Some Properties of Certain Subsets of Infinite Dimensional Spaces.

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A dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mathematics

by

William Philip Barit
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ABSTRACT

In this dissertation we investigate some properties of certain subsets of infinite dimensional spaces. Results in three general areas are obtained.

The first area concerns the extension of homeomorphisms between special subsets of Hilbert space. An ε-homeomorphisms extension theorem is proved which shows that small homeomorphisms between these subsets can be extended to space homeomorphisms.

Results in the second area concern properties which most of the commonly studied infinite dimensional spaces do not have. These properties include having points which are not negligible, being non-homogeneous, and having no non-degenerate factors. Certain remainders of compactifications of infinite dimensional spaces are shown to possess these properties.

The last area concerns contractibility. Certain subsets of the space of homeomorphisms of some infinite dimensional spaces are shown to be contractible.
CHAPTER I
TERMINOLOGY AND DEFINITIONS

In this dissertation certain infinite dimensional spaces will be studied. This first chapter provides some necessary background and definitions. The symbols $X$, $Y$, and $Z$ will represent arbitrary topological spaces unless otherwise noted. Taking the topological product of infinitely many suitable spaces is one of the most common ways of producing an infinite dimensional space. Let $X^\infty$ denote the space $\prod_{i=0}^\infty X_i$, where each $X_i$ is homeomorphic to $X$. In the case $X$ is an $n$-dimensional manifold, the term basic neighborhood refers to a finite product of open $n$-cells cross the rest of the factors. The projection map from $X^\infty$ onto $X_i$ is denoted by $\pi_i$. A point $x$ in $X$ will be commonly written $(x_1)$ or $(x_1,x_2,x_3,...)$ where $x_i = \pi_i x$. If $d'$ is a bounded metric on $X$, then the following metric is used for $X$.

$$d(x,y) = \left[ \sum_{i=0}^\infty (2^{-i}d'(x_i,y_i))^2 \right]^{\frac{1}{2}}.$$  

Such metric spaces will be called infinite product spaces.

Sometimes subsets of infinite product spaces are given different topologies than the subspace topology. In this case they will be called infinite coordinate spaces. The coordinate representation of points and notation for projection will be the same as for infinite product spaces. A subset $K$ of either such space is said to be deficient in the $i^{th}$ direction if $\pi_i K$ consists of a single point. $K$ is said to be infinitely deficient if it is deficient in the $i^{th}$ direction for infinitely many $i$. Let $0$ be some specified point of $X$, then $X_0$ denotes $\{x \in X^\infty | \text{all but}$
finitely many coordinates are 0). Let 0 be the point in $X^\infty$ with each coordinate equal to 0.

Let $I$ be the closed interval $[-1,1]$ and $I$ be the open interval $(-1,1)$. The interior of the $n$-cell $I^n(\text{int } I^n)$ is $I^n$, and the boundary ($\partial I^n$) is $I^n \setminus \text{int } I^n$. The $n$-1 sphere, $S^{n-1}$, is another name for $\partial I^n$.

Let $\mathbb{R}$ be the real line.

The following infinite product space has been widely studied, and certain subsets of it play an important role in infinite dimensional topology. The compact space, $I^\infty$, is called the Hilbert cube and is customarily denoted by $Q$. The following subsets of $Q$ are analogous to faces in finite dimensional cubes. $W_i^+ = \{ x \in Q \mid x_i = \pm 1 \}$. These are called the end faces in the (positive, negative respectively) $i$-th direction. Note that each endface is itself homeomorphic to $Q$. We call $B(Q) = \bigcup_i > 0 (W_i^+ \cup W_i^-)$ the pseudo-boundary of $Q$. We may equivalently define $B(Q)$ as $\{ x \in Q \mid$ at least one coordinate of $x$ is $\pm 1 \}$. The complement of $B(Q)$ in $Q$ will be called the pseudo-interior of $Q$ and is denoted by $s$. Note that $s$ is the infinite product space $I^\infty$.

For $b$ belonging to $(0,1)$, call $[-b,b]^\infty$ a standard subcube is $s$.

Two other subsets of $Q$ that have been useful are: $\delta = \{ x \in s \mid$ all but finitely many coordinates of $x$ belong to $[-b,b] \}$ and $\sigma = \{ x \in s \mid$ all but finitely many coordinates of $x$ are 0\}. Note $\sigma$ is $I_f^\infty$.

Hilbert space, defined below, is an infinite dimensional coordinate space which has played an important role in infinite dimensional topology. Let $l_2 = \{ x \in \mathbb{R}^\infty \mid \sum_i > 0 x_i^2 < \infty \}$. (i.e. all square sumable sequences of real numbers) with the following metric.

$$d(x,y) = [\sum_i > 0 (x_i - y_i)^2]^\frac{1}{2}.$$
Sometimes it is useful to regard $Q$ and $s$ as subspaces of $l_2$. It is easy to verify that the convex set $\{x \in l_2 \mid |x_i| \leq 2^{-1}\}$ is naturally isometric to $Q$.

The following definitions deal with functions. Let $H(Y)$ be the set of all homeomorphisms of $Y$ onto $Y$. Let $C(X,Y)$ be the set of all continuous functions from $X$ into $Y$, and let $E(X,Y)$ be the set of all embeddings of $X$ into $Y$. If $Y$ has a bounded metric $d$, we define the distance between $f$ and $g$, $\rho(f,g)$, as $\sup \{d(f(x),g(x)) \mid x \text{ in the domain of } f \text{ and } g\}$. The identity function from $X$ onto $X$ will be denoted by $id_X$, or if no confusion is possible by $id$. If $f$ belongs to $C(X,Y)$ and $A$ is a subset of $X$, then $f \mid A$ represents the restriction of $f$ to $A$. Also if $g$ belongs to $C(A,Y)$ and $f \mid A = g$, then we say $f$ extends $g$. Let $H_A(X) = \{h \in H(X) \mid h \mid A = id\}$. Let $f$ belong to $C(X,Y)$ and $g$ belong to $C(X,Z)$, then $(f,g)$ represents the function in $C(X,Y \times Z)$ where $(f,g)(x) = (f(x),g(x))$. Let $f$ belong to $C(X,Y)$ and $g$ belong to $C(X',Y')$ then $f \times g$ represents the function in $C(X \times X', Y \times Y')$ where $(f \times g)(x,x') = (f(x),g(x'))$. A map $F$ in $C(X \times J,Y)$, where $J$ is a closed interval, is called a homotopy. For each $t$ in $J$, the map $F_t$ in $C(X,Y)$, where $F_t(x) = F(x,t)$, is called a level of the homotopy. If each level is a homeomorphism, $F$ is called an isotopy.

For $U$ an open subset of $X$, $h$ in $H(X)$ is said to be supported in $U$ if $h \mid X \setminus U = id$. Given $(h_i)_i > 0$ in $H(X)$, $h$ in $H(X)$ is called the infinite left product of $(h_i)$ and is denoted by $\Lambda_i \circ h_i$ if for each $p$ in $X$, $h(p)$ is the limit of the sequence $(h_i \circ h_{i-1} \circ \ldots \circ h_1(p))_i > 0$. Let $K$ be a subset of $X$. A homeomorphism, $h$, that is an infinite left product, $\Lambda_i h_i$, is said to be locally finite mod $K$ if the following is satisfied.
For each $p$ in $X \setminus K$, there is a neighborhood $U$ of $p$ and an integer, $N$, such that for each $n \geq N$,

$$h_n|h_{n-1}\ldots h_1(U) = \text{id}.$$  

If $K = \emptyset$, $h$ is called locally finite. For $h$ in $H(X^\infty)$ we use the following terminology. We say $h$ is independent of the $k$th coordinate, if $h = h' \times \text{id}_x$ where $h'$ belongs to $H(X, X_k)$. The homeomorphism is simple if it is independent of all but a finite number of coordinates. The concept of locally finite mod $K$ in the case of infinite product spaces is particularly useful when each $h_i$ is simple. Then all points outside of $K$ are subjected to motions which affect only a finite number of coordinates.

Given $K_1$ and $K_2$ subsets of $X$, $K_1$ is said to be equivalent to $K_2$ if there exists an $h$ in $H(X)$ such that $h(K_1) = K_2$. Two points $x$ and $y$ are equivalent if $\{x\}$ and $\{y\}$ are. If $x$ is equivalent to no other point of $X$, then $x$ is said to be rigid in $X$. If every pair of points are equivalent, then $X$ is called homogeneous. If $X \setminus K_i$ is homeomorphic to $X$, then $K_i$ is called negligible. These properties are studied for various spaces in later chapters.

Some terms are given for homeomorphisms of $\mathbb{Q}$. Let $h$ be in $H(\mathbb{Q})$. If $h(s) = s$, then $h$ is called $\beta^*$, and if $h(s) \neq s$, then $h$ is $\beta$. Since a $\beta^*$ homeomorphism when restricted to $s$ belongs to $H(s)$, such homeomorphisms have helped in the study of $s$.

The following types of sets have played a very important role in the study of the Hilbert cube and $s$, but are also useful in other infinite dimensional spaces. Let $X$ be a metric space. An open subset $U$ of $X$ is called homotopically trivial if for each $n > 0$, each map of
$S^{n-1}$ into $U$ can be extended to a map of $I^N$ into $U$. A subset $K$ of $X$ is called a **Z-set in $X$** if $K$ is closed and for each non-empty, homotopically trivial open set $U$ in $X$, $U \setminus K$ is non-empty and homotopically trivial. The following definition concerns sets with the compact absorption property and the finite dimensional compact absorption property. (cap and fd cap). In this definition it is understood that finite dimensional (f.d.) is to be read in all cases or not at all. A subset $M$ of $X$ is an **(f.d.) cap set** in $X$ of the following is satisfied.

$$M = \bigcup_{i>0} M_i,$$

with each $M_i$ an (f.d.) compact Z-set in $X$, $M_i \subseteq M_{i+1}$, and for each $\epsilon > 0$, $n > 0$, and each $K$ which is an (f.d.) compact Z-set in $X$, there exists an integer $m > 0$ and $h$ belonging to $H(X)$ such that $\rho(h, id) < \epsilon$, $h(K) \subseteq M_m$, and $h|_{M_n \cap K} = id$.

Basic properties of Z-sets and (f.d.) cap sets are discussed in chapter two.

The remaining chapters of this dissertation will be devoted to various results and problems pertaining to the Hilbert cube and other spaces. In chapter 2 certain theorems are quoted that will be needed throughout the paper. Chapter 3 is concerned with the $\epsilon$-homeomorphism extension theorem which was first proved by the author and has been useful in the development of the theory of (f.d.) cap sets. Chapter 4 deals with other naturally arising infinite product spaces that can be thought of as compactifications of $X$. Chapter 5 is concerned with a particular subset of $Q$ which is related to some spaces discussed in Chapter 4. This subset is extensively studied. Various other generali-
zations of it are discussed in Chapter 6. The last chapter extends some results of Wong concerning the contractibility of certain spaces of homeomorphisms.
CHAPTER II
PRELIMINARY RESULTS AND OBSERVATIONS

Certain results are presented in this chapter that will be used later. These results are representative of the work done by others in the study of infinite dimensional topology. The first result provides a method of determining when a sequence of homeomorphisms yields a left product. Anderson [2] first used this result for infinite product spaces, and it has been extensively exploited.

Theorem 2.1 (Convergence criterion). Let $X$ be compact metric.
For each $h$ in $H(X)$ and each $\epsilon > 0$ let
$$n(h, \epsilon) = \text{g.l.b.} \{d(h(x), h(y)) | d(x, y) < \epsilon\}.$$
If $(h_i)_1 \rightarrow 0$ belongs to $H(X)$ and
$$\rho(h_i, id) < \min \{2^{-i}, 3^{-i}n(h_{i-1}, \ldots, h_1, 1/i)\},$$
for each $i$, then there is an $h$ in $H(X)$ equal to $Lm h_i$.

Note that in practice the homeomorphisms $h_i$ are constructed inductively to perform certain tasks. If it is possible to construct $h_i$ arbitrarily close to the identity, then clearly it is possible to satisfy the conditions of the theorem.

The following theorem first appeared in Keller's paper [16] and also appears in [2].

Theorem 2.2 $Q$ is homogeneous.
The apparatus developed in [2] gives the following special result.

**Theorem 2.3** Let \( x \) and \( y \) both be points in \( s \), or both be points in \( B(Q) \). For each connected neighborhood \( U \) of \( Q \) that contains \( x \) and \( y \), there is an \( h \) in \( H(Q) \) such that \( h \) is supported in \( U \), \( h \) is \( \delta^* \), and \( h(x) = y \).

The previous theorem implies easily that \( B(Q) \) and \( s \) are also homogeneous, and indeed that they have this stronger property of homogeneity under elements of \( H(Q) \).

The following theorems provide different representations of \( s \). Bessaga and Pelczynski [10] as well as Anderson have developed apparatus for proving these theorems. Anderson [11] was the first to show the following.

**Theorem 2.4** Hilbert space, \( l^2 \), is homeomorphic to \( s \).

These next two theorems give other representations.

