: Yes.

RCH: Okay…let’s look at the solution of 4a, the algebraic process. Tell me more about it?

V-2: I did that to get $P$ by itself and multiply…that is where I went wrong.

RCH: In 4b, you have the solutions as 2 and $\frac{1}{2}$. Did you think these are the only solutions?

V-2: (short silence)…probably not.

RCH: In quadratic equations, the variable has two values, then what would happen to $y$?

V-2: You plug in the values of $x$ to get $y$.

RCH: Would each value of $x$ generate another value for $y$?

V-2: It should.

RCH: But probably you didn’t care much to do that?

V-2: Yes…I didn’t care to put it down.

RCH: Did you have the idea before, or you just skip it?

V-2: I mean if they said…you know…actually give the points…I just solve the equation.
RCH: So you solve the equation without actually knowing exactly what the question requires you to do?

V-2: Yes.

RCH: How about 4c. You got the first step correct, what happens to the rest part?

V-2: Yes that was the easiest part.

RCH: How would you transform the equation and get rid of the radical?

V-2: If you divide by the cosine to get the tangent…

RCH: But if you divide by the cosine, you are going back to here? Is there any way you can transform this $\cos \theta + 1 = \sqrt{3} \sin \theta$, so that it would look simple?

V-2: (short silence)…uhm…I don’t know.

RCH: Okay, lets look at 5a. I agree with you that the volume is (pointing at what she wrote), and the surface area is (pointing at what she wrote). Does this represent the information? Is the diagram here represent the information we have?

V-2: (long silence)…uhmmm…probably not.

RCH: What make the difference with the information you wrote?

V-2: It seems I couldn’t get it into a single variable or something of that sort.

RCH: What you have is a close tank, but the question expresses an open tank. How can you reconcile the two?

V-2: I have to get rid of that…probably this one.

RCH: Tell me about 5b? You have the perimeter of a square as $4r$, area of the circle would be $\pi r^2$, but I am really wandering, would the area of a square going to be $4s$?

V-2: It is side square.

RCH: So what does the $4s$ represent?
V-2: Is more of a perimeter. I don’t know why I did that. I think I must have forgotten the formula.

RCH: You use similar triangle concept in your solution of 5c, tell me more about the solution?

V-2: Yeah I used it because…

RCH: Okay, is this easy…right…

V-2: Yeah.

RCH: I know you differentiate term-by-term here, how did you get the derivative of the middle term? Did you use direct rule (power rule) or…

V-2: I used the quotient rule?

RCH: And how did you use the quotient rule?

V-2: The top derivative would be zero, and then you multiply by the bottom …minus that derivative and the bottom squared. Oh this cancels.

RCH: The 6b is also straight forward. It seems you use quotient rule again here?

V-2: Yes.

RCH: But here, you have the square root of 400, and the square root normally comes…

V-2: Is it positive or negative?

RCH: Right, which one did you think would be most appropriate?

V-2: I think the positive…

RCH: You choose the positive value, why did you choose the positive value?

V-2: The largest possible volume…probably wrong.

RCH: Did you normally get negative dimensions?

V-2: No.
RCH: Let us look at 7a, what happens here? Did you think this diagram represent the information?

V-2: (long silence)…uhm…I thought too…it says the width is twice the height, so is $2h$. I don’t why I did it wrong.

RCH: A close rectangular box, is it a 2-D or 3-D?

V-2: 3-D.

RCH: Does this represent a 3-D or a 2-D? (Pointing at what she draw)

V-2: It is a 2-D.

RCH: What can you do to make it a 3-D.

V-2: I am not good at drawing but…

RCH: You can visualize it mentally but you have some difficulty writing it?

V-2: Yes.

RCH: If you want to construct something, is the material required in volume or in area?

V-2: Area…volume…uhmm volume.

RCH: For example, I want to construct a box, and have this sheet of paper. Is this paper in 2-D or 3-D before the construction?

V-2: 2-D.

RCH: What is the size of this material (paper)?

V-2: Area.

RCH: So now what is wrong with your work?

V-2: I use the area formula.

RCH: So did you need a surface area or volume formula?

V-2: Surface area formula.

RCH: What is the surface area of a close box?
V-2: It is like $2lw + 2lh + 2wh$.

RCH: In 7b, what happens?

V-2: It is confusing.

RCH: What really confuses you?

V-2: (long silence)…uhmmm.

RCH: Does the $L$ represent side of the inner rectangle or the outer one?

V-2: The outer one. I wasn’t sure how to find the area, once you took this portion out.

RCH: What happens in 7c? Seems you are not sure of how the picture looks like.

V-2: Yes.

W-2 Interview Protocol/Transcription

W-2 Transcription

RCH: Tell me about each of the solutions of question 1?

W-2: This is just basic geometric problem…right; they are just conceptual if I remember. What did you want to know about them?

RCH: I want to know how you interpret the question.

W-2: You have cardboard box and in making it (demonstrating with hand how to fold it).

RCH: So you visualize it before you write?

W-2: Yeah

RCH: How about the second one?

W-2: For this one I wasn’t quite sure how to do the top…kind of a cut off. The visual representation doesn’t need to be perfect. But again, you just need to visualize, because it is purely conceptual.

RCH: How about the last one?
W-2: (long silence)…

RCH: This look like a triangle…right? How is this different from (drawing and comparing a diagram of a sector and triangle). Are they the same?

W-2: No…not quite the same.

RCH: What make the difference?

W-2: The top of this (showing the sector).

