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The Mach Disc in Axisymmetric Rocket Plumes.

Ricardo Jesus Jofre
Louisiana State University and Agricultural & Mechanical College

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A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mechanical, Aerospace and Industrial Engineering

by

Ricardo Jesús Jofre
B.S., Louisiana Polytechnic Institute, 1965
M.S., Louisiana Polytechnic Institute, 1967
May, 1971
DEDICATION

To Lourdes for her love,
understanding and encouragement.
ACKNOWLEDGMENT

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Test Case. Mach No. Axial Plot. Time Dependent Solution. Iteration = 6,609. r/De = 0.050

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ABSTRACT

The strong shock wave found in some axisymmetric plumes was studied. The location of the strong shock, called the Mach disc, was found by solving the time-dependent servation equations. These equations were integrated forward in time in a high speed computer until the solution converged to the steady-state solution.

The Mach disc location was determined within the accuracy of reported experimental measurements. A damping scheme was necessary to maintain stability. It is recommended that additional logic be incorporated into the reported computer program in order to obtain more accurate flow conditions downstream of the Mach disc.
CHAPTER 1

INTRODUCTION

When compressed gases are expanded through nozzles and exhaused into a quiescent or moving environment, jet plumes are created. These plumes are primarily inviscid and non-turbulent, except in a thin, free-shear-layer region which separates the plume from its environment and in the region far downstream from the nozzle exit. These plumes are supersonic except for possible locally imbedded subsonic regions. These plumes may vary in three spatial dimensions and time. However, there are enough steady axisymmetric and two-dimensional examples of these phenomenons to make them an interesting subset of general plumes. The fluid mechanics of this subset may be completely analyzed with existing methods, except for the locally subsonic regions. This region occurs behind a strong shock which is called the Mach disc. The purpose of this research is to analyze the Mach disc and verify the analysis with available data.

The Mach disc has not been successfully described in the mathematical study of axisymmetric supersonic plumes. Supersonic plumes without a Mach disc have been predicted by using the method of characteristics, see References 1.1-1.3. The formation of the oblique shocks is detected and compares very well with experimental data; however, the location and flow downstream of the Mach disc have not been completely described. Several semi-empirical techniques have been developed over the years, References 1.4-1.8, but none
applies for a complete range of pressure ratios. Turbulent and
viscous phenomena associated with these supersonic flows are
described adequately in References 1.9-1.10.

The appearance of a strong normal shock or Mach disc in a
supersonic flow field creates many analytical problems. Axisymmetric
and two-dimensional jet flows can be described by the use of
the inviscid conservation equations in the characteristic coordinates.
The governing equations are hyperbolic and, therefore, the solution
may be obtained by using nozzle exit conditions as initial condi-
tions and integrating them downstream from the exit plane. The
oblique shock waves are detected by the crossing of the characteris-
tics of like family. Properties across the oblique shock waves are
calculated using the Rankine-Hugoniot equation. The solution down-
stream of the shock is continued with the method of characteristics,
since the flow stays supersonic. The appearance of a strong normal
shock in the flow field is not detected by crossing of like character-
istics. Even after the location is known, the method of character-
istics cannot be used. Since the flow field downstream of the normal
shock is subsonic the characteristic network becomes imaginary.
This behavior means that the classification of the governing equations
changes from one point to another in the flow field, i.e. hyperbolic
for supersonic flow, parabolic for sonic flow, and elliptic for sub-
sonic flow.

Normal shocks not only appear in plumes, but they may develop
in other supersonic flow fields. A great deal of experimental data
are available for the supersonic jet, References 1.2, 1.5, and 1.11-1.15. Schlieren and shadowgraph photographs show clearly the location of the Mach disc. Usually together with the pictures, estimates of the ambient exhaust pressure, total jet-pressure, exit-plane static jet-pressure and Mach number, and nozzle configuration are available. The study of the Mach disc in a supersonic plume has been chosen because of the great amount of experimental data available and because it typifies imbedded subsonic behavior.

The solution method used consists of leaving the time dependent terms in the conservation equations. Including these terms makes the equations hyperbolic in time. The solution was started by giving an initial guess to the entire flow field and maintaining steady-state boundary conditions. The equations were integrated by using a forward difference scheme on time proposed by Lax and refined by MacCormack, References 1.16-1.19. The equations were integrated in a high speed computer until the solution converged to the steady-state solution.
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CHAPTER 2

PHYSICAL DESCRIPTION OF THE MACH DISC AND SUPERSONIC PLUME

The prediction of supersonic flow emanating from a nozzle is very important for the design of rocket powered vehicles. Rocket engines are designed to operate at a given altitude and back pressure. The operation of the rocket engines at any other condition reduces propulsion efficiency and can create serious heating problems. Figures 2.1-2.8 show the different plume structures that are found and the nomenclature used in the following chapters. These structures have been drawn for an inviscid plume model. Figures 2.9 and 2.10 show a more realistic picture of a supersonic plume where the viscous effect has been shown. The major structural components will be discussed in the sections that follow.

SHOCK STRUCTURE

The operation of a rocket engine at a pressure greater or less than the design pressure creates shocks in the plume.

External Shock

Figure 2.3 shows the shock structure of an under-expanded plume of a missile as it flies at supersonic speed. The exhaust pressure of the plume is higher than the ambient pressure. As the under-expanded exhaust gas exits the nozzle, the gas expands out and displaces the outside flow. The displacement of the outside supersonic flow is accomplished by a curved shock which is attached to the lip of the nozzle. The strength of the external shock and the expansion of the exhaust gas must match at the lip so that pressure and gas flow direction are the
Figure 2.1. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Slightly Under-expanded Jet; $P_e > P_a$
Figure 2.2. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Moderate Under-expanded Jet, $P_e > P_a$. 
Figure 2.3. Highly Underexpanded Jet Exhausting into Supersonic Stream.
Figure 2.4. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Slightly or Moderate Overexpanded Jet, $P_e < P_a$
Figure 2.5. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Highly Overexpanded Jet, $p_e \ll p_a$. 

Jet Boundary
Slip-Line
Mach Disc
Triple point
Expansion Waves
Reflected Shock
Internal Lip Shock
Figure 2.6. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Highly Underexpanded Jet, $P_e \gg P_a$. 
Figure 2.7. Sonic or Supersonic Jet Exhausting into Quiescent Atmosphere. Overexpanded Jet, $P_e < P_a$
Figure 2.8. Rocket Exhaust Plume
Figure 2.9. Underexpanded Plume
Figure 2.10. A Typical Plume
same. As the ratio of the exit jet pressure to ambient pressure increases, the displacement of the outside flow, caused by the expansion of the jet, becomes so great that the external lip shock detaches from the lip of the nozzle. Figure 2.8 shows a detached external shock. The strong curved shock causes part of the outside flow field to become subsonic. This subsonic flow is accelerated to supersonic flow in the region between the detached external shock and the plume boundary.

**Internal Shocks**

Several shock structures are observed within the boundaries of a plume. These shocks allow the exhaust gas to adjust to the outside boundary conditions.

**Underexpanded Internal Lip Shock**

Figures 2.1-2.3, 2.6, 2.8 and 2.10 show the different shock structure of supersonic jets exhausting into a medium where the pressure is lower than the exit plane pressure. Physically as the gas exhausts from the nozzle it "feels" the pressure differential at the lip. If the gas around the lip were to follow a tangent to the boundary the increase in area would cause the pressure to decrease very rapidly and thus to fall below the ambient pressure. In order to match the pressure at the boundary, the gas curves inward causing a small compression wave to form. The coalescence of these compression waves, see Figure 2.1-2.3, causes the formation of the internal lip shock. The strength of this shock is zero at the lip and the strength increases downstream as more compression waves coalesce.
The flow field between the center line and internal lip shock is one of expansion, and the flow between the internal lip shock and the boundary is one of compression. At a given axial position a radial plot of the flow angle shows the flow to have an inclination of zero at the center line and to deflect away from the center line up the internal lip shock. Between the internal lip shock and the jet boundary the flow tends to turn towards the centerline.

Eventually the exhaust gas turning in will make the slope of the boundary equal to zero and the jet will start decreasing in diameter, except for very large degrees of underexpansion. From this axial station onward the flow between the boundary and the shock will be pointing toward the centerline.

Overexpanded Internal Lip Shock

Figures 2.4, 2.5, 2.7, and 2.9 show the different shock structures found in an overexpanded jet. As the gas leaves the nozzle it "feels" the compression at the lip of the nozzle. This compression takes place through an oblique shock attached to the lip of the nozzle. The oblique shock propagates downstream toward the centerline. The flow between the exit plane and the internal lip shock expands. In this region the flow direction is away from the centerline. As the flow crosses the oblique shock it is compressed, then it is further compressed by the outside boundary.

Internal Nozzle Shock

Nozzle walls are contoured for maximum performance and not for ideal expansion, and therefore nozzle shocks are common. Figure 2.7
shows an overexpanded jet exhausting into a quiescent atmosphere. In the figure two types of internal nozzle shocks can be seen. The one closer to the centerline, originates at the throat of the nozzle and it is generally the weaker of the two shown. It reflects as a weak shock wave. The other shock one is formed downstream of the throat. This latter shock is the stronger of the two and it can coalesce with the internal lip shock. The formation of this shock is caused by zeroth and first order discontinuities in the nozzle boundary equation. Rapid changes in the higher order derivatives may also cause shocks to form. Often such discontinuities are part of the design and no attempt is made to remove them. In fact they may serve important functions, such as using a bell shape design to give good expansion ratios with minimum weight.

**JET BOUNDARY**

As the exhaust gas exits from the nozzle, it experiences intense shearing stress and starts mixing with the outside gas. This region of mixing is called the free shear layer.

Figure 2.11 shows the prominent features of an underexpanded jet in an isopycnic picture as reported in Reference 2.1. As can be seen from Figure 2.11 there is a "valley" emanating from the lip of the nozzle. This "valley" marks the innermost region influenced by the outside atmosphere. Landenburg, VanVoohis and Winckler, Reference 2.1, found that by assuming sonic velocity at the exit plane, one finds from a Prandtl-Meyer flow equation that along a radius in the direction of the "valley" the pressure is less than
Figure 2.11. Density Contours for an Axisymmetric Jet (Reference 2.1)
the external atmospheric pressure. Some recompression must occur between the "valley" and the outer jet boundary in order that jet pressure may be built up to atmospheric pressure, commonly referred to as the free shear layer phenomena. This phenomena is shown more clearly in Figure 2.12 which presents a radial density plot of this region.

The free shear layer is very thin at the lip of the nozzle and grows downstream. The free shear layer dissipation effects eventually become dominant and causes the supersonic flow to become subsonic and eventually there is no flow. In the region where the free shear layer is thin the pressure across it is very constant. For this region the free shear layer can be assumed to be a slip plane. A slip plane is a discontinuity in the flow field where entropy, velocity, and temperature are discontinuous but pressure and flow direction are the same on both sides of the discontinuity.

**INTERNAL SHOCK REFLECTION**

The function of the shock wave in the plume is to adjust the flow field to the boundary conditions. A repeated shock structure acts as the mechanism by which the pressure gradients in the jet are smoothed out until a balanced condition is obtained downstream. There are two basic shock patterns observed in supersonic plumes, namely: (a) The diamond shape. (b) The Mach reflection. These two shock patterns will be explained in the following two sections.
0 - Data Point From Figure 2.11.
- Fitted Curve Through Data Point.
Distance From Orifice 10 mm.

Figure 2.12. Radial Plot of Density Underexpanded Jet
Diamond Shape Reflection

Figure 2.9 shows a sketch of the repeating diamond pattern for an underexpanded jet. The internal lip shock grows from the lip of the nozzle and gathers strength as it proceeds downstream. If the jet is slightly underexpanded the internal lip shock meets in the axis of symmetry and is "reflected". The reflected shock extends back to the constant pressure boundary. At the intersection of the reflected shock and the constant pressure boundary the gas is expanded and an envelop shock is again produced. Sometimes the free shear layer has grown to such an extent on the boundary that the reflected shock pierces into the free shear layer. This process is repeated until a balanced jet is produced. The same pattern is observed for slightly overexpanded jets.

Mach Reflection

Figures 2.3, 2.5, 2.6, 2.8, and 2.10 show the shock pattern observed when a supersonic jet is highly underexpanded or overexpanded. For highly underexpanded jets the internal lip shock gathers strength as it proceeds downstream. The center flow is expanded very rapidly and the pressure behind the internal lip shock is decreasing, although the strength of the shock is increasing. For highly underexpanded jets an oblique shock may not be enough to produce the necessary pressure increase and a strong shock forms. This strong shock, referred to as the Mach disc, creates a subsonic flow which lies imbedded in the supersonic plume.
For a very highly underexpanded jet the internal lip shock increases the pressure of the jet and turns the flow inward. After the flow crosses the shock it is further compressed by the outside conditions. Further the lip shock may gather strength by the coalescence with a nozzle shock. If the internal lip shock becomes strong enough, a reflected shock cannot turn the flow away from the centerline and the Mach disc appears. At the intersection of the Mach disc and the internal lip shock a reflected shock occurs which extends back to the boundary. The entire shock structure of the internal lip shock, Mach disc, and the reflected shock is called the Mach reflection.

MACH DISC STRUCTURE

The Mach disc is located along the internal lip shock. The intersection of the Mach disc and the internal lip shock is called the "triple point". Figure 2.13 shows the structure around the triple point. Regions I, II, III, and V are supersonic. Region IV is the subsonic region and the region that makes the solution to this problem difficult. Between regions IV and V there is a slip line, i.e., static pressure and flow direction are the same on both sides of the slip line. Regions I and II are separated by the internal lip shock. Regions II and III are separated by a reflected shock. Regions V and IV are separated by the Mach disc. The Mach disc formed in overexpanded flows concaves sometimes away from the nozzle, sometimes towards the nozzle, as can be seen in Reference 2.2. If the degree of over or under expansion is not too large, the disc
Figure 2.13. Triple Point Structure
might be very flat and parallel to the exit plane.

The subsonic region has a velocity considerably less than the supersonic region adjacent to it. It is to be expected that there will be shearing forces trying to accelerate the subsonic flow back to supersonic, Figure 2.11 shows this phenomena. The velocity differences are greater immediately after the Mach disc and the shearing forces will be greater at this point. Schlieren photographs of the Mach disc reflection show the mixing region due to the shearing forces to be very small, and to be mostly a slip line. D'Attorre and Harshbarger, Reference 2.2, have found that the subsonic regions behind the Mach disc to be very unstable and that oscillations occur very easily. The position of the Mach disc also oscillated and was not stable. An axial plot of density, Figure 2.14, obtained from Figure 2.11 show that in the subsonic region there is recompression of the flow.
Figure 2.14. Axial Plot of Density. Underexpanded Jet
REFERENCES


CHAPTER 3

MATHEMATICAL DESCRIPTION OF PLUMES

THE NATURE OF THE PROBLEM

Mach discs have been observed in plane, two-dimensional and in axisymmetric exhaust plumes. Some of the plumes are created by blowing chemically inert gases through a nozzle, while others are actual exhausts of rocket engines. If, as is usually done, the dissipative action of shock waves is considered to occur in planes of discontinuity, the remaining viscous and/or turbulent dissipation in the plumes is separated from the Mach disc by regions of inviscid, supersonic flow. Hence, the Mach disc phenomena may be completely explained by piecewise-continuous, inviscid conservation equations. Most of the pertinent experimental data which exist are for flows which are in mechanical (pressure) equilibrium and in thermal (internal) equilibrium. The cold, inert-gas flows are, therefore, also in chemical equilibrium, whereas, the hot, rocket-exhaust flows may or may not be in chemical equilibrium. Since there are abundant axisymmetrical cold-flow data available to describe Mach disc phenomena, such flows were selected for this study. The effect of more complex chemical behavior may be evaluated with the same methodology as that used in this study, but, undoubtedly, the calculations will be more difficult.

In summary, the conservation laws which are used must describe axisymmetric, single-component, continuum flows which are in thermal-
equilibrium. The gases involved are believed to obey the perfect gas equation of state and to have constant specific heats. Even though viscous effects are not thought to be important, they will be included in the discussion because the solution technique employed inadvertently adds analogous terms.

THE CONSERVATION EQUATIONS

There are two mathematical ways of describing matter, the Lagrangian or system analysis, and the Eulerian or control volume method. In thermodynamics and solid mechanics, the fundamental laws are usually stated from a system viewpoint. A system is defined as a fixed mass upon which attention is focused. The properties then are stated as functions of time. In the Eulerian form, a fixed volume relative to a coordinate system is specified. Properties are then specified as functions of both time and space. The Eulerian form is customarily used in fluid mechanics. Both methods are united by the vector theorem which states, see Reference 3.1,

\[
\frac{DM_s}{Dt} = \iiint_A (\rho / \Delta t + \nabla \cdot \mathbf{u}) dA 
\]

(3.1)

where \( M_s \) is the mass of the system, \( \rho \) is the density, \( \mathbf{u} \) is the velocity, \( dA \) is the differential element of volume, and \( dS \) is the differential element of area. The overbar denotes a vector. Notice that Equation 3.1 defines the volume, \( A \), and the corresponding surface, \( S \), which contains the mass \( M_s \) at any time \( t \).

Conservation of Mass

For a system in which matter is neither created nor destroyed

\[
\frac{DM_s}{Dt} = 0 
\]

(3.2)
or

\[ \iiint_{\Lambda} (\partial \rho / \partial t + \nabla \cdot \mathbf{V}) d\Lambda = 0 \]  

(3.3)

Since this equation is valid for an arbitrary volume, it follows that

\[ \partial \rho / \partial t + \nabla \cdot \mathbf{V} = 0 \]  

(3.4)

This is the conservation of mass for a single component fluid, in differential form.

Various forms of Equation 3.4 are shown in Table 3.1.

**Conservation of Momentum**

The conservation of momentum states that the rate of change of momentum of a system is equal to the sum of the external forces on the system.

\[ \frac{D}{Dt} \iiint_{\Lambda} \rho \mathbf{V} d\Lambda = \mathbf{L} \]  

(3.5)

Typical components of the external forces, \( \mathbf{L} \), are viscous forces, pressure forces, gravity or other body forces, and forces arising because solid boundaries are contained in the control volume of interest.

Consider first the left hand side of this equation. Assume an incremental volume \( \Lambda \) over which an averaged value of \( \mathbf{V} \) and \( \rho \) may be assumed such that the integration in Equation 3.5 can be replaced by \( \rho \mathbf{V} \Lambda \).
TABLE 3.1

The Conservation of Mass

Plane, 1-dimensional flow.
\[ \begin{align*} 
\mathbf{r} &= x\mathbf{i} \quad \text{-The position vector} \\
\mathbf{V} &= u\mathbf{i} \quad \text{-The velocity vector} \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0 \quad \text{-The continuity equation (3.4a)} 
\end{align*} \]

Plane, 2-dimensional flow. (Cartesian Coordinates)
\[ \begin{align*} 
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\
\mathbf{V} &= u\mathbf{i} + v\mathbf{j} \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} &= 0 \quad (3.4b) 
\end{align*} \]

3-dimensional flow. (Cartesian Coordinates)
\[ \begin{align*} 
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\
\mathbf{V} &= u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} &= 0 \quad (3.4c) 
\end{align*} \]

Axisymmetric flow. (Cylindrical Coordinates)
\[ \begin{align*} 
\mathbf{r} &= x\mathbf{i} + r\mathbf{j} \\
\mathbf{V} &= u\mathbf{i} + v\mathbf{j} \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \left(\frac{1}{r}\right)\frac{\partial (\rho vr)}{\partial r} &= 0 \quad (3.4d) 
\end{align*} \]
\[ \frac{D(\rho \vec{V})}{Dt} = \vec{L} \]  
(3.6)

\[ (\rho \vec{V}) \vec{A} \frac{DA}{Dt} + \frac{D(\rho \vec{V})}{Dt} = \vec{L}/\vec{A} = \vec{l} \]  
(3.7)

Where \( \vec{l} \) is the force vector per unit volume. Now this equation could have been written:

\[ \vec{V} D(\rho \vec{A})/Dt + (\rho \vec{A})D\vec{V}/Dt = \vec{L} \]  
(3.8)

If \( \rho \vec{A} \) is interpreted as the mass \( M_s \) of the system, then by using the continuity equation, Equation 3.2, the first term of Equation 3.8 becomes

\[ \frac{D(\rho \vec{A})}{Dt} = \frac{D M_s}{Dt} = 0 \]  
(3.9)

Therefore,

\[ \rho D\vec{V}/Dt = \vec{l} \]  
(3.10)

Equation 3.10 is an adequate statement of the conservation of momentum, but consider adding Equation 3.4 to it so that the momentum equation is obtained in terms of \( \rho \vec{V} \), i.e., in terms of the momentum per unit volume.

\[ \vec{V} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] + \rho D\vec{V}/Dt = \vec{l} \]  
(3.11)

Also consider these rearrangement of Equation 3.11:

\[ \nabla \rho/\partial t + \rho \vec{V}/\partial t + \vec{V} \left[ \nabla \rho + \rho \vec{V} \right] + \rho (\vec{V} \cdot \nabla) \vec{V} = \vec{l} \]  
(3.12)
\[
\frac{\partial (\rho \vec{v})}{\partial t} + \vec{v} \cdot \nabla (\rho \vec{v}) + \rho \vec{v} (\nabla \cdot \vec{v}) = \vec{f} 
\]

(3.13)

\[
\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = \vec{f} 
\]

(3.14)

By comparing Equation 3.7 and 3.13, it follows that:

\[
\frac{DA}{Dt} = (\nabla \cdot \vec{v}) A 
\]

(3.15)

This is a correct interpretation of the change in the volume of the system.

