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The Ring Theory and the Representation Theory of Quantum Schubert Cells

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Notational Conventions

Symbol	Definition
$\Bbbk$, $q \in \Bbbk^\times$	arbitrary field and a nonzero, non root of unity
$\mathfrak{g} \supseteq \mathfrak{h}$	complex semisimple Lie algebra and choice of Cartan subalgebra
Δ, $\Delta^\vee$, Δ_+	root system of rank r, coroots, and positive roots
$\Pi = \{\alpha_1, \ldots, \alpha_r\}$, $\alpha^\vee$	choice of simple roots, coroot of α
$\langle -, - \rangle$	symmetric bilinear form on $\mathbb{R}\Pi$ normalized to $\langle \alpha, \alpha \rangle = 2$ for short roots
$Q \subset P$, $Q_{\geq 0}$, $P_{\geq 0}$	roots lattice, weight lattice, dominant integral part of root and weight lattice
$I \subset \Pi$, Q_I	parabolic subset of simple roots and corresponding root lattice
$\{\varpi_1, \ldots, \varpi_r\}$	fundamental weights of $\mathfrak{g}$
$w \in W$, $s_{i_1}, \ldots, s_{i_r}$	Weyl group of $\mathfrak{g}$ and generating simple reflections
$\mathbf{i}$, $\mathbf{w}$, $W^{\leq w}$	reduced word for w, reduced decomposition of w, Bruhat order interval
$\mathcal{B}$, $T_{i_1}, \ldots, T_{i_r}$	braid group associated to the W and generators
$U_q(\mathfrak{g})$	quantized universal enveloping algebra of $\mathfrak{g}$
$\mathscr{M}$	semisimple category of finite dimensional integrable $U_q(\mathfrak{g})$-modules
$V(\lambda) \in \mathscr{M}$	irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ
$U^-[w]$, Spec$U^-[w]$	quantum Schubert cell algebra associated to w, n.c. prime spectrum of $U^-[w]$
$\mathbb{H}$, $\mathbb{H}$-Spec$U^-[w]$	algebraic torus $(\Bbbk^\times)^r$ and the $\mathbb{H}$-prime spectrum of $U^-[w]$
$\varphi : U^-[w] \hookrightarrow \overline{U^-[w]}$	Cauchon's deleting derivations map to the Cauchon quantum affine space
G	connected, simply connected, algebraic group G with Lie algebra $\mathfrak{g}$
$R_q[G] \supset R$	quantized coordinate ring of G, dominant part of $R_q[G]$
$R_w \supset R_w^0$	localization of R, Cartan-invariant subalgebra of R_w (under *right hit*)
$\phi_w : R_w^0 \twoheadrightarrow U^-[w]$	Yakimov's surjection from the representation theoretic approach to $U^-[w]$

Abstract

In recent years the quantum Schubert cell algebras, introduced by Lusztig and De Concini–Kac, and Procesi, have garnered much interest as this versatile class of objects are furtive testing grounds for noncommutative algebraic geometry. We unify the two main approaches to analyzing the structure of the torus-invariant prime spectra of quantum Schubert cell algebras, a ring theoretic one via Cauchon's deleting derivations and a representation theoretic characterization of Yakimov via Demazure modules. As a result one can combine the strengths of the two approaches. In unifying the theories, we resolve two questions of Cauchon and Mériaux , one of which involves the Cauchon diagram containment problem. Moreover, we discover explicit quantum-minor formulas for the final generators arising from iterating the deleting derivation method on any quantum Schubert cell algebras. These formulas will play a large role in subsequent research. Lastly, we provide an independent and elegant proof of the Cauchon–Mériaux classification. The main results in this thesis appear in [GY] and are joint with Milen Yakimov.

Introduction

Approximately twenty years ago, Joseph [Jos94, Jos95] and Hodges–Levasseur–Toro [HLT97] obtained a number of important results on them the spectra of quantum groups for generic parameter q,

One of their main goals was to understand these spectra in terms of symplectic foliations in an attempt to extend the orbit method [Dix96] to more general classes of noncommutative algebras. This lead to the study of various quantum analogs of universal enveloping algebras of solvable Lie algebras. The quantum Schubert cell algebras $U^-[w]$, defined by De Concini–Kac–Procesi [DCKP95] and Lusztig [Lus93], comprise a large and versatile such class algebras. For every simple Lie algebra $\mathfrak{g}$ and an element w, one constructs $U^-[w]$, which is a deformation of the universal enveloping algebra $U(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$ and a subalgebra of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ Alternatively, we can view the algebra $U^-[w]$ as a deformation of the coordinate ring of the Schubert cell corresponding to w of the full flag variety of $\mathfrak{g}$ equipped with the standard Poisson structure [GY09]. These algebras played important roles in many different contexts in recent years such as the study of coideal subalgebras of $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{g})$ [HS, HK12] and quantum cluster algebras [GLS].

There are two very different approaches to the study of the spectra of $U^-[w]$. One is purely ring theoretic and is based on the Cauchon procedure of deleting derivations [Cau03a]. The second is a representation theoretic one and builds on the above mentioned methods of Joseph, Hodges, Levasseur, and Toro [Jos95, HLT97]. Each of these methods has a number of advantages over the other, and relating them was an important open problem with many potential applications. Previously there were no connections between them even for special cases of the algebras $U^-[w]$, such as the algebras of quantum matrices.

In this paper we unify the ring theoretic and the representation theoretic approaches to the study of Spec$U^-[w]$. Furthermore, we resolve several other open problems on the deleting derivation procedure and the spectra of $U^-[w]$, two being questions posed by Cauchon and Mériaux [MC10]. Before we proceed with the statements of these results, we need to introduce some additional background.

There is a canonical action of the torus $\mathbb{H} = (\mathbb{k}^*)^{\times r}$ on $U^-[w]$ by algebra automorphisms, where $\mathbb{k}$ is the base field and r is the rank of $\mathfrak{g}$. By a general stratification result of Goodearl and Letzter [GL00], one has a partition

$$\mathrm{Spec}U^-[w] = \bigsqcup_I \mathrm{Spec}_I U^-[w] \text{ over } I \in \mathbb{H}\text{-}\mathrm{Spec}U^-[w]$$

Here $\mathbb{H}$-Spec$U^-[w]$ denotes the set of $\mathbb{H}$-invariant prime ideals. By two general results of [GL00] $\mathbb{H}$-Spec$U^-[w]$ is finite and each stratum

$$\mathrm{Spec}_I U^-[w] = \left\{ L \in \mathrm{Spec}U^-[w] \mid \bigcap_{t \in \mathcal{H}} t(L) = I \right\}$$

is homeomorphic to the spectrum of a (commutative) Laurent polynomial ring. The problem of the description of the Zariski topology of Spec$U^-[w]$, however, is wide open.

The Cauchon method of deleting derivations is a multi-stage recursive procedure [Cau03a] beginning with an iterated Ore extension A of length l (of a certain general type) equipped with a compatible $\mathbb{H}$-action and ending with a quantum affine space algebra $\overline{A}$ with a $\mathbb{H}$-action. Cauchon constructed in [Cau03a] a set-theoretic embedding of Spec$A$ into Spec$\overline{A}$. It restricts to a set-theoretic embedding $\mathbb{H}$-Spec$A \hookrightarrow \mathbb{H}$-Spec$\overline{A}$. The $\mathbb{H}$-invariant prime ideals of A are then parametrized by some of the subsets of $[1, l]$, called Cauchon diagrams. The $\mathbb{H}$-prime ideal of A corresponding to a Cauchon diagram $D \subseteq [1, l]$ will be denoted by J_D. The problem of determining which subsets of $[1, l]$ arise in this way (i.e., are Cauchon diagrams), is the essence of the method and is very difficult for each particular class of algebras. It was solved for the algebras of quantum matrices by Cauchon [Cau03a] and for all algebras $U^-[w]$ by Cauchon and Mériaux [MC10]. To state the latter result, we denote the set of simple roots of $\mathfrak{g}$ by Δ and the corresponding simple reflections of W by s_α, $\alpha \in \Delta$. A word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ in the alphabet Δ will be called a reduced word for w if $s_{\alpha_1} \ldots s_{\alpha_l}$ is a reduced expression of w. Each reduced word $\mathbf{i}$ for w gives rise to a presentation of $U^-[w]$ as an iterated Ore extension of length l. The subsets of $[1, l]$ are index sets for the subwords of $\mathbf{i}$ by the assignment $\{j_1 < \ldots < j_n\} \longmapsto (\alpha_{j_1}, \ldots, \alpha_{j_n})$. We will denote by $\leq$ the (strong) Bruhat order on W and set $W^{\leq w} = \{y \in W \mid y \leq w\}$. For each $y \in W^{\leq w}$ there exists a unique left positive subword of $\mathbf{i}$ corresponding to y (see Section 2.2 for its definition and details on Weyl group combinatorics). Its index set will be denoted by $\mathcal{D}^+_{\mathbf{i}}(y)$. The Cauchon–Mériaux classification theorem states the following:

For all Weyl group elements $w \in W$ and reduced words $\mathbf{i}$ for w, consider the presentation of $U^-[w]$ as an iterated Ore extension correspodning to $\mathbf{i}$. The Cauchon diagrams of the $\mathbb{H}$-prime ideals of $U^-[w]$ are precisely the index sets $\mathcal{D}^+_{\mathbf{i}}(y)$ for $y \in W^{\leq w}$.

The representation theoretic approach [Yak10] to the spectra Spec$U^-[w]$ relies on a family of surjective $\mathbb{H}$-equivariant antihomomorphisms $\phi_w \colon R_0^w \longrightarrow U^-[w]$, where R_0^w are certain quotients of subalgebras of the quantum groups $R_q[G]$. The algebras R_0^w were introduced by Joseph [Jos95] as quantizations of the coordinate rings of w-translates of the open Schubert cell of the flag variety of $\mathfrak{g}$, see Section 4.2 for details. Via these maps one can transfer back and forward questions on the spectra of $U^-[w]$ to questions on the spectra of quantum function algebras. The latter can be approached via representation theoretic methods, building on the works of Joseph [Jos94, Jos95], Gorelik [Gor00], and Hodges–Levasseur–Toro [HLT97]. This leads to an explicit picture for $\mathbb{H}$-Spec$U^-[w]$. First, the $\mathbb{H}$-invariant prime ideals of $U^-[w]$ are parametrized by $W^{\leq w}$, and the ideal $I_w(y)$ corresponding to $y \in W^{\leq w}$ is explicitly given in terms of Demazure modules using the maps ϕ_w, see 4.2.2 for a precise statement. Second, each of the strata $\mathrm{Spec}_{I_w(y)} U^-[w]$ consists of ideals constructed by contractions from localizations of $U^-[w]/I_w(y)$ by explicit small multiplicative sets of normal elements.

Each of the above two methods has many advantages over the other. Using the representation theoretic approach, it was proved that all ideals $I_w(y)$ are polynormal, it was established that $U^-[w]$ are catenary and satisfy Tauvel's height formula, the containment problem for $\mathbb{H}\text{-Spec} U^-[w] = \{I_w(y) \mid y \in W^{\leq w}\}$ was solved, and theorems for separation of variables for $U^-[w]$ were established (see [Yak10, Yakb, Yakc]). In the special case of the algebras of quantum matrices, catenarity and ideal containment was proved earlier [Cau03b, Lau07] within the framework of the ring theoretic approach (though with more complicated arguments), but there was no progress on polynormality or proofs of these results for more general $U^-[w]$ algebras. On the other hand using the ring theoretic approach, it was proved that for all $\mathbb{H}$-primes J_D of $U^-[w]$ the factor $U^-[w]/J_D$ always has a localization that is a quantum torus, its center (which is closely related to the structure of the stratum $\mathrm{Spec}_{J_D} U^-[w]$) was described, and in the case of quantum matrices $\mathbb{H}$-primes were related to total positivity (see [Cau03a, BCL, GLL11]).

Our first result resolves Question 5.3.3 of Cauchon and Mériaux [MC10] and unifies the two approaches to $\mathbb{H}\text{-Spec} U^-[w]$

Theorem 0.0.1. *Let $\Bbbk$ be an arbitrary base field, $q \in \Bbbk^*$ not a root of unity, $\mathfrak{g}$ a simple Lie algebra, w an element of the Weyl group of $\mathfrak{g}$, and $\mathbf{i}$ a reduced word for w. Consider the presentation of the quantum Schubert cell algebra $U^-[w]$ as an iterated Ore extension corresponding to $\mathbf{i}$.*

Then for all Weyl group elements $y \leq w$ the Cauchon diagram of the $\mathbb{H}$-prime ideal $I_w(y)$ of $U^-[w]$ (from the representation theoretic approach from 4.2.2 (i)) is equal to $\mathcal{D}^+_{\mathbf{i}}(y)$, the index set of the left positive subword of $\mathbf{i}$ whose total product is y.

Thus the $\mathbb{H}$-prime ideals of $U^-[w]$ from the representation theoretic approach are related to the ideals J_D from the ring theoretic approach via

$$I_w(y) = J_{\mathcal{D}^+_{\mathbf{i}}(y)}.$$

Furthermore, we prove a theorem that explicitly describes the behavior of the representation theoretic ideals $I_w(y)$ of $U^-[w]$ in each stage of the Cauchon deleting derivation procedure. This appears in 5.2.5 below and will not be stated in the introduction since it requires additional background.

With the help of 0.0.1, one can now combine the strengths of the two approaches to the spectra of the quantum Schubert cell algebras. We expect that the combination of the two methods will lead to substantial progress in the study of the topology of Spec$U^-[w]$. We use 0.0.1 and previous results of the second author to resolve Question 5.3.2 of Cauchon and Mériaux [MC10], thereby solving the containment problem for the ideals

$$\{J_{\mathcal{D}^+_{\mathbf{i}}(y)} \mid y \in W^{\leq w}\}$$

of the classification of [MC10].

Theorem 0.0.2. *In the setting of Theorem 0.0.1, the map*

$$W^{\leq w} \longrightarrow \mathbb{H}\text{-Spec}U^-[w] \text{ given by } y \longmapsto J_{\mathcal{D}^+_{\mathbf{i}}(y)}$$

is an isomorphism of posets with respect to the (strong) Bruhat order and the inclusion order on ideals.

Finally, 0.0.1 also gives a new, independent proof of the Cauchon–Meriaux classification [MC10] described above. (The proof of 0.0.1 does not use results from [MC10].)

Let us return to the general case of Cauchon's method of deleting derivations. It relates the prime ideals of an initial iterated Ore extension A to the prime ideals of the final algebra $\overline{A}$, the Cauchon quantum affine space algebra associated to A. In order to study these ideals, one needs an explicit description of $\overline{A}$ as a subalgebra of the ring of fractions of A. We obtain such for all algebras $U^-[w]$, establishing yet another relationship between the two approaches to the structure of the algebras $U^-[w]$. Given a reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for w, define a successor function $s\colon [1,l] \sqcup \{\infty\} \longrightarrow [1,l] \sqcup \{\infty\}$ by

$$s(j) = \min\{k \mid k > j, \alpha_k = \alpha_j\}, \text{ if } \exists k > j \text{ such that } \alpha_k = \alpha_j, \ s(j) = \infty, \text{ otherwise.}$$

For $j \in [1,l]$ denote by $\Delta_{\mathbf{i},j} \in U^-[w]$ the element obtained by evaluating the quantum minor corresponding to the fundamental weight ϖ_{α_j} and the Weyl group elements $s_{\alpha_1} \dots s_{\alpha_{j-1}}$, $w \in W$ on the R-matrix $\mathcal{R}^w$ corresponding to w. We refer to Section 4.2 and Section 5.1 for details and the description of these elements in the framework of the antiisomorphisms $\phi_w\colon R_0^w \longrightarrow U^-[w]$.

Theorem 0.0.3. *In the setting of 0.0.1, for all Weyl group elements w and reduced words $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for w, the generators $\overline{x}_1, \dots, \overline{x}_l$ of the corresponding Cauchon quantum affine space algebras are given by*

$$\overline{x}_j = \begin{cases} (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},s(j)}^{-1} \Delta_{\mathbf{i},j}, & \text{if } s(j) \neq \infty \\ (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},j}, & \text{if } s(j) = \infty \end{cases}$$

for the standard powers $q_{\alpha_j} \in \Bbbk^*$ of q, see Section 3.1.

This theorem establishes a connection between the initial cluster for the cluster algebra structure on $U^-[w]$ of Geiß–Leclerc–Schröer and Cauchon's method of deleting derivations. We will present a deeper study of this in a forthcoming publication. 0.0.3 is also an important ingredient in a very recent proof [Yakd] of the second author of the Andruskiewitsch–Dumas conjecture [AD08].

The paper is organized as follows. Section 3.4 contains background on the quantum Schubert cell algebras and the representation theoretic and ring theoretic approaches to the study of their spectra. Section 5.1.2 contains the proof of 0.0.3. Theorems 0.0.1 and 0.0.2 are proved in Section 5.2, where we also establish a theorem describing the behavior of the ideals $I_w(y)$ under the iterations of the deleting derivation procedure.

Chapter 1
Lie theory background

1.1 Lie algebra preliminaries

Throughout this introduction $\Bbbk$ will denote a field of characteristic 0. All vector spaces will be over $\Bbbk$ and all $\Bbbk$-algebras will be unital and associative. We begin by constructing several important algebras associated to V. Let V be a vector space and $T^iV := V^{\otimes i}$. We adopt the convention $T^0V := \Bbbk$.

Definition 1.1.1. The tensor algebra $T(V)$ of V is the vector space

$$\bigoplus_{i=0}^{\infty} T^iV$$

with multiplication defined by the concatenation isomorphism $T^mV \otimes T^nV \longrightarrow T^{m+n}V$.

By this definition it is clear that $T(V)$ is a graded algebra if we take the k-th graded piece to be T^kV.

Definition 1.1.2. The symmetric algebra $S(V)$ is the quotient $T(V)/I$, where I is the ideal of $T(V)$ generated by $x \otimes y - y \otimes x$ for all x and y in V.

Here and after we identify V with its canonical image in $T(V)$.

Definition 1.1.3. The exterior algebra $\bigwedge(V)$ is the quotient $T(V)/I$, where I is the ideal of $T(V)$ generated by $x \otimes x$ for all x in V.

The exterior algebra inherits a natural grading from $T(V)$. We denote its k-th graded piece by $\bigwedge^k(V)$.

Definition 1.1.4. A Lie algebra is a vector space $\mathfrak{g}$ endowed with an alternating bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

The Jacobi identity fulfills the role of an associativity condition for the (Lie) bracket $[-,-]$, which is a nonassociative product. If the bracket is 0, then the Lie algebra is called *abelian*. We record the canonical example of a lie algebra. If A is any associative $\Bbbk$-algebra, we may define $[x, y] = xy - yx$, the commutator bracket. Then, A equipped with this product is a Lie algebra. This procedure defines a functor $\mathfrak{Lie}(-)$ from the category associative algebras to the category of Lie algebras. Given any vector space V, we can form the Lie algebra $\mathfrak{Lie}(\mathrm{End}(V))$. We will simplify notation and write $\mathfrak{gl}(V)$ to denote the aforementioned Lie algebra. A Lie algebra map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a linear map preserving the bracket, that is, satisfying $\varphi([x, y]) = [\varphi(x), \varphi(y)]$. If a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is closed under the bracket, then it is Lie *subalgebra*. To encode this we write $\mathfrak{h} \leq \mathfrak{g}$. If $\mathfrak{h}$ satisfies the stronger condition $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, then we write $\mathfrak{h} \trianglelefteq \mathfrak{g}$ and call $\mathfrak{h}$ an *ideal*. We should note that if $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is an ideal and $\mathfrak{b} \leq \mathfrak{g}$ is a subalgebra, then $\mathfrak{a} + \mathfrak{b}$ is a subalgebra. An ideal is called *proper* if it is not the zero ideal or $\mathfrak{g}$ itself. A nonabelian Lie algebra with no proper ideals is called simple. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, we may construct the coproduct $\mathfrak{g} \oplus \mathfrak{g}'$. It is the usual categorical vector space coproduct with bracket

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, y_1], [x_2, y_2]) .$$

If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{i}$ for some subalgebra $\mathfrak{a}$ and some ideal $\mathfrak{i}$ then we write $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{i}$. If $\mathfrak{a}$ is also an ideal, then $\mathfrak{g}$ is a coproduct $\mathfrak{a} \oplus \mathfrak{i}$, which is also a product $\mathfrak{a} \times \mathfrak{i}$. For any derivation action $\varphi : \mathfrak{a} \longrightarrow \mathrm{Der}(\mathfrak{i})$ mapping a to φ_a, we can give $\mathfrak{a} \ltimes \mathfrak{i}$ a Lie algebra structure by defining

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, y_1], [x_2, y_2] + \varphi_{x_1}(y_2) - \varphi_{y_1}(x_2)) .$$

A representation of $\mathfrak{g}$ or a $\mathfrak{g}$-*module* (when $\Bbbk$ is merely a commutative ring) is a $\Bbbk$-module M and a bilinear product $\mathfrak{g} \otimes_{\Bbbk} M \longrightarrow M$ satisfying $[x,y]\, m = x(ym) - y(xm)$. Equivalently, a representation of $\mathfrak{g}$ is a Lie algebra morphism $\varphi : \mathfrak{g} \longrightarrow \mathrm{End}(M)$. We will denote the category of all $\mathfrak{g}$ modules by $\mathfrak{g}$-Mod. Define a map

$$\mathbf{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$$

by $x \mapsto \mathbf{ad}_x$, where $\mathbf{ad}_x(-) = [x, -]$. This is called the *adjoint representation* of $\mathfrak{g}$. The kernel of $\mathbf{ad}$ is called the *center* of $\mathfrak{g}$ denoted by $\mathfrak{z}(\mathfrak{g})$. It is clear by construction that $\ker \mathbf{ad} = \mathfrak{g}$ if and only if $\mathfrak{g}$ is abelian. Let R be a (not necessarily associative) graded ring. A derivation ∂ is an endomorphism of R satisfying

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a \partial(b),$$

where $|a|$ denotes the degree of a. Let $\mathrm{Der}(R)$ denote the derivation algebra of R. The Jacobi identity and alternating property of the bracket give immediately that $\mathbf{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ is in $\mathrm{Der}(\mathfrak{g})$; here we consider $\mathfrak{g}$ with the trivial grading, that is, $\mathfrak{g}$ is concentrated in degree zero. A Lie algebra derivation ∂ is called an *inner* derivation if $\partial = \mathbf{ad}_x$ for some x in $\mathfrak{g}$.

Although Lie algebras are not associative in general, we can construct an associative algebra by forcing the bracket to be a commutator. This construction gives much insight into the representation theory of $\mathfrak{g}$. The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient $T(\mathfrak{g})/I$, where I is the ideal of $T(V)$ generated by $[x,y] - x \otimes y - y \otimes x$ for all x and y in $\mathfrak{g}$. By construction we see that a $\mathfrak{g}$-module is precisely a module for $U(\mathfrak{g})$. More precisely, for every $\Bbbk$-algebra A there is a natural isomorphism

$$\mathrm{Hom}_{\Bbbk\text{-Alg}}(U(\mathfrak{g}), A) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{Lie}(A)).$$

The statement follows from setting $A = \mathrm{End}(\mathfrak{g})$.

We briefly give an alternate definition of a Lie algebra in terms of a construction of Chevalley and Eilenberg [CE48]. Given a vector space V with an alternating bilinear map $[-,-] : V \times V \to V$, we can construct the sequence of vector spaces

$$\mathcal{C}V := \ldots \longrightarrow \bigwedge^{n+1}(V) \longrightarrow \bigwedge^{n}(V) \longrightarrow \ldots \longrightarrow V \longrightarrow \Bbbk \longrightarrow 0$$

with maps $\partial : \bigwedge^{n+1}(V) \longrightarrow \bigwedge^{n}(V)$ defined by

$$\partial(x_1 \wedge \cdots \wedge x_{n+1}) = \frac{1}{n+1} \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{n+1}$$

A routine inductive argument shows

$$\partial(\omega_1 \wedge \omega_2) = \partial\omega_1 \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge \partial\omega_2,$$

where $|x|$ denotes the degree of x.

Lemma 1.1.5. *Suppose $\mathfrak{g}$ is a vector space equipped with such a bracket. The sequence $\mathcal{C}\mathfrak{g}$ is a chain complex if and only if $\mathfrak{g}$ is a Lie algebra.*

Proof. Consider the first nontrivial case $\partial^2 : \bigwedge^3(\mathfrak{g}) \longrightarrow \mathfrak{g}$. A direct computation yields

$$\begin{aligned}
\partial^2(x \wedge y \wedge z) &= \frac{1}{3}\partial([x,y] \wedge z - [x,z] \wedge y + [y,z] \wedge x) \\
&= \frac{1}{6}\,[[x,y],z] - [[x,z],y] + [[y,z],x] \\
&= \frac{1}{6}\,[[x,y],z] + [[z,x],y] + [[y,z],x]
\end{aligned}$$

which is 0 if and only if $[-,-]$ satisfies Jacobi. The general case follows from applying ∂^2 to $x_1 \wedge \cdots \wedge x_n = x_1 \wedge \omega$ to obtain

$$\begin{aligned}
\partial\left[\partial(x_1 \wedge \omega)\right] &= \partial\left[\partial(x_1) \wedge \omega - x_1 \wedge \partial(\omega)\right] \\
&= \partial^2(x_1) \wedge \omega + \partial(x_1) \wedge \partial(\omega) - \left[\partial(x_1) \wedge \partial(\omega) - x_1 \wedge \partial^2(\omega)\right] \\
&= \partial^2(x_1) \wedge \omega + x_1 \wedge \partial^2(\omega) \\
&= x_1 \wedge \partial^2(\omega),
\end{aligned}$$

which is zero by the inductive hypothesis. □

By the previous lemma, we may define a Lie algebra to be a vector space $\mathfrak{g}$ with a linear map $\varphi : \bigwedge^2(\mathfrak{g}) \longrightarrow \mathfrak{g}$ such that $(\mathcal{C}\mathfrak{g}, \varphi)$ is a chain complex. The abelian group $H_n(\mathfrak{g}) := \ker \partial_{n+1} / \operatorname{im} \partial_n$ is called the n-th homology group of $\mathfrak{g}$. If M is any $\mathfrak{g}$-module we apply the functor $\operatorname{Hom}(-, M)$ to $\mathcal{C}\mathfrak{g}$ to obtain the analogously defined cohomology groups $H^*(\mathfrak{g}, M)$ with values in M.

The following collections of Lie algebras will play a major role in Lie algebra representation theory. A Lie algebra $\mathfrak{g}$ is called solvable if the derived series

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \dots \mathfrak{g}^{(n)} \supseteq \dots$$

terminates. Here $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = \left[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}\right]$. A less restrictive property than solvability is nilpotence. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \dots \mathfrak{g}^n \supseteq \dots$$

terminates. Here $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$.

Proposition 1.1.6. *Consider the short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \longrightarrow 0$$

We have $\mathfrak{g}$ is solvable (resp. nilpotent) if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable (resp. nilpotent).

If $\mathfrak{g}$ has basis $\{X_1, \dots X_n\}$, then the Lie algebra structure is completely determined by the *structure constants*— elements $c_{i,j}^k \in \Bbbk$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{i,j}^k X^k$$

Proposition 1.1.7. *Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are solvable (resp. nilpotent) ideals of $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ is a solvable (resp. nilpotent) ideal.*

We define $\mathfrak{rad}(\mathfrak{g})$ the radical of a Lie algebra to be the sum of all the solvable ideals of $\mathfrak{g}$. Similarly, the nilradical $\mathfrak{nil}(\mathfrak{g})$ is the sum of all the nilpotent ideals. We call $\mathfrak{g}$ *semisimple* if $\mathfrak{rad}(\mathfrak{g}) = 0$ and *reductive* if $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ is a semisimple Lie algebra and $\mathfrak{a}$ is an abelian Lie algebra.

Reductive Lie algebras are generalizations of semisimple Lie algebras in the sense that every semisimple Lie algebra is reductive. This follows immediately from the straightforward observation

$$0 = \mathfrak{rad}(\mathfrak{s} \oplus \mathfrak{a}) \supseteq \mathfrak{z}(\mathfrak{s} \oplus \mathfrak{a}) = \mathfrak{a}.$$

Fixing $\mathfrak{g}$ gives a non-split (in general) exact sequence

$$0 \longrightarrow \mathfrak{nil}(\mathfrak{g}) \xrightarrow{\ i\ } \mathfrak{g} \xrightarrow{\ \pi\ } \mathfrak{b} \longrightarrow 0$$

and a split exact sequence

$$0 \longrightarrow \mathfrak{rad}(\mathfrak{g}) \xrightarrow{\ i\ } \mathfrak{g} \xrightarrow{\ \pi\ } \mathfrak{l} \longrightarrow 0.$$

By construction we see that $\mathfrak{l}$ is semisimple. This subalgebra is called the *levi* subalgebra and the decomposition $\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{rad}(\mathfrak{g})$ is called the levi decomposition.