RCH: And normally what did you call this?

W-2: Again it is aa…amm…(a section of the circle).

RCH: Here in 2a there is nothing much…you did what is expected, but what can you say about the dimensions of a cylinder? What are they?

W-2: Ammm…height, radius.

RCH: Okay…height and radius are the dimensions of a cylinder. Would the variable change from w to something else?

W-2: Yes.

RCH: Here you got what is expected…what did you call these letters that you use?

W-2: They are just variables.

RCH: You have a question here, what does that mean in question 3?

W-2: It is a rectangular….I don’t really remember.

RCH: How did you figure out the formulas?

W-2: Well…for most of them I remember…some of them I use the process of elimination.

RCH: So you recall from your geometry… right?

W-2: Yes.
RCH: You started up by clearing the fraction and collecting like terms in 4a. Can you tell me more about that process?

W-2: You add 4 to both sides, and the on the LHS, the negative and positive 4 would cancel each other.

RCH: On question 4b, you got two solutions for x, are they the only solutions?

W-2: That is what the question is asking right?

RCH: Solve the system of equations, did think these are the only solutions? Besides how did you get $x = 2$, and $y = 8$?

W-2: Well…amm $x = 2$ because ….that is the only thing that goes in there right.

RCH: You substitute some numbers and see whether they match up?

W-2: Yes…If it is a complicated problem, I could have probably gone with something more regiment, but because when I see that it just…

RCH: What did you think of this equation?

W-2: Well, it could be simplified more.

RCH: If you simplify, what did you get?

W-2: You can separate the $x$ out…

RCH: What type of equation is this?

W-2: Polynomial… (Quadratic).

RCH: If you solve a quadratic equation, how many solutions did you normally got?

W-2: Two.

RCH: So if you solve for $x = 2$, did you think it would be the only solution?

W-2: Probably not (attempting to solve the quadratic equation).

RCH: Initially, you have some difficulties seeing the two solutions of the quadratic equation?
W-2: Yes.

RCH: When last did you solve equations involving two variables?

W-2: Amm…we solved the two variables…well I don’t think of ever seeing two variables in that being quadratic.

RCH: Okay…you got the first step of 4c, what did think would be the next thing to do?

W-2: Amm…(long silence)…this is simplifying something.

RCH: You couldn’t recall solving trigonometric quadratic equation…

W-2: It has been a while.

RCH: Lets look at the question 5, how did you got the equation for 5a without drawing the diagram?

W-2: Well..amm…is just a normal box right…I visualize it, no need to draw.

RCH: Since it is an open box, did you think there would be 2 in all?

W-2: (demonstrating with hand)…ahh there would be only one.

RCH: You visualize the total length of 10cm and you take away $x$, did you consider $x$ or $4x$ to be the total perimeter of the square?

W-2: Just the area…oh no the perimeter of the square.

RCH: Then what happens in 5c?

W-2: We go back to trigonometry…that is where I have some trouble.

RCH: In question 6, you got everything right in 6a, as well as the derivative of 6b. How did you lost the 300 in your computation?

W-2: Well we take the derivative…I am not sure, it look like it just disappear.

RCH: If you set this equation to zero, what did you need to do?

W-2: I need to solve for $b$.

RCH: So how would you solve for $b$?
W-2: You put the 300 on the other side…and…

RCH: Which of the two solutions make sense here?

W-2: Positive.

RCH: Why?

W-2: Because you are looking for the volume of the box.

RCH: Tell me more about your solution of question 7b, since you got 7a correct? You got the area of the smaller portion right, which is that. What is this area again, is it the area of the whole poster?

W-2: Maybe, that is what I am looking for.

RCH: Which means you substitute for x in here right, so it would give a derivative of a product function.

W-2: Yeah.

RCH: And if you are finding the derivative of a product function, what did you have here?

W-2: (long silence)…trying to recall.

RCH: You actually set it up right…but getting the derivative seems difficulty.

W-2: Yeah.

RCH: Tell me about your thought here (question 7c).

W-2: Attempt to solve it during the interview session.

**J-23 Interview Protocol/Transcription**

**J-23 Protocol**

**Question 1.**
**Hypotheses:** The geometric concepts required are presented.
**Probes:** Explain how did you get the diagrams?
Question 2.
**Hypotheses:**
- Couldn’t establish relationship between L and circumference, and making connection with circumference formula is vague.
- The concept of dimensions of a cylinder is vague.
- The formula of the circumference of a circle is obscured.

**Probes:**
- Explain to me the relationship between L in the rectangle and C, circumference of the cylinder. Can they be represented symbolically?
- What is the formula for computing the circumference of a circle or top of a cylinder?
- Tell more about the dimensions of a cylinder?
- Explain how you label the sides of the rectangular box?

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Question 3.
**Hypotheses:**
- Misinterpretation and geometric weakness.

**Probes:**
- Explain how you got the formulas for the listed geometric figures.

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Question 4.
**Hypotheses:**
- Skills are not connected with conceptual understanding in solving trigonometric equation.
- Misunderstanding what the question is asking.

**Probes:**
- How do you solve a quadratic equation?
- What can you say about this trigonometric equation?

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Question 5.
**Hypotheses:**
- Misconception and interpretation of the concept of variables as it is connected to geometric diagrams.
- Difficulty relating concept of perimeter and ides of squares using variables.
- Difficulty labeling the newly formed geometric shape.

**Probes:**
- Can you explain how you got the surface area equation?
- Tell me more about your area formula of circle and square?
- How do you label the complex diagram?