Consider the force term. Only viscous forces and pressure forces will be assumed to be significant, see Reference 3.2.

\[
\vec{f} = \nabla \cdot \hat{T} - \nabla p = \nabla \cdot (\hat{T} - \hat{I} p) 
\]

(3.16)

\(\hat{T}\) is the viscous stress tensor; its elements are chosen to be positive when the stress component and surface normal of like sign and negative when they are unlike. Where \(\hat{I}\) is the unit tensor, and \(p\) is the thermodynamic pressure. All tensors used herein are second order.

The final form of the momentum equation is

\[
\frac{\partial (\rho \vec{v})}{\partial t} + \vec{v} \cdot \nabla (\rho \vec{v}) + \rho \vec{v} (\nabla \cdot \vec{v}) = \nabla \cdot (\hat{T} - \hat{I} p) 
\]

(3.17)

Its \(x\) component, where \(x\) is any orthogonal coordinate, is

\[
\frac{\partial (\rho u)}{\partial t} + \vec{v} \cdot \nabla (\rho u) + \rho u (\nabla \cdot \vec{v}) = \nabla \cdot (\hat{T} - \hat{I} p) \big|_{x \text{-component}} 
\]

(3.18)
or

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{v}) = \left\{ \nabla \cdot (\vec{f} - \vec{f}_p) \right\}_{x\text{-component}}$$  \hspace{1cm} (3.19)$$

Forms of Equation 3.18 specific to the coordinate system used in this study are shown in Table 3.2. In the construction of Table 3.2 the gas is assumed to obey the perfect gas equation of state. For a perfect gas, the second coefficient of viscosity is negligible small, and the first coefficient of viscosity may be assumed constant.

Various other methods of representing the momentum equation are available in the literature. A free-body approach presented in Reference 3.3 and a general balance approach is given in Reference 3.2. All of these methods yield the same equations, although sign conventions on some terms vary between references. Forms of the equations quite similar to those given in Table 3.2 are presented in Reference 3.3, 3.4, and 3.5.

Conservation of Energy

The conservation of energy states the rate of change of the total energy of a system is equal to the rate heat is transferred into the system plus the rate of doing work on the system by viscous and pressure forces. Other energy affects are assumed negligible. Using a similar approach as that used in discussing the momentum equation, the conservation of energy relation is

$$\frac{D}{Dt} \iiint_A \rho \vec{e} dA = \text{heating plus viscous dissipation}$$
TABLE 3.2

The Momentum Equations

All definitions given in Table 3.1 apply to this table also.

Plane, 1-dimensional flow.

\[ \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{m^2}{\rho} + p \right) + \frac{\partial}{\partial y} \left( \frac{mn}{\rho} \right) = \frac{\mu}{(4/3)} \frac{\partial^2 u}{\partial x^2} \]

(3.18a)

where \( m = \rho u \) is momentum per unit volume

Plane, 2-dimensional flow.

\[ \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{m^2}{\rho} + p \right) + \frac{\partial}{\partial y} \left( \frac{mn}{\rho} \right) + \frac{\partial}{\partial z} \left( \frac{nm}{\rho} \right) = \frac{\mu}{(4/3)} \frac{\partial^2 u}{\partial x^2} \]

(3.18b)

where \( n = \rho v \) is momentum per unit volume

\( y \) = component

\[ \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} \left( \frac{n^2}{\rho} + p \right) + \frac{\partial}{\partial y} \left( \frac{nm}{\rho} \right) + \frac{\partial}{\partial z} \left( \frac{nm}{\rho} \right) = \frac{\mu}{(4/3)} \frac{\partial^2 n}{\partial x^2} \]

(3.18c)

Plane, 3-dimensional flow.

\( x \) = component

\[ \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{m^2}{\rho} + p \right) + \frac{\partial}{\partial y} \left( \frac{mn}{\rho} \right) + \frac{\partial}{\partial z} \left( \frac{mn}{\rho} \right) = \frac{\mu}{(4/3)} \frac{\partial^2 u}{\partial x^2} \]

\[ + \frac{\partial^2}{\partial y^2} \left( \frac{n^2}{\rho} + p \right) + \frac{\partial^2}{\partial z^2} \left( \frac{nm}{\rho} \right) \]

(3.18d)
TABLE 3.2 (continuation)

\( y \) - component

\[ \begin{align*}
\alpha_n / \alpha t + \alpha (n^2 / \rho + \rho) / \alpha y + \alpha (nq / \rho) / \alpha z + \alpha (nm / \rho) / \alpha x = \\
\mu \left[ 2 \alpha^2 (n / \rho) / \alpha y^2 - 2 / 3 \{ \alpha^2 (n / \rho) / \alpha y^2 + \alpha^2 (q / \rho) / \alpha y z + \\
+ \alpha^2 (m / \rho) / \alpha y \alpha x \} + \alpha^2 (n / \rho) / \alpha z^2 + \alpha^2 (q / \rho) / \alpha z \alpha y + \\
+ \alpha^2 (m / \rho) / \alpha \alpha x \alpha y + \alpha^2 (n / \rho) / \alpha x^2 \right] \tag{3.18e} 
\end{align*} \]

\( z \) - component

\[ \begin{align*}
\alpha q / \alpha t + \alpha (q^2 / \rho + \rho) / \alpha z + \alpha (qm / \rho) / \alpha x + \alpha (qn / \rho) / \alpha y = \\
= \mu \left[ 2 \alpha^2 (n / \rho) / \alpha z^2 - 2 / 3 \{ \alpha^2 (q / \rho) / \alpha z^2 + \alpha^2 (m / \rho) / \alpha z \alpha x + \\
+ \alpha^2 (n / \rho) / \alpha z \alpha y \} + \alpha^2 (q / \rho) / \alpha x^2 + \alpha^2 (m / \rho) / \alpha \alpha x \alpha z + \\
+ \alpha^2 (n / \rho) / \alpha \alpha x \alpha y + \alpha^2 (q / \rho) / \alpha x^2 \right] \tag{3.18f} 
\end{align*} \]

where \( q = \rho w \) is momentum per unit volume in the \( z \) - direction

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Axisymmetric flow

\( x \) - component

\[ \begin{align*}
\alpha (rm) / \alpha t + \alpha (rm^2 / \rho + rp) / \alpha x + \alpha (rmn / \rho) / \alpha r = \\
= \mu \left[ (1 / 3) \alpha (n / \rho) / \alpha r + (r / 3) \alpha^2 (n / \rho) / \alpha r \alpha x + \\
+ r \alpha^2 (m / \rho) / \alpha r^2 + (4r / 3) \alpha^2 (m / \rho) / \alpha x^2 \right] \tag{3.18g} 
\end{align*} \]

\( r \) - component

\[ \begin{align*}
\alpha (rn) / \alpha t + \alpha (rmn / \rho) / \alpha x + \alpha (rn^2 / \rho + rp) / \alpha r - p = \\
= \mu \left[ (4r / 3) \alpha^2 (n / \rho) / \alpha r^2 + (2 / 3) \alpha (n / \rho) / \alpha r + \\
- (2 / 3) \alpha (n / \rho) / \alpha x + r \alpha^2 (n / \rho) / \alpha x^2 + \\
+ (r / 3) \alpha^2 (m / \rho) / \alpha x \alpha y \right] \tag{3.18h} 
\end{align*} \]
where $e$ is the total internal energy per unit mass. Again considering an incremental element

$$D(\rho e\Lambda)/Dt = eD(\rho\Lambda)/Dt + \rho\Lambda De/Dt$$

(3.20)

Where $D(\rho\Lambda)/Dt = 0$ by continuity

Therefore, the heating and dissipation are evaluated for an incremental volume to yield

$$\rho De/Dt = \nabla \cdot (\tilde{\mathbf{f}} - \tilde{\mathbf{f}}_p) \cdot \nabla - \nabla \cdot \mathbf{q}$$

(3.21)

where $\mathbf{q}$ is the heat conduction vector.

Reintroducing the $e$ times the continuity equation produces

$$e(\partial \rho/\partial t + \nabla \cdot \rho \mathbf{V}) + \rho De/Dt = \nabla \cdot (\tilde{\mathbf{f}} - \tilde{\mathbf{f}}_p) \cdot \nabla - \nabla \cdot \mathbf{q}$$

(3.22)

and rearranging

$$\partial (\rho e)/\partial t + \nabla \cdot (\rho e \mathbf{V} + \tilde{\mathbf{f}}_p \cdot \mathbf{V}) = -\nabla \cdot \mathbf{q} + (\nabla \cdot \mathbf{q}) \cdot \mathbf{V}$$

(3.23)

Various forms of Equation 3.23 specific to the coordinates systems used in this study are shown in Table 3.3. Note $\rho e = E$ is the total energy per unit volume, and $p = \rho RT$ is the perfect gas equation of state.

**Ideal Gas Relations**

An ideal gas with constant specific heats is one that obeys the equation of state

$$p = \rho RT$$

(3.25)
TABLE 3.3

The Energy Equation

Plane, 1-dimensional flow.

\[ \frac{\partial E}{\partial t} + \lambda (m(E + p)/\rho)/\partial x = [k/R_1] \left( \frac{\partial^2 (p/\rho)/\partial x^2 \right) + \\
+ \left[ \mu /3 \right] \left( \frac{\partial^2 (m^2/\rho^2)/\partial x^2 \right) \] 

(3.25a)

Plane, 2-dimensional flow.

\[ \frac{\partial E}{\partial t} + \lambda (m(E + p))/\partial x + \lambda(n(E + p)/\rho)/\partial y = \\
[k/R_1] \left( \frac{\partial^2 (p/\rho)/\partial x^2 + \partial^2 (p/\rho)/\partial y^2 \right) + \\
+ \mu(\lambda /\partial x) \left[ \partial^2 (m/\rho)/\partial x^2 - \left(1/3\right) (\partial(m/\rho)/\partial x + \\
+ \partial(n/\rho)/\partial x) \right] + \left[ n/\rho \right] \left[ \partial(m/\rho)/\partial y + \partial(n/\rho)/\partial x \right] + \\
+ \mu(\lambda /\partial y) \left[ \partial^2 (m/\rho)/\partial y^2 + \partial(n/\rho)/\partial x \right] - \\
- \left[ 2n/3 \rho \right] \left[ \partial(m/\rho)/\partial x + \partial(n/\rho)/\partial y \right] \] 

Plane, 3-dimensional flow.

\[ \frac{\partial E}{\partial t} + \lambda (m(E + p)/\rho)/\partial x + \lambda(n(E + p)/\rho)/\partial y + \lambda(q(E + p)/\rho)/\partial z = \\
[k/R_1] \left( \frac{\partial^2 (p/\rho)/\partial x^2 + \partial^2 (p/\rho)/\partial y^2 + \partial^2 (p/\rho)/\partial z^2 \right) + \\
+ \mu(\lambda /\partial x) \left[ \partial^2 (m/\rho)/\partial x^2 - \left(2m/3 \rho \right) \left[ \partial^2 q - \partial q \right] + \left[ n/\rho \right] \left[ \partial(m/\rho)/\partial y + \\
+ \partial(n/\rho)/\partial x \right] + \left[ q/\rho \right] \left[ \partial(q/\rho)/\partial x + \partial(m/\rho)/\partial y \right] + \\
+ \mu(\lambda /\partial y) \left[ \partial^2 (m/\rho)/\partial y^2 + \partial(n/\rho)/\partial x \right] - \\
- \left[ 2n/3 \rho \right] \left[ \partial^2 q - \partial q \right] + \left[ q/\rho \right] \left[ \partial(q/\rho)/\partial y + \\
+ \partial(n/\rho)/\partial x \right] + \left[ q/\rho \right] \left[ \partial(q/\rho)/\partial z + \partial^2 q - \partial q \right] + \\
+ \left[ n/\rho \right] \left[ \partial(q/\rho)/\partial x + \partial(m/\rho)/\partial z \right] + \\
+ \left[ n/\rho \right] \left[ \partial(q/\rho)/\partial y + \partial(m/\rho)/\partial x \right] + \\
+ \left[ n/\rho \right] \left[ \partial(q/\rho)/\partial z + \partial(m/\rho)/\partial y \right] + \left[ q/\rho \right]^2 \partial z \] 

where

\[ \nabla \cdot \mathbf{V} = \left( \frac{\partial (m/\rho)}{\partial x} + \frac{\partial (n/\rho)}{\partial y} + \frac{\partial (q/\rho)}{\partial z} \right) \]
Axisymmetric flow.

\[
\frac{\partial (rE)}{\partial t} + \frac{\partial [E + p]m}{\partial \rho} + \frac{\partial [r(E + p)n]}{\partial r} + \frac{\partial r}{\partial t} = \\
+ \left[ \frac{k}{R} \right] \left[ \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} (p / \rho) \right) + r \frac{\partial^2 (p / \rho)}{\partial x^2} \right] + \\
+ \mu \left[ \frac{\partial^2 (E / \rho)}{\partial r^2} \right] + \frac{\partial}{\partial x} \left( \frac{\partial (E / \rho)}{\partial x} \right) + \mu \left[ \frac{\partial^2 (E / \rho)}{\partial x^2} \right] + \\
+ \frac{\partial (m / \rho)}{\partial x} \left( \frac{\partial (m / \rho)}{\partial r} \right) + 2 \frac{\partial \theta}{\partial x} \frac{\partial (m / \rho)}{\partial x} + \\
- \frac{1}{3} \left( \nabla \cdot V \right) \left( m / \rho \right)
\]

where

\[
\nabla \cdot V = \frac{\partial (m / \rho)}{\partial x} + \frac{\partial (n / \rho)}{\partial r} + \frac{n}{\rho r}
\]
and

\[ E = \frac{p}{(\gamma-1)} + \rho \mathbf{u}^2 / 2 \]  

(3.26)

Equation 3.26 is derived in Reference 3.6.

**Summary of Conservation Equations**

The conservation equation as used in this study are of the form

\[ \mathbf{u}_t = \mathbf{F}_x(\mathbf{u}) + \mathbf{G}_r(\mathbf{u}) + \mathbf{H}(\mathbf{u}) \]

where the subscripts denote partial differentiation, i.e.,

\[ \mathbf{F}_x(\mathbf{u}) = \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} \]

and the quantities \( \mathbf{u}, \mathbf{F}(\mathbf{u}), \mathbf{G}(\mathbf{u}), \) and \( \mathbf{H}(\mathbf{u}) \) are n-space vectors. \( \mathbf{F}(\mathbf{u}) \) means that the vectors \( \mathbf{F} \) is a function of the elements of \( \mathbf{u} \). The conservation equations, Equations 3.4, 3.17, and 3.23 can be put in the above form if the assumption of negligible viscous and heating effects are made. Notice the viscous terms which have been dropped are function of \( \mathbf{u} \) and its derivatives. Various forms of the conservation equation in vector form are shown in Table 3.4.

**CHARACTERISTICS ANALYSIS**

Before presenting a discussion of the method of characteristics solution an explanation of the basic terminology is in order.

**Mach Lines, Mach Waves, Mach Angle**

Liepmann and Roshko, Reference 3.7, define the Mach angle, Mach wave, and Mach lines as follow:

The angle \( \alpha \) is simply a characteristic angle associated with the Mach number \( M \) by the selection

\[ \alpha = \sin^{-1} \left( \frac{1}{M} \right) \]  

(3.27)
| \( \rho \frac{r^e}{t} \) | \[-mr^e \] | \[-nr^e \] | \[-q \] | \[0\] |
| \( mr^e \) | \[-\frac{\alpha^2}{\rho} \] | \[-\frac{mn}{\rho} \] | \[-\frac{mq}{\rho} \] | \[0\] |
| \( \delta'nr^e \) | \[-\beta \frac{mn}{\rho} \] | \[-\beta \frac{n^2}{\rho} \] | \[-\frac{ng}{\rho} \] | \[\epsilon \rho \] |
| \( \zeta \) | \[-\zeta \frac{ng}{\rho} \] | \[-\zeta \frac{g^2}{\rho} \] | | \[0\] |
| \( \frac{Er^e}{\rho} \) | \[-m \frac{(E+p)}{\rho} \] | \[-m \frac{(E+p)}{\rho} \] | \[-m \frac{(E+p)}{\rho} \] | | |

Where:

1. \( \epsilon = 0, \beta = 0, \) and \( \zeta = 0 \) for 1-D.
2. \( \epsilon = 0, \beta = 1, \) and \( \zeta = 0 \) for 2-D.
3. \( \epsilon = 0, \beta = 1, \) and \( \zeta = 1 \) for 3-D.
4. \( \epsilon = 1, \beta = 1, \) and \( \zeta = 0, \) and \( y = r \) for axisymmetric.
It is called the Mach angle.

The line of inclination $\alpha$ which may be drawn at any position in the flow field are called the Mach lines and sometimes, Mach waves. The latter name, however, is misleading for it is often used, ambiguously, for the weak but finite waves that are produced by small disturbances.

The term Mach wave is used for convenience to identify the very weak waves that are often found in shadowgraph pictures. When boundaries cause the coalescence of Mach lines a shock wave is eventually formed. Notice when the flow is subsonic the Mach angle is undefined.

The Method of Characteristics

The conservation law for 2-D and axisymmetric flow are represented by a system of partial differential equations, References 5.8. Putting the conservation equation in a characteristic coordinate system reduces the partial differential equation to systems of ordinary differential equations, Reference 3.6. A method of characteristics solution of sufficient generality to solve the supersonic plume analyzed in this investigation is described in Reference 3.9.

Prozan's description of the method of characteristics solution (Reference 3.9) is given below.

The resulting ordinary differential equation which describes the flow properties along characteristics lines is called the compatibility equation. The compatibility equation for 2-dimensional or axisymmetric flow is, where $\theta$ is flow direction,

\[
(d\theta)_{I,II} + \frac{\cot \alpha}{V} (dy)_{I,II} + \frac{\sin \theta \cos \alpha}{y} \frac{(dy)_{I,II}}{\sin(\theta + \alpha)} \epsilon + \frac{\sin \gamma \cos \alpha \, ds}{\gamma R \sin(\theta + \alpha) \, dn} (dy)_{I,II} = 0
\]  

(3.28)
Where the subscripts I and II refer to right and left running characteristics, and \( c = 0 \) for 2-dimensional flows and \( c = 1 \) for axisymmetric flows. The characteristics are lines

\[
\frac{\text{d}y}{\text{d}x_{I,II}} = \tan(\theta + \alpha)
\]

Notice that equation, Equation 3.28 includes entropy gradients normal to the streamline, \( \frac{\text{d}s}{\text{d}n} \); however, it does not account for entropy gradients along a streamline.

**Shock Wave Equation**

The method of characteristics solution is valid upstream and downstream of oblique shocks; however, the crossing of the shock wave must be accomplished with an additional calculation. Prozan further reports that this calculation was performed by using the following oblique shock relations

\[
\rho_1 v_{n1} = \rho_2 v_{n2}
\]

(3.30)

\[
\rho_1 + \rho_1 v_{n1}^2 = \rho_2 + \rho_2 v_{n2}^2
\]

(3.31)

\[
v_{t1} = v_{t2}
\]

(3.32)

\[
C_p(T_1 - T_2) = \frac{(v_2^2 - v_1^2)}{2}
\]

(3.33)
where the subscript 1 and 2 refer to conditions before and after the shock, and the subscript n and t to normal and tangential components to the shock.

In performing these calculations Prozan derived and made use of the following relations.

The Rankine-Hugoniot Equation:

\[
\frac{\rho_2}{\rho_1} = \frac{V_{n1}}{V_{n2}} = \frac{p_2}{p_1} \frac{T_1}{T_2} = \frac{l + \frac{\gamma+1}{\gamma-1} \frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}
\]

(3.34)

The Prandtl Relation:

\[
\frac{V_{n1}}{V_{n2}} = \frac{2\gamma}{\gamma-1} \frac{R}{\gamma-1} - \frac{\gamma-1}{\gamma+1} V_{t1}^2
\]

(3.35)

The Pressure Ratio:

\[
\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 \sin^2 \beta \frac{\gamma-1}{\gamma+1}
\]

(3.36)

Flow Deflection Angle:

\[
\delta = \tan^{-1} \left[ 2 \cot \beta \frac{M_1^2 \sin^2 \beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2} \right]
\]

(3.37)

where \(\beta\) is the local inclination of the shock surface with respect to the upstream flow direction, and \(\delta\) is the flow deflection angle through the shock.
The Entropy Change:

\[
\frac{s_2 - s_1}{\gamma} = \ln \left\{ \left[ 1 + \frac{2\gamma}{\gamma + 1} \left( M_1^2 \sin^2 \beta - 1 \right) \right] \frac{1}{\gamma - 1} \left[ \frac{(\gamma - 1)M_1^2 \sin^2 \beta + 2}{(\gamma + 1)M_1^2 \sin^2 \beta} \right] \right\}
\]

Boundary Conditions

When a plume is free of shock waves the flow field can be determined by a step-by-step numerical calculation of the characteristics equations. Initial information at the exit plane of the nozzle and the pressure outside of the plume is usually assumed to be known.