Example 1.1.8. Let $\mathfrak{sl}$ and $\mathfrak{t}$ be the subalgebras of $\mathfrak{gl} = \mathfrak{gl}(V)$ given by traceless endomorphisms and scalar multiples of the identity transformation, respectively. We have $\mathfrak{gl} = \mathfrak{sl} \oplus \mathfrak{t}$.

1.2 The Cartan–Killing Form

Let $\langle -, - \rangle$ be the usual inner product on $\mathfrak{gl}(V)$ given by

$$\langle A, B \rangle = \operatorname{tr} AB.$$

For any lie algebra representation $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, we define the symmetric bilinear form

$$\langle x, y \rangle_\varphi = \langle \varphi_x, \varphi_y \rangle = \operatorname{tr} \varphi_x \varphi_y.$$

When $\varphi = \mathbf{ad}$ we obtain the Cartan–Killing form, which we will denote by $\langle -, - \rangle$ in unambiguous circumstances. For any subspace $\mathfrak{a} \subset \mathfrak{g}$, we define the orthogonal complement $\mathfrak{a}^{\perp}$ to be all the elements of $\mathfrak{g}$ whose bracket with every element of $\mathfrak{a}$ is 0. In symbols

$$\mathfrak{a}^{\perp} = \{x \in \mathfrak{g} \mid \langle x, \mathfrak{a} \rangle = 0\}\,.$$

Such a product is called *nondegenerate* on $\mathfrak{g}$ if $\mathfrak{g}^{\perp} = 0$.

Proposition 1.2.1. *The Killing form is invariant in the sense that*

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle.$$

Proof. A straightforward computation yields

$$\begin{aligned}
\langle [X, Y], Z \rangle &= \operatorname{tr} \mathbf{ad}_{[X,Y]}\, \mathbf{ad}_Z \\
&= \operatorname{tr} [\mathbf{ad}_X, \mathbf{ad}_Y]\, \mathbf{ad}_Z \\
&= \operatorname{tr} \mathbf{ad}_X\, \mathbf{ad}_Y \operatorname{tr} \mathbf{ad}_Z - \operatorname{tr} \mathbf{ad}_Y\, \mathbf{ad}_X\, \mathbf{ad}_Z \\
&= \operatorname{tr} \mathbf{ad}_X\, \mathbf{ad}_Y\, \mathbf{ad}_Z - \operatorname{tr} \mathbf{ad}_X\, \mathbf{ad}_Y\, \mathbf{ad}_Z \\
&= \operatorname{tr} \mathbf{ad}_X\, [\mathbf{ad}_Y, \mathbf{ad}_Z] \\
&= \operatorname{tr} \mathbf{ad}_X\, \mathbf{ad}_{[Y,Z]} = \langle X, [Y, Z] \rangle.
\end{aligned}$$

The equality $\mathbf{ad}_{[A,B]} = [\mathbf{ad}_A, \mathbf{ad}_B]$ follows immediately from the Jacobi identity. $\square$

An immediate corollary is if $\mathfrak{a}$ is an ideal, then so is $\mathfrak{a}^{\perp}$.

Proposition 1.2.2. *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the Cartan form of $\mathfrak{a}$ coincides with the Cartan form on $\mathfrak{g}$ restricted to $\mathfrak{a}$.*

Proof. We have $\mathbf{ad}_{\mathfrak{a}}(\mathfrak{g}) \subset \mathfrak{a}$. Thus, we choose a basis for $\mathfrak{g}$ so that for all $z \in \mathfrak{a}$

$$\mathbf{ad}_z = \begin{pmatrix} \mathbf{ad}_z\big|_{\mathfrak{a}} & 0 \\ 0 & 0 \end{pmatrix}$$

We then have that for all $x, y \in \mathfrak{a}$

$$\langle x,y\rangle = \operatorname{tr}\begin{pmatrix} \mathbf{ad}_x\big|_{\mathfrak{a}} & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{ad}_y\big|_{\mathfrak{a}} & * \\ 0 & 0 \end{pmatrix} = \operatorname{tr}\begin{pmatrix} \mathbf{ad}_x\big|_{\mathfrak{a}}\mathbf{ad}_y\big|_{\mathfrak{a}} & 0 \\ 0 & 0 \end{pmatrix} = \langle x,y\rangle\big|_{\mathfrak{a}}$$

□

Proposition 1.2.3. *A Lie algebra is semisimple if and only if it is a direct sum of simples.*

Proof. Suppose $\mathfrak{a}$ is a proper ideal of $\mathfrak{g}$, then

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$$

If $\mathfrak{a}'$ is an ideal of $\mathfrak{a}$, then $\mathfrak{a}'$ is also an ideal in $\mathfrak{g}$. This is clear since

$$[\mathfrak{g}, \mathfrak{a}'] = \left[\mathfrak{a} \oplus \mathfrak{a}^{\perp}, \mathfrak{a}'\right]$$

and $\left[\mathfrak{a}^{\perp}, \mathfrak{a}'\right]$ vanishes.

□

Corollary 1.2.4. *The lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$*

Theorem 1.2.5 (Cartan's Criterion)**.** *The Killing form is nondegenrate on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is semisimple.*

1.3 Classification of semisimple Lie algebras

A linear map $\varphi \in \operatorname{End}(\mathfrak{g})$ is called *diagonalizable* or *semisimple* if $\mathfrak{g}$ has a basis consisting of eigenvectors $v_1, v_2, \ldots, v_n$ of φ. This is equivalent to $\mathfrak{g}$ being semisimple as a $\Bbbk[\varphi]$-module.

$$\mathfrak{g} = \Bbbk v_1 \oplus \Bbbk v_2 \oplus \ldots \oplus \Bbbk v_n$$

An element $x \in \mathfrak{g}$ is called semisimple with respect to a representation φ if φ_x is a semisimple endomorphism. Subablgebras of $\mathfrak{g}$ consisting entirely of semisimple elements are often called *toral.*

Let $S \subset \mathfrak{g}$ be a subset. The normalizer of S in $\mathfrak{g}$ is the set stabilzer under the action of the bracket, that is,

$$N(S) = \{x \in \mathfrak{g} \mid [x, s] \in S \text{ for all } s \in S\}$$

By construction if S is a subalgebra of $\mathfrak{g}$, then $N(S)$ it is the largest Lie subalgebra of $\mathfrak{g}$ containing S as an ideal. A nilpotent subalgebra $\mathfrak{h} \leq \mathfrak{g}$ that is self-normalizing ($N(\mathfrak{h}) = \mathfrak{h}$) is called a *Cartan* subalgebra. Since the center $\mathfrak{z}(\mathfrak{g})$ is a solvable ideal, it is clear that a semisimple Lie algebra is centerless. If $\mathbb{k}$ is algebraically closed, however, then Cartan subalgebras are maximal abelian, toral subalgebras.

A weight of $\mathfrak{g}$ is a 1-dimensional representation $\lambda : \mathfrak{g} \longrightarrow \mathbb{k}$. Thus, $\varphi\big|_{[\mathfrak{g},\mathfrak{g}]} = 0$. Moreover, since semisimple Lie algebras coincide with their derived subalgebras, semisimple Lie algebras have no nontrivial weights. Restricted to a Cartan, however, weights $\lambda \in \mathfrak{h}^*$ need not be zero.

Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then, $\varphi(\mathfrak{h})$ is a commutative subalgebra of $\mathfrak{gl}(V)$ consisting entirely of semisimple operators. Let $\lambda \in \mathfrak{h}^*$ be a weight, and define the λ-weight space V_λ with respect to φ (really $\varphi\big|_{\mathfrak{h}}$) to be

$$V_\lambda = \{v \in V \mid \varphi_h(v) = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$$

When φ is the adjoint representation restricted to $\mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$ is a weight, the definition becomes

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [h, v] = \alpha(h)v \text{ for all } h \in \mathfrak{h}\}$$

Immediately, we see $\mathfrak{g}_0 = \mathfrak{h}$, and by construction $\mathfrak{g}$ is a semisimple $\mathbb{k}[\mathbf{ad}_h]$-module for all $h \in \mathfrak{h}$. That is,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

Elements in $\mathfrak{g}_\alpha$ are called weight vectors. The nonzero weights α for which $\mathfrak{g}_\alpha$ are nonzero are called *roots* of $\mathfrak{g}$. The set of roots of $\mathfrak{g}$ is denoted by Δ.

It is clear that if $\mathfrak{g}$ is nilpotent, then $\mathbf{ad}_x$ will be nilpotent for any $x \in \mathfrak{g}$ since

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots$$

terminating in n steps implies $(\mathbf{ad}_x)^n(y) = 0$ for all $x, y \in \mathfrak{g}$. Engel's theorem is the converse statement.

Theorem 1.3.1 (Engel)**.** *Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a dimension $n > 0$ representation of $\mathfrak{g}$, and suppose φ_x is nilpotent for all $x \in \mathfrak{g}$, then there is a flag in V*

$$0 = V_0 \subset V_1 \subset \cdots \subset \cdots \subset V_n = V$$

with $\varphi_{\mathfrak{g}}(V_i) \subset V_{i-1}$ for $1 \leq i \leq n$.

Corollary 1.3.2. *If $\mathbf{ad}_x$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.*

Proof. Applying Engel's theorem to $\varphi = \mathbf{ad}$, we have

$$\mathbf{ad}_{x_1} \mathbf{ad}_{x_2} \dots \mathbf{ad}_{x_n}(\mathfrak{g}) = [x_1, [x_2, [x_n, \mathfrak{g}]]] = 0$$

for any $x_i \in \mathfrak{g}$. Therefore, $\mathfrak{g}^{n+1} = 0$ since (in particular) $\mathfrak{g}^n \subset \mathfrak{g}$. □

Two remarks: A nonzero nilpotent lie algebra $\mathfrak{g}$ must have nonzero center since the last $\mathfrak{g}^i$ nonzero in the derived series is contained within $\mathfrak{z}(\mathfrak{g})$. Also, $\mathbf{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is a faithful representation when $\mathfrak{z}(\mathfrak{g}) = 0$, e.g., when $\mathfrak{g}$ is semisimple. For centerless Lie algebras this observation proves the simplest case of

Theorem 1.3.3 (Ado's Theorem)**.** *Any finite dimensional Lie algebra has a faithful finite dimensional representation.*

If $\mathfrak{g}$ is semisimple and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ is a root space decomposition, define the subalgebras

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha, \qquad \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$$

Proposition 1.3.4. *The subalgebras* $\mathfrak{n}_\pm$ *are nilpotent and* $\mathfrak{b}_\pm$ *are solvable.*

We call $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ a triangular decomposition. Triangular decompositions gives rise to a corresponding decompostion of universal enveloping algebras

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$$

1.4 Root systems and Cartan matrices

We have seen the definition of roots in the classification of semisimple Lie algebras. Collections of roots, however, are intrinsically important as the following classical theorem suggests:

Theorem 1.4.1. *Simple Lie algebras are completely determined by the irreducible crystallographic root systems.*

We proceed by defining root systems as abstract combinatorial objects.

Definition 1.4.2. Let V be a finite dimensional real vector space with standard inner product $\langle -, - \rangle$. A (reduced, crystallographic) root system in V is a finite set Δ of vectors such that

(i) the $\mathbb{R}$-span of Δ is V

(ii) if $r\alpha \in \Delta$, then $r = \pm 1$

(iii) for all $\alpha_i, \alpha_j \in \Delta$, we have $s_{\alpha_i}(\alpha_j) = \alpha_j - c_{ij}\alpha_i$ is in Δ, where $c_{ij} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$

(iv) each c_{ij} is an integer.

The invertible maps $\varphi : V \longrightarrow V$ satisfying $\varphi(\Delta) \subset \Delta$ form a finite subgroup O_Δ of $\mathrm{Aut}(V)$. The subgroup of O_Δ generated by all the *reflections* s_{α_i} is called the Weyl group of Δ and is denoted W_Δ. A root system Δ in V is reducible if $\Delta = \Phi_1 \sqcup \Phi_2$, where Φ_i is a root system for the nonempty W_i, and $V = W_1 \oplus W_2$. An isomorphism of root systems is an isomorphism of vector spaces $\varphi : V \longrightarrow V'$ such that $\varphi(\Delta) = \Delta'$ and the matrix entries c_{ij} are preserved. The *rank* of a root system is the dimension of V

Definition 1.4.3. A *generalized Cartan matrix* is a matrix $C = (c_{ij})$ with integral entries such that

(i) all diagonal entries are 2

(ii) all other entries are not positive.

(iii) the matrix C is the product of a symmetric matrix and an invertible diagonal matrix

Generalized Cartan matrices with positive determinant are exactly Cartan matrices.

Theorem 1.4.4 (Serre)**.** *A Cartan matrix C uniquely determines a semisimple Lie algebra $\mathfrak{g}$ isomorphic to $\Bbbk\langle f_1, \dots, f_n, h_1, \dots, h_n, e_1, \dots, e_n\rangle$ modulo the following relations:*

(i) $[h_i, h_j] = 0$

(ii) $[e_i, f_j] = \delta_{ij} h_i$

(iii) $[h_i, e_j] = c_{ij} e_j$

(iv) $[h_i, f_j] = -c_{ij} f_j$

(v) $\mathbf{ad}_{e_i}^{1-c_{ij}}(e_j) = 0$

(vi) $\mathbf{ad}_{f_i}^{1-c_{ij}}(f_j) = 0$

Recall the reflection s_α adds a multiple of the root α to any root β. In particular,

$$s_\alpha(\beta + k\alpha) = \beta - c_{ij}\alpha - k\alpha = \beta - (c_{ij} + k)\alpha.$$

The Serre relations (v) and (vi) reflect the fact that for $\alpha \neq \pm\beta$, the chain

$$\beta - s\alpha, \quad \beta - (s-1)\alpha, \quad \dots, \quad \beta + t\alpha$$

is comprised of all roots, where s, t are the largest integers such that $\beta - s\alpha$ and $\beta + t\alpha$ are roots. To see this, note that $e_i - e_j$ is not a root so $s = 1$ and then $t = -c_{ij}$. Thus, $e_i + (1 - c_{ij})e_j$ is not a root, and we have the anologous property

$$\mathbf{ad}_{e_i}^{1-c_{ij}}(e_j) = 0.$$

Since $[x, y]$ equals $xy - yx \in U(\mathfrak{g})$, we obtain the following simple formula whose straightforward proof we leave to the reader

Proposition 1.4.5. *For all $x, y \in \mathfrak{g}$, the formula for $\mathbf{ad}_x^n(y)$ in $U(\mathfrak{g})$ is given by*

$$\mathbf{ad}_x^n(y) = \sum_{k=0}^{n} \binom{n}{k} x^k y x^{n-k}$$

Corollary 1.4.6. *In $U(\mathfrak{g})$ the Serre relations are*

$$\sum_{k=0}^{1-c_{ij}} \binom{1-c_{ij}}{k} e_i^k e_j e_i^{1-c_{ij}-k} = 0, \quad (i \neq j)$$
$$\sum_{k=0}^{1-c_{ij}} \binom{1-c_{ij}}{k} f_i^k f_j f_i^{1-c_{ij}-k} = 0, \quad (i \neq j)$$

Example 1.4.7. Let C be the Cartan matrix define by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The corresponding Lie algebra is then isomorphic to $\mathfrak{sl}_3$. The generators e_1 and e_2 satisfy the Serre relations

$$[e_1, [e_1, e_2]] = 0 \text{ and } [e_2, [e_2, e_1]] = 0.$$

In $U(\mathfrak{sl}_3)$ the corresponding relations are

$$[e_1, [e_1, e_2]] = e_1^2 e_2 - 2e_1 e_2 e_1 + e_2 e_1^2 = 0$$
$$[e_2, [e_2, e_1]] = e_2^2 e_1 - 2e_2 e_1 e_2 + e_1 e_2^2 = 0$$

Quantized analogues of the Serre relations in $U(\mathfrak{g})$ will play a major role in the study of the positive part U_q^+ of the quantized universal enveloping algebra of $\mathfrak{g}$.

We conclude this section with the following well-known classification which highlights the deep interplay between Lie theory and root system combinatorics.

Theorem 1.4.8. *There are bijective correspondences between any two of the following:*

(i) irreducible Cartan matrices

(ii) isomorphism classes of simple Lie algebras

(iii) isomorphism classes of irreducible root systems

(iv) connected Dynkin diagrams

Furthermore, the following exhaust all connected Dynkin diagrams

$$A_n \qquad \begin{array}{ccccccc} \circ & \text{—} & \circ & \text{—}\ \cdots\ \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array} \qquad (n+1)$$

$$B_n \qquad \begin{array}{ccccccc} \circ & \text{—} & \circ & \text{—}\ \cdots\ \text{—} & \circ & \Rightarrow & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array} \qquad (2)$$

$$C_n \qquad \begin{array}{ccccccc} \circ & \text{—} & \circ & \text{—}\ \cdots\ \text{—} & \circ & \Leftarrow & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array} \qquad (2)$$

$$D_n \qquad \begin{array}{ccccccc} & & & & \circ & & \\ & & & & \mid \ \alpha_n & & \\ \circ & \text{—} & \circ & \text{—}\ \cdots\ \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} \end{array} \qquad (4)$$

$$E_6 \qquad \begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & \mid \ \alpha_6 & & & & \\ \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \end{array} \qquad (3)$$

$$E_7 \qquad \begin{array}{ccccccccccc} & & & & \circ & & & & & & \\ & & & & \mid \ \alpha_7 & & & & & & \\ \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array} \qquad (2)$$

$$E_8 \qquad \begin{array}{ccccccccccccc} & & & & & & & & \circ & & & & \\ & & & & & & & & \mid \ \alpha_8 & & & & \\ \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \end{array} \qquad (1)$$

$$F_4 \qquad \begin{array}{ccccccc} \circ & \text{—} & \circ & \Rightarrow & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array} \qquad (1)$$

$$G_2 \qquad \begin{array}{ccc} \circ & \Rrightarrow & \circ \\ \alpha_1 & & \alpha_2 \end{array}$$

Chapter 2
Stratifying Spec$R$

2.1 Goodearl–Letzter Stratification

Following the notational conventions in [BG02], we outline a procedure of Goodearl and Letzter for analyzing the prime spectrum of a ring R. First, we say a proper ideal $P \lhd R$ is *prime* if the product of ideals IJ being in P implies I or J is contained in P. We will call P completely prime if it is prime with respect to ideals, that is, if P is prime is the usual sense of commutative algebra. Primes need not be completely prime. The zero ideal in $M_n(\mathbb{R})$, for instance, is prime but not completely prime. If $\mathbb{H}$ is a group acting on R, we say the ideal I is $\mathbb{H}$-*invariant* (or an $\mathbb{H}$-*ideal*) if $h(I) = I$ for all $h \in \mathbb{H}$, and we call a proper $\mathbb{H}$-ideal P an $\mathbb{H}$-*prime* ideal of R if the product of two $\mathbb{H}$-ideals being in P implies at least one is in P. We pause to make the important observation that $\mathbb{H}$-invariant prime ideals are $\mathbb{H}$-primes, but the converse is false. Note that $\mathcal{H}$-prime need not be prime. For instance,

Example 2.1.1. Let A be any simple ring. Set $R = \prod_1^n A$, let G be the symmetric group on n symbols S_n, and suppose G acts on R by permuting coordinates. Clearly, 0 is an $\mathbb{H}$-prime, but is clearly not prime.

If 0 is an $\mathbb{H}$-prime in R, we shall call R and $\mathbb{H}$-prime ring. Let us denote by $\mathbb{H}$-Spec$R$ the collection of $\mathbb{H}$-prime ideals of R. We can endow $\mathbb{H}$-Spec$R$ with the Zariski topology, but this will not play a role in this paper.

The action of $\mathbb{H}$ on R induces an action on Spec$R$, thereby partitioning Spec$R$ into $\mathbb{H}$-orbits. An investigation of spectra with respect to $\mathbb{H}$-orbits, however, may be impractical as there are often more $\mathbb{H}$-orbits than $\mathbb{H}$-primes. We take the simple example of a Manins quantum plane

$$R_q[\mathbb{k}^2] = \mathbb{k}\langle x, y \mid xy = qyx\rangle$$

to observe the difference between $\mathbb{H}$-primes and $\mathbb{H}$-orbits. Let $\mathbb{k}$ be a field. The affine plane $\mathbb{H} = \mathbb{k}^2$ acts on this quantum affine space by

$$(\mathbb{k}^\times)^2 \curvearrowright R_q[\mathbb{k}^2]$$
$$(s,t)\cdot x^i y^j = s^i t^j x^i y^j.$$

There are six $\mathcal{H}$-orbits

(i) $\langle 0\rangle$

(ii) $\langle x, y\rangle$

(iii) $\langle x\rangle$

(iv) $\langle y\rangle$

(v) $\langle x-\alpha, y \mid \alpha \in \mathbb{k}^\times\rangle$

(vi) $\langle x, y-\beta \mid \beta \in \mathbb{k}^\times\rangle$,

but only the first four are $\mathbb{H}$-primes. With this in mind, we define $(I : \mathbb{H})$ to be the largest $\mathbb{H}$-ideal contained in I, that is,

$$(I : \mathbb{H}) = \bigcap_{t\in\mathbb{H}} t(I).$$

Observe the following: If P is prime, then $(P : \mathbb{H})$ is an $\mathbb{H}$-prime, and if P_1 and P_2 are in the same $\mathbb{H}$-orbit, then $(P_1 : \mathbb{H}) = (P_2 : \mathbb{H})$ Following Goodearl and Letzter [GL00], we can now define the $\mathbb{H}$-*stratum* of $\mathrm{Spec}R$ corresponding to J defined by

$$\mathrm{Spec}_J R = \{P \in \mathrm{Spec}R \mid (P : \mathbb{H}) = J\}$$

and, thus, refine the orbit-stratification of $\mathrm{Spec}R$ to the $\mathbb{H}$-stratification

$$\mathrm{Spec}R = \bigsqcup_J \mathrm{Spec}_J R,$$

where the union is over all $\mathbb{H}$-primes in R. In this theory $\mathbb{H}$-primes parametrize the strata of $\mathrm{Spec}R$. Moreover, we have $\mathbb{H}$-prime ideal containment represented in the following diagram.

$$
\begin{tikzcd}
\{\langle x, y-\beta \mid \beta \in \Bbbk^{\times}\rangle\} \arrow[d, dashed, no head] & \{\langle x, y\rangle\} \arrow[dl, no head] \arrow[dr, no head] & \{\langle x-\alpha, y \mid \alpha \in \Bbbk^{\times}\rangle\} \arrow[d, dashed, no head] \\
\{\langle x\rangle\} \arrow[dr, no head] & & \{\langle y\rangle\} \arrow[dl, no head] \\
& \{\langle 0\rangle\} &
\end{tikzcd}
$$

We will later observe that this poset is isomorphic to the *initial Bruhat order interval* $W^{\leq s_1 s_2}$. By restricting a result of Lorenz [Lor] if $\mathbb{H}$ is an algebraic torus acting rationally (see [[BG02] Section II.2.6]) on R, then each $\mathbb{H}$-stratum is $\mathbb{H}$-equivariantly isomorphic to a torus—Spec of a commutative Laurent polynomial algebra. The $\mathbb{H}$-stratification theory is especially powerful when the number of $\mathbb{H}$-strata is known to be finite, as will be our case.

2.2 Cauchon diagrams

We follow the conventions of Cauchon [Cau03a, Cau03b] and very briefly outline a procedure for obtaining . We later require more precise statement; see Section 4.1 for a proper treatment of Cauchon's algorithm.

Let A be a unital $\Bbbk$-algebra, let $A[X;\sigma,\delta]$ be an *Ore extension* of A, and let $S = \{X^n \mid n \in \mathbb{N}\}$. The deleting derivation homomorphism is a map $\phi : A \longrightarrow A[X;\sigma,\delta]S^{-1}$ given by

$$a \longmapsto \textstyle\sum_{n=0}^{\infty} \frac{(1-q)^{-n}}{(n)_q!} \delta^n \sigma^{-n}(a) X^{-n};$$

here $(n)_q! = (1)(1+q)\ldots(1+q+q^2+\cdots+q^{n-1})$. Under appropriate hypotheses, for instance the ones in [Cau03a], this sum gives well-defined algebra map ϕ satisfying

$$X\phi(a) = \phi(\sigma(a))X \text{ for all } a \in A.$$

By the universal property of Ore extensions, we therefore have a unique map φ extending ϕ and mapping Y to X.

$$\begin{tikzcd}
A[Y;\sigma] \arrow[dr, dashed, "\varphi"] & \\
A \arrow[u, hook] \arrow[r, "\phi"] & A[X;\sigma,\delta]S^{-1}
\end{tikzcd}$$

The image of φ is the algebra obtained from the algorithm. Cauchon's method is an process that starts with some *iterated Ore extension* $R^{(N+1)} = \mathbb{k}[X_1][X_2;\sigma_2,\delta_2]\dots[X_N;\sigma_N,\delta_N]$ and yields a sequence of algebras

$$R^{(j)} \cong \mathbb{k}[Y_1]\dots[Y_{j-1};\sigma_{j-1},\delta_{j-1}][Y_j;\tau_j]\dots[Y_N;\tau_N]$$

terminating with $R^{(2)} = \overline{R}$ — the *Cauchon (quantum affine) space*

To illustrate this method, we restrict to the case of *quantum matrices.* Let $\mathbb{k}$ be an algebraically closed field (for simplicity), and assume $q \in \mathbb{k}^\times$ is not a root of unity. The $\mathbb{k}$-algebra of $n \times n$ quantum matrices A is given by n^2 generators $X_{i,j}$ $(1 \le i, j \le n)$ subject to relations: If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a 2×2 submatrix of $(X_{i,j})$, then

$$ab = qba, \quad ac = qca,$$

$$bd = qdb, \quad cd = qdc,$$

$$bc = cb, \text{ and}$$

$$da = ad + (q - q^{-1})bc.$$

Another realization of A due to Faddeev, Reshetikhin, and Takhtadzhyan [RTF89] is as the associative algebra over $\mathbb{k}[q, q^{-1}]$ generated by formal entries X_{ij} of a matrix X such that

$$R(X \otimes I)(I \otimes X) = (I \otimes X)(X \otimes I)R$$

holds for the *R-matrix*

$$\mathcal{R} = q^{-1} \sum_{1 \le i \le n} e_{ii} \otimes e_{ii} + \sum_{1 \le i \ne j \le n} e_{ii} \otimes e_{jj} + (q^{-1} - q) \sum_{1 \le j < i \le n} e_{ij} \otimes e_{ji}.$$

We denote this algebra by $R_q[M_n]$ and note that the name "quantum matrices" or quantized coordinate ring of matrices is warranted, though we will not justify it here. Furthermore, it is easy to generalize this construction to obtain m $\times$ n quantum matrices.

There is a natural action of the algebraic torus $\mathbb{H} = (\Bbbk^{\times})^{2n}$ by $\Bbbk$-algebra automorphisms on A given by

$$(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \cdot X_{i,j} = \alpha_i \beta_j X_{i,j}.$$

This induces an action on $\mathrm{Spec} R_q[M_n]$. It is well known that $R_q[M_n]$ is an iterated Ore extension of $\Bbbk$ and thus is a noetherian domain. In fact, the automorphisms and derivations can be read off by the relations in $R_q[M_n]$. A result of Goodearl and Letzter [GL00] gives that all $\mathbb{H}$-primes of A are completely prime and that $\mathbb{H}\text{-}\mathrm{Spec} R_q[M_n]$ is finite.