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Question 6.
**Hypotheses:**
- Transformation difficulty and non-reflective/conceptual limitation of algebra skills.

**Probes:**
- Can you write \( \sqrt{\frac{10}{t}} \) in index form?
- What are the likely value(s) for b and which one make sense to you in the context of the problem?

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Question 7.
**Hypotheses:**
- Struggling relating concept of surface area with volume, setting up relationship between sides of geometric figures and variables.
- Difficulty differentiating the derive equation.

**Probes:**
- Can you represent this problem in a diagrammatic form and label its sides using variables?
- How can you generate the required equation?
J-23 Transcription

RCH: Tell me about how you solve these first three questions?

J-23: Okay... actually we have some of these stuff in our calculus class. I am a bit familiar with...

RCH: Okay...

J-23: Aaa... this one definitely... and then aa... about... the others. This one... cut out the squares from the four corners and bending up the sides. So... I just...

RCH: So these are the four squares... right?

J-23: Right... these are the squares cut out and then the dotted line show where it would bend. This one I wasn’t sure... but I knew that this is a hemisphere and a right circular cylinder... and I wasn’t sure if this is how it is suppose to be drawn... inscribed in a hemisphere. I just went up, this is a cylinder, a hemisphere and inscribe in it.

RCH: Inscribe means inside right?

J-23: Uhm.

RCH: How about this one?

J-23: Ammm... and so you have a circle and cut out the sector and joining the edges... I... I don’t know. I guess is from the fast a little cone cut out a piece of the triangle, you can wrap it around...and this is what it is form (showing the drawn cone).

RCH: So this comes from your previous experience?

J-23: Just by... I don’t know (laughter). This is a little like a Kenwood water drinking fountains. You have cut out a triangle piece for it to work.

RCH: How about this labeling of diagram?
J-23: Okay . . . am . . . (reading the question). So I knew that this is the length, width and the height. So …if it is giving everything in terms of width, then you have to use a single variable. So the length is twice the width, so 2w, and the height is a third of the width, so one-third w. That is how I label it.

RCH: Okay.

J-23: The length is L and it width . . . I wasn’t sure about this one . . . I knew that when you wrap it around …this circumference is going to equal whatever distance you have, because you are not going to be adding or subtracting anything. With width . . . you are not cutting anything . . . so you didn’t lose anything.

RCH: When you transform the rectangle into a cylinder, you generate a circumference here.

J-23: Right.

RCH: And then you generate a height so, what are the dimensions of a cylinder in your new object?

J-23: . . . no response.

RCH: Lets look at here, what are the dimensions of a rectangle?

J-23: Length times width.

RCH: Then what are the dimensions of the cylinder?

J-23: Aaa . . . are going to take into account volume or not necessarily? Is that what you are asking?

RCH: No . . . no. The dimensions . . . you know what the dimensions means?

J-23: Yeah . . . are the . . . would the circumference, plus the width . . .

RCH: If you move from here to here (showing the two different figures), then this is no longer width . . . right. It becomes . . .
J-23: The height.

RCH: So height is one of the dimensions.

J-23: Right.

RCH: The circumference generate two things, one of them is the longest distance and is call….

J-23: Diameter.

RCH: Half the diameter is…..

J-23: Radius.

RCH: Normally, the radius and height are the dimensions of the cylinder. Can you represent the circumference using a formula?

J-23: Aaa . . . is $2\pi r$.

RCH: Right . . . so now $L$ becomes what?

J-23: $2\pi r$, right.

RCH: Is it possible to write the radius in terms of $L$?

J-23: Yes.

RCH: And what should that be?

J-23: And that . . . so $r = L/2\pi$.

RCH: Right . . . so that is what I am looking for.

J-23: So that is what you are looking for, okay. I didn’t…when I read the question it says in terms of these variables …that seems just…

RCH: What happen here?

J-23: Four variables representing all six sides. Well the distance of this figure (drawing dotted lines), so this amm…separate this like a third that y would be equal to 3x, with this being x.
RCH: Why did you represent these two different distances with x? did you assume them to be the same or…

J-23: Aa….

RCH: Any reason for that?

J-23: Long silence….I….because we can only use four variables. So I wasn’t sure if this length and this length were the same.

RCH: But you have the four variables…1…2…3…4.

J-23: Right…and it just appeared that this was the same length with this….this would be a rectangle (showing some space).

RCH: Is that your assumption?

J-23: Yes,…I don’t know if it is right?

RCH: You mean this distance equal to that? (pointing at the two distances on the test)

J-23: Right.

RCH: That is what you assume?

J-23: When I did this problem, but probably now I can see it is not correct, because it is rectangle ….but I didn’t know another way to express it with a different variable.

RCH: Okay…if suppose you have the four variables w, x, y and z, and you didn’t named this to be x,

J-23: It would be z – w.

RCH: Why?

J-23: Because it is the difference between this entire areas (length) is z, this area (length) is going to be z – w. That makes more sense.

RCH: Why did you write 2x here…did you assume this distance is twice that?

J-23: Yes, … but again in the same way you can do y – x.
RCH: Talk to me about your understanding/misunderstanding of associating geometric
diagrams with appropriate formula?

J-23: So following each formula with it figure...for area...perimeter...okay.

This...this...this and this (circling the 3-D) have surface areas...they are the only
3-D shapes.

RCH: Okay.

