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Complementary group resolution of the SU(n) outer multiplicity problem. II. Recoupling approach for SU(3)⊃U(2) reduced Wigner coefficients

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A complementary group technique for the resolution of the outer multiplicity problem of SU(n). (II) A recoupling approach to the solution of $SU(3) \supset U(2)$ reduced Wigner coefficients

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Abstract

A general procedure for the derivation of $SU(3) \supset U(2)$ reduced Wigner coefficients for the coupling $(\lambda_1\mu_1) \times (\lambda_2\mu_2) \downarrow (\lambda\mu)^\eta$, where η is the outer multiplicity label needed in the decomposition, is proposed based on a recoupling approach according to the complementary group technique given in (I). It is proved that the non-multiplicity-free reduced Wigner coefficients of $SU(n)$ are not unique with respect to canonical outer multiplicity labels, and can be transformed from one set of outer multiplicity labels to another. The transformation matrices are elements of $SO(m)$, where m is the number of occurrence of the corresponding irrep $(\lambda\mu)$ in the decomposition $(\lambda_1\mu_1) \times (\lambda_2\mu_2) \downarrow (\lambda\mu)$. Thus, a kind of the reduced Wigner coefficients with multiplicity is obtained after a special $SO(m)$ transformation. New features of this kind of reduced Wigner coefficients and the differences from the reduced Wigner coefficients with other choice of the multiplicity label given previously are discussed. The method can also be applied to the derivation of general $SU(n)$ Wigner or reduced Wigner coefficients with multiplicity. Algebraic expression of another kind of reduced Wigner coefficients, the so-called reduced auxiliary Wigner coefficients for $SU(3) \supset U(2)$, are also obtained.

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I. Introduction

Wigner Coefficients (WCs), or Reduced Wigner Coefficients (RWCs) of $SU(3)$ in the canonical basis, i.e. the basis adapted to $SU(3) \supset U(2)$, were discussed by many authors, for example, by Biedenharn et al[1−10] using the canonical unit tensor operator method, Moshinsky et al[11−14] using the infinitesimal approach and a complementary group method, Ališauskas et al using the symmetric group approach[15−17] and paracanonical and pseudo-canonical coupling schemes.[18−21] A large class of the RWCs were also considered by Hecht,[22] Resnikoff,[23] Shelepin and Karasev,[24−25] Klimyk and Gavrilik,[26] Le Blanc and Rowe,[27] and many others. Among these approaches, only the outer multiplicity labeling scheme of the unit tensor operator method is canonical, which leads to the usual orthogonalities of the RWCs. Other methods for labeling the outer multiplicity are noncanonical, i. e. the RWCs obtained will be non-orthogonal with respect to the outer multiplicity label. Therefore, the Gram-Schmidt process will be adopted, which depends upon an arbitrary choice of order to the elements to be orthogonalized. In this case, only numerical algorithm is possible, as have been done so by Draayer and Akiyama,[28−29] and Kaeding and Williams.[30−32] In [33], WRCs associated to 27-plet operator, which is a multiplicity three case, were discussed.

Very recently, Parkash and Sharatchandra[34] have worked out an algebraic formula for the general Wigner coefficients for $SU(3)$ in the canonical basis. However, the final results are expressed in terms of free summations over 33 variables under some restrictions, and there is a normalization factor needed to be determined in the expression, which will not be easy to compute values of the RWCs by using their formula either algebraically or numerically.

In this paper, we will use the complementary group technique proposed in (I) to compute the RWCs of $SU(3) \supset U(2)$ by using the multiplicity-free RWCs known previously with a recoupling approach. It should be noted that this approach, in principle, is labeling scheme independent. For example, the RWCs of $SU(3) \supset U(2)$ in another labeling scheme proposed by Biedenhran et al can also be derived by using this method. However, the values of the RWCs with different canonical outer multiplicity labeling scheme will be different, which are actually within a $SO(m)$ group transformation among the RWCs from one set of outer multiplicity label to another.

In Sec. II, we will discuss the non-uniqueness of the canonical resolution of RWCs with multiplicity, and prove that the transformation group from one set of multiplicity label to another can be chosen as $SO(m)$, where m is the number of occurrence of the resultant irrep considered in the decomposition $(\lambda_1\mu_1) \times (\lambda_2\mu_2)$. Then, a special $SO(m)$ transformation is made for the RWCs. In such a case, the RWCs with multiplicity can be derived recursively. In Sec. III, we will propose a recoupling procedure for the evaluation of the RWCs from known multiplicity-free RWCs of $SU(n) \supset U(n-1)$ given previously by Ališauskas et al.[15] New feature of this kind of RWCs and differences from ones with other choice of the outer multiplicity label will be discussed in IV. In Sec. V, an analytical expression of another kind of RWCs, the so-called reduced auxiliary WCs proposed by Brody, Moshinsky, and Renero[11] will also be derived by using this recoupling approach. Discussions will be given in VI.

II. Relations among different choices of the multiplicity labels

Let $|[\lambda]\xi, \rho> \equiv |[\lambda_1][\lambda_2][\lambda](\xi), \rho>$ be coupled basis vectors of $U(n) \times U(n) \downarrow U(n)$, where ξ denotes a set of multiplicity labels needed in the decomposition $[\lambda_1] \times [\lambda_2] \downarrow [\lambda]$, ρ is the sublabels for the resultant irrep $[\lambda]$. The completeness condition for $|[\lambda]\xi, \rho>$ is

$$\sum_{(\xi)[\lambda]\rho} |[\lambda]\xi, \rho>< [\lambda]\xi, \rho| \;=\; 1. \tag{2.1}$$

Actually, the choice of the multiplicity labels is not unique. One can use transformations from one set of multiplicity labels to another. If (ξ) can be taken m different values denoted as $(\xi) = \xi_1,\ \xi_2,\ \cdots,\ \xi_m$, the transformation matrix group is $SU(m)$. Therefore,

$$|[\lambda]\eta, \rho >= \sum_{(\xi)} y(\xi,\eta)|[\lambda]\xi,\ \rho >, \tag{2.2}$$

where (η) is another set of multiplicity labels for $U(n)$, $y(\xi,\eta)$ is a matrix element of $SU(m)$. One can verify that $y(\xi,\eta)$ should satisfy

$$\sum_{(\xi)} y(\xi,\eta)y^*(\xi,\eta') = \delta_{\eta\eta'},$$

$$\sum_{(\eta)} y(\xi,\eta)y^*(\xi',\eta) = \delta_{\xi\xi'}. \tag{2.3}$$

Therefore, $\{y(\xi,\eta)\}$ defines a unitary transformation according to the basic representation of $SU(m)$. Hence, the choice of the outer multiplicity labels for the Kronecker product $[\lambda_1] \times [\lambda_2] \downarrow [\lambda]$ is not unique. There always exists a unitary transformation $\mathbf{Y} \in SU(m)$, where m is the number of occurrence of $[\lambda]$ in the decomposition $[\lambda_1]\times[\lambda_2]$, which transforms from one set of outer multiplicity labels to another. Usually, the WCs of $U(n)$ are taken to be real. In this case, the internal symmetry group for the transformation of the outer multiplicity labels can be chosen as $SO(m)$. It is obvious that the dimension of the transformation group is the number of occurrence of the resultant irrep $[\lambda]$ dependent on the decomposition $[\lambda_1] \times [\lambda_2] \downarrow [\lambda]$, and the basis vectors of $[\lambda]$ are still orthonormal with each other if they are transformed from a set of orthonormal basis vectors of $[\lambda]$. That is why there are different forms of RWCs or WCs with respect to the outer multiplicity labels after a Gram-Schmidt transformation from noncanonical resolutions or obtained from canonical resolutions. In the following, we always assume that the RWCs considered are real.

We restrict our discussion to $SU(3) \supset U(2)$ case only, which can easily be extended to the general $SU(n)$ case. We adopt the usual physical notation for $SU(3)$ irrep $(\lambda\mu) \equiv [\lambda+\mu, \mu]$, where $[\lambda+\mu, \mu]$ is a usual two-rowed irrep corresponding to the irrep described by two-rowed Young diagram with $\lambda+\mu$ boxes in the first row, and μ boxes in the second row. We have proved that there is only one multiplicity label needed in the decomposition $(\lambda_1\mu_1)\times(\lambda_2\mu_2)\downarrow[m_1m_2m_3]$, which is assigned as ξ with $\xi=\xi_1,\ \xi_2,\ \cdots,\ \xi_m$, where m is the number of occurrence of $[m_1m_2m_3]$ in the decomposition.

The key step to evaluate $SU(3)$ WCs or RWCs of $SU(3)\supset U(2)$ is to use the transforamtion

$$\sum_{\xi}\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle y(\xi,\eta)=\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\eta\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle, \quad (2.4)$$

where $\rho_1,\ \rho_2$, and ρ are the corresponding sublabels of $SU(3)$ if

$$\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle$$

is WCs, or RWCs if $\rho_1,\ \rho_2,\ \rho$ are the corresponding U(2) labels, and $y(\xi,\eta)$ are chosen to be a special set of matrix elements of the special orthogonal group $SO(m)$, which are chosen as follows.

Assume

$$\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle,\quad \xi=\xi_1,\ \xi_2,\ \cdots,\ \xi_m, \quad (2.5)$$

is a set of WCs satisfying the orthogonality relation

$$\sum_{\rho_1\rho_2}\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi'\ [m_1'm_2'm_3']\\ \rho'\end{array}\right\rangle=\delta_{m_im_i'}\delta_{\xi\xi'}\delta_{\rho\rho'}, \quad (2.6a)$$

$$\sum_{\xi\rho m_i}\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1' & \rho_2'\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle=\delta_{\rho_1\rho_1'}\delta_{\rho_2\rho_2'}, \quad (2.6b)$$

or RWCs satisfying

$$\sum_{\xi m_i}\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1' & \rho_2'\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle=\delta_{\rho_1\rho_1'}\delta_{\rho_2\rho_2'}. \quad (2.6c)$$

$$\sum_{\rho_1\rho_2}\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ \rho\end{array}\right\rangle\left\langle\begin{array}{cc}(\lambda_1\mu_1) & (\lambda_2\mu_2)\\ \rho_1 & \rho_2\end{array}\middle|\begin{array}{c}\xi'\ [m'_1m'_2m'_3]\\ \rho\end{array}\right\rangle=\delta_{m_im'_i}\delta_{\xi\xi'}, \tag{2.6d}$$

According to the Schur-Weyl duality relation, the RWCs for $SU(3)\supset U(2)$ given by (2.5) are also RWCs for $SU(4)\supset U(3)$ for the same coupling with the same set of outer multiplicity labels $\{\xi_i\}$. Then, according to the complementary group technique given by (I), the complementary group for the above $SU(3)$ coupling is $\mathcal{U}(4)$. We need to consider the same coupling of $\mathcal{U}(4)$ in the special Gel'fand basis according to the Littlewood rule, namely

$$\left\langle\begin{array}{cc}[\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2]\\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0]\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta]\end{array}\right\rangle. \tag{2.7}$$