**Theorem 2.5** Let each \( X_i \) be an interval (open, half-open, or closed). Then \( \prod_{i=0}^{n} X_i \) is homeomorphic to \( s \) if and only if infinitely many of the \( X_i \) are either open or half-open.

**Theorem 2.6** Hilbert space, \( l^2 \), is homeomorphic to its closed unit ball \( \{ x \in l^2 \mid d(x,0) \leq 1 \} \).

In the study of the Hilbert cube and \( s \), closed sets of infinite deficiency play an important role, since in many respects they can be treated like points. The class of \( Z \)-sets is useful because it turns out to be the class of all sets equivalent to closed sets of infinite deficiency [3]. Some examples of \( Z \)-sets are endfaces of \( Q \), compact subsets of \( s \), and closed sets of infinite deficiency. Some elementary facts
about Z-sets are: 1) being a Z-set is a topological property; 2) the
finite union of Z-sets is a Z-set; and 3) a closed subset of a Z-set is
a Z-set. The following theorem provides a characterization.

Theorem 2.6 [3] Let X be s or Q. A closed set K is a Z-set in X if and
only if for every $\epsilon > 0$ there exists an $h$ in $H(X)$ such that $h(K)$ is of
infinite deficiency and $\rho(h, id) < \epsilon$, and in the case $X = Q$, $h$ is $\mathfrak{B}^*$.

The following theorem which follows easily from 2.6 provides an useful
alternative definition for Z-set.

Theorem 2.7 Let X be s or Q. A closed set K is a Z-set in X if and only
if for each $\epsilon > 0$ there exists an $\epsilon$-homotopy off K (i.e. $H$ belongs to
$C(X \times I, X)$ and $H(x,-1) = x$, $H(X \times \{1\}) \subseteq X \setminus K$, and $\rho(H, \pi_x) < \epsilon$.)

In s, and more generally in manifolds modeled on s, Z-sets are negli­
gible [4]. This next theorem shows that in fact a stronger condition is
satisfied for compact subsets of s.

Theorem 2.8 Let K be a compact subset of s. Then for each $\epsilon > 0$ there
exists an $h$ in $H(Q)$ such that $h$ is $\mathfrak{B}$, $\rho(h, id) < \epsilon$, and $h(B(Q) \cup K) = B(Q)$.

Cap sets and f.d. cap sets in Q arose in the characterization
of sets equivalent to $B(Q)$ [5]. Independently Bessaga, Pelczynski [10],
and Torunczyk [19] developed the similar concept of skeletonized sets.
Chapman [12] has used cap sets in the study of Hilbert cube manifolds.
The following four theorems establish some important properties of cap
and f.d. cap sets.

Theorem 2.9 $B(Q)$ and $\mathfrak{O}$ are cap sets in Q, and $\mathfrak{O}$ is an f.d. cap set.
Theorem 2.10 Let $X$ and $Y$ be (f.d.) cap sets in $Q$. Then given $\epsilon > 0$ there exists an $h$ in $H(Q)$ such that $h(X) = Y$ and $\rho(h, \text{id}) < \epsilon$.

Theorem 2.11 Let $X$ be an (f.d.) cap set in $Q$ and $K$ be a Z-set in $Q$. Then $X \setminus K$ is an (f.d.) cap set in $Q$.

Theorem 2.12 If $X$ is a countable union of Z-sets in $Q$, and $X$ contains a cap set, then $X$ is a cap set in $Q$. 
CHAPTER III

AN $\varepsilon$-HOMEOMORPHISM EXTENSION THEOREM

The extension theorem proved in this chapter has been useful in the development of the theory of (f.d.) cap sets. It was first proved by the author [9]. Later generalizations have been proved for manifolds [7,8].

Theorem 3.1 ($\varepsilon$-homeomorphism extension) Let $X$ equal $\mathbb{Q}$, $s$, or $l_2$, and let $K_1$ and $K_2$ be closed sets in $s$ (for $X = \mathbb{Q}$ or $s$) or in $l_2$ (for $X = l_2$) such that $K_1 \cup K_2$ is of infinite deficiency. Let $h$ be a homeomorphism from $K_1$ onto $K_2$ with $\rho(h, id) < \varepsilon$. Then there exists an $H$ in $H(X)$ such that $H$ extends $h$ and $\rho(H, id) < \varepsilon$.

The following definitions and lemmas are given before the proof of 3.1. Note that $l_2$ is a normed linear space [see [14] for a definition], and so addition of points and multiplication by scalars is well defined.

A subset $C$ of $l_2$ is convex if for each pair of points $a$, $b$ in $C$, the segment from $a$ to $b = \{x \in l_2 \mid x = ta + (1-t)b, \ t \in [0,1]\}$ is contained in $C$. Let $A$ be a subset of $l_2$, the convex hull of $A$, $\text{CH}(A)$, is the intersection of all convex sets containing $A$. This first lemma is Dugundji's extension theorem stated for normed linear spaces [14].

Lemma 3.2 Let $A$ be a closed subset of a normal space $X$ and let $f$ belong to $C(A,L)$, where $L$ is a normed linear space. Then there exists a function $F$ in $C(X,L)$ such that $F$ extends $f$ and $F(X) \subset \text{CH}(f(A))$. 

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These next two lemmas relate the convexity condition on $F$ to a metric condition.

**Lemma 3.3** Let $A$ be a subset of $X$, and let $f$ and $g$ belong to $C(A, l_2)$ where for each $a$ in $A$, $d(f(a), g(a)) \geq \epsilon$. If $F$ in $C(A, l_2 \times l_2)$ is an extension of $(f, g)$ such that $F(X) \subset \text{CH}((f, g)(A))$, then for each $x$ in $X$, $d(\pi_1 F(x), \pi_2 F(x)) \leq \epsilon$.

Proof. It is easy to check that $D_\epsilon = \{(x, y) \in l_2 \times l_2 \mid d(x, y) \leq \epsilon\}$ is convex. The metric condition on $f$ and $g$ implies that $(f, g)(A) \subset D_\epsilon$ and so $\text{CH}((f, g)(A)) \subset D_\epsilon$. Therefore $F(X) \subset D_\epsilon$ and this implies the result. □

In the special case where $g$ is constantly $0$, we obtain the following.

**Lemma 3.3**' Let $A$ be a subset of $X$. Let $F$ belong to $C(X, l_2)$ with $F(X) \subset \text{CH}(F(A))$. If for each $a$ in $A$, $d(F(a), 0) \leq \epsilon$, then for each $x$ in $X$, $d(F(x), 0) \leq \epsilon$.

The proof of theorem will now be given for the case $X = Q$. The proof for the other two cases is similar. This argument is a sharpening of Klee's method [17].

Proof. Consider $Q$ as the convex set in $l_2$ equal to $\{x \in l_2 \mid |x_i| \leq 2^{-i}\}$. Let $l_o, Q_o, s_o, l_e, Q_e, \text{ and } s_e$ be the subsets of $l_2, Q$ and $s$ which are zero in the even (for index $o$) or odd (for index $e$) coordinates. These spaces are just copies of their namesakes. In this argument, $l_2$ will be identified with $l_o \times l_e$, and likewise $Q$ with $Q_o \times Q_e$ and $s$ with $s_o \times s_e$ (i.e. $(x_1, x_2, \ldots)$ corresponds to $(x_1, \overline{0}, \overline{0}, \overline{0}, \ldots, (\overline{0}, x_2, \overline{0}, \ldots))$.

We can assume without loss of generality that $K_1 \cup K_2 \subset s_o$. Let $h$ be the homeomorphism between $K_1$ and $K_2$. Let $\epsilon = \rho(h, \text{id})$. It is given that
\[ \epsilon_1 < \epsilon, \text{ so pick } \delta > 0 \text{ such that } \epsilon_1 + 2\delta < \epsilon. \text{ Let } f \text{ be an embedding of } K_2 \text{ into } S_e \text{ such that } f(K_2) \text{ is closed in } Q_e \text{ and } d(f(K_2), \delta) \leq \delta. \] A sketch of the argument is first given.

Let \( K_1^* = \{(x_0, f^o h(x_0)) \in l_0 \times l_e | x_0 \in K_1\} \) be the graph of the map \( f \circ h \), and let \( K_2^* = \{(x_0, f(x_0)) \in l_0 \times l_e | x_0 \in K_2\} \) be the graph of \( f \). Notice that the following composition of maps is \( f \).

\[
K_1 \xrightarrow{(id,f^o h)} K_1^* \xrightarrow{h \cdot id} K_2^* \xrightarrow{(id,f)^{-1}} K_2
\]

All of the spaces in the above diagram lie in \( Q_o \times Q_e \). The goal is to extend each of the three maps to space homeomorphisms of \( Q_o \times Q_e \).

For notation let \( f_1 = f \circ h \) and \( f_2 = f \). Let \( h^* \) be the map \( (f^{-1}, h \circ f^{-1}) \) which takes \( f(K_2) \) into \( l_0 \times l_0 \). Notice in fact that \( h^* \) is into the convex set \( s_o \times s_o \) and that for each \( x_0 \in f(K_2) \),

\[ d(\pi_1 h^*(x_0), \pi_2 h^*(x_0)) \leq \rho(f^{-1}, h \circ f^{-1}) = \epsilon_1. \] Lemmas 3.2 and 3.3 provide an extension of \( h^* \), \( H' \) from \( Q_e \) into \( S_o \times S_o \) with \( d(\pi_1 H'(x_0), \pi_2 H'(x_0)) \leq \epsilon_1 \) for \( x_0 \in Q_e \). Similarly let \( F_i^1 \) from \( Q_o \times Q_o \) into \( S_o \times S_o \). A sketch of \( F_i \) with \( d(F_i^1(x), \delta) \leq \delta \).

It will now be shown how \( H' \) induces a homeomorphism of \( Q_o \times Q_e \) which extends \( h \times id \). In a similar way \( F_i^1 \) will induce the other two necessary homeomorphisms. Let \( m \) be an odd integer and for each \( x \) in \( Q_e \) consider the \( m \)th coordinate of \( \pi_1 H'(x) \) and \( \pi_2 H'(x) \), \( a_1, a_2 \) respectively. Since \( H'(Q_e) \subset S_o \times S_o \), \( a_1 \) and \( a_2 \) are interior points of \( I_m \). Let \( \phi_{m,x} \) denote the order preserving piecewise linear homeomorphism of \( I_m \) which takes \( a_1 \) to \( a_2 \). Define \( H^* \) in \( H(Q_o \times Q_e) \) as follows:

\[ H^*((x_1, x_3, x_5, \ldots), x) = ((\phi_{1,x}(x_1), \phi_{3,x}(x_3), \ldots), x). \]

It is easy to check that \( H^* \) is a \( \beta^* \) homeomorphism, \( H^* \) extends \( h \times id \), and \( \rho(H^*, id) \leq \epsilon_1 \). Now let \( F_i \) be the \( \beta^* \) homeomorphism of \( Q_o \times Q_e \) induced
by $F_i'$ in this way. $F_i$ extends $(id, f^i)$ and $\rho(F_i', id) \leq \delta$. Therefore the desired homeomorphism is:

$$H = F_2^{-1} \circ H' \circ F_1$$

and $\rho(H, id) \leq \epsilon_1 + 2\delta < \epsilon$.

The following more general extension theorem can be obtained from theorem 3.1 from the apparatus in [3].

Theorem 3.4 Let $K_1$ and $K_2$ be $Z$-sets in $Q$, and let $h$ be a homeomorphism from $K_1$ onto $K_2$ with $\rho(h, id) < \epsilon$. Then there exists an $H$ in $H(Q)$ such that $H$ extends $h$ and $\rho(H, id) < \epsilon$. Furthermore if $h(K_1 \cap s) = K_2 \cap s$, then $H$ can be specified to be $\beta^*$.

A general extension theorem for $Z$-sets in $l_2$ is a corollary to the main results in [8].

Theorem 3.5 Let $K_1$ and $K_2$ be $Z$-sets in $l_2$, and let $h$ be a homeomorphism from $K_1$ onto $K_2$ with $\rho(h, id) < \epsilon$. Then there exists an extension of $h$, $H$ in $H(l_2)$ such that $\rho(H, id) < 5\epsilon$. 
CHAPTER IV
OTHER COMPACTIFICATIONS OF \( s \) AND RELATED SPACES

Q is compact and contains \( s \) as a dense subset. In standard terminology, Q is said to be a compactification of \( s \). Since the closed interval is a natural compactification of the open interval, Q can reasonably be considered as a natural way of compactifying \( s \). In this chapter other compact spaces will be discussed which can also be considered as natural compactifications of \( s \).

The term remainder will represent the difference between the compactification and the set being compactified. Q as a compactification of \( s \) has been very useful. One reason is that the remainder, B(Q), is a nice space in many respects. First of all, points of B(Q) are easily distinguished from those of \( s \). Also certain topological properties of B(Q) in relationship to Q provide for the existence of many \( \beta^* \) homeomorphisms. The fact that Q is contractible also has been useful.

Four properties possessed by B(Q) are listed below. In the study of some other compactifications, these properties will be examined. Let X be a compactification of some space \( f \), and let \( R = X \setminus Y \) be the remainder.