J-23: ...and like I cross them out, and I use ...for some of them I use process of
elimination. But for the circle I knew...the area and the perimeter...and for the
rectangle I knew the area and perimeter. Then, from there...the cube was the only
one you can use L, because the cube all the sides do the same. The only way you
can do length times width times height, all of them be the same. So that the way
you do the cube...and then for it surface area, there is six sides and each side is L
times L to get the area of it and then times six, because there are six of them. For
the rectangle (prism), I knew that the area (volume) is length times width times
height. For the surface area is the same thing, two of the length times height (2lh),
two of the width times height (2wh), and two of length times width (2lw). The
cone...cone and cylinder ...am...I actually I don’t remember how I got those,
only because.....let me think (silently). I won’t say I knew this was the volume
because we just done that...in class we just went over about the volume of the
cylinder.

RCH: Okay.

J-23: So I knew the volume was this (pointing at the formula). This has to be the
volume by default because there are no other volumes. (RCH interject with
laughter). And so...
RCH: So you mean by default….since there are no any other options, this has to be the volume?

J-23: Right,…this has to be the volume.

RCH: You have already eliminated…..

J-23: Three out of four….and then amm…let see …this is the surface area that seems to go with…(long silence) this I guess ….because…I think about that, but this…if actually this work together …then surface area would be smaller than the volume …..that is correct. Am so…I choose …..

RCH: $\pi r L$…..

J-23: Yes….so the cone and the cylinder had a little bit of trouble and it was more of a kind of process of elimination …actually like figuring it out. And because this was two and two, this would be one and the bottom would be one, this side of it and the back side of it would be one.

RCH: What are your solution process of question 4a, b, and c? Okay tell me about the 4a?

J-23: Okay (take a deep breath)….so I distributed the one-half and then the one-third…aa…and simplify and express $P$ …so you want to find what $P$ is in terms of a (and the constant).

RCH: Okay.

J-23: Okay (take a deep breath)….so I distributed the one-half and then the one-third…aa…and simplify and express $P$ …so you want to find what $P$ is in terms of a (and the constant).

RCH: Okay.

J-23: …and so I then ….let see combine like terms from one side to the other. Aa…find what $P$ was and then amm…multiply by six to find everything in terms of a and then simplify. This one (4b)…I found ….i set this equation …let see so…$3x - y = -2, \ldots$ so $y = 3x + 2$ so then I use this to plug-in into this equation. So that I plug-
in that and then solve for x, and then I plug each number into the original equation to solve for y.

RCH: What happens here (4c).

J-23: And because this are single terms and … am added, I combine them so that I can multiply them by the sine and then subtract \((1 - \sqrt{3} \sin \theta)\). And then…..i don’t remember trigonometric properties either now in Calculus, I don’t remember trigonometric properties. So since we couldn’t use the calculator…I didn’t know how to find out exactly what the angle was.

RCH: How did you approach the solution of question 5?

J-23: So with an open top, it has a volume of this (showing the given volume), the length of the base is two times the width, so the length is \(2w\), and the total surface area in terms of a single variable. So I write everything in terms of \(w\). I knew for a rectangle ….let see (long silence) ….and so this is the surface area of a normal rectangle (close). We are missing the top so it is going to be \(SA = 2lw + 2wh + 2lh\). So earlier I found what \(h\) was in terms of \(w\).

RCH: Since you are missing the top, do you think the formula you gave would be the right one?

J-23: Oh…since the top is missing, then…..it would \(lw\) and not…then I solve (volume) in terms \(h\) and so I had to \(h\) in terms of \(w\). so I use that and replace \(L\) with \(2w\), and plug it in the formula.

RCH: How did get to solve 5b?

J-23: (Read the question again). And so I solve in terms of \(L\)….I guess. Area equal to one over….
RCH: How did you relate the perimeter (10 cm) with dimensions of the new figure?

J-23: …..and so (after long silence). I factor the L and set it this….

RCH: How did you handle the last part?

J-23: (Re-read the question again). I think I am heading toward the end so I have no idea what this was. Though…..after I draw the diagram and label the sides with length of the leg of the triangle.

RCH: Tell about the procedures in this section 6a, b?

J-23: Okay…amm…so re-wrote it. Amm… so that everything was the same level to get the derivative. And then you multiple this (exponent of the variable) with and add one to it, that gives - 7/5, and then you multiple – 6 and then added one to it and you get negative five not negative seven.

RCH: Is it plus one or minus one, adding or subtracting?

J-23: It would be – 6 + 1.

RCH: Are you sure?

J-23: Yeah…I thought so because it would be ….  

RCH: So what did you do here (pointing at the first term)?

J-23: So let see ….it would be negative three-fifth.

RCH: I think you are right…because you are subtracting one…by the way how did you get the one here (the last term of the function)?

J-23: Because the derivative of t is one.

RCH: If you go by the power rule….

J-23: Oh…okay.. 

RCH: If you have $a^n$ …. 
J-23: Okay…okay…oaky, I am thinking about what we are doing in class right now…never mind.

RCH: What you are doing…are now doing integration?

J-23: Yes. (Laughter together with the researcher). X plus one over….okay.

RCH: The last part of question 6, tell me about it?

J-23: So volume in terms of the base,…find the largest possible volume. We also just done what is it call….optimization…problem. So to do that if I find the derivative…and so find the derivative set that equal to zero, find what b was then find the second derivative and plug in each of this values. Because this give you a negative value that is going to be the maximum volume at $b = 20$.

RCH: So you use the second derivative test?

J-23: Yes.