The deleting derivations algorithm gives rise to an inclusion of sets

$$\varphi : \mathrm{Spec} R_q[M_n] \hookrightarrow \mathrm{Spec}\overline{R_q[M_n]},$$

where $\overline{R_q[M_n]}$ is given by the generators and relations of A, except that now the relation $da = ad + (q - q^{-1})bc$ is modified to $da = ad$. Moreover, this map descends to an inclusion on the respective $\mathbb{H}$-prime spectra. If we denote the generators of $\overline{R_q[M_n]}$ by $\overline{X_1}, \dots, \overline{X_{n^2}}$, then it is well-known that the $\mathbb{H}$-primes in the quantum affine space $\overline{R_q[M_n]}$ have the form

$$Q_D = \overline{R_q[M_n]} \langle \overline{X_i} \mid i \in D \subset \{1, \dots, n^2\} \rangle.$$

Cauchon deduced the preimage of φ is a disjoint union of strata indexed by elementary combinatorial objects called Cauchon diagrams, which also parametrize $\mathbb{H}$-orbits of symplectic leaves in M_n [GLL11] and restricted permutations, $\sigma \in S_{2n}$ such that $-n+i \leq \sigma(i) \leq n+i$ for $1 \leq i \leq 2n$ [BCL].

Definition 2.2.1. The Cauchon diagram of $J \in \mathbb{H}\text{-}\mathrm{Spec} R_q[M_n]$ is the unique set D such that $\varphi(J) = Q_D$.

For $n \times n$ quantum matrices, the governing relation

$$da - ad = (q - q^{-1})bc \tag{2.2.1}$$

implies that if d is in a completely prime ideal, then either b or c must be as well. With some small additional argument, we conclude that the Cauchon diagrams for $R_q[M_n]$ can be described combinatorially by $n \times n$ boxes such that if one square is filled, then every square to its left or every square above it is also filled. Explicitly, the general theory of deleting derivations yields the decomposition

$$\varphi(\mathrm{Spec} R_q[M_2]) = \bigsqcup_D \mathrm{Spec}_D \overline{R_q[M_2]},$$

where the union is over all Cauchon diagrams. Moreover, Cauchon proves [Cau03a] that the collections $\varphi^{-1}(\mathrm{Spec}_D \overline{R_q[M_n]})$ coincide with the Goodearl–Letzter stratification of $\mathrm{Spec} R_q[M_n]$.

In the spirit of a running example, we observe the poset of Cauchon diagrams, which does *not* correspond to the poset of $\mathbb{H}$-primes in $R_q[M_n]$. The Cauchon diagram poset is missing two $\mathbb{H}$-prime inclusions, which we denote in Figure 2.2 by dashed lines. We will later observe that the full poset of $\mathbb{H}$-primes, with the dashed lines honestly included, is isomorphic to (for instance) the initial Bruhat order interval $W^{\leq s_2 s_1 s_3 s_2}$. This consideration from the basis for [[MC10] Question 5.3.2], which is the simple Corollary 5.2.4 following from one of our main theorems. We shall make this precise in subsequent sections.

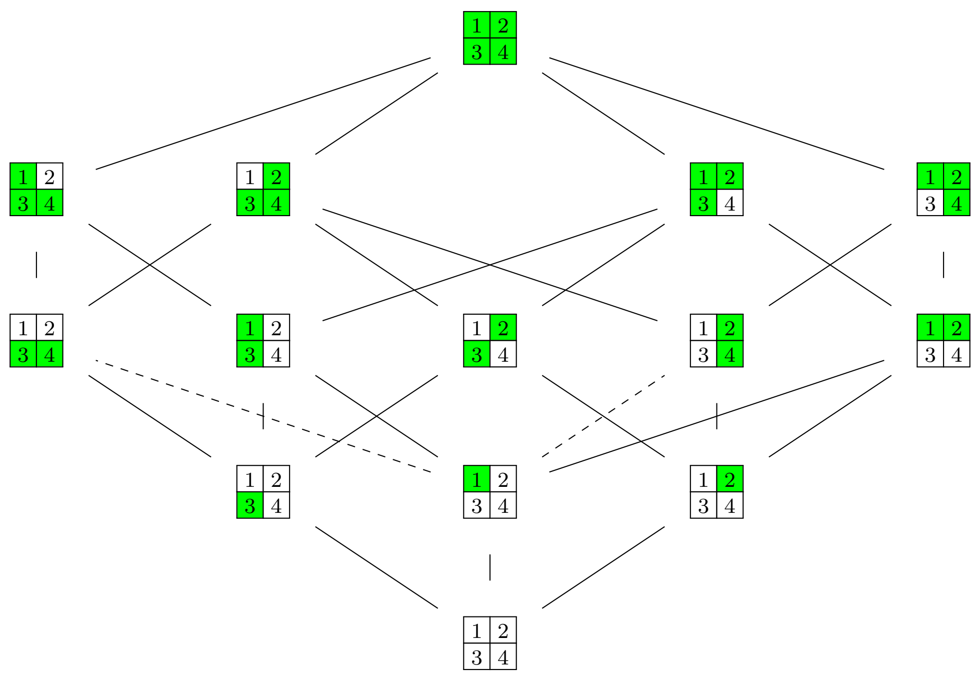

FIGURE 2.1. The fourteen Cauchon diagrams of $R_q[M_2]$ with missing inclusions

Chapter 3
Constructing quantum Schubert cells

3.1 Quantized Universal Enveloping Algebras

Let $\mathfrak{g}$ be complex semisimple Lie algebra of rank r with root system Δ, Weyl group W, Cartan matrix $C = (c_{ij}) = 2\langle\alpha_j, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$ corresponding to the choice of Cartan subalgebra $\mathfrak{h}$ and with simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$; here $\langle -, -\rangle$ is the invariant bilinear form on the real vector space $\mathbb{R}\Pi$ normalized by $\langle\alpha, \alpha\rangle = 2$ for short roots. Denote by $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$ and $s_\alpha \in W$ the corresponding coroot and reflection. Let $\mathfrak{g}$ have the associated triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with corresponding universal enveloping algebra decompostion

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+).$$

Let $\{\varpi_1, \dots, \varpi_r\}$ be the fundamental weights of $\mathfrak{g}$. These are the weights dual to the simple coroots, i.e. those given by the condition

$$\langle\varpi_i, \alpha_j\rangle = \delta_{i,j} d_j, \text{ where } d_j = \frac{\langle\alpha_j, \alpha_j\rangle}{2} \in \{1, 2, 3\}. \tag{3.1.1}$$

Note that $d_i c_{ij} = d_j c_{ji}$, where $D = \mathrm{diag}(d_1, \dots, d_n)$ is the *symmetrizer* of C. We will denote the root and weight lattices of $\mathfrak{g}$ by Q and P and their dominant integral counterparts by

$$Q_{\geq 0} = \sum_{i=1}^{r} \mathbb{Z}_{\geq 0}\alpha_i, , \text{ and } P_{\geq 0} = \sum_{i=1}^{r} \mathbb{Z}_{\geq 0}\varpi_i.$$

By definition of the fundamental weights. we have that for any $\lambda \in Q$ and root α, the number $\langle\lambda, \alpha\rangle$ is an integer. Therefore, by manipulating 3.1.1, we define $\langle\langle -, -\rangle\rangle : P \times Q \longrightarrow \mathbb{Q}$ by

$$\langle\langle\lambda, \alpha^\vee\rangle\rangle = 2\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle.} \tag{3.1.2}$$

When no confusion exists we will drop the double-angle brackets in favor of single-angle brackets. Note, in particular, that $\langle\alpha_j, \alpha_i^\vee\rangle$ is the Cartan matrix entry c_{ij}. The simple reflection s_α acts on P by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

For a *parabolic* subset $I \subseteq \Pi$ set $Q_I = \sum_{\alpha_i \in I} \mathbb{Z}\alpha_i$. Recall the standard partial order on P given by

$$\lambda_1 \geq \lambda_2 \text{ if } \lambda_1 = \lambda_2 + \gamma \text{ for some } \gamma \in Q_{\geq 0} \text{ and}$$
$$\lambda_1 > \lambda_2 \text{ if } \lambda_1 \geq \lambda_2 \text{ and } \lambda_1 \neq \lambda_2.$$

If $\lambda = \sum_{i \in I} n_i \omega_i \in P_{\geq 0}$, $n_i > 0$, for all i in I we will say that the support of λ is I.

Henceforth, $\Bbbk$ shall be a field of arbitrary characteristic and q will be a nonzero element in $\Bbbk$ which is not a root of unity. Set $q_i = q^{d_i}$. We will use the following notation in the subsequent sections:

$$[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \qquad (n)_{q_i} = \frac{q_i^n - 1}{q_i - 1} \qquad \exp_{q_i}(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_{q_i}!}$$

$$[n]_{q_i}! = [1]_{q_i} \dots [n]_{q_i} \qquad (n)_{q_i}! = (1)_{q_i} \dots (n)_{q_i} \qquad \widehat{X_i} = X_i - X_i^{-1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_i} = \frac{[n]_{q_i}!}{[k]_{q_i}![n-k]_{q_i}!} \qquad \binom{n}{k}_{q_i} = \frac{(n)_{q_i}!}{(k)_{q_i}!(n-k)_{q_i}!} \qquad \widehat{cX} = \widehat{c}\widehat{X}$$

Define the quantized universal enveloping algebra $U_q(\mathfrak{g})$ to be the associative, unital $\Bbbk$-algebra with generators $E_1, \dots, E_r, K_1^{\pm 1}, \dots, K_r^{\pm 1}, F_1, \dots, F_r$ and the following relations:

$$K_i E_j K_i^{-1} = q_i^{c_{ij}} E_j \qquad K_i F_j K_i^{-1} = q_i^{-c_{ij}} F_j \tag{3.1.3}$$

$$K_i K_j = K_j K_i, \qquad E_i F_j - F_j E_i = \delta_{i,j} \widehat{q_i}^{-1} \widehat{K_i} \tag{3.1.4}$$

$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-c_{ij}-k} = 0, \quad (i \neq j) \tag{3.1.5}$$

$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-c_{ij}-k} = 0, \quad (i \neq j) \tag{3.1.6}$$

The algebra $U_q(\mathfrak{g})$ is graded by the root lattice Q by setting

$$\deg F_i = -\alpha_i, \qquad \deg E_i = \alpha_i, \qquad \deg K_i^{\pm 1} = 0, \tag{3.1.7}$$

We label the γ component of $U_q(\mathfrak{g})$ by $U_q(\mathfrak{g})_\gamma$. There is a unique (noncommutative, noncocommutative) Hopf algebra structure on $U_q(\mathfrak{g})$ determined by the following comultiplication, antipode, and counit:

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \qquad S(F_i) = -F_iK_i \qquad \epsilon(F_i) = 0 \tag{3.1.8}$$

$$\Delta(K_i) = K_i \otimes K_i \qquad S(K_i) = K_i^{-1} \qquad \epsilon(K_i) = 1 \tag{3.1.9}$$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \qquad S(E_i) = -K_i^{-1}E_i \qquad \epsilon(E_i) = 0 \tag{3.1.10}$$

For any subset S of $U_q(\mathfrak{g})$, let $\Bbbk\langle S\rangle$ be the subalgebra of $U_q(\mathfrak{g})$ generated by S. When confusion should not arise, we will drop the set braces for notational convenience. We set notation for the following important subalgebras to be utilized throughout this paper:

$$U_q^- = \Bbbk\langle \{F_i\}_{i=1}^r\rangle \qquad U_q^0 = \Bbbk\langle \{K_i\}_{i=1}^r\rangle \qquad U_q^+ = \Bbbk\langle \{E_i\}_{i=1}^r\rangle$$

$$U_q^{\geq 0} = \Bbbk\langle \{K_i, E_i\}_{i=1}^r\rangle \qquad U_q^{\leq 0} = \Bbbk\langle \{F_i, K_i\}_{i=1}^r\rangle \qquad U_q^i = \Bbbk\langle F_i, K_i^{\pm 1}, E_i\rangle.$$

For any simple root α_i, the corresponding algebra U_q^i is canonically isomorphic to $U_{q_i}(\mathfrak{sl}_2)$. The isomorphism being $X_i \mapsto X_1$. Many arguments regarding $U_q(\mathfrak{g})$ are simple extensions of those for U_q^i. For instance,

Lemma 3.1.1. *There is a unique algebra automorphism ω, and a unique algebra anti-automorphism τ of $U_q(\mathfrak{g})$ such that $\omega^2 = 1 = \tau^2$ and*

$$\omega(E_i) = F_i \qquad \omega(F_i) = E_i \qquad \omega(K_i) = K_i^{-1}$$

$$\tau(E_i) = E_i \qquad \tau(F_i) = F_i \qquad \tau(K_i) = K_i^{-1}$$

We call ω the *Cartan involution* and τ the *principal involution*. The algebras $U_q^{\geq 0}$ and $U_q^{\leq 0}$ are Hopf subalgebras of $U_q(\mathfrak{g})$. There is a unique *Hopf pairing*

$$\langle -,-\rangle : (U_q^{\geq 0})^{\mathrm{op}} \times U_q^{\leq 0} \longrightarrow \Bbbk \tag{3.1.11}$$

called the *Rosso–Tanisaki form* satisfying the following for all $i,j \in [1,l]$:

$$\langle K_i, K_j\rangle = q^{-\langle \alpha_i, \alpha_j\rangle} \qquad \langle K_i, F_j\rangle = 0$$

$$\langle E_i, F_j\rangle = -\delta_{ij}(\widehat{q_i})^{-1} \qquad \langle E_i, K_j\rangle = 0.$$

Let $\mathfrak{g}$ have Weyl group W. The Artin braid group of $\mathfrak{g}$, in abstract terms, is the group B with generators $T_1,\dots,T_r$ and relations $T_iT_jT_iT_j\cdots = T_jT_iT_jT_i\dots$ if and only if we have the braid relation $s_is_js_is_j\cdots = s_js_is_js_i\dots$ in W.

Let $X_i^{(r)} = \frac{X_i^r}{[r]_{q_i}!}$. The braid group $\mathcal{B}$ of $\mathfrak{g}$ acts on $U_q(\mathfrak{g})$ by algebra automorphisms

$$T_iE_i = -F_iK_i, \quad T_iF_i = -K_i^{-1}E_i, \quad T_iK_j = K_jK_i^{-c_{ij}}, \tag{3.1.12}$$

$$T_iE_j = \sum_{k=0}^{-c_{ij}} (-1)^k (q_i)^{-k} E_i^{(-c_{ij}-k)} E_j E_i^{(k)} \quad (j \neq i), \tag{3.1.13}$$

$$T_iF_j = \sum_{k=0}^{-c_{ij}} (-1)^k (q_i)^{k} F_i^{(k)} F_j F_i^{(-c_{ij}-k)} \quad (j \neq i). \tag{3.1.14}$$

Suppose that A is a commutative $\Bbbk$-algebra, M is an A-module and $\lambda : A \to \Bbbk$ is a weight. In staunch analogy with Lie algebra representation theory, we define the λ-weight spaces of M by

$$M_\lambda = \{v \in M \mid a \cdot v = \lambda(a)v \text{ for all } a \in A\}$$

All modules are left modules unless declared otherwise. Let M be a 1-dimensional representation of $U_q(\mathfrak{g})$. Alternatively, consider the weight $\lambda : U_q(\mathfrak{g}) \to \Bbbk$. The relations in 3.1.3 imply that $\lambda(E_i)M = \lambda(F_i)M = 0$ and so $\lambda(K_i^2)M = M$. It is only necessary to consider the case in which K_i acts by 1. See, for instance, [[BG02] Section I.6.12] for a summary.

The weight spaces of $U_q(\mathfrak{g})_q$ as a module over itself are all trivial. Thus we let $U_q(\mathfrak{g})$ act on itself by the adjoint representation, or Hopf-algebra conjugation as follows: Let H be a Hopf algebra with comultiplication $\Delta(X) = \sum A_i \otimes B_i$. Then $\mathbf{ad}_X : H \longrightarrow H$ is defined by

$$\mathbf{ad}_X(Y) = \sum A_i Y S(B_i)$$

Recalling the Hopf algebra structure of from 3.1.8, we observe the adjoint action of $U_q(\mathfrak{g})$ on itself is

$$\mathbf{ad}_{K_i}(x) = K_i x K_i^{-1}$$

To keep track of the weights, we will write λ for the weight satisfying $\lambda(K_i) = q^{\langle \lambda, \alpha_i \rangle}$ The (quantized) λ-weight space of the $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})$ is then given by

$$U_q(\mathfrak{g})_\lambda = \left\{ v \in M \mid K_i \cdot v = K_i x K_i^{-1} = q^{\langle \lambda, \alpha_i \rangle} v \text{ for all } i \text{ in } [1, r] \right\}. \tag{3.1.15}$$

In analogy with this, we define the λ-weight space of any $U_q(\mathfrak{g})$-module M by

$$M_\lambda = \left\{ v \in M \mid K_i \cdot v = q^{\langle \lambda, \alpha_i \rangle} v \text{ for all } i \text{ in } [1, r] \right\}. \tag{3.1.16}$$

Definition 3.1.2. Let M be a $U_q(\mathfrak{g})$-module. We call M *integrable* (or type 1) if

$$M = \bigoplus_{\lambda \in \mathcal{P}} M_\lambda.$$

Let $\mathscr{M}$ denote the category of finite dimensional, integrable $U_q(\mathfrak{g})$-modules. The first important fact is that $\mathscr{M}$ is a semisimple category when q is not a root of unity (see [Jan96, Theorem 5.17] and the remark on p. 85 of [Jan96]). Furthermore, $\mathscr{M}$ is closed under taking tensor products and duals (defined as left modules using the antipode of $U_q(\mathfrak{g})$). Denote by $V(\lambda)$ the irreducible integrable $U_q(\mathfrak{g})$-module of highest weight $\lambda \in P_{\geq 0}$. We have the well-known theorem from [[BG02] Section I.6.12], which shows that $\mathscr{M}$ is the quantized analogue of the category of finite dimensional , irreducible $U(\mathfrak{g})$-modules.

Theorem 3.1.3. *The following hold:*

(i) There is a bijection between dominant integral weights and highest weight modules in $\mathscr{M}$ given by

$$\lambda \longmapsto V(\lambda)$$

(ii) Each $V(\lambda)$ is the quotient of the Verma module

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q^{\geq 0}} \Bbbk(\lambda)$$

(iii) Each $V(\lambda)$ has a highest weight vector of weight λ, and the Weyl character formula gives the weights of $V(\lambda)$

(iv) Every module in $\mathscr{M}$ is completely reducible.

Now, the braid group $\mathcal{B}$ acts on all $U_q(\mathfrak{g})$-modules in a manner that is compatible with the fact that $U_q(\mathfrak{g})$ is a left module over itself; specifically, for all simple roots α, u in $U_q(\mathfrak{g})$ and v in M_λ we have

$$T_\alpha(x \cdot v) = T_\alpha(x) \cdot T_\alpha(v). \tag{3.1.17}$$

(see [Jan96, eq. 8.14 (1)]). Let $m = \langle \lambda, \alpha^\vee \rangle$ and explicitly we define the braid automorphisms T_α and T'_α by

$$T_\alpha(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q_\alpha^{b-ac} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} \tag{3.1.18}$$

$$T'_\alpha(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q_\alpha^{ac-b} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} \tag{3.1.19}$$

cf. [Jan96, Section 8.6] and [Lus93, Section 5.2]. Let T_w denote the automorphism $T_{i_1} \dots T_{i_l}$ for some reduced decomposition of w. When indices are irrelevant, we will label the simple braid automorphisms by T_i for $i \in [1, r]$. The notation T_w is well-defined according to the following theorem.

Theorem 3.1.4 (Lusztig, De Concini–Kac–Procesi)**.** *The T_i satisfy the braid group relations, hence the braid automorphism T_w is independent of the choice of reduced decomposition of w.*

In particular we have

$$T_w(x \cdot v) := (T_w x) \cdot (T_w v), \text{ for all } v \in V(\lambda) \text{ and } \lambda \in P_{\geq 0}, \tag{3.1.20}$$

For all $w \in W$, $\lambda \in P_{\geq 0}$, and $\mu \in P$, the braid group action satisfies

$$T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu} \tag{3.1.21}$$

From this, the remark [Jan96, Section 5. Remark 1], and the fact that the weights of $V(\lambda)^*$ are the negatives of the weights of $V(\lambda)$ we have

$$\dim V(\lambda)_{w\lambda} = \dim V(\lambda)^*_{-w\lambda} = 1, \tag{3.1.22}$$

and $w_0\lambda$ is the unique lowest weight of $V(\lambda)$. Such weight spaces $V(\lambda)_{w\lambda}$ in the Weyl group orbit of a highest weight are called *extremal* weight spaces. In subsequent sections we will make use of

Lemma 3.1.5. *Let v_λ be a highest weight vector of highest weight $\lambda = n\varpi_i$ for an irreducible, finite dimensional, integrable U_q^i-module. For all $m, n \in \mathbb{Z}_{\geq 0}$ such that $m \leq n$ we have*

(i) $T_i(v_\lambda) = (-q_i)^n F_i^{(n)} v_\lambda$, *and* $T_i^{-1}(v_\lambda) = F_i^{(n)} v_\lambda$,

(ii) $E_i^m F_i^m v_\lambda = \dfrac{[m]_i! [n]_i!}{[n-m]_i!} v_\lambda$

Proof. By [Jan96, eqs. 8.6 (6), (7)] we have

$$\begin{aligned} T_i^{-1}(v_\lambda) &= (-q_i)^n T_i'(v) \\ &= (-q_i)^n \sum_{\substack{a,b,c\geq 0 \\ -a+b-c=m}} (-1)^b q_i^{ac-b} E_i^{(a)} F_i^{(b)} E_i^{(c)} v_\lambda \\ &= (-q_i)^n (-q_i)^{-n} F_i^{(n)} v_\lambda \\ &= F_i^{(n)} v_\lambda \end{aligned}$$

The proof of $T_i(v_\lambda) = (-q_i)^n F_i^{(n)} v_\lambda$ is equally straightforward and omitted . Now, observe

$$\begin{aligned} E_i^m F_i^m v_\lambda &= [m]_{q_i}! \prod_{j=1}^{m} (q^{j-m} - q^{-(j-m)}) \widehat{K_i} v_\lambda \\ &= [m]_{q_i}! \prod_{j=1}^{m} [n-m+j]_i \, v_\lambda \\ &= [m]_{q_i}! [n-m+1]_i \dots [n]_i v_\lambda \\ &= \frac{[m]_{q_i}! [n]_{q_i}!}{[n-m]_{q_i}!} v_\lambda \end{aligned}$$

The first equality follows from [Jan96, Lemma 1.7] and the second from [Jan96, eq. 5.12 (7)]. The third and forth follow from common sense. $\square$

We will also use the following fact regarding the previously defined 3.1.1 principal involution τ: For all $x \in U_q(\mathfrak{g})$ and $w \in W$, τ satisfies

$$\tau(T_w x) = T_{w^{-1}}^{-1}(\tau(x)). \tag{3.1.23}$$

In particular, we have $\tau(T_i x) = T_i^{-1}(\tau(x))$ see [Jan96, eq. 8.18(6)].

3.2 Quantized Coordinate Rings

We construct $R_q[G]$ the (single parameter) q-quantized coordinate ring of a connected, simply connected, complex semisimple algebraic group G with Lie algebra $\mathfrak{g}$. Let A be a

$\Bbbk$-algebra. We define the finite or *restricted dual* A° to be the algebra with underlying set

$$\{f \in \mathrm{Hom}(A, \Bbbk) \mid f(I) = 0 \text{ on some ideal of } A \text{ satisfying } \dim_{\Bbbk} A/I < \infty\} \tag{3.2.1}$$

Intuitively, these are the functionals which vanish on almost all of A (in an algebraic sense). It is well-known that A° is a coalgebra with comultiplication and counit induced by $\mathrm{Hom}(A, -)$.

We pause here to draw an analogy with the universal enveloping algebra . For the moment allow G to be *any* algebraic group over $\Bbbk$ with Lie algebra $\mathfrak{g}$. Another interpretation of $U(\mathfrak{g})$ is the algebra of k-valued algebraic functions on the coordinate ring $\mathcal{O}(G)$ that die on a formal neighborhood of the identity of G. Explicitly, let $\mathfrak{m}_e$ be the ideal of functions in $\mathcal{O}(G)$ which vanish at the identity of G. It is well known that

$$U(\mathfrak{g}) \cong \{f \in \mathcal{O}(G)^* \mid f(\mathfrak{m}_e^n) = 0 \text{ for some } n > 0\}.$$

In general $U(\mathfrak{g})^\circ$ is not isomorphic to $\mathcal{O}(G)$, but we have the following classical result:

Theorem 3.2.1 (Cartier)**.** *For a semisimple, connected and simply connected algebraic group G over an algebraically closed field of characteristic* 0

$$\mathcal{O}(G) \cong U(\mathfrak{g})^\circ$$

We wish to mimic the behavior Theorem 3.2.1 in the non-classical setting. To this end let H be a Hopf algebra, and suppose M is an H-module. For any $f \in \mathrm{Hom}(M, \Bbbk)$ and $v \in M$, we define the *coordinate function* $c^M_{f,v} \in \mathrm{Hom}(H, \Bbbk) = H^*$ determined by pairing f against xv for any $x \in H$. Explicitly,

$$x \longmapsto c^M_{f,v}(x) = f(xv). \tag{3.2.2}$$

Lemma 3.2.2. *If M is a finite dimensional H-module, then $c^M_{f,v}(x)$ is in H°.*

Proof. Note that if s is the annihilator of M, then $c^M_{f,v}(s) = 0$. Thus, we may take $I = \mathrm{Ann} M$ in the notation of 3.2.1. □

Recall $\mathscr{M}$—the category of finite dimensional, integrable $U_q(\mathfrak{g})$-modules. Let $\mathscr{M}_+$ be the full subcategory of $\mathscr{M}$ consisting of modules M satisfying

$$M = \bigoplus_{\lambda \in P_{\geq 0}} V(\lambda) \tag{3.2.3}$$

We define $R_q[G]$ to be the Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ generated by coordinate functions of the modules in $\mathscr{M}_+$. Equivalently, $R_q[G]$ is generated by the coordinate functions of the highest weight modules $V(\lambda) \in \mathscr{M}$ for $\lambda \in P_{\geq 0}$. We abbreviate these functions to $c^\lambda_{f,v}$ and summarize:

$$c^\lambda_{f,v}(x) = f(xv) \qquad v \in V(\lambda) \qquad f \in V(\lambda)^* \qquad x \in U_q(\mathfrak{g})$$

For any bialgebra H we may endow all of H^* with a ring structure by setting

$$fg(x) = \sum f(x_1) \otimes g(x_1), \text{ where } \Delta(x) = \sum x_1 \otimes x_2 \tag{3.2.4}$$

Here we are using Sweedler's notation for $\Delta(x)$. Thus, $R_q[G]$ inherits this standard ring structure. For the sake of clarity we investigate multiplication in $R_q[G]$. For all x we have the string of equalities

$$\begin{aligned} c_{f,u}c_{g,v}(x) &= \sum c_{f,u}(x_1)c_{g,v}(x_2) = \sum f(x_1u)g(x_2v) \\ &= (f \otimes g)(\sum x_1u \otimes x_2v) = f \otimes g(x(u \otimes v)) \\ &= c_{f\otimes g, u \otimes v}(x) \end{aligned}$$

Thus, we have shown $c_{f,u}c_{g,v} = c_{f\otimes g,u\otimes v}(x)$.

It will be useful to keep in mind an alternative formulation of $R_q[G]$ that is independent from the coordinate functions. If H is a bialgebra, then H^* has a natural H–H-bimodule structure given by

$$(a.f)(x) = f(xa) \text{ and } (f.a)(x) = f(ax),$$

where $a, x \in H$ and $f \in H^*$. A straightforward check shows H° is a sub-bimodule of H^*. Thus, $U_q(\mathfrak{g})^\circ$ is naturally a $U_q(\mathfrak{g})$–$U_q(\mathfrak{g})$-bimodule with structure arising from the bimodule structure on $U_q(\mathfrak{g})$. We can express the left and right actions of $x \in U_q(\mathfrak{g})$, on $c \in U_q(\mathfrak{g})^\circ$ given by the following: If $\Delta(f) = \sum f_1 \otimes f_2$, then set

$$f.a(-) = \sum f_2(a) f_1(-) \text{ and} \tag{3.2.5}$$

$$a.f(-) = \sum f_1(a) f_2(-) \tag{3.2.6}$$

Define the double weight space (compatible with the construction in 3.1.15) by

$$U_q(\mathfrak{g})^\circ_{\lambda_2\lambda_1} = \left\{ v \in U_q(\mathfrak{g})^\circ \mid K_i \cdot v = q^{\langle \lambda_1, \alpha_i \rangle} v \;\text{ and }\; v \cdot K_i^{-1} = q^{\langle \lambda_2, \alpha_i \rangle} \text{ for all } i \text{ in } [1, r] \right\} \tag{3.2.7}$$

We then have

$$R_q[G] = \bigoplus_{\lambda_1, \lambda_2 \in P} U_q(\mathfrak{g})^\circ_{\lambda_2\lambda_1}$$

We remark that the notation $R_q[G]$ is common although it does not include the field $\Bbbk$ on which the construction of $U_q(\mathfrak{g})$ depends. Thus, G is more of a suggestive symbol.