1) Strong neighborhood homogeneity: For each connected open set \( U \) in X and each pair of points \( x, y \) in \( U \cap R \), there exists an \( h \) in \( H(X) \) such that \( h(x) = y \), \( h \) is supported in \( U \), and \( h(R) = R \).

(This is related to locally setwise homogeneous defined in [11].)

2) Space negligibility of points: For each point \( x \) in \( R \), there exists an \( h \) in \( H(X) \) such that \( h(R) = R \setminus \{x\} \).

The two properties above concern \( R \) as a subspace of \( X \). The two below concern as abstract space \( R \).
1') **Homogeneity**: For each pair of points \( x, y \) in \( R \), there exists an \( h \) in \( H(R) \) such that \( h(x) = y \).

2') **Negligibility** of points: For each point \( x \) in \( R \), \( R \) is homeomorphic to \( R \setminus \{x\} \).

Clearly 1 implies 1' and 2 implies 2'. Theorems 2.2 and 2.3 establish that \( B(Q) \) satisfies all the properties above.

Denote by \( X(n) \) the infinite product of \( n \)-spheres, \((S^n)^\infty\).

For \( n \geq 1 \), \( X(n) \) is a compactification of \( s \), since each factor is the one point compactification of the open \( n \)-cell, and \( s \) can be regarded as the infinite product of open cells. Let \( 0 \) be the north pole of \( S^n \).

\( R(n) = \{x \in X(n)\mid \text{at least one coordinate is 0}\} \) is the remainder in \( X(n) \).

Clearly \( R(n) \) plays a role with respect to \( X(n) \) that \( B(Q) \) plays with respect to \( Q \), in the sense that points of \( R(n) \) are easily distinguished from those in \( s \). \( X(1) \) which is the infinite product of circles will sometimes be denoted by \( T \) and be called the infinite torus. It is easy to see that \( X(n) \) is homogeneous, and that homotopically it is very "bad".

The next two theorems show that \( R(n) \) does not have properties 1 and 2.

**Theorem 4.1** \( R(n) \) **does not have property 2**.

**Proof.** Given \( p \) in \( R(n) \), suppose that there were an \( h \) in \( H(X(n)) \) such that \( h(R(n)) = R(n) \setminus \{p\} \). This implies that \( h(s) = s \cup \{p\} \). It will be shown that \( s \cup \{p\} \), as a subspace of \( X(n) \), is not homotopically trivial and therefore contradicts the existence of \( h \).

Let \( S_0 \) be the non-empty set of indices where \( p_i = 0 \) for \( i \) in \( S_0 \). Consider the map \( f \) from \( S^n \) into \( s \cup \{p\} \) where for \( i \) in \( S_0 \)

\[ n_i f(x) = x \] and \[ n_k f(x) = v_k \] for \( k \) not in \( S_0 \). Observe that this maps
the north pole of $S^n$ to $p$ and the rest of $S^n$ into $s$. By definition

\[ \pi_i f = \text{id} \text{ for } i \in S_0. \]

If $f$ were homotopic to a constant map via the homotopy $F_t$, then $\pi_1 F_t$ would be a homotopy of the id on $S^n$ to a constant which is impossible. \(\square\)

**Theorem 4.2** For $R(n)$, property 1 implies property 2.

**Proof.** This method is the essence of the one used in [2] to prove homogeneity of $Q$. Let $p^0$ belong to $R(n)$. Using the convergence criterion (Theorem 2.1), we construct a sequence, $(h_i)$, in $H(X(n))$ whose left product, $h$, takes $R$ onto $R \cup \{ pt \}$ with $h(p^0) = pt$. Then $h^{-1}$ is a homeomorphism which takes $R$ onto $R \setminus p^0$.

The construction is as follows. Let $(U_j)$ be a nested family of basic neighborhoods in $X(n)$ of $p^0$ such that $\cap_{j \geq 0} U_j = p^0$. Let $p'$ be a point in $U_1 \cap R(n)$ with first coordinate different from 0, and let $h_1$ belong to $H(X(n))$ where $h_1$ is supported in $U$, and takes $p^0$ to $p'$. The remaining homeomorphisms are constructed inductively.

Let $V_{i+1}$ be a small basic neighborhood of $p^i$ contained in $h_1 \ldots h_i(U_{i+1})$. Let $p^{i+1}$ be a point in $V_{i+1} \cap R(n)$ with the first $i$ coordinates the same as $p^i$ and the $(i+1)$th coordinate different from 0. Let $h_{i+1}$ belong to $H(X(n))$ where $h_{i+1}$ is supported in $V_{i+1}$ and takes $p^i$ to $p^{i+1}$. Since $V_{i+1}$ can be chosen arbitrarily small, $h_{i+1}$ can be made arbitrarily close to the identity.

Therefore $h = \lim_{j \to 0} h_j$ exists and is in fact locally finite mod $(p^0)$. Clearly $h(p^0)$ does not belong to $R(n)$ since no coordinate is 0. All other points of $R(n)$ or $s$ have their images in the same space, since they are affected by only finitely many of the $h_j$. \(\square\)
Corollary 4.3 R(n) does not have property 1.

The rest of this chapter is devoted to considering properties 1' and 2' for R(n). It is conjectured that R(n) does not have either of these properties. We will show that certain spaces analogous to R(n) do not have 1' and 2', and that R(\(\mathbb{N}\)) fails to have property 1'.

Consider the following subsets of \(\mathbb{Q}^n\): \(\mathbb{Q}^n = \{ (x_1, \ldots, x_n) : x_i \in \mathbb{Q} \},\) at least one coordinate is the origin of \(\mathbb{I}^n\). We may consider R'(n) as analogous to R(n), since the coordinate definitions are similar. Also, R'(n) is a countable union of copies of \(\mathbb{Q}\), each of which n-disconnects \(\mathbb{Q}\) (i.e. \(\mathbb{Q}\) with the copy removed is not n-connected), and R(n) in an analogous way is a countable union of copies of X(n), each of which n-disconnects certain open subsets of X(n). Under the appropriate identification R'(n) becomes R(n). R'(n) is the remainder in \(\mathbb{Q}\) of a compactification of X(n-1) \(\times s\), since \(\mathbb{Q}\setminus R'(n)\) is homeomorphic to X(n-1) \(\times s\). (This follows from theorem 2.5) Before discussing properties 1' and 2', it will be necessary to develop some apparatus. Some definitions are given first.

A closed set A in a connected space X is called a \textit{separating set} if \(X \setminus A\) is not connected. A is a \textit{minimal separating set} if \(X \setminus A\) is not connected, but for any closed proper subset B of A, \(X \setminus B\) is connected. For a point \(p\) in X, let \(S_p\) denote the intersection of all compact minimal separating sets that contain \(p\). It is possible that there are no such sets containing \(p\), and in that case, following the standard convention, we have that \(S_p = X\). These sets will be called \(\textit{min-sets}\), and \(S_p\) is the \(\textit{min-set for } p\). Let \(B(X)\) denote \(\{ p \in X \mid S_p = X \}\). Let Y be a metric space, and X be a subset of Y.
We say that $X$ is a **dense-deformation** of $Y$ if for every $\epsilon < 0$ there is a homotopy $H$ in $C(Y \times [0,1], Y)$ such that 1) $H_0 = \text{id}$, 2) for $t > 0$ $H_t(Y) \subseteq X$, and 3) for each $t \rho(H_0, H_t) < \epsilon$. Note that if $Y$ is compact, the third condition is not necessary. In this case, given $\epsilon > 0$ there exists $t_0$ such that $\rho(H_0, H_t) < \epsilon$ for $t \in [0, t_0]$, and so condition 3 can be satisfied by parameterizing $[0, t_0]$ as $[0, 1]$. Observe that the definition implies that a dense-deformation is dense. The next few lemmas provide examples of dense-deformations of $Q$.

**Lemma 4.4** If $X$ is an f.d. cap set in $Q$, then $X$ is a dense-deformation of $Q$.

*Proof.* First it will be shown that $\sigma$ is a dense-deformation. Recall that $\sigma = \{x \in s| \text{all but finitely many coordinates of } x \text{ are zero}\}$. Define a homotopy $H$ in $C(Q \times [0,1], Q)$ as follows:

$$H_0 = \text{id} \text{ and for } t \text{ in } (0,1] \text{ let }$$

$$H_t(x_1, x_2, x_3, ...) = ((1-t)x_1, (1-t)x_2, ..., (1-t)x_{k-1}, r(1-t)x_k, 0, 0, ...),$$

where $k$ is the greatest integer less than or equal to $1/t$, and $r = 1/t - k$.

For $t > 0$, $H_t(Q) \subseteq \sigma$ since the image is in $s$ and all coordinates past the $k^{th}$ are zero. Continuity is easily checked for values of $t$ where $1/t$ is an integer and at $t = 0$. Elsewhere it is clearly continuous.

For the general case where $X$ is an arbitrary f.d. cap set, let $h$ in $H(Q)$ be a homeomorphism such that $h(X) = \sigma$ (Theorem 2.10). Then $h^{-1}oH^{-1}(h \times \text{id})$ is the desired homotopy. □

**Lemma 4.5** If $X$ is a subset of $Y$, and $X$ contains a dense-deformation of $Y$, then $X$ is a dense-deformation of $Y$. 
Proof. This lemma follows immediately from the definition. □

Since the cap set $\mathcal{C}$ contains $\sigma$, we have that cap sets are also examples of dense-deformations of $Q$. The author is not aware of an example of a dense deformation of $Q$ which does not contain an f.d. cap set. An important property of dense-deformations is made clear in the following.

Lemma 4.6 Let $X$ be a dense-deformation of $Y$, $U$ open in $X$, and $U'$ open in $Y$ where $U = U' \cap X$. A map $f$ in $C(S^{n-1}, U)$ can be extended to a map of $I^n$ into $U$ if and only if $F$ can be extended to a map of $I^n$ into $U'$.

Proof. Only if: Immediate since a map into $U$ is also into $U'$.

If: Let $F'$ in $C(I^n, U')$ be an extension of $f$. Since $F'(I^n)$ is compact there is a positive distance, $\epsilon$, between $F'(I^n)$ and $Y \setminus U'$. Let $H$ in $C(X \times [0,1], Y)$ be a homotopy with $H_0 = \text{id}$, $H_t(Y) \subseteq X$ for $t > 0$, and $\rho(H_0, H_t) < \epsilon/2$. Consider the map $F$ from $I^n$ where $F(x,t) = H_{1-t} \circ F'(x,t)$. (Here $I^n$ is thought of as the cone over $S^{n-1}$ with the $t = 0$ level being the cone point). $F$ extends $f$ because $F'$ extends $f$ and $H_0 = \text{id}$. Since $H_t$ is close enough to the identity, and $H_t(Y) \subseteq X$ for $t > 0$, we have that $F(I^n) \subseteq U' \cap X = U$. □

This next lemma is a corollary of 4.6.

Lemma 4.7 Let $X$ be a dense-deformation of $Y$, and let $Y$ be locally path connected. Then a compact subset of $X$ separates $X$ if and only if it separates $Y$.

Proof. Let $A$ be a compact subset of $X$.

Only if: Suppose $X \setminus A$ is not connected, then $X \setminus A$ is not path connected. Lemma 4.6 implies that $Y \setminus A$ is not path connected as
follows. Let $U = X \setminus A$ and $U' = Y \setminus A$. Note that the compactness of $A$ guarantees that $U'$ is open. Let $f$ be a map of $S^0$ into $U$ that cannot be extended to a map of $I$ into $U$. This map cannot be extended into $U'$ either, establishing that $Y \setminus A$ is not path connected. Together with the local path connectedness of $Y$, this implies that $Y \setminus A$ is not connected.

If: Suppose that $Y \setminus A$ is not connected. Let $Y \setminus A = U_1 \cup U_2$ where $U_1$ and $U_2$ are non-empty disjoint open sets. Since $X$ is dense, $U_1 \cap X$ and $U_2 \cap X$ are non-empty and hence form a separation of $X \setminus A$. □

This following modification of 4.7 is stated without proof, since it follows immediately from the definition of minimal separating set and 4.7.

Lemma 4.7' Let $X$ be a dense-deformation of $Y$, and let $Y$ be locally path connected. Then a compact subset of $X$ minimally separates $Y$ if and only if it minimally separates $Y$.

The next lemma provides an useful property of minimal separating sets.

Lemma 4.8 If $K$ is a closed subset of a connected space $X$, and $K$ minimally separates $X$, then $K$ separates any connected open set in $X$ which meets $K$.

Proof. $K$ separates $X$, so let $X \setminus K = U_1 \cup U_2$, where $U_1$ and $U_2$ are disjoint non-empty open sets. Suppose that $V$ is a connected open set such that $V \cap K \neq \emptyset$ but $V \setminus K$ is connected. This will lead to a contradiction. If $V \setminus K = \emptyset$, then $V$ is a subset of $K$ and meets neither $U_1$ nor $U_2$. On the other hand, if $V \setminus K \neq \emptyset$, then $V \setminus K$ belongs to only
one of $U_1$ or $U_2$ since it is connected. We can assume then without loss of
generality that $V$ does not intersect $U_1$. Now $K \setminus V$ is a proper closed
subset of $K$, and yet $U_1 \cup (U_2 \cup V)$ is a separation of $X \setminus (K \setminus V)$. This
contradicts the minimal separation by $K$. □

Next lemma identifies elementary topological properties of
$X(n)$, $R(n)$, and $R'(n)$. It is stated without proof.