RCH: Tell me about question seven?

J-23: Re-read the question. So this is also the …hmmm the optimization. So I found the …we knew the volume was eight, I found $L$ in terms of $h$, because we need to find …a …operate in terms of what h would be. Amm…

RCH: Did you remember what the question is asking?

J-23: Re-read the question …I guess I should have done the derivative?

RCH: Okay..am..what did you think the question was asking you…?

J-23: So…the same thing with the previous one…but you wants to find…the length which is going to be the least volume.

RCH: The previous one asked about what?

J-23: About the most volume…

RCH: Are you talking about this problem (opening the previous page)?
J-23: Right.

RCH: What is your own interpretation of “the use of least material”? what are thinking about?

J-23: So the smallest dimension…because that use the smallest material.

RCH: Right. If you are going to use the least material, are you using volume or surface area?

J-23: I use the surface area, but I use the volume here.

J-23: On this part, this is 384, okay so… you know the length and width of the area is 384. Amm…this length from here to here is going to be the length minus eight (L – 8), and the width the same thing (w – 12). So I foil this and solve for L and plug it into the equation where it is needed and simplifying and set it equal to zero.

Then I added, I simplify it again until I got this equation.

RCH: The last part of question.

J-23: Again…trigonometric stuff ….I really didn’t have a clue on how to get started.

### L-7 Interview Protocol/Transcription

#### L-7 Protocol

**Question 1.**

**Hypotheses:** Difficulty understanding what the question requires. The concepts of hemisphere and inscription are elusive.

**Probes:**
- Can you represent what the question requires here?
- Can you think of an example of a hemisphere in outside world?
- Explain what you understand by inscribing an object into another.

**Question 2.**

**Hypotheses:** Couldn’t establish relationship between L and circumference, as well as between width and height from rectangle and cylinder respectively. The concept of dimensions of a cylinder is vague. The formula of the circumference of a circle is obscured.

**Probes:**
- Explain to me the relationship between L in the rectangle and C, circumference of the cylinder. Can they be represented symbolically?
What is the formula for computing the circumference of a circle or top of a cylinder?
Tell more about the dimensions of a cylinder?

**Question 3.**
**Hypotheses:** Misinterpretation and geometric weakness.
**Probes:** Make a shape and ask what it is. Ask its associated formula.

**Question 4.**
**Hypotheses:** Skills not connected with conceptual understanding.
**Probes:** How do you solve a quadratic equation?
What can you say about the solution of this trigonometric equation?

**Question 5.**
**Hypotheses:** Misconception and interpretation of the concept of variables as it is connected to geometric diagrams.
**Probes:** Can you explain how you got the surface area equation?
Tell me more about your area formula of circle and square?
Can you inscribe a rectangle in a right triangle whose sides are on the leg of the triangle?

**Question 6.**
**Hypotheses:** Transformation difficulty and non-reflective/conceptual limitation of algebra skills.
**Probes:** Can you write $\sqrt{10^t}$ in index form?
What are the likely value(s) for b and which one make sense to you in the context of the problem?

**Question 7.**
**Hypotheses:** Struggling with incorporating geometric, algebraic skills into the solution procedures.
**Probes:** Can you represent this problem in a diagrammatic form and label its sides using variables?
Explain how you can set up the required equation.

**L-7 Transcription**

**RCH:** Lets look at what you have wrote in the test,…actually…like in the first 1a you didn’t …you left the space blank…what happens?

**L-7:** I just ah…I wasn’t really familiar with it…I skip it.

**RCH:** Okay…so you are not familiar meaning that you didn’t understand the content of the question?
L-7: I didn’t understand the question or whatever. I know it…(trying to look at the question again, reading it for the second time). I guess I didn’t understand the question.

RCH: Okay…still when you read it now for the second time you didn’t understand again?

L-7: Ah…I could probably just fed with it ….but I am not sure about that.

RCH: How about the second one?

L-7: I didn’t know if it made ah…(short silence), is it made of right circular cylinder inscribe into a hemisphere, …I know if it is on top of a hemisphere or how that was situated so.

RCH: Okay so the issue of inscription is creating a problem for you?

L-7: Yeah…I don’t know if it was attach on top of it or.

RCH: But if I say one object is inscribe inside another one.

L-7: Yeah, I don’t know.

RCH: Here you got the sector, the folded cone and the labeling of 2a correct. Tell me what happen here in 2b?

L-7: Long silence. So the length would be the circumference of the aa… of the top…I just don’t know how to write on amm.

RCH: What did you mean by you don’t know how write? What is the circumference of the circle?

L-7: Is it \(2\pi r\)…so \(L\) equals to \(2\pi r\).

RCH: Initially, you didn’t know \(L\) is equal to \(2\pi r\) or you are…?
L-7: I wasn’t sure the circumference….am… I wasn’t sure about the circumference. I thought about $2\pi r$, but I don’t know how one may write it (making some gestures to express his inability). I guess I was just thinking…..

RCH: What are the dimensions of a cylinder?

L-7: Length times width times height, but I don’t know if…

RCH: No but I mean the dimensions of a cylinder?

L-7: Long silence…am…what did you mean by the dimensions?

RCH: If you pick a rectangle, its dimensions are length and width, right.

L-7: Uhmm.

RCH: So what are the dimensions of a cylinder?

L-7: It could probably be a… one of the radius of the top and then the height.

RCH: What the question want you to do is to label the dimensions in term of the variables.

L-7: Okay.