3.3 Weyl group combinatorics

Let W be the Weyl group of a fixed rank r Lie algebra $\mathfrak{g}$, and let $S = \{s_1, \ldots, s_r\}$ be a set of generators for W corresponding to a choice of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ for $\mathfrak{g}$, that is, s_j is the simple reflection s_{α_j}. Let $[m, n]$ denote the set of integers from m to n. For simplicity of notation, we define the following: If $\mathbf{i} = i_1 \ldots i_l$ is an *expression* or string of numbers in $[1, r]$, then the corresponding Weyl group element $s_{\mathbf{i}}$ in the symbols of S is given by mapping a subword to its *index set*.

$$s_{\mathbf{i}} := s_{i_1} \ldots s_{i_l}$$

By construction there is a natural bijection $\mathcal{D} : \mathcal{S}_{\mathbf{w}} \longrightarrow \mathcal{P}_l$ between subwords of $s_{\mathbf{i}}$ and the 2^l subsets of $[1, l]$ given by

$$s_{i_{j_1}} \ldots s_{i_{j_m}} \longmapsto \{j_1, \ldots, j_m\} \tag{3.3.1}$$

We suggestively denote the empty subword of $s_{\mathbf{i}}$ by 1. Let us pause to stress that we consider elements in $\mathcal{S}_{\mathbf{w}}$ to merely be subwords and *not* Weyl group elements; therefore, for example, $s_{i_j}s_{i_j} \neq 1$ as elements of $\mathcal{S}_{\mathbf{w}}$. Observe $1 \leq j_1 < \cdots < j_m \leq l$ by construction. With this bijection in mind, we set the following notation: For each expression $\mathbf{i} = i_1 \ldots i_n$ and each subset $D \subset [1, l]$ set

$$s_j^D = \begin{cases} s_j & \text{if } j \in D \\ 1 & \text{if } j \notin D \end{cases}$$

we define $s_{\mathbf{i}}^D := \mathcal{D}^{-1}(D)$. Explicitly, we have

$$s_{\mathbf{i}}^D = s_1^D \ldots s_n^D. \tag{3.3.2}$$

Let $\mathcal{D}_l$ be the quotient of $\mathcal{P}_l$ defined by $D \sim D'$ if $s_{\mathbf{i}}^D$ and $s_{\mathbf{i}}^{D'}$ are equal as Weyl group elements. Elements in $\mathcal{D}_l$ are called *diagrams*. Note that in general $\mathcal{D}$ does not descend to a bijection $\mathcal{S}_{\mathbf{w}}$ onto $\mathcal{D}_l$. In the spirit of a running example, observe that if $\mathbf{i} = 2132$, then $|\mathcal{P}_l| = 16$, while $|\mathcal{D}_l| = 14$. To see this we simply observe

$$\{1\} \sim \{4\} \text{ and } \{1, 4\} \sim \varnothing.$$

We will be interested in the subset of $\mathcal{S}_{\mathbf{w}}$ that $\mathcal{D}$ maps bijectively onto $\mathcal{D}_l$.

To this end we define the standard length function on W; it is a map

$$\ell : W \longrightarrow \mathbb{Z}_{\geq 0}$$

where $\ell(w)$ is defined to be the smallest number l so that w is a product of l simple reflections.

Example 3.3.1. Let $\mathbf{i} = 12121$ so that $s_{\mathbf{i}}$ determines an element in Σ_3, the symmetric group on three symbols. We have $\ell(s_{\mathbf{i}}) = 1$ since $s_{12121} = s_{12212} = s_2$.

By definition it is clear that

(i) $\ell(1) = 0$

(ii) $\ell(w) = \ell(w^{-1})$

(iii) $\ell(ww') \leq \ell(w) + \ell(w')$

If a length l word $\mathbf{i} = i_1 \dots i_l$ in the elements of $[1, r]$ has the property $\ell(s_{\mathbf{i}}) = l$, then $\mathbf{i}$ is called a *reduced expression* for the Weyl group element represented by $s_{\mathbf{i}}$. When $\mathbf{i}$ is a reduced expression for w, we write $\mathbf{w}$ to denote the representative $s_{\mathbf{i}}$. By finiteness W contains an element of maximal length. Now, if $w \in W$ is any element and $s_i \in S$ a simple reflection, then

$$\ell(s_i w) = \ell(w) \pm 1$$

Theorem 3.3.2. *There is a unique maximal length element w_0 of W called the* longest word*. It is characterized by the property*

$$\ell(s_i w_0) < \ell(w_0) \textit{ for all simple reflections } s_i$$

Proof. Existence follows immediately from finiteness of W. Such a word can be constructed from any maximally long sequence $\mathbf{i}$ satisfying

$$\ell(s_{i_1}) \leq \ell(s_{i_1 i_2}) \leq \cdots \leq \ell(s_{\mathbf{i}})$$

Let w_1 be any other maximal length word. Uniqueness follows from showing $\ell(w_0 w_1^{-1}) = 0$. If not then $w_0 w_1^{-1} = s_{\mathbf{j}}$ for some reduced expression. We proceed by induction on the length of $\mathbf{j}$. Note that the length of $\mathbf{j}$ cannot be 1, otherwise $\ell(w_0) = \ell(w_1) - 1$. Similarly, $\mathbf{j}$ cannot be length two. because $w_0 = s_{i_1} s_{i_2} w_1$ □

Henceforth, let the longest word have length N. Multiple reduced expressions for the longest word exist for Weyl groups containing at least two simple generators. In general many reduced expressions may exist for a Weyl group element. With this in mind, we define a partial order on W. Let $\mathbf{i}$ be a reduced expression for w. The *Bruhat order* on W is defined by

$$w' \leq w \text{ if some reduced word for } w' \text{ is a subword of } \mathbf{i}.$$

The *intial Bruhat order interval* $W^{\leq w}$ is defined by

$$W^{\leq w} = \{v \in W \mid v \leq w\}$$

By construction $\exp : \mathcal{D}_l \longrightarrow W^{\leq w}$ (determined by $D \mapsto w^D$) is a bijection. For m and n in $[0, l]$, we define the truncations of $\mathbf{w}$ by

$$\mathbf{w}_{[m,n]} = \begin{cases} s_{i_m} s_{i_{m+1}} \dots s_{i_{m-1}} s_{i_n} & \text{if } m \leq n \quad (3.3.3) \\ s_{i_m} s_{i_{m-1}} \dots s_{i_{n+1}} s_{i_n} & \text{if } m > n \quad (3.3.4) \end{cases}$$

We set $\mathbf{w}_{[j,j]} = \mathbf{w}_j$ for notational simplicity, and we adopt the convention $\mathbf{w}_0 := 1$. Observe the simple (but useful) facts $\mathbf{w}_{[1,l]} = w$ and $\mathbf{w}_{[m,n]}^{-1} = \mathbf{w}_{[n,m]}$. Furthermore, observe that truncation commutes with diagram exponentiation so the notation $\mathbf{w}_{[1,j]}^D$, for instance, is unambiguous.

Following [MR04] we call a subexpression $\mathbf{y}$ of $\mathbf{w}$ *right positive* if its diagram $D = \mathcal{D}(\mathbf{y})$ has the property that

$$\mathbf{w}_{[1,j]}^D s_{i_{j+1}} > \mathbf{w}_{[1,j]}^D \text{ for all } j \in [1, l-1].$$

and *left positive* if

$$s_{i_j} \mathbf{w}_{[j+1,l]}^D > \mathbf{w}_{[j+1,l]}^D \text{ for all } j \in [1, l-1]. \tag{3.3.5}$$

Let $\mathcal{L}_{\mathbf{w}}$ denote the collection of left positive subexpressions of the reduced expression $\mathbf{w}$ corresponding to the reduced word $\mathbf{i}$.

Example 3.3.3. Let $\mathbf{i} = 2132$ so that $\mathbf{w} = s_2 s_1 s_3 s_2$. Then every subset of $\{1, 2, 3, 4\}$ except for $\{1, 4\}$ and $\{4\}$ corresponds to left positive subexpressions. Both sets fail immediately since

$$1 = s_2(s_1 s_3 s_2)^{\{1,4\}} \not> (s_1 s_3 s_2)^{\{1,4\}} = s_2$$

Analogously, we see that that every subexpression of $\mathbf{w}$ is right positive except (again) $s_2 s_2$ and s_2, this time though with index sets $\{1, 4\}$ and $\{1\}$.

The following lemma is the first step to establishing a bijection $W^{\leq w} \longrightarrow \mathcal{L}_{\mathbf{w}}$.

Lemma 3.3.4 (Marsh–Rietsch). *For any reduced expression* $\mathbf{i}$ *for* w *and any element* $y \in W^{\leq w}$, *there is a unique right positive subexpression for* y.

Corollary 3.3.5. *For any* $y \in W^{\leq w}$, *there is a unique left positive subword* $\mathbf{y} \in \mathcal{L}_{\mathbf{w}}$ *whose diagram* $D = \mathcal{D}(\mathbf{y})$ *satisfies* $\mathbf{w}^D = y$.

Proof. The *inversion* map $\mathbf{i} = i_1 \dots i_l \mapsto i_l \dots i_1 = \mathbf{i}^{-1}$ gives rise to a bijection between the left positive subwords of $\mathbf{w}$ and the right positive subwords of $\mathbf{w}^{-1}$. The claim follows because the induced (honest inversion) map $W^{\leq w} \longrightarrow W^{\leq w^{-1}}$ defined by $y \mapsto y^{-1}$ is a bijection. $\square$

The bijection $\mathcal{L}$ arising from corollary is represented by the diagram

$$\begin{array}{ccccccc} W^{\leq w} & \xrightarrow{\mathcal{L}} & \mathcal{L}_{\mathbf{w}} & \xrightarrow{\mathcal{D}} & \mathcal{D}_l & \xrightarrow{\exp} & W^{\leq w} \\ \\ y & \longmapsto & \mathbf{y} & \longmapsto & D & \longmapsto & \mathbf{w}^D = y \end{array}$$

We will denote the composition $\mathcal{D} \circ \mathcal{L}$ by $\mathcal{D}^+$; this is the *left positive diagram* bijection to be heavily used in subsequent sections. We now summarize the results and observations of this section. For any reduced word $\mathbf{i}$ we have the commutative diagram

$$\begin{tikzcd} \mathcal{S}_{\mathbf{w}} \arrow[r, "\mathcal{D}"] & \mathcal{P}_l \arrow[r, two heads, "\sim"] & \mathcal{D}_l \arrow[d, "\exp"] \\ \mathcal{L}_{\mathbf{w}} \arrow[u, hook] \arrow[urr, "\mathcal{D}"] & & W^{\leq w}. \arrow[ll, "\mathcal{L}"] \end{tikzcd}$$

3.4 Explicit construction

In this section we outline the original construction of the upper and lower quantum *Schubert cell algebras* $U^{\pm}[w]$ due to De Concini–Kac–Procesi[DCKP95] and Lusztig [Lus93, Section 40.2]. We also gather relevant facts regarding these algebras. The naming suggests that, from the quantized ring of functions vantage-point, that these algebras are deformations of the coordinate ring of Schubert cells BwB/B. This is compatible with the underlying theme *quantum groups are quantized coordinate rings of algebraic groups.*

For every $w \in W$ set

$$\Delta_w = \Delta_+ \cap w\Delta_-.$$

This is the set of positive roots α such that $w^{-1}\alpha < 0$; in particular $\Delta_{w_0} = \Delta_+$, where w_0 denotes the longest word. For each reduced word $\mathbf{i} = i_1 \dots i_l$ for w, we obtain an ordered list of distinct positive roots $\beta_1, \dots, \beta_l$ defined by

$$\beta_j = \mathbf{w}_{[1,j-1]}\alpha_{i_j} \tag{3.4.1}$$

It is well known that $\Delta_w = \{\beta_1, \dots, \beta_l\}$. Recall the Lusztig braid group action on $U_q(\mathfrak{g})$ 3.1.12. In complete analogy with the previous construction of positive roots, we define an ordered list of positive (negative) Lusztig root vectors. Following Lusztig [Lus93, Section 39.3] we define the positive root vectors $E_{\beta_1}, \dots, E_{\beta_l}$ and the negative ones $F_{\beta_1}, \dots, F_{\beta_l}$ by

$$E_{\beta_j} = T_{\mathbf{w}_{[1,j-1]}} E_{i_j} \text{ and } F_{\beta_j} = T_{\mathbf{w}_{[1,j-1]}} E_{i_j}. \tag{3.4.2}$$

For every $\mathbf{m} = (m_1, \dots, m_l) \in (\mathbb{Z}_{\geq 0})^l$ and $i, j \in [1, l]$, define the standard monomials

$$\mathbf{X}^{\mathbf{m}_{[i,j]}} = \begin{cases} X_{\beta_i}^{m_i} X_{\beta_{i+1}}^{m_{i+1}} \dots X_{\beta_j}^{m_{j-1}} X_{\beta_j}^{m_j} & \text{if } i \leq j \\ X_{\beta_i}^{m_i} X_{\beta_{i+1}}^{m_{i-1}} \dots X_{\beta_j}^{m_{j+1}} X_{\beta_j}^{m_j} & \text{if } i > j \end{cases}$$

Let us adopt the conventions $\mathbf{X}^{\mathbf{m}} = \mathbf{X}^{\mathbf{m}_{[1,l]}}$ and $\mathbf{X}^{\mathbf{m}_{(i,j)}} = \mathbf{X}^{\mathbf{m}_{[i+1,j-1]}}$ or $\mathbf{X}^{\mathbf{m}_{[i-1,j+1]}}$ (according to whether $i < j$).

Theorem 3.4.1 (Lusztig, De Concini–Kac–Procesi)**.** *Let $\ell(w_0) = N$ and suppose w has reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$, then*

(i) We have the containment $E_{\beta_1}, \dots, E_{\beta_N} \subset U_q^+$

(ii) If $\beta_j = \alpha_i$, then $E_{\beta_j} = E_i$. Thus, $\{E_{\beta_1}, \dots, E_{\beta_N}\}$ contains all the generators of U_q^+.

(iii) Define $U^+[w] = \Bbbk\langle E_{\beta_1}, \dots, E_{\beta_l}\rangle$, the subalgebra of U_q^+ generated by $E_{\beta_1}, \dots, E_{\beta_l}$. The collection of monomials $\{\mathbf{E}^{\mathbf{m}}\}$ forms a vector-space basis for $U^+[w]$.

(iv) $U^+[w_0] = U_q^+$.

We have analogous results for $U^-[w]$—the subalgebra of U_q^- generated by $F_{\beta_1}, \dots, F_{\beta_l}$. The basis $\{\mathbf{F}^{\mathbf{m}}\}$ is called a *Poincaré–Birkhoff–Witt* basis because specializing to $q = 1$, we obtain the usual PBW basis for $U(\mathfrak{n}^+ \cap w\mathfrak{n}_-)$. The following lemma [DCKP95, Prop. 2.2] and [Lus93, Prop. 40.2.1] shows that the notation $U^+[w]$ is appropriate.

Proposition 3.4.2 (De Concini–Kac–Procesi, Lusztig)**.** *The algebras $U^\pm[w]$ do not depend on the choice of reduced word* $\mathbf{i}$ *for w.*

The Levendorskiĭ–Soibelman *straightening law* will play a major role in our study of $U^-[w]$.

Proposition 3.4.3 (Levendorskiĭ–Soibelman)**.** *For $i < j$ there are structure constants $\mathbf{c}_{\mathbf{m}_{[j,i]}} \in \Bbbk$ such that*

$$F_{\beta_j} F_{\beta_i} - q^{-\langle \beta_i, \beta_j \rangle} F_{\beta_i} F_{\beta_j} = \sum_{\mathbf{m}_{[j,i]}} \mathbf{c}_{\mathbf{m}_{[j,i]}} \mathbf{F}^{\mathbf{m}_{(j,i)}}. \tag{3.4.3}$$

The support of $w \in W$ is defined by

$$\mathbb{S}(w) = \{i \in [1, r] \mid s_i \leq w\}. \tag{3.4.4}$$

Its complement is given by

$$\Pi \backslash \mathbb{S}(w) = \{i \in [1, r] \mid w\varpi_i = \varpi_i\}, \tag{3.4.5}$$

see [Yaka, Lemma 3.2 and equation (3.2)]. Now observe that the Q-grading on $U_q(\mathfrak{g})$ 3.1.7 induces a Q-grading on $U^\pm[w]$. We will label the γ graded component by $U^\pm[w]_\gamma$; we have the following easy fact

$$\mathbb{Z}\{\gamma \in Q \mid U^\pm[w]_\gamma \neq 0\} = Q_{\mathbb{S}(w)}, \tag{3.4.6}$$

see e.g. [Yaka, equation (2.44) and Lemma 3.2 (ii)].

Recall the unique ω involution 3.1.1 of $U_q(\mathfrak{g})$. It satisfies

$$\omega(T_\alpha(u)) = (-1)^{\langle \alpha^\vee, \gamma\rangle} q^{-\langle \alpha, \gamma\rangle} T_\alpha(\omega(u)), \text{ for all } \gamma \in Q \text{ and } u \in U_q(\mathfrak{g})_\gamma,$$

see [Jan96, equation 8.14(9)]. In other words, if ρ is the sum of all fundamental weights of $\mathfrak{g}$ and $\rho^\vee$ is the sum of all fundamental coweights of $\mathfrak{g}$, then $\omega(T_\alpha(u)) = (-1)^{\langle s_\alpha(\gamma)-\gamma, \rho^\vee\rangle} q^{-\langle s_\alpha(\gamma)-\gamma, \rho\rangle} T_\alpha(\omega(u))$ for $u \in U_q(\mathfrak{g})_\gamma$. Thus

$$\omega(T_y(u)) = (-1)^{\langle y(\gamma)-\gamma, \rho^\vee\rangle} q^{-\langle y(\gamma)-\gamma, \rho\rangle} T_y(\omega(u)), \text{ for all } y \in W, \gamma \in Q, u \in U_q(\mathfrak{g})_\gamma,$$

see [Jan96, equation 8.18(5)] for an equivalent formulation of this fact. In particular, the restrictions of ω induce the isomorphisms

$$\omega : U^+[w] \to U^-[w], \ \omega(E_{\beta_j}) = (-1)^{\langle \beta_j - \alpha_j, \rho^\vee\rangle} q^{-\langle \beta_j - \alpha_j, \rho\rangle} F_{\beta_j} \text{ for all } j \in [1, l]. \tag{3.4.7}$$

To each $\gamma \in Q$ associate a character as follows: Let $t \in \mathbb{H}$ be given by $(t_1, \dots, t_r)$, and define the multiplicative character $X_\gamma \in \mathrm{Hom}(\mathbb{H}, \Bbbk^\times)$ by

$$X_\gamma(t) = t_1^{\langle \gamma, \varpi_1\rangle} \dots t_r^{\langle \gamma, \varpi_r\rangle} \tag{3.4.8}$$

The torus $\mathbb{H}$ acts rationally on $U_q(\mathfrak{g})$ by characters. Specifically, if $\in U_q(\mathfrak{g})_\gamma$, then we set

$$t \cdot x = X_\gamma(t)x = t_1^{\langle \gamma, \varpi_1\rangle} \dots t_r^{\langle \gamma, \varpi_r\rangle} x. \tag{3.4.9}$$

This torus-action preserves the subalgebras $U^\pm[w]$. We will denote by $\mathbb{H}$-Spec$U^-[w]$ the set of $\mathbb{H}$-prime ideals of $U^-[w]$. We can endow this $\mathbb{H}$-spectrum with the Zariski topology, but this topology will play no role in this paper.

Fix a length l reduced word $\mathbf{i}$ for w and recall the positive roots β_j from equation (3.4.1). Equation (3.4.6) implies that for all $j \in [1, l]$ there exists a unique $t_j = (t_{j,1}, \dots, t_{j,r}) \in \mathbb{H}$ such that for all $k \le j$ and $i \in [1, r] \backslash \mathbb{S}(\mathbf{w}_{[1,j]})$, respectively, we have

$$X_{\beta_k}(t_j) = t_{j,1}^{\langle \beta_k, \varpi_1\rangle} \dots t_{j,r}^{\langle \beta_k, \varpi_r\rangle} = q^{\langle \beta_k, \beta_j\rangle}, \text{ and } t_{j,i} = 1. \tag{3.4.10}$$

See, for example, [BG02, Proposition I.6.10] for details. The following lemma is a direct consequence of Theorem 3.4.1 (iii), equation (3.4.10), and the Levendorskiĭ–Soibelman straightening law in Proposition 3.4.3.

Lemma 3.4.4. *For all fields $\Bbbk$ with $q \in \Bbbk^{\times}$ not a root of unity, Weyl group elements w, reduced words $\mathbf{i} = (i_1, \ldots, i_l)$ for w, and $j \in [1, l]$ we have:*

(i) The subalgebra of $U^{-}[w]$ generated by $F_{\beta_1}, \ldots, F_{\beta_j}$ is equal to $U^{-}[\mathbf{w}_{[1,j]}]$.

(ii) There is an algebra isomorphism $U^{-}[\mathbf{w}_{[1,j]}] \cong U^{-}[\mathbf{w}_{[1,j-1]}][x_j, \sigma_j, \delta_j]$ given by the identity on $U^{-}[\mathbf{w}_{[1,j-1]}]$ and $F_{\beta_j} \mapsto x_j$. Furthermore, $U^{-}[\mathbf{w}_0] = U^{-}[1] \cong \Bbbk$, $\sigma_1 = \mathrm{id}$, and $\delta_1 = 0$.

(iii) The eigenvalues $t_j \cdot F_{\beta_j} = q_{\alpha_j} F_{\beta_j}$ are not roots of unity.

In Lemma 3.4.4 (ii) we have $\sigma_j = (t_j \cdot)$ as automorphisms of $U^{-}[\mathbf{w}_{[1,j-1]}]$, and δ_j is a locally nilpotent σ_j-derivation of $U^{-}[\mathbf{w}_{[1,j-1]}]$ satisfying $\sigma_j \delta_j = q_{i_j}^{-2} \delta_j \sigma_j$. This σ_j-derivation δ_j is explicitly given by

$$\delta_j(x) = F_{\beta_j} x - q^{\langle \beta_j, \gamma \rangle} x F_{\beta_j}, \text{ for } x \in (U^{-}[\mathbf{w}_{[1,j-1]}])_{\gamma} \tag{3.4.11}$$

and is computed using the Levendorskiĭ–Soibelman straightening law. The isomorphisms in Lemma 3.4.4 (ii) give rise to the Ore extension presentations

$$U^{-}[\mathbf{w}_{[1,j]}] = U^{-}[\mathbf{w}_{[1,j-1]}][F_{\beta_j}, \sigma_j, \delta_j], \text{ for all } 1 \leq j \leq l.$$

Inductively, we obtain a presentation for $U^{-}[w]$ as an iterated Ore extension for any reduced word $\mathbf{i}$ for w given by

$$U^{-}[w] = \Bbbk[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \ldots [F_{\beta_l}; \sigma_l, \delta_l].$$

Chapter 4

Ring vs representation theoretic approaches to $\mathrm{Spec}U^-[w]$

4.1 Cauchon's effacement des dérivations

We outline Cauchon's ring theoretic approach to the study of $\mathrm{Spec}U^-[w]$ via the method of deleting derivations. We follow [Cau03a, MC10] and the review in [BL10, Section 2].

Fix an iterated Ore extension

$$A = \mathbb{k}[x_1][x_2; \sigma_2, \delta_2] \ldots [x_l; \sigma_l, \delta_l], \tag{4.1.1}$$

For all $j \in [2, l]$, σ_j is an automorphism and δ_j is a (left) σ_j-skew derivation of

$$A_{[1,j-1]} = \mathbb{k}[x_1][x_2; \sigma_2, \delta_2] \ldots [x_{j-1}; \sigma_{j-1}, \delta_{j-1}].$$

Definition 4.1.1. An iterated Ore extension A as in equation (4.1.1) is called a Cauchon–Goodearl–Letzter (CGL) extension if it is equipped with an action of an algebraic torus $\mathbb{H} = (\mathbb{k}^*)^l$ by algebra automorphisms satisfying the following conditions for all $1 \leq i < j \leq l$:

(i) $\sigma_j(x_i) = q_{j,i}x_i$ for some $q_{j,i} \in \mathbb{k}$

(ii) δ_j is a locally nilpotent σ_j-skew derivation of $A_{[1,j-1]}$

(iii) The elements $x_1, \ldots, x_l$ are $\mathbb{H}$-eigenvectors and the following set is infinite:

$$\{p \in \mathbb{k} \mid \text{ there is a } t \text{ in } \mathbb{H} \text{ satisfies } t \cdot x_1 = px_1\}.$$

(iv) There exists $t_j \in \mathbb{H}$ such that $t_j \cdot x_j = q_j x_j$ for some $q_j \in \mathbb{k}^*$, which is not a root of unity, and $t_j \cdot x_i = q_{j,i}x_i$, for all $i \in [1, j-1]$

A length l CGL extension is called torsion-free if the subgroup of $\mathbb{k}^*$ generated by all $q_{j,i}$, $1 \leq i < j \leq l$ is torsion-free. Condition iv is merely the statement that $\sigma_j = (t_j \cdot)$ as automorphism of $A_{[1,j-1]}$.

Lemma 4.1.2. *For all CGL extensions $A = R[x; \sigma, \delta]$, we have $\sigma\delta = q\delta\sigma$.*

Proof. For all $r \in R$ we have

$$\delta(r) = xr - \sigma(r)x. \tag{4.1.2}$$

This follows directly from the multiplication in A. Upon applying σ, we obtain

$$\begin{aligned}\sigma(\delta(r)) &= \sigma(x)\sigma(r) - \sigma^2(r)\sigma(x)\\ &= qx\sigma(r) - q\sigma^2(r)x\end{aligned}$$

On the other hand, equation (4.1.2) is valid in particular for $r = \sigma(r)$ yielding

$$\delta(\sigma(r)) = x\sigma(r) - \sigma^2(r)x.$$

Thus, $\sigma\delta(r) = q\delta\sigma(r)$. $\square$

By applying the lemma to any length l Ore extension, we obtain $\sigma_j\delta_j = q_j\delta_j\sigma_j$ for all $j \in [2, l]$. Given a CGL extension A as in (4.1.1), for $j = l+1, l, \dots, 2$, Cauchon iteratively constructed in [Cau03a] l-tuples of nonzero elements

$$(x_1^{(j)}, \dots, x_l^{(j)})$$

and families of subalgebras of $\mathrm{Frac}(A)$ given by

$$A^{(j)} = \Bbbk\langle x_1^{(j)}, \dots, x_l^{(j)}\rangle.$$

We begin by setting

$$(x_1^{(l+1)}, \dots, x_l^{(l+1)}) = (x_1, \dots, x_l) \text{ and } A^{(l+1)} = A.$$

For $j = l, \dots, 2$, the l-tuple $(x_1^{(j)}, \dots, x_l^{(j)})$ is determined from $(x_1^{(j+1)}, \dots, x_l^{(j+1)})$ by

$$x_i^{(j)} = \begin{cases} x_i^{(j+1)} & \text{if } i \geq j \quad (4.1.3)\\ \displaystyle\sum_{m=0}^{\infty} \frac{(1-q_j)^{-m}}{(m)_j!}\left[\delta_j^m\sigma_j^{-m}\left(x_i^{(j+1)}\right)\right]\left(x_j^{(j+1)}\right)^{-m} & \text{if } i < j. \quad (4.1.4)\end{cases}$$

For $j \in [2, l+1]$, Cauchon constructed an algebra isomorphism

$$A^{(j)} \xrightarrow{\cong} \Bbbk[y_1]\dots[y_{j-1};\sigma_{j-1},\delta_{j-1}][y_j;\tau_j]\dots[y_l;\tau_l], \tag{4.1.5}$$

where τ_k denotes the automorphism of $\Bbbk[y_1]\dots[y_{j-1};\sigma_{j-1},\delta_{j-1}][y_j;\tau_j]\dots[y_{k-1};\tau_{k-1}]$ such that $\tau_k(y_i) = q_{k,i}y_i$ for all $i \in [1, k-1]$. This variable substitution isomorphism is given by $x_i^{(j)} \longmapsto y_i$, $i = 1, \dots, l$. For each $j \in [2, l]$, define

$$S_j = \left\{ \left(x_j^{(j+1)} \right)^m \,\middle|\, m \in \mathbb{Z}_{\geq 0} \right\}.$$

Then S_j is an Ore subset of $A^{(j)}$ and $A^{(j+1)}$. Moreover, we have $A^{(j)}[S_j^{-1}] = A^{(j+1)}[S_j^{-1}]$ by [[Cau03a] Théorèm 3.2.1. (1)].