When shock waves appear in the flow field, a given initial condition at the nozzle exit and a constant boundary pressure are sufficient to determine the flow fields upstream and downstream of the incident shock by applying the method of characteristics. At the shock surface, shock relations like those given in Equations 3.30-3.33 can be used to find the jumps in flow properties.

MACH DISC LOCATION TECHNIQUES

As previously stated the location of the triple point and the shape of the Mach disc cannot be located with a method of characteristics analysis. A complete theory which can predict the existence or non-existence of the Mach disc has not been developed. There are several criteria to locate the triple point once the existence of the Mach disc is known. None of these techniques give a complete solution to the subsonic region.
Method of Abdelhamid and Dosanjh

Abdelhamid and Dosanjh, Reference 3.10, have proposed the first technique where consideration is given to the conservation equations in order to determine where the Mach disc will form. They theorized that the Mach disc will form where the conservation equations would be satisfied when applied to a section perpendicular to the center-line. First they solved the supersonic plume with a method of characteristics solution. In their computational technique they located the Mach disc at several positions along the internal lip shock and then assumed the mass flow rate per unit area through the Mach disc to be constant. They also calculated the mass flow rate through the reflected shock from the triple point, and compared the total calculated mass flow rate with the mass flow rate coming out from the nozzle.

This technique fails to provide a solution for the subsonic region. Since a method of characteristics solution will detect the boundary by assuming a constant pressure on the boundary the mass flow rate could be decreasing as one travels downstream from the exit plane. Also, the assumption of constant mass flow per unit area for the subsonic core will be a good approximation for slightly underexpanded jets, but will become less accurate as the degree of underexpansion increases. These approximations could accumulate to make the results questionable.

Abdelhamid and Dosanjh, Reference 3.10, show a comparison of results obtained with their technique. Abdelhamid and Dosanjh
found that their technique located the Mach disc downstream of its actual location. Their error in locating the Mach disc increased as the jet increased in being underexpanded.

**Method of Adamson and Nicholls**

Adamson and Nicholls, Reference 3.11, postulated that the pressure on the axis behind the Mach disc equals the atmospheric pressure. They determined from nozzle test that the Mach reflection is similar to the lambda shock observed within a nozzle, and that the viscous effects are rather insignificant for the location of the Mach disc.

This method will work for highly underexpanded jets, i.e., jets that have a large diameter disc. Also, this method fails to provide a solution for the subsonic region. Adamson and Nicholls fail to accurately determine the position of the Mach disc at small pressure ratios where the Mach disc is much smaller than the exit nozzle diameter.

**Method of Bowyer, D'Attorre, and Yoshihara**

In seeking a solution for the Mach disc, Bowyer, D'Attorre, and Yoshihara, Reference 3.12, hypothesized that the Mach disc will form where the flow is locally normal to the incident flow.

This method gives very good results for the cases where the Mach disc is almost a straight line and the flow upstream of the triple point is almost parallel to the center-line and consequently perpendicular to the Mach disc. The assumption of one-dimensional flow for the subsonic region does not take into account any viscous effect.
Reference 3.13 used Bowyer, D'AttoRe, and Yoshihara technique to determine the location of the triple point. It was found that for moderately underexpanded jets, i.e., Mach disc diameter small compared to the exit plane diameter, the method gives good results.

These investigators were the first to suggest a pressure iteration procedure to determine the inclination of the reflected shock from the triple point. This iteration procedure is useful in the flow model which will subsequently be recommended; therefore, this procedure will be reviewed in detail.

If the location of the triple point were known the solution around the triple point could be determined. It is known that the triple point occurs on the internal lip shock. There must be no discontinuity in flow angularity or pressure across a slip line, which separated the supersonic and subsonic regions behind the triple point. Various shock locations are assumed and the condition of pressure continuity and flow angle continuity across the slip line is sought. Notice that \( \theta \) is the flow angle in the region with the corresponding Roman numeral and subscript and \( \delta \) is the turning angle caused by downstream shock bounding the numbered region, i.e.,

\[
\delta_I = \text{the total turning angle between I and II due to the internal shock wave.}
\]

These functional relations are from Figure 3.1,

\[
\theta_I + \delta_I = \theta_{II} \quad (3.39)
\]

\[
\theta_{II} + \delta_{II} = \theta_{III} \quad (3.40)
\]
Figure 3.1. Triple Point Angular Relations

Sign Convention
therefore from Equations 3.39 and 3.40

\[ \theta_{III} = \theta_{I} + \delta_{I} + \delta_{II} \]  

(3.41)

also

\[ \delta_{V} + \theta_{V} = \theta_{IV} \]  

(3.42)

and from the slip line conditions

\[ \theta_{III} = \theta_{IV} \]  

(3.43)

Combining Equations 3.41-3.43 gives

\[ \delta_{I} + \delta_{II} - \delta_{V} = 0 \]  

(3.44)

Notice that \( \theta_{I} = \theta_{V} \)

From oblique shock relations

\[ P_{II} = P_{I} + \Delta P_{I} \]  

(3.45)

and

\[ P_{III} = P_{II} + \Delta P_{II} \]  

(3.46)

or

\[ P_{III} = P_{I} + \Delta P_{I} + \Delta P_{II} \]  

(3.47)
also

\[ P_{IV} = P_V + \Delta P_V \]  \hspace{1cm} (3.48)

and from the slip line condition

\[ P_{IV} = P_{III} \]  \hspace{1cm} (3.49)

then

\[ \Delta P_I + \Delta P_{II} - \Delta P_{IV} = 0 \]  \hspace{1cm} (3.50)

If the location of the triple point was known, Equations 3.44 and 3.50 determine a unique solution. The computational procedure would proceed as follows:

1. \( \delta_I \) and \( \Delta P_I \) of Equations 3.44 and 3.50 are known from a method of characteristics analysis.

2. The slope of the Mach disc at the triple point is assumed.

3. \( \Delta P_V \) is evaluated from data upstream of the disc obtained with a method of characteristics. \( \Delta P_{II} \) is evaluated from Equation 3.50.

4. The angle of the reflected shock from the triple point is assumed and perturbated until a solution is found. If a solution cannot be found then steps 2-4 are repeated.

**Method of Eastman and Radtke**

Eastman and Radtke, Reference 3.14, proposed that the triple point would appear at a point along the internal lip shock where the pressure behind the shock has reached a minimum. This method is highly empirical and with no physical explanation; however, Reference
3.13 found that the method of Eastman and Radtke gives good results for disc diameter small compared with the exit nozzle diameter.

Method of Edelman, Abbott, Weilerstein, Fortune, and Genovese

These investigators hypothesized, Reference 3.15, that the triple point location is determined by the requirement that the flow in the subsonic core just downstream of the Mach disc pass smoothly through a singular throat-like region becoming supersonic. This region is assumed to be mostly one-dimensional.

This is the first method that incorporated some of the physical reasons for the formation of the Mach disc. However, the subsonic flow is mostly one-dimensional for slightly, underexpanded or overexpanded flows only. From figure 2.14 it can be seen that the flow in the subsonic regions is compressed, therefore the assumption of the flow being expanded smoothly is not true.
REFERENCES


CHAPTER 4
TIME DEPENDENT TECHNIQUES

INTRODUCTION

The basic idea behind the time dependent techniques is that one is able to find steady-state solutions by assuming the problem to be an unsteady one. The nature of the equations which describe gas dynamics flows are: hyperbolic in the supersonic region, elliptic in the subsonic region, and parabolic in the sonic region, Reference 4.1. When gas dynamics flows vary such that the nature of the flow changes within the flow field, they are said to be mixed flows. Hyperbolic, parabolic, and elliptic problems have different mathematical solutions each one requiring a prior knowledge of the type of flow under consideration. This means mixed flow problems are especially difficult to solve. It has been observed that if one takes the variables to be a function of time and space, the steady equations which were mixed in nature can be solved as asymptotic solutions to the corresponding unsteady problem, References 4.2 and 4.3. One could then analyze the problem by imposing steady-state boundary conditions and an arbitrary distribution of the variables at an initial time, t = 0, and let the flow evolve in time. This is the idea proposed by Lax and Wendroff, References 4.4-4.6.

CONSERVATION LAW FORM

The Lax-Wendroff, References 4.4-4.6, idea of integrating the time dependent equations depends strongly on putting the equation
in conservation form or divergence free form. The meaning of these terms will be explained in the following section.

**Conservation Equations**

Assume that a vector density \( \mathbf{Q} \) is defined in a domain \( D \) bounded by a surface \( S \) with outward normal \( \mathbf{n} \); then by the Gaussian divergence theorem

\[
\int_D \nabla \cdot \mathbf{Q} \, dA = \int_S \mathbf{Q} \cdot \mathbf{n} \, dS = \varphi
\]  

(4.1)

where \( dA \) is the element of volume, \( dS \) is an element of surface, and \( \varphi \) is the scalar flux of \( \mathbf{Q} \) through the surface \( S \). If the surface \( S \) is moving in space, the surface can be specified by the position vector \( \mathbf{r} \), which moves with velocity \( \mathbf{v}(\mathbf{r}, t) \) then it can be shown, Reference 4.7, that

\[
\frac{D\varphi}{Dt} = \int_S \left\{ (\nabla \cdot \mathbf{Q}) \mathbf{v} + \frac{\partial \mathbf{Q}}{\partial t} + \mathbf{v} \times (\mathbf{Q} \times \mathbf{v}) \right\} \cdot \mathbf{n} \, dS.
\]  

(4.2)

If a scalar \( U \) is defined as \( U = \mathbf{v} \cdot \mathbf{Q} \) then differentiating Equation 4.1 by \( t \) and letting

\[
-\mathbf{F} = \frac{\partial \mathbf{Q}}{\partial t} + \nabla \times (\mathbf{Q} \times \mathbf{v})
\]  

(4.3)

one obtains

\[
\frac{D}{Dt} \int_D U \, dA = \int_S (U \mathbf{v} - \mathbf{F}) \cdot \mathbf{n} \, dS.
\]  

(4.4)

Equation 4.4 is the integral form of a conservation law. The rate of charge of a quantity \( U \) contained in a domain \( D \) is equal to the flux entering \( D \) through the moving boundary \( S \). Notice that the
scalar $U$ and $\bar{F}$ are not independent of each other. Taking the divergence of Equation 4.3 and the definition of $U$ one obtains

$$-\nabla \cdot \bar{F} = \nabla \cdot (\partial \bar{Q} / \partial t) + \nabla \cdot \{(\nabla \times (\bar{Q} \times \bar{V}))\}$$

where

$$\nabla \cdot \{(\nabla \times \bar{Q} \times \bar{V})\} = 0$$

resulting in

$$\partial / \partial t (\nabla \cdot \bar{Q}) = -\nabla \cdot \bar{F}$$

or

$$U_t = -\nabla \cdot \bar{F}$$

Equation 4.7 is in divergence free form or conservation form.

Discontinuities in Conservation Equations

Hyperbolic systems such as the one described in Equation 4.7 could develop discontinuities in time even though the initial data are smooth. Suppose that $\bar{F}$ and $U$ have discontinuities across a hypersurface $\sigma(\bar{r}, t) = \text{const.}$, taking the differential of such a surface gives

$$\frac{\partial \sigma}{\partial t} dt + \frac{\partial \sigma}{\partial x^1} dx^1 + \frac{\partial \sigma}{\partial x^2} dx^2 + \ldots + \frac{\partial \sigma}{\partial x^m} dx^m = 0,$$

or

$$\frac{\partial \sigma}{\partial t} dt + \nabla \cdot \sigma \cdot d\bar{r} = 0$$

and in the limit

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \sigma \cdot \bar{V} = 0$$
Where $\vec{V}$ is the local velocity of propagation of the discontinuity

Figure 4.1 is a picture of a cylindrical element of volume crossed by the hypersurface $\sigma(\vec{r},t) = \text{const}$. The surface divides the element of volume into two parts surfaces $dS_1$ and $dS_2$ are both parallel to the hypersurface. Equation 4.4 becomes

$$\frac{D}{Dt} dV dA = \int_{dS_1} (UV-\vec{F})_1 \cdot \vec{n}_1 dS + \int_{dS_1} (UV-\vec{F})_2 \cdot \vec{n}_2 dS$$

$$+ \int (UV-\vec{F}) \cdot \vec{n} dS'$$

side of cylinder

(4.11)

In the limit as $dA \to 0$ the surfaces $dS_1$ and $dS_2$ coincide with the surface discontinuity then $\vec{n}_1 = -\vec{n}_2$, and since $dS \to 0$, then Equation 4.11 becomes

$$(UV-\vec{F})_1 \cdot \vec{n}_1 + (UV-\vec{F})_2 \cdot \vec{n}_2 = 0.$$  

(4.12)

or

$$[UV\cdot \vec{n}_1 - \vec{F} \cdot \vec{n}_1] = 0.$$  

(4.13)

where the notation, $[X]$, signifies a jump in the quantity $X$ across the discontinuity surface. The unit vector $\vec{n}_1$ normal to the hypersurface is given by

$$\vec{n}_1 = \frac{\nabla_\sigma}{|\nabla_\sigma|}$$  

(4.14)

and by Equation 4.10

$$\frac{\partial \sigma}{\partial t} + |\nabla_\sigma| \vec{n}_1 \cdot \vec{V} = 0$$  

(4.15)
Figure 4.1 Element of Volume Crossed by Discontinuity
or
\[ \tilde{V} \cdot \tilde{n}_1 = \frac{-\partial \sigma / \partial t}{|\nabla \sigma|} = \tilde{\lambda} \]

(4.16)

where \( \tilde{\lambda} \) is the local velocity of propagation of the discontinuity along
the normal. Equations 4.13 and 4.16 give the generalized Rankine-Hugoniot relation

\[ [\tilde{\lambda}u - \tilde{F} \cdot \tilde{n}_1] = 0 \]

(4.17)

or
\[ \tilde{\lambda}[u] = [\tilde{F} \cdot \tilde{n}_1]. \]

(4.18)

for stationary discontinuities Equation 4.18 becomes

\[ [\tilde{F} \cdot \tilde{n}_1] = 0 \]

(4.19)

Weak Solution

Hyperbolic equations can develop discontinuities after a finite
time has elapsed. To overcome this difficulty the concept of weak
solution is introduced. In essence the weak solution concept con­
sists of satisfying the partial differential equations on both sides
of a discontinuous surface and obeying the Rankine-Hugoniot condition
across the shock. Reference 4.8 gives a mathematical description of
weak solution theory.

From the previous paragraphs, placing the conservation equa­
tions in a divergence free form is essential to derive the generalized
Rankine-Hugoniot conditions.

**Lax-Wendroff Difference Technique**

Lax and Wendroff, Reference 4.5, presented a scheme to
difference the conservation equations in one dimension. This scheme is of second order accuracy. Using the following scheme Lax and Wendroff postulated that weak solutions are obtained that satisfy the Rankine-Hugoniot conditions. The conservation equations can be written as follows

\[ \ddot{U}_t + \dddot{F}_x [\dot{U}] = 0 \] (4.20)

where \( \ddot{U} \) and \( \dddot{F} \) are vectors in the mathematical sense. [ ] indicates the preceding term is a function. The enclosed term will state the coordinates of the point in space and time, where \( \dot{U} \) is evaluated in the order \( x, r, t \). Subscripts indicate partial differentiation from this point on. The system of equations represented in Equation 4.20 is a system of quasi-linear partial differential equation. The vector \( \ddot{U} \) is a physical quantity per unit volume. Lax and Wendroff expanded the vector in a Taylor series expansion retaining terms up to second order accuracy.

\[ \ddot{U} [J,N+1] = \ddot{U} [J,N] + \Delta t \ddot{U}_t [J,N] \]
\[ + \left( (\Delta t)^2 / 2 \right) \dddot{U}_{ttt} [J,N] \] (4.21)

where \( \dddot{U}[N,J] \) represents the vector \( \ddot{U} \) evaluated at the grid point \( [N,J] \) in the grid network, see Figure 4.2. Note \( \ddot{U} \) is a function of \( x \) and \( t \) and that the \( N,J \) grid point represents a particular \( x \) and \( t \) denoted by \( N,J \).

The derivatives in Equation 4.21 are approximated as follows. From the conservation equation, Equation 4.20,
Figure 4.2 Grid Network
\[ \ddot{u}_t [J,N] = -\dddot{F}_x [J,N] \] (4.22)

The second order derivative is approximated as follows

\[ \ddot{u}_{tt} [J,N] = -({\ddot{u}_t}_t) [J,N] \] (4.23)

Substituting Equation 4.22 into Equation 4.23 gives

\[ \ddot{u}_{tt} [J,N] = (-\dddot{F}_x)_t [J,N] \] (4.24)

Changing the order of integration gives

\[ \ddot{u}_{tt} [J,N] = -({\ddot{F}_t}_x) [J,N] \] (4.25)

and

\[ \ddot{F}_t = A \dddot{u}_t \] (4.26)

where the matrix A is the Jacobian of \( \ddot{F}_t \).

Substituting Equation 4.26 into Equation 4.25 and the result into Equation 4.21 and also substituting Equation 4.22 into Equation 4.21 gives

\[ \ddot{u} [J,N+1] = \ddot{u} [J,N] - (\Delta t) \dddot{F}_x [J,N] + \] 
\[ + (\Delta t^2/2) \left( A\dddot{F}_x \right) [J,N] \] (4.27)

The finite difference scheme for Equation 4.27 is obtained by approximating space derivative with central difference, as follows

\[ \ddot{F}_x = (\Delta x) (\ddot{F} [J+1,N] - \ddot{F} [J-1,N]) \] (4.28)
and

\[(A\tilde{F}_x)_x[J,N] = \frac{1}{\Delta x}(A\tilde{F}_x[J+\frac{1}{2},N] - A\tilde{F}_x[J-\frac{1}{2},N])\]  \quad (4.29)

and expanding again

\[(A\tilde{F}_x)_x[J,N] = \frac{1}{\Delta x} \left[ A[J+\frac{1}{2},N] \left( 1/\Delta x \right) \left( \tilde{F}[J+1,N] - \tilde{F}[J,N] \right) \right.

\[+ A[J-1,N] \left( 1/\Delta x \right) \left( F[J,N] - F[J-1,N] \right) \]  \quad (4.30)

where the matrix are evaluated as follows

\[A[J+\frac{1}{2},N] = A(\frac{1}{2}) \left( \tilde{U}[J+1,N] + \tilde{U}[J,N] \right) \]  \quad (4.31)

and

\[A[J-\frac{1}{2},N] = A(\frac{1}{2}) \left( \tilde{U}[J,N] + \tilde{U}[J-1,N] \right) \]  \quad (4.32)

Substituting Equation 4.30 and 4.28 into Equation 4.27 and using Equation 4.31 and 4.32 to evaluate the matrix A at the midpoint the difference scheme is obtained

\[\tilde{U}[J,N+1] = \tilde{U}[J,N] - (\Delta t/2\Delta x) \left[ \tilde{F}[J+1,N] - \tilde{F}[J-1,N] \right.

\[- (\frac{1}{2}) (\Delta t/\Delta x)^2 \left( A[J+\frac{1}{2},N] \left( \tilde{F}[J+1,N] - \tilde{F}[J,N] \right) \right.

\[+ A[J-\frac{1}{2},N] \left( \tilde{F}[J,N] - \tilde{F}[J-1,N] \right) \]  \quad (4.33)

Even though Equation 4.33 shows an explicit equation to evaluate the vector \(\tilde{U}\) it requires much computation in order to evaluate the matrix A. In order to circumvent the problem the two-step Lax-Wendroff scheme has been developed. The two-step techniques have the advantage that the transformation matrix, A, does not appear in the computation, as it will be shown in the next section.
MacCORMACK DIFFERENCE SCHEME

After Lax and Wendroff presented their scheme for numerically integrating the conservation equations many investigators have proposed different two-step techniques. The idea of the two-step technique is to evaluate provisional values, forward in time, of the vector $\bar{U}$ and then use this predicted value in order to evaluate a new value of the vector $\bar{U}$. This two-step technique is of second order accuracy and eliminates the evaluation of the transformation matrix. Moretti, Reference 4.9, has a survey of two-step techniques. There is no clear advantage of using one technique over any other. The technique chosen in this work is the one proposed by MacCormack Reference 4.10. It was chosen because of the ease of computation and it is of second order accuracy. The MacCormack scheme was used to solve the conservation equations for an axisymmetric plume. In such a system the conservation equations take the form

$$\bar{U}_t = \bar{F}_x + \bar{G}_r + \bar{H}$$

(4.34)

where $\bar{U}$, $\bar{F}$, and $\bar{H}$ are column vectors, see Equation 3.24.

First Step

The vector $\bar{U}$ is expanded in a Taylor series expansion around the point $[I,J,N]$ where $I,J,N$ refer to the $x,r,t$, coordinates, see Figure 4.3.

$$\bar{U}^* [I,J,N+1] = \bar{U} [I,J,N] + \Delta t \bar{U}_t [I,J,N]$$

(4.35)

where the asterisk, $*$, refers to provisional values of $\bar{U}$ computed at an extrapolated time. Substituting the differential equation
Figure 4.3 Axisymmetric Grid Network
into Equation 4.35 gives

$$\tilde{U}^* [i,j,N+1] = \tilde{U} [i,j,N] + \Delta t \left( \tilde{F}_x + \tilde{G} + \tilde{H} \right)$$

(4.36)

Using forward difference to approximate the space derivatives in Equation 4.36 gives

$$\tilde{U}^* [i,j,N+1] = \tilde{U} [i,j,N] + (\Delta t / \Delta x) \left( \tilde{F} [i+1,j,N] - \tilde{F} [i,j] \right) + (\Delta t / \Delta r) \left( \tilde{G} [i,j,N+1] - \tilde{G} [i,j,N] \right) + \Delta t \tilde{H} [i,j,N]$$

(4.37)

Equation 4.37 is used to evaluate the provisional values of \( \tilde{U} \) at a time \( t_0 + n\Delta t \).