RCH: If you set $2\pi r = L$, then you are moving from $L$ to $2\pi r$,

L-7: Then the radius would be …. 

RCH: Would be what?

L-7: $L$ over $2\pi$.

RCH: And then what would be the height?

L-7: The width.

RCH: So you having trouble with the concept of dimensions?

L-7: Yes.

RCH: Here also you assume four sides and label the rest in terms of others. Okay, let’s look at the geometric diagrams and their formulas.
L-7: This is like confused me…aa. I know if we are suppose to (pointing at a particular geometric diagram) use this or. The way the problem was presented ….I don’t want to spend too much time …kind of a figured it out. I want to get to rest of the questions. So I fill out basically what I knew and went on to aaa…

RCH: Okay. So like if you have a cylinder you think the volume of a cylinder is $\frac{1}{2}\pi r^2 h$?

L-7: I don’t know the volume of a cylinder. I guess it would be ….the amm…area of a circle times the height.

RCH: Right.

L-7: That volume would be the volume of that (pointing at the diagram of a cylinder), I think… $\frac{1}{2}\pi r^2 h$ would be the volume of E, or it is one-third.

RCH: If you look at the surface area of a cylinder…?

L-7: I don’t know.

RCH: Okay.

L-7: I guess the surface area (pointing at a geometric diagram) of B (cylinder) is $2\pi rh + 2\pi r^2$.

RCH: Here you have cube?

L-7: Volume is is cube.

RCH: And the surface area is?

L-7: The surface area would be $6l^2$

RCH: How about this? You left it blank.

L-7: Amm…(long silence).

RCH: Even though you wrote the volume is….
L-7: I just...I ...at that time...I didn’t see this formulas, I was just doing it from memory. Whenever I saw this I change a couple of them, but I had already spend a lot of time, so decide to come back to them later.

RCH: Here your solution process $a$ equals to....lets look at what the question says.

Express $p$ in terms of $a$. what does that....

L-7: You have to have $p$ on one side...you have to solve for $a$.

RCH: Right.

L-7: $P$ is equal to something. I guess I was just trying to rush.

RCH: On the second solution, here you have one set of solution; did you think is the only solution you are supposed to have?

L-7: Amm...I got those and plug them back in ...I thought that there was more but $x$ is equal to two things.

RCH: Yeah...because it is a quadratic equation. Okay, tell me about your solution of the trigonometric equation?

L-7: Amm...(long silence)...I change the terms and then ...I just got confuse.

RCH: When last did you solve this type of problems or come across it?

L-7: This year we kind of solved, not really this stuff, probably in the Fall semester.

RCH: You mean you took trigonometry in Fall?

L-7: Yes. I guess it would have been....can I write?

RCH: Yes.

L-7: It would be one over cosine of theta...(long silence). I think there is a trig. Identity for that...may be ...no.

RCH: Okay you are not sure?

L-7: Yes.
RCH: One of the trig. identities, $\sin^2 \theta + \cos^2 \theta = 1$, is this similar to what you wrote?
L-7: No, because this is squared. I knew it was squared. I did it this way but I couldn’t remember that was one, then I erased it and did it this way, but still couldn’t get it.
RCH: How would you clear the fraction from here?
L-7: Multiply sine times ..... 
RCH: What is the next thing to do?
L-7: Uhm….long silence)…..I don’t know.
RCH: To get rid of the radical you have to square both sides.
L-7: Uhm.
RCH: What happens here (5a)…you succeeded in setting up but you couldn’t proceed?
L-7: Rectangular …. (reading the question again, later long silence).
RCH: You are able to set up the volume in terms of the variables given…right. So what is the next thing to do if you are looking….if you want to find out the formula for the surface area?
L-7: I just aa…(long silence). I had l there …I couldn’t figure out what it is.
RCH: Okay…Is the l addition or multiplication?
L-7: I think it is multiplication.
RCH: So you have $10 = 2hw$, but you can write l in terms of $2w$, right? What is the next thing to do?
L-7: I don’t know.
RCH: If you reflect back….lets visualize (referring to rectangular box). Since the top is open, what is the area of the bottom?
L-7: The area of the bottom is $w$ times $2w$
RCH: This face plus this face would give you what?

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L-7: This (pointing at the faces on the rectangle) and this one is 2l times w.

RCH: This face plus this one?

L-7: 2 times 2w times l.

RCH: Total surface area if you put the pieces is?

L-7: Sum of ……

RCH: In 5b, you split the two lengths into x and 10-x, and one is transform into squared and the other into a circle. If you relate total distance to 2\(\pi\) …if 10 – x = 2\(\pi\)r, what is r?

L-7: \(r = \frac{10 - x}{2\pi}\).

RCH: The total distance around the square is x, what is size of each side of the square?

L-7: x over 4.

RCH: Total area is what?

L-7: \(\left(\frac{x}{4}\right)^2 + \pi\left(\frac{10 - x}{2\pi}\right)^2\)

RCH: Tell me about the solution of 5c?

L-7: (Long silence)…I did it the wrong way. I inscribe the triangle inside the rectangle, I don’t know what I was thinking. So the other way round would be …(picked a pen and start to re-draw the diagram again).

RCH: You got the correct derivative in 6a, but it looks like…what happens?

L-7: Uhm….lets see (long silence) I just brought…I did the amm…

RCH: …..the quotient rule?

L-7: Yeah.

RCH: Is it possible without using the quotient rule to transform this into index form?

L-7: Yeah…amm (long silence) …
RCH: If you transform this into index form, what did you get?