Next, there is a $m\times m$ orthonormal matrix Y that transforms the RWCs or WCs between two sets of multiplicity labels (ξ) and $(\tilde{\eta})$, where the range of ξ, and $\tilde{\eta}$ is the same as the sublabels η in the $SU(3)$ subirrep $[m_1,\ m_2-\eta,\ m_3-\mu_2+\eta]$. It has been proved that the number of occurrence of $[m_1,m_2,m_3]$ in the Kronecker product $[\lambda_1+\mu_1,\mu_1]\times[\lambda_2+\mu_2,\mu_2]$ can be described exactly by η within the following ranges

$$\eta_{\min}\le\eta\le\eta_{\max}, \tag{2.8a}$$

where

$$\eta_{\max}=\min(m_1-\lambda_1-\mu_1,\ \mu_2,\ m_2-\mu_1,\ \lambda_2+\mu_2-m_3,\ \mu_1+\mu_2-m_3,\ m_2-m_3),$$

$$\eta_{\min}=\max(0,\ \mu_2-m_3,\ m_2-\lambda_1-\mu_1). \tag{2.8b}$$

Thus, we require

$$\sum_{\xi}y(\xi,\tilde{\eta})\left\langle\begin{array}{cc}[\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2]\\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0]\end{array}\middle|\begin{array}{c}\xi\ [m_1m_2m_3]\\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta]\end{array}\right\rangle$$
$$=\left\langle\begin{array}{cc}[\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2]\\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0]\end{array}\middle|\begin{array}{c}\tilde{\eta}\ [m_1m_2m_3]\\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta]\end{array}\right\rangle', \tag{2.9}$$

where $\tilde{\eta},\ \eta\ =\eta_1,\ \eta_2,\cdots,\ \eta_m$, and the prime indicates a new RWC, which is different from the old one. The RWCs

$$\left\langle \frac{\xi}{\eta} \right\rangle \equiv \left\langle \begin{matrix} [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2] \\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0] \end{matrix} \middle| \begin{matrix} \xi\ [m_1 m_2 m_3] \\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta] \end{matrix} \right\rangle \tag{2.10}$$

with fixed η can be regarded as a vector in $\mathbf{R}^m$ space.

$$\left\langle \frac{\xi}{\eta} \right\rangle, \quad \xi = \xi_1,\ \xi_2, \cdots,\ \xi_m. \tag{2.11}$$

We choose the following special transformation such that

$$\mathbf{Y} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_1} \right\rangle \\ \left\langle \frac{\eta_2}{\eta_1} \right\rangle \\ \vdots \\ \left\langle \frac{\eta_m}{\eta_1} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_1} \right\rangle' \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.12}$$

Therefore, all the components of the old vectors $\left\langle \frac{\xi}{\eta_1} \right\rangle$ can be expressed by the following relation

$$\left\langle \frac{\xi}{\eta_1} \right\rangle = y(\eta_1, \xi) \left\langle \frac{\eta_1}{\eta_1} \right\rangle. \tag{2.13}$$

While other $m-1$ vectors $\left\langle \frac{\xi}{\eta_i} \right\rangle$, $i = 2,\ 3, \cdots,\ m$, also undergo the same transformation. We choose

$$\mathbf{Y} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_2} \right\rangle \\ \left\langle \frac{\eta_2}{\eta_2} \right\rangle \\ \vdots \\ \left\langle \frac{\eta_m}{\eta_2} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_2} \right\rangle' \\ \left\langle \frac{\eta_2}{\eta_2} \right\rangle' \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{Y} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_3} \right\rangle \\ \left\langle \frac{\eta_2}{\eta_3} \right\rangle \\ \left\langle \frac{\eta_3}{\eta_3} \right\rangle \\ \vdots \\ \left\langle \frac{\eta_m}{\eta_3} \right\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_3} \right\rangle' \\ \left\langle \frac{\eta_2}{\eta_3} \right\rangle' \\ \left\langle \frac{\eta_3}{\eta_3} \right\rangle' \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots\cdots,$$

$$\mathbf{Y}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_{m-1}}\right\rangle\\ \left\langle\frac{\eta_2}{\eta_{m-1}}\right\rangle\\ \vdots\\ \left\langle\frac{\eta_m}{\eta_{m-1}}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_{m-1}}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_{m-1}}\right\rangle'\\ \vdots\\ \left\langle\frac{\eta_{m-1}}{\eta_{m-1}}\right\rangle'\\ 0\end{pmatrix},\quad \mathbf{Y}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_m}\right\rangle\\ \left\langle\frac{\eta_2}{\eta_m}\right\rangle\\ \vdots\\ \left\langle\frac{\eta_m}{\eta_m}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_m}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_m}\right\rangle'\\ \vdots\\ \left\langle\frac{\eta_m}{\eta_m}\right\rangle'\end{pmatrix}, \tag{2.14}$$

where the zero components (the special RWCs) have clearly been written out after the transformation. Other components are non-zero in general.

In the following, we give an example for $m = 3$ case to show such transformation is always possible. In $m = 3$ case, one can make the following special rotation A_1 around the third axis such that

$$\begin{pmatrix}cos\theta_1 & -sin\theta_1 & \\ sin\theta_1 & cos\theta_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_1}\right\rangle\\ \left\langle\frac{\eta_2}{\eta_1}\right\rangle\\ \left\langle\frac{\eta_3}{\eta_1}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_1}\right\rangle'\\ 0\\ \left\langle\frac{\eta_3}{\eta_1}\right\rangle\end{pmatrix},\quad \begin{pmatrix}cos\theta_1 & -sin\theta_1 & \\ sin\theta_1 & cos\theta_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_2}\right\rangle\\ \left\langle\frac{\eta_2}{\eta_2}\right\rangle\\ \left\langle\frac{\eta_3}{\eta_2}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_2}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_2}\right\rangle'\\ \left\langle\frac{\eta_3}{\eta_2}\right\rangle'\end{pmatrix},$$

$$\begin{pmatrix}cos\theta_1 & -sin\theta_1 & \\ sin\theta_1 & cos\theta_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_3}\right\rangle\\ \left\langle\frac{\eta_2}{\eta_3}\right\rangle\\ \left\langle\frac{\eta_3}{\eta_3}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_3}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_3}\right\rangle'\\ \left\langle\frac{\eta_3}{\eta_3}\right\rangle'\end{pmatrix}, \tag{2.15}$$

Then, make another rotation A_2 around the second axis with

$$\begin{pmatrix}cos\theta_2 & & -sin\theta_2\\ 0 & 1 & 0\\ sin\theta_2 & & cos\theta_2\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_1}\right\rangle'\\ 0\\ \left\langle\frac{\eta_3}{\eta_1}\right\rangle\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_1}\right\rangle''\\ 0\\ 0\end{pmatrix},\quad \begin{pmatrix}cos\theta_2 & & -sin\theta_2\\ 0 & 1 & 0\\ sin\theta_2 & & cos\theta_2\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_2}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_2}\right\rangle'\\ \left\langle\frac{\eta_3}{\eta_2}\right\rangle'\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_2}\right\rangle''\\ \left\langle\frac{\eta_2}{\eta_2}\right\rangle''\\ \left\langle\frac{\eta_2}{\eta_2}\right\rangle''\end{pmatrix},$$

$$\begin{pmatrix}cos\theta_2 & & -sin\theta_2\\ 0 & 1 & 0\\ sin\theta_2 & & cos\theta_2\end{pmatrix}\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_3}\right\rangle'\\ \left\langle\frac{\eta_2}{\eta_3}\right\rangle'\\ \left\langle\frac{\eta_3}{\eta_3}\right\rangle'\end{pmatrix}=\begin{pmatrix}\left\langle\frac{\eta_1}{\eta_3}\right\rangle''\\ \left\langle\frac{\eta_2}{\eta_3}\right\rangle''\\ \left\langle\frac{\eta_2}{\eta_3}\right\rangle''\end{pmatrix}, \tag{2.16}$$

Finally, make a special rotation A_3 around the first axis with

$$\begin{pmatrix} 1 & & \\ & cos\theta_3 & -sin\theta_3 \\ & sin\theta_3 & cos\theta_3 \end{pmatrix} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_1} \right\rangle'' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_1} \right\rangle'' \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & cos\theta_3 & -sin\theta_3 \\ & sin\theta_3 & cos\theta_3 \end{pmatrix} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_2} \right\rangle'' \\ \left\langle \frac{\eta_2}{\eta_2} \right\rangle'' \\ \left\langle \frac{\eta_3}{\eta_2} \right\rangle'' \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_2} \right\rangle'' \\ \left\langle \frac{\eta_2}{\eta_2} \right\rangle''' \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & \\ & cos\theta_3 & -sin\theta_3 \\ & sin\theta_3 & cos\theta_3 \end{pmatrix} \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_3} \right\rangle'' \\ \left\langle \frac{\eta_2}{\eta_3} \right\rangle'' \\ \left\langle \frac{\eta_3}{\eta_3} \right\rangle'' \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\eta_1}{\eta_3} \right\rangle''' \\ \left\langle \frac{\eta_2}{\eta_3} \right\rangle''' \\ \left\langle \frac{\eta_2}{\eta_3} \right\rangle''' \end{pmatrix}, \tag{2.17}$$

Hence, the special orthogonal matrix **Y** is

$$Y = A_3 A_2 A_1. \tag{2.18}$$

It is obvious that the angle θ_1 is fixed by $\left\langle \frac{\eta_1}{\eta_1} \right\rangle$ and $\left\langle \frac{\eta_2}{\eta_1} \right\rangle$, θ_2 is fixed by $\left\langle \frac{\eta_1}{\eta_1} \right\rangle'$ and $\left\langle \frac{\eta_3}{\eta_1} \right\rangle$, and θ_3 is fixed by $\left\langle \frac{\eta_2}{\eta_2} \right\rangle''$ and $\left\langle \frac{\eta_3}{\eta_2} \right\rangle''$. One can extend this special rotation to m-dimemnsional space, and find that there are indeed unique solutions to m rotational angles if the final form (2.14) is selected. But we should point out that the choice of the rotation is not unique because there are infinite number of solutions to the RWCs or WCs with outer multiplicity, which are all within the $SO(m)$ transformations, and the number of elements in $SO(m)$ is infinite.

However, once the special rotation given in (2.14) is chosen, the resolution to the outer multiplicity is thus fixed. There will no longer be arbitrariness for the RWCs with multiplicity except an over all phase factor. We shall show that the over all phase factor can be chosen as

$$\left\langle \frac{\eta_i}{\eta_i} \right\rangle \geq 0 \tag{2.19}$$

for $i = 1,\ 2,\ \cdots,\ m$. Thus, the structure of the RWCs is determined completely.