Second Step

The differential equation, Equation 4.34, is centered, i.e., evaluated at a point \([i,j,N+\frac{1}{2}]\).

$$\tilde{U}_t [i,j,N+\frac{1}{2}] = \tilde{F}_x [i,j,N+\frac{1}{2}] + \tilde{G} [i,j,N+\frac{1}{2}] + \tilde{H} [i,j,N+\frac{1}{2}]$$

(4.38)

The time derivative is approximated using central difference.

The spatial partial derivatives at the half point in time are averaged from the values at time \( t_0 \) and the provisional value at time \( t_0 + n\Delta t \). Equation 4.38 becomes

$$\tilde{U} [i,j,N+1] = \tilde{U} [i,j,N] + (\Delta t / 2) \left( \tilde{F}_x [i,j,N] + \tilde{F}_x^* [i,j,N+1] \right) +$$

$$+ (\Delta t / 2) \left( \tilde{G}_r [i,j,N] + \tilde{G}_r^* [i,j,N+1] \right) +$$

$$+ (\Delta t / 2) \left( \tilde{H} [i,j,N] + \tilde{H}^* [i,j,N+1] \right)$$

(4.39)
where $\tilde{F}_x^*[I,J,N+1]$ means the partial of $F$ with respect to $x$ evaluated at a time $t+\Delta t$, $x+\Delta x$, and $r+\Delta r$ using the values of $\tilde{U}$ to evaluate the vector $\tilde{F}$ since $F$ is a function of $\tilde{U}$, it can be evaluated by substituting $\tilde{U}$ into $\tilde{F}$. To evaluate $\tilde{F}_x^*[I,J,N+1]$ the value of $\tilde{U}^*[I,J,N+1]$ is used. Rearranging Equation 4.39 gives

\[
\tilde{U}^*[I,J,N+1] = \left\{ \frac{1}{2} \left\{ \tilde{U}^*[I,J,N] + \Delta t \tilde{F}_x[I,J,N] + \right. \right.
\]

\[
+ \Delta t \tilde{G}_r[I,J,N] + \Delta t \tilde{H}[I,J,N] \} +
\]

\[
+ (\Delta t/2) \tilde{F}_x^*[I,J,N+1] + (\Delta t/2) \tilde{G}_r^*[I,J,N+1] +
\]

\[
+ (\Delta t/2) \tilde{H}^*[I,J,N+1]
\]

\[ (4.40) \]

The terms in braces equation 4.40 is simply just Equation 4.36. Equation 4.36 into 4.40 and using backward difference to evaluate $\tilde{F}_x^*$, $\tilde{G}_r^*$ gives the second step.

\[
\tilde{U}^*[I,J,N+1] = \left\{ \frac{1}{2} \left\{ \tilde{U}^*[I,J,N] + \tilde{U}^*[I,J,N+1] \right. \right.
\]

\[
(\Delta t/\Delta x) \left\{ \tilde{F}_x^*[I,J,N+1] - \tilde{F}_x^*[I-1,J,N+1] \right. \} +
\]

\[
+ (\Delta t/\Delta r) \left\{ \tilde{G}_r^*[I,J,N+1] - \tilde{G}_r^*[I-1,J,N+1] \right. \} +
\]

\[
+ \tilde{H}^*[I,J,N+1]\}
\]

\[ (4.41) \]

MacCormack suggested that greater stability might be obtained by using a cyclic procedure of using a forward and backward difference in the first computation and in the second computation using a backward and forward difference, for the space derivatives. For the backward and forward difference technique the equation takes the form:
**First Step**

\[
\tilde{u}^*[I,J,N+1] = \tilde{u} [I,J,N] + (\Delta t/\Delta x)\{\tilde{F} [I,J,N] - \tilde{F}[I-1,J,N]\} + \\
+ (\Delta t/\Delta r)\{\tilde{G} [I,J,N] - \tilde{G} [I,J-1,N]\} + \\
+ \Delta t \tilde{h} [I,J,N] \\
(4.42)
\]

**Second Step**

\[
\tilde{u} [I,J,N+1] = (\hat{\tilde{u}}) \{\tilde{u} [I,J,N] + \tilde{u}^* [I,J,N+1]\} + \\
+ (\Delta t/\Delta r)\{\hat{\tilde{F}} [I+1,J,N+1] - \hat{\tilde{F}} [I,J,N+1]\} + \\
+ (\Delta t/\Delta r)\{\hat{\tilde{G}} [I,J+1,N+1] - \hat{\tilde{G}} [I,J,N+1]\} + \\
+ \Delta t \hat{\tilde{h}}^* [I,J,N+1] \\
(4.43)
\]

**DAMPING TERM**

Several investigators have found the Lax and Wendroff technique to be unstable where sharp discontinuities occur. The philosophy of using damping mechanisms to control these instabilities was reported by Lax and Wendroff. References 4.11 and 4.12 used this damping scheme to perform calculations. Later, these schemes were improved specifically to get better resolution in the vicinity of shock waves, Reference 4.13. A more recent work, Reference 4.14, has proposed a damping mechanism which is very simple to compute.

**Lax and Wendroff Damping Coefficient**

Lax and Wendroff, Reference 4.5, proposed a method for improving the stability of their difference scheme. They suggested adding an extra term to Equation 4.21 which would be second order. This extra term would not have any effect on the stability of the
scheme where the solution varies smoothly, but would come into
effect where the solution is varying very rapidly. Equation 4.33
is rewritten as follows

\[ \tilde{U} [J,N+1] = \tilde{U} [J,N] - (\Delta t/\Delta x) \Delta' F + \\
+ (\tilde{F}) (\Delta t/\Delta x)^2 \Delta\Delta\tilde{F} + (\Delta t/2\Delta x) \Delta Q \Delta\tilde{U} \]

\[ (4.44) \]

where \( \Delta' \) denotes the operator (\( \tilde{F} \)) \((T[\Delta x] - T[-\Delta x])\) and \( \Delta \) the
operator (\( \tilde{F} \)) \((T[\Delta x/2] - T[-\Delta x/2])\). \( T[S] \) denoting translation of the
independent variable by the amount \([S]\). \( Q \) is a matrix that will
be determined later. This discussion is restricted to a one space
system. Notice that when Equation 4.44 is expanded it takes the
form

\[ \tilde{U} [J,N+1] = \tilde{U} [J,N] - (\Delta t/2\Delta x) \{ \tilde{F} [J,N+1] - \\
- \tilde{F} [J,N-1] \} + \\
+ (\tilde{F})(\Delta t/\Delta x)^2 \Delta[A (\tilde{F} [I,J+1/2] - \\
- \tilde{F} [I,J-1/2])] + (\Delta t/2\Delta x) \Delta Q \Delta\tilde{U} \]

\[ (4.45) \]

or

\[ \tilde{U} [J,N+1] = \tilde{U} [J,N] - (\Delta t/2\Delta x) (\tilde{F} [J,N+1] - \\
- \tilde{F} [J,N-1]) + (\tilde{F})(\Delta t/\Delta x)^2 \{ A[I,J+1/2] \tilde{F} [I,J+1] - \\
+ A[I,J-1/2] \tilde{F} [I,J-1]) + (\Delta t/\Delta x) \Delta Q \Delta\tilde{U} \]

\[ (4.46) \]
rearranging

\[
\bar{U}[J,N+1] = \bar{U}[J,N] - (\Delta t/2\Delta x) \left\{ \bar{F}[J,N+1] - \bar{F}[J,N-1] + + \left( \frac{1}{2} \right) (\Delta t/\Delta x)^2 \{ A[I,J+1/2] \bar{F}[I,J+1] - - \bar{F}[I,J] - A[I,J-1/2] \bar{F}[I,J] - - \bar{F}[I,J-1]\} + (\Delta t/2\Delta x) \Delta Q \Delta \bar{U} \right\}
\]

Equation 4.41 is the same equation as Equation 4.33 except for the term \((\Delta t/2\Delta x) \Delta Q \Delta \bar{U}\). In order to determine the form of the matrix \(Q\), the second term of the right hand side of Equation 4.41 is changed as follows:

\[
A \bar{F}_x = A^2 \bar{U}_x A^2 (\Delta \bar{U}/\Delta x)
\]

and

\[
(\Delta t)(\Delta F/\Delta x) \approx \Delta t \bar{F}_x = \Delta t A \bar{U}_x \approx (\Delta t/\Delta x) A \Delta \bar{U}
\]
or

\[
\bar{U}[J,N+1] = \bar{U}[J,N] - (\Delta t/\Delta x) A \Delta \bar{U} + + \left( \frac{1}{2} \right) (\Delta t/\Delta x)^2 \Delta A^2 \Delta \bar{U} + (\Delta t/2\Delta x) \Delta Q \Delta \bar{U}
\]

The second term and the last term on the right hand side of Equation 4.48 are of the same form which suggest that the matrix \(Q\) and \(A\) must be dimensionally the same. Since the term involving \(Q\) appears in the equation in fashion similar to the viscous term, it is reasonable to assume \(Q\) to be definite positive. A detailed comparison of this damping term to the viscous terms shown in Equations 3.18 and 3.23 would indicate that although they are similar, they are not identical. In fact, an exact calculation of
their difference could be made; this is not a simple calculation to perform and it was not attempted. Also since the term involving Q must be of second order, then Q must approach zero as the grid points approach each other. In conclusion, Lax and Wendroff require the matrix A to have the following properties:

a) Q must be positive
b) Q \([U_1, U_2]\) = 0 when \(U_1 = U_2\)
c) Q must have the dimension of A.

For the case of \(\bar{F}\) and \(\bar{U}\) being scalars the above restrictions give Q in the form

\[
Q = (\bar{A}) B \left| A [U_1] - A [U_2] \right|
\]

for the case of \(\bar{F}\) and \(\bar{U}\) being scalars the above restrictions give Q in the form

\[
Q = (\bar{A}) B \left| A [U_1] - A [U_2] \right|
\]

where B is a dimensionless function which would depend on \(U_1\) and \(U_2\).

For the case of conservation laws \(\bar{F}\) and \(\bar{U}\) are vectors valued functions and the three properties of Q mentioned before are not sufficient to determine the value of Q, Reference 4.6, and one more restriction is necessary. For steady-state solution Equation 4.44 takes the form

\[
- \Delta' \bar{F} + (\Delta t/2\Delta x) \Delta A^2 \Delta \bar{U} + (\Delta Q \Delta \bar{U}/2) = 0
\]

Denoting by \(\bar{U}_1\), \(\bar{U}_2\), and \(\bar{U}_3\), three consecutive values of \(\bar{U}\) and making the following approximation

\[
\Delta' \bar{F} = \bar{F} [\bar{U}_3] - \bar{F} [\bar{U}_2] =
\]

\[
= \bar{F} [\bar{U}_3] - \bar{F} [\bar{U}_2] + \bar{F} [\bar{U}_2] - \bar{F} [\bar{U}_1] \]

\[
A [\bar{U}_3, \bar{U}_2] (\bar{U}_3 - \bar{U}_2) + A [\bar{U}_2, \bar{U}_1] (\bar{U}_2 - \bar{U}_1)
\]
where

\[ A [\tilde{U}_1, \tilde{U}_j] = (\frac{\Delta t}{\Delta x}) (A [\tilde{U}_1] - A [\tilde{U}_j]) \]  (4.52)

Substituting these approximations into Equation 4.49 gives

\[ \{-A + (\Delta t/\Delta x) A^2 + Q\} (\tilde{U}_3 - \tilde{U}_2) + \]

\[ + \{-A - (\Delta t/\Delta x) A^2 - Q\} (\tilde{U}_2 - \tilde{U}_1) = 0 \]  (4.53)

where A in the first bracket is evaluated between \( U_3 \) and \( U_2 \) and A
in the second bracket is evaluated between \( U_2 \) and \( U_1 \). It is
desired to reduce the above difference equation for a vector valued
function to a scalar equation. Such a reduction is rigorously
possible if all the coefficient matrices commute; in this case
difference equations similar to Equation 4.53 are obtained, where
the role of A and Q is played by the eigenvalues of A and Q. This
dictates the following choice for Q:

a) Q should be a matrix commuting A

b) The eigenvalues of Q are equal to the absolute values of
the difference of the corresponding eigenvalues of \( A(U_2) \) and \( A(U_1) \)
times dimensionless factors \( B_1, \ldots, B_n \) of the order of
magnitude 1.

c) The requirement that Q commute with A implies, according
to a theorem of matrix theory, see Reference 4.15, that Q is a
function of A of the form

\[ Q = g_0 I + g_1 A + \ldots + g_{n-1} A^{n-1} \]  (4.54)

whose coefficients \( g_0, g_1, \ldots, g_{n-1} \) are uniquely determined by the
proposed choice for the eigenvalues of \( Q \). Finding the coefficient \( g_0, g_1, \ldots, g_n \) leads to the Lagrange interpolation problem which is easily solved, see Reference 4.16.

L. D'Attorre and H. U. Thommen Damping Technique.

D'Attorre and Thommen, Reference 4.13, proposed a damping technique different from the one used by Lax and Wendroff. Their difference technique is of the 2-step form.

D'Attorre and Thommen used a conservation equation of the form

\[
\tilde{W}_x = \tilde{F}_y + \tilde{G}_z
\quad (4.55)
\]

to describe three-dimensional supersonic flow. \( \tilde{W} \) is a column vector with four elements. D'Attorre and Thommen considered the elements of \( \tilde{W} \) to be dependent variables and then they expressed \( \tilde{F} \), and \( \tilde{G} \) as function of the elements of \( \tilde{W} \). Since this equation, Equation 4.55, is of the same form as the unsteady conservation equation for 2-dimensions, the cartesian coordinate \( x \) is interpreted as the coordinate time, \( t \), in the unsteady equations, and the vector \( \tilde{W} \), \( \tilde{F} \), \( \tilde{G} \) as the vector \( \tilde{U} \), \( \tilde{F} \), and \( \tilde{G} \) as given by Equation 3.24. The differential equation is then the unsteady, 2-dimensional conservation equation

\[
\tilde{U}_t = \tilde{F}_x + \tilde{G}_y
\quad (4.56)
\]

Notice that in this equation, Equation 4.56, \( \tilde{F} \) and \( \tilde{G} \) are also functions of the elements of \( \tilde{U} \). The difference approximation to Equation 4.56 is based on a Taylor series expansion of \( \tilde{U} \).
\[ \tilde{u} [I,J,N+1] = \tilde{u} [I,J,N] + \Delta t \tilde{u}_L [I,J,N] + \\
+ (\Delta t)^2/2 \tilde{u}_{tt} [I,J,N] \] (4.57)

The terms on the right hand side of Equation 4.57 are calculated in two steps.

First step: Calculate temporary values of \( \tilde{u} \) at time \( t_0 + \frac{3}{2}\Delta t \) as follows

\[ \tilde{u} [I+\frac{1}{2},J,N+\frac{1}{2}] = \tilde{u} [I+\frac{1}{2},J,N] + \\
+ \Delta t/2 \tilde{u}_L [I,\frac{1}{2},J,N] \] (4.58)

\[ \tilde{u} [I,J+\frac{1}{2},N+\frac{1}{2}] = \tilde{u} [I,J+\frac{1}{2},N] + \\
+ \Delta t/2 \tilde{u}_L [I,J+\frac{1}{2},N] \] (4.59)

where \( \tilde{u}_L \) is evaluated from the differential equation, Equation 4.56, then Equation 4.58 and 4.59 become

\[ \tilde{u} [I+\frac{1}{2},J,N+\frac{1}{2}] = \tilde{u} [I+\frac{1}{2},J,N] + \\
+ \Delta t/2 \{ \tilde{F}_x [I+\frac{1}{2},J,N] + \tilde{G}_y [I+\frac{1}{2},J,N] \} \] (4.60)

\[ \tilde{u} [I,J+\frac{1}{2},N+\frac{1}{2}] = \tilde{u} [I,J+\frac{1}{2},N] + \\
+ \Delta t/2 \{ \tilde{F}_x [I,J+\frac{1}{2},N] + \tilde{G}_y [I,J+\frac{1}{2},N] \} \] (4.61)

The temporary values calculated in Equation 4.60 and 4.61 are used to evaluate temporary values of \( \tilde{F} [I+\frac{1}{2},J,N+\frac{1}{2}] \) and \( \tilde{G} [I,J+\frac{1}{2},N+\frac{1}{2}] \).

Second step: Evaluate \( \tilde{u}_{tt} [I,J,N] \) using temporary values as follows
\[ \ddot{U}_{t} \left[ I,J,N \right] = \frac{1}{\Delta t} \left( \ddot{U}_{t} \left[ I,J,N+\frac{1}{2} \right] - \ddot{U}_{t} \left[ I,J,N-\frac{1}{2} \right] \right) \quad (4.62) \]

and using the differential equation, Equation 4.56, Equation 4.62 becomes

\[ \ddot{U}_{t} \left[ I,J,N \right] = \frac{1}{\Delta t} \left( \ddot{F}_{x} \left[ I,J,N+\frac{1}{2} \right] + \ddot{G}_{y} \left[ I,J,N+\frac{1}{2} \right] \right. \]
\[ \left. - \ddot{F}_{x} \left[ I,J,N-\frac{1}{2} \right] - \ddot{G}_{y} \left[ I,J,N-\frac{1}{2} \right] \right) \quad (4.63) \]

where

\[ \ddot{F}_{x} \left[ I,J,N+\frac{1}{2} \right] = \frac{1}{\Delta x} \left( \ddot{F} \left[ I+\frac{1}{2},J,N+\frac{1}{2} \right] - \ddot{F} \left[ I-\frac{1}{2},J,N+\frac{1}{2} \right] \right) \quad (4.64) \]

Figure 4.4 shows a typical nodal point at time \( t_{0} - \frac{1}{2} \Delta t \), \( t_{0}, t_{0} + \frac{1}{2} \Delta t \). Values of \( \ddot{U} \) are known at time \( t_{0} \). First provisional values of \( \ddot{U} \) are generated at time \( t_{0} + \frac{1}{2} \Delta t \) at the half space, see Figure 4.4. Second the first derivative with respect to time of \( \ddot{U}_{tt} \) is approximated with a central difference approximation to these intermediate points in time, Equation 4.62. Using the partial differential equation, Equation 4.56, the time derivatives are changed to space derivatives, Equation 4.63. These space derivatives are evaluated using central differences to the half points, Equation 4.61. The first derivative in time in Equation 4.63 is evaluated from the partial differential equation and values of \( \ddot{U} \) at time \( t_{0} \) as follows

\[ \ddot{U}_{t} \left[ I,J,N \right] = \ddot{F}_{x} \left[ I,J,N \right] + \ddot{G}_{y} \left[ I,J,N \right] \quad (4.65) \]
Figure 1: D'Attorre and Thommen Grid Network
where
\[ F_x[I,J,N] = \frac{1}{2\Delta x}[F[I+1,J,N] - F[I-1,J,N]] \quad (4.66) \]

and
\[ G_r[I,J,N] = \frac{1}{2\Delta x}[G[I,J+1,N] - G[I,J-1,N]] \quad (4.67) \]

Values of \( \ddot{u} \) at time \( t_0 + \Delta t \) are evaluated from Equation 4.57. Substituting Equations 4.65, 5.66, and 4.67 for the first derivative, and substituting for the second derivative as it was outlined above.

D'Attorre and Thommen improved the damping of the difference technique by multiplying the second order term, \( \dddot{u} \), of Equation 4.57 by a coefficient "D", as follows

\[ \ddot{u}[I,J,N+1] = \ddot{u}[I,J,N] + \Delta t \ddot{u}_t[I,J,N] + \\
+ D'[(\Delta t)^2/2] \dddot{u}_tt[I,J,N] \quad (4.68) \]

D'Attorre and Thommen used the damping coefficient two different ways: a) One that they called uniform damping. b) A second one that they called local damping.

Uniform damping: For uniform damping

D'Attorre and Thommen used a constant value of "D" for each plane at \( t=\text{const.} \) but not necessarily the same value for each plane of \( t=\text{const.} \).

Local damping: For local damping the value of "D" varies at each point in the plane \( t=\text{const.} \). The criteria used by these authors for choosing local values of "D" was quite complex.
D'Attorre and Thommen Damping Applied to MacCormack Scheme

In order to apply the technique for damping suggested by D'Attorre and Thommen to the MacCormack scheme, the MacCormack scheme must be restated in a different manner.

(a) The vector $\tilde{U}$ is expanded in a Taylor series expansion about $[I,J,N+1]$ as follows

$$
\tilde{U}[I,J,N+1] = \tilde{U}[I,J,N] + \Delta t \tilde{U}_t[I,J,N] + \\
+ D'((\Delta t)^2/2) \tilde{U}_{tt}[I,J,N]
$$

Where "D'" is the damping coefficient.