L-7: You take the derivative of the top….

RCH: No what I am saying is, before you took the derivative, can you write this into index form?

L-7: Oh yeah …yeah…yeah. It would be.

RCH: It does not come to your mind?

L-7: Oh yeah, I wasn’t thinking like that when I solve the problem.

RCH: Similarly you have it here…everything is okay. Okay similarly in 7a, you set the volume equals to given dimensions. How would the surface area be different from the previous case?

L-7: It would be 2 times top face and bottom…

RCH: Let’s label the dimensions…length is \(l\), height is \(h\), width is \(2h\). So what is the area of top and bottom?

L-7: 2 times \(2hl\),

RCH: Then plus?

L-7: 2 times \(hl\) and then 2 times \(2h^2\)

RCH: What happens in 7b? Photo framing…you have the picture but something is missing when you set the equation.

L-7: (long silence)…Uhmmm…I didn’t know how to this one…I draw the margin and the stuff.

RCH: The margins are top and bottom …so the total margin is…?

L-7: You have to double…

RCH: Okay…What is the area of the printed portion?

L-7: \(x \text{ times } y \ldots\)
RCH: Equals to what?

L-7: 384.

RCH: Is it possible to write x in terms of y or y in terms of x?

L-7: Uhmmm.

RCH: So what next?

L-7: (Long silence)…..

RCH: Nothing comes to your mind?

L-7: Umm…uhm.

RCH: Lets look at 7c again.

L-7: (Reading the question again). I…don’t know. It is like triangular something…I didn’t….

T-20 Interview Protocol/Transcription

T-20 Protocol

Question 1.
Hypotheses: The concepts of hemisphere and inscription are elusive.
Probes: Can you think of an example of a hemisphere in outside world?

Question 2.
Hypotheses: Couldn’t establish relationship between L and circumference, as well as between width and height from rectangle and cylinder respectively.
The concept of dimensions of a cylinder is vague.
The formula of the circumference of a circle is obscured.
Probes: Explain to me the relationship between L in the rectangle and C, circumference of the cylinder. Can they be represented symbolically?
What is the formula for computing the circumference of a circle or top of a cylinder?
Tell more about the dimensions of a cylinder?

Question 3.
Hypotheses: Misinterpretation and geometric weakness.
Probes: Make a shape and ask what it is. Ask its associated formula.

Question 4.
Hypotheses: Skills not connected with conceptual understanding.
Probes: How do you solve a quadratic equation?
What can you say about this trigonometric equation?

Question 5.
Hypotheses: Misconception and interpretation of the concept of variables as it is connected to geometric diagrams.
   Couldn’t relate concept perimeter to that of an area using variables.
Probes: Can you explain how you got the total area equation?
   Tell me more about your area of the rectangle inscribe in a triangle?

Question 6.
Hypotheses: Transformation difficulty and non-reflective/conceptual limitation of algebra skills.
Probes: What are the likely value(s) for b and which one make sense to you in the context of the problem?

Question 7.
Hypotheses: Struggling representing the word problem in diagram and conceptual difficulty with derivatives?
Probes: Can you represent this problem in a diagrammatic form and label its sides using variables?
   Explain how you got this derivative?

T-20 Transcription

RCH: Okay, let start. Are you from Louisiana?
T-20: Yes.
RCH: Where?
T-20: Iberia.
RCH: Tell about 1b?
T-20: This one?
RCH: Yes…even though it looks like a square, but if you fold it you can get a square.
   What happens here, you have only a cylinder, but the question requires inscribing a cylinder in a hemisphere?
T-20: I didn’t understand what the question says. All what I know was circular cylinder and the rest, I didn’t know.
RCH: Okay...so all what you know is a cylinder, but the concepts of inscription and hemisphere are vague, right?

T-20: Yes.

RCH: What did you think is the shape of the earth?

T-20: Oh...I got it now that you are saying, but when I am taking it...it is just (demonstrating shook of wander).

RCH: And inscribing means inside. Tell about what you get in 2a?

T-20: Amm...and so I said this is the width, this was length,...and I put the width as \( x \), and since the length is twice the width, the length is \( 2x \), and the height is one-third width, I have this one.

RCH: What can you tell about the dimensions of a cylinder? Did you know what the dimensions of a cylinder are?

T-20: No...we didn't do much with cylinder.

RCH: How much geometry...when last did take geometry class?

T-20: In the tenth grade.

RCH: And how many years back?

T-20: Am...three.

RCH: What can say about the relationship between this side of the rectangle and the newly constructed side?

T-20: It is twice this or ...

RCH: Is that what I am asking you?

T-20: Oh...this is this folded?

RCH: So are they equal?

T-20: Yes.
RCH: If they are equal, then \( l \) right, then what is this distance around the circle?

T-20: Did you ask as a definition?

RCH: Yeah, you can give a definition, you can give me a formula, anything you can think of?

T-20: (long silence)…circumference of a circle?

RCH: Yeah…did you normally represent it by a formula?

T-20: (Short silence)….uhmmm 2 \( \pi r \) (\( 2\pi r \)).

RCH: If they are equal, can you express \( r \) in terms of \( l \)?

T-20: What did you mean?

RCH: Can you write \( r \) alone?

T-20: Oh divide \( l \) by \( 2\pi \).

RCH: Okay…so now what did you call this distance?

T-20: The height,…oh so that is what the question says?

RCH: You get it now?

T-20: Yeah…I did understand.

RCH: So understanding the question might cause some trouble, right?