One can easily prove that the WCs or RWCs after such transformation will not change the orthogonality conditions. i. e., the orthogonality conditions given by (2.6) are still valid after transformation for both $\mathcal{U}(4) \supset \mathcal{U}(3)$ and $SU(3) \supset U(2)$ cases. Firstly, all $\mathcal{U}(4) \supset \mathcal{U}(3)$ or $SU(3) \supset U(2)$ RWCs will undergo the same transformation **Y**. One can verify that the orthogonality conditions are still valid for them. $SU(3)$ WCs or $SU(3) \supset U(2)$ RWCs are a sub-set of those of $\mathcal{U}(4)$ or $\mathcal{U}(4) \supset \mathcal{U}(3)$ according to the Schur-Weyl duality relation. Hence, the same conclusion applies to WCs of $SU(3)$ or RWCs of $SU(3) \supset U(2)$ as well.

However, unlike Biedenharn's definition for WCs or RWCs , some symmetry properties of the WCs or the RWCs will be changed. For example, the new RWCs do not satisfy the symmetry property for $1 \leftrightarrow 2$ exchange of Biedenharn's due to the special orthogonal transformation $\mathbf{Y}$. We will discuss this later in Sec. IV.

Finally, we want to show what have been achieved after the special rotation $\mathbf{Y}$. If we arrange the RWCs of $\mathcal{U}(4) \supset \mathcal{U}(3)$ in terms of $m \times m$ matrix. The column is set by the outer multiplicity label $\tilde{\eta} = \eta_1,\ \eta_2, \cdots,\ \eta_m$, while the row is set by the label η in the irrep $[m_1,\ m_2 - \eta,\ m_3 - \mu_2 + \eta]$ for $\mathcal{U}(3)$, the RWCs will have the following structure

$$\left(\left\langle \frac{\tilde{\eta}}{\eta} \right\rangle\right) = \begin{pmatrix} \times & 0 & \cdots & \cdots & 0 \\ \times & \times & 0 & \cdots & 0 \\ \times & \times & \times & 0\cdots & 0 \\ \cdots & \cdots & & & \\ \times & \times & \cdots & \times & 0 \\ \times & \times & \cdots & \cdots & \times \end{pmatrix}. \tag{2.21}$$

i.e., it just reflects the lower triangular structure of the RWCs with multiplicity postulated by Braunschweig[35]. Here, however, we have show that it is indeed possible to choose such structure. Hecht in [22] argued that one can resolve the $SU(3)$ multiplicity problem simply by requiring a similar lower triangular structure. Le Blanc and Rowe also pointed out that such resolution will becomes *ipso facto* equivalent to a canonical labeling scheme[27]. Actually, One can also choose a upper triangular structure for these RWCs. Therefore, the structure of the RWCs is also not unique, which depends on what kind of special transformation is chosen.

Now, let us recall Biedenharn's definition for canonical resolution to the outer multiplicity problem. In [6], 'canonical' was used in the sense that there are no free choices involved in the solution of the multiplicity. This explanation seems incorrect because the choice involved in the resolution of the multiplicity is not unique. This situation is quite the same as the definition of canonical basis for $U(n)$. A canonical construction has to be explained as an equivalent class corresponding to the designation of a particular $U(1)$, out of the set of all equivalent $U(1)$ groups, at each stage of the decomposition. While the canonical resolution to the outer multiplicity has also to be explained as an equivalent class of solutions with respect to the outer multiplicity labels to be chosen, with which the WCs are mutually orthogonal.

The special transformation given by (2.14) makes it possible to evaluate all RWCs of $SU(3)$ in the canonical or noncanonical basis by using the recoupling approach. In the following, we will only discuss the RWCs for $SU(3)$ in its canonical basis. The RWCs for $SU(3)$ in its noncanonical basis will be discussed elsewhere.

III. A recoupling approach to the resolution of $SU(3) \supset U(2)$ RWCs

In this section, we want to demonstrate that the RWCs with multiplicity for $SU(3) \supset U(2)$, or $SU(n) \supset U(n-1)$ in general, can be evaluated by using the accumulated results on the subject together with the special choice of the transformation for a special set of RWCs, especially the analytical expression for RWCs of $U(n) \supset U(n-1)$ with one irrep symmetric[15] given by删šauskas et al. The $U(3) \supset U(2)$ RWCs of the same type was also obtained at the same period by Chacon et al[5] based on the canonical unit tensor operator method proposed by Biedenharn et al. The result of Chacon's and that of Ališauskas et al's are the same including the phase factor. In order to make every step clear, we will divided this section into several subsections.

(a) A recoupling approach

Using the analytical expressions of RWCs given by [15], one can construct the following expression for $\mathcal{U}(4)$, and $SU(3)$, respectively, with the help of the building-up principle.[36−37]

$$\sum_{\xi} U_{\xi}\begin{pmatrix} (\lambda_1\mu_1) & [\lambda_2+\mu_2\ 0] & [\bar{m}] \\ [\mu_2\ 0] & [m_1m_2m_3] & (\lambda_2\mu_2) \end{pmatrix} \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \middle| \begin{matrix} \xi\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle =$$

$$\sum \left\langle \begin{matrix} (\lambda_1\mu_1) & [\lambda_2+\mu_2\ 0] \\ \rho_1 & \rho_2' \end{matrix} \middle| \begin{matrix} [\bar{m}] \\ \bar{\rho} \end{matrix} \right\rangle \left\langle \begin{matrix} [\bar{m}] & [\mu_2\ 0] \\ \bar{\rho} & \rho_2'' \end{matrix} \middle| \begin{matrix} [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_2+\mu_2\ 0] & [\mu_2\ 0] \\ \rho_2' & \rho_2'' \end{matrix} \middle| \begin{matrix} (\lambda_2\mu_2) \\ \rho_2 \end{matrix} \right\rangle, \tag{3.1}$$

where U is unitary form of Racah coefficient for $SU(3)$ if $SU(3)$ case is considered, or is for $\mathcal{U}(4)$ if we discuss the $\mathcal{U}(4)$ coupling case, and the sum on the rhs. is over ρ_2', ρ_2'' and $\bar{\rho}$.

We will frequently use the following abbreviated notations.

$$U_{\xi}([\bar{m}]) \equiv U_{\xi}\begin{pmatrix} (\lambda_1\mu_1) & [\lambda_2+\mu_2\ 0] & [\bar{m}] \\ [\mu_2\ 0] & [m_1m_2m_3] & (\lambda_2\mu_2) \end{pmatrix}$$

for the Racah coefficient,

$$\left\langle \frac{\tilde{\eta}}{\eta} \right\rangle \equiv \left\langle \begin{matrix} [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2] \\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0] \end{matrix} \middle| \begin{matrix} \tilde{\eta}\ [m_1m_2m_3] \\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta] \end{matrix} \right\rangle$$

for a special set of $\mathcal{U}(4) \supset U(3)$ RWCs,

$$\left\langle \frac{\tilde{\eta}}{\rho_1\rho_2\rho} \right\rangle \equiv \left\langle \begin{matrix} [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2] \\ \rho_1 & \rho_2 \end{matrix} \middle| \begin{matrix} \tilde{\eta}\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle$$

for $SU(3)$ WCs, or $SU(3) \supset U(2)$ RWCs if ρ_1, ρ_2, and ρ are referred to as the corresponding $U(2)$ labels,

$$\begin{pmatrix} \eta' \\ \eta \end{pmatrix} \equiv \begin{pmatrix} [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ 0] & [m_1\ m_2-\eta'\ m_3-\mu_2+\eta'] \\ [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ \mu_2] & [m_1\ m_2\ m_3] \\ [\lambda_1+\mu_1] & [\lambda_2+\mu_2\ 0] & [m_1\ m_2-\eta\ m_3-\mu_2+\eta] \end{pmatrix}$$

for $(\mathcal{U}_4 \supset \mathcal{U}(3)) \star (\mathcal{U}(4) \supset \mathcal{U}(3))$ reduced coupling coefficient, and

$$\begin{pmatrix} \eta' \\ \rho_1\rho_2\rho \end{pmatrix} \equiv \begin{pmatrix} [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ 0] & [m_1\ m_2-\eta'\ m_3-\mu_2+\eta'] \\ [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ \mu_2] & [m_1\ m_2\ m_3] \\ \rho_1 & \rho_2 & \rho \end{pmatrix}$$

for $(\mathcal{U}(4) \supset \mathcal{U}(3)) \star SU(3)$ coupling coefficient.

In $\mathcal{U}(4) \supset \mathcal{U}(3)$ case, we only need to consider a simpler case

$$\sum_{\xi} U_{\xi}([\bar{m}]) \left\langle \begin{matrix} \xi \\ \eta \end{matrix} \right\rangle = G([\bar{m}], \eta), \tag{3.2}$$

where

$$G([\bar{m}], \eta) = \sum_{p_1 p_2 [\bar{m}']} \left\langle \begin{matrix} [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ 0] \\ [\lambda_1+\mu_1\ \mu_1] & p_1 \\ [\lambda_1+\mu_1\ \mu_1] & 0 \end{matrix} \middle| \begin{matrix} [\bar{m}] \\ [\bar{m}'] \\ [\lambda_1+\mu_1\ \mu_1] \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} [\bar{m}] & [\mu_2\ 0] \\ [\bar{m}'] & p_2 \\ [\lambda_1+\mu_1\ \mu_1] & 0 \end{matrix} \middle| \begin{matrix} [m_1 m_2 m_3] \\ [m_1\ m_2-\eta\ m_3-\mu_2+\eta] \\ [\lambda_1+\mu_1\ \mu_1] \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_2\ 0] & [\mu_2\ 0] \\ p_1 & p_2 \end{matrix} \middle| \begin{matrix} [\lambda_2+\mu_2\ \mu_2] \\ [\lambda_2+\mu_2\ 0] \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} [\lambda_1+\mu_1\ \mu_1] & [\lambda_2+\mu_2\ 0] \\ [\lambda_1+\mu_1\ \mu_1] & 0 \end{matrix} \middle| \begin{matrix} [m_1\ m_2-\eta\ m_3-\mu_2+\eta] \\ [\lambda_1+\mu_1\ \mu_1] \end{matrix} \right\rangle^{-1}. \tag{3.3}$$

While in $SU(3)$ case, the following expression is of importance

$$\sum_{\xi} U_{\xi}([\bar{m}]) \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \middle| \begin{matrix} \xi\ [m_1\ m_2\ m_3] \\ \rho \end{matrix} \right\rangle = G([\bar{m}], \rho_1\rho_2\rho), \tag{3.4}$$

where

$$\left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \middle| \begin{matrix} \xi\ [m_1\ m_2\ m_3] \\ \rho \end{matrix} \right\rangle$$

is the WCs of $SU(3)$, and $G([\bar{m}], \rho_1\rho_2\rho)$ is the expression given by the rhs. of (3.1).