Set $q_{i,i} = 1$ for $i \in [1, l]$ and $q_{i,j} = q_{j,i}^{-1}$ for $1 \leq i < j \leq l$. The multiparameter quantum affine space algebra $R_{\mathbf{q}}[\mathbb{A}^l]$ associated to the multiplicatiively antisymmetric $l \times l$ matrix $\mathbf{q} = (q_{i,j})$ is the $\Bbbk$-algebra with generators $X_1, \dots, X_l$ and relations

$$X_iX_j = q_{i,j}X_jX_i, \text{ for all } i, j \in [1, l].$$

We will call the algebra $\overline{A} = A^{(2)}$ obtained at the end of the Cauchon deleting derivation procedure *the Cauchon space of* A; the final l-tuple of generators of $\overline{A}$ will be denoted by $(\overline{x}_1, \dots, \overline{x}_l) = (x_1^{(2)}, \dots, x_l^{(2)})$. For $j = 2$ equation (4.1.5) gives an isomorphism

$$\overline{A} \xrightarrow{\cong} R_{\mathbf{q}}[\mathbb{A}^n] \text{ determined by } \overline{x}_i \longmapsto X_i. \tag{4.1.6}$$

We now proceed by describing the induced set-theoretic embeddings

$$\varphi_j \colon \operatorname{Spec} A^{(j+1)} \hookrightarrow \operatorname{Spec} A^{(j)} \tag{4.1.7}$$

for $j \in [2, l]$. By Goodearl and Letzter [GL00, Proposition 4.2] all $\mathbb{H}$-prime ideals of a CGL extension are completely prime (recall equation (4.1.5)), and by another result of Goodearl and Letzter [GL94, Theorem 2.3] all prime ideals of a torsion-free CGL extension are completely prime. Thus, the above mentioned condition is satisfied for all torsion-free CGL extensions A because of equation (4.1.5). Now, suppose J is in $\mathbb{H}$-Spec$A^{(j+1)}$ or Spec$A^{(j+1)}$. As an immediate corollary of the preceeding observation, we have

$$x_j^{(j+1)} \notin J \Rightarrow J \cap S_{j+1} = \varnothing.$$

Let $g_j \colon A^{(j)} \longrightarrow A^{(j+1)}/(x_j^{(j+1)})$ be the homomorphism

$$g_j(x_i^{(j)}) = x_i^{(j+1)} + (x_j^{(j+1)}) \text{ for } i \in [1, l]. \tag{4.1.8}$$

The induced spectrum map in (4.1.7) is given by

$$\varphi_j(J) = \begin{cases} JS_j^{-1} \cap A^{(j)} & \text{if } x_j^{(j+1)} \notin J \quad (4.1.9) \\ g_j^{-1}\Big(J/(x_j^{(j+1)})\Big) & \text{if } x_j^{(j+1)} \in J \quad (4.1.10) \end{cases}$$

This constructions applies, in particular, when $A = U^-[w]$ as these algebras are torsion-free CGL extensions when $q \in \mathbb{k}^*$ is not a root of unity by 3.4.4. We summarize this conditional set theoretic embedding following diagram:

$$\begin{tikzcd}
\mathrm{Spec}A^{(j+1)}/(x_j^{(j+1)}) \arrow[dr, "g^{-1}"] & \mathrm{Spec}A^{(j+1)} \arrow[l] \arrow[r] \arrow[d, "\varphi_j"] & \mathrm{Spec}A^{(j+1)}S_j^{-1} \arrow[d, equal] \\
& \mathrm{Spec}A^{(j)} & \mathrm{Spec}A^{(j)}S_j^{-1} \arrow[l]
\end{tikzcd}$$

Setting $\varphi = \varphi_2 \dots \varphi_l$ we have the set inclusion

$$\mathrm{Spec}A^{(l+1)} \overset{\varphi_l}{\hookrightarrow} \mathrm{Spec}A^{(l)} \overset{\varphi_{l-1}}{\hookrightarrow} \dots \overset{\varphi_2}{\hookrightarrow} \mathrm{Spec}A^{(2)}$$

from Spec$A$ to Spec$\overline{A}$. This map restricts to an inclusion

$$\varphi : \mathbb{H}\text{-Spec}A \hookrightarrow \mathbb{H}\text{-Spec}\overline{A}, \tag{4.1.11}$$

which we also label φ. Now, $\overline{A}$ is a quantum affine space by (4.1.6). Hence, the $\mathbb{H}$-prime ideals of $\overline{A}$ are in bijection with the subsets of $[1, l]$, that is, they are of the form

$$K_D = \overline{A}\langle \overline{x}_i \mid i \in D\rangle \text{ for } D \subseteq [1, l].$$

The Cauchon diagram of $J \in \mathbb{H}\text{-Spec}A$ is the unique set $D \subseteq [1,l]$ such that $\varphi(J) = K_D$. We will denote the Cauchon diagram of J by $\mathcal{C}(J)$. If $D \subseteq [1,l]$ is the Cauchon diagram of a $\mathbb{H}$-invariant prime ideal of A, then this prime ideal will be denoted by

$$J_D = \varphi^{-1}(K_D). \tag{4.1.12}$$

To define the leading terms of an ideal in A, we set

$$A_{[1,l-1]} = \Bbbk\langle x_1, \ldots, x_{l-1}\rangle = \Bbbk[x_1][x_2; \sigma_2, \delta_2] \ldots [x_{l-1}; \sigma_{l-1}, \delta_{l-1}] \text{ and} \tag{4.1.13}$$

$$A^{(l)}_{[1,l-1]} = \Bbbk\langle x^{(l)}_1, \ldots, x^{(l)}_{l-1}\rangle$$

so that we have the equalities

$$A = A_{[1,l-1]}[x_l; \sigma_l, \delta_l], \quad A^{(l)} = A^{(l)}_{[1,l-1]}[x_l; \tau_l], \quad \text{and} \quad A^{(l)}S_l^{-1} = A^{(l)}_{[1,l-1]}[x_l^{\pm 1}; \tau_l].$$

It is clear that $A_{[1,l-1]}$ and $A^{(l)}_{[1,l-1]}$ are $\mathbb{H}$-stable subalgebras of A and $A^{(l)}$, respectively. Furthermore, the deleting derivations map establishes an $\mathbb{H}$-equivariant algebra isomorphism

$$\theta : A_{[1,l-1]} \xrightarrow{\cong} A^{(l)}_{[1,l-1]} \tag{4.1.14}$$

determined by

$$x_i \longmapsto x_i^{(l)} = \sum_{m=0}^{\infty} \frac{(1-q_l)^{-m}}{(m)_{q_l}!}[\delta_l^m \sigma_l^{-m}(x_i)]x_l^{-m}. \tag{4.1.15}$$

for $i \in [1, l-1]$. Suppose J an ideal of A. Each nonzero element of J can be written in the form

$$j = a_m x_l^m + a_{m-1}x_l^{m-1} + \cdots + a_1 x_l + a_0,$$

where $a_m \neq 0$ and each a_i is in $A_{[1,l-1]}$. We let $\widetilde{J}$ denote the collection of all such a_m as j ranges over J. We call this the *leading part* of J. We will make use of the equality of the left and right leading parts of J, that is,

$$\widetilde{J} = \{a \in A_{[1,l-1]} \mid \text{ there exists } b \in J \text{ and } m \in \mathbb{Z}_{\geq 0} \text{ such that } a - bx_l^m \in A_{[1,l-1]}x_l^{m-1} + \ldots + A_{[1,l-1]}\} \tag{4.1.16}$$

$$= \{a \in A_{[1,l-1]} \mid \text{ there exists } b \in J \text{ and } m \in \mathbb{Z}_{\geq 0} \text{ such that } a - x_l^m b \in x_l^{m-1}A_{[1,l-1]} + \ldots + A_{[1,l-1]}\}.$$

This follows because σ_l is locally finite.

Recall a regular element in a ring R is an element that is neither a right nor a left zero divisor. The proof of the following lemma is analogous to [KL00, Lemma 4.7] and is left to the reader.

Lemma 4.1.3. *Let x be a regular element of the $\Bbbk$-algebra A for which there exist two $\Bbbk$-linear maps $\sigma, \delta \colon A \longrightarrow A$ such that σ is locally finite, δ is locally nilpotent, $\sigma\delta = q\delta\sigma$ for some $q \in \Bbbk^*$, and*

$$xa = \sigma(a)x + \delta(a), \text{ for all } a \in A.$$

Then the set $\Omega = \{1, x, x^2, \ldots\}$ is an Ore subset of A and

$$\mathrm{GKdim} A[\Omega^{-1}] = \mathrm{GKdim} A.$$

Suppose J is an $\mathbb{H}$-prime ideal of a CGL extension such as in equation (4.1.1), there are two possibilities—either x_l is in J or it is not. This simple observation paired with the following two important proposition will afford us methods for recursive computation.

Proposition 4.1.4 (G., Yakimov)**.** *Let A be a length l CGL extension, and assume J is in $\mathbb{H}$-Spec$A$. If $x_l \notin J$, then*

(i) $JS_l^{-1} = \bigoplus_{m \in \mathbb{Z}} \theta(\widetilde{J})x_l^m$ *and* $\varphi_l(J) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \theta(\widetilde{J})x_l^m$

(ii) $\mathcal{C}(J) = \mathcal{C}(\widetilde{J})$, *and*

(iii) $\mathrm{GKdim}(A/J) = \mathrm{GKdim}(A_{[1,l-1]}/\widetilde{J}) + 1$.

Proof. Part i: By [LLR06, Lemma 2.2] every $\mathbb{H}$-invariant ideal L of $AS_l^{-1} = A_{[1,l-1]}^{(l)}[x_l^{\pm 1}; \tau_l]$ has the form

$$L = \bigoplus_{m \in \mathbb{Z}} L_0 x_l^m \text{ for some ideal } L_0 \text{ of } A_{[1,l-1]}^{(l)}. \tag{4.1.17}$$

Indeed, if $a = \sum_m a_m x_l^m$ is in the $\mathbb{H}$-invariant ideal L, then

$$t_l^k \cdot (x_l^{-k} a x_l^k) = \sum_m q_l^{km} a_m x_l^m \text{ is in } L \text{ for all } k \in \mathbb{Z}_{\geq 0},$$

where $t_l \in \mathbb{H}$ is the element from Defeinition 4.1.1 iv. Thus, $a_m x_l^m \in L$, for all $m \in \mathbb{Z}$, which proves equation (4.1.17).

We apply this to the ideal $L = JS_l^{-1}$. Equation (4.1.15) implies that for all $a \in A_{[1,l-1]}$ and $m \in \mathbb{Z}$

$$\theta(a) x_l^m = a x_l^m + \sum_{k=m-n}^{m-1} b_k x_l^k$$

for some $n \geq 0$, and $b_k \in A_{[1,l-1]}$. Now, every nonzero element of $JS_l^{-1} \subset A_{[1,l-1]} S_l^{-1}$ has the form

$$j = a x_l^m + \sum_{k=-n}^{m-1} a_k x_l^k \text{ for some } a \in \widetilde{J} \backslash \{0\}, \text{ and } a_k \in A_{[1,l-1]}.$$

It should also have the form

$$j = \theta(a) x_l^m + \sum_{k=m-n}^{m-1} a'_k x_l^k \text{ for some } a \in \widetilde{J} \backslash \{0\}, \text{ and } a'_k \in A_{[1,l-1]}^{(l)}.$$

Now the two equalities in i follow from equation (4.1.17). Claim ii that $\mathcal{C}(J)$ equals $\mathcal{C}(\widetilde{J})$ is immediate from the definition of Cauchon diagrams, and claim iii easily follows from Lemma 4.1.3 and the isomorphism

$$(A/J)[S_l^{-1}] \cong \theta(A_{[1,l-1]}/\widetilde{J})[x_l^{\pm}, \tau_l].$$

□

We will also make use of the following proposition, which is complementary to Proposition 4.1.4:

Proposition 4.1.5 (G., Yakimov)**.** *Let A be a length l CGL extension, and assume J is in $\mathbb{H}$-Spec$A$. If $x_l \in J$, then*

(i) $\varphi_l(J) = \theta(J \cap A_{[1,l-1]}) + A^{(l)}x_l$,

(ii) $\mathcal{C}(J) = \mathcal{C}(J \cap A_{[1,l-1]}) \sqcup \{l\}$, *and*

(iii) there are $\mathbb{H}$-equivariant algebra isomorphisms

$$A/J \cong A^{(l)}/\varphi_l(J) \cong A_{[1,l-1]}/(J \cap A_{[1,l-1]}) \cong A^{(l)}_{[1,l-1]}/(\varphi_l(J) \cap A^{(l)}_{[1,l-1]}).$$

In particular, $\mathrm{GKdim}(A/J) = \mathrm{GKdim}(A_{[1,l-1]}/J \cap A_{[1,l-1]})$.

Proof. Statements i and ii follow from the definition of φ_l. The latter also implies that

$$g_l \colon A^{(l)} \longrightarrow A^{(l+1)}/(x_l^{(l+1)}) = A/(x_l)$$

induces the $\mathbb{H}$-equivariant algebra isomorphism $A^{(l)}/\varphi_l(J) \cong (A/(x_l))\,/(J/(x_l)) \cong A/J$. Since $x_l \in J$ and $x_l \in \varphi_l(J)$ the inclusions

$$A_{[1,l-1]} \hookrightarrow A \quad \text{and} \quad A^{(l)}_{[1,l-1]} \hookrightarrow A^{(l)}$$

induce the $\mathbb{H}$-equivariant algebra isomorphisms

$$A_{[1,l-1]}/(J \cap A_{[1,l-1]}) \xrightarrow{\cong} A/J \quad \text{and} \quad A^{(l)}_{[1,l-1]}/(\varphi_l(J) \cap A^{(l)}_{[1,l-1]}) \xrightarrow{\cong} A^{(l)}/\varphi_l(J)$$

The claim follows by the $\mathbb{H}$-equivariant isomorphism $A^{(l)}_{[1,l-1]} \cong A_{[1,l-1]}$ from 4.1.14. □

All quantum Schubert cell algebras $U^-[w]$ over arbitrary fields $\mathbb{k}$ such that $q \in \mathbb{k}^*$ is not a root of unity are torsion-free CGL extensions by equation 3.4.4. Moreover, each reduced word $\mathbf{i}$ for w gives rise to a presentation (5.1.1) of $U^-[w]$ as a CGL extension. We conclude this section by recalling the following classification of $\mathbb{H}$-Spec$U^-[w]$.

Theorem 4.1.6. *(Cauchon–Mériaux , [MC10]) For any simple Lie algebra $\mathfrak{g}$, Weyl group element w, reduced word $\mathbf{i}$ for w, and any field $\mathbb{k}$ such that $q \in \mathbb{k}^*$ not a root of unity, let $U^-[w]$ have the presentation given in equation (5.1.1). In this presentation of $U^-[w]$ as a torsion-free CGL extension, the $\mathbb{H}$-prime ideals of $U^-[w]$ are the ideals $J_{\mathcal{D}^+(y)}$ for the elements $y \in W^{\leq w}$ (recall equation (4.1.12)), where $\mathcal{D}^+(y) \subseteq [1, l]$ is the index set of the left positive subword of $\mathbf{i}$ whose total product is y, cf. Section 3.3.*

This theorem gives a bijection between Cauchon diagrams of the $\mathbb{H}$-invariant prime ideals of $U^-[w]$ and the index sets of all left positive subwords of $\mathbf{i}$. We note that in [MC10] Theorem 4.1.6 was formulated for the upper quantum Schubert cells $U^+[w]$. However, these formulations are equivalent by the Cartan involution from (3.4.7). We give a second, independent proof of this theorem in Section 5.2.

4.2 The prime spectrum of $U^-[w]$ via Demazure modules

We proceed with the realization of the algebras $U^-[w]$ in terms of quantum function algebras and the description of the spectra of $U^-[w]$ via Demazure modules from [Yak10].

Recall that, for $\lambda \in P_{\geq 0}$ and $w \in W$, the extremal weight spaces $V(\lambda)_{w\lambda}$ are all one dimensional and that the braid group permutes the weight spaces of the $V(\lambda)$ (see 3.1.15 and 3.1.21). We then have $v_\lambda^w = T_{w^{-1}}^{-1} v_\lambda$ is in $V(\lambda)_{w\lambda}$, and there is a unique $f_w^\lambda \in (V(\lambda)^*)_{-w\lambda}$ such that

$$\langle f_w^\lambda, v_\lambda^w \rangle = 1. \tag{4.2.1}$$

Also, recall the previously defined coodinate functions 3.2. For each dominant integral weight λ, fix a highest weight vector v_λ of $V(\lambda)_\lambda$, and let $c_f^\lambda \in U_q(\mathfrak{g})^\circ$ be given by

$$c_f^\lambda(x) = f(xv_\lambda)$$

Note that $c_f^\lambda = c_{f,v_\lambda}^\lambda$. Now we define the subalgebra R of $R_q[G]$ by

$$R = \mathrm{Span}\{c_f^\lambda \mid \lambda \in P_{\geq 0} \text{ and } f \in V(\lambda)^*\}$$

For $y, w \in W$ and $\lambda \in P_{\geq 0}$, define the elements $e^{\lambda}_{w,y} \in R_q[G]$ and $e^{\lambda}_w \in R$ by

$$e^{\lambda}_{w,y}(x) = f^{\lambda}_w(xT_y v_\lambda) \tag{4.2.2}$$

$$e^{\lambda}_w(x) = f^{\lambda}_w(xv_\lambda). \tag{4.2.3}$$

Note that $e^{\lambda}_w = e^{\lambda}_{w,1}$. When ϖ is a fundamental weight, the element e^{ϖ}_w is called a *quantum minor*, cf. [BZ05].

Proposition 4.2.1. *For all $\lambda_1, \lambda_2 \in P_{\geq 0}$ and $w \in W$, we have*

$$e^{\lambda_1}_w e^{\lambda_2}_w = e^{\lambda_1+\lambda_2}_w = e^{\lambda_2}_w e^{\lambda_1}_w. \tag{4.2.4}$$

Proof. Proved analogously to [Yaka, eq. (2.18)] using the second equality in (i). □

Joseph proved the multiplicative subsets $E_w = \left\{e^{\lambda}_w \mid \lambda \in P_{\geq 0}\right\} \subset R$ are Ore sets, see [Jos95, Lemma 9.1.10]. We remark Joseph's proof works for all base fields $\Bbbk$, $q \in \Bbbk^{\times}$ not a root of unity, see [Yakb, Section 2.2]. For a fixed w, define R_w to be the localization of R

$$R_w = R[E_w^{-1}] \tag{4.2.5}$$

Following Joseph [Jos95, Section 10.4.8] we define subalgebras $R^0_w \subset R_w$ by

$$R^0_w = \operatorname{Span}\left\{c^{\lambda}_f (e^{\lambda}_w)^{-1} \mid \lambda \in P_{\geq 0}, f \in V(\lambda)^*\right\}, \tag{4.2.6}$$

The subalgebras R^0_w are invariant subalgebras of R_w with respect to the left action 3.2.5 of K_i for all i. We remark that we do not need to take span in the right hand side of the above formula, cf. [Jos95, Section 10.4.8] or [Yakb, eq. (2.18)]. For $\lambda_1, \lambda_2 \in P_{\geq 0}$, set

$$e^{\lambda_1-\lambda_2}_w = e^{\lambda_1}_w (e^{\lambda_2}_w)^{-1} \in R^0_w. \tag{4.2.7}$$

It follows from (4.2.4) that this does not depend on the choice of λ_1, λ_2 and that $e^{\mu_1}_w e^{\mu_2}_w = e^{\mu_1+\mu_2}_w$ for all $\mu_1, \mu_2 \in P$. The algebra R^0_w is Q-graded by

$$(R^0_w)_\gamma = \left\{c^{\lambda}_f e^{-\lambda}_w \mid \lambda \in P_{\geq 0}, \text{ and } f \in (V(\lambda)^*)_{\gamma+w(\lambda)}\right\}.$$

For any Weyl group element y there is a corresponding *Demazure module* $U_q^+V(\lambda)_{y\lambda} \subset V(\lambda)$. These are extremal $U_q^{\geq 0}$-submodules of $V(\lambda)$. Because the extremal weight spaces of $V(\lambda)$ are all one-dimensional, we may also write $U_q^+T_y v_\lambda$ for the Demazure module corresponding to y. For example, note that $U_q^+T_{w_0}v_\lambda = V(\lambda)$ since $T_{w_0}v_\lambda$ is a lowest weight vector for $V(\lambda)$. Similarly, we have $U_q^{\leq 0}$-submodules of $V(\lambda)$ given by $U_q^-T_y v_\lambda$. These upper and lower Demazure modules give rise to the upper and lower *quantum Schubert cell ideals* $S^\pm(y)$ of R paramentrized by $y \in W$. These ideals are defined by

$$S^\pm(y) = \text{Span}\left\{c_f^\lambda \mid \lambda \in P_{\geq 0} \text{ and } f \in V(\lambda)^* \text{ such that } f \perp U^\pm T_y v_\lambda\right\}.$$

By extension to R_w and contraction along R_w^0, we obtain the counterparts of $S^\pm(y)$ in R_w^0. These ideals are given by

$$S_w^\pm(y) = \left\{c_f^\lambda e_w^{-\lambda} \mid \lambda \in P_{\geq 0}, f \in V(\lambda)^* \text{ such that } f \perp U^\pm T_y v_\lambda\right\} = S^\pm(y)E_w^{-1} \cap R_w^0. \quad (4.2.8)$$

Analogously to (4.2.6) one does not need to take a span in (4.2.8), see [Gor00, Yak10]. For $\gamma \in Q_{\geq 0} \setminus \{0\}$, we have $\dim U^+[w]_\gamma = \dim U^-[w]_{-\gamma}$. Denote this dimension by m_γ, and choose dual bases $\{u_{\gamma,i}\}_{i=1}^{m_\gamma}$ and $\{u_{-\gamma,i}\}_{i=1}^{m_\gamma}$ of $U^+[w]_\gamma$ and $U^-[w]_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [Jan96, Ch. 6]. The quantum R matrix corresponding to w is given by

$$\mathcal{R}^w = 1 \otimes 1 + \sum_{\substack{\gamma \neq 0 \\ \gamma \in Q_{\geq 0}}} \sum_{i=1}^{m_\gamma} u_{\gamma,i} \otimes u_{-\gamma,i} \in U^+ \widehat{\otimes} U^-, \quad (4.2.9)$$

where $U^+ \widehat{\otimes} U^-$ is the completion of $U^+ \otimes U^-$ with respect to the descending filtration [Lus93, Section 4.1.1].

The following far-reaching theorem summarizes the representation theoretic approach to $\text{Spec} U^-[w]$ via quantum function algebras and Demazure modules.

Theorem 4.2.2 (Yakimov)**.** *For any field $\Bbbk$ with $q \in \Bbbk^\times$ not a root of unity, any simple Lie algebras $\mathfrak{g}$, and Weyl group elements $w \in W$, the following hold:*

(i) The maps $\phi_w \colon R_w^0 \to U^-[w]$ given by

$$c_f^\lambda e_w^{-\lambda} \longmapsto \left(c_{f,v_\lambda^w}^\lambda \otimes \mathrm{id}\right)(\tau \otimes \mathrm{id})\mathcal{R}^w, \ \textit{with } \lambda \in P_{\geq 0}, f \in V(\lambda)^* \tag{4.2.10}$$

are well-defined surjective Q-graded algebra homomorphisms with $\ker \phi_w = S_w^+(w)$.

(ii) For $y \in W^{\leq w}$ the ideals

$$I_w(y) = \phi_w(S_w^+(w) + S_w^-(y)) = \phi_w(S_w^-(y))$$

are distinct, $\mathbb{H}$-invariant, completely prime ideals of $U^-[w]$. Moreover, all $\mathbb{H}$-prime ideals of $U^-[w]$ are of this form.

(iii) The map $y \in W^{\leq w} \mapsto I_w(y) \in \mathbb{H}\text{-Spec}U^-[w]$ is an isomorphism of posets with respect to the Bruhat order on $W^{\leq w}$ and the inclusion order on $\mathbb{H}\text{-Spec}U^-[w]$.

Part (i) is [Yaka, Theorem 2.6]. It was first proved in [Yak10] for another version of the Hopf algebra $U_q(\mathfrak{g})$ equipped with the opposite coproduct, and a different braid group action. Theorem 2.6 in [Yaka] used $T_w v_\lambda$ in place of $v_\lambda^w = T_{w^{-1}}^{-1}$ in equation (4.2.1) and Theorem 4.2.2 (i). The two formulations are equivalent since $\dim V(\lambda)_{w\lambda} = 1$ and $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$ for all $w \in W$, $\lambda \in P_{\geq 0}$, $\mu \in P$. Parts (ii) and (iii) of Theorem 4.2.2 are proved in [Yakb, Theorem 3.1 (a)] relying on results of Gorelik [Gor00] and Joseph [Jos94]. These statements were earlier proved in [Yak10, Theorem 1.1 (a)-(b)] under slightly stronger conditions on $\Bbbk$ and q.

For $\lambda \in P_{\geq 0}$ define the elements $b_{y,w}^\lambda \in U^-[w]$ by

$$b_{y,w}^\lambda = \phi_w\left(e_y^\lambda e_w^{-\lambda}\right),$$

and let $b_{y,w}^\lambda$ also denote the canonical images of $b_{y,w}^\lambda \in U^-[w]/I_w(y)$. We will keep this notational convention throughout. For the sake of transparency, we trace through the definition of $b^{\lambda}_{y,w}$ to obtain

$$\begin{aligned} b^{\lambda}_{y,w} &= \left(e^{\lambda}_{y,w}\tau \otimes \mathrm{id}\right)\mathcal{R}^w \\ &= \left(f^{\lambda}_y(T_w v_\lambda)\tau \otimes \mathrm{id}\right)\left(1 \otimes 1 + \sum_{\substack{\gamma\neq 0 \\ \gamma\in Q_{\geq 0}}} \sum_{i=1}^{m_\gamma} u_{\gamma,i} \otimes u_{-\gamma,i}\right) \end{aligned}$$

For all $\lambda \in P_{\geq 0}$ and $\gamma \in Q_{\mathbb{S}(w)}$, these are nonzero normal elements of $U^-[w]/I_w(y)$ by [Yakb, Theorem 3.1(b) and eq. (3.1)]:

$$b^{\lambda}_{y,w} x = q^{-\langle (w+y)\lambda, \gamma\rangle} x b^{\lambda}_{y,w}. \text{ for all } x \in (U^-[w]/I_w(y))_\gamma. \tag{4.2.11}$$

The R-matrix commutation relations in R (see e.g. [BG02, Theorem I.8.15]) and eq. (4.2.4) imply that for all $\lambda_1, \lambda_2 \in P_{\geq 0}$

$$b^{\lambda_1}_{y,w} b^{\lambda_2}_{y,w} = q^{-\langle \lambda_1, \lambda_2 - y^{-1}w\lambda_2\rangle} b^{\lambda_1+\lambda_2}_{y,w}.$$

Thus, the multiplicative subset of $U^-[w]/I_w(y)$ given by

$$B_{y,w} = \left\{ k b^{\lambda}_{y,w} \mid \lambda \in P_{\geq 0} \text{ and } k \in \Bbbk^\times \right\}$$

consists entirely of normal elements. Let $R_{y,w}$ be the localized quotient of $U^-[w]$

$$R_{y,w} = (U^-[w]/I_w(y))[B^{-1}_{y,w}]$$

This quotient is $\mathbb{H}$-simple, its center $Z(R_{y,w})$ is a Laurent polynomial ring of dimension equal to $\dim \ker(w+y)$, and the prime spectrum of $U^-[w]$ is partitioned into

$$\mathrm{Spec} U^-[w] = \bigsqcup_{y \in W^{\leq w}} \mathrm{Spec}_{I_w(y)} U^-[w].$$

The strata $\mathrm{Spec}_{I_w(y)} U^-[w]$ are given by

$$\left\{ P \in \mathrm{Spec} U^-[w] \mid P \supseteq I_w(y) \text{ and } P \cap B_{y,w} = \varnothing \right\}.$$

Moreover, extension and contraction establishes the homeomorphisms:

$$\mathrm{Spec} Z(R_{y,w}) \xrightarrow{\cong} \mathrm{Spec} R_{y,w} \xrightarrow{\cong} \mathrm{Spec}_{I_w(y)} U^-[w].$$

We refer to [Yakb, Theorem 3.1 and Proposition 4.1] for details and proofs of the above statements. The dimensions of the Laurent polynomial rings $Z(R_{y,w})$ were determined in [BCL, Yakb]. The previous results are compatible with the general Goodearl–Letzter $\mathbb{H}$-stratification theory[GL00]. The above framework for $\mathrm{Spec} U^-[w]$ is much more explicit. It deals with specific $\mathbb{H}$-prime ideals and localizations by small sets of normal elements.