T-20: Yeah.

RCH: Tell me how you get this \( 2c \)?

T-20: I took the two longest (show the sides) sides this side and this side, and we can only use four variables. So this small ones made this longer side, similarly the other side.

RCH: Okay, almost everything here is perfect (question 3) except amm…what is this a rectangle right? The first diagram here is a rectangle right, so here you say volume, does a rectangle has a volume?
T-20: (Laughter)…..no.
RCH: So what happens?
T-20: I have no idea.
RCH: You just probably write.
T-20: Because I was just doing the process of elimination. I don’t know why I am doing this.
RCH: Here you have a box and left it blank, probably you want to write it here but you didn’t. Since you follow the process of elimination, can you describe how you get the volume of a cylinder?
T-20: Amm…Okay…I didn’t know …but I know this is a circle…
RCH: You simplify 4a and a collect like terms; right… tell me more about how you got the solution?
T-20: Yeah…I did that but don’t know how.
RCH: Tell me about question 4b?
T-20: Oh I didn’t do it right…I solve for y and it wasn’t coming up easy as I thought. I change and solve for x and put it in there
RCH: If you solve for y from here it could have been easier.
T-20: I think the first time I did it; I must have done something wrong.
RCH: What happens in 4c?
T-20: I couldn’t figure out where to go, what to start with.
RCH: Okay…if you have \(\frac{1}{a} + \frac{b}{a}\), how can you add this?
T-20: It would be \(\frac{1+b}{a}\).
RCH: Is it possible to apply it here?
T-20: Yeah.
RCH: And that would give us what?

T-20: \( \frac{1 + \cos \theta}{\sin \theta} \).

RCH: Then how would you clear the fraction?

T-20: Multiply each side by \( \sin \theta \).

RCH: And it would give us what?

T-20: \( 1 + \cos \theta = \sqrt{3} \sin \theta \).

RCH: How can you remove the radical?

T-20: (short silence)...I don’t know.

RCH: Tell me how you get the right model in 5a?

T-20: I draw the rectangle with open top...the volume...I used everything they give me. The length is two times the width. I put that in there. The area of this is \( 2w \) times \( w \), ....

RCH: What happens in 5b?

T-20: (Reading the question)...I wrote the sum of the areas for the square and the circle.

RCH: And this is \( \pi r^2 \), is it s square or s cube?

T-20: I don’t know.

RCH: Area of a square...is it length times length or

T-20: Oh yeah, it is length square.

RCH: So it should be squared here, right?

T-20: Yeah.

RCH: You are unable to make the relationship between perimeter and area?
T-20: Yeah, ... because I didn’t know how this length was a well this one (referring to the perimeter of a square and a circle). I couldn’t figure out that.

RCH: On the last part, you have….you inscribe the rectangle inside the rectangle…right.

What is the next thing to do?

T-20: (long silence)...basically the same thing right...

RCH: Yes…here you have to create some variables to represent these unknown sides of the rectangle. Now, tell me more about your solution of 6a. What did you use, I see the sign of t, was it negative 6. Are you moving t to the numerator there?

T-20: Am…this suppose to be t?

RCH: This is what you wrote..right? just explain to me how you got?

T-20: I think I did something wrong here. I move it up here, and the 6 comes into the front and it would be t to the negative seven.

RCH: What happens here?

T-20: (Laughter)...I have no idea...uhmmm...(long silence). You couldn’t found the volume without finding the base. So I am trying to look for b.

RCH: If you want to maximize a variable in a given equation, how would do that?

T-20: Take the derivative?

RCH: Uhm. Okay the last question...you didn’t do anything there?

T-20: Yeah I didn’t make it.

RCH: Because of time...

T-20: Yeah because of time.

RCH: lets us review and see.

T-20: Did I have to draw it?

RCH: I think it would be helpful to draw.
T-20: What did they mean by a different length?

RCH: Width, length and height are different.
VITA

Ahmed Ibrahim Usman was born in Gagarawa Railway Station, Jigawa State, Nigeria in May 1964, and the son of Late Ibrahim Danbaba Mai Gyaran Agogo and Hauwa Ahmadu. He later move to Maiduguri after the Nigerian civil war and began elementary education, and afterward graduated from Maigatari Teachers’ College in 1983.

Ahmed received state scholarship to attend Gumel Advanced Teachers’ College for three years, and graduated in 1987 as certified junior and secondary school teacher. He worked as classroom teacher at Senior Secondary School, Rano, under Kano State Ministry of Education for two years. He joined Bayero University Kano, Nigeria and received his Bachelor of Science degree in Mathematics and Education in 1993. He enrolled for compulsory national service for a period of one year. The following year after the national service in 1994, Ahmed was hired as instructor and began teaching at Federal College of Education, Kano, Nigeria in the Department of Mathematics. He taught there for seven years.

In 2001, Ahmed took a leave from teaching to begin his masters degree in teaching of mathematics with an assistantship from College of Education at the Northwest Missouri State University, Maryville, United States of America. Ahmed graduated in 2002. He later enrolled for another masters degree in applied mathematics and received graduate teaching assistantship at the University of Missouri-Rolla, which he completed in 2005. Later in 2005, he began doctoral studies at Louisiana State University in Baton Rouge, Louisiana. Ahmed held teaching and research assistantships in the department of Educational Theory, Policy and Practice.

Ahmed is happily married to Hafsat Yusuf Othman, and they have four children: Maryam, Abdulwarith, Hamida and Hanifa.