(b) The partial derivative $\tilde{U}_t[I,J,N]$ is evaluated from the partial differential equation, Equation 4.34, as follows

$$
\tilde{U}_t[I,J,N] = \tilde{F}_x[I,J,N] + \tilde{G}_r[I,J,N] + \tilde{H}[I,J,N]
$$

(c) The second order partial derivative $\tilde{U}_{tt}[I,J,N]$ is evaluated as follows

$$
\tilde{U}_{tt}[I,J,N] = (1/\Delta t) \{\tilde{U}_t[I,J,N+1] - \tilde{U}_t[I,J,N]\}
$$

and using the differential equation, Equation 4.34, gives

$$
\tilde{U}_{tt}[I,J,N] = (1/\Delta t) \{[\tilde{F}_x^*[I,J,N+1] + \tilde{G}_r^*[I,J,N+1]] + \\
\tilde{H}^*[I,J,N+1] - [\tilde{F}_x[I,J,N] + \tilde{G}_r[I,J,N] + \tilde{H}[I,J,N]]\}
$$

(d) Substituting Equations 4.72 and 4.70 into Equation 4.69 gives the interpolation formula
The above equation is rearranged as follows:

\[\bar{U} [I,J,N+1] = \bar{U} [I,J,N] + \Delta t \left[ \bar{F}_x [I,J,N] + \bar{G}_y [I,J,N] + \bar{H} [I,J,N] + \frac{D'}{(\Delta t/2)} \left( (1/\Delta t) \bar{F}_x [I,J,N+1] + \bar{G}_y [I,J,N+1] + \bar{H} [I,J,N+1] \right) \right] - \frac{(1/\Delta t)}{2} \left( \bar{F}_x [I,J,N] - \bar{F}_x [I,J,N+1] \right) + \frac{(1/\Delta t)}{2} \left( \bar{G}_y [I,J,N] - \bar{G}_y [I,J,N+1] \right) + \frac{H} {1} \left( \bar{G}_y [I,J,N+1] + \bar{H} [I,J,N+1] \right) \]

The calculation procedure is as follows:

(a) Calculate provisional values of \(\bar{U}\) at time \(t = t_0 + \Delta t\) using forward difference

\[\bar{U}^* [I,J,N+1] = \bar{U} [I,J,N] + \Delta t \left[ \frac{1}{\Delta x} \right] \left( \bar{F}[I+1,J,N] - \bar{F}[I,J,N] \right) + \left( \frac{1}{\Delta t} \right) \left( \bar{G}[I,J+1,N] - \bar{G}[I,J,N] \right) + \bar{H} [I,J,N] \]

(b) Calculate \(\bar{U} [I,J,N+1]\) using Equation 4.74 where derivatives at the provisional values in time are evaluated with backwards differences as follows:

\[\bar{U} [I,J,N+1] = \frac{\Delta t}{2} \left( \bar{U} [I,J,N] + \bar{U}^* [I,J,N+1] \right) + \left( \frac{\Delta t}{2} \right) \frac{1}{\Delta x} \left( \bar{F}[I+1,J,N] - \bar{F}[I,J,N] \right) + \left( \frac{\Delta t}{2} \right) \frac{1}{\Delta t} \left( \bar{F}_x [I,J,N+1] + \bar{G}_y [I,J,N+1] + \bar{H} [I,J,N+1] \right)\]
The procedure just outlined could be reversed so that one computation in time is done with a backward forward difference and the next computation in time with a forward backward difference.

**Arnold Lapidus Damping Technique**

Lapidus, Reference 4.13, proposed to use a damping technique somewhat similar to the one used by Lax and Wendroff, with the advantage of being much faster to compute. Lapidus proposed to substitute the matrix $Q$ used in Lax and Wendroff scheme, see Equation 4.47, by the difference in the absolute values of the velocities instead of the difference of the absolute values of the eigenvalues of the matrix "A" and "B",

where

$$ A \frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} $$

(4.77)

and

$$ B \frac{\partial u}{\partial t} = \frac{\partial G}{\partial t} $$

(4.78)

The difference equation used by Lapidus for the damping is

$$ u' [I,J,N+1] = u [I,J,N+1] + \Delta t/\Delta x \, \xi' [I+1,J,N+1] $$

(4.79)
\[ \ddot{u}'' [I, J, N+1] = \ddot{u} [I, J, N+1] + \Delta t/\Delta x \cdot C_6'' \{ |\delta'' v'| [I+1, J, N+1] \}
\]
\[ \delta'' U' [I+L, J, N+1] \]  

Where

\[ \delta' \ddot{u} [I, J, N] = \ddot{u} [I, J, N] - \ddot{u} [I-1, J, N] \]  
\[ \delta'' \ddot{u} [I, J, N] = \ddot{u} [I, J, N] - \ddot{u} [I, J-1, N] \]  

where \( u \) and \( v \) are horizontal and vertical fluid velocity components.

The method of Lapidus was used in this work to describe damping. Details of this application will be given in the next chapter.
REFERENCES


CHAPTER 5

PROBLEM AND COMPUTER PROGRAM DESCRIPTION

STATEMENT OF SPECIFIC PROBLEM

An axisymmetric supersonic plume emanating from a nozzle was investigated. This chapter describes the computer program which was written to calculate the plume flow properties. The flowing gases were assumed to be represented by the mathematical model stated in Chapter 3. The MacCormack scheme for integrating the time dependent equations was used, Equations 4.42 and 4.43.

The plume shown in Figure 5.1 was chosen for study because the Mach disc diameter is of the same magnitude as the exit nozzle diameter. For this specific type plume, the techniques described in Chapter 3 fail to accurately describe the Mach disc.

NON DIMENSIONAL EQUATIONS IN AXISYMMETRIC COORDINATES

The conservation equations as given by Equation 3.14 were used to describe axisymmetric plumes. Prior to programming the conservation equations, the pressure in Equation 3.24 was replaced with the value given by Equation 3.26. The conservation equations take the following form:
Exit Plane Conditions

\[ M_0 = 0.0, \frac{P_e}{P_d} = 1.4, v = 1.1, \]

Nozzle Divergence Angle 10°

From Reference 1

---

Figure 5.1. Shadowgraph of Jet Exhausting Into Still Air
Notice that Equation 5.1 treats $(\rho r), (mr), (nr)$, and $(Er)$ as the dependent variable and that if Equation 5.1 is written in symbolic form as

$$\mathbf{\ddot{U}}_t = \mathbf{F}_x [\mathbf{U}] + \mathbf{G}_r [\mathbf{U}] + \mathbf{H} [\mathbf{U}]$$

then the vectors $\mathbf{F}, \mathbf{G}$, and $\mathbf{H}$ are functions of the elements of the conservation variables represented by the vector $\mathbf{U}$.

The above equations were put in non-dimensional form by

$$\tilde{x} = x/D_e, \quad \tilde{r} = r/D_e, \quad \tilde{t} = t(\rho_e D_e/m_e)$$

$$\tilde{E} = E/E_e, \quad E_e = (m_e)^2/(\rho_e \varepsilon_e), \quad m_e = \rho_e \varepsilon_e$$

$$\tilde{\rho} = \rho/\rho_e, \quad \tilde{m} = m/m_e, \quad \tilde{n} = n/m_e$$
where the subscript e means the value at the exit plane center line, and where (~) means a non-dimensional variable, and all other terms are defined in Chapter 3.

In non-dimensional form the axisymmetric conservation equations take the same form as the above equation (Equation 5.1).

**SOLUTION METHOD**

The integrating scheme used was the one proposed by MacCormack, and the damping scheme the one proposed by Lapidus.

The equations were integrated by using first the backward and forward difference scheme, Equations 4.37 and 4.39, and then in the next calculation (in time) the forward and backward procedure. This cyclic scheme was used as suggested by MacCormack permitting greater permissible values of Δt to be used.

After each iteration in time the damping proposed by Lapidus, Equation 4.79, was used. Equation 4.79 is expanded in terms of the variables evaluated at time \( \bar{U}[I,J,N+1] \). First the conservation variables are smoothed out in the x-direction as follows

\[
\bar{U}'[I,J,N+1] = \bar{U}[I,J,N+1] + C(\Delta t/\Delta x)[|u[I+1,J,N+1] - u[I,J,N+1]| \cdot (\bar{U}[I+1,J,N+1] - \bar{U}[I,J,N+1]) - |u[I,J,N+1] - u[I-1,J,N+1]| \cdot (\bar{U}[I,J,N+1] - \bar{U}[I-1,J,N+1])]
\]

and then using the above smoothed values in the x-direction the values of the vector \( \bar{U} \) are smoothed out in the y-direction as follows
\[
\begin{align*}
\tilde{u}'[I,J,N+1] &= \tilde{u}'[I,J,N+1] + C(\Delta t/\Delta x)[v'[I,J+1,N+1] - \text{ } \text{ } \text{ } v'[I,J,N+1] - (\tilde{u}'[I,J,N+1]) - \\
&\quad - v'[I,J,N+1] - v'[I,J-1,N+1] \cdot (\tilde{u}'[I,J,N+1] - \tilde{u}'[I,J-1,N+1])
\end{align*}
\]

where \( u \) and \( v \) are \( x \) and \( r \) components of velocity and are evaluated as follows

\[
u = \frac{m}{p} \tag{5.5}
\]

\[
v = \frac{n}{p} \tag{5.6}
\]

**BOUNDARY AND INITIAL CONDITIONS**

Boundary and initial conditions must be given in order to solve the conservation equations for a given axisymmetric plume. A rectangular grid as shown in Figure 5.2 was used in the solution. A boundary data line along AB (see Figure 5.2) is necessary in order to have a solution. A one-dimensional or a method of characteristics solution is suitable for the initial condition. A method of characteristics program was run with the nozzle exit conditions, and then, by interpolating between characteristics lines, data points in the rectangular grid network were obtained.

The ambient conditions were set along a fixed row of data points, lines BC and CD (see Figure 5.2). The conservation variables \((pr)\), \((ur)\), \((vr)\), and \((Er)\) were kept fixed as the solution progressed in time.

Points on the centerline are not calculated. The variables are of the form \((pr)\), \((mr)\), \((nr)\) and \((Er)\) at the centerline \( r = 0 \) and the dependent variables have the value at zero at any time.
Figure 5.2 Rectangular Grid For Time Dependent Program
Notice that in the difference equations, Equations 4.42 and 4.43, it is not necessary to evaluate the term $\tilde{H}$ at the centerline. This is the only term which is undetermined at the centerline. The rest of the vectors at the centerline take the values of zero since $r = 0$.

\[
\begin{align*}
\tilde{F} [I, l, N] &= 0 \\
\tilde{G} [I, l, N] &= 0 \\
\tilde{F}^* [I, l, N+1] &= 0 \\
\tilde{G}^* [I, l, N+1] &= 0
\end{align*}
\] (5.7)

A second order polynomial was used to evaluate values of the conservation at the last column of data points in the rectangular grid network, line DE Figure 5.2. Values of $\bar{U}$ at the last column are obtained as follows, see Figure 5.2,

\[
\begin{align*}
\bar{U}[\text{IMAX}, J, N+1] &= 3.0(\bar{U}[\text{IMAX}-1, J, N+1]-\bar{U}[\text{IMAX}-2, J, N+1]) + \\
&\quad \bar{U}[\text{IMAX}-3, J, N+1]
\end{align*}
\] (5.8)

**COMPUTER PROGRAM LOGIC**

Two computer programs were used in this investigation:

(a) A time dependent solution of the conservation equations which is listed in Appendix A.

(b) A method of characteristics program that was developed at Lockheed Missiles and Space Company, Huntsville, Alabama, see Reference 5.1.

**Time Dependent Program**

The time dependent program was written such that a rectangular grid could be analyzed. The maximum number of grid points in the
x-direction was set to be 180 points, and the maximum number of points in the r-direction was set to be 40 grid points. The maximum number of points was limited first by the computer physical storage capacity and then by the time factor involved in the calculations. Notice that a grid network of \((40 \times 180)\) results in a minimum requirement of \((4 \times 40 \times 180) \times (4 \text{ bytes}) = 115.2 \text{k}\) since there are four variables to be stored for each grid point. The size of the rectangular grid can be set by the operator of the program. The rectangular grid size is a compromise between resolution and time per iteration. The number of grid points involved in the calculations is directly proportional to the resulting resolution and is inversely proportional to running time.

A listing of the program is given in Appendix A. The flow chart of the calculation is given in Figure 5.3. The logic of the program is as follows.

**Initialization:**

The conservation variables, \(\bar{U}\), are given values at each grid point. The damping coefficient "C" in Equations 5.3 and 5.4 is given a value. The spacing between grid points is given a value. Thus \(\bar{U}[I,J,N]\).

**Starting Line Conditions:**

The values of the vector, \(\bar{F}\) and \(\bar{G}\) are evaluated at the initial data line, line AC Figure 5.2. Thus \(\bar{F}, \bar{G}[I,J,N]\).

**Evaluation of \(U[I,J,N+1]\) by the MacCormack Integration Scheme:**

Figures 5.4 and 5.5 are flow charts of the calculation used to evaluate values of \(U[I,J,N+1]\) using the
Figure 5.3 Flow Chart. Time Dependent Program
Figure 5.4  Flow Chart for Backward Forward Scheme
Figure 5.5 Flow Chart for Forward Backward Scheme
MacCormack scheme. The figures are read as follows:

a) The flow of the calculations is indicated by the sequence of numbers in the boxes; i.e., the first calculation is the one indicated in box number 1, the second is indicated in box number 2, and so on. The calculation at each box is evaluated for a complete column of values of J for which each set of calculations performed represents the x location, i.e., x1, x2, etc. But note the x iteration number is a dummy in the expressions for \( \tilde{F}, \tilde{G}, \tilde{H}, \tilde{F}^*, \tilde{G}^*, \tilde{H}^* \), because at any one time only two values of \( \tau \) are needed in any given calculation. Thus \( \tilde{F}[2,J,N] \) in the computer program does not represent the value of \( \tilde{F} \) at column two. For example, in Figure 5.4, box number 11 the values of \( \tilde{F}, \tilde{G}, \tilde{H} \) have iteration numbers \([1,J,N+1]\), but this vector or functions are evaluated for x location number 4, i.e., for an \( x = 3(\Delta x) \). This bookkeeping was done in order to be able to store the largest array \( \bar{U} \) possible, reducing down on the size of each of the \( \tilde{F}, \tilde{G}, \tilde{H} \) arrays. b) The arrows between boxes indicate the source of the variables or functions which were used to evaluate the variable of function to which the arrows are pointing.

**Downstream Extrapolation:**

After values of \( \bar{U}[I,J,N+1] \) are evaluated then an extrapolation is performed for the last column of \( \bar{U} \), i.e., \( \bar{U}[\text{IMAX},J,N+1] \) (see Figure 5.2).
Damping of Variables:

Damping is applied using the previously discussed Lapidus scheme, Equations 5.3 and 5.4. Damping is applied in the x-direction first; then, the damping in the y-direction is applied. This produces the values for \( \tilde{U}'[I,J,N+1] \), then the values for \( \tilde{U}''[I,J,N+1] \).

Check for Stability:

If the solution is stable, i.e., no negative values of densities, then the solution is either restarted or the values stored and printed out. If the solution is unstable then a decision must be made on the type of corrective action required.

Restart:

If the solution is restarted then the values just evaluated of \( \tilde{U}''[I,J,N+1] \) are used as an initial guess for the next calculation.

Evaluation of each variable and function in the program was performed in as economical a manner as could be devised. The running time on the computer was so extensive that it was necessary to examine each part of the program with both storage capacity and economy in mind.

Method of Characteristics Program

The method of characteristics program used in this work had the following capabilities, see Reference 5.1,

(1) Either two-dimensional or axisymmetric problem geometry can be used.
Damping of Variables:

Damping is applied using the previously discussed Lapidus scheme, Equations 5.3 and 5.4. Damping is applied in the x-direction first; then, the damping in the y-direction is applied. This produces the values for \( \bar{U}^{'}[I,J,N+1] \), then the values for \( \bar{U}^{''}[I,J,N+1] \).

Check for Stability:

If the solution is stable, i.e., no negative values of densities, then the solution is either restarted or the values stored and printed out. If the solution is unstable then a decision must be made on the type of corrective action required.

Restart:

If the solution is restarted then the values just evaluated of \( \bar{U}[I,J,N+1] \) are used as an initial guess for the next calculation.

Evaluation of each variable and function in the program was performed in as economical a manner as could be devised. The running time on the computer was so extensive that it was necessary to examine each part of the program with both storage capacity and economy in mind.

Method of Characteristics Program

The method of characteristics program used in this work had the following capabilities, see Reference 5.1,

1. Either two-dimensional or axisymmetric problem geometry can be used.
Ideal gas thermodynamics or real gas data tapes may be used.

An input guide for this program is given in Appendix B.
REFERENCES


CHAPTER 6
RESULTS AND CONCLUSIONS

MACH DISC UNDER STUDY

The Mach disc studied by Adamson, Reference 5.2, is shown in Figure 5.1, which is reproduced from Adamson's report. The picture is a shadowgraph of a supersonic jet exhausting into still air. The Mach number at the exit plane was 2. The ratio of the static pressure at the exit plane to the ambient pressure was 3.3. The cold jet had a ratio of specific heats of 1.4. The nozzle had a divergence angle of 10°. Most of the data available on exhaust plumes are of this type. More desirable data would be to have temperature, pressure, velocity, flow angle, and Mach number profiles. But such data at the exit plane condition are difficult to obtain.

A statistical analysis was made of the position and thickness of the Mach disc. The Mach disc was located at an x/Dₜ distance of 3.99 ± .20. The thickness of the shock was determined to be 0.10 x/Dₜ. The numerical value of the location of the Mach disc was taken as 3.99. Figure 6.1 shows the shock wave and the inviscid boundary of the jet as scaled from Figure 5.1.

METHOD OF CHARACTERISTICS

The method of characteristics was used in order to obtain a starting line for the time dependent program. The method of characteristics program was started at the exit plane of the nozzle by assuming a constant Mach number at the exit plane. The flow angle
Figure 6.1. Test Case. Scaled Drawing From Photograph
was assumed parallel to the centerline and equal to $10^\circ$ at the lip of the nozzle and then to vary linearly from the centerline to the value of the lip. Figure 6.2 shows a flow picture obtained with the method of characteristics. Notice that the internal lip shock continues downstream towards the centerline, and that it is unable to detect any normal shock.

Figure 6.2 shows the location of the internal lip shock and the inviscid boundary as obtained with the method of characteristics program. From Figure 6.2 it can be seen that the method of characteristics detects very well the position of the internal lip shock and the inviscid boundary. Notice, however, that both the internal lip shock and the inviscid boundary continue to curve in the far flow field and that the solution fails to properly locate the boundary downstream of the intersection of the reflected shock and the inviscid boundary.

**TIME DEPENDENT SOLUTION**

The time dependent program described in Chapter 5 was used to calculate the axisymmetric plume. A grid network with a spacing $\Delta x = \Delta r = .050$ was used.

**Starting Line**

In order to start the time dependent solution a starting line downstream of the exit plane was chosen. This starting line was obtained from the method of characteristics solution at a position where the internal lip shock had already obtained some strength; such that the "jump" in properties, i.e., the Rankine-Hugoniot
Figure 6.2. Test Case. Method of Characteristics Solution
conditions (Equations 3.30-3.33), were explicitly defined.

Figure 6.3 shows the characteristics lines crossing the chosen starting line. The starting line was chosen at an axial position of $x/D_e = 1.250$. The grid network is shown in the figure as black squares. As can be seen from the picture, points on the characteristic lines do not completely agree with the grid network for the time dependent solution. An interpolation was done between two points on the characteristic line so that an interpolated value of the conservation variables were obtained on the starting line, i.e., point A and B Figure 6.3. Then, an interpolation was done between points on the starting line to obtain points on the grid network of the time dependent solution, i.e., point C and D Figure 6.3. The internal lip shock and the inviscid boundary in the method of characteristics program were located by interpolation also.

Figure 6.4 shows a pressure profile on the starting line ABC in Figure 5.2 and Figure 6.5 shows a Mach number profile. Notice the smoothing effect due to the interpolation around the internal lip shock. Even if the utmost care is taken in obtaining jump conditions, there will be smoothing effects due to the interpolations. From Figure 6.5 it can be seen that the Mach number drops very rapidly to a value of zero at the inviscid boundary. Figure 6.4 clearly shows that the pressure for the inviscid boundary is the slip line condition imposed on the boundary with the method of characteristics program. A starting line at the exit plane could also be used to start the solution but then a higher damping could have been needed in order to damp out the oscillation occurring at the
Figure 6.3. Characteristics Lines Used in Starting Time Dependent Solution
Figure 6.4. Test Case. Pressure Radial Plot. Starting Line. $x/D_e = 1.250$

Method of Characteristics
Figure 6.5. Test Case. Mach No. Radial Plot. Starting Line. $x/D_e = 1.250$

Method of Characteristics
nozzle lip. This higher damping tends to smear or spread the shock transition to such a degree that the detection of the internal lip shock becomes very difficult.

Initial Guess

As an initial guess to the flow field, the starting line conditions were assumed to hold at any axial position. Figures 6.6 and 6.7 show the progression of the solution after 346 iterations. Recall that the values of the variables evaluated for each step in time become the new guess for the next iteration. Figures 6.6 and 6.7 show that the location of the Mach disc, recognized in the figures by the very steep slopes, is nearly the correct position as shown in Figure 6.8 and 6.9 which are plots of Mach number and pressure after 6,609 iterations.