Lemma 4.9 The spaces $X(n)$, $R(n)$, and $R'(n)$ are connected and locally
arcwise connected.

It will now be shown that $R'(1)$ does not have properties 1' and 2'. In the next chapter a more complete discussion of this space
is given. Let $\bar{0}$ be the point of $R'(1)$ with all coordinates zero. It
will be shown that this point is rigid in $R'(1)$, and also that it is
not negligible. The two lemmas below are needed.

Lemma 4.10 The spaces $R'(n)$ and $R'(n) \setminus \{p\}$ for $p$ in $R'(n)$ are
dense-deformations of $Q$.

Proof. $R'(n)$ contains $0$. For any point $p$, $0 \setminus \{p\}$ is an f.d. cap set
(theorem 2.11), so both $R'(n)$ and $R'(n) \setminus \{p\}$ contain f.d. cap sets.
Lemma 4.4 and 4.5 establish that the spaces are dense-deformations. □

Consider $Q_i = \{(x_j) \in R'(1)|x_i = 0\}$. These are compact subsets
of $R'(1)$ which separate the space. Clearly $\bar{0}$ belongs to each $Q_i$.
The next lemma shows that every compact separating set is like $Q_i$
in this respect.
Lemma 4.11 If A is a compact separating set in $R'(1)$, then $\delta$ belongs to A.

Proof. From lemma 4.7, it follows that A separates Q. If $\delta$ does not belong to A, then $Q \setminus R'(1) \cup \{\delta\}$ is a subset of $Q \setminus A$. But $Q \setminus R'(1) \cup \{\delta\}$ is dense in $Q \setminus A$, and $Q \setminus R'(1) \cup \{\delta\}$ is connected since it is contractible (The contraction is multiplication by $t$ in $[0,1]$). This implies that $Q \setminus A$ is connected which is a contradiction. □

The desired results can now be stated.

Theorem 4.12 The point $\delta$ is rigid in $R'(1)$.

Proof. It is necessary to show that $\delta$ is not equivalent to any other point. Consider the following topological property for a point x: x belongs to all compact separating sets of $R'(1)$. Lemma 4.11 says that $\delta$ has this property. It suffices to observe that no other point of $R'(1)$ has this property. Let $(p,)$ in $R'(1)$ be different from $\delta$. There is some coordinate $j$ such that $p_j \neq 0$. The compact set $Q_j$ separates $R'(1)$ and does not contain $(p,)$.

Theorem 4.13 $R'(1) \setminus \{\delta\}$ is not homeomorphic to $R'(1)$.

Proof. Since many compact subsets of $R'(1)$ separate the space, it suffices to show that no compact subset of $R'(1) \setminus \{\delta\}$ separates $R'(1) \setminus \{\delta\}$. Suppose A were such a compact set in $R'(1) \setminus \{0\}$. From lemmas 4.7 and 4.10 A separates Q, and thus separates $R'(1)$. However, lemma 4.11 then implies that $\delta$ belongs to A which is a contradiction. □
The questions concerning properties 1' and 2' for \( R'(1) \) have been answered by theorems 4.12 and 4.13. These questions for \( R'(n) \) are answered in chapter 6. The rest of this chapter establishes that the remainder in the infinite torus, \( R(1) \), fails to have property 1'.

These next two lemmas are modifications of lemmas 4.6 and 4.11.

**Lemma 4.14** Let \( U \) be a connected open set in \( R(1) \), and let \( U' \) be open in \( X(1) \) such that \( U' \cap R(1) = U \). If a compact set \( A \) in \( R(1) \) separates \( U \), then it separates \( U' \).

**Proof.** Suppose \( U' \setminus A \) is connected, then it is arcwise connected since \( X(1) \) is locally arcwise connected. Let \( p \) and \( g \) be points in \( U \setminus A \). Let \( C \) be an arc between \( p \) and \( g \) in \( U' \setminus A \). Let \( \epsilon_1 \) be the distance from \( C \) to \( X(1) \setminus U' \cup A \). Pick arcwise connected neighborhoods \( U_p \) and \( U_g \) of \( p \) and \( g \) such that \( U_p \) and \( U_g \) are subsets of \( U' \) and disjoint from \( A \). Choose \( \epsilon > 0 \) smaller than \( \epsilon_1 \) so that the \( \epsilon \)-balls about \( p \) and \( g \) stay inside \( U_p \) and \( U_g \) respectively. There exists a map \( f \) in \( C(X(1), R(1)) \) such that \( \rho(f(id)) < \epsilon \). (This map can be obtained by sending a high index factor to the north pole). Now \( f(C) \) is a subset of \( U \setminus A \) and contains an arc from \( f(p) \) to \( f(g) \). Since \( f(p) \) is within \( \epsilon \) of \( p \), both belong to \( U_p \), and there is an arc from \( p \) to \( f(p) \) inside \( U \setminus A \). Similarly there is an arc from \( g \) to \( f(g) \) in \( U \setminus A \). We have then that \( U \setminus A \) is connected.

Let \( \delta \) denote the point in \( R(1) \) with all coordinates the north pole of \( S^1 \).

**Lemma 4.15** Any compact subset \( A \) of \( R(1) \) that minimally separates some connected neighborhood of \( \delta \), must contain \( \delta \).

**Proof.** Let \( A \) minimally separate some connected neighborhood, \( U \), of \( \delta \).
Let $U'$ be open in $X(I)$ such that $U' \cap R(I) = U$. Let $V'$ be a basic open set in $X(I)$ that contains $0$ and is contained in $U'$. Let $V = V' \cap R(I)$. Lemma 4.8 implies that $A$ separates $V$, and 4.14 implies then that $A$ separates $V'$. If $A$ did not contain $0$, then $V' \setminus A$ would contain the connected and dense set $V' \setminus R(I) \cup \{0\}$. This contradicts the fact that $V' \setminus A$ is not connected. □

The final desired result will only be stated. It follows from lemma 4.15, just as theorem 4.12 follows from lemma 4.11.

Theorem 4.16 The point $0$ is rigid in $R(I)$. 


CHAPTER V

A DISCUSSION OF R'(1)

The space R'(1) defined in a previous chapter is perhaps the most natural model of a sigma compact dense-deformation of Q which is different from a cap or r.d. cap set. We now give further properties of this space. Points of R'(1) will be completely characterized with respect to equivalence under space homeomorphisms; negligibility of points discussed; and various products of spaces with R'(1) investigated.

The techniques heavily use min-sets. Recall that for a connected space $X, S_p = \cap \ldots, K$ is a compact subset of $X$ that minimally separates $X$ and contains $p$. The next lemma follows immediately from the definition.

Lemma 5.1 If $S_p$ is the min-set for $p$ in $X$, then 1) $p$ belongs to $S_p$,
2) $q$ in $S_p$ implies $S_q$ is contained in $S_p$, and 3) for any $h$ in $H(X)$ $S_{h(p)}$ equals $h(S_p)$.

It should be noted that 3) gives a necessary condition for the existence of a homeomorphism taking $p$ to $q$. If there exists an $h$ in $H(X)$ such that $h(p) = q$, then $S_p$ is homeomorphic to $S_q$. We now characterize the min-sets of $R'(1)$. It will be easy to determine when two min-sets of $R'(1)$ are homeomorphic from this characterization.

Theorem 5.2 Let $p = (p_1)$ belong to $R'(1)$. Then $S_p$ equals $\mathbb{N}_\infty \cdot 0^\infty$.
where \( K_i = \{0\} \) if \( p_i = 0 \); \( K_i = [-1,0] \) if \( p_i < 0 \); and \( K_i = [0,1] \) if \( p_i > 0 \).

Proof. Step 1: it will be shown that any compact set \( K \in \mathbb{R}^n \) which minimally separates \( \mathbb{R}^n \) and contains \( p \) must contain \( \Pi K_i \).

Case 1: suppose that \( p_i = 0 \) for all but finitely many indices.
Let \( U_p \) be the following neighborhood of \( p \) in \( Q \). \( U_p = \Pi \bigcap_{j \neq \hat{p}_i} U_j \) where
\( U_j = [-1,1] \) if \( p_j = 0 \); \( U_j = [-1,0) \) if \( p_j < 0 \); and \( U_j = (0,1] \) if \( p_j > 0 \).
Lemma 4.7' implies that \( K \) minimally separates \( \mathbb{R}^n \), and so lemma 4.8 implies that \( K \) separates \( \mathbb{R}^n \). It is only necessary to show that \( \Pi K_i \cap U_p \) is contained in \( K \) since \( \Pi K_i \) is the closure of \( \Pi K_i \cap U_p \).
Suppose that there is a point \( p' \) in \( \Pi K_i \cap U_p \) which is not in \( K \). It is easy to see that there is an arc from any point of \( \mathbb{R}^n \cup \{p'\} \) to \( p' \). This means that \( \mathbb{R}^n \cup \{p'\} \) is a connected dense subset of \( \mathbb{R}^n \cap U_p \) which is a contradiction.

Case 2: suppose that \( p_i \neq 0 \) for infinitely many indices.
Consider \( K = \Pi_{j=1}^{n} K_i \times 0 \times 0 \times \ldots \). The argument in case 1 implies that \( K_i \) is contained in \( K \) for each \( i \). Therefore \( \Pi_{i=1}^{n} K_i \) is contained in \( K \) since \( \Pi K_i \) equals the closure of \( \cup_{i=1}^{n} K_i \).

Step 2: it is now necessary to find a collection of compact minimal separating sets each containing \( p \) whose intersection is \( \Pi K_i \).
Let \( S_0 \) be the coordinate indices such that for \( j \) in \( S_0 \), \( p_j = 0 \). Let \( S_1 \) be the indices such that \( p_j \neq 0 \) for \( j \) in \( S_1 \). For \( j \) in \( S_0 \) let \( Q_j = \{x \in Q \mid x_j = 0\} \). Note that \( S_0 \) is non-empty, so let \( k \) belong to \( S_0 \). Now for \( j \) in \( S_1 \), let \( \bar{Q}_j = \{x \in Q \mid x_k = 0 \text{ and } x_j \text{ is zero or has the same sign as } p_j \} \cup \{x \in Q \mid x_k < 0 \text{ and } x_j = 0\} \). It is easy to verify that \( \{Q_j\} \cup S_0 \) is the desired collection. \( \square \)
This theorem indicates the structure of all min-sets in $\mathbb{R}'(1)$. It will be useful to make some observations here and develop some notation. Since all min-sets are products of intervals, an $n$-dimensional min-set is just an $n$-cell. Such min-sets will be called $n$-min-sets. Clearly the intersection of min-sets is a min-set. Also if $M$ is a min-set, then $-M$ is one also. Given a collection of min-sets, $\{M_i\}$, we say that $\{M_i\}$ spans a min-set $M$, if $M$ is the smallest min-sets which contains each member of the collection. There are some collections which are not contained in any min-set, and hence cannot span. Such a collection will be called incompatible. For example, let $p_1 = (1,0,0,\ldots)$, $p_2 = (-1,0,0,\ldots)$, and $p_3 = (0,1,0,0,\ldots)$.

$S_{p_1}$, $S_{p_2}$, and $S_{p_3}$ are all 1-min-sets. $S_{p_1}$ and $S_{p_3}$ span a 2-min-set. $S_{p_1}$ and $S_{p_2}$ are incompatible (i.e. $\{S_{p_1}, S_{p_2}\}$ is incompatible). It should be noted that spanning or incompatible collections are preserved under space homeomorphisms of $\mathbb{R}'(1)$.

**Classification of Points in $\mathbb{R}'(1)$**

Theorem 5.2 above provides a very obvious necessary condition that there exists a space homeomorphism carrying $p$ to $q$. This condition is that both points have the same number of non-zero coordinates, for then it is clear that $S_p$ is homeomorphic to $S_q$. This condition is clearly not sufficient, as the following example shows. Let $p = (1,0,0,\ldots)$ and $q = (1/2,0,0,\ldots)$. Here both $S_p$ and $S_q$ are the same 1-min-set, $p$ is an endpoint of the interval, and $q$ is an interior point. Since any homeomorphism taking $p$ to $q$ would have to take this 1-min-set onto itself, it would induce an homeomorphism of the interval taking an endpoint to an interior point which is impossible. This example
shows that since homeomorphisms of $R'(1)$ must preserve min-sets, all homeomorphisms must be consistent with homeomorphisms of finite cells. The following lemma makes this explicit and is given without proof.

Lemma 5.3 Let $p$ be a point in $R'(1)$ such that $S_p$ is finite-dimensional. Let $h$ belong to $H(R'(1))$. Then both $p$ and $h(p)$ belong to the boundary (i.e. $S^{n-1}$ of the $n$-cell) of their min-sets, or neither do.