Using these G polynomials, one can easily construct the following coupling coefficients.

$$\begin{pmatrix} \eta' \\ \eta \end{pmatrix} = \sum_{[\bar{m}]} G([\bar{m}],\eta)G([\bar{m}],\eta'), \tag{3.5}$$

which is a symmetric function with respect to the labels $\eta,\ \eta'$. What (3.5) gives is nothing but reduced coupling coefficient of $\mathcal{U}(4)\star\mathcal{U}(4)$

$$\left\langle \begin{pmatrix} [\lambda_1+\mu_1\ \mu_1] \\ [\lambda_1+\mu_1\ \mu_1] \\ [\lambda_1+\mu_1\ \mu_1] \end{pmatrix} ; \begin{pmatrix} [\lambda_2+\mu_2\ 0] \\ [\lambda_2+\mu_2\ \mu_2] \\ [\lambda_2+\mu_2\ 0] \end{pmatrix} \middle| \begin{pmatrix} [m_1\ m_2-\eta'\ m_3-\mu_2+\eta'] \\ [m_1 m_2 m_3] \\ [m_1\ m_2-\eta\ m_3-\mu_2+\eta] \end{pmatrix} \right\rangle . \tag{3.6}$$

Similarly, we have

$$\begin{pmatrix} \eta \\ \rho_1\rho_2\rho \end{pmatrix} = \sum_{[\bar{m}]} G([\bar{m}],\eta)G([\bar{m}],\rho_1\rho_2\rho), \tag{3.7}$$

which is the following $\mathcal{U}(4)\star SU(3)$ coupling coefficient

$$\left\langle \begin{pmatrix} [\lambda_1+\mu_1\ \mu_1] \\ [\lambda_1+\mu_1\ \mu_1] \\ \rho_1 \end{pmatrix} ; \begin{pmatrix} [\lambda_2+\mu_2\ 0] \\ [\lambda_2+\mu_2\ \mu_2] \\ \rho_2 \end{pmatrix} \middle| \begin{pmatrix} [m_1\ m_2-\eta\ m_3-\mu_2+\eta] \\ [m_1 m_2 m_3] \\ \rho \end{pmatrix} \right\rangle . \tag{3.6}$$

As pointed out by Biedenharn et al[2], there are close relations between RWCs of $U(n)$ and the coupling coefficients of $U(n)\star U(n)$. They have also proved that such coupling coefficients can be expressed in terms of the product of two $U(n)$ WCs with summation over the outer multiplicity labels. One can directly verify by using the recoupling technique that

$$\begin{pmatrix} \eta' \\ \eta \end{pmatrix} = \sum_{[\bar{m}]} G([\bar{m}],\eta)G([\bar{m}],\eta') = \sum_{\xi} \left\langle \frac{\xi}{\eta} \right\rangle \left\langle \frac{\xi}{\eta'} \right\rangle \tag{3.9}$$

and

$$\begin{pmatrix} \eta \\ \rho_1\rho_2\rho \end{pmatrix} = \sum_{[\bar{m}]} G([\bar{m}],\eta)G([\bar{m}],\rho_1\rho_2\rho) = \sum_{\xi} \left\langle \frac{\xi}{\eta} \right\rangle \left\langle \frac{\xi}{\rho_1\rho_2\rho_3} \right\rangle \tag{3.10}$$

(b) Explicit expressions of G polynomials

Using the recoupling technique, we can evaluate explicit expressions for the G polynomials defined in the above subsection with the known multiplicity-free RWCs of $U(n)\supset U(n-1)$ given by Ališauskas et al.

$$G(k_1k_2,\eta) = \Bigg[\frac{(m_3-\mu_2+\eta)!^2(m_2-\eta+1)!^2(m_1+2)!\lambda_1!(\lambda-\mu_1-2k_1-k_2+2)(\mu_1+k_1-k_2+1)}{(\lambda_2+\mu_2)!^2(\mu_1+\mu_2-m_3-\eta)!(\lambda_2+\mu_2+1)} \times$$

$$\frac{(\lambda-k_1-k_2-\lambda_1-\mu_1)!(\lambda-k_1-k_2-\mu_1+1)!(m_1-m_2+1)!(m_1-m_3+2)!(m_2-m_3+1)}{(\lambda_1-k_1)!(\lambda_1+\mu_1-k_2+1)!(\mu_1-k_2)!(\mu_2-\eta)!(\lambda_1+\mu_1-m_2+\eta)!(\lambda_1+\mu_1+\mu_2-m_3-\eta+1)!} \times$$

$$\frac{(\lambda-k_1-k_2-m_2)!(\lambda-k_1-k_{[2}-m_3+1)!(\lambda-k_1-k_2+2)!(\mu_1+k_1-m_3)!(\mu_1+k_1+1)!k_1!k_2!}{(m_1-\lambda_1+k_1+k_2)!(m_1-\mu_1-k_1+1)!(m_1-k_2+2)!(m_2-\mu_1-k_1)!(m_2+1)!m_3!(m_3-k_2)!} \times$$

$$\frac{(\lambda_2+1)!\mu_2!(m_2-m_3-\eta)!(m_1-\lambda_1-\mu_1)!(m_1-\mu_1+1)!(m_2-\mu_1-\eta)!}{(m_1-m_2+\eta+1)!(m_1-m_3+\mu_2-\eta+2)!\eta!(m_2-m_3+\mu_2-\eta+1)!}\Bigg]^{1/2} \times$$

$$\sum_{p=0}^{\mu_2} \sum_{m_{23}=\min(m_{23})}^{\max(m_{23})} \sum_{m_{33}=\min(m_{33})}^{\max(m_{33})} (-)^p \frac{p!(\lambda_1-m_{23})!(\lambda_1+\mu_1-m_{33}+1)!(\mu_1-m_{33})!(\lambda_2+\mu_2-p)!}{(\lambda-p-m_{23}-m_{33}-\lambda_1-\mu_1)!(\lambda-p-m_{23}-m_{33}-\mu_1+1)!m_{23}!} \times$$

$$\frac{(\lambda-p-2m_{23}-m_{33}-\mu_1+1)(\lambda-p-m_{23}-2m_{33}+2)(m_{23}-m_{33}+\mu_1+1)}{(\lambda-p-m_{23}-m_{33}+2)!(\mu_1+m_{23}+1)!m_{33}!} \times$$

$$F_3\left(\begin{matrix}[\lambda-k_1-k_2,\mu_1+k_1,k_2] \\ [\lambda-p-m_{23}-m_{33},m_{23}+\mu_1,m_{33}]\end{matrix};\ \begin{matrix}[m_1m_2m_3] \\ [m_1\ m_2-\eta\ m_3-\mu_2+\eta]\end{matrix}\right), \tag{3.11}$$

where

$$\lambda = \lambda_1+\mu_1+\lambda_2+\mu_2, \qquad (3.12a)$$

$$[\bar{m}] = [\lambda-k_1-k_2,\mu_1+k_1,k_2], \qquad (3.12b)$$

the boundary conditions for m_{23}, and m_{33} can be obtained by using the Littlewood rule for the Kronecker products involved and the betweenness conditions for the decomposition $U(n)\downarrow U(n-1)$, from which we get

$$\max(m_{23}) = \min(\lambda_1,k_1,\lambda_2+\mu_2-p,m_2-\eta-\mu_1),$$

$$\min(m_{23}) = \max(0,m_2-\eta-\mu_1-p,m_3-\mu_1-\mu_2+\eta), \qquad (13.12c)$$

for fixed p, while

$$\max(m_{33}) = \min(\lambda - p - m_{23} - m_2 + \eta, m_3 - \mu_2 + \eta, \mu_1,$$

$$\lambda_2 + \mu_2 - p - m_{23}, \lambda - p - m_{23} - \mu_1 - k_1, \lambda - p - m_{23} - \lambda_1 - \mu_1),$$

$$\min(m_{33}) = \max(0, m_2 + m_3 + \mu_1 - \mu_2 - p + m_{23}, k_1 + k_2 - p - m_{23}, k_2) \qquad (3.12d)$$

for fixed p and m_{23}, and

$$F_3 \left(\begin{matrix} [h_1 h_2 h_3] \\ [q_1 q_2 q_3] \end{matrix} ; \begin{matrix} [m_1 m_2 m_3] \\ [n_1 n_2 n_3] \end{matrix} \right) = \sum_{xyz} (-)^{x+y+z-q_1-q_2-q_3} \times$$

$$\frac{(x-y+1)(x-z+2)(y-z+1)(x-h_3+1)!}{(h_2-y)!(n_2-z+1)!(h_3-z)!(h_1-z+2)!(h_1-y+1)!} \times$$

$$\frac{(q_1-y)!(q_1-z+1)!(q_2-z)!(x-n_2)!(x-n_3+1)!(y-n_3)!(y-h_3)!}{(x-q_1)!(x-q_2+1)!(x-q_3+2)!(y-q_2)!(y-q_3+1)!(z-q_3)!(n_1-x)!(n_1-y+1)!(n_1-z+2)!} \times$$

$$\frac{(m_1-x)!(m_1-y+1)!(m_1-z+2)!(m_2-y)!(m_2-z+1)!(m_3-z)!(x-h_2)!}{(n_2-y)!(n_2-z+1)!(n_3-z)!(x-m_2)!(x-m_3+1)!(y-m_3)!(h_1-x)!}. \qquad (3.13)$$

Similarly, we have

$$G(k_1 k_2; [m_{12} m_{22}] \; [m'_{12} m'_{22}][m''_{12} m''_{22}]) = \sum_{\xi} U_{\xi}(k_1 k_2) \left\langle \begin{matrix} (\lambda_1 \mu_1) \\ [m_{12} m_{22}] \end{matrix} \quad \begin{matrix} (\lambda_2 \mu_2) \\ [m'_{12} m'_{22}] \end{matrix} \middle| \begin{matrix} \xi \; [m_1 m_2 m_3] \\ [m''_{12} m''_{22}] \end{matrix} \right\rangle =$$

$$\left[\frac{(\lambda_2+1)!(\mu_2-m'_{22})!(m_1-m_2+1)(m_1-m_3+2)(m_2-m_3+1)(m''_{12}-m_2)!}{(\lambda_2+\mu_2-m'_{12})!(\lambda_2+\mu_2-m'_{22}+1)!(m'_{12}-\mu_2)!(m_2-k_2+1)!(m_1-m''_{12})!(m_2-m''_{22})!} \times \right.$$

$$\frac{(\lambda-k_1-k_2-m_2)!(\mu_1+k_1-m_3)!(\lambda-k_1-k_2-m_3+1)!(m''_{22}-m_3)!(m''_{12}-m_3+1)!}{(m_1-\lambda+k_1+k_2)!(m_2-\mu_1-k_1)!(m_3-k_2)!(m_1-\mu_1-k_1+1)!(m_1-k_2+2)!} \times$$

$$\left. \frac{(\mu_1-k_2)!(\lambda_1+\mu_1-k_2+1)!(\lambda_1+\mu_1-m_{12})!(\mu_1-m_{22})!(\lambda_1+\mu_1-m_{22}+1)!}{(\mu_1+k_1+1)!(m_{12}-\mu_1)!m_{22}!(m_{12}+1)!(m_1-m''_{22}+1)!} \right]^{1/2} \sum_{p=0}^{\mu_2} \sum_{q=\min(q)}^{\max(q)} \times$$

$$(-1)^{p-m'_{22}}(\lambda_2+\mu_2+p-m'_{12}-m'_{22})!((m'_{12}-p)!(m''_{12}+m''_{22}-p-q-m_{12})!(q-m_{22})!)^{1/2}\times$$