For all $a, b, j \in [1, l]$ and $\mathbf{m}_{[a,b]} = (m_a, \dots, m_b) \in \mathbb{N}^{b-a}$, define the symbols p_j and $\mathbf{p_m}$ by

$$p_j = \frac{(q_{i_j}^{-1} - q_{i_j})^{-1}}{q_{i_j}^{m_j(m_j-1)/2}[m_j]_{i_j}!}, \text{ and} \tag{4.2.12}$$

$$\mathbf{p}_{\mathbf{m}_{[a,b]}} = \begin{cases} p_a p_{a+1} \dots p_{b-1} p_b & \text{if } a \leq b \\ p_a p_{a-1} \dots p_{b+1} p_b & \text{if } a > b \end{cases}$$

Proposition 4.2.3. *Fix a Weyl group element w. For all $\lambda \in P_{\geq 0}$ and $f \in V(\lambda)^*$ the antihomomorphisms $\phi_w \colon R_0^w \to U^-[w]$ are explicitly given by*

$$\phi_w(c_f^\lambda e_w^{-\lambda}) = \sum_{\mathbf{m} \in \mathbb{N}^l} \mathbf{p_m} \langle f, \tau(\mathbf{E}^{\mathbf{m}_{[l,1]}}) v_\lambda^w \rangle \mathbf{F}^{\mathbf{m}_{[l,1]}} \tag{4.2.13}$$

$$= \sum_{\mathbf{m} \in \mathbb{N}^l} \mathbf{p_m} \langle f, (\tau \mathbf{E})^{\mathbf{m}} v_\lambda^w \rangle \mathbf{F}^{\mathbf{m}_{[l,1]}} \tag{4.2.14}$$

Proof. This follows from Theorem 4.2.2 (i) and the standard formula [Jan96, eqs. 8.30 (1) and (2)] for the inner product of the pairs of monomials (iii) with respect to the the Rosso–Tanisaki form. □

Chapter 5
Main results and implications

5.1 Statement of main result

We recall from Section 4.1 that each length l reduced word $\mathbf{i}$ for a Weyl group element w gives a presentation of the quantum Schubert cell algebra $U^-[w]$ as a torsion free CGL extension

$$U^-[w] = \Bbbk[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \dots [F_{\beta_l}; \sigma_l, \delta_l]. \tag{5.1.1}$$

By iteratively deleting derivations, we obtain a presentation of the associated Cauchon space

$$\overline{U^-[w]} = \Bbbk[\overline{F}_{\beta_1}][\overline{F}_{\beta_2}; \tau_2] \dots [\overline{F}_{\beta_l}; \tau_l]. \tag{5.1.2}$$

For brevity we write $\overline{F}_i$ in place of $\overline{F}_{\beta_i}$. In this section we explicitly describe of each of these quantum affine spaces using the antiisomorphisms from 4.2.2 (i); the generators of the Cauchon spaces $\overline{U^-[w]}$ are quantum minors or quotients of two quantum minors.

Given $\mathbf{i} = i_1 \dots i_l$, we define the *successor* function $s\colon [1, l] \sqcup \{\infty\} \to [1, l] \sqcup \{\infty\}$ by

$$s(j) = \begin{cases} \min \{k \mid j < k \text{ and } i_k = i_j\} & \text{if such a } k \text{ exists} \\ \\ \infty & \text{otherwise} \end{cases}$$

Example 5.1.1. Let $\mathbf{i} = 121321$, then

$$\begin{array}{lll} s(1) = 3 & s(2) = 5 & s(3) = 6 \\ \\ s(4) = \infty & s(5) = \infty & s(6) = \infty \end{array}$$

It is clear that s is locally nilpotent in the sense that for any j, we have $s^n(j) = \infty$ for some n. Thus, we define the *order* of $j \in [1, l]$ by

$$|j| = \max \{n \in \mathbb{Z}_{\geq 0} \mid s^n(j) \neq \infty\}. \tag{5.1.3}$$

Example 5.1.2. Let $\mathbf{i} = 121321$, then

$|1| = 2$ $\quad |2| = 1$ $\quad |3| = 1$

$|4| = 0$ $\quad |5| = 0$ $\quad |6| = 0$

Recall the notation for the quantum minors e_w^{ϖ}, the R-matrix $\mathcal{R}^w$, the tau involution τ, and Yakimov's surjection $\phi_w \colon R_w^0 \to U^-[w]$ from equations (4.2.2), (4.2.9), (3.1.1), and Theorem 4.2.2 (i). For each $j \in [1, l]$ define the quantum minor $\Delta_j \in U^-[w]$ by

$$\Delta_j = b^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}, w} \tag{5.1.4}$$

This notation is extremely compact; upon unraveling it we obtain

$$\begin{aligned}\Delta_j &= \phi_w\left(e^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}} e_w^{-\varpi_{i_j}}\right) = \left(e^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}, w}\tau \otimes \mathrm{id}\right)\mathcal{R}^w \\ &= \left(f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}}(T_w v_{\varpi_{i_j}})\tau \otimes \mathrm{id}\right)\left(1 \otimes 1 + \sum_{\substack{\gamma \neq 0 \\ \gamma \in Q_{\geq 0}}} \sum_{i=1}^{m_\gamma} u_{\gamma,i} \otimes u_{-\gamma,i}\right)\end{aligned}$$

We stress that "Δ_j" is abbreviated notation as Δ_j is heavily dependent upon $\mathbf{i}$. We have the following main theorem:

Theorem 5.1.3 (G., Yakimov)**.** *[G., Yakimov] Assume that $\Bbbk$ is an arbitrary base field, $q \in \Bbbk^\times$ is not a root of unity, $\mathfrak{g}$ is a simple Lie algebra, $w \in W$ is a Weyl group element, and $\mathbf{i}$ is a reduced word for w. Then the generators $\overline{F}_1, \ldots, \overline{F}_l$ of the Cauchon space $\overline{U^-[w]}$ are given by*

$$\overline{F}_j = \begin{cases} \widehat{q_{i_j}}^{-1} \Delta_{s(j)}^{-1} \Delta_j & \text{if } s(j) \leq l \\ \widehat{q_{i_j}}^{-1} \Delta_j & \text{if } s(j) = \infty \end{cases}$$

Theorem 5.1.3 is equivalent to the the following theorem to be proved in Section 5.1.2.

Theorem 5.1.4. *In the setting of Theorem 5.1.3 the quantum minors $\Delta_1, \ldots, \Delta_l$* (5.1.4) *are explicitly given by*

$$\Delta_j = \widehat{q_{i_j}}^{|j|} \overline{F}_{s^{|j|}(j)} \ldots \overline{F}_j. \tag{5.1.5}$$

As an illustration of the generality of this theorem, consider the following very special situation: Given two non-negative integers m and n, let $\mathfrak{g}$ be $\mathfrak{sl}_{m+n}$ and set

$$w = (12 \ldots m+n)^m \in S_{m+n} \tag{5.1.6}$$

By [MC10, Proposition 2.1.1] and [Yakb, Lemma 4.1] the algebras $U^{\pm}[w]$ is then isomorphic to $m \times n$ quantum matrices $R_q[M_{m,n}]$. By [Yakb, Lemma 4.3] the elements $b_{y,w}^{\varpi_{i_j}} \in U^-[w]$ correspond (under this isomorphism) to scalar multiples of quantum minors of $R_q[M_{m,n}]$ for all $j \in [1, m+n]$ and y in the Bruhat-order interval $S_{m+n}^{\leq w}$ (depending on the normalization from 4.2.1). The special case of this theorem for the algebras of quantum matrices $R_q[M_{m,n}]$ is due to Cauchon [Cau03b]

5.1.1 Leading terms of quantum minors

The presentation of $U^-[w]$ as iterated Ore extension in the generators $F_{\beta_1}, \ldots, F_{\beta_l}$ is by no means unique. The Levendorskiĭ–Soibelman straightening law implies that we may adjoin the generators in the opposite order as well. *Translation* from one presentation to the *reverse* presentation will play a major role in our proof of Theorem 5.1.4 in Section 5.1.2. In Sections 5.1.1 and 5.1.2 we examine reversed iterated Ore extensions and prove a leading term result for the elements Δ_j.

For a length l reduced expression $\mathbf{w}$, we define the upper and lower *truncated* algebras

$$U^-[\mathbf{w}]_{[j,k]} = \Bbbk\big[F_{\beta_j}, F_{\beta_{j+1}}, \ldots, F_{\beta_{k-1}}, F_{\beta_k}\big]$$

These algebras do, in general, depend on the reduced expression of $\mathbf{w}$. The Levendorskiĭ–Soibelman straightening law gives that $U^-[\mathbf{w}]_{[j,k]}$ are subalgebras of $U^-[w]$. The truncated algebras are closely related to truncated reduced expressions (cf. (3.3.3)) as the following proposition suggests.

Proposition 5.1.5. *For all length l reduced expressions $\mathbf{w}$ and $j \leq k \in [1,l]$, we have*

$$T^{-1}_{\mathbf{w}_{[1,j-1]}} U^-\big[\mathbf{w}_{[j,k]}\big] = U^-[\mathbf{w}]_{[j,k]}$$

Proof. Tracing through the definitions of the inherent symbols, we obtain

$$\begin{aligned}
T^{-1}_{\mathbf{w}_{[1,j-1]}} U^-\big[\mathbf{w}_{[j,k]}\big] &= T^{-1}_{\mathbf{w}_{[1,j-1]}} \Bbbk\Big[F_{i_j}, T_{i_j}F_{i_{j+1}}, \ldots, T_{\mathbf{w}_{[j,k-2]}}F_{i_{k-1}}, T_{\mathbf{w}_{[j,k-1]}}F_{i_k}\Big] \\
&= \Bbbk\Big[T^{-1}_{\mathbf{w}_{[1,j-1]}}F_{i_j}, T_{\mathbf{w}_{[1,j]}}F_{i_{j+1}}, \ldots, T_{\mathbf{w}_{[1,k-2]}}F_{i_{k-1}}, T_{\mathbf{w}_{[1,k-1]}}F_{i_k}\Big] \\
&= \Bbbk\big[F_{\beta_j}, F_{\beta_{j+1}}, \ldots, F_{\beta_{k-1}}, F_{\beta_k}\big] \\
&= U^-[\mathbf{w}]_{[j,k]}
\end{aligned}$$

□

By noting that $(\mathbf{w}_{[1,j-1]})^{-1}\mathbf{w}_{[1,k]} = \mathbf{w}_{[j,k]}$ we obtain the immediate and useful corollary

$$T^{-1}_{\mathbf{w}_{[1,j-1]}} U^-\big[(\mathbf{w}_{[1,j-1]})^{-1}\mathbf{w}_{[1,k]}\big] = T^{-1}_{\mathbf{w}_{[1,j-1]}} U^-\big[\mathbf{w}_{[j,k]}\big] \tag{5.1.7}$$

Towards the main result we prove the following theorem.

Theorem 5.1.6. *For all fields $\Bbbk$, $q \in \Bbbk^\times$ not a root of unity, simple Lie algebras $\mathfrak{g}$, length l reduced expressions $\mathbf{w}$, and $j \in [1,l]$, we have*

$$\Delta_j = \begin{cases} \widehat{q_{i_j}}\Delta_{s(j)}F_{\beta_j} \bmod U^-[\mathbf{w}]_{[j+1,l]} & s(j) \neq \infty \\ \widehat{q_{i_j}}F_{\beta_j} \bmod U^-[\mathbf{w}]_{[j+1,l]} & s(j) = \infty \end{cases} \tag{5.1.8}$$

Proof. For all $k \in [1,l]$ we construct a twisted algebra $R^k = \tau T_{\mathbf{w}_{[1,k-1]}}(U^k)$. Recall this algebra is isomorphic to $U_{q_{i_k}}(\mathfrak{sl}_2)^{\mathrm{op}}$, see (3.1). Set $1 \leq k \leq j-1$, and consider the R^k-submodule of

$V(\varpi_{i_j})$ generated by $x = T^{-1}_{\mathbf{w}_{[j-1,1]}} v_{\varpi_{i_j}}$. It is irreducible since

$$(\tau F_{\beta_k})x = \Big(\tau(T_{\mathbf{w}_{[1,k-1]}} F_k)\Big) T^{-1}_{\mathbf{w}_{[j-1,1]}} v_{\varpi_{i_j}} \tag{5.1.9}$$

$$= \Big(T^{-1}_{\mathbf{w}_{[k-1,1]}} F_k\Big) T^{-1}_{\mathbf{w}_{[j-1,1]}} v_{\varpi_{i_j}} \tag{5.1.10}$$

$$= T^{-1}_{\mathbf{w}_{[j-1,1]}} \Big((T_{\mathbf{w}_{[j-1,1]}} T^{-1}_{\mathbf{w}_{[k-1,1]}} F_k) v_{\varpi_{i_j}}\Big) \tag{5.1.11}$$

$$= T^{-1}_{\mathbf{w}_{[j-1,1]}} \Big((T_{\mathbf{w}_{[j-1,k]}} F_k) v_{\varpi_{i_j}}\Big) = 0; \tag{5.1.12}$$

see equations (3.1.17) and (3.1.23). In the last equation we used that $T_{\mathbf{w}_{[j-1,k]}}(F_k) \in U^+$. Therefore, there is a R^k-module V_k such that

$$V(\varpi_{i_j}) = R^k x \oplus V_k.$$

It follows that V_k is also U^0-stable. From this, equation (5.1.9), and the fact that $\dim V(\varpi_{i_j})_{\mathbf{w}_{[1,j-1]}\varpi_{i_j}} = 1$ it follows that

$$\langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}}, (\tau E_{\beta_k})v\rangle = 0, \text{ for all } v \in V(\varpi_{i_j}) \text{ and } 1 \le k < j, \tag{5.1.13}$$

recall (4.2.1).

Next, we consider the R^j-submodule of $V(\varpi_{i_j})$ generated by x. Using (i)–(ii), we obtain:

$$(\tau E_{\beta_j})\Big(T^{-1}_{\mathbf{w}_{[j,1]}} v_{\varpi_{i_j}}\Big) = \Big(\tau(T_{\mathbf{w}_{[1,j-1]}} E_j)\Big)\Big(T^{-1}_{\mathbf{w}_{[j,1]}} v_{\varpi_{i_j}}\Big) \tag{5.1.14}$$

$$= \Big(T^{-1}_{\mathbf{w}_{[j-1,1]}} E_j\Big)\Big(T^{-1}_{\mathbf{w}_{[j,1]}} v_{\varpi_{i_j}}\Big)$$

$$= T^{-1}_{\mathbf{w}_{[j-1,1]}}\Big(E_j T_j^{-1} v_{\varpi_{i_j}}\Big) = x. \tag{5.1.15}$$

Analogously one shows that

$$(\tau E_{\beta_j})x = 0 \text{ and } \Big(\tau(T^{-1}_{\mathbf{w}_{[j-1,1]}} K_j)\Big) x = q_{i_j}^{-1} x.$$

Therefore,

$$R^j x = \Bbbk x \oplus \Bbbk T^{-1}_{\mathbf{w}_{[j,1]}} v_{\varpi_{i_j}}.$$

Let v be in $V(\varpi_{i_j})$. Using the preceeding fact, the complete reducibility of finite dimensional type one U_q^i-modules, and equation (5.1.14), we obtain :

$$\langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}}, (\tau E_{\beta_j})^m v\rangle = \begin{cases} \langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j]}}, v\rangle & m=1 \\ 0 & m>1 \end{cases}$$

In a similar way one proves that for all $j < k \leq \min\{l, s(j)-1\}$

$$\left(R^k\right) T^{-1}_{\mathbf{w}_{[1,k-1]}} v_{\varpi_{i_j}} = \Bbbk T^{-1}_{\mathbf{w}_{[1,k-1]}} v_{\varpi_{i_j}} \text{ and } T^{-1}_{\mathbf{w}_{[1,k-1]}} v_{\varpi_{i_j}} = T^{-1}_{\mathbf{w}_{[1,k]}} v_{\varpi_{i_j}}.$$

From this one obtains that for all $j < k \leq \min\{l, s(j)-1\}$

$$\langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,k-1]}}, (\tau E_{\beta_k})v\rangle = 0 \text{ and } f^{\varpi_{i_j}}_{\mathbf{w}_{[1,k-1]}} = f^{\varpi_{i_j}}_{\mathbf{w}_{[1,k]}}. \tag{5.1.16}$$

□

Proof of the Theorem 5.1.3. Equation (5.1.8) is deduced from from equations (5.1.13), (5.1.1), and (5.1.16) as follows. Using (4.2.13), (5.1.13), and (5.1.1), we obtain:

$$\begin{aligned}\Delta_j &= \sum_{\mathbf{m}_{[j,l]}} \mathbf{p}_{\mathbf{m}_{[j,l]}} \langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}}, \tau(\mathbf{E}^{\mathbf{m}_{[l,j]}})\, T^{-1}_{w^{-1}} v_\lambda\rangle \mathbf{F}^{\mathbf{m}_{[l,j]}} \\ &= \widehat{q_{i_j}} \sum_{\mathbf{m}_{[j+1,l]}} \mathbf{p}_{\mathbf{m}_{[j+1,l]}} \langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,j]}}, \tau(\mathbf{E}^{\mathbf{m}_{[l,j+1]}})\, T^{-1}_{w^{-1}} v_\lambda\rangle \mathbf{F}^{\mathbf{m}_{[l,j+1]}} \bmod U^-[\mathbf{w}]_{[j+1,l]}\end{aligned}$$

If $s(j) \leq l$,we apply equation (5.1.16) to the above expression to obtain

$$\begin{aligned}\Delta_j &= \widehat{q_{i_j}} \sum_{\mathbf{m}_{[s(j),l]}} \mathbf{p}_{\mathbf{m}_{[s(j),l]}} \langle f^{\varpi_{i_j}}_{\mathbf{w}_{[1,s(j)-1]}}, \tau(\mathbf{E}^{\mathbf{m}_{[l,sj]}})\, T^{-1}_{w^{-1}} v_\lambda\rangle \mathbf{F}^{\mathbf{m}_{[l,s(j)]}} F_{\beta_j} \bmod U^-[\mathbf{w}]_{[j+1,l]} \\ &= \widehat{q_{i_j}} \Delta_{s(j)} F_{\beta_j} \bmod U^-[\mathbf{w}]_{[j+1,l]}\end{aligned}$$

This proves equation (5.1.8). The proof of equation (5.1.6) is analogous, requiring only a small modification of the last argument. It is left to the reader. □

Starting from a reduced word $\mathbf{i} = i_1, \ldots, i_l$ for $w \in W$, one can construct a presentation of $U^-[w]$ as an iterated Ore extension by adjoining the elements $F_{\beta_1}, \ldots, F_{\beta_l}$ (recall (3.4.2)) in the opposite order. For all $j \in [1, l]$ we have the Ore extension presentation

$$U^-[\mathbf{w}]_{[j,l]} = U^-[\mathbf{w}]_{[j+1,l]}[F_{\beta_j}; \sigma_j^{\leftarrow}, \delta_j^{\leftarrow}], \tag{5.1.17}$$

where $\sigma_j^{\leftarrow}$ and $\delta_j^{\leftarrow}$ are defined as follows. Let t_j' be an element of $\mathbb{H}$ such that

$$(t_j')^{\beta_k} = q^{-\langle \beta_k, \beta_j \rangle}, \text{ for all } k \geq j$$

(cf. (3.4.8) and (3.4.10)). We have that $\sigma_j^{\leftarrow}(x) = (t_j' \cdot x)$ in terms of the restriction of the $\mathbb{H}$-action (3.4.9) to $U^-[\mathbf{w}]_{[j+1,l]}$. The skew derivation $\delta_j^{\leftarrow}$ of $U^-[\mathbf{w}]_{[j+1,l]}$ is defined by

$$\delta_j^{\leftarrow}(x) = F_{\beta_j} x - q^{-\langle \beta_j, \gamma \rangle} x F_{\beta_j}, \text{ where } x \in (U^-[\mathbf{w}]_{[j+1,l]})_\gamma, \text{ and } \gamma \in Q,$$

cf. (3.4.11). The Levendorskiĭ–Soibelman straightening law (3.4.3) implies that $\delta_j^{\leftarrow}$ preserves $U^-[\mathbf{w}]_{[j+1,l]}$, where $\sigma_l^{\leftarrow} = \mathrm{id}$, and $\delta_l^{\leftarrow} = 0$.) Equations (iii) and (3.4.3) imply (5.1.17). Iterating (5.1.17) with the convention $U^-[\mathbf{w}]_{[l+1,l]} = \mathbb{k}$ leads to the presentation

$$U^-[w]^{\leftarrow} = \mathbb{k}[F_{\beta_l}][F_{\beta_{l-1}}; \sigma_{l-1}^{\leftarrow}, \delta_{l-1}^{\leftarrow}] \ldots [F_{\beta_1}; \sigma_1^{\leftarrow}, \delta_1^{\leftarrow}],$$

which is reverse to the presentation (5.1.1). It is straightforward to show $U^-[w]^{\leftarrow}$ is a torsion free CGL extension for the action (3.4.9). Theorem 5.1.6 gives that the quantum minors Δ_j are in $U^-[\mathbf{w}]_{[j,l]}^{\leftarrow}$ and provides a formula for its leading term as a left polynomial with respect to the Ore extension (5.1.17), for all $j \in [1, l]$, cf. Section 4.1.

5.1.2 Proof of Theorem 5.1.4

We keep the notation for $\mathbf{i}$, w, and l from the previous two subsections. For $j \in [1, l]$ consider the chain of extensions

$$\mathbb{k} = U^-[\mathbf{w}]_{[j,j-1]} \subset U^-[\mathbf{w}]_{[j,j]} \subset U^-[\mathbf{w}]_{[j,j+1]} \subset \ldots \subset U^-[\mathbf{w}]_{[j,l]}.$$

It follows from the Levendorskiĭ–Soibelman straightening law (3.4.3) and the definition of the $\mathbb{H}$-action in equation (3.4.9) that the maps δ_k and σ_k from 3.4.4 (ii) preserve the subalgebra $U^-[\mathbf{w}]_{[j,k-1]}$ of $U^-[\mathbf{w}_{[1,k-1]}] = U^-[\mathbf{w}]_{[1,k-1]}$ for all $1 \le j \le k \le l$. For brevity set

$$ {}_j\delta_k = \delta_k|_{U^-[\mathbf{w}]_{[j,k-1]}} \text{ and } {}_j\sigma_k = \sigma_k|_{U^-[\mathbf{w}]_{[j,k-1]}}. $$

In particular, ${}_1\sigma_k = \sigma_k$ and ${}_1\delta_k = \delta_k$. By 3.4.4 (ii) we have the presentation

$$ U^-[\mathbf{w}]_{[j,k]} = U^-[\mathbf{w}]_{[j,k-1]}[F_{\beta_k}; {}_j\sigma_k, {}_j\delta_k], \text{ for } 1 \le j \le k \le l. $$

Using that $U^-[\mathbf{w}]_{[j,j-1]} = \Bbbk$, ${}_j\sigma_j = \mathrm{id}$, and ${}_j\delta_j = 0$, we obtain

$$ U^-[\mathbf{w}]_{[j,l]} = \Bbbk[F_{\beta_j}][F_{\beta_{j+1}}; {}_j\sigma_{j+1}, {}_j\delta_{j+1}] \ldots [F_{\beta_l}; {}_j\sigma_l, {}_j\sigma_l]. \tag{5.1.18} $$

It follows now from 3.4.4 that $U^-[\mathbf{w}]_{[j,k]}$ is a CGL extension. Since $\{0\}$ is an $\mathbb{H}$-prime ideal of $U^-[\mathbf{w}]_{[j,k]}$, we can apply the strong rationality theorem of Goodearl [BG02, Theorem II.6.4] to obtain

$$ Z(\mathrm{Fract}(U^-[\mathbf{w}]_{[j,l]}))^{\mathbb{H}} = \Bbbk. \tag{5.1.19} $$

where $Z(A)$ denotes the center of an algebra A. As in Section 4.1, $\mathrm{Fract}(A)$ denotes the division ring of fractions of a domain A, and $(-)^{\mathbb{H}}$ refers to the fixed point subalgebra with respect to the action of $\mathbb{H}$.

Fixing $\mathbf{i}$ as before, we denote by $\mathbb{T}$ the quantum torus algebra generated by $\overline{F}_1^{\pm 1}, \ldots, \overline{F}_l^{\pm}$. The Q-grading on $U_q^{\ge 0}$ induces a grading on $\mathbb{T}$. Equations (3.4.3) and (4.1.6) imply that

$$ \overline{F}_j \overline{F}_k = q^{\langle \beta_j, \beta_k \rangle} \overline{F}_k \overline{F}_j, \text{ for all } 1 \le j < k \le l. \tag{5.1.20} $$

For $j, k \in [1, l]$ denote by $\mathbb{T}_{[j,k]}$ the quantum subtorus of $\mathbb{T}$ generated by $\overline{F}_i^{\pm 1}$ for $j \le i \le k$.

Using that

$$ \delta_k(F_{\beta_j}) \in U^-[\mathbf{w}]_{[k+1,j-1]}, $$

by a simple induction argument one proves the following lemma:

Lemma 5.1.7. *In the above setting, the following hold for all $j \in [1, l]$:*

(i) $F_{\beta_j} - \overline{F}_j \in \mathbb{T}_{[j+1,l]}$

(ii) The generators for the Cauchon space of $U^-[\mathbf{w}]_{[j,l]}$ (with presentation as in equation (5.1.18)*) are $\overline{F}_j$, ..., $\overline{F}_l$. Therefore,*

$$\overline{U^-[\mathbf{w}]_{[j,l]}} = \overline{U^-[\mathbf{w}]}_{[j,l]}$$

.

The lemma implies that

$$U^-[\mathbf{w}]_{[j,l]} \subset \mathbb{T}_{[j,l]} \subset \mathrm{Fract}(U^-[\mathbf{w}]_{[j,l]}).$$

Therefore, the strong rationality result (5.1.19) gives that

$$Z(\mathbb{T}_{[j,l]})_0 = \Bbbk. \tag{5.1.21}$$

Next, we apply a theorem of Berenstein and Zelevinsky [BZ05, Theorem 10.1], to obtain that for all $1 \leq j < k \leq l$ there exists an integer n_{jk} such that

$$e^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}} e^{\varpi_k}_{\mathbf{w}_{[1,k-1]}} = q^{n_{jk}} e^{\varpi_k}_{\mathbf{w}_{[1,k-1]}} e^{\varpi_{i_j}}_{\mathbf{w}_{[1,j-1]}}.$$

We remark that the setting of [BZ05] is for $\Bbbk = Q(q)$, but the proof of Theorem 10.1 in [BZ05] only uses the R-matrix commutation relations in $R_q[G]$ and the left and right actions of $U_q^{\geq 0}$ on $R_q[G]$ (from (3.2.5)), which are defined for all fields $\Bbbk$ and $q \in \Bbbk^\times$ not a root of unity. Moreover, the R-matrix commutation relations in $R_q[G]$ (see e.g. [BG02, Theorem I.8.15]) imply that for all $\nu \in P$, $f \in V(\lambda)_\nu$, and $\mu, \lambda \in P_{\geq 0}$

$$e^\mu_w c^\lambda_f = q^{-\langle \mu, \lambda + w^{-1}\nu \rangle} c^\lambda_f e^\mu_w \mod Q(w)^+.$$

Using (5.1.4) and the fact that the maps $\phi_w \colon R_w^0 \to U^-[w]$ are antihomomorphisms by 4.2.2 (i), we obtain that there is some $n'_{jk} \in \mathbb{Z}$ for which

$$\Delta_j \Delta_k = q^{n'_{jk}} \Delta_k \Delta_j, \text{ for all } 1 \le j < k \le l. \tag{5.1.22}$$

Proof of 5.1.4. By Lemma 5.1.7 (ii)

$$U^-[\mathbf{w}]_{[j,l]} \subseteq \mathbb{T}_{[j,l]}, \text{ for all } j \in [1,l].$$

Combining this, Theorem 5.1.6, and Lemma 5.1.7 (i), we obtain

$$\Delta_j = \begin{cases} \widehat{q_{i_j}} \Delta_{s(j)} \overline{F}_j \bmod \mathbb{T}_{[j+1,l]} & \text{if } s(j) \le l \quad (5.1.23) \\ \widehat{q_{i_j}} \overline{F}_j \bmod \mathbb{T}_{[j+1,l]} & \text{if } s(j) = \infty \quad (5.1.24) \end{cases}$$

We prove equation (5.1.5) by induction on j, from l to 1. By equation (5.1.24), $\Delta_l - \widehat{q_l} \overline{F}_l \in \Bbbk$. Since Δ_l is a homogeneous element of nonzero degree (equal to β_l), this implies equation (5.1.5) for $j = l$.