The necessary condition for $p$ to go to $q$ under a space homeomorphism is in fact stronger than the condition that $S_p$ be homeomorphic to $S_q$. Actually if $h(p) = q$, then $h(S_p) = S_q$. That is to say that $S_p$ and $S_q$ are homeomorphic under a space homeomorphism. The existence of a space homeomorphism between min-sets $S_p$ and $S_q$ can be determined by comparing the number of zero and non-zero coordinates of $p$ and $q$. Before stating this lemma, we consider the following definitions. A homeomorphism $h$ in $H(R'(1))$ will be called a standard homeomorphism if $h$ is the restriction to $R'(1)$ of a homeomorphism of $Q$ obtained by rearranging the order of some coordinates, changing the sign of some coordinates, or both. For $p$ in $R'(1)$, let $Z(p)$ equal the number of zero coordinates, and let $N(p)$ be the number of non-zero coordinates. (Either can be infinite).

Lemma 5.4 For $p$ and $q$ in $R'(1)$, there exists an $h$ in $H(R'(1))$ such that $h(S_p) = S_q$ if and only if $N(p) = N(q)$ and $Z(p) = Z(q)$.

Proof. If: It is clear from the structure of $S_p$ and $S_q$ that there is a standard homeomorphism which takes $S_p$ onto $S_q$. 
Only if: Since \( h(S_p) = S_q \), \( S_p \) and \( S_q \) have the same dimension. Therefore \( N(p) = N(q) \) since these numbers are the dimensions. Suppose that \( Z(p) \) is different from \( Z(q) \). We can assume that \( Z(q) < Z(p) \). There is a standard homeomorphism \( f \) such that \( f(S_p) = S_q \). Since \( f \) takes \( S_p \) to \( S_f(p) \), \( g \) equal to \( h \circ f^{-1} \) is a homeomorphism such that \( g(S_f(p)) = S_q \). This implies that \( S_q = g(S_f(p)) \neq g(S_p) \) which implies the existence of the infinite chain of proper inclusions: \( S_q \neq g(S_p) \neq g^2(S_p) \neq \ldots \). From the fact that \( Z(q) \) is finite and the structure of \( S_q \), there is clearly no such chain, so \( Z(q) \) must equal \( Z(p) \). □

This next theorem gives the necessary and sufficient conditions for two points of \( R'(1) \) to be equivalent.

**Theorem 5.5** Let \( p \) and \( q \) be points of \( R'(1) \). Then there is an \( h \) in \( H(R'(1)) \) such that \( h(p) = q \) if and only if 1) \( Z(p) = Z(q) \) and \( N(p) = N(q) \) and 2) if \( N(p) \) and \( N(q) \) are finite, then both \( p \) and \( q \) belong to the boundary of \( S_p \) and \( S_q \) respectively, or neither do.

**Proof.** Only if: Immediate from lemmas 5.3 and 5.4.

If: Lemma 5.4 asserts the existence of a \( g \) in \( H(R'(1)) \) such that \( g(S_p) = S_q \). All that is necessary is to show the existence of a homeomorphism taking \( g(p) \) to \( q \).

**Case 1:** \( N(q) \) is finite. Without loss of generality we can assume that \( S_q = [0,1]^n \times 0 \times 0 \times \ldots \). It will be shown that there is a homeomorphism \( h' \) of \( S_q \) onto itself which takes \( g(p) \) to \( q \) and is the identity on \( B_0 = \{ x \in S_q | \text{one of the first } n \text{ coordinates of } x \text{ is zero} \} \). This homeomorphism can then be extended to a homeomorphism
h" of \([-1,1]^n \times 0 \times 0 \times \ldots\) which is the identity on those points with one of the first \(n\) coordinates zero, and \(h"\) clearly extends to a homeomorphism of \(R'(1)\) with the required property. If \(g(p)\) and \(q\) are interior points, \(h'\) clearly exists. If they both belong to the boundary, note that they do not belong to \(B_0\), and again \(h'\) exists.

Case 2: \(N(q)\) is infinite. It will be shown that there is a space homeomorphism of \(Q\) which taken \(R'(1)\) onto itself and carries \(g(p)\) to \(q\). If both \(g(p)\) and \(q\) belong to the intersection of \(s\) and \(R'(1)\), then the homeomorphism can be constructed coordinatewise. So it is sufficient to show that any point \(q\) of \(B(Q) \cap R'(1)\) can be pushed into \(R'(1) \cap s\) with a space homeomorphism \(f\) of \(Q\) which preserves \(R'(1)\). The argument which will only be sketched here is just a modification of the proof of theorem 4.2. The homeomorphism \(f\) will be constructed as a left product \(L\prod f_i\). Let \(j_1\) be the first coordinate index where \(g_{j_1} = \pm 1\), and let \(j_2\) be the next index where \(g_{j_2} = \pm 1\) if there is one, and otherwise some large number bigger than \(j_1\) with \(g_{j_2} \neq 0\). Consider the homeomorphism of \(I_{j_1} \times I_{j_2}\) which is the identity on \(\{x \in I_{j_1} \times I_{j_2} | one \ coordinate \ of \ x \ is \ zero\}\) and moves the projection of \(q\) so that the \(j_1^{th}\) coordinate of the image is not equal to 1. The first homeomorphism, \(f_1\), is induced by this homeomorphism of the two cell. The remaining homeomorphisms are defined in a similar manner. □

It is also possible to give some indication of where pairs of points can go using these same techniques. It can easily be seen that a necessary condition for the pair \((p,p')\) to go to \((q,q')\) under
a space homeomorphism is that the pairs of min-sets; $S_p$ and $S_p'$, $S_q$ and $S_q'$, and $S_p \cap S_q$ and $S_p' \cap S_q'$ are all homeomorphic under the same space homeomorphism of $R'(1)$. This is not sufficient as some of the examples for single points show. There are still other reasons though. Let $p = (1/3,0,0,...)$ and $q = (2/3,0,0,...)$. Although it is possible to take $p$ to $q$, it is not possible to take the pair $(p,q)$ to $(q,p)$. (i.e. $p$ and $q$ can not be interchanged under a space homeomorphism). If this were possible it would induce a homeomorphism of the interval which keeps the endpoints fixed but interchanges two interior points.

A special situation will be described concerning the pair $(p,-p)$. $R'(1)$ clearly has symmetry about $0 = (0,0,...)$. (i.e. $p$ belongs to $R'(1)$ if and only if $-p$ belongs to $R'(1)$). It will be shown that all homeomorphisms preserve this symmetry to a certain degree. While it need not be true that $f(p)$ and $f(-p)$ are opposite each other, it is true that they lie in opposite min-sets. For example consider $p = (1/2,0,0,...)$. Any homeomorphism of $p$ is a point with only one coordinate different from zero and also different from $1$.

Suppose $f(p) = (0,0,...,a,0,0,...)$, then $f(-p) = (0,0,...,b,0,0,...)$ where $a,b < 0$, $b = -1$, and $b$ is in the same coordinate position as $a$.

Theorem 5.6 Let $p$ belong to $R'(1)$ and $h$ be in $H(R'(1))$. Then $S_{h(p)}$ equals $-S_{h(-p)}$.

Proof. Consider all 1-min-sets in $S_p$, $\{M_i\}$. This collection spans $S_p$, and $\{-M_i\} spans S_p$. Therefore $\{h(M_i)\}$ spans $S_{h(p)}$, and $\{h(-M_i)\}$ spans $S_{h(-p)}$. Each $M_i$ is incompatible with $-M_i$, and so $h(M_i)$ and
are incompatible. For 1-min-sets the only incompatible pairs are opposites, so \( h(-M_1) = -h(M_1) \). Therefore \( S_h(p) = S_h(-p) \). □

Homeomorphisms of \( R'(1) \) that cannot be extended to \( Q \).

Many of the homeomorphisms of \( R'(1) \) that may be produced are obtained by restricting homeomorphisms of \( Q \) which preserve \( R'(1) \). An example is given to show that there are homeomorphisms of \( R'(1) \) which do not extend to homeomorphisms of \( Q \). In fact some do not even extend to maps of \( Q \).

Let \( f \) be the piecewise linear map of \( I \) onto itself which takes \([-1, -2/3] \) linearly onto \([-1, -1/2], [-2/3, -1/3] \) to \([-1/2], [-1/3, 0] \) linearly onto \([-1/2, 0], [0, 1] \) identically onto itself. Let \( F_t \) be the linear homotopy to the identity. (i.e. \( F_t = t \cdot \text{id} + (1-t)f \)). For \( t > 0 \) \( F_t \) is a homeomorphism, and for all \( t, F_t(0) = 0 \). For \( x = (x_1, x_2, \ldots) \) in \( Q \), let \( t(x) = 1/2 \cdot d((x_2, x_3, \ldots), (1, 1, 1, 1, \ldots)) \).

Consider the following map of \( Q \).

\[
H(x_1, x_2, x_3, \ldots) = (F_t(x)(x_1), x_2, x_3, x_4, \ldots).
\]

It can be verified that \( H|_{R'(1)} \) is a homeomorphism. Since \( R'(1) \) is dense in \( Q \), the only extension of \( H|_{R'(1)} \) is \( H \) which is not a homeomorphism. Also, \([H|_{R'(1)}]^{-1}\) has no extension at all.

Non-negligibility of points.

An argument will now be given which shows that no point of \( R'(1) \) is negligible. It should be noted that this implies that abstractly no point can be adjoined to \( R'(1) \) without changing its topological nature. For if \( h(R'(1) \cup \text{pt}) = R'(1) \), then
h(R'(1)) = R'(1)\{pt\} which says that some point is negligible.

Recall that for a connected space X, we have defined

\[ B(X) = \{ p \in X | S_p = X \} \]. We characterize these sets for X equal to R'(1)\{pt\}.

**Theorem 5.7**  Let \( p = (p_1, p_2, \ldots) \) belong to \( R'(1) \), and let X equal \( R'(1)\{p\} \). Then \( B(X) = \prod_i \mathbb{I}_{L_i} \cap X \) where \( L_i = (0,1) \) \( \text{if } p_i > 0 \), \([-1,0) \) \( \text{if } p_i = 0 \), and \([-1,1] \) \( \text{if } p_i = 0 \).

Proof. From lemma 4.10, X is a dense-deformation of \( Q \). It can easily be seen that each point of X not in \( \prod L_i \) belongs to a compact minimally separating set in X, and hence does not belong to \( B(X) \). So it is just necessary to show that each point of \( \prod L_i \cap X \) has no compact minimal separating set containing it. If A is a compact minimal separating set in X that contains a point \( q \) of \( \prod L_i \), then A also minimally separates \( R'(1) \). This implies that \( S_q \) is contained in A, but \( p \) belongs to \( S_q \) which yields a contradiction. □

**Corollary 5.8** For each \( p \) in \( R'(1) \), \( R'(1)\{p\} \) is not homeomorphic to \( R'(1) \).

Proof. \( B(R'(1)) = \emptyset \) and \( B(R'(1)\{p\}) \neq \emptyset \). These sets are topological invariants, hence the spaces are different. □

**Theorem 5.7** can also be used to show that for some points \( p \) and \( q \), \( R'(1)\{p\} \) is not homeomorphic to \( R'(1)\{q\} \). The following corollary provides an example.

**Corollary 5.9** Let \( p = (1,0,0,0,\ldots) \) and \( q = (0,1,1,1,\ldots) \). Then \( R'(1)\{p\} \) is not homeomorphic to \( R'(1)\{q\} \).
Proof. Let $X_p = R'(1) \setminus \{p\}$ and $X_q = R'(1) \setminus \{q\}$. $B(X_p) = [-1,1] \times (0,1)^\infty \cap X_p = \{(0) \times (0,1)^\infty\}\{\text{point}\}$ which by theorem 2.5 and 2.8 is homeomorphic to $s$. $B(X_q) = (0,1] \times [-1,1)^\infty \cap X_q$ and this is $\sigma$-compact. Since $s$ is nowhere locally compact, the Baire category theorem implies that $s$ is not $\sigma$-compact, hence $B(X_q)$ and $B(X_p)$ are different. Therefore $X_p$ and $X_q$ are different. □

Products with $R'(1)$.

In the study of infinite dimensional spaces, products are frequently examined. West [20] has shown that any contractible polyhedron cross $Q$, $S$, or $I$ is still the same space. This type of result indicates that these spaces are preserved under multiplying by suitable nice objects. $R'(1)$ definitely does not have this property. For example since $R'(1)$ has a rigid point it is easy to see that $R'(1)$ cross a space that does not have a rigid point cannot be $R'(1)$. In fact it will be shown that $R'(1)$ has no non-degenerate factors.

For a space $X$, it is clear that $R'(1) \times X$ is a dense-deformation of $Q \times X$. It is possible to compute the min-sets of $R'(1) \times X$ using the same technique of proof as in theorem 5.2. The following is given without proof.

Theorem 5.10 Let $X$ be a connected locally arcwise connected space. Then the min-set for a point $(p,x)$ in $R'(1) \times X$ is $S_p \times X$ if $X$ is compact or $R'(1) \times X$ if $X$ is not compact.

We have the following corollary which shows that $R'(1)$ has no non-degenerate factors.
Corollary 5.11 If $X$ is not a point, then $R'(1) \times X$ is not homeomorphic to $R'(1)$.