$$\left(\frac{(m''_{12}+m''_{22}-p-q-m_{22}+1)!(m''_{12}+m''_{22}-p-2q+1)(p+q-m''_{22})!(m''_{22}-q+1)!}{(p-m'_{22})!(m''_{12}-p-q)!(m''_{12}+m''_{22}-p-q-\mu_1-k_1)!(m_{12}-q)!}\right)^{1/2}\times$$

$$U\begin{pmatrix}[m_{12}m_{22}] & [m'_{12}+m'_{22}-p\ 0] & [m''_{12}+m''_{22}-p-q,q] \\ [p0] & [m''_{12}\ m''_{22}] & [m'_{12}\ m'_{22}]\end{pmatrix}\times$$

$$F_2\begin{pmatrix}[\lambda-k_1-k_2,\mu_1+k_1,k_2] & [m_1m_2m_3] \\ [m''_{12}+m''_{22}-p-q,q] & [m''_{12}\ m''_{22}]\end{pmatrix}\times$$

$$F_2\begin{pmatrix}[\lambda_1+\mu_1\ \mu_1] & [\lambda-k_1-k_2,\mu_1+k_1,k_2] \\ [m_{12}+m_{22}\ 0] & [m''_{12}+m''_{22}-p-q,q]\end{pmatrix}, \qquad (3.14)$$

where

$$\max(q)=\min(\mu_1+k_1,m''_{12}+m''_{22}-p-\mu_1-k_1),$$

$$\min(q)=\max(k_2,m''_{12}+m''_{22}-p-\lambda+k_1+k_2), \qquad (3.15a)$$

U is $SU(2)$ Racah coefficient in unitary form, and

$$F_2\begin{pmatrix}[h_1h_2h_3] & [m_1m_2m_3] \\ [q_1q_2] & [n_1n_2]\end{pmatrix}=\sum_{xy}(-)^{x+y-q_1-q_2}\ (x-y+1)\times$$

$$\frac{(x-n_2)!(m_1-x)!(x-h_2)!(x-h_3+1)!}{(x-q_1)!(x-q_2+1)!(n_1-x)!(x-m_2)!(x-m_3+1)!(h_1-x)!}\times$$

$$\frac{(q_1-y)!(m_1-y+1)!(m_2-y)!(y-h_3)!}{(y-q_2)!(n_2-y)!(n_1-y+1)!(y-m_3)!(h_1-y+1)!(h_2-y)!}. \qquad (3.15b)$$

Using these expressions, (3.9) and (3.10) can be expressed explicitly as

$$\begin{pmatrix}\eta' \\ \eta\end{pmatrix}=\sum_{k_1=\min(k_1)}^{\max(k_1)}\sum_{k_2=\min(k_2)}^{\max(k_2)}G(k_1,k_2,\eta)G(k_1,k_2,\eta'), \qquad (3.16a)$$

$$\begin{pmatrix}\eta \\ \rho_1\rho_2\rho\end{pmatrix}=\sum_{k_1=\min(k_1)}^{\max(k_1)}\sum_{k_2=\min(k_2)}^{\max(k_2)}G(k_1,k_2,\eta)G(k_1k_2,\rho_1\rho_2\rho), \qquad (3.16b)$$

where

$$\max(k_1) = \min(m_2 - \mu_1, \lambda_1, \lambda_2 + \mu_2),$$

$$\min(k_1) = \max(m_3 - \mu_1, m_2 - \mu_1 - \mu_2, 0),$$

$$\max(k_2) = \min(\mu_1, \lambda_2 + \mu_2 - k_1, \lambda_2 + \mu_2 - k_1 + \lambda_1 + \mu_1 - m_2),$$

$$\min(k_2) = \max(m_2 + m_3 - \mu_1 - \mu_2 - k_1). \tag{3.16c}$$

(D) A recursive procedure for evaluation of all RWCs of $SU(3) \supset U(2)$

We can now use the G polynomials to construct $\mathcal{U}(4) \star \mathcal{U}(4)$ as well as $\mathcal{U}(4) \star SU(3)$ reduced coupling coefficients as given by (3.16a) and (3.16b), which can be used to evaluate all RWCs of $SU(3) \supset U(2)$ after the special transformation given in Sec. II. It can be proved that

$$\sum_{\xi} \left\langle \frac{\xi}{\eta} \right\rangle \left\langle \frac{\xi}{\eta} \right\rangle \neq 0. \tag{3.17}$$

for any η. Firstly, if there exists an η such that

$$\sum_{\xi} \left\langle \frac{\xi}{\eta} \right\rangle \left\langle \frac{\xi}{\eta} \right\rangle = 0, \tag{3.18}$$

which requires that

$$\left\langle \frac{\xi}{\eta} \right\rangle = 0 \tag{3.19}$$

for $\xi = \eta_1,\ \eta_2, \cdots,\ \eta_m$ because the condition (3.18) can be written as $\vec{\eta} \cdot \vec{\eta} = 0$ in the $\mathbf{R}^m$ space, which implies its components are all zero. i. e.,

$$\left\langle \begin{matrix} [\lambda_1 + \mu_1,\ \mu_1] & [\lambda_2 + \mu_2,\ \mu_2] \\ [\lambda_1 + \mu_1,\ \mu_1] & [\lambda_2 + \mu_2,\ 0] \end{matrix} \right| \left. \begin{matrix} \xi\ [m_1 m_2 m_3] \\ [m_1,\ m_2 - \eta,\ m_3 - \mu_2 + \eta] \end{matrix} \right\rangle = 0 \tag{3.20}$$

for any multiplicity label ξ and fixed η. However, using the building up principle, one can deduce that (3.20) is valid if and only if

$$G([\bar{m}],\eta)=0 \tag{3.21}$$

where the expression of $G([\bar{m}],\eta)$ is given by (3.3). In this case, we can always use unitarity condition for $U(n)\supset U(n-1)$ RWCs to prove that (3.21) is satisfied only when one type of the WCs or RWCs involved are zero. However, one can verify from the explicit expressions for these WCs or RWCs given by Ališauskas et al[15] that (3.21) is satisfied only when the irreps involved in the coupling do not satisfy Littlewood rule for the Kronecker products involved or betweenness conditions for the decomposition. But these are all trivial cases and will not be considered. Hence, (3.17), generally, is valid for all non-trivial cases.

Hence, after the special transformation given in Sec. II, the rhs. of (3.9), generally, has k non-zero terms in the summation when a smaller η which is a label in $[m_1\; m_2-\eta\; m_3-\mu_2+\eta]$ equals to η_k, namely

$$\begin{pmatrix}\eta_k\\ \eta_l\end{pmatrix}=\sum_{\xi=\eta_1}^{\eta_k}\left\langle\frac{\xi}{\eta_k}\right\rangle\left\langle\frac{\xi}{\eta_l}\right\rangle,\quad \text{for}\;\; k\le l,\;\; k\le m. \tag{3.22}$$

Using (3.22), we have

$$\left\langle\frac{\eta_k}{\eta_k}\right\rangle=+\left[\begin{pmatrix}\eta_k\\ \eta_k\end{pmatrix}-\sum_{\xi=\eta_1}^{\eta_{k-1}}\left\langle\frac{\xi}{\eta_k}\right\rangle^2\right]^{1/2}, \tag{3.23}$$

where the sign of (3.23) is always chosen positive for any k. The over all phase is thus fixed. Then,

$$\left\langle\frac{\eta_k}{\eta_k}\right\rangle\left\langle\frac{\eta_k}{\eta_l}\right\rangle=\begin{pmatrix}\eta_k\\ \eta_l\end{pmatrix}-\sum_{\xi=\eta_1}^{\eta_{k-1}}\left\langle\frac{\xi}{\eta_k}\right\rangle\left\langle\frac{\xi}{\eta_l}\right\rangle \tag{3.24}$$

for $l>k$. It can be proved that (3.23) can not be zero. Firstly, $\left\langle\frac{\eta_k}{\eta_k}\right\rangle^2$ is square of the component of the vector $\vec{\eta_k}$ in $\mathbf{R}^m$. Therefore,

$$\left\langle\frac{\eta_k}{\eta_k}\right\rangle^2\ge 0,\quad \text{for}\;\; k\le m. \tag{3.25a}$$

Secondly, if (3.23) is zero, (3.24) should also be zero for any l values. This only occurs when the multiplicity equals to $k-1$. However, we assumed that the multiplicity $m\ge k$. Hence, (3.24) can not be zero for $k\le m$. Hence,

$$\left\langle \frac{\eta_k}{\eta_k} \right\rangle^2 > 0, \quad \text{for} \quad k \leq m. \tag{3.25b}$$

Thus, (3.23) and (3.24) allow us to calculate all the special RWCs of $\mathcal{U}(4) \supset \mathcal{U}(3)$ recursively,

$$\left\langle \frac{\eta_1}{\eta_1} \right\rangle = \begin{pmatrix} \eta_1 \\ \eta_1 \end{pmatrix}^{1/2},$$

$$\left\langle \frac{\eta_1}{\eta} \right\rangle = \begin{pmatrix} \eta_1 \\ \eta_1 \end{pmatrix}^{-1/2} \begin{pmatrix} \eta_1 \\ \eta \end{pmatrix}, \tag{3.26a}$$

$$\left\langle \frac{\eta_2}{\eta} \right\rangle = \left[\begin{pmatrix} \eta_2 \\ \eta \end{pmatrix} - \begin{pmatrix} \eta_1 \\ \eta_1 \end{pmatrix}^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta \end{pmatrix} \right] / \left\langle \frac{\eta_2}{\eta_2} \right\rangle, \tag{3.26b}$$

where

$$\left\langle \frac{\eta_2}{\eta_2} \right\rangle = \left[\begin{pmatrix} \eta_2 \\ \eta_2 \end{pmatrix} - \begin{pmatrix} \eta_1 \\ \eta_1 \end{pmatrix}^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^2 \right]^{1/2}. \tag{3.26c}$$

$$\cdots \quad \cdots$$

Once $\left\langle \frac{\eta_{k-1}}{\eta} \right\rangle$ for any η are known from the $k-1$'th step, $\left\langle \frac{\eta_k}{\eta} \right\rangle$ can be obtained by using (3.23) and (3.24). Thus, one obtains all the special RWCs of $\mathcal{U}(4) \supset \mathcal{U}(3)$, which are important in determining the RWCs of $SU(3) \supset U(2)$.