Unstable Parameters

The time dependent solution was started with a step size \( \Delta t/\Delta x = .300 \), this ratio proved to make the solution unstable because step size was so large that convergence did not occur. Changing the ratio of \( \Delta t/\Delta x \) from .300 to .200 and using the same initial guess for the \( \tilde{U}'s \) the solution became stable. This value of \( \Delta t/\Delta x = .200 \) was retained for the remainder of the calculation.

The length of the grid network was varied, since steep gradient occurred for some of the downstream extrapolations. These steep gradients made the extrapolation unstable, i.e., negative values of density appeared when using the extrapolation. Figure 6.10 is a plot of the downstream points at iteration number 1450 showing the steep gradients. The length of the calculated
Figure 6.6 Test Case. Pressure Axial Plot.

Time Dependent Solution. Iteration = 346.

Damping = 0.50

$\Delta t/\Delta x = 0.30$

$r/D_e = 0.050$
Oscillation

Damping = 0.50

$\Delta t/\Delta x = 0.300$

Figure 6.7. Test Case. Mach No. Axial Plot. Time Dependent Solution.

Iteration = 346, $r/D_e = 0.050$
Figure 6.5 Test Case. Pressure Axial Plot. Time Dependent Solution.
Iteration = 6,009. r/D_e = 0.050

Damping = 4.0

\[ \Delta t/\Delta x = 0.200 \]
Figure 6.9 Test Case. Mach No. Axial Plot. Time Dependent Solution.

Damping = 4.0

$\Delta t/\Delta x = 0.200$

Iteration = 5,609, $r/D_e = 0.050$
Figure 6.10. Test Case. Downstream Condition. Unstable Extrapolation

Iteration = 1,450
Damping = 0.50
\( \Delta t/\Delta x = 0.200 \)
\( r/D_e = 0.050 \)
variables was begun with a length of 0.0 diameters downstream of the exit plane and was later changed to 8.0 diameters downstream of the exit plane. This value was retained for the remainder of the calculations. Figure 6.11 shows a plot of pressure at iteration number 6,609 for the downstream condition.

**Time Dependent Solution and Method of Characteristics Solution**

Figures 6.12 and 6.13 show two radial plots of Mach number and pressure at an axial station of 2.5 diameters as obtained from the method of characteristics program and the time dependent solution at an iteration number 6,473 and a damping of 1.0. From the method of characteristics program, the Mach number profile is fairly constant (\(M = 4.5\)) along the internal lip shock where it drops to a value of \(M = 3.2\) and then remains constant to the inviscid boundary where it drops to a value of zero. The pressure, Figure 6.12, also remains fairly constant up to the internal lip shock where it suddenly increases due to the shock wave. It is further compressed up to the boundary where the pressure is then equal to the ambient pressure and remains constant.

From Figures 6.12 and 6.13 using the time dependent solution, it can be seen that the Mach number is fairly constant (\(M = 4.5\)) up to the internal lip shock where it drops off to a value of zero. The value of \(M = 4.5\) agrees very well with the one calculated by the method of characteristics program. The pressure also stays fairly constant up to the internal lip shock and then increases up to the boundary where it becomes and remains constant. Comparing the method of characteristics and the time dependent solution it
Pressure (lbf./in.²)

Non-Dimensional Distance x/Dₑ

Figure 6.11. Test Case. Downstream Condition. Stable Extrapolation

Iteration = $6,473$

Damping = 1.0

$\Delta t/\Delta x = 0.20$

$r/Dₑ = 0.050$
Figure 6.12 Test Case. Method of Characteristics vs Time Dependent Solution Plot of Pressure. $x/D_e = 2.50$

- Method of Characteristics Solution
- Time Dependent Solution

Iteration = 6,473
Damping = 1.0
$\Delta t/\Delta x = 0.200$
Figure 6.13. Test Case. Method of Characteristics vs Time Dependent Solution Plot of Mach Number
can be seen that the greatest difference between the two solutions is that the time dependent solution "smears" the velocity at the boundary. More specifically, the time dependent solution does not place a slip line condition on the boundary. If the inviscid boundary (in the time dependent solution) is assumed to occur where the pressure remains constant and is equal to the ambient pressure; then, the Mach number at the boundary calculated by the time dependent solution is equal to 2.0 while for the method of characteristics the value Mach number is 3.1. Using the pressure boundary and the time dependent solution has the effect of making the plume appear smaller in diameter. The principal difference between the method of characteristics solution and the time dependent solution is attributed to the "smearing" of the shock by the time dependent solution. Note that both levels of damping shown produce comparable "smearing".

**Damping Effects**

Several runs were made with different values of the damping coefficient. Figure 6.7 shows the axial plot of the Mach number at a radial position of $r = 0.050$. Around the Mach disc there is noticeable oscillation shown in the sudden increase of the Mach number prior to the sudden reduction. Figure 6.9 shows a similar plot with a damping coefficient of 4.00 and it is obvious that the oscillation, around the Mach disc have been damped out. This illustrates the favorable result of utilizing damping.

Figure 6.14 shows radial plots of pressure for two values of the damping coefficient. Figure 6.15 shows similar plots for Mach
Figure 6.14 Test Case. Time Dependent Solution Damping = 1.0 vs Time Dependent Solution Damping = 4 for Pressure.
Figure 6.15. Test Case. Time Dependent Solution Damping = 1.0 vs Time Dependent Solution Damping = 4.0 for Mach Number.
number. It can be seen that the effect of the greater damping has resulted in a "spreading" or "smearing" of the internal lip shock over more grid points.

Figures 6.16 and 6.17 show the plots of Mach number and pressure for a damping coefficient 1.0. Figure 6.16 shows a very high pressure region around 6 diameter downstream of the exit plane. Figure 6.8 is a similar plot to that of Figure 6.16 but with increased damping. The high pressure region has been decreased, but note that with increased damping the internal lip shock is "smear"ed" more.

Figure 6.18 shows a flow picture drawn from an iteration number 6,473 for the solution. The high pressure region observed on Figure 6.16 is associated with the reverse flow shown on Figure 6.18. This compression in the subsonic region is qualitatively similar to the compression shown on Figure 2.16. The flow on Figure 6.19 is well behaved, since the reversed flow shown on Figure 6.18 is not a realistic phenomena. The implied compression with this damping level resembles much more closely that shown in Figure 2.16. Indications are that damping levels and also damping schemes, may cause unrealistic flows to be created by surfaces of discontinuity. However, some damping levels produce calculated results which are qualitatively reasonable and typical. The results are shown on Figures 6.9, 6.19, and 6.20. "Smearing" of the internal lip shock and of the plume boundary is greater than that desired. Also, the Mach number behind the disc is supersonic over an appreciable region of the flow field; therefore, the damping level of 4.0 is unrealistic. In addition, the indicated axial extent of the subsonic
Figure 6.16. Test Case. Pressure Axial Plot. Time Dependent Solution

Damping = 1.0

Iteration = 6,473. r/D_e = 0.050
Figure 6.17. Test Case. Mach No. Axial Plot. Time Dependent Solution, r/De = 0.050

Iteration = 6,473

Damping = 1.0
Figure 6.16 Test Case. Iteration = 6,473. Damping = 1.0
Figure 6.19  Test Case. Iteration = 6,609. Damping = 4.0
Figure 6.20. Test Case. Mach No. Axial Plot. Time Dependent Solution
Iteration = 6,609. r/D_e = 0.250
region is very small. These observations lead one to speculate as to the desirability of significantly complicating the calculation by including sufficient logic to cross shocks without using a time dependent scheme. This matter will be discussed at greater length in the section of this report entitled RECOMMENDATIONS AND CLOSURE.

The discrepancy in the subsonic region may be due to an increment "jump" condition imposed on the flow by the difference scheme. The conservation equations in axisymmetric form are not in divergence or conservation form, i.e., as shown in Equation 4.7. The "jump" conditions for the internal lip shock do not possess this problem since the "jump" conditions are established by the starting line.

Mach Disc Analysis

Figures 6.21 - 6.30 show axial plots of Mach number at several different values of radial position, $r/D_e$. The location of the shock wave is determined by the continuation of the smooth curve by a dashed line and the intersection with the curve as it drops, point $E$ in Figure 6.21. Figure 6.31 is a plot of the shock position as calculated from the time dependent solution. The line above the figure shows the experimental values.

The curvature of the Mach disc as determined by the time dependent solution is convex toward the exit plane. From Figure 5.1 it is impossible to determine the curvature of the disc, since in axisymmetric plumes a shadowgraph picture does not produce resolution in depth sufficient to distinguish between concave towards or away from the exit plane. It is important to note that the Mach disc location as determined by the calculations is well within the
Figure 6.21. Test Case. Mach Disc Location. Axial Plot
Mach No. $r/D_e = 0.05$. Iteration = 6,473
Damping = 1.00

Figure 6.22. Test Case. Mach Disc Location. Axial Plot
Mach No. $x/D_e = 0.150$. Iteration = 6,473
Figure 6.23. Test Case. Mach Disc Location. Axial Plot
Mach No. r/De = 0.200. Iteration = 6,473
Figure 6.21*. Test Case. Mach Disc Location. Axial Plot
Mach No. r/De = -250. Iteration = 6,473

Damping = 1.00
Damping = 1.00

Figure 6.26. Test Case. Mach Disk Location. Axial Plot

Mach No. r/De = .350. Iteration = 6,473
Figure 6.27. Test Case. Mach Disc Location. Axial Plot
Mach No. r/De = .400. Iteration = 6,473
Damping = 1.0

Figure 6.28. Test Case. Mach Disc Location. Axial Plot

Mach No. r/De = .450. Iteration = 6,472
Figure 6.29. Test Case. Mach Disc Location. Axial Plot

Mach No. $r/D_e = 0.50$. Iteration = 6.473
Damping = 1.0

Figure 6.30. Test Case. Mach Disc Location: Axial Plot

Mach No. r/De = .550. Iteration = 6,473
accuracy of the measurements that can be made from the photographs. It is felt that the accuracy of the calculation method has been adequately demonstrated.

Sinha, Zakkay and Erdos (Reference 6.1) and D'Attorre (Reference 6.2) have reported solutions using time dependent techniques to flows containing a Mach disc. The first of these references shows calculations for a plane, two-dimensional underexpanded jet. For these calculations, damping was necessary and the "jumps" across the disc were not consistent with one-dimensional expectations. The two-dimensional experimental data which corresponded to this set of calculations had some unexplained extraneous shocks in the flow field and are therefore considered to be of limited value. It is well to note that the two-dimensional flow equations, Equation 3.24, are of the conservative form, whereas the axisymmetric equations are not of this form. The second reference describes a disc caused by impingement on a surface; therefore, a direct comparison is not considered appropriate. In general, both of these works support the observations made in this study. Sinha, Zakkay, and Erdos reported that their method required seven hours of computer time to reach a steady solution. The computer time required for the first 360 iterations was approximately equal to 15 minutes. Referring to Figure 6.6 and 6.8 it is seen that the shock location and the supersonic region had already converged.

CONCLUSIONS

The Mach disc was determined within the accuracy of the experimental measurement made. Figure 6.31 shows the location and shape
Figure 6.31. Mach Disc Calculated Geometry
of the Mach disc as determined from the time dependent solution. The thickness and position of the Mach disc as determined from a statistical analysis is shown on Figure 6.31 by the shaded area. Notice from the picture that the width of the shock as determined from the time dependent solution is within the expected values.

Flow properties downstream of the Mach disc were found to be strongly dependent on the damping coefficient used; however, the location of the Mach disc was found to be relatively independent of these values. Since retention of damping is necessary to maintain stability, it appears desirable to include additional logic for the purpose of crossing all surfaces of discontinuity. Additional difficulties may be encountered because non-conservative forms of the equations have been solved; however, similar difficulties have been observed in conservative form solutions.

Since the internal lip-shock and the Mach disc are "smeared" by damping, a compromise must be made between obtaining valid subsonic solutions and retaining adequate resolution for determining shock locations. The supersonic core bounded by the internal lip-shock and the disc is essentially independent of the damping coefficient used.

Pressures shown in Figure 6.12 may be in error because a stagnation temperature was assumed. Had this been reported, better comparisons of pressure profiles should have been obtained.
REFERENCES


CHAPTER 7
RECOMMENDATIONS AND CLOSURE

Based on the results presented in this study, the following recommendations are made.

1. The time dependent solution described herein for the mixed-flow, axisymmetric plume problem may be used to obtain a crude estimate of the flow properties for the entire plume.

2. To estimate mixed-flow, axisymmetric-plume properties as accurately as possible the following procedure is suggested.

   A. A method-of-characteristics program, which contains the logic necessary to include a Mach disc with associated one-dimensional subsonic flow for a flow field in which the triple point has been located by one of the available empirical techniques should be used to calculate an initial set of conservation variables. The characteristics program should have an internal interpolation capability to provide flow property data for an axial and radial grid array. This array should be prepared in such a manner that it will be suitable for input to the time dependent program.

   B. The time dependent program described herein should be run for as low a damping level as possible to determine the location of the Mach disc. Additions to the program should include: logic for calculating the crossing of this disc with the Rankine-Hugoniot relations, logic for performing the pressure iteration necessary to establish the initial reflected shock angle and slip line angle for
the given Mach disc location, and logic for restarting the time
dependent program from a new initial data plane downstream of the disc.
When these additions have been made, the remainder of the plume may
be calculated. The initial estimate for conservation variables in
this second stage of the calculation must be determined. No changes
are believed necessary to the present method of treating boundary
conditions nor to the present damping scheme. The recompressions
indicated in both experimental and calculated flow field data
indicate that a one-dimensional subsonic model will not accurately
replace this calculation step.

4. Real gas fluid properties should ultimately be included
in these analyses.
APPENDIX A

TIME DEPENDENT COMPUTER PROGRAM LISTING

A listing of the time dependent program that was developed in this research is given in the following pages. The computer program was started by initializing the following variables:

- \( C, C_Y \): damping coefficient
- \( \Gamma \): specific heat ratio
- \( D_T X \): \( \Delta t/\Delta x \)
- \( D_E L T X \): grid spacing
- \( IM \): maximum number of grid points in the \( \text{x-direction} \)
- \( JM \): maximum number of grid points in the \( \text{r-direction} \)
- \( U(K, I, J) \): \( \bar{\rho}, \bar{m}, \bar{n}, \bar{E} \) for every \( K, I, J \).
COMMON / VISCO / C, CDTX, CY, COTY

READ INITIAL GUESS
READ FREE BOUNDARY
READ IN SPACE DATA

READ (5,1,9911)
1000 FORMAT(15)
    GO TO (100,200,300,400), I
100 CALL DATA1
    GO TO 10
200 CALL DATA2
    GO TO 10
300 CALL DATA3
    GO TO 10
400 CALL DATA4
10 CONTINUE
    IER = C

BOUNDARY CONDITIONS FOR INITIAL DATA

EVALUATION OF INITIAL LINE FOR FACING A CONSTANT
CALL F0V0C
I CONTINUE

EVALUATION OF U(I,J,J+1)
CALL ILAX4F

BOUNDARY CONDITIONS
CALL DOWEXT

VIS COSITY X-DIRECTION
CALL VSCOX

BOUNDARY CONDITION
CALL VSCQY

VIS COSITY Y-DIRECTION
CALL DOWEXT

TEST FOR NEGATIVE DENSITIES
CALL TESTDE
IF(ITER.EQ.135)GO TO 4
ITER=ITER+1
GO TO 1
4 WRITE(4)((U(I,J,K),I=1,4),J=1,180),K=1,40),KOUNT
3001 FORMAT(1UX,'ITERATION=',16//1UX,'C=',F10.6//1UX,'DELTA=',F10.6//1UX,'DELTY/DELTX=',F10.6//1UX,'CY=',F10.6)
WRITE(5,3001)KOUNT,C,DELT,DTX,CY
CALL ABC
STOP
END
EXECUTING FIV,FAC
DIMENSION U(4,15,4,1), UBAR(4,2,4,1)
DIMENSION F(4,2,4,1), G(4,3,4,1), H(2,4,1)
DIMENSION FBAR(4,2,4,1), GBAR(4,2,4,1), HBAR(2,4,1)
DIMENSION FC(4,4,1)
COMMON/UVECT,U
COMMON/JINTER/UBAR
COMMON/FGH,F,G,H
COMMON/FGHINT/FBAR,GBAR,HBAR
COMMON/FGHCON/FC
COMMON/TIMECO/DELT,DTX,DTX5
COMMON/SPCECU/JM,JML1,JML2,JML3,IM,IML1,IML2,IML3,IML4
COMMON/BKKEEP/N1,N2
COMMON/LIMIT1/JMIN,JMX
COMMON/LIMIT2/JMIBAR,JMXBAR
COMMON/LIMIT3/JMXNEW
COMMON/EXTRA/IXTRAP

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

X
X EVALUATION OF VECTOR U(IZ,I,J,N+1) BY USING A
X TWO STEP LAX WENDROFF WITH FORWARD
X BACKWARD DIFFERENCE SCHEME
X

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

SETTING INDEXES
I IS THE COLUMN BEING EVALUATED
J IS THE ROW BEING EVALUATED

JMIN=2
JMX=JML1
JMIBAR=2
JMXBAR=JML2
JMXNEW=JML2
"I=1?

749
CALL FGHVE(I)
I=2
N2=MOD(N2,2)+1
EVALUATION OF F,G,H FOR COLUMN 3
CALL FGHVE(IFURKH)
EVALUATION OF UBAR USING FORWARD DIFFERENCE FOR COLUMN 2
CALL UINFOR(I)
EVALUATION OF F, G, H BAR FOR COLUMN 2
CALL FIGIHI
EVALUATION OF U(Z,2,J,N+1) USING BACKWARD DIFFERENCE
DO 1 J=2,ML2
DO 2 IVAlUE=1,4
1(U(IVAlUE,I,J)=C.50+(U(IVAlUE,I,J)+UBAR(IVAlUE,N1,J)+
2*TXS*(FRAK(IVAlUE,N1,J)+GRAR(IVAlUE,N1,J)+
2*GR(IVAlUE,J)+HAR(IVAlUE,N1,J-II))
2 CONTINUE
1 U(3,I,J)=U(3,I,J)+2TXS*HAR(N1,J)
EVALUATION OF U(Z,1,J,N+1) FOR I GREATER THAN 2.
DO = 1,2,ML2
n1=n2
n2=n0*(n2+2)+1
IF(n0=n1+1

C EVALUATION OF F,G,H AT I+1

CALL FGHVE(IFD&W)

C EVALUATION OF UBAR AT I USING FORWARD DIFFERENCE

CALL UNFOR(I)

C EVALUATION OF F,G,H BAR AT I.

CALL FIGIHI

C EVALUATION OF U(Z,I,J,N+1) USING BACKWARD DIFFERENCE

CALL UZCNP(I)

3 CONTINUE
IXTRAP=0
RETURN
END
SUBROUTINE EVALU
DIMENSION U(4,16,4), JBAK(4,2,4)
DIMENSION S(4,2,4), H(4,2,4)
DIMENSION FBAK(4,2,4), GBAK(4,2,4), HBAK(2,4)
DIMENSION FC(4,4)
COMMON/UVECT0/U
COMMON/UINDEX/UBAR
COMMON/FGH/F,G,H
COMMON/FGHCON/FC
COMMON/TIMECO/D,DTX,DT,X5
COMMON/SPCECO/JM,JML1,JML2,JML3,IM,IM1,IM2,IM3,IM4
COMMON/FGHINT/FBAK,GBAK,HBAR
COMMON/KEEP/N1,N2
COMMON/LIMIT1/JMIN,JMX
COMMON/LIMIT2/JMIBAR,JMXBAR
COMMON/LIMIT3/JMXNEW
COMMON/EXTRA/IXTRAP

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

XX
XX EVALUATION OF VECTOR U(Z,1,J,N+1) BY USING A
XX TWO STEP LAX-WENDROFF WITH BACKWARD
XX FORWARD DIFFERENCE SCHEME.
XX
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

SETTING INDEXES
I IS THE COLUMN UNDER EVALUATION.
J IS THE ROW UNDER EVALUATION
N1=1
N2=2
1->
JMIN=2
JMX=JML1
JMIBAR=2
JMXBAR=JML2
CALL FGHVE(I)

EVALUATION OF UBAR AT COLUMN 2 USING BACKWARD DIFFERENCE

DO 1 J=2, JML2
DO 2 IVALUE=1,4
A=F(IVALUE,N1,J)+G(IVALUE,N1,J)
B=F(IVALUE+1,N1,J)+C(IVALUE,N1,J-1)
A=A-B
A=DTX*A
UBAR(IVALUE,N1,J)=A+U(IVALUE,I,J)
1 UBAR(3,N1,J)=UBAR(3,N1,J)+DELT*H(N1,J)

EVALUATION OF FBAR, GBAR, HBAR AT COLUMN 2

CALL FIGIMI
DO 3 I=2, IML2
N1=N2
N2=MOD(N2,2)+1
IFROMW=I+1

EVALUATION OF F,G,H AT COLUMN I+1

CALL FGHVE(IFROMW)

EVALUATION OF UBAR AT COLUMN I+1 USING BACKWARD DIFFERENCE

CALL UINBCA(IFROMW)

EVALUATION OF FBAR, GBAR, HBAR AT COLUMN I+1
CALL EVALUATE

EVALUATION OF U(Z,I,J,N+1) USING FORWARD DIFFERENCE UP TO TWO RUNS
AFTER THE FREE SURFACE SINCE THE LAST RUN BEFORE THE FREE SURFACE
IS A SPECIAL CASE.