Proof. If $X$ is not connected or not locally arcwise connected, then neither is $R'(1) \times X$. If $X$ is connected and locally arcwise connected then the smallest min-set in $R'(1) \times X$ is not homeomorphic to a point in $\Omega_f$. The smallest min-set in $R'(1)$ is a point.

Related to products of a space $X$, is the space $X_f^\infty$. It can be shown using techniques in [5] that $\mathbb{Q}_f^\infty$ and $\mathbb{Q}_f^\infty$ are homeomorphic to $\mathbb{Q}$, and that $\sigma_f^\infty$ is homeomorphic to $\sigma$. We have the following theorem.

Theorem 5.12 $[R'(1)]_f^\infty$ is homeomorphic to $\mathbb{Q}^\infty$.

Proof. Let $R_f = [R'(1)]_f^\infty$. Theorem 2.9 states that $\mathbb{Q}$ is a cap set in $\mathbb{Q}$. Because of Theorem 2.10, it suffices to show that $R_f$ can be embedded as a cap set in $\mathbb{Q}$. Let $Q_i$ be a copy of $\mathbb{Q}$, and $R_i$ be the copy of $R'(1)$ in $Q_i$. Let $\delta$ be the point in $R_i$ with all coordinates zero. By definition $R_f = \{(x_i) \in \bigcap_i \mathbb{R}_i \setminus \mathbb{Q}_i \mid x_i = \delta \text{ for all but finitely many coordinates}\}$, and this space lies in $\bigcap_i \mathbb{Q}_i$, which is a Hilbert cube. From theorem 2.12 it suffices to verify that (1) that $R_f$ is a countable union of Z-sets, and (2) that $R_f$ contains a cap set.

(1) $R_f$ is a countable union of Z-sets. Let $S = \{(n_i) \mid (n_i)\}$ is a sequence of non-negative integers with $n_i = 0$ for all but finitely many $i$. $S$ is a countable set. For each $p = (p_i)$ in $S$, define $M_p$ to be $\{(x_i) \in \bigcap_i \mathbb{R}_i \setminus \mathbb{Q}_i \mid x_i = \delta \text{ if } p_i = 0 \text{ and otherwise the } p_i^{th} \text{ coordinate of } x_i \text{ is zero}\}$. Each $M_p$ is a closed set of infinite deficiency in $\bigcap_i \mathbb{Q}_i$, and hence a Z-set. It is easy to check that $R_f = \bigcup_{p \in S} M_p$. 


(2) $R_f$ contains a cap set. Write each $Q_i$ as the countable product of copies of $Q$, $Q_i = Q_1^1 \times Q_2^2 \times Q_3^3 \times \ldots$. Consider the following sets.

$$
N_1 = (Q_1^1 \times 0 \times 0 \times \ldots) \times 0 \times 0 \times \ldots
$$

$$
N_2 = (Q_1^1 \times Q_2^2 \times 0 \times 0 \times \ldots) \times (Q_2^2 \times 0 \times 0 \times \ldots) \times 0 \times 0 \times \ldots
$$

$$
N_3 = (Q_1^1 \times Q_2^2 \times Q_3^3 \times 0 \times 0 \times \ldots) \times (Q_2^2 \times Q_3^3 \times 0 \times 0 \times \ldots) \times (Q_3^3 \times 0 \times 0 \times \ldots) \times 0 \times 0 \times \ldots
$$

Each $N_j$ is a Z-set Hilbert cube contained in $R_f$. The following property will be established.

Given $\epsilon > 0$, $m > 0$ there exists an $n > 0$ and a homeomorphism $h$ of $\cap Q_i$ onto $N_n$ with $\rho(h, \text{id}) < \epsilon$ and $h |_{M_m} = \text{id}$. This can be achieved by choosing $n$ so large that any homeomorphism of $\cap Q_i$ which is independent of the first $n-1$ coordinates of $Q_i$ through $Q_{n-1}$ is within $\epsilon$ of the identity. Let $h$ be independent of these coordinates and send the rest of the coordinates of $\cap Q_i$ homeomorphically onto $Q_1^n \times Q_2^n \times \ldots \times Q_{n-1}^n \times Q_n^1 \times \ldots \times Q_n^n \times 0 \times 0 \times \ldots$ in the natural way which takes $0$ to $0$.

It is now easy to verify that $\cup N_i$ is a cap set. Given $K$ a compact Z-set, $\epsilon > 0$, and $m > 0$. To satisfy the definition of cap set, we must produce a homeomorphism $\hat{h}$ in $H(\cap Q_i)$ such that $\rho(\hat{h}, \text{id}) < \epsilon$, $\hat{h}(K) \subset N_n$ for some $n$, and $\hat{h} |_{K \cap N_m} = \text{id}$. Let $h$ be the homeomorphism of $\cap Q_i$ onto $N_n$ as in the preceding paragraph. Now $h |_{K}$ is a homeomorphism of $K$ onto $h(K)$. Use the $\epsilon$-homeomorphism extension (theorem 3.4) to get $\hat{h}$. $\Box$
CHAPTER VI
OTHER SPACES RELATED TO R(n)

In this chapter two groups of spaces related to R(n) are discussed. Many of the techniques used here are similar to those in chapter 5, so the proofs will only be sketched.

I. Some compactifications of the cantor set cross s.

These first spaces are generalizations of R'(1). It was remarked earlier that Q\R'(1) is homeomorphic to X(0) \times s. Note that X(0) is the infinite product of 0-spheres, hence is homeomorphic to the cantor set, C. All of these spaces to be introduced also can be regarded as remainders of various compactifications of C \times s. These spaces are discussed because they show how the apparatus of min-sets can be used to distinguish spaces topologically. They arise from an alternative definition of R'(1).

Consider the following subset of I^2. Let D(4) equal 
\{(x_1, x_2) \in I^2 \mid x_1 = 0 \text{ or } x_2 = 0\}. D(4) is the union of four straight intervals each of which joins (0,0) to the boundary of I^2. Another description of R'(1) is \{(x_1) \in (I^2)_\omega \mid \text{there is an index } j \text{ such that } x_j \text{ belongs to } D(4)\}. Now consider the following generalization of this definition.

For n \geq 3, let D(n) be the union of n straight intervals joining (0,0) to the boundary of I^2, with the condition that consecutive intervals have equal angular separation, and one of the
intervals contain \((1,0)\). Let \(J_1, J_2, \ldots, J_n\) be the clockwise consecutive intervals of \(D(n)\), and let \(W_i\) in \(I^2\) be the two cell "wedge" bounded by \(J_i\) and \(J_{i+1}\) \((W_n\) is bounded by \(J_n\) and \(J_1\)). Call a sequence of positive integers \((n_i)\) an acceptable sequence if \(3 \leq n_i \leq n_{i+1} \leq n_{i+2} \leq \ldots\). If \(S = (n_i)\) is an acceptable sequence define \(R(S)\) to be \\
\{\(x_i \in (I^2)^\infty\) | there is an index \(j\) such that \(x_j\) belongs to \(D(n_j)\)\}.

Note when \(S = (4,4,4,\ldots)\), \(R(S) = R'(1)\).

Using min-sets it will be shown that each acceptable sequence yields a different space. The min-sets of these spaces are identified, and it is shown that they "fit together" differently for each different acceptable sequence. This next theorem provides the characterization of min-sets. It is analogous to theorem 5.2 and will be stated without proof.

**Theorem 6.1** Let \(S = (n_i)\) be an acceptable sequence, and let \(p = (p_i)\) be a point of \(R(S)\). Then the min-set for \(p\) in \(R(S)\) is \(\Pi_i K_i\) where

\(K_i = \{(0,0)\}\) if \(p_i = (0,0)\), \(K_i = J_i^k\) if \(p_i\) belongs to \(J_i^k\setminus\{(0,0)\}\), or

\(K_i = W_i^k\) if \(p_i\) belongs to \(W_i^k\). Here we denote the intervals of \(D(n_i)\) by \(J_i^k\) and the wedges in \(I^2\) by \(W_i^k\).

An illustration is now given to indicate how the above theorem can be used to distinguish between two spaces. Let \(S_1 = (4,4,4,\ldots)\) and \(S_2 = (3,3,3,\ldots)\). In \(R(S_1)\) (which is just \(R'(1)\)) every 1-min-set has a corresponding opposite which is incompatible. In \(R(S_2)\) every two 1-min-sets span a 2-min-set, so these two spaces are different.

The following is a discussion of some properties which provide the means to distinguish these spaces. Let \(S\) be an acceptable
sequence, \( (n_i) \). For each \( i \), there is a collection of \( n_i \) 1-min-sets, 
\( \{ M_{i,k} \}_{k=1}^{n_i} \), which are just the \( n_i \) intervals of \( D(n_i) \) cross zero in all the other coordinates. This collection has property \( P(n_i) \) where we define this property as follows.

A collection of 1-min-sets \( \{ L_1, L_2, \ldots, L_k \} \) has property \( P(n) \) if 0) the collection has \( n \) distinct members, 1) the collection can be ordered so that each consecutive pair spans a 2-min-set (here the first and last are considered consecutive), 2) every three members are incompatible, and 3) if \( L \) is any other 1-min-set, then 
\( \{ L, L_1, L_2, \ldots, L_n \} \) contains \( n \) subsets which span \( n \) different 3-min-sets.

If the number \( n \) appears \( n \) times in a sequence \( S \), then we have \( n \) mutually disjoint collections of 1-min-sets with property \( P(n) \) in \( R(S) \). (If \( n_{k+1}, \ldots, n_{k+m} \) all equal \( n \), then consider the collections \( \{ M_{k+j, 1}, \ldots, M_{k+j, n} \} \). The following lemma states that this is the best number. The proof is a straightforward handling of cases and is not given.

**Lemma 6.2** If \( S \) is an acceptable sequence with the number \( n \) appearing \( m \) times, then there are no more than \( n \) mutually disjoint collections of 1-min-sets in \( R(S) \) with property \( P(n) \).

The existence of these collections is the topological property necessary for the main result stated here.

**Theorem 6.3** If \( S_1 \) and \( S_2 \) are different acceptable sequences, then 
\( R(S_1) \) is not homeomorphic to \( R(S_2) \).

II. The spaces \( R'(n) \) for \( n \) greater than one.
This second group of spaces was defined in chapter 4. Recall that \( R'(n) = \{ (x_i) \in (I^n)^\infty | \text{there is an index } j \text{ such that } x_j \text{ is the origin of } I^n \} \). Unlike \( R'(1) \), these spaces for \( n > 1 \) have complements in \( Q \) which are connected. This means that no compact subsets can separate them. However, it is possible to obtain some information about sets which destroy higher dimensional connectivity and to show that these spaces do not have properties 1' and 2' (\( \emptyset \) is rigid and non-negligible). Consider the following lemma about the \( n \)-cell.

**Lemma 6.4** Let \( f \) belong to \( C(S^{n-1}, I^n) \) and \( \epsilon \) be greater than zero. Then there is a homotopy \( F \) in \( C(S^{n-1} \times [0,1], I^n) \) such that \( F_0 = f \), \( F_1(S^{n-1}) = I^n \setminus \{\emptyset\} \), and \( \rho(F_0, F_t) < \epsilon \) for each \( t \).

**Proof.** A sketch is given using standard techniques of simplicial approximation. Choose a finite triangulation, \( T \), of \( S^{n-1} \) such that the image of every \( n-1 \) simplex under \( f \) is sufficiently small. We construct a new map \( g \) by defining the values on the vertices of \( T \), and then extending to a piecewise linear map. If \( v \) is a vertex of \( T \), choose \( g(v) \) in \( I^n \setminus \{\emptyset\} \) close to \( f(v) \). Also choose these values in such a way that for \( v_0, \ldots, v_n \) the vertices of any \( n-1 \) simplex of \( T \), \( \text{CH}(g(v_0), \ldots, g(v_n)) \) misses \( \emptyset \). Let \( F_t \) be the linear homotopy between \( f \) and \( g \). ☐

As a corollary to this lemma, we have the following.

**Lemma 6.5** Let \( f \) belong to \( C(S^{n-1}, Q) \) and \( \epsilon \) be greater than zero. Then there is a homotopy \( F \) in \( C(S^{n-1} \times [0,1], Q) \) such that \( F_0 = f \), \( F_1(S^{n-1}) = Q \setminus R'(n) \), and \( \rho(F_0, F_t) < \epsilon \) for each \( t \).
Proof. Note that $Q \setminus R'(n)$ equals $(I^n \setminus \overline{\emptyset})^\infty$. □

This next lemma is the analog of lemma 4.11 and is sufficient to establish that $R'(n)$ does not have property 1' and 2'.

Lemma 6.6 Every compact subset of $R'(n)$ that (n-1) disconnects $R'(n)$ must contain $\overline{\emptyset}$.