Using (3.10) and all known special RWCs of $\mathcal{U}(4) \supset \mathcal{U}(3)$, we can obtain $SU(3)$ WCs or $SU(3) \supset U(2)$ RWCs

$$\left\langle \frac{\eta_1}{\rho_1 \rho_2 \rho} \right\rangle = \left\langle \frac{\eta_1}{\eta_1} \right\rangle^{-1} \begin{pmatrix} \eta_1 \\ \rho_1 \rho_2 \rho \end{pmatrix},$$

$$\left\langle \frac{\eta_2}{\rho_1 \rho_2 \rho} \right\rangle = \left[\begin{pmatrix} \eta_2 \\ \rho_1 \rho_2 \rho \end{pmatrix} - \begin{pmatrix} \eta_1 \\ \eta_1 \end{pmatrix}^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \rho_1 \rho_2 \rho \end{pmatrix} \right] / \left\langle \frac{\eta_2}{\eta_2} \right\rangle,$$

$$\cdots \quad \cdots$$

$$\left\langle \frac{\eta_k}{\rho_1 \rho_2 \rho} \right\rangle = \left[\begin{pmatrix} \eta_k \\ \rho_1 \rho_2 \rho \end{pmatrix} - \sum_{\eta' = \eta_1}^{\eta_{k-1}} \left\langle \frac{\eta'}{\rho_1 \rho_2 \rho} \right\rangle \left\langle \frac{\eta'}{\eta_k} \right\rangle \right] / \left\langle \frac{\eta_k}{\eta_k} \right\rangle, \quad \text{for} \quad k \leq m. \tag{3.28}$$

It should be noted that (3.28) determines not only $SU(3)$ WCs in the canonical basis, and $SU(3) \supset U(2)$ RWCs, but also $SU(4) \supset U(3)$ RWCs for the same coupling.

(E) Some algebraic expressions

In this subsection, we will work out some algebraic expressions for $SU(3) \supset U(2)$ RWCs and related Racah coefficients of $SU(3)$. Starting from $\eta = \eta_1$, and using (3.23), and (3.24), we can obtain

$$\left\langle \frac{\eta_1}{\eta_1} \right\rangle = \sum_{k_1 k_2} G^2(k_1, k_2, \eta_1),$$

$$\left\langle \frac{\eta_1}{\eta_{1+i}} \right\rangle = \left\langle \frac{\eta_1}{\eta_1} \right\rangle^{-1} \sum_{k_1 k_2} G(k_1 k_2, \eta_{i+1}) G(k_1 k_2, \eta_1),$$

$$\left\langle \frac{\eta_2}{\eta_2} \right\rangle^2 = \left\langle \frac{\eta_1}{\eta_1} \right\rangle^{-2} \sum_{i=0}^{1} \sum_{k_1 k_2 k_1' k_2'} (-)^{[i/2]} G(k_1 k_2, \eta_2) G(k_1 k_2, \eta_{2-i}) G(k_1' k_2', \eta_{1+i}) G(k_1' k_2', \eta_1),$$

$$\cdots \quad \cdots$$

$$\left\langle \frac{\eta_3}{\eta_3} \right\rangle^2 = \left\langle \frac{\eta_1}{\eta_1} \right\rangle^{-5} \left\langle \frac{\eta_2}{\eta_2} \right\rangle^{-1} \sum_{i=0}^{2}{}' \sum_{j=0}^{1}{}' \sum_{k=0}^{1}{}' \sum_{k_l, p_l, q_l, n_l} (-)^{[i/2]+[j/2]+[k/2]} G(k_1 k_2, \eta_3) G(k_1 k_2, \eta_{3-i-j}) \times$$

$$G(p_1 p_2, \eta_{1+i}) G(p_1 p_2, \eta_1) G(q_1 q_2, \eta_{2+j}) G(q_1 q_2, \eta_{2-k}) G(n_1 n_2, \eta_{1+k}) G(n_1 n_2, \eta_1), \tag{3.29}$$

where the primes on the summation signs indicate that the sums should be restricted by

$$i + j + k \leq 2, \tag{3.30}$$

and

$$[x] = \begin{cases} x & \text{if } x \text{ is an integer}, \\ 2x & \text{if } x \text{ is an half} - \text{integer}. \end{cases} \tag{3.31}$$

The expression will become more complicated with increasing of k in η_k. Once $\left\langle \frac{\eta_k}{\eta_k} \right\rangle$ for $k = 1,\ 2,\ \cdots,\ m$ are known, one can similarly get the WCs or RWCs of $SU(3) \supset U(2)$ with multiplicity.

$$\left\langle \frac{\eta_k}{\rho_1 \rho_2 \rho} \right\rangle = \left(\prod_{t=0}^{k-1} \left\langle \frac{\eta_t}{\eta_t} \right\rangle^{-1} \sum_{i_t=0}^{k-t}{}' \right) \sum_{k_1 k_2} G(k_1, k_2, \rho_1 \rho_2 \rho) G(k_1, k_2, \eta_{k - \sum_{j=1}^{k-1} i_j}) \times$$

$$(-)^{\sum_{p=1}^{k-1}[i_p/2]}\prod_{q=1}^{k-1}\left\langle\frac{\eta_q}{\eta_{q+i_p}}\right\rangle, \tag{3.32}$$

where the prime on the summation sum indicates that the condition

$$\sum_{t=0}^{k-1} i_t \le k-1 \tag{3.33}$$

should be satisfied. Similarly, we define

$$V(\frac{\eta_i}{\eta_j}) = \left\langle\frac{\eta_i}{\eta_i}\right\rangle^{-1}\left\langle\frac{\eta_i}{\eta_j}\right\rangle. \tag{3.34a}$$

The Racah coefficient of $SU(3)$ can be expressed as

$$R_{\eta_1}([\bar{m}])\left\langle\frac{\eta_1}{\eta_1}\right\rangle = G([\bar{m}],\eta_1),$$

$$R_{\eta_2}([\bar{m}])\left\langle\frac{\eta_2}{\eta_2}\right\rangle = G([\bar{m}],\eta_2) - V(\frac{\eta_1}{\eta_2})G([\bar{m}],\eta_1),$$

$$\cdots \quad \cdots$$

$$R_{\eta_j}([\bar{m}])\left\langle\frac{\eta_j}{\eta_j}\right\rangle = G([\bar{m}],\eta_j) + \sum_{i_1=1}^{j-1}\sum_{k=1}^{j-1}(-)^k\sum_{i_1\le i_2\le\cdots\le i_k<j} G([\bar{m}],\eta_{i_1})\times$$
$$V(\frac{\eta_{i_k}}{\eta_j})\prod_{l=1}^{k-1}V(\frac{\eta_{i_l}}{\eta_{i_{l+1}}}). \tag{3.34b}$$

IV. Some features of the new RWCs for $SU(3)\supset U(2)$

In this section, we will discuss symmetry properties of WCs of $SU(3)$ in the canonical basis. In Sec. III, we have chosen the absolute phase with

$$\left\langle\begin{matrix}[\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ \mu_2] \\ [\lambda_1+\mu_1,\ \mu_1] & [\lambda_2+\mu_2,\ 0]\end{matrix}\middle|\begin{matrix}\eta\ [m_1m_2m_3] \\ [m_1,\ m_2-\eta,\ m_3-\mu_2+\eta]\end{matrix}\right\rangle \ge 0. \tag{4.1}$$

Then, the relative phase is completely determined by the recursion relations (3.23) and (3.24). When $\mu_2=0$, one can check that our phase choice is consistent with that of [15], which is the same as that of [5] with the same phase structure defined by Biedenharn et al[10] as it should be because we use multiplicity-free RWCs of [5,15]. Hence, in order to discuss symmetry properties of the SU(3) WCs, we can expand the WCs obtained in this paper in terms of those defined in [10]

$$\left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \right| \left. \begin{matrix} \eta\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle = \sum_{\xi} y(\xi,\eta) \left[\begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \right| \left. \begin{matrix} \xi\ [m_1m_2m_3] \\ \rho \end{matrix} \right], \tag{4.2}$$

where the coupling coefficients on the rhs. in square brackets are the WCs defined in [10], and $y(\xi,\eta)$ is the corresponding special transformation matrix elements. Using the orthogonality relations for $y(\xi,\eta)$'s, and symmetry properties of the WCs discussed in [10], we can prove that

$$\left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \right| \left. \begin{matrix} \eta'\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle = \sum_{\eta}\sum_{\xi} y(\xi,\eta)y(\xi,\eta')(-)^{\phi(\xi)} \left\langle \begin{matrix} (\lambda_2\mu_2) & (\lambda_1\mu_1) \\ \rho_2 & \rho_1 \end{matrix} \right| \left. \begin{matrix} \eta\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle, \tag{4.2}$$

where the phase factor $(-)^{\phi(\xi)}$ comes from

$$\left[\begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ \rho_1 & \rho_2 \end{matrix} \right| \left. \begin{matrix} \xi\ [m_1m_2m_3] \\ \rho \end{matrix} \right] = (-)^{\phi(\xi)} \left[\begin{matrix} (\lambda_2\mu_2) & (\lambda_1\mu_1) \\ \rho_2 & \rho_1 \end{matrix} \right| \left. \begin{matrix} \xi\ [m_1m_2m_3] \\ \rho \end{matrix} \right] \tag{4.3}$$

given in [10],

$$\phi(\xi) = \Gamma_{12} - \Gamma_{11} + \mu_1 + \mu_2 - m_2 + m_1. \tag{4.4}$$

The multiplicity label ξ in this case can be regarded as

$$\xi = \Gamma_{12} - \Gamma_{11}, \tag{4.5}$$

where Γ_{ij} are multiplicity labels from the upper pattern of $SU(3)$ defined by Biedenharn et al.

It is obvious that

$$Z_{\eta\eta'} = Z \begin{pmatrix} (\lambda_1\mu_1) & (0) & (\lambda_1\mu_1)\ \eta' \\ (\lambda_2\mu_2) & [m_1m_2m_3] & (\lambda_2\mu_2)\ \eta \end{pmatrix} = \sum_{\xi} y(\xi,\eta)y(\xi,\eta')(-)^{\phi(\xi)}, \tag{4.6}$$

where $Z_{\eta\eta'}$ is a special Z coefficient[38] defined by Millener, which transforms the coupling coefficients from the coupling $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$ to $(\lambda_2\mu_2)\times(\lambda_1\mu_1)$. (4.2) can only be simplified when the coupling is multiplicity-free. In this case, the transformation matrix Y is 1×1 with $y(\xi, \eta) = 1$ for fixed ξ and η, and

$$Z_{\eta\eta'} = \delta_{\eta\eta'}(-)^{\phi(\xi)}, \tag{4.7}$$

where ξ can be expressed in terms of $\lambda_1,\ \mu_1,\ \lambda_2,\ \mu_2$, and m_i with $i = 1,\ 2,\ 3$.