Now assume that for some $j \in [1, l-1]$

$$\Delta_k = \widehat{q_k}^{|k|} \overline{F}_{s^{|k|}(k)} \ldots \overline{F}_k \text{ for all } k \in [j+1, l]. \tag{5.1.25}$$

If

$$\Delta_j = \widehat{q_{i_j}}^{|j|} \overline{F}_{s^{|j|}(j)} \ldots \overline{F}_j, \tag{5.1.26}$$

then we are done with the inductive step. Assume the opposite, that (5.1.26) is not satisfied. Combining the inductive hypothesis with (5.1.23) and (5.1.24) (whichever applies for the particular j), we get that

$$\Delta_j - \widehat{q_{i_j}}^{|j|} \overline{F}_{s^{|j|}(j)} \ldots \overline{F}_j \in \mathbb{T}_{[j+1,l]}. \tag{5.1.27}$$

It follows from equations (5.1.20), (5.1.22), and (5.1.25), that

$$\Delta_j \overline{F}_k = q^{m_k} \overline{F}_k \Delta_j, \text{ for all } k = j+1, \ldots, l$$

for some $m_{j+1}, \dots, m_l \in \mathbb{Z}$. Quantum tori have bases consisting of Laurent monomials in their generators. By comparing the coefficients of $\overline{F}_{s^{|j|}(j)} \dots \overline{F}_j \overline{F}_k$ in the two sides of the above equality and using (5.1.27), we get that

$$(\overline{F}_{s^{|j|}(j)} \dots \overline{F}_j)\overline{F}_k = q^{m_k} \overline{F}_k (\overline{F}_{s^{|j|}(j)} \dots \overline{F}_j), \text{ for all } k = j+1, \dots, l$$

for the same collection of integers $m_{j+1}, \dots, m_l$. From the last two equalities it follows that

$$y := (\overline{F}_{s^{|j|}(j)} \dots \overline{F}_j)^{-1} \Delta_j$$

commutes with $\overline{F}_{j+1}, \dots, \overline{F}_l$:

$$y\overline{F}_k = \overline{F}_k y, \text{ for all } k = j+1, \dots, l. \tag{5.1.28}$$

Since (5.1.26) is not satisfied, (5.1.27) implies that

$$y = \widehat{q_{i_j}} + y' \overline{F}_j^{-1} \text{ for some } y' \in \mathbb{T}_{[j+1,l]} \backslash \{0\}. \tag{5.1.29}$$

But y commutes with itself and by (5.1.28) it commutes with $y' \neq 0$. Thus y also commutes with $\overline{F}_j$. Combining this with (5.1.28) leads to the fact that y belongs to the center of $\mathbb{T}_{[j,l]}$. Since Δ_j is a homogeneous element of $U^-[w]$ with respect to its Q-grading, (5.1.27) implies

$$y \in Z(\mathbb{T}_{[j,l]})_0.$$

At the same time $y \notin \Bbbk$ by (5.1.29), which contradicts the strong rationality result (5.1.21). Thus equation (5.1.26) holds. This completes the proofs of the inductive step and the theorem. $\square$

5.2 Unification of the two approaches to $\mathbb{H}$-Spec$U^-[w]$

5.2.1 Solutions of two questions of Cauchon and Mériaux

In this section we establish a dictionary between the representation theoretic and ring theoretic approaches to $\mathbb{H}$-Spec$U^-[w]$, see Section 4.2 and Section 4.1. Theorem 5.2.5 explicitly describes the behavior of all $\mathbb{H}$-prime ideals $I_w(y)$, from Theorem 4.2.2, under the deleting derivations. We also describe the Cauchon diagrams of all ideals $I_w(y)$ and use this to resolve [MC10, Question 5.3.3] of Cauchon and Mériaux ; see Theorem 5.2.1. We use the combination of Theorems 4.2.2 and 5.2.1 to give an independent and elegant proof of the Cauchon and Mériaux classification in our proof of Theorem 5.2.3. Finally, we also settle [MC10, Question 5.3.2] of Cauchon and Mériaux , solving the poset containment problem in the classification of 4.1.6, see 5.2.4.

Recall Section 3.3 and Corollary 3.3.5 for definitions and details of the left positive diagram bijection

$$\mathcal{D}^+ : W^{\leq w} \xrightarrow{\mathcal{L}} \mathcal{L}_{\mathbf{w}} \xrightarrow{\mathcal{D}} \mathcal{D}_l$$

and its counterpart the Cauchon diagram bijection

$$\mathcal{C} : \mathbb{H}\text{-Spec}U^-[w] \xhookrightarrow{\varphi} \mathbb{H}\text{-Spec}\overline{U^-[w]} \xrightarrow{\cong} \mathcal{P}_l \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow \mathcal{D}_l.$$

Also, recall the quantum Schubert cell ideals $S_w^+(y) \in R_w^0$ and the algebra isomorphism $\phi_w : R_w^0/S_w^+(w) \xrightarrow{\cong} U^-[w]$. For a fixed Weyl group element w, set

$$\mathcal{S}_w^- = \left\{ S_w^-(y) + S_w^+(w) \mid y \in W^{\leq w} \right\}.$$

By the Yakimov's classification Theorem, 4.2.2 iii, the map ϕ_w induces a poset isomorphism

$$I_w : W^{\leq w} \xrightarrow{S_w^-} \mathbb{S}_w^- \xrightarrow{\phi_w} \mathbb{H}\text{-Spec}U^-[w]$$

The following main theorem summarizes and relates all known approaches to analyzing the $\mathbb{H}$-prime spectrum of $U^-[w]$:

Theorem 5.2.1 (G., Yakimov)**.** *For any field $\mathbb{k}$ with $q \in \mathbb{k}^\times$ not a root of unity, simple Lie algebra $\mathfrak{g}$, weyl group element w, and reduced word $\mathbf{i}$ for w representing $\mathbf{w}$, we have a* *commutative diagram*

$$\begin{tikzcd}
\mathcal{S}_{\mathbf{w}} \arrow[r, "\mathcal{D}"] & \mathcal{P}_l \arrow[r, two heads, "\sim"] & \mathcal{D}_l & \mathbb{H}\text{-Spec}U^-[w] \arrow[l, "\mathcal{C}"'] \\
\mathcal{L}_{\mathbf{w}} \arrow[u, hook] \arrow[urr, "\mathcal{D}"] & & W^{\leq w} \arrow[ll, "\mathcal{L}"] \arrow[u, "\mathcal{D}^+"'] \arrow[ur, "I_w"'] \arrow[r, "S_w^-"'] & \mathbb{S}_w^- \arrow[u, "\phi_w"']
\end{tikzcd}$$

The left hand square commutes by the analysis conducted in Section 3.3, and commutativity of the right hand square is proved in subsection 5.2.2. This main theorem is the bridge between the representation theoretic and the ring theoretic approach to analyzing $\mathbb{H}$-Spec$U^-[w]$. In particular, the theorem implies

$$\mathcal{C}(I_w(y)) = \mathcal{D}^+(y). \tag{5.2.1}$$

The following is, therefore, an immediate consequence of Theorem 5.2.1. It settles Question 5.3.3 of Cauchon and Mériaux [MC10]; the Cauchon–Mériaux [MC10] and Yakimov [Yak10] classifications of $\mathbb{H}$-Spec$U^-[w]$ of described in 4.1.6 and Theorem 4.2.2 coincide.

Theorem 5.2.2 (G., Yakimov)**.** *For all base fields* $\Bbbk$*,* $q \in \Bbbk^\times$ *not a root of unity, simple Lie algebras* $\mathfrak{g}$*, Weyl group elements* w*, and reduced words* $\mathbf{i}$ *for* w*,*

$$I_w(y) = J_{\mathcal{D}^+(y)} \textit{ for all } y \in W^{\leq w}. \tag{5.2.2}$$

Proof. By Theorem 5.2.1 we have $I_w(y) = \mathcal{C}^{-1}(\mathcal{D}^+(y))$, but $\mathcal{C}^{-1}(\mathcal{D}^+(y)) = J_{\mathcal{D}^+(y)}$ by definition. Therefore, we obtain the result. $\square$

The Cauchon–Mériaux classification [MC10]—a result whose abridged version we restate for convenience—is now a simple corollary of Theorem 5.2.1.

Theorem 5.2.3. *(Cauchon–Mériaux ,* [MC10]*) The* $\mathbb{H}$*-prime ideals of* $U^-[w]$ *are the ideals* $J_{\mathcal{D}^+(y)}$ *for the elements* $y \in W^{\leq w}$*.*

Proof. By an application of Yakimov's classification Theorem 4.2.2 ii and our unification result, Theorem 5.2.2, we have

$$\mathbb{H}\text{-Spec}U^-[w] = \{I_w(y) \mid y \in W^{\leq w}\} = \{J_{\mathcal{D}^+(y)} \mid y \in W^{\leq w}\}.$$

$\square$

Finally, the next theorem answers Question 5.3.2 of Cauchon and Mériaux [MC10].

Theorem 5.2.4 (G., Yakimov)**.** *For all base fields* $\mathbb{k}$*,* $q \in \mathbb{k}^\times$ *not a root of unity, simple Lie algebras* $\mathfrak{g}$*, Weyl group elements* w*, and reduced words* $\mathbf{i}$ *for* w*, the map* $W^{\leq w} \longrightarrow \mathbb{H}\text{-Spec}U^-[w]$

$$y \longmapsto J_{\mathcal{D}^+(y)}$$

is an isomorphism of posets with respect to the Bruhat order and inclusion of ideals.

Proof. By Theorem 4.2.2 iii, the map sending y to $I_w(y)$ is a poset isomorphism. Thus, by Theorem 5.2.2, the result follows. $\square$

We provide a complete description of the behavior of the ideals $I_w(y)$ under the deleting derivation procedure. Recall the definition of leading part $\widetilde{J}$ of an ideal in an Ore extension. According to Proposition 4.1.4, Cauchon's method relies on taking leading parts of ideals or contracting ideals. For the remainder of this section assume that $\mathbf{i} = (i_1, \ldots, i_l)$ is a reduced word for $w \in W$. It follows immediately that

$$\mathbf{w}_{[1,l-1]} = ws_{i_l}, \tag{5.2.3}$$

and Theorem 3.4.4 (i) and (ii) imply

$$U^-[\mathbf{w}_{[1,l-1]}] = U^-[\mathbf{w}_{[1,l-1]}] \subset U^-[w] \text{ and } U^-[w] = U^-[\mathbf{w}_{[1,l-1]}][F_{\beta_l}; \sigma_l, \delta_l]. \tag{5.2.4}$$

Theorem 5.2.5 (G., Yakimov)**.** *Let $\Bbbk$ be an arbitrary base field, $q \in \Bbbk^{\times}$ not a root of unity, $\mathfrak{g}$ a simple Lie algebra, $w \in W$ a Weyl group element, and $\mathbf{i} = (i_1, \ldots, i_l)$ a reduced word for w. Then the following hold for all $y \in W^{\leq w}$ inside $U^-[w] = U^-[\mathbf{w}_{[1,l-1]}][F_{\beta_l}; \sigma_l, \delta_l]$ (from equation (5.2.4)):*

(i) If $l \notin \mathcal{D}^+(y)$, then $\widetilde{I_w(y)} = I_{\mathbf{w}_{[1,l-1]}}(y)$

(ii) If $l \in \mathcal{D}^+(y)$, then $I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}] = I_{\mathbf{w}_{[1,l-1]}}(ys_{i_l})$

We prove Theorem 5.2.1 using 5.2.5 in this subsection, and we establish 5.2.5 in Section 5.2.3–5.2.4. With the goal of proving Theorem 5.2.1, we first establish an auxiliary lemma.

Lemma 5.2.6. *In the setting of Theorem 5.2.5, if y is ssuch that $l \in \mathcal{D}^+(y)$, then*

$$T_{\mathbf{w}_{[1,l-1]}} v_{\varpi_{i_l}} \notin U^- T_y v_{\varpi_{i_l}}. \tag{5.2.5}$$

Proof. We proceed by induction on $\ell(w)$. If $l = 1$, then $T_{ws_{i_1}} v_{\varpi_{i_1}} = v_{\varpi_{i_1}}$ Now, since $1 \in \mathcal{D}^+(y)$ eliminates the possibility that $y = id$. Thus, $y(\varpi_{i_1}) < \varpi_{i_1}$ and the result follows.

Now assume the statement for $\ell(w) = l - 1$, and suppose that (5.2.5) does not hold, that is,

$$T_{\mathbf{w}_{[1,l-1]}} v_{\varpi_{i_l}} \in U^- T_y v_{\varpi_{i_l}}. \tag{5.2.6}$$

There are two possibilities: $1 \in \mathcal{D}^+(y)$ or $1 \notin \mathcal{D}^+(y)$.

Case (A): $1 \in \mathcal{D}^+(y)$. We begin by noting that $\mathbf{i}_{[2,l]} = (i_2, \ldots, i_l)$ is a reduced word for $s_{i_1}w$. By the definition of positivity $\mathcal{D}^+(y)$, we have

$$y = s_{i_1} \mathbf{w}_{[2,l]}^{\mathcal{D}^+(y)} > \mathbf{w}_{[2,l]}^{\mathcal{D}^+(y)} = s_{i_1} y. \tag{5.2.7}$$

Moreover, we have $s_{i_1} y \leq s_{i_1} w$ and $\mathcal{D}^+(s_{i_1} y) = \mathcal{D}^+(y) \backslash \{1\}$. Recall the definition (3.1) of the subalgebras U_q^i of $U_q(\mathfrak{g})$ for $i \in [1, r]$. Equation (5.2.6), (5.2.7) and [Jos95, Lemma 4.4.3

(iii)–(iv)] imply

$$T_{s_{i_1}\mathbf{w}_{[1,l-1]}}v_{\varpi_{i_l}} \in U^{i_1}T_{\mathbf{w}_{[1,l-1]}}v_{\varpi_{i_l}} \subseteq U^{i_1}U^-T_yv_{\varpi_{i_l}} = U^-U^{i_1}T_yv_{\varpi_{i_l}} = U^-T_{s_{i_1}y}v_{\varpi_{i_l}},$$

which contradicts the inductive hypothesis for $s_{i_1}y$, $s_{i_1}w$, and $\mathbf{i}_{[2,l]}$.

Case (B)': $1 \notin \mathcal{D}^+(y)$. The argument in this case is similar to the previous one. From the left positivity of the index set $\mathcal{D}^+(y)$ we have

$$s_{i_1}y = s_{i_1}\mathbf{w}_{[2,l]}^{\mathcal{D}^+(y)} > \mathbf{w}_{[2,l]}^{\mathcal{D}^+(y)} = y. \tag{5.2.8}$$

Furthermore, $y < s_{i_1}w$ and $\mathcal{D}^+_{\mathbf{i}_{[2,l]}}(y) = \mathcal{D}^+(y)$. Equations (5.2.6), (5.2.8) and [Jos95, Lemma 4.4.3 (iii)–(iv)] imply

$$T_{s_{i_1}\mathbf{w}_{[1,l-1]}}v_{\varpi_{i_l}} \in U^{i_1}T_{\mathbf{w}_{[1,l-1]}}v_{\varpi_{i_l}} \subseteq U^{i_1}U^-T_yv_{\varpi_{i_l}} = U^-U^{i_1}T_yv_{\varpi_{i_l}} = U^-T_yv_{\varpi_{i_l}},$$

but this contradicts the inductive hypothesis on y, $s_{i_1}w$, and $\mathbf{i}_{[2,l]}$. □

One can similarly show $T_{\mathbf{w}_{[1,l-1]}}v_\lambda \notin U^-T_yv_\lambda$ for $\lambda \in \sum_{\alpha\in\Pi}\mathbb{Z}_{>0}\varpi_\alpha$, which follows from [Jos95, Lemma 4.4.5] and the simple fact that $y \not\leq \mathbf{w}_{[1,l-1]}$.

5.2.2 Proof of Theorem 5.2.1

Before proceeding, we establish the following lemma to aid our inductive arguments:

Lemma 5.2.7. *Suppose* $\mathbf{y}$ *is a left positive subexpression for* $y \in W^{\leq w}$ *with diagram* D.

(i) If $l \notin D$*, then* D *is left positive diagram for* $y \in W^{\leq ws_{i_l}}$*. Thus,* $ys_{i_l} = \mathbf{w}^D_{[1,l-1]}$.

(ii) If $l \in D$*, then* $D' = D\backslash\{l\}$ *is a left positive diagram for* $ys_{i_l} \in W^{\leq ws_{i_l}}$*. Therefore,* $y = \mathbf{w}^{D'}_{[1,l-1]}$.

Proof. Let $D = D^+(y)$. By assumption $s_{i_j}\mathbf{w}^D_{[j+1,l]} > \mathbf{w}^D_{[j+1,l]}$ for all $j \in [1, l-1]$.

Part i: If $l \notin D$, then we obtain for all $j \in [1, l-2]$ that D satisfies

$$s_{i_j}\mathbf{w}^D_{[j+1,l-1]} = s_{i_j}\mathbf{w}^D_{[j+1,l]} > \mathbf{w}^D_{[j+1,l]} = \mathbf{w}^D_{[j+1,l-1]},$$

Part ii: If $l \in D$, then for all $j \in [1, l-1]$ we have that D' satisfies

$$s_{i_j} \mathbf{w}^{D'}_{[j+1,l-1]} s_{i_l} = s_{i_j} \mathbf{w}^{D}_{[j+1,l]} > \mathbf{w}^{D}_{[j+1,l]} = \mathbf{w}^{D'}_{[j+1,l-1]} s_{i_l}.$$

Upon right multiplication by s_{i_l}, we obtain the result. □

Proof of Theorem 5.2.1. We prove Theorem 5.2.1 by induction on the length $\ell(w)$. The case $\ell(w) = 0$ is trivial. Therefore, we assume the hypothesis for $\ell(w) = l-1$. Note that

$$\mathbf{i}_{[1,l-1]} = (i_1, \ldots, i_{l-1})$$

is a reduced word for the the reduced expression $\mathbf{w}_{[1,l-1]} = w s_{i_l}$. In the setting of Section 4.1, $\overline{x}_l = x_l$. 5.1.3 implies that, for some $p_l \in \Bbbk^{\times}$,

$$F_{\beta_l} = p_l \Delta_l = p_l b^{\varpi_{i_l}}_{\mathbf{w}_{[1,l-1]},w}.$$

Again, we have two cases: (A) $l \notin \mathcal{D}^+(y)$ and (B) $l \in \mathcal{D}^+(y)$.

Case (A): $l \notin D$. It follows from Lemma 5.2.7 (i) that $y = \mathbf{w}^D_{[1,l-1]} \le \mathbf{w}_{[1,l-1]}$ and $\mathcal{D}^+_{\mathbf{i}_{[1,l-1]}}(y) = D$. By the inductive hypothesis, we therefore have

$$\mathcal{C}(I_{\mathbf{w}_{[1,l-1]}}(y)) = D. \tag{5.2.9}$$

Recall from Section 4.2 that $b^{\varpi_{i_l}}_{\mathbf{w}_{[1,l-1]},w} \notin I_w(\mathbf{w}_{[1,l-1]})$; see [Yakb, Theorem 3.1 (b)]. Moreover, $y \le \mathbf{w}_{[1,l-1]}$ implies $I_w(y) \subseteq I_w(\mathbf{w}_{[1,l-1]})$ by 4.2.2 ii. Thus,

$$F_{\beta_l} = p_l b^{\varpi_{i_l}}_{\mathbf{w}_{[1,l-1]},w} \notin I_w(y) \subset I_w(\mathbf{w}_{[1,l-1]}).$$

Now, we can apply Proposition 4.1.4 i to the $J = I_w(y)$. By Theorem 5.2.5 i, $\widetilde{I_w(y)} = I_{\mathbf{w}_{[1,l-1]}}(y)$. Finally, by combining Proposition 4.1.4 i and equation (5.2.9) we obtain

$$\mathcal{C}(I_w(y)) = \mathcal{C}(I_{\mathbf{w}_{[1,l-1]}}(y)) = D.$$

Case (B): $l \in D$. By Lemma 5.2.7 (i), we have $ys_{i_l} = \mathbf{w}^D s_{i_l} = \mathbf{w}^{D'}_{[1,l-1]}$ and hence $D' = \mathcal{D}^+_{\mathbf{i}_{[1,l-1]}}(ys_{i_l})$. The inductive hypothesis, applied to $ys_{i_l} \leq \mathbf{w}_{[1,l-1]}$, implies

$$\mathcal{C}(I_{\mathbf{w}_{[1,l-1]}}(ys_{i_l})) = D'. \tag{5.2.10}$$

Lemma 5.2.6 gives that $T_{\mathbf{w}_{[1,l-1]}} v_{\varpi_{i_l}} \notin U^- T_y v_{\varpi_{i_l}}$; therefore, by definition $f_{\mathbf{w}_{[1,l-1]}, \varpi_{i_l}} \in (U^- T_y v_{\varpi_{i_l}})^{\perp}$, and we have $F_{\beta_l} = p_l b^{\varpi_{i_l}}_{\mathbf{w}_{[1,l-1]}, w} \in I_w(y)$. Now we are in a position to apply Proposition 4.1.4 ii (again) to $J = I_w(y)$. Theorem 5.2.5 ii implies $I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}] = I_{\mathbf{w}_{[1,l-1]}}(ys_{i_l})$. It follows from Proposition 4.1.4 ii and equation (5.2.10) that

$$\mathcal{C}(I_w(y)) = \mathcal{C}(I_{\mathbf{w}_{[1,l-1]}}(ys_{i_l})) \sqcup \{l\} = D' \sqcup \{l\} = D.$$

$\square$

5.2.3 Proof of Theorem 5.2.5 (i)

Let $U^-[w]$ have the prescribed Ore extension presentation

$$U^-[w] = U^-[\mathbf{w}_{[1,l-1]}][F_{\beta_l}; \sigma_l, \delta_l].$$

The following result, important in its own right, will help us prove Part i of the theorem. Theorem 5.2.8 computes the leading term of the *right* polynomial in F_{β_l}

$$\phi_w(c^\lambda_f e^{-\lambda}_w) \in \sum_{m=0}^{N} F^m_{\beta_l} U^-[\mathbf{w}_{[1,l-1]}],$$

where $N = \langle \lambda, \alpha^\vee_{i_l} \rangle$. We remark that this result is strongly analogous to Theorem 5.1.6.

Proposition 5.2.8 (G., Yakimov)**.** *For all base fields $\mathbb{k}$, $q \in \mathbb{k}^\times$ not a root of unity, Weyl group elements $w \in W$, reduced words $\mathbf{i} = (i_1, \ldots, i_l)$ for w, dominant integral weights $\lambda \in P_{\geq 0}$, and $f \in V(\lambda)^*$, we have*

$$\phi_w(c^\lambda_f e^{-\lambda}_w) = \frac{(q^{-1}_{i_l} - q_{i_l})^N}{q^{N(N-1)/2}_{i_l}} F^N_{\beta_l} \phi_{\mathbf{w}_{[1,l-1]}}(c^\lambda_f e^{-\lambda}_{\mathbf{w}_{[1,l-1]}}) + \textit{lower order terms.}$$

Proof of Theorem 5.2.8. Recall (3.1). The vector v_λ is a highest weight vector for U^{i_l} of highest weight $N\varpi_{i_l}$. Lemma 3.1.5 (i) and (ii) (respectively) imply that for all $m > N$

$$E_{i_l}^N T_{i_l}^{-1} v_\lambda = \frac{1}{[N]_{i_l}!} E_{i_l}^N F_{i_l}^N v_\lambda = [N]_{i_l}! v_{i_l}$$

and

$$E_{i_l}^m T_{i_l}^{-1} v_\lambda = 0.$$

Therefore, for all $m > N$

$$\begin{aligned}
(\tau E_{\beta_l})^N T_{w^{-1}}^{-1} v_\lambda &= \left(T_{\mathbf{w}_{[l-1,1]}}^{-1}(E_{i_l}^N)\right)\left(T_{\mathbf{w}_{[l-1,1]}}^{-1} T_{i_l}^{-1} v_\lambda\right) \\
&= T_{\mathbf{w}_{[l-1,1]}}^{-1}\left(E_{i_l}^N T_{i_l}^{-1} v_\lambda\right) \\
&= [N]_{i_l}! T_{\mathbf{w}_{[l-1,1]}}^{-1} v_\lambda
\end{aligned}$$

and similarly

$$(\tau E_{\beta_l})^m T_{w^{-1}}^{-1} v_\lambda = 0.$$

Using the formula (4.2.13) for the antihomomorphism $\phi_w \colon R_0^w \to U^-[w]$, we obtain that for all $\lambda \in P_{\geq 0}$, $f \in V(\lambda)^*$

$$\begin{aligned}
\phi_w(c_f^\lambda e_w^{-\lambda}) &= \frac{(q_{i_l}^{-1} - q_{i_l})^N}{q_{i_l}^{N(N-1)/2}} \sum_{\mathbf{m}_{[1,l-1]} \in \mathbb{N}^{l-1}} \mathbf{p}_{\mathbf{m}_{[1,l-1]}} \langle f, \tau(\mathbf{E}^{\mathbf{m}_{[1,l-1]}}) T_{\mathbf{w}_{[l-1,1]}}^{-1} v_\lambda \rangle F_{\beta_l}^N \mathbf{F}^{\mathbf{m}_{[l-1,1]}} \\
&= \frac{(q_{i_l}^{-1} - q_{i_l})^N}{q_{i_l}^{N(N-1)/2}} F_{\beta_l}^N \phi_{\mathbf{w}_{[1,l-1]}}(c_f^\lambda e_{\mathbf{w}_{[1,l-1]}}^{-\lambda}) \bmod \sum_{m=0}^{N-1} F_{\beta_l}^m U^-[\mathbf{w}_{[1,l-1]}].
\end{aligned}$$

$\square$

We proceed by showing $I_{\mathbf{w}_{[1,l-1]}}(y) \subset \widetilde{I_w(y)}$ by Theorem 5.2.8. We will then will compare the Gelfand–Kirillov dimensions of $U^-[w]/I_w(y)$ and $U^-[\mathbf{w}_{[1,l-1]}]/\widetilde{I_w(y)}$ to obtain the reverse inclusion

Proof of Theorem 5.2.5 (i). In the proof of Theorem 5.2.1 we showed that $l \notin D$ implies $F_{\beta_l} \notin I_w(y)$. Now, $I_w(y)$ is an $\mathbb{H}$-invariant completely prime ideal of $U^-[w]$. Applying Proposition 4.1.4 i to $J = I_w(y)$, we see that $\widetilde{I_w(y)}$ is an $\mathbb{H}$-invariant, completely prime ideal of $U^-[\mathbf{w}_{[1,l-1]}]$. By Theorem 4.2.2 i, there is a $y' \in W^{\leq ws_\alpha}$ such that

$$\widetilde{I_w(y)} = I_{\mathbf{w}_{[1,l-1]}}(y').$$

Proceeding, we let $\lambda \in P_{\geq 0}$ and $f \in (U^- T_y v_\lambda)^\perp \subset V(\lambda)^*$. By definition and by Theorem 5.2.8, respectively, we have

$$\phi_w(c_f^\lambda e_w^{-\lambda}) \in I_w(y) \text{ and } \phi_{\mathbf{w}_{[1,l-1]}}(c_f^\lambda e_{\mathbf{w}_{[1,l-1]}}^{-\lambda}) \in \widetilde{I_w(y)}.$$

Therefore, $I_{\mathbf{w}_{[1,l-1]}}(y) \subset \widetilde{I_w(y)}$, which is true if and only if $y \leq y'$. To prove the opposite containment note that by (iii),

$$\begin{aligned} \operatorname{GKdim}(U^-[w]/I_w(y)) &= \operatorname{GKdim}(U^-[\mathbf{w}_{[1,l-1]}]/\widetilde{I_w(y)}) + 1 \\ &= \operatorname{GKdim}(U^-[\mathbf{w}_{[1,l-1]}]/I_w(y')) + 1. \end{aligned}$$

On the other hand, [Yakb, Theorem 5.8] gives that

$$\operatorname{GKdim}(U^-[w]/I_w(y)) = l - \ell(y) \text{ and } \operatorname{GKdim}(U^-[\mathbf{w}_{[1,l-1]}]/I_w(y')) = l - 1 - \ell(y').$$

Therefore, $\ell(y') = \ell(y)$, but since $y \leq y'$ we must have $y' = y$, which yields the result

$$\widetilde{I_w(y)} = I_{\mathbf{w}_{[1,l-1]}}(y).$$

□

5.2.4 Proof of Theorem 5.2.5 (ii)

A straightforward computation of the contraction $I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}]$ is complicated and impractical. We instead deduce

$$I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}] = I_{\mathbf{w}_{[1,l-1]}}(y') \text{ for some } y' \in W^{\leq \mathbf{w}_{[1,l-1]}}$$

and analyze the homogeneous, nonzero P-normal elements in

$$U^-[w]/I_w(y) \cong U^-[\mathbf{w}_{[1,l-1]}]/(I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}]) \tag{5.2.11}$$

to conclude $y' = ys_{i_l}$.