Proof. Let $K$ be a compact subset $R'(n)$, and suppose $R'(n) \setminus K$ is not (n-1) connected. Lemma 4.6 implies that $Q \setminus K$ is also not (n-1) connected. Let $f$ in $C(S^{n-1}, Q \setminus K)$ be a map that cannot be extended. Lemma 6.5 guarantees the existence of a homotopy $F$ in $C(S^{n-1}, [0,1], Q \setminus K)$ with $F_0 = f$ and $F_1$ into $Q \setminus R'(n)$. If $\overline{\emptyset}$ does not belong to $K$, then $Q \setminus R'(n) \cup \{\overline{\emptyset}\}$ does not meet $K$. $Q \setminus R'(n) \cup \{\overline{\emptyset}\}$ is contractible so there is a homotopy which takes $F_1$ to a constant map in $Q \setminus K$. This contradicts the fact that $f$ cannot be extended. □

The next two theorems follow from lemma 6.6 just as theorems 4.12 and 4.13 follow from lemma 4.11.

Theorem 6.7 The point $\overline{\emptyset}$ is rigid in $R'(n)$.

Theorem 6.8 $R'(n) \setminus \{\overline{\emptyset}\}$ is not homeomorphic to $R'(n)$.
CHAPTER VII
CONTRACTION OF SPACES OF HOMEOMORPHISMS

In this chapter a discussion is given about certain subsets of the space of homeomorphisms of $Q$, $s$, and $l_2$. As background to this problem, Wong [21] showed that any homeomorphism of $Q$ or $s$ is isotopic to the identity. Renz [18] observed that Wong's process is continuous, and so in fact contracts the space of homeomorphisms to the identity. In a later paper [22], Wong showed that any homeomorphism of $Q$ or $s$ which is the identity restricted to some compact $Z$-set, $K$, is isotopic to the identity, with each level of the isotopy also being the identity on $K$. In this chapter it will be shown that the space of such homeomorphisms is contractible. (i.e. for $K$ is a compact $Z$-set of $X = Q, s$, or $l_2$, $H_K(X)$ is contractible). The proof of this generalization uses a result of Chapman [13] concerning canonical extensions of homeomorphisms. The technique needed to prove the theorem for $X = Q$, requires a non-trivial modification of Wong's technique. The condition that $K$ be a $Z$-set is necessary as some examples at the end of this chapter show.

In order to discuss contractibility, it is necessary to put topologies on these spaces of homeomorphisms. Two topologies will now be described for $C(X,Y)$, the set of continuous functions from $X$ into $Y$. Since all the various function spaces discussed in this chapter are subsets of the appropriate space of continuous functions, they will be given the inherited topology.
For each compact subset $K$ of $X$ and each open set $U$ in $Y$, let $Q(K,U)$ be the \{ $f \in C(X,Y) | f(K) = U$ \}. The collection of all such sets form the subbase for the **compact-open topology**, $T_{co}$, of $C(X,Y)$.

If $Y$ has a bounded metric, $d$, then the metric $\rho$ generates the **sup metric topology**, $T_d$. Recall that $\rho(f,g) = \sup \{ d(f(x),g(x)) | x \in X \}$ and note that $T_d$ depends on the choice of $d$. Unless specifically indicated, function spaces will be given the topology $T_{co}$.

A number of preliminary lemmas are now given. They are necessary to justify steps in the proofs of the main results. The reader may first wish to read the discussion which begins after lemma 7.8. Some of these lemmas are standard, and the proofs are omitted (see chapter XII in [14]).

**Lemma 7.1** Let $X$ be compact, and $Y$ have bounded metric $d$. Then $T_{co}$ is equivalent to $T_d$ for $C(X,Y)$.

**Lemma 7.2** Let $X$ have bounded metric $d$. Then the map $c$ from $\langle H(X), T_d \rangle \times \langle H(X), T_{co} \rangle$ into $\langle H(X), T_{co} \rangle$ is continuous, where $c$ is the composition map defined by $c(g,f) = g \circ f$.

**Proof.** Suppose $c(g,f)$ belongs to $Q(K,U)$. Let $K'$ equal $g \circ f(K)$. There is an $\epsilon > 0$ such that $g \circ f(K) \subset N_{\epsilon/2}(K') \subset N_{\epsilon}(K') \subset U$. Clearly any $f'$ in $Q(K,g^{-1}N_{\epsilon/2}(K'))$ and $g'$ in $N_{\epsilon/2}(g)$ have the required property that $c(g',f')$ belongs to $Q(K,U)$. \( \square \)

**Lemma 7.3** Let $X$ be any space, $Y$ be compact, and $Z$ have bounded metric. Then given a function $F$ in $C(X,Y,Z)$, then the induced map $F$ is continuous, where $F$ is the map from $X$ into $C(Y,Z)$ defined by $F(x)(y) = F(x,y)$. 
Lemma 7.4 Let $A$ be a subset of $X$. Then the restriction map $R$ from $C(X,Y)$ into $C(A,Y)$ defined by $R(f) = f|_A$ is continuous, where $C(A,Y)$ and $C(X,Y)$ can both have either $T_{co}$ or $T_d$ (if $d$ is a bounded metric for $Y$).

Proof. Follows immediately from the facts that $\rho(f,g) \leq \rho(f|_A,g|_A)$ and $f(K) \subseteq U$ implies that $f(K \cap A) \subseteq U$. □

Homotopies on the space of homeomorphisms of $X$ can be generated by producing homotopies on the space of homeomorphisms of some copy. The following lemma indicates how this is done.

Lemma 7.5 Let $h$ be a homeomorphism from $Y$ onto $Z$, and let $G$ from $X \times H(Z)$ into $H(Z)$ be continuous. Then $\bar{G}$ from $X \times H(Y)$ into $H(Y)$ is also continuous where $\bar{G}(x,f) = h^{-1} \cdot G(x,h \circ f \circ h^{-1}) \cdot h$.

Proof. It can easily be verified that the conjugation map, $k$, sending $f$ in $H(Y)$ to $h \circ f \circ h^{-1}$ in $H(Z)$ is a homeomorphism. The result follows from the fact that $\bar{G}$ is the composition indicated below.

\[
\begin{array}{ccc}
X \times H(Y) & \xrightarrow{id \times k} & X \times H(Z) & \xrightarrow{G} & H(Z) & \xrightarrow{k^{-1}} & H(Y).
\end{array}
\]

Lemma 7.6 Let $K$ be a compact subset of $X$, and $C$ be a compact subset of $Y$. Let $f$ be a continuous function from $X$ into $C(Y,Z)$. Then $f(K)(C) = \bigcup_{g \in f(K)} g(C)$ is compact.

Proof. Let $e$ be the evaluation map from $C(Y,Z) \times Y$ into $Z$ defined by $e(f,y) = f(y)$. It is standard that $e|_{C(Y,Z) \times C}$ is continuous. Note that $f(K)(C)$ is the image under $e$ of the compact set $f(K) \times C$ and hence compact. □
These next two lemmas provide a means of producing homotopies on spaces of homeomorphisms.

Lemma 7.7 Let $X$ be locally compact, and $Y$ have bounded metric $d$. Let $\psi$ from $X$ into $(H(Y), T_{co})$ and $\phi$ from $X$ into $(H(Y), T_d)$ be continuous maps. Then $F$ from $H(Y) \times X$ into $H(Y)$ is continuous, where

$$F(h, x) = \phi(x) \circ h \circ \psi(x).$$

Proof. Suppose $F(h, x)$ belongs to $Q(K, U)$. A neighborhood of $(h, x)$ will be produced which maps into $Q(K, U)$ under $F$.

Let $K' = \phi(x) \circ h \circ \psi(x)(K)$ and choose $\epsilon > 0$ such that

1. $\phi(x) \circ h \circ \psi(x)(K) \subset N_{\epsilon/2}(K') \subset N_{\epsilon}(K') \subset U$.

Statement (1) implies that $\psi(x)$ belongs to $Q(K, h^{-1} \circ \phi^{-1}(x) N_{\epsilon/2}(K'))$.

From the continuity of $\psi$ and $\phi$, and the local compactness of $X$, there exists a neighborhood of $x$, $N(x)$, with $\overline{N(x)}$ compact,

2. $\rho(\phi(x), \phi(x')) < \epsilon/2$ for $x'$ in $N(x)$, and

3. $N(\overline{N(x)})$ contained in $Q(K, h^{-1} \circ \phi^{-1}(x) N_{\epsilon/2}(K'))$.

Lemma 7.6 implies that $K''$ equal to $\psi(\overline{N(x)})(K)$ is compact. The claim is that $Q(K'', \phi^{-1}(x) N_{\epsilon/2}(K')) \times N(x)$ is the desired neighborhood of $(h, x)$. The fact that this is a neighborhood of $(h, x)$ follows from (3).

Let $(h', x')$ belong to this neighborhood. We must verify that

$$F(h', x') \subset (h', x')(K) \subset U.$$

We have that $\psi(x')(K)$ is contained in $K''$, and $h'(K'')$ is contained in $\phi^{-1}(x) N_{\epsilon/2}(K')$. Therefore $\phi(x) \circ h' \circ \psi(x')(K) \subset N_{\epsilon/2}(K')$. Since $\phi(x)$ is within $\epsilon/2$ of $\phi(x)$ (fact (2)) we have $F(h', x')$ contained in $N_{\epsilon}(K')$ which is contained in $U$ (fact (1)). □

Lemma 7.8 Let $X_1$ and $X_2$ be metric spaces, and let $d$ be a bounded
metric for $X_1 \times X_2$. Let $i_1$ be a homeomorphism from $X_1$ onto $X_1 \times X_2$.
Let $\phi$ be a continuous map from $[0,1]$ into $C(X_1 \times X_2, X_1 \times X_2, T_d)$ with the property that $\phi((0,1]) \subset H(X_1 \times X_2)$ and $\phi(0) = i_1 \circ \pi_1$. Let $\pi$ and $\psi$ be a continuous maps, where $\pi$ is from $(0,1]$ into $(H(X_1 \times X_2), T_{co})$, $\psi$ is from $[0,1]$ into $C(X_1 \times X_2, X_1)$, $\psi(0) = i_1^{-1}$, and for each $t$ in $(0,1)$, $\pi_1 \psi(t)$ equals $\pi(t)$.

Then $F$ from $H(X_1 \times X_2) \times [0,1]$ into $H(X_1 \times X_2)$ is continuous, where

$F(h,t) = \pi(t) \circ (i_1^{-1} \cdot h \cdot i_1 \cdot \text{id}_2) \cdot \psi(t)$ for $t$ in $(0,1)$, and

$F(h,0) = h$.

Proof. It is straightforward to verify that the map, $\alpha$, of $H(X_1 \times X_2)$ into itself which sends $h$ to $i_1^{-1} \cdot h \cdot i_1 \cdot \text{id}_2$ is continuous. Notice that $F|_{H(X_1 \times X_2) \times (0,1]}$ is just a composition of the $\alpha \times \text{id}$ followed by a map as defined in lemma 7.7, and hence is continuous. So it is only necessary to verify continuity at points $(h,0)$.

Suppose $F(h,0) = h$ belongs to $Q(K,U)$. We will produce a neighborhood of $(h,0)$ which maps into $Q(K,U)$ under $F$. Let $K' = h(K)$ and choose $\epsilon > 0$ such that

$$(1) \quad h(K) \subset N_{\epsilon/2}(K') \subset N_{\epsilon}(K') \subset U.$$ 

Since $h$ equals $h \cdot i_1^{-1}$ and $\psi(0) = i_1^{-1}$, $(1)$ implies that $\psi(0)$ belongs to $Q(K, i_1^{-1} h^{-1} N_{\epsilon/2}(K'))$. From the continuity of $\psi$ and $\phi$ we get an interval $[0,t_0]$ such that

$$(2) \quad \rho(\pi(t), \psi(0)) < \epsilon/2 \text{ for } t \text{ in } [0,t_0] \text{ and,}$$
$$
(3) \quad \psi([0,t_0]) \text{ belongs to } Q(K, i_1^{-1} h^{-1} N_{\epsilon/2}(K')).$$

$K''$ equal to $\psi([0,t_0])(K)$ is compact. The claim is that $Q(i_1 K'', N_{\epsilon/2} K') \times [0,t_0]$ is the desired neighborhood of $(h,0)$. This is neighborhood from $(3)$. Let $(i',t)$ belong to this neighborhood. We must verify
that $F(h',t)(K) \subset U$. If $t = 0$, then $F(h',t) = h'$, and $h'(K)$ is clearly contained in $U$ since $K \subset \iota_1 K''$ and $N_{\varepsilon/2} K' \subset U$. In the other case, consider $C = \psi(0) = (i_1^{-1} : h' - i_1 \times \text{id}_2) \circ \psi(K)$. Since $\psi(0)$ equals $i_1 \circ \iota_1$, $C$ equals $h'i_1\psi(t)(K)$ which is contained in $h'i_1 K''$, and this from the definition of the neighborhood belongs to $N_{\varepsilon/2}(K')$. Statement (2) gives that $F(h',t)(K)$ is within $\varepsilon/2$ of $C$, and hence $F(h',t)(K)$ is contained in $N_{\varepsilon}(K')$ which is in $U$ by (1). □

Before proving the main results, it will be necessary to develop some notation for certain isotopies of $R$ and $s$. Consider the $\beta$ homeomorphism of $R$, $f$, where $f(x_1,x_2,x_3,\ldots) = (x_2,x_1,x_3,x_4,\ldots)$. An informal discussion will be giving showing that this homeomorphism is nicely isotopic to the identity. The details of this appear in Wong's paper [21].