Similarly, we obtain the following symmetry properties for the WCs of $SU(3)$

$$\left\langle \begin{matrix} (\mu_1\lambda_1) \\ \tilde{\rho}_1 \end{matrix} \quad \begin{matrix} (\mu_2\lambda_2) \\ \tilde{\rho}_2 \end{matrix} \right| \left. \begin{matrix} \eta'\ [-m_3\ \ -m_2\ \ -m_1] \\ \tilde{\rho} \end{matrix} \right\rangle =$$

$$\sum_\eta Z_{\eta\eta'}(-)^{m_1-m_3} \left\langle \begin{matrix} (\lambda_1\mu_1) \\ \rho_1 \end{matrix} \quad \begin{matrix} (\lambda_2\mu_2) \\ \rho_2 \end{matrix} \right| \left. \begin{matrix} \eta\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle, \tag{4.8a}$$

where $\tilde{\rho}$ is the conjugation of ρ defined by

$$|\tilde{\rho} >\equiv |\tilde{m} >= \left| \begin{matrix} -m_{22}\ \ -m_{12} \\ -m_{11} \end{matrix} \right\rangle, \tag{4.8b}$$

and

$$\left\langle \begin{matrix} (\lambda_1\mu_1) \\ \rho_1 \end{matrix} \quad \begin{matrix} (\lambda_2\mu_2) \\ \rho_2 \end{matrix} \right| \left. \begin{matrix} \eta'\ [m_1m_2m_3] \\ \rho \end{matrix} \right\rangle = \left[\frac{\dim([m_1m_2m_3])}{\dim((\lambda_1\mu_1))} \right]^{1/2} (-)^{\phi_3-\phi_1} \times$$

$$\sum_\eta Z_{\eta\eta'} \left\langle \begin{matrix} [m_1m_2m_3] \\ \rho \end{matrix} \quad \begin{matrix} (\mu_2\lambda_2) \\ \tilde{\rho}_2 \end{matrix} \right| \left. \begin{matrix} \eta\ (\lambda_1\mu_1) \\ \rho_1 \end{matrix} \right\rangle, \tag{4.9}$$

where

$$\phi_3 = m''_{11} - m''_{12} - m''_{22} + m_1,$$

$$\phi_1 = m_{11} - m_{12} - m_{22} + \lambda_1 + \mu_1, \tag{4.10}$$

and

$$\rho = \left(\begin{matrix} [m''_{12}\ m''_{22}] \\ m''_{11} \end{matrix} \right), \quad \rho_1 = \left(\begin{matrix} [m_{12}\ m_{22}] \\ m_{11} \end{matrix} \right). \tag{4.11}$$

Now, we give some examples for the coupling $[21] \times [21] \downarrow [321]$, which show us main features of the new RWCs. We use the following notations.

$$< \frac{0}{0} >= \left\langle \begin{matrix} [21] & [21] \\ [21] & [20] \end{matrix} \right| \left. \begin{matrix} \eta = 0 \; [321] \\ [32] \end{matrix} \right\rangle, \quad < \frac{1}{0} >= \left\langle \begin{matrix} [21] & [21] \\ [21] & [20] \end{matrix} \right| \left. \begin{matrix} \eta = 1 \; [321] \\ [32] \end{matrix} \right\rangle$$

$$< \frac{0}{1} >= \left\langle \begin{matrix} [21] & [21] \\ [21] & [20] \end{matrix} \right| \left. \begin{matrix} \eta = 0 \; [321] \\ [311] \end{matrix} \right\rangle, \quad < \frac{1}{1} >= \left\langle \begin{matrix} [21] & [21] \\ [21] & [20] \end{matrix} \right| \left. \begin{matrix} \eta = 1 \; [321] \\ [311] \end{matrix} \right\rangle. \tag{4.12}$$

Using the recursion relations given by (3.23), and (3.24), we can easily get

$$\begin{pmatrix} < \frac{0}{0} > & < \frac{1}{0} > \\ < \frac{0}{1} > & < \frac{1}{1} > \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{7}{10}} & 0 \\ -\sqrt{\frac{1}{42}} & \sqrt{\frac{10}{21}} \end{pmatrix}. \tag{4.13}$$

Then, Tables I and II can be worked out by using (3.28). In these tables, the upper parts are taken from de Swart [39], while the lower parts are derived by the new method. These two types of RWCs with multiplicity two can be transformed with each other by a two dimensional rotation. Furthermore, one can check that the new RWCs still satisfy the orthogonality conditions for $SU(3)$ RWCs given by (2.6).

V. Reduced auxiliary Wigner coefficients for $SU(3) \supset U(2)$

In [11], they used also $U(4)$ complementary group to label the multiplicity labels of $SU(3)$. However, the $U(4)$ group in that case is labeled in a noncanonical $U(2) \times U(2)$ chain. They also pointed out that the so-called Auxiliary Wigner Coefficient (AWC) of $SU(3)$ can be calculated from a $U(4) \star SU(3)$ scalar product. The reduced AWCs satisfy

$$\sum_{\lambda_2\mu_2 \; q'_i q''_i} \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ [q'_1 q'_2] & [q''_1 q''_2] \end{matrix} \right| \left. \begin{matrix} [m_1 m_2 m_3] \\ [q_1 q_2] \end{matrix} ; [u_1 u_2 u_3] \right\rangle \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ [q'_1 q'_2] & [q''_1 q''_2] \end{matrix} \right| \left. \begin{matrix} [\bar{m}_1 \bar{m}_2 \bar{m}_3] \\ [q_1 q_2] \end{matrix} ; [\bar{u}_1 \bar{u}_2 \bar{u}_3] \right\rangle$$

$$= \prod_i \delta_{\bar{u}_i u_i} \delta_{\bar{m}_i m_i}, \tag{5.1}$$

$$\sum_{u_i m_i} \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ [q'_1 q'_2] & [q''_1 q''_2] \end{matrix} \right| \left. \begin{matrix} [m_1 m_2 m_3] \\ [q_1 q_2] \end{matrix} ; [u_1 u_2 u_3] \right\rangle \left\langle \begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) \\ [\bar{q}'_1 \bar{q}'_2] & [\bar{q}''_1 \bar{q}''_2] \end{matrix} \right| \left. \begin{matrix} [m_1 m_2 m_3] \\ [q_1 q_2] \end{matrix} ; [u_1 u_2 u_3] \right\rangle$$

$$= \prod_i \delta_{\bar{q}''_i q''_i} \delta_{\bar{q}_i q'_i}, \tag{5.2}$$

where $[u_1 u_2 u_3]$ are labels of $\mathcal{U}(3)$ which is the subgroup of $U(4)$. We can show that such AWCs are equivalent to what will be constructed by using the recoupling approach. Firstly, we assume the irrep $(\lambda_2\mu_2)$ is a coupled irrep, namely

$$\left|\begin{matrix}(\lambda_2\mu_2)\\ \rho_2\end{matrix}\right\rangle = \sum_{\rho_2'\rho_2''}\left\langle\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right|\left.\begin{matrix}(\lambda_2\mu_2)\\ \rho_2\end{matrix}\right\rangle\left|\begin{matrix}\lambda_2'+\mu_2'\\ \rho_2'\end{matrix};\begin{matrix}\mu_2'\\ \rho_2''\end{matrix}\right\rangle, \tag{5.3}$$

where $\rho_2,\ \rho_2',\ \rho_2''$ are sublabels of $SU(3)$ in the canonical basis. Then, a special type of final coupled basis vectors can be written as

$$\left|((\lambda_1\mu_1),(\lambda_2'+\mu_2'0))[u_1u_2u_3],(\mu_2'0),\begin{matrix}[m_1m_2m_3]\\ \rho\end{matrix}\right\rangle \equiv \left|\begin{matrix}[m_1m_2m_3]\\ \rho\end{matrix};[u_1u_2u_3]\right\rangle =$$

$$\sum\left\langle\begin{matrix}(\lambda_1\mu_1)\\ \rho_1\end{matrix}\quad\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix}\right|\left.\begin{matrix}[u_1u_2u_3]\\ \bar\rho\end{matrix}\right\rangle\left\langle\begin{matrix}[u_1u_2u_3]\\ \bar\rho\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right|\left.\begin{matrix}[m_1m_2m_3]\\ \rho\end{matrix}\right\rangle\times$$

$$\left\langle\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right|\left.\begin{matrix}(\lambda_2\mu_2)\\ \rho_2\end{matrix}\right\rangle\left|\begin{matrix}(\lambda_1\mu_1)\\ \rho_1\end{matrix};\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix};\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right\rangle, \tag{5.4}$$

where the sum is over $\rho_1,\ \rho_2',\ \rho_2'',\ \bar\rho$, and the WCs involved in the summation are all given by删išauskas et al [15] and Chacon et al [5].

The AWCs can be evaluated by the norm

$$\left\langle\begin{matrix}(\lambda_1\mu_1)\\ \rho_1\end{matrix};\begin{matrix}(\lambda_2\mu_2)\\ \rho_2\end{matrix}\right|\left.\begin{matrix}[m_1m_2m_3]\\ \rho\end{matrix};[u_1u_2u_3]\right\rangle =$$

$$\sum_{\rho_2'\rho_2''\bar\rho}\left\langle\begin{matrix}(\lambda_1\mu_1)\\ \rho_1\end{matrix}\quad\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix}\right|\left.\begin{matrix}[u_1u_2u_3]\\ \bar\rho\end{matrix}\right\rangle\left\langle\begin{matrix}[u_1u_2u_3]\\ \bar\rho\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right|\left.\begin{matrix}[m_1m_2m_3]\\ \rho\end{matrix}\right\rangle\times$$

$$\left\langle\begin{matrix}(\lambda_2'+\mu_2'0)\\ \rho_2'\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ \rho_2''\end{matrix}\right|\left.\begin{matrix}(\lambda_2\mu_2)\\ \rho_2\end{matrix}\right\rangle. \tag{5.5}$$

One can check that the labels u_1, u_2, u_3 take the same values as those given by Brody et al [11]. One can also get reduced AWCs as follows.

$$\left\langle\begin{matrix}(\lambda_1\mu_1)\\ [q_1q_2]\end{matrix}\quad\begin{matrix}(\lambda_2\mu_2)\\ [q_1'q_2']\end{matrix}\right|\left.\begin{matrix}[m_1m_2m_3]\\ [q_1''q_2'']\end{matrix};[u_1u_2u_3]\right\rangle = \sum_{p\ \bar q_1\bar q_2}\left\langle\begin{matrix}(\lambda_1\mu_1)\\ [q_1q_2]\end{matrix}\quad\begin{matrix}(\lambda_2'+\mu_2'0)\\ [\lambda_2'+\mu_2'-p0]\end{matrix}\right|\left.\begin{matrix}[u_1u_2u_3]\\ [\bar q_1\bar q_2]\end{matrix}\right\rangle\times$$

$$\left\langle\begin{matrix}[u_1u_2u_3]\\ [\bar q_1\bar q_2]\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ [p0]\end{matrix}\right|\left.\begin{matrix}[m_1m_2m_3]\\ [q_1''q_2'']\end{matrix}\right\rangle\left\langle\begin{matrix}(\lambda_2'+\mu_2'0)\\ [\lambda_2'+\mu_2'-p0]\end{matrix}\quad\begin{matrix}(\mu_2'0)\\ [p0]\end{matrix}\right|\left.\begin{matrix}(\lambda_2\mu_2)\\ [q_1'q_2']\end{matrix}\right\rangle\times$$

$$U\begin{pmatrix}[q_1q_2] & [\lambda_2'+\mu_2'-p0] & [\bar q_1\bar q_2]\\ [p0] & [q_1''q_2''] & [q_1'q_2']\end{pmatrix}. \tag{5.6}$$

One can verify that the reduced AWCs given by (5.5) indeed satisfy the orthogonality relations (5.1) and (5.2). Using the explicit expression of the multiplicity -free RWCs involved in the sum, one can get a closed algebraic expression for the reduced AWCs. Such AWCs may also be useful, especially in two-particle coupling problems. It should be pointed out that though the result (5.6) is the same as required by Brody et al, the labeling scheme is, after all, not the same. The multiplicity labels of their AWCs are specified by a $U(4)$ subgroup $U(3)$, while they are now specified by the irrep of $SU(3)$ coupled from the first two irreps.