Observe there is a natural inclusion $P \hookrightarrow \mathbb{H}$ of the weight lattice given by

$$\lambda \longmapsto (q^{\langle \lambda, \alpha_1 \rangle}, \ldots, q^{\langle \lambda, \alpha_r \rangle}).$$

This in turn gives rise to a P-action on $U_q(\mathfrak{g})$, $U^-[w]$, and $U^-[w]/I_w(y)$, given by

$$\lambda \cdot x = q^{\langle \lambda, \gamma \rangle} x \text{ for } x \in (U_q(\mathfrak{g}))_\gamma.$$

Suppose a group G acts on a ring R. An element r of R is called G-*normal* or *equivariantly normal* with respect to G if there exists $g \in G$ such that

$$rx = (g \cdot x)r \text{ for all } x \in R.$$

For all $y \in W^{\leq w}$ and $\lambda \in P$, the elements $b^\lambda_{y,w} \in U^-[w]/I_w(y)$ are nonzero, homogeneous, and P-normal by equation (4.2.11). Alternatively, The next proposition is a result in the opposite direction concerning the possible weights of all homogeneous P-normal elements of $U^-[w]/I_w(y)$.

Proposition 5.2.9 (G., Yakimov)**.** *For all base fields $\Bbbk$, $q \in \Bbbk^\times$ not a root of unity, Weyl group elements $y \in W^{\leq w}$, and nonzero, homogeneous, P-normal elements $u \in U^-[w]/I_w(y)$, there exists $\rho \in (1/2)P$ such that $(w-y)\rho \in Q_{\mathbb{S}(w)}$, $u \in (U^-[w]/I_w(y))_{(w-y)\rho}$, $(w+y)\rho \in P$, and*

$$ux = q^{-\langle (w+y)\rho, \gamma \rangle} xu, \text{ for all } \gamma \in Q \text{ and } x \in (U^-[w]/I_w(y))_\gamma.$$

Proof. Let $u \in (U^-[w]/I_w(y))_\beta$ be a homogeneous P-normal element such that

$$ux = q^{\langle \rho', \gamma \rangle} xu \text{ for all } \gamma \in Q \text{ and } x \in (U^-[w]/I_w(y))_\gamma \tag{5.2.12}$$

for some $\rho' \in P$; here β is an element in the support lattice $Q_{\mathbb{S}(w)}$. Equations (4.2.11) and (5.2.12) imply that for all $\lambda \in P_{\geq 0}$

$$\begin{aligned} b^{\lambda}_{y,w} u &= q^{-\langle (w+y)\lambda, \beta \rangle} u b^{\lambda}_{y,w} \\ &= q^{-\langle (w+y)\lambda, \beta \rangle} q^{\langle \rho', (w-y)\lambda \rangle} b^{\lambda}_{y,w} u. \end{aligned}$$

Now $q \in \mathbb{k}^{\times}$ is not a root of unity and $U^{-}[w]/I_w(y)$ is a domain, so the preceding equality is only possible if

$$-\langle (w+y)\lambda, \beta \rangle + \langle \rho', (w-y)\lambda \rangle = 0 \text{ for all } \lambda \in P_{\geq 0}.$$

Therefore,

$$\langle w\lambda, (wy^{-1}+1)\beta \rangle + \langle w\lambda, (wy^{-1}-1)\rho' \rangle = 0 \text{ for all } \lambda \in P_{\geq 0},$$

that is,

$$(wy^{-1}+1)\beta = (wy^{-1}-1)(-\rho') = 0.$$

By a standard linear algebra argument for Cayley transforms (see[Yaka, Theorem 3.6])), we obtain that there exits $\rho \in Q\Pi$ satisfying

$$\beta = (wy^{-1}-1)y\rho = (w-y)\rho \text{ and } -\rho' = (wy^{-1}+1)y\rho = (w+y)\rho \tag{5.2.13}$$

Thus, we have

$$(w-y)\rho \in Q_{\mathbb{S}(w)}, u \in (U^{-}[w]/I_w(y))_{(w-y)\rho}, \text{ and } (w+y)\rho \in P.$$

Combining the equalities in (5.2.13) and solving for ρ gives

$$\rho = \frac{1}{2} w^{-1}(\beta - \rho').$$

Finally, substituting $-(w+y)\rho$ for ρ in (5.2.12) gives the final claim

$$ux = q^{-\langle (w+y)\rho, \gamma \rangle} xu \text{ for all } \gamma \in Q \text{ and } x \in (U^{-}[w]/I_w(y))_{\gamma}.$$

□

Proof of 5.2.5 (ii). In the proof of Theorem 5.2.1, we saw that $l \in D$ implies $F_{\beta_l} \in I_w(y)$. Recall equation (5.2.3). Since $I_w(y)$ is a $\mathbb{H}$-invariant, completely prime ideal of $U^-[w]$, we have that $I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}]$ is a $\mathbb{H}$-invariant completely prime ideal of $U^-[\mathbf{w}_{[1,l-1]}]$. It follows from Theorem 4.2.2 (i) that there is some $y' \in W^{\leq \mathbf{w}_{[1,l-1]}}$ such that

$$I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}] = I_{\mathbf{w}_{[1,l-1]}}(y').$$

The $\mathbb{H}$-eigenvectors of $U_q(\mathfrak{g})$ are precisely the homogeneous vectors of the Q-grading of $U_q(\mathfrak{g})$. Thus, by Proposition 4.1.5 ii we have a Q-graded algebra isomorphism

$$U' := U^-[\mathbf{w}_{[1,l-1]}]/I_{\mathbf{w}_{[1,l-1]}}(y') \cong U^-[w]/I_w(y).$$

Denote the support of the Q-grading of the above algebras:

$$Q' = \mathbb{Z}\{\gamma \in Q \mid U'_\gamma \neq 0\} \subseteq Q.$$

For $\lambda \in P$ equation (4.2.11) implies that $b^\lambda_{y,w}$ is a nonzero, homogeneous, P-normal element of $U'_{(w-y)\lambda}$ and

$$b^\lambda_{y,w} x = q^{-\langle (w+y)\lambda, \gamma\rangle} x b^\lambda_{y,w} \text{ for all } \gamma \in Q' \text{ and } x \in U'_\gamma. \tag{5.2.14}$$

Applying Proposition 5.2.9 to the algebra $U' = U^-[\mathbf{w}_{[1,l-1]}]/I_{\mathbf{w}_{[1,l-1]}}(y')$ and the homogeneous, P-normal element $b^\lambda_{y,w}$, we have there exists $\mu' \in (1/2)P$ such that $b^\lambda_{y,w} \in U'_{(\mathbf{w}_{[1,l-1]}-y')\mu}$, $(\mathbf{w}_{[1,l-1]} + y')\mu \in P$, and

$$b^\lambda_{y,w} x = q^{-\langle (\mathbf{w}_{[1,l-1]}+y')\mu, \gamma\rangle} x b^\lambda_{y,w} \text{ for all } \gamma \in Q' \text{ and } x \in U'_\gamma. \tag{5.2.15}$$

As before we use the fact that $q \in \mathbb{k}^\times$ is not a root of unity, that U' is a domain, and we combine equations (5.2.14) and (5.2.15) to obtain that for all $\gamma \in Q'$

$$(w-y)\lambda = (\mathbf{w}_{[1,l-1]} - y')\mu \text{ and } \langle (w+y)\lambda, \gamma\rangle = \langle (\mathbf{w}_{[1,l-1]} + y')\mu, \gamma\rangle. \tag{5.2.16}$$

Therefore, for all $\gamma \in Q'$, we have

$$\begin{aligned}
\langle w\lambda, \gamma\rangle &= \langle (w-y)\lambda + (w+y)\lambda, \gamma\rangle \\
&= \langle (\mathbf{w}_{[1,l-1]} - y')\mu + (\mathbf{w}_{[1,l-1]} + y')\mu, \gamma\rangle \\
&= \langle \mathbf{w}_{[1,l-1]}(\mu), \gamma\rangle.
\end{aligned}$$

For all $\nu \in P_{\geq 0}$ Since $0 \neq b^{\nu}_{\mathbf{w}_{[1,l-1]},y'} \in U'_{(\mathbf{w}_{[1,l-1]}-y')\nu}$, we have $(\mathbf{w}_{[1,l-1]} - y')\nu \in Q'$; hence, by the previous string of equalities,

$$\langle \mathbf{w}_{[1,l-1]}(s_{i_l}\lambda - \mu), (\mathbf{w}_{[1,l-1]} - y')\nu\rangle = 0,$$

that is,

$$\langle (y' - \mathbf{w}_{[1,l-1]})(s_{i_l}\lambda - \mu), y'\nu\rangle = 0.$$

Thus, $(y' - \mathbf{w}_{[1,l-1]})\mu = (y' - \mathbf{w}_{[1,l-1]})s_{i_l}\lambda$. The first part of (5.2.16) then gives

$$(w-y)\lambda = (\mathbf{w}_{[1,l-1]} - y')s_{i_l}\lambda.$$

and so $y\lambda = y's_{i_l}(\lambda)$ for all $\lambda \in P_{\geq 0}$. This, however, is only possible if $y' = ys_{i_l}$; hence,

$$I_w(y) \cap U^-[\mathbf{w}_{[1,l-1]}] = I_{\mathbf{w}_{[1,l-1]}}(ys_{i_l}),$$

and our proof is complete. $\square$

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Appendix: Relations in $U^+[w_0]$ for type A_3

Consider the A_3 quantum Schubert cells $U_q^+[w_0]$ for the lexicographically minimal reduced decomposition $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 = s_{121321}$. From this we obtain an order on Δ_+ given by

$$\alpha < \alpha + \beta < \beta < \alpha + \beta + \gamma < \beta + \gamma < \gamma.$$

Recall that a theorem of Levendorskiĭ and Soibelman gives that

$$X_{\beta_j} X_{\beta_i} - q^{-\langle \beta_i, \beta_j \rangle} X_{\beta_j} X_{\beta_j} = \sum \mathbf{c_n} X_{\beta_{i+1}}^{n_{i+1}} \dots X_{\beta_{j-1}}^{n_{j-1}} \text{ for } j > i. \tag{A.17}$$

In general the right hand side of (A.17) is unknown. Abbreviate X_{β_i} to X_i. We investigate a very specific case with further analysis to come in forthcoming publications. Let us note that the relation immediately gives that the root vectors X_i and X_{i+1} commute if $\langle \beta_{i+1}, \beta_i \rangle = 0$ and q-commute otherwise. Due to the prevalence of relations of the form $-XY + q^{-1} YX = Z$, we denote the *q-commutator* by $[X_i, X_j]_q = X_i X_j - q^{-1} X_j X_i$. In particular $[X_i, X_j]_1$ is the usual commutator bracket. Note that $[X, Y]_q$ is related to $[Y, X]_q$ by

$$-[X, Y]_q = [-X, Y]_q = [X, -Y]_q = q^{-1} [Y, X]_{q^{-1}}.$$

Consider the A_3 quantum Schubert cells $U_q^+[w_0]$ for the lexicographically minimal reduced decomposition $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 = s_{121321}$. From this we obtain an order on Δ_+ given by

$$\alpha < \alpha + \beta < \beta < \alpha + \beta + \gamma < \beta + \gamma < \gamma.$$

To this order we associate the generators

$$
\begin{aligned}
E_\alpha &= E_1 \\
E_{\alpha+\beta} &= -E_1E_2 + q^{-1}E_2E_1 \\
&= -[E_1, E_2]_q \\
E_\beta &= E_2 \\
E_{\alpha+\beta+\gamma} &= E_1E_2E_3 - q^{-1}E_2E_3E_1 - q^{-1}E_3E_1E_2 + q^{-2}E_3E_2E_1 \\
&= [E_1, [E_2, E_3]_q]_q = [[E_1, E_2]_q, E_3]_q \\
E_{\beta+\gamma} &= -E_2E_3 + q^{-1}E_3E_2 \\
&= -[E_2, E_3]_q \\
E_\gamma &= E_3
\end{aligned}
$$

Note that we are using the braid group action in [[BG02] Section I.6.7 equation (6)] and not the previously one from (3.1.12). We construct a diagram to encode the following relations among these generators:

$$
\begin{tikzcd}
X \arrow[r, dashed] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr] & & Z \\
& & \mathrm{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, leftrightarrow, blue] & Y
\end{tikzcd}
$$

$$
\begin{array}{ccc}
-[X, Y]_q = 0 & -[X, Y]_q = Z & [X, Y] = 0 \\
YX = qXY & YX = qXY + qZ & YX = XY
\end{array}
$$

$$\begin{tikzcd} X & Y \arrow[l, dashed] \\ Z \arrow[u, dashed] \arrow[ur, leftrightarrow, blue] & W \arrow[l, dashed] \arrow[u, dashed] \end{tikzcd}$$

$$[X, W] = (q - q^{-1})YZ$$

$$XW + q^{-1}YZ = qYZ + WX$$

Here are explicit relations (in positive form) for $U^+[s_{121321}]$:

$$qX_2X_1 = X_1X_2 \qquad qX_3X_2 = X_2X_3 \qquad qX_4X_2 = X_2X_4$$

$$qX_5X_3 = X_3X_5 \qquad qX_5X_4 = X_4X_5 \qquad qX_6X_5 = X_5X_6$$

$$q^{-1}X_3X_1 = X_1X_3 + X_2 \qquad q^{-1}X_6X_3 = X_3X_6 + X_5$$

$$q^{-1}X_5X_1 = X_1X_5 + X_4 \qquad q^{-1}X_6X_2 = X_2X_6 + X_4$$

$$X_2X_5 + q^{-1}X_3X_4 = X_5X_2 + qX_3X_4$$

For any subset D of $\{1, \ldots, 6\}$, we can consider the subalagebra of U^+ genrated by $\{X_i \mid i \in D\}$. Denote this by U^+_D. More details later. If $D = \{1, 2, 4\}$, then U^+_D is a quantum affine space on three generators. If $D = \{1, 2, 3\}$, then U^+_D is a q-deformed version of the Heisenberg Lie algebra. If $D = \{2, 3, 4, 5\}$, then U^+_D is 2×2 quantum matrices. There are obvious relation subdiagrams corresponding to these subalgebras.

We conclude by providing the relation diagrams for all sixteen reduced decompositions of w_0. These diagrams will play a major role in our future results on $U^-[w]$.

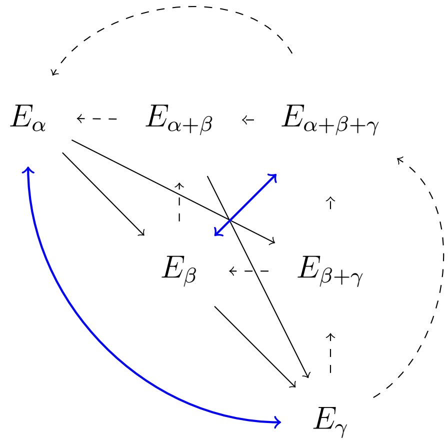

FIGURE A.1. Relation diagram for $\mathbf{i} = 121321$

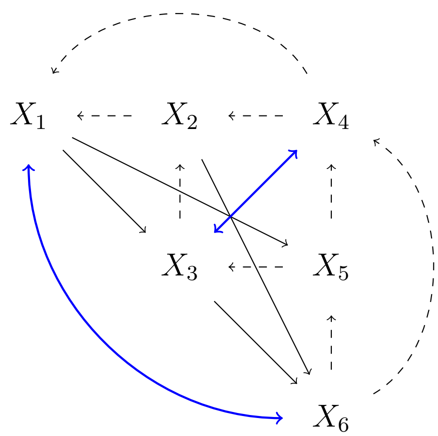

FIGURE A.2. Relation diagram for $\mathbf{i} = 121321$

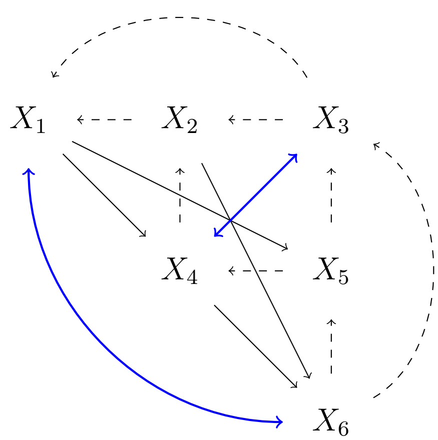

FIGURE A.3. Relation diagram for $\mathbf{i} = 123121$

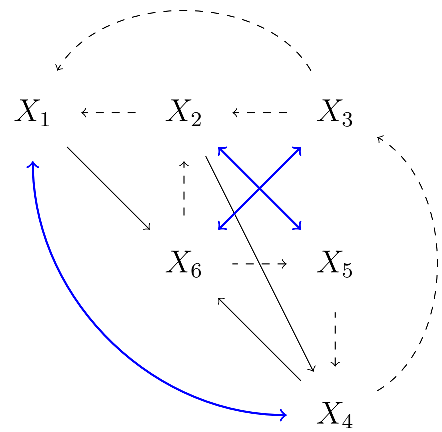

FIGURE A.4. Relation diagram for $\mathbf{i} = 123212$

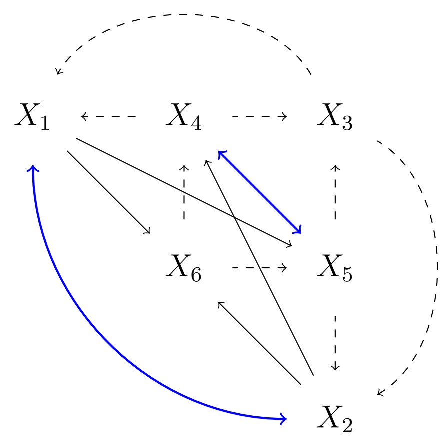

FIGURE A.5. Relation diagram for $\mathbf{i} = 132312$

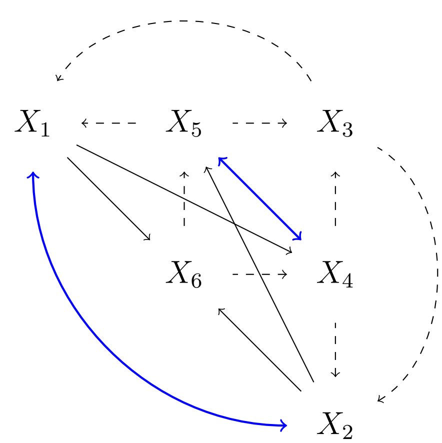

FIGURE A.6. Relation diagram for $\mathbf{i} = 132132$

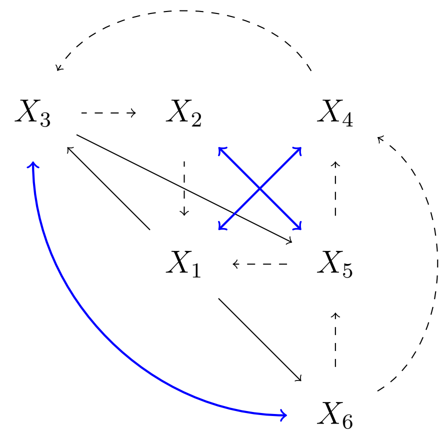

FIGURE A.7. Relation diagram for $\mathbf{i} = 212321$

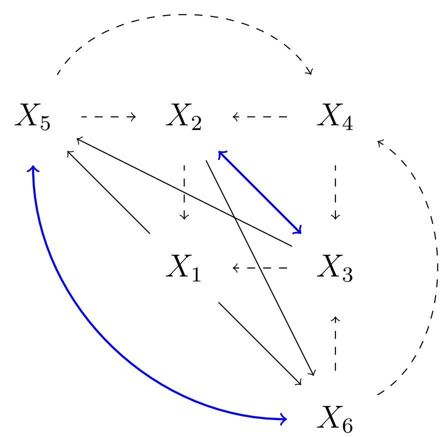

FIGURE A.8. Relation diagram for $\mathbf{i} = 213231$

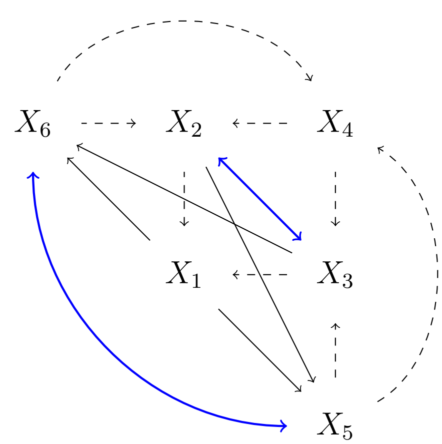

FIGURE A.9. Relation diagram for $\mathbf{i} = 213213$

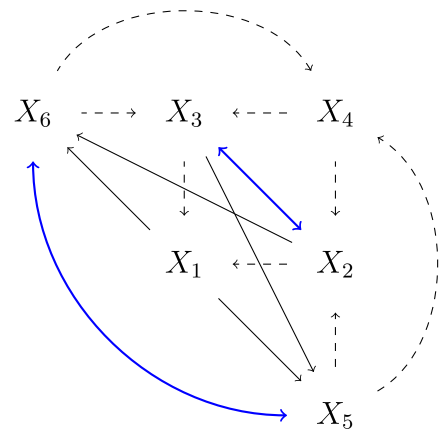

FIGURE A.10. Relation diagram for $\mathbf{i} = 231213$

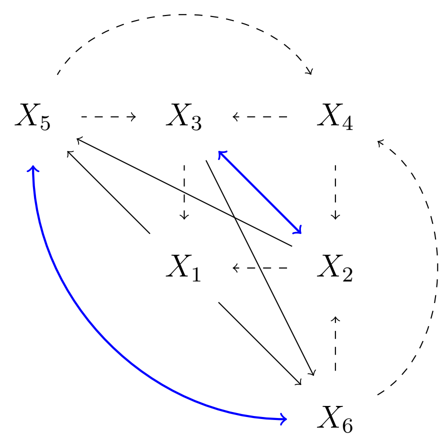

FIGURE A.11. Relation diagram for $\mathbf{i} = 231231$

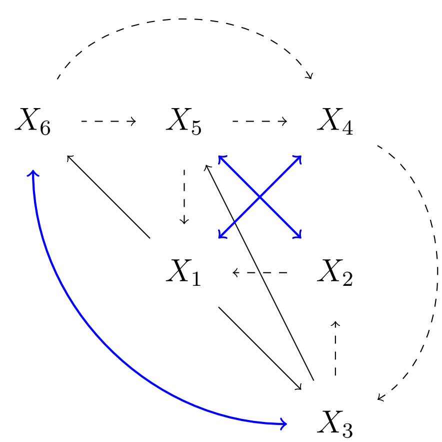

FIGURE A.12. Relation diagram for $\mathbf{i} = 232123$

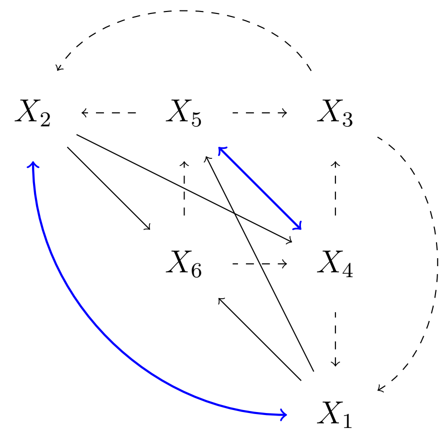

FIGURE A.13. Relation diagram for $\mathbf{i} = 312132$

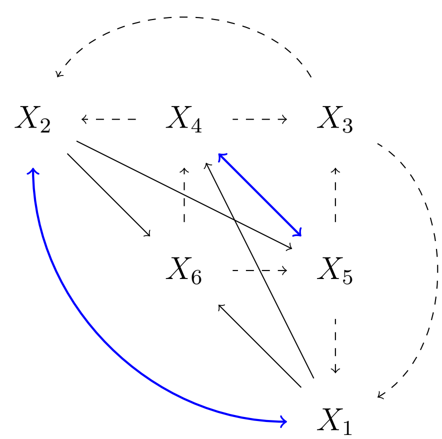

FIGURE A.14. Relation diagram for $\mathbf{i} = 312312$

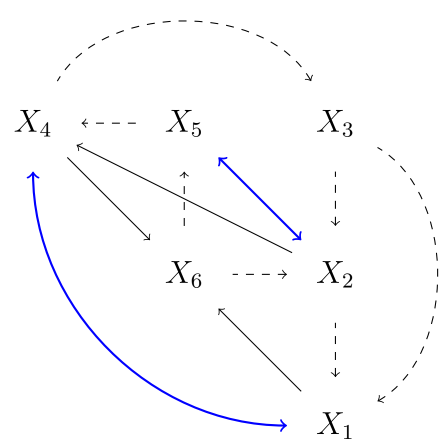

FIGURE A.15. Relation diagram for $\mathbf{i} = 321232$

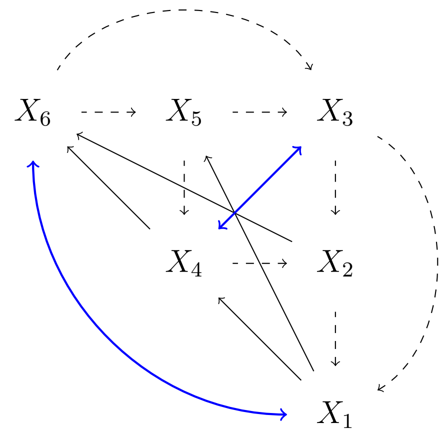

FIGURE A.16. Relation diagram for $\mathbf{i} = 321323$

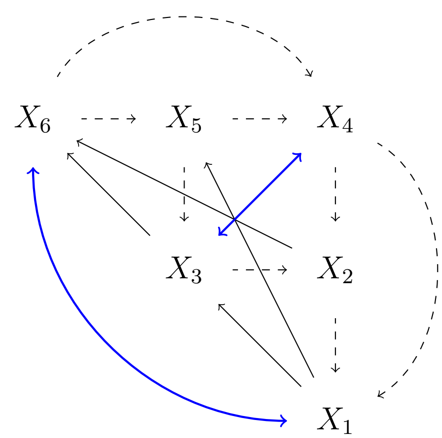

FIGURE A.17. Relation diagram for $\mathbf{i} = 323123$

Vita

Joel Benjamin Geiger was born in Houma, Louisiana where he earned his high school diploma from H.L. Bourgeois High School. In May 2005, he obtained his Bachelor of Science degree in Mathematics at Nicholls State University in Thibodaux, Louisiana. In August 2008 Geiger initiated his graduate studies at Louisiana State University and earned his Master of Science in Mathematics in May 2010. He is currently is a candidate for the degree of Doctor of Philosophy in Mathematics to be awarded in August 2013.