We wish to construct an isotopy $F$ from $R \times [0,1]$ onto $R$ such that $F_0$ equals $f$, and $F_1$ equals $\text{id}$. In addition it is desired that for each $t$ and each standard subcube $R'$, $F_t(R') = R'$. This condition also implies that $F_t$ is $\beta$ for each $t$. Consider a rotation of $I_1 \times I_2$ about the origin. As $t$ varies from 1 to $1/2$, let $F_t$ be the homeomorphism of $R$ which rotates the coordinates in $I_1 \times I_2$ by an angle varying from 0 to 90 degrees. We have $F_1$ is the identity, and $F_{1/2}(x_1,x_2,x_3,\ldots) = (x_2,-x_1,x_3,\ldots)$. As $t$ varies from $1/2$ to $1/4$, follow $F_{1/2}$ by rotations in the $I_2 \times I_4$ coordinates varying from 0 to 180 degrees. Note then that $F_{1/4}(x_1,x_2,x_3,\ldots) = (x_2,x_1,x_3,-x_4,x_5,\ldots)$. Continuing with rotations of 180 degrees in $I_4 \times I_6$, $I_6 \times I_8$, etc., it is possible to "twist" the identity to $f$. Note that this isotopy is independent of the odd coordinates past the first, $\beta$ is taken
onto $\delta$ at all levels, and also that standard subcubes are preserved. In general this type of isotopy can be constructed as long as infinitely many coordinates can be used. Since transpositions generate all finite permutations, this same technique can be applied to any homeomorphism which permutes finitely many coordinates.

Some notation will now be introduced to describe isotopies on $Q$. Consider the following diagram.

$$
\begin{array}{cccccccc} 
 t & x \\
1 & 1 & 2 & 3 & 4 & 5 & \ldots & 2n & \ldots \\
1/2 & 2 & \underline{1} & 3 & 4 & 5 & \ldots & 2n & \ldots \\
1/4 & 3 & 1 & 4 & 2 & 5 & \ldots & 2n & \ldots \\
1/2^n & n+1 & 1 & n+2 & 2 & n+3 & 3 & \ldots & n & 2n+1 & 2n+2 & \ldots \\
\end{array}
$$

This diagram will be understood to represent $\psi$ from $Q \times (0,1]$ onto $Q$ where:

1) For the values of $t = 1/2^n$, $\psi_t(x)$ is the value implied by the table. (e.g. $\psi_{1/8}(x_1,x_2,x_3,\ldots) = (x_4,x_1,x_5,x_2,x_6,x_3,x_7,x_8,x_9,\ldots)$).

2) If for an indicated value of $t$, a coordinate position is underlined, then the rest of the levels below have the same value in that position. (e.g. for $t \leq 1/4$, the forth coordinate of $\psi_t(x)$ is always $x_2$).

3) The transition between indicated levels is done via the transposition isotopies described above. This means that for all $t$, $\psi_t$ is $\delta^*$, $\psi_t$ preserves standard subcubes, and $\psi_t(\delta) = \delta$. It is sometimes possible to extend $\psi$ to $Q \times [0,1]$. In this case $\psi_0$ will also
be specified (sometimes this is not a homeomorphism).

The following theorem was proved by Renz [18], and is stated here for completeness.

**Theorem 7.9** Let $X$ equal $Q$ or $s$. Then there exists an $F$ from $H(X) \times [0,1]$ into $H(X)$ such that $F(h,0) = h$ and $F(h,1) = id$. Also, if $Q'$ is a standard subcube of $X$ and $h(Q') = Q'$, then $F(h,t)(Q') = Q'$ for all $t$.

The following result follows from the apparatus on canonical extensions of homeomorphisms due to Chapman [13]. This result provides a key step in the proof of the main theorem.

**Theorem 7.10** Let $K$ be a compact subset of $s$ in $Q$. Then there exists a continuous function $\phi$ from $E(K,s)$ into $H(Q)$ where 1) $\phi(f)|_{f(K)} = f^{-1}$, 2) $\phi(i) = id$ where $i$ is the inclusion map of $K$ into $X$, and 3) $\phi(f)$ is $s^*$ for each $f$.

It is now possible to prove the main result for $X = s$ or $l_2$.

**Theorem 7.11(1)** Let $K$ be a compact subset of $X( = l_2$ or $s$). Then $H_K(X)$ is contractible.

**Proof.** Since $l_2$ is homeomorphic to $s$, lemma 7.5 implies it is sufficient to prove the theorem for $X = s$.

Let $F$ from $H_K(s) \times [0,1]$ into $H(s)$ be the restriction of the homotopy produced in theorem 7.9. Let $\phi$ be as in theorem 7.10. Consider the map $G$ from $H_K(s) \times [0,1] \to (H(s), T_d)$ defined by,

$$G(f,t) = \phi(F(f,t)|_{K'})|s'.$$
This is continuous by lemmas 7.1 and 7.4. It is easily checked that
$H$ from $H_K(s) \times [0,1]$ into $H(s)$ is the desired contraction, where
\[ H(f,t) = G(f,t) \circ F(f,t). \]

The above method of proof fails for $X = Q$. While it can be
assumed that $K$ is a compact set in $s$, there are still some homeo-
morphisms of $H_K(Q)$ where the contraction of theorem 7.9 does not keep
the image of $K$ in $s$ at all levels. Chapman's theorem cannot be applied
then. In Wong's paper [22] it was shown that each homeomorphism $h$ in
$H_K(Q)$ is isotopic to the identity. A crucial step was to show that if
$h|_K = id$ and $K$ is in $s$, then there exists a subcube $Q'$ in $s$, such that
$K \subset Q'$ and $h(Q') \subset s$. $Q'$ was produced by a Baire category argument and
hence does not vary continuously with $h$. The key lemma that is presented
next provides a way of deforming $H_K(Q)$, so that at the end of the de-
formation, all homeomorphisms in the image take a given Z-set cube onto
itself.

**Lemma 7.12** Let $X = Q$ or $s$, and let $K$ be a Z-set in $X$. Then there exists
a homotopy $F$ from $H(X) \times [0,1]$ into $H(X)$ such that:

1. $F(h,0) = h$,
2. If $h|_K = id$, then $F(h,t)|_K = id$ for all $t$,
3. $F(h,1)(K) = \tilde{K}$ where $\tilde{K}$ is a Z-set containing $K$ and $\tilde{K}$ is homeomorphic to $X$.

**Proof.** Let $X = X_e = X_1 = X_3$. Since $X$ is homeomorphic to $X_e \times X_1 \times X_3$, lemma 7.5 implies that we can consider the space $H(X_e \times X_1 \times X_3)$, and consider $K$ to lie in $X_e \times \emptyset \times \emptyset$. Given $h$ in $H(X_e \times X_1 \times X_3)$, let $\bar{h}$
be the homeomorphism of $\times X_1 \times X_3$ equal to $(i_1^{-1} \circ h \circ i_1 \circ \text{id}_3)$ where $i_1$ is the homeomorphism from $\times X_1$ onto $\times X_1 \times X_3$ with formula $i_1((x_i),(y_i)) = ((x_i),(y_1,y_3,y_5,\ldots),(y_2,y_4,y_6,\ldots))$. The desired homotopy will be constructed using lemma 7.8.

We define $F$ from $H(\times X_1 \times X_3) \times [0,1]$ to $H(\times X_1 \times X_3)$ as follows.

$F(h,0) = h$, and

$F(h,t) = \phi(t) \circ h \circ \psi(t)$ for $t$ in $(0,1]$.

The functions $\phi$ and $\psi$ are constructed to satisfy the conditions of lemma 7.8. In addition $\psi(1)$ and $\phi(1)$ are the identity, and for each $t$, $\psi(t)|_{K}$ and $\phi(t)|_{K}$ equal the identity. Notice that if $h|_{K} = \text{id}$, then $\tilde{h}|_{K} = \text{id}$. This implies that $F$ satisfies condition 2). We also have that $F(h,1) = \tilde{h}$. Since $\tilde{h}$ is independent of the $X_3$ coordinate, $\tilde{h}(\tilde{K}) = \tilde{K}$ where $\tilde{K} = \times X_1$ is a Z-set homeomorphic to $X$, and $\tilde{K}$ contains $K$. An intuitive discussion about $\psi$ and $\phi$ is given first.

The three factors of $\times X_1 \times X_3$ are thought of as representing groups of coordinates in $X$, those with indices that are even, those with indices congruent to 1 mod 4, and those with indices congruent to 3 mod 4. The function $\psi$ has the property that $\psi(t)$ is a homeomorphism which is independent of the even coordinates, and, as $t$ approaches 0, more and more of the odd coordinates occupy the coordinate positions congruent to 1 mod 4. The function $\phi$ also has the property that $\phi(t)$ is a homeomorphism which is independent of the even coordinates, and, as $t$ approaches 0, more and more of the coordinates with indices congruent to 1 mod 4 occupy all the odd coordinate positions. The functions are produced by the twisting type homeomorphisms previously described above. Because the two
homeomorphisms $\phi(t)$ and $\psi(t)$ are independent of the even coordinates, and the twisting motion in the odd coordinates preserves $0$, we have that $K$ which lies in $X_e \times 0 \times 0$ is not changed. We now define these functions. First consider the functions $\bar{\phi}$ and $\bar{\psi}$ given below.

\[
\begin{array}{ccccccccccc}
\text{t} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 2n & \ldots & \frac{1}{2^n} & \ldots \\
\phi & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 2n & \ldots & \frac{1}{2^n} & \ldots \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
\text{t} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 2n+1 & \ldots & \frac{1}{2^n} & \ldots \\
\psi & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 2n & \ldots & \frac{1}{2^n} & \ldots \\
\end{array}
\]

Let $\psi$ from $[0,1]$ into $H(X_e \times X_1 \times X_3)$ be given by

$\psi(t)(x_e, x_1, x_3) = i\bar{\psi}(i^{-1}(x_e, x_1, x_3), t)$, and $\phi$ from $[0,1]$ into $H(X_e \times X_1 \times X_3)$ be given by $\phi(t)(x_e, x_1, x_3) = i\bar{\phi}(i^{-1}(x_e, x_1, x_3), t)$. Here $i$ is the canonical homeomorphism from $X$ onto $X_e \times X_1 \times X_3$ with formula $i(x_1, x_2, x_3, \ldots) = ((x_2, x_6, x_8, \ldots), (x_1, x_5, x_9, \ldots), (x_3, x_7, x_{11}, \ldots))$. Lemmas 7.1, 7.3, and 7.4 imply that $\phi$ and $\psi$ satisfy the conditions of lemma 7.8. 

The main theorem is now given for $Q$. 

Theorem 7.11(2) Let \( K \) be a Z-set in \( Q \). Then \( H^*_K(Q) \) is contractible.

Proof. Lemma 7.12 deforms \( H^*_K(Q) \) into \( H^*_K(Q) = \{ h \in H^*_K(Q) \mid h(\bar{K}) = \bar{K} \} \), where \( \bar{K} \) is a Z-set Hilbert cube containing \( K \). All that is necessary is to contract \( H^*_K(Q) \). \( \bar{K} \) is homeomorphic to a standard subcube, so lemma 7.5 and theorem 7.9 produce a map \( F \) from \( H^*_K(Q) \times [0,1] \) into \( H^*(Q) = \{ h \in H(Q) \mid h(\bar{K}) = \bar{K} \} \), with \( F(h,0) = h \) and \( F(h,1) = \text{id} \). Note that for each \( t \), \( F(h,t)(K) = \bar{K} \subseteq s \). Hence the desired contraction is constructed just as in the proof of part one of this theorem because Chapman's theorem can be applied. □

In the above proof it should be noted that a stronger condition is satisfied by \( F \) than is necessary to apply Chapman's theorem. We have in fact that for each \( t \), \( F(h,t) \) is a homeomorphism of a fixed Z-set in \( s \). A canonical homeomorphism extension theorem for homeomorphisms of a fixed Z-set can be shown directly by methods similar to those used in the proof of theorem 3.1.

A family of examples is now given which shows that the requirement that \( K \) be a Z-set is necessary. This example for \( n = 1 \), appeared in [22] also.

Consider \( Q \) equal to \( I^n \times Q' \). Let \( \bar{0} \) be the origin of \( I^n \). Let \( K \) equal \( \bar{0} \times Q' \). \( K \) is not a Z-set since its complement is not homotopically trivial. Let \( h \) be the homeomorphism of \( I^n \times Q' \) induced by the antipodal map on \( I^n \). Suppose \( H^*_K(Q) \) were contractible. This would imply that \( h \) is isotopic to the identity via \( H_t \) where \( H_t|_K = \text{id} \) for all \( t \). This implies that for all \( t \), \( H_t|(I^n \setminus \{0\}) \times Q' \) is a homeomorphism.
Let \( r \) be a retraction of \( I^n \setminus \{0\} \times Q' \) onto \( S^{n-1} \times \emptyset \). Now \( r \circ H_{t \mid S^{n-1} \times \emptyset} \) is a homotopy of \( S^{n-1} \) that takes the antipodal map to the identity. This is impossible for \( n \) an odd integer [see 15].
BIBLIOGRAPHY


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