Using analytical expressions for the multiplicity-free RWCs given in [15], we obtain the following algebraic expression for the reduced AWCs of $SU(3) \supset U(2)$

$$\left\langle \begin{matrix} (\lambda_1\mu_1) \\ [m_1 m_2] \end{matrix} \quad \begin{matrix} (\lambda_2'+\mu_2';\mu_2')(\lambda_2\mu_2) \\ [m_1' m_2'] \end{matrix} \,\middle|\, \begin{matrix} [m_1 m_2 m_3] \\ [m_1'' m_2''] \end{matrix} ; [u_1 u_2 u_3] \right\rangle = \delta_{\lambda_2'+2\mu_2',\lambda_2+2\mu_2}\delta_{\lambda_1+2\mu_1+\lambda_2+2\mu_2,m_1+m_2+m_3} \times$$

$$\left[\frac{(\lambda_2+1)(\lambda_2'+\mu_2'-\mu_2)!\mu_2!(m_1-m_2+1)(m_1-m_3+2)(m_2-m_3+1)(m_{12}''-m_2)!}{(\lambda_2+\mu_2-\lambda_2'-\mu_2')!(\mu_2-\mu_2')!(\lambda_2+\mu_2-m_{12}')!(m_2-k_2+1)!(m_1-m_{12}'')!(m_2-m_{22}'')!} \times \right.$$

$$\frac{(\lambda-k_1-k_2-m_2)!(\mu_1'+k_1-m_3)!(\lambda-k_1-k_2-m_3+1)!(m_{22}''-m_3)!(m_{12}''-m_3+1)!}{(m_1-\lambda+k_1+k_2)!(m_2-\mu_1-k_1)!(m_3-k_2)!(m_1-\mu_1-k_1+1)!(m_1-k_2+2)!} \times$$

$$\left. \frac{(m_{12}'-\mu_2)!(\mu_1-k_2)!(\lambda_1+\mu_1-k_2+1)!(\lambda_1+\mu_1-m_{12})!(\mu_1-m_{22})!(\lambda_1+\mu_1-m_{22}+1)!}{(\mu_2-m_{22}')!(\lambda_2+\mu_2-m_{22}'+1)!(\mu_1+k_1+1)!(m_{12}-\mu_1)!m_{22}!(m_{12}+1)!(m_1-m_{22}''+1)!} \right]^{1/2} \times$$

$$\sum_{p=0}^{\mu_2'} \sum_{q=\min(q)}^{\max(q)} (\mu_2'-p)!(\lambda_2'+\mu_2'+p-m_{12}'-m_{22}')! \left(\frac{(m_{12}''+m_{22}''-p-q-m_{12})!(p-m_{22}')!(q-m_{22})!}{(m_{12}'-p)!} \right)^{1/2} \times$$

$$\left(\frac{(m_{12}''+m_{22}''-p-q-m_{22}+1)!(m_{12}''+m_{22}''-p-2q+1)(p+q-m_{22}'')!(m_{22}''-q+1)!}{(m_{12}''-p-q)!(m_{12}''+m_{22}''-p-q-\mu_1-k_1)!(m_{12}-q)!} \right)^{1/2} \times$$

$$U\begin{pmatrix} [m_{12}m_{22}] & [m_{12}'+m_{22}'-p\ 0] & [m_{12}''+m_{22}''-p-q,q] \\ [p0] & [m_{12}''\ m_{22}''] & [m_{12}'\ m_{22}'] \end{pmatrix} \times$$

$$F_2\begin{pmatrix} [\lambda-k_1-k_2,\mu_1+k_1,k_2] & [m_1 m_2 m_3] \\ [m_{12}''+m_{22}''-p-q,q] & [m_{12}''\ m_{22}''] \end{pmatrix} \times$$

$$F_2\begin{pmatrix} [\lambda_1+\mu_1\ \mu_1] & [\lambda-k_1-k_2,\mu_1+k_1,k_2] \\ [m_{12}+m_{22}\ 0] & [m_{12}''+m_{22}''-p-q,q] \end{pmatrix} \times$$

$$F\begin{pmatrix} \lambda_2'+\mu_2' \\ m_{12}'+m_{22}'-p \end{pmatrix}; \begin{matrix} \mu_2' \\ p \end{matrix}; \begin{matrix} [\lambda_2+\mu_2\ \mu_2] \\ [m_{12}'\ m_{22}'] \end{matrix} \Bigg), \tag{5.7}$$

where we have written explicitly that the irrep $(\lambda_2\mu_2)$ is coupled from $(\lambda_2'0)\times(\mu_2'0)$ with $\lambda_2'+2\mu_2'=\lambda_2+2\mu_2$,

$$\lambda=\lambda_1+\mu_1+\lambda_2'+\mu_2',$$

$$[u_1,u_2,u_3]=[\lambda-k_1-k_2,\mu_1+k_1,k_2], \tag{5.8}$$

and

$$F\left(\begin{matrix}\lambda_2'+\mu_2'\\ m_{12}'+m_{22}'-p\end{matrix};\begin{matrix}\mu_2'\\ p\end{matrix};\begin{matrix}[\lambda_2+\mu_2\ \mu_2]\\ [m_{12}'\ m_{22}']\end{matrix}\right)=$$

$$\sum_t(-)^t\frac{(m_{12}'-p+t)!(\lambda_2+\mu_2-m_{12}'-m_{22}'+p-t)!}{t!(\lambda_2'+\mu_2'-m_{12}'-m_{22}'+p-t)!(p-m_{22}'-t)!(m_{12}'+m_{22}'-p-\mu_2+t)!}. \tag{5.9}$$

IV. Discussions

In this paper, based on the complementary group technique for the resolution of outer multiplicity problem of $SU(n)$ proposed in (I), a general procedure for the derivation of $SU(3)\supset U(2)$ RWCs with multiplicity is outlined by using the recoupling approach after a special transformation. It is proved that outer multiplicity labeling scheme is not unique, and can be transformed from one another within $SO(m)$, where m is the multiplicity of $[m_1m_2m_3]$ in the decomposition $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$. From this point of view, Biedenharn's resolution for the outer multiplicity of $SU(n)$ can also be regarded as a complementary group resolution. In their works, the complementary group is $U(n)$. Unlike the resolution proposed in (I), in which only special Gel'fand basis was considered according to the Littlewood rule, the subirreps of the complementary group $U(n)$ will all be considered in order to resolve the multiplicity of $SU(n)\times SU(n)$. That is why multiplicity formulae can not easily be worked out from their outer multiplicity labeling scheme. Actually, all these canonical resolutions are equivalent, and can be transformed from each other. Therefore, all canonical resolutions form an equivalent class.

Using this method, RWCs of $SU(3)\supset U(2)$ with multiplicity can be derived recursively. It is obvious that computer code based on this procedure can easily be compiled, which will make numerical calculation possible in practical applications. Furthermore, one can also obtain closed algebraic expressions of these RWCs for small m values. However, the expression is still cumbersome with summation over many variables. The complexity increases with increasing the multiplicity. Therefore, a further simplification is till expected.

Only the RWCs of $SU(3)$ in the canonical chain $SU(3)\supset U(2)$ are discussed. In fact, this method can also be applied to noncanonical basis of $SU(3)$, for example $SU(3)\supset SO(3)$, by using RWCs of $SU(3) \supset SO(3)$ with one irrep symmetric in the coupling, which is multiplicity-free, and may be easily obtained.[40−41] In addition, the procedure outlined in this paper can also be extended to general $SU(n)$ case. We will discuss $SU(4) \supset U(3)$ RWCs in the next paper.

A closed algebraic expression of reduced AWCs proposed in [11] is also obtained by using the recoupling approach. These coefficients satisfy different orthogonality relations, and may be useful in many-particle coupling problems.

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Table I. RWCs of $SU(3) \supset U(2) \left\langle \begin{matrix} [21] & [21] \\ [q_1q_2] & [q'_1q'_2] \end{matrix} \middle| \begin{matrix} \eta\ [m_1m_2m_3] \\ [32] \end{matrix} \right\rangle$

$\downarrow$ $[q_1q_2]$ $[q'_1q'_2]$ / $[m_1m_2m_3]$	[420]	$[321]_1$	$[321]_2$	[300]
[21] [20]	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
[20] [21]	$\sqrt{\frac{1}{20}}$	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
[21] [11]	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
[11] [21]	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$
[21] [20]	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{7}{10}}$	0	$\sqrt{\frac{1}{4}}$
[20] [21]	$\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{2}{35}}$	$\sqrt{\frac{9}{14}}$	$\sqrt{\frac{1}{4}}$
[21] [11]	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{70}}$	$-\sqrt{\frac{2}{7}}$	$\sqrt{\frac{1}{4}}$
[11] [21]	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{8}{35}}$	$\sqrt{\frac{1}{14}}$	$-\sqrt{\frac{1}{4}}$
$\uparrow$ $[q_1q_2]$ $[q'_1q'_2]$ / $[m_1m_2m_3]$	[420]	$[321]_{\eta=0}$	$[321]_{\eta=1}$	[300]

Table II. RWCs of $SU(3) \supset U(2)$ $\left\langle \begin{matrix} [21] & [21] \\ [q_1 q_2] & [q_1' q_2'] \end{matrix} \middle| \begin{matrix} \eta\ [m_1 m_2 m_3] \\ [22] \end{matrix} \right\rangle$

$\downarrow$ $[q_1 q_2]$ $[q_1' q_2']$ / $[m_1 m_2 m_3]$	[420]	$[321]_1$	$[321]_2$	[330]
[10] [21]	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{4}}$
[21] [10]	$-\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{4}}$
[20] [20]	$-\sqrt{\frac{1}{40}}$	$-\sqrt{\frac{3}{5}}$	0	$\sqrt{\frac{3}{8}}$
[11] [11]	$\sqrt{\frac{27}{40}}$	$-\sqrt{\frac{1}{5}}$	0	$-\sqrt{\frac{1}{8}}$

[10] [21]	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{16}{35}}$	$-\sqrt{\frac{1}{7}}$	$\sqrt{\frac{1}{4}}$
[21] [10]	$-\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{35}}$	$-\sqrt{\frac{4}{7}}$	$-\sqrt{\frac{1}{4}}$
[20] [20]	$-\sqrt{\frac{1}{40}}$	$-\sqrt{\frac{27}{70}}$	$-\sqrt{\frac{3}{14}}$	$\sqrt{\frac{3}{8}}$
[11] [11]	$\sqrt{\frac{27}{40}}$	$-\sqrt{\frac{9}{70}}$	$-\sqrt{\frac{1}{14}}$	$-\sqrt{\frac{3}{8}}$
$\uparrow$ $[q_1 q_2]$ $[q_1' q_2']$ / $[m_1 m_2 m_3]$	[420]	$[321]_{\eta=0}$	$[321]_{\eta=1}$	[330]