CALL UNIFcell()

EVALUATION OF U(Z,I,J-2,N+1). J-2 IS A SPECIAL CASE.

DO 4 VALUE=1,4
U(I,VALUE,I,JML2)=C.500*(U(I,VALUE,I,JML2)*U(I,VALUE,N2,JML2)+
1DTX*(FBAR(I,VALUE,N1,JML2)+G(I,VALUE,N2,JML1)-
2FBAR(I,VALUE,N2,JML2)-GBAR(I,VALUE,N2,JML2))
4 CONTINUE
U(3,1,JML2)=U(3,1,JML2)+DTX*HBAR(N2,JML2)
3 CONTINUE
IXTRAP=0
RETURN
END
SUBROUTINE FQUAD

DIMENSION U(4,1:N1,1:N2)
DIMENSION RADIUS(1:N1)
COMMON/VECTOR/U
COMMON/FOUR/F,G,H
COMMON/TIMELT/L,N,T,LX,DT,X
COMMON/LIMIT/L,JMIN,JMX
COMMON/GAMAS/GAMA,GAMA1,GAMA2,GAMA3,CONST
COMMON/KEEP/N1,N2
COMMON/RADIUS

C

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USING VISCOSITY IN THE X-DIRECTION

DC 1 J=2, JML2
F(1,1,J)=U(2,1,J)/U(1,1,J)
F(2,1,J)=U(2,2,J)/U(1,2,J)
G(1,1,J)=ABS(F(2,1,J)-F(1,1,J))*CDT X
DO 2 IVALUE=1, 4
2 UBAR(IVALUE,1,J)=G(1,1,J)*(U(IVALUE+2,J)-U(IVALUE+1,J))
1 CONTINUE
N1=1
N2=2
DC 3 I=2, IML1
IFORW1=I+1
DC 4 J=2, JML2
F(N1,1,J)=U(2,1,JML1)/U(1,1,JML1)
G(N2,1,J)=ABS(F(N1,1,J)-F(N2,1,J))*CDT X
5 IVALUE=2, 4
UBAR(IVALUE,N2,J)=G(N2,1,J)*(U(IVALUE+2,J)-U(IVALUE+1,J))
U(IVALUE,1,J)=U(IVALUE,1,J)+UBAR(IVALUE,N2,J)-JBAR(IVALUE,N1,J)
1 CONTINUE
N1=2
N2=MIN(N2,2)+1
1 CONTINUE
SUBROUTINE U1VAL2(I)
    INT, INT*4(J1,J2,J3,J4),I,J2,J3,J4
    INT = INT(I,J2,J3,J4)
    COMM/JTV/TC/0
    COMM/JT/TEP/UBAR
    COMMON/IT,TEP/UBAR
    COMMON/IM,VE/DELT,DTX,DTX5
    COMMON/SI,EP/NI,N2
    COMMON/LIMIT2/JMBAR,JMXBAR
    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
    X
    X    EVALUATION OF PRELITTAL U1Z,I,J,I*11 BY USING
    X
    X    BACKWARD DIFFERENCE 
    X
    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

N1 IS THE COLUMN BEING EVALUATED
N2 IS THE COLUMN BEHIND THE ONE BEING EVALUATED

DO 1 J=JMBAR,JMXBAR
   DO 2 IV=1,4
      A=F(IVALE,N1,J)+G(IVALE,N1,J)
      R=F(IVALE,N2,J)+G(IVALE,N1,J-I)
      C=A-R
      C=C+DTX
   2   UBAR(IVALE,N1,J)=C+U(IVALE,I,J)
1   UBAR(3,N1,J)=UBAR(3,N1,J)+H(N1,J)*DELT
    END
DIMENSIONS: ETA= (4,1), \( \mu \) = (4,2), \( \mu \) = (2,4,1)

DIMENSIONS: \( \mu \) = (4,2,2), \( \mu \) = (4,2,1), \( \mu \) = (2,4,1)

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COMMON / JAUS / ETA= \( \mu \) = (4,2,2), \( \mu \) = (4,2,1), \( \mu \) = (2,4,1)
C : \text{EVALUATION OF } U(1,1,1,1,1) \text{ USING FORWARD DIFFERENCE}

\text{N1 IS THE COLUMN WHICH IS FORWARD TO THE ONE BEING EVALUATED}
\text{N2 IS THE COLUMN BEING EVALUATED}

\text{DO 1 J=2, JMXNE}
\text{DO 2 IVALUE=1,4}
U(IVALUE, I, J) = 0.5 \times (U(IVALUE, I, J) + \text{UBAR}(IVALUE, N2, J) + 10 \times (\text{FBAR}(IVALUE, N1, J) + \text{GBAR}(IVALUE, N2, J) + 1) - 2 \times \text{FBAR}(IVALUE, N2, J) - \text{GBAR}(IVALUE, N2, J))
\text{CONTINUE}
U(3, I, J) = U(3, I, J) + DTX \times \text{HBAR}(N2, J)
\text{CONTINUE}
\text{RETURN}
\text{END}
VALUATION of the difference

N1 IS THE COLUMN BEING EVALUATED
N2 IS THE COLUMN BEHIND THE ONE BEING EVALUATED

01 J=2, JM1, EM
02 DO IVALUE=1, 4
U(IVALUE, I, J) = 0.5000 * (U(IVALUE, I, J) + UBAR(IVALUE, N1, J) +
10*FBAR(IVALUE, N1, J) + GBAR(IVALUE, N1, J) -
2*FBAR(IVALUE, N2, J) - GBAR(IVALUE, N1, J-1))
2 CONTINUE
U(3, J) = U(3, J) + DTX5*HBAR(N1, J)
1 CONTINUE
RETURN
END
COMPUTATION CONTINUES...

DO 1 J=2, JNL2
DO 1 IVALUE=1, 4
U(IVALUE, M1, J) = 3.0 * (U(IVALUE, M2, J) - U(IVALUE, M3, J)) +
U(IVALUE, M4, J)
1 CONTINUE
RETURN
END
**C**

**DIMENSION RADIUS(4)**

**COMMON / RADIUS**

**COMMON / DELTA**

**COMMON / C, CO, JML1, JML2, ML, IC, IC1, IC2, IL1, IL2, IL3, ILM**

*XX...*  \( \text{EVALUATION OF THE RADIUS FROM THE ORIGIN TO THE POINTS} \)

*XX...*  \( \text{THE GRID POINTS} \)

**DELTA = DELTA**

**DO 1 J = 1, JMAX**

1 **RADIUS(J) = (J-1) * DELTA**

**RETURN**

**END**
SUBROUTINE FIVE

DIMENSION U(4,1,J,4), FC(4,4)

COMMON/VECTR/U
COMMON/HCN, FC
COMMON/FLK/(JL1, JL2, JL3, IM1, IM2, IM3, IM4)
COMMON/GAMAS/GAMAS, GAMAS1, GAMAS2, GAMAS3, CONST

C

EVALUATION OF VECTOR F FOR INITIAL DATA LINE

C

X

DC 1, J=2, J
X1=U(1,1,1,J)
X2=X1*U(2,1,J)
X3=X1*U(3,1,J)
WQUAD=X2*X2*X3
Y2=X2*U(2,1,J)
Y3=X3*U(3,1,J)
W5=GAMAS1*U(4,1,J)
FC(1,J)=-U(2,1,J)
FC(2,J)=GAMAS3*Y2+GAMAS2*Y3-W5
FC(3,J)=-X2*U(3,1,J)
FC(4,J)=-GAMAS*X2*U(4,1,J)+GAMAS*WQUAD*U(2,1,J)

1 CONTINUE
FC(1,1)=W5
FC(2,1)=W5
FC(3,1)=W5
FC(4,1)=W5
RETURN.
END
EXECUTIVE UGAMAT

COMMON /TEST/ITEST

XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX XXX X

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THIS SUBROUTINE SETS-UP THE INTERPOLATION SEQUENCE AND CALLS FOR THE INTERPOLATION SCHEME TO BE USED

CALL FORAC
ITEST=2
RETURN

CALL BACKFOR
ITEST=1
RETURN

END
COMMON/VECT/V
COMMON/JINTER/UBAR
COMMON/FGH/F,5,H
COMMON/TIME/DEL,T,OTX,OTX5
COMMON/KEEP/N1,N2
COMMON/LIMIT2/JMIBAR,JMXBAP

EVALUATION OF PROVITIONAL U(T,1,J,N+1) BY USING
FORWARD DIFFERENCE

N1 IS THE COLUMN FORWARD TO THE ONE BEING EVALUATED
N2 IS THE COLUMN BEING EVALUATED

DO 1 J=JMIBAR,JMXBAP
DC 2 VALUE=1,4
A=F(VALUE,N1,J)+G(VALUE,N2,J+1)
B=F(VALUE,N2,J)+G(VALUE,N2,J)
A=A-B
A=OTX*A
2 UBAR(VALUE,N1,J)=A+U(VALUE,I,J)
1 UBAP(3,N1,J)=UBAR(3,N1,J)+DELTH(N2,J)
RETURN
END
DO 1 I=2,IML1
DO 2 J=2, JML1
2 F(I,1,1)=U(3,1,1)/U(1,1,1)
   G(1,1,1)=ABS(F(1,1,2))*CDTY
   G(1,1,2)=ABS(F(1,1,3)-F(1,1,2))*CDTY
   DC 3 IVALUE=2,4
   UBAR(IVALUE,1,1)=G(1,1,1)*U(IVALUE,1,2)
   UBAR(IVALUE,1,2)=G(1,1,2)*U(IVALUE,1,3)-U(IVALUE,1,2))
3 U(IVALUE,1,2)=U(IVALUE,1,2)+UBAR(IVALUE,1,2)-UBAR(IVALUE,1,1)
   DC 4 J=3, JML2
   G(1,1,1)=ABS(F(1,1,J+1)-F(1,1,J))*CDTY
   DC 5 IVALUE=2,4
   UBAR(IVALUE,1,J)=G(1,1,J)*U(IVALUE,1,J+1)-U(IVALUE,1,J))
5 U(IVALUE,1,J)=U(IVALUE,1,J)+UBAR(IVALUE,1,J)-UBAR(IVALUE,1,J-1)
4 CONTINUE
1 CONTINUE
RETURN
END
SUBROUTINE TEST
DIMENSION U(4,14,4,4)
COMMON/SPACE/DELTX
COMMON/TIMECEDELTX,DTX
COMMON/SC18/MTEST
COMMON/VISCO/C,CDTX,CY,CDTY
COMMON/SC18/MTEST
COMMON/SC18/MTEST
COMMON/SC18/MTEST

C XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX X
C TEST FOR NEGATIVE DENSITIES X
C XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX X

C DO 1 I=2,1M
C DC 1 J=2,JML2
1 IF(U(I,J+1,J)=0.0,CONC=0)2,1,1
2 IF(I-1=M.L(1))6,5,6
6 IF(I.LT.1M.AND.1.GT.75)GO TO 3
C GO TO A
1 CONTINUE
KOUNT=KOUNT+1
RETURN
C 3 CONTINUE
C DO 11 J=2,JML1
C DC 11 K=1,4
A=K*0
DC 10 I=1,JML1
10 A=A+U(K,J,J)/I(I,J,J)-1.0
DC 12 I=81,JML1
12 U(J,J,J,J)=A
11 CONTINUE
C DO 2 10 J=2,JML1
C DC 2 K=1,4
\begin{verbatim}
D = A(J1, J2, J3) = A(J1) / c \\
U(K, J1) = \Phi \\
U(K, J2) = \Psi \\
U(K, J3) = \Theta \\
\theta = \phi \\
N = 1 = 2m, n, m \\
D = a + D \\
2 - U(K, I, J1) = a + AHC \\
RETURN \\
5 CONTINUE \\
   J0: J = 1, 4 \\
   J1: J = 1, J \\
20 U(K, IML1, J) = U(K, IML2, J) \\
   GO TO 3 \\
5 WRITE (6, 200) KUNIT, I, J, U(K, IML2, J) \\
200 FORMAT (NEGATIVE DENSITIES FOUND!, 3I5, E15.6) \\
STOP \\
END 
\end{verbatim}
SUBROUTINE DATA
DIMENSION U(4,1M,4C)
DIMENSION RADIUS(4C)
DIMENSION D(4)
COMMON/UVEC/0/U
COMMON/Κ/RA DIUS
COMMON/SPCCO/JM, JML1, JML2, JML3, IM, IML1, IML2, IML3, IML4
COMMON/TIME/DELT, DTX, DTX5
COMMON/SPACE/DELTX
COMMON/ESCRIB/MTES
COMMON/VISCO/C, LDTX, CY, CDTY
COMMON/GAMAS/GAMA, GAMA1, GAMA2, GAMA3, CONST
COMMON/DIME/ROJ, XMOMEJ, ENERJ
C
XXX
XXX
XXX
XXX
READ IN AND EVALUATION OF NECESSARY CONSTANT
XXX
XXX
READ(5,1000) MTES, C, GAMA, DTX, DELTX, XM, RM
1000 FORMAT(15,6F10.4)
READ(5,2000) ROJ, XMOMEJ
2000 FORMAT(2F10.4)
READ(5,3000) PINF, TINF, RINF, VELINF
READ(5,4000) U(1), D(1), D(2), D(3), D(4), IS, JS, IF, JF
4000 FORMAT(4F10.0,4I5)
3000 FORMAT(4F10.0)
GAMA1 = GAMA - 1.0
GAMA2 = (GAMA - 1.0) * 5000
GAMA3 = (GAMA - 3.0) * (5000)
CONST = 0.5000
DELT = DTX * DELTX
DTX5 = DELT * (5.0)
CDTX = C * DTX
JK = 40
JML2 = JM - 1
END
RETURN
2907 U(K*1.4)=0.64100
DO 2907 K=1,4
2907 J=1,4J
DO 2907 I=1,4*J
DO 44(D(4)S(4)ENERY
DO 43(D(3)=C(3)
D(2)=D(2)/XWOME
D(1)=D(1)/R*3
CONTINUE
444 U(4.14)=U(1.14)*J
443 U(3.14)=U(3.14)
442 U(2.14)=U(2.14)
441 U(1.14)=U(1.14)
401 I=I+1.0
DO 1 DO 1
ENER=ENER+ENER
DERN=DERN+DERN
ENER=ENER+DERN
ENER=ENER+DERN
ENER=ENER+DERN
ENER=ENER+DERN
ENER=ENER+DERN
ENER=ENER+
CALL P31D0.1
IML=4 IM=4
IML=3 IM=3
IML=2 IM=2
IML=1 IM=1
IM=1G
JML=3 JML=3
JML=2 JML=2
JML=1 JML=1
JML=0 JML=0
JML=-1 JML=-1
JML=-2 JML=-2
SUBROUTINE DATA
DIMENSION U(4,180,4)
COMMON/VECTC/U
COMMON/SPCECC/JM,JML1,JML2,JML3,IM,IML1,IML2,IML3,IML4
COMMON/TIMECO/DEL,T,DTX,DTX5
COMMON/SPACE/DELTX
COMMON/SPAC/C,CDTX,CY,COTY
COMMON/GAMAS/GAMA,GAMA1,GAMA2,GAMA3,CONST
COMMON/CONTAD/KGUNT
C
X
C
RESTART PROBLEM
C
X
C
READ(5,1000),C,GAMA,DTX,DELTX,CY
1000 FORMAT(5F10.4)
READ(5,2000),JM,IM,ITETA,ISTART,ITETAL,ITETA2
2000 FORMAT(6I5)
DO 37 I=1,IM
DO 37 K=1,4
37 U(K,1,1)=0.0
GAMA1=GAMA-1.0
GAMA2=(GAMA-1.0)*0.5C00
GAMA3=(GAMA-3.0)*0.5C00
CONST=0.5C00
DELTX=DTX*DELTX
DTX5=DELTX*1.510C
CDTX=C*DTX
COTY=CY*DTX
JML1=JM-1
JML2=JM-2
JML3=JM-3
IML1=IM-1
IML2=IM-2
IML3=IM-3
X
X
CALL RADI
READ (4) ((U(I,J,K),I=1,4),J=1,150,K=1,40),KUNT
REWIND 4
IF(ITETA.EQ.0) GO TO 300
DO 110 J=1,150
   DO 110 K=1,4
       DO 110 I=1,4
110 U(K,I,J)=U(K,1START,J)
CONTINUE
IF(ITETA1.EQ.0) GO TO 301
DO 10 J=2,11
   DO 10 I=40,55
      DO 10 K=1,4
10  U(K,I,J)=U(K,39,J)
CONTINUE
IF(ITETA2.EQ.0) GO TO 302
DO 20 J=2,11
   DO 20 K=1,4
      A=0.0
      DO 30 I=51,IMB
30  A=A+U(K,I,J)/(IMB-50)
      DO 31 I=51,IMB
31  U(K,I,J)=A
CONTINUE
RETURN
END
SUBROUTINE DATA
DIMENSION U(4,10,46)
DIMENSION RADIUS(46)
COMMON/CONTAD/KOUNT
COMMON/UVECTO/U
COMMON/R/RADIUS
COMMON/SPCECO/JM,JML1,JML2,JML3,IM,IML1,IML2,IML3,IML4
COMMON/TIMECC/DELT,DTX,DTX5
COMMON/SPACE/DELTX
COMMON/SCREIB/MTEST
COMMON/SPC/C,CTX,CY,CDTY
COMMON/GAMAS/GAMA,GAMA1,GAMA2,GAMA3,CONST
COMMON/DIME/ROJ,XMOMEJ,ENERJ
C
C
C
C
C
2000 FORMAT(2F10.0)
READ(5,2000)ROJ,XMOMEJ
3000 FORMAT(4F10.0)
GAMA1=GAMA-1.0
GAMA2=GAMA-1.0+0.500
GAMA3=GAMA-2.0+0.500
CONST=0.500
DELT=DTX*DELTX
TX5=DELT+(0.5*TX5)
CTX=C+DTX
JM=40
JML1=JM-1
JML2=JM-2
JML3=JM-3
CALL RADIO

ENEF = RJ0 * 32.2 / XMDEM / XMDEM

PIN = PIN * 14400

DENINF = PIN / ( RINF * TINF )

ENEIF = PIN / GAMAL + ( DENINF * VELINF * VELINF ) / ( 2.0 * 32.2 )

DENINF = DENINF / RJ0

ENEINF = ENEINF * ENERJ

READ(4) ((U(I,J,K),I=1,4),J=1,180),K=1,40),KOUNT

REWIND 4

DO 4 I = 1, 180

K = 1

DO 4 J = 1, 20

DO 5 K = 1, 4

U(K, I, J) = DENINF * RADIUS(J)

U(2, I, J) = U(1, I, J) * VELINF

U(3, I, J) = 0.000

U(4, I, J) = ENEINF * RADIUS(J)

5 CONTINUE

4 K ++ = K ++ + 2

CONTINUE

RETURN

END
SUBROUTINE DATA
DIMENSION A(100,2,0),B(100)
DIMENSION U(4,18C,4C)
DIMENSION RADIUS(4C)
COMMON/VECTU/U
COMMON/INTER/X1,Y1,X2,Y2,X1,Y1,RAT,COORD,XGURY
COMMON/INLIN/V1,V2,V1
COMMON/R/RADIUS
COMMON/CONTAD/KOUNT
COMMON/SPCEO/JM,JML1,JML2,JML3,IM,IML1,IML2,IML3,IML4
COMMON/TIMEDE/DELTX,DTX,DTX5
COMMON/SPACE/DELTX
COMMON/ESCRIB/HTEST
COMMON/SPCEC/CM,CDTX,XY,CDTY
COMMON/GAMAS/GAMA,GAMA1,GAMA2,GAMA3,CONST
COMMON/SPACEM/ROJ,XMOMEJ,ENERJ

C XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C X
C X READ IN AND EVALUATION OF NECESSARY CONSTANT X
C X
C XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

READ(5,1000) IMAX,XI,DELT,JMAX
1000 FORMAT(5,2F16.3,15)
  KOUNT=0
  DO 1 J=1,1MAX
    DO 1 K=1,2
      READ (5,2000) (A(J,K,L),L=1,6)
1000   CONTINUE
2000 FORMAT(6F16.3)
  DO 2 J=1,1MAX
    X1=A(J,1,1)
    Y1=A(J,1,2)
    X2=A(J,2,1)
    Y2=A(J,2,2)
    Y1=0.0
  CALL MATIC
CALL LINIT
A(J+1,L)=VI
3 CONTINUE
A(J,1,1)=CGORX
A(J,1,2)=COORY
2 CONTINUE
8(LI)=C$0
DO 4 K=1,JMAX
4 B(K+1)=B(K)+DETY
DO 5 K=2,JMAX
DO 6 J=1,1MAX
IF(A(J+1,2).LT.B(K).AND.A(J+1,1,2).GT.B(K))GO TO 7
6 CONTINUE
7 CONTINUE
KOUNT=KOUNT+1
X1=A(J,1,1)
Y1=A(J,1,2)
X2=A(J+1,1,1)
Y2=A(J+1,1,2)
Y1=B(K)
CALL RATIO
DO 8 KL=3,6
V1=A(J,1,KL)
V2=A(J+1,1,KL)
CALL LINIT
8 A(KOUNT,2,KL)=VI
A(KOUNT,2,11)=CGORX
A(KOUNT,2,21)=COORY
5 CONTINUE
FORMAT (5F10.4)
A=1-COUNT-1.
READ (5, 5, 60) PLF, TINF, RINF, VELINF
555.
FORMAT (4F10.6)
ENERJ = ROJ * 32.2 / XMOMEJ / XMOMEJ
PINF = PINF * 144.0
DENINF = PINF / (RINF * TINF)
ENEINF = PINF / GAMAI * (DENINF * VELINF * VELINF) / (2.0 * 32.2)
DENINF = DENINF / ROJ
ENEINF = ENEINF * ENERJ
DO 10 I = 1, JMAX
   A(I, 2, 3) = A(I, 2, 3) * 32.2
   A(I, 2, 5) = (A(I, 2, 5) / GAMAI) * A(I, 2, 3) * A(I, 2, 4) * A(I, 2, 4) / (2.0 * 32.2) *
   ENERJ
   A(I, 2, 2) = A(I, 2, 2) / (3 * 141593 / 180.0)
   ACOS = COS (A(I, 2, 2))
   ASIN = SIN (A(I, 2, 2))
   A(I, 2, 1) = A(I, 2, 4) * ASIN
   A(I, 2, 4) = A(I, 2, 4) * ACOS
   A(I, 2, 6) = A(I, 2, 3) * A(I, 2, 6) / XMOMEJ
   A(I, 2, 4) = A(I, 2, 3) * A(I, 2, 4) / XMOMEJ
10   A(I, 2, 3) = A(I, 2, 3) / ROJ
   GAMAI = (GAMA - 1.0) * C.50G
   GAMA3 = (GAMA - 3.0) * C.5000G
  CONST = 0.50CG
   DELT = DTX * DELTX
   DTX5 = DELT * C.50CG
   CCTX = C * DTX
   J... = 4
   JML1 = JM - 1
   JML2 = JM - 2
   JML3 = JM - 3
   JM = 10G
   IML1 = IM - 1
   IML2 = IM - 2
   IML3 = IM - 3
   IM = ...
CALL RADIUS

DO 242 J = 1, 4
U(1,1,J) = RADIUS(J)*VARIABLES
J(1,1,J) = VALUE
U(3,1,J) = RADIUS(J)*VALUE
242 U(4,1,J) = RADIUS(J)*VARIABLES

DO 243 J = 2, JMAX
U(1,1,J) = RADIUS(J)*A(J-1,2,3)
U(2,1,J) = RADIUS(J)*A(J-1,2,4)
U(3,1,J) = RADIUS(J)*A(J-1,2,6)
243 U(4,1,J) = RADIUS(J)*A(J-1,2,5)

DO 244 I = 2, 100
DO 244 J = 1, 4
DO 244 K = 1, 4
244 U(K,1,J) = U(K,1,J)
RETURN
END
SUBROUTINE LININT
COMMON/INTER/X1,Y1,X2,Y2,X1,Y1,PAR,COURX,COORY
COMMON/INTLIN/V1,V2,V1
V1=V1+AT*(V2-V1)
RETURN
END
SUBROUTINE RATIG
COMMON/INTER/X1,Y1,X2,Y2,X1,Y1,RAT,CORRX,CORRY
A=(X2-X1)
B=(Y2-Y1)
C=A*A
D=R*R
E=C+D
E=SQRTE(E)
IF(X1.EQ.X2)GO TO 20
IF(Y1.EQ.Y2)GO TO 10
IF(X1.EQ.Y1)GO TO 40
IF(Y1.EQ.X1)GO TO 30
20 RAT=(Y1-Y1)/(Y2-Y1)
CORRX=X1
CORRY=Y1
RETURN
10 RAT=(X1-X1)/(X2-X1)
CORRX=X1
CORRY=Y1
RETURN
40 ASIN=B/E
RAT=(Y1-Y1)/ASIN
CORRY=Y1
CORRX=X1+(A/E)*RAT
RAT=RAT/E
RETURN
20 ACOS=E/E
RAT=(X1-X1)/ACOS
CORRX=X1
CORRY=Y1+(C/E)*RAT
RAT=RAT/E
RETURN
N.)
\[
\begin{align*}
J &= \frac{\pi}{4} \\
U(4, i, j) &= \text{DIME} \\
\text{IF}(U(3, i, j) \leq 0 \land U(2, i, j) \leq 0) \text{ GO TO 5} \\
\text{IF}(U(2, i, j) \leq 0) \text{ GO TO 6} \\
\text{ANGLE}(i, j) &= \text{ATAN} \left( \frac{U(3, i, j)}{U(2, i, j)} \right) \text{ RADCRA} \\
\text{GO TO 7} \\
5 \quad \text{ANGLE}(i, j) &= 0 \\
\text{GO TO 7} \\
6 \quad \text{ANGLE}(i, j) &= 90^\circ \\
7 \quad \text{CONTINUE} \\
\text{DOMY}(i, j) &= \sqrt{C/B} \\
U(1, i, j) &= U(1, i, j) \times \text{DENSj} \\
U(3, i, j) &= U(4, i, j) \times \text{CONST} / U(1, i, j) \\
U(2, i, j) &= \text{VELj} \times \sqrt{C/B} \\
2 \quad \text{CONTINUE} \\
\text{DO } 3 \text{ } i = 1, IM \\
3 \quad \text{WRITE}(6, 10C1)X(I), (R(J), U(4, I, J), U(1, I, J), U(3, I, J), U(2, I, J), \\
\text{DOMY}(I, J), \text{ANGLE}(I, J), J = 2, 40) \\
1C01 \quad \text{FORMAT}(6C \times \text{X}=\text{F6.3/2(1X, \text{R}, \text{F6.3/2(1X, \text{P}, \text{F6.3/2(1X, \text{DENS}, \text{3X, TEM}}, \text{3X, TEM}}, \text{1X, TEM}}, \text{1X, TEM}}, \text{2X})/ \\
\text{RETURN} \\
\text{END}
\end{align*}
\]
BLOCK DATA
COMMON/EXTRA/I XTRAP
COMMON/TEST/ITEST
COMMON/CONTAD/KOUNT
DATA I XTRAP/1/
DATA ITEST/1/
DATA KOUNT/1/
END
APPENDIX B

METHOD OF CHARACTERISTICS INPUT GUIDE

A method of characteristics program which was used to describe supersonic plumes was obtained from Lockheed and Missiles and Space Company. An input data guide for this program follows, Reference B.1.

<table>
<thead>
<tr>
<th>CARD NO. 1</th>
<th></th>
<th>Problem Title or Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Format:</td>
<td>12A6</td>
<td></td>
</tr>
<tr>
<td>Cols. 1-72</td>
<td>HOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CARD NO. 2</th>
<th>Run Control Card</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Format:</td>
<td>16I5</td>
<td>1 Read cards for gas properties</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 Read tape 10 (A6) for gas properties</td>
</tr>
<tr>
<td>Col 5</td>
<td>ICON(1)</td>
<td>0 Normal start line</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 Right running characteristic start line</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 Left running characteristic start line</td>
</tr>
<tr>
<td>Col 9</td>
<td>ICON(2)</td>
<td>0 Straight start line M given</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 Source start line A/A* given</td>
</tr>
<tr>
<td>Col 10</td>
<td>ICON(2)</td>
<td>2 Starting line input</td>
</tr>
</tbody>
</table>
Col 14, 15  ICON(3)  Number of starting line points

Col 19, 20  ICON(4)  Number of upper boundary equations

Col 24, 25  ICON(5)  Number of lower boundary equations

Col 30  ICON(6)  No. SC-4020 plots

Col 35  ICON(7)  Two dimensional solution

Col 39  ICON(8)  Full output

Col 40  

Col 41-43  ICON(9)  No. of left running points up to and including upstream shock point. Used when ICON(2) ≥ 20 and shock crosses starting line.

Col 44-45  ICON(9)  Number of regular start line points if ICON(2) ≥ 20

3 Starting line calculated by conservation of mass
| Col 50 | ICON(10) | 0 Radiance tape not desired  
|       |          | 1 Radiance tape desired (1 tape)  
|       |          | 2 Radiance tape desired (2 tapes)  
| Cols 51-55 | ICON(11) | Case number (prints at top of each page)  
| Col 60 | OCON(12) | 0 Calculate shock wave  
|         |          | 1 No rotation option  
| Cols 61-65 | ICON(13) | 1 Use viscous boundary layer  
| Cols 66-80 | ICON(14-16) | Not presently used  
|        | CARD(S) NO. 3 | Describes physical boundaries of the flow field  
| Format | 11, 3X, 11, 5X, 6E10.6 |  
| Col 1 | IWALL | 1, Conic equation  
|       |       | R=A*(SQRT(B+C*X+D*X**2)+E)  
|       | IWALL | 2 Polynomial equation  
|       |       | R=A*X**4+B*X**3+C*X**2+D*X+E  
|       | IWALL | 3 Free boundary equation  
|       |       | P=PINF*(1+GAMMAINF*(MINF*SIN*(THETAB-THETAINF))**2)  
| Col 5 | ITRANS | 0 No discontinuity follows this equation  
|       | ITRANS | 1 Expansion corner follows this equation  
|       | ITRANS | 2 Compression corner follows this equation  
| Cols 11-20 | WALLCO | A(If IWALL = 1 or 2). PINF (If IWALL=3)  
|         | WALLCO | B(If IWALL = 1 or 2), GAMMAINF |
(If IWALL = 3)

31-40 WALLCO C(IF IWALL = 1 or 2), MINF (If IWALL = 3)

41-50 WALLCO D(IF IWALL = 1 or 2), THETAINF (IF IWALL = 3)

51-60 WALLCO E(IF IWALL = 1 or 2), 1. (IF IWALL = 3)

61-70 XMAX Maximum X value for which this equation applies.

NOTE - The coefficients of each equation are contained on a single card. As many cards, i.e., equations, as necessary to describe the boundaries are input. The units of physical dimensions affect only the thrust calculations in which units of feet are assumed. Upper boundary information is given first. Program assumes that starting line is bounded by solid walls and that the equations are ordered with XMAX monotonically increasing.

CARD NO. 4

Gas Identification and Gas Property Input Control

Format: 4A6, 5X, A3, 7X, II

Cols 1-24 ALPHA Gas name, identification for real gas properties on tape. May be any name when gas properties are input via cards.

Cols 30-32 UNITS ENG English units are to be input (cards only)

MKS Metric Units (cards or tape)
Col 40  IS  Number of entropy cuts (ignored for tape, 1 for ideal gas, 9 max for real gas via cards).

Cards No. 5

Format:  E10.6, 8X, 12

Col 1-10  STAB  Entropy value

Col 19, 20  IVTAB  Number of Mach numbers for this entropy value 13 max, (1 if ideal gas)

Cards No. 6

This card(s) gives the Mach number and associated gas properties at that Mach number and entropy.

Format:  5E10.6

Cols 1-10  TAB  Mach number

Cols 11-20  TAB  Gas constant (R) if UNITS = ENG,
Molecular weight (MWT) If Units = MKS.

Cols 21-30  TAB  GAMMA

Cols 31-40  TAB  Static temperature (TO) at this Mach number

Cols 41-50  TAB  Static pressure (TO) at this Mach number

NOTE - Cards 5 and 6 are omitted if gas properties are input via tape.
CARD NO. 7

This card specified the necessary information for the starting line.

Format: 5E10.6

Cols 1-10 CORLIP Axial coordinate of upper limit of start line.

Cols 11-20 CORLIP Axial coordinate of lower limit of start line.

Cols 21-30 CORLIP Mach number or A/A* for start line.

Cols 31-40 CORLIP Entropy level of start line.

Cols 41-50 CORLIP Area of nozzle throat (units consistent with boundary equations).

Cols 61-70 STEP(3) Point insert criteria

CARD NO. 8

These cards are used to read in a known starting line (ICON(2) = 2), otherwise omitted.

Format: 5E10.6

Cols 1-10 PSI Radial coordinate of this point

Cols 11-20 PSI Axial coordinate of this point

Cols 21-30 PSI Mach number of this point

Cols 31-40 PSI Flow angle of this point

Cols 41-50 PSI Entropy level of this point

Cols 51-60 PSI Shock angle of downstream shock point when ICON(9) ≥ 1.0

NOTE - As many of these cards are input as specified by ICON(3) on the run control card.
CARD NO. 9

This card contains the necessary information to limit the calculations to those areas of interest. An unusual scheme is employed in order to make these limits efficient for the many problem orientations which are possible.

Format: \(6\times10.6\)

Cols 1-10 CUTDAT Radial coordinate defining upper cutoff.
Cols 11-20 CUTDAT Axial coordinate defining upper cutoff.
Cols 21-30 CUTDAT Angle cutoff line makes with horizontal.
Cols 31-40 CUTDAT Radial coordinate defining downstream cutoff.
Cols 41-50 CUTDAT Axial coordinate defining downstream cutoff.
Cols 51-60 CUTDAT Angle cutoff line makes with horizontal

CARD NO. 10

This card contains the input information for the viscous boundary layer option

Format: \(11, 2X, 12, 5X, 6\times10.6\)

Col 1 NPOWER Exponent of the velocity profile in the boundary layer
Cols 4-5 NBLPTS Number of boundary layer points
**CARD NO. 11**

A characteristic length (usually nozzle length)

Conversion factor for mixed units of length

This card is used when ICON (10) equals 1 or 2. Its purpose is to "juggle" the method of characteristics output such that gas properties in the flow field can be determined at designated points and saved on binary output tapes.

| Col 5 | LASTC | 0 Juggle case other than last or only case  
| Col 10 | ISETR | 0 Radial value at intersection of juggled axial location and left running characteristic.  
| Col 15 | MAXNOR | Maximum number of radial increments (used only if ISETR = 1, maximum number allowable is 10) DELTA RS)  

Col 20  ITIME(1)  Number of times first DELTA X is used

ITIME(2)  Number of times second DELTA X is used

ITIME(3)  Number of times third DELTA X is used

ITIME(4)  Number of times fourth DELTA X is used

ITIME(5)  Number of times fifth DELTA X is used

ITIME(6)  Number of times sixth DELTA X is used

NOTE - Up to a maximum of six Delta X changes are allowed but not always necessary.

CARD No. 12  Continuation of Card No. 10

Format:  7E10.6

Col 1-10  DELT(1)  Length of first Delta X increment.

Col 11-20  DELT(2)  Length of second DELTA X increment.

Col 21-30  DELT(3)  Length of third DELTA X increment.

Col 31-40  DELT(4)  Length of fourth DELTA X increment.

Col 41-50  DELT(5)  Length of fifth DELTA X increment.

Col 51-60  DELT(6)  Length of sixth DELTA X increment.

Col 61-70  DELTAR  Radial increment used if ISETR = 1

NOTE - Total length of summation of DELTA X must be greater than, or equal to axial cutoff value.
CARD No. 13

Format: 3012

Col 2 \( (J) \) 1 Mach No.

Col 4 2 Press. static

Col 6 3 Temp. static

Col 8 4 Press (stag norm shock)

Col 10 5 GAMMA

Col 12 6 Mole wt

Col 14 7 Density

Col 16 8 Entropy

NOTE - The order of these parameters is arbitrary

NOTE - All units used are ENG. (except entropy) which may be either ENG. or MKS. depending on gas property units. Following is a list of units for specific parameters.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Thrust calc. assumes ft. Otherwise any scale acceptable</td>
</tr>
<tr>
<td>R</td>
<td>Deg.</td>
</tr>
<tr>
<td>All angles</td>
<td>FT<strong>2/SEC</strong>2DEG.R=ENG.</td>
</tr>
<tr>
<td>S</td>
<td>CAL/GRAMDEG.K=MKS</td>
</tr>
<tr>
<td>Density</td>
<td>LBM/FT**3</td>
</tr>
<tr>
<td>T</td>
<td>DEG.R=ENG, DEG,K=MKS</td>
</tr>
<tr>
<td>V</td>
<td>FT/SEC</td>
</tr>
<tr>
<td>P</td>
<td>PSFA=ENG, ATM=MKS</td>
</tr>
<tr>
<td>R</td>
<td>1545.4*32.17/MWT</td>
</tr>
</tbody>
</table>
REFERENCE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>damping coefficient</td>
</tr>
<tr>
<td>( C_p )</td>
<td>specific heat at constant pressure</td>
</tr>
<tr>
<td>D</td>
<td>diameter</td>
</tr>
<tr>
<td>( D' )</td>
<td>damping coefficient</td>
</tr>
<tr>
<td>E</td>
<td>internal energy per unit volume</td>
</tr>
<tr>
<td>e</td>
<td>internal energy per unit mass</td>
</tr>
<tr>
<td>( \vec{F}, \vec{G}, \vec{H} )</td>
<td>vector function of conservation variable:</td>
</tr>
<tr>
<td>I</td>
<td>unit tensor</td>
</tr>
<tr>
<td>( \vec{i} )</td>
<td>unit vector x-direction</td>
</tr>
<tr>
<td>( \vec{j} )</td>
<td>unit vector y or r-direction</td>
</tr>
<tr>
<td>( \vec{k} )</td>
<td>unit vector z-direction</td>
</tr>
<tr>
<td>( \hat{L} )</td>
<td>external forces</td>
</tr>
<tr>
<td>( \hat{L} )</td>
<td>force vector per unit volume</td>
</tr>
<tr>
<td>M</td>
<td>Mach number</td>
</tr>
<tr>
<td>( M_\infty )</td>
<td>mass</td>
</tr>
<tr>
<td>m</td>
<td>momentum per unit volume x-direction</td>
</tr>
<tr>
<td>n</td>
<td>momentum per unit volume y or r-direction</td>
</tr>
<tr>
<td>( \vec{n} )</td>
<td>unit vector</td>
</tr>
<tr>
<td>p</td>
<td>pressure</td>
</tr>
<tr>
<td>Q</td>
<td>matrix</td>
</tr>
<tr>
<td>( \vec{Q} )</td>
<td>vector density</td>
</tr>
<tr>
<td>q</td>
<td>momentum per unit volume z-direction</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathbf{q}$</td>
<td>heat conduction vector</td>
</tr>
<tr>
<td>$R$</td>
<td>ideal gas constant</td>
</tr>
<tr>
<td>$r$</td>
<td>radial coordinate</td>
</tr>
<tr>
<td>$S$</td>
<td>surface</td>
</tr>
<tr>
<td>$s$</td>
<td>entropy per unit mass</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
</tr>
<tr>
<td>$T_c$</td>
<td>stagnation temperature</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$\mathbf{U}$</td>
<td>conservation variable vector</td>
</tr>
<tr>
<td>$u$</td>
<td>$x$-component of velocity</td>
</tr>
<tr>
<td>$\mathbf{V}$</td>
<td>velocity vector</td>
</tr>
<tr>
<td>$v$</td>
<td>$y$ or $r$ component of velocity</td>
</tr>
<tr>
<td>$w$</td>
<td>$z$-component of velocity</td>
</tr>
<tr>
<td>$x$</td>
<td>axial coordinate</td>
</tr>
</tbody>
</table>

**Greek**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Mach angle</td>
</tr>
<tr>
<td>$\beta$</td>
<td>shock angle</td>
</tr>
<tr>
<td>$\beta'$</td>
<td>constant equals zero for 1-dimensional; equals one otherwise</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>specific heat ratio</td>
</tr>
<tr>
<td>$\delta$</td>
<td>shock deflection angle</td>
</tr>
<tr>
<td>$\delta', \delta''$</td>
<td>difference operator</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>constant equals one for axisymmetric flows; equals zero otherwise</td>
</tr>
<tr>
<td>$\xi$</td>
<td>constant, equals one for 3-dimensions; equals zero otherwise</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>θ</td>
<td>flow angle</td>
</tr>
<tr>
<td>Λ</td>
<td>volume</td>
</tr>
<tr>
<td>λ̃</td>
<td>local velocity of discontinuity</td>
</tr>
<tr>
<td>µ</td>
<td>viscosity</td>
</tr>
<tr>
<td>ρ</td>
<td>density</td>
</tr>
<tr>
<td>n(r,t)</td>
<td>hypersurface</td>
</tr>
<tr>
<td>Ψ</td>
<td>viscous stress tensor</td>
</tr>
<tr>
<td>ψ</td>
<td>scalar flux</td>
</tr>
</tbody>
</table>

**Subscripts**

| a      | ambient |
| e      | nozzle exit conditions |
| n,t    | normal and tangential directions to shock wave |
| r,t,x,y,z | partial derivative symbolic notations |
| l, l'  | upstream, downstream to shock |
| I,II   | right and left running characteristics |

**Superscripts**

| ^      | tensor |
| -      | vector |
| °      | estimated values on time |
| o      | degree |
VITA

Ricardo Jesús Jofre was born in Havana, Cuba, on February 16, 1943, the son of Carlos Jofre and Georgina Jofre. He attended school in Havana, Cuba, and was graduated from Colegio de Belen in June of 1960. In February, 1961, he entered Louisiana Polytechnic Institute. In June, 1965, he received the Bachelor of Science Degree in Mechanical Engineering. In 1967, he received the Master of Science Degree in Mechanical Engineering from Louisiana Polytechnic Institute. He is married to the former Lourdes Silvia Navarro of Santa Clara, Cuba. He is now a candidate for the degree of Doctor of Philosophy in Mechanical Engineering.
Candidate: Ricardo Jesus Jofre

Major Field: Mechanical Engineering

Title of Thesis: The Mach Disc in Axisymmetric Rocket Plumes

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

May 19, 1971