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Analysis and classification of nonlinear dispersive evolution equations in the potential representation

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Abstract

A potential representation for the subset of traveling solutions of nonlinear dispersive evolution equations is introduced. The procedure involves a reduction of a third order partial differential equation to a first order ordinary differential equation. In this representation it can be shown that solitons and solutions with compact support only exist in systems with linear or quadratic dispersion, respectively. In particular, this article deals with so the called $K(n,m)$ equations. It is shown that these equations can be classified according to a simple point transformation. As a result, all equations that allow for soliton solutions join the same equivalence class with the Korteweg-deVries equation being its representative.

1 Introduction

The description of a physical system under extreme conditions, e.g. large amplitude excitations, requires more than a linear theory. However, since dissipation is normally only effective over large time scales, it can be neglected, at least in a first approximation, leaving a nonlinear dispersive partial differential equation for describing the behavior of such a system. In particular, one typically has to deal with third-order as well as first-order spatial derivatives of a wave function u or powers of u, i.e. $(u^m)_{xxx}$, which describe (nonlinear) dispersion, and $(u^n)_x$, which describes nonlinear convection, plus a dynamical term, namely, the time derivative of the wave function, u_t.

A special class of nonlinear dispersive evolution equations in $1+1$ dimensions are the so-called $K(n,m)$ equations, $(u^m)_{xxx} - A(u^n)_x + u_t = 0$ [1]. The most prominent examples of these are the Korteweg-deVries (KdV) equation, $K(2,1)$, and the modified KdV equation, $K(3,1)$. These forms are of particular interest because they can be used to describe the motion of stable and localized solitary waves (solitons) that are observed in a variety of physical systems. These forms are also of particular import for various specialized applications, such as data transfer in fiber [2] or as an analytical tool for an explanation of cluster radioactivity [3] or nuclear fission [4].

The considerations in this article will be carried out for the specific case of traveling wave solutions. It will be shown that this restriction leads to a special representation of nonlinear dispersive evolution equations called the potential representation. (This concept has been used previously, for example, in [5, 6]). This representation resembles an energy conservation law with a nonrelativistic kinetic energy term as well as a potential energy. The potential representation, being a first order ordinary differential equation, constitutes an enormous simplification of the original problem. Gross properties of the solutions can be read directly from the potential function without the actual need to solve a differential equation. The conditions for solitary waves and solitons can thus be easily stated qualitatively. Moreover, the investigation of the potential picture directly reveals that solitons can only emerge in systems with linear dispersion. Compactons, i.e. solitary waves with compact support, only exist in systems with quadratic dispersion. These results are discussed in more detail for the specific cases of systems that are modeled by $K(n,m)$ equations.

By restricting our consideration to a particular subset of solutions, a $K(n,m)$ equation can be transformed into another $K(N,M)$ equation by means of a simple point transformation. This point transformation defines an equivalence relation between the various $K(n,m)$ equations and in so doing divides these equations into equivalence classes of connected equations.

2 Potential Representation

2.1 Reformulation of the nonlinear evolution equation

We consider a general nonlinear dispersive evolution equation of an autonomous one-dimensional non-dissipative dynamical system

$$(u^m)_{xxx} = V(u)u_x - u_t \quad , \tag{1}$$

where $V(u)$ is an arbitrary integrable and continuous function of u. The subscripts denote partial differentiation with respect to the index. We focus on traveling solutions with the space-time dependence $u(x,t) = u(x-vt) = u(\xi)$, with v denoting the speed, so that the partial differential equation (1) is reduced to the ordinary differential equation

$$(u^m)_{\xi\xi\xi} = V(u)u_\xi + vu_\xi = \mathcal{V}(u)u_\xi \quad , \tag{2}$$

Equation (2) can be integrated once

$$(u^m)_{\xi\xi} = \int_0^u \mathrm{d}t\mathcal{V}(t) - C_1 \quad , \tag{3}$$

and by using an integrating factor $(u^m)_\xi$ (see Appendix A) eq. (3) turns into

$$\frac{m^2}{2}\left[u^{2m-2}\left(u_\xi\right)^2\right]_\xi = \left[\int_0^{u(\xi)} \mathrm{d}t\; m\; t^{m-1}\int_0^t \mathrm{d}s\mathcal{V}(s)\right]_\xi - C_1\left(u^m\right)_\xi \quad , \tag{4}$$

allowing a further integration and leading finally to

$$\left(u_\xi\right)^2 = -\mathcal{F}(u) \quad , \tag{5}$$

where $\mathcal{F}(u)$ is given by

$$\mathcal{F}(u) = C_1 u^{2-m} + C_2 u^{2-2m} - \frac{2}{m^2}u^{3-m}\int_0^1 \mathrm{d}x(1-x^m)\mathcal{V}(ux) \quad . \tag{6}$$

In the following, eq. (5) is referred to as the *potential representation* of eq. (1). This notation is inspired by the fact that if ξ and u are identified with *time* and *space*, respectively, the l.h.s. of eq. (5) may be associated with a nonrelativistic 'kinetic' energy, and, accordingly, the r.h.s. with the negative value of a 'potential' energy $\mathcal{F}(u)$. u_ξ is then the 'velocity' of a particle moving along the u-axis. Only nonlinear evolution equations of the form (1) can be reduced to this potential form. This integrability property is related to the existence of an 'energy' conservation law. The evolution of u proceeds on the zero-energy hypersurface in the phase space belonging to $\mathcal{F}(u)$. $\mathcal{V}(u)$ can now be expressed as

$$\mathcal{V}(u) = -m(m-1)(m-2)u^{m-3}\mathcal{F}(u) - \frac{3}{2}m(m-1)\mathcal{F}'(u) - \frac{1}{2}mu^{m-1}\mathcal{F}''(u) \quad , \tag{7}$$

where the prime indicates differentiation with respect to the argument.

The problem of integrating eq. (1) is thus reduced to a quadrature that can be solved by separation of variables yielding ξ as a function of u. One is now left with the challenging task of inverting this function, which in many cases may not be possible.

For later purposes we briefly mention that the potential function for evolution equations of the form

$$\left((u+\gamma)^m\right)_{xxx} = \mathcal{V}(u)u_x \tag{8}$$

with an arbitrary constant γ is given by

$$\mathcal{F}(u) = C_1(u+\gamma)^{2-m} + C_2(u+\gamma)^{2-2m} - \frac{2}{m^2}(u+\gamma)^{3-m}\int_0^1 \mathrm{d}x(1-x^m)\mathcal{V}(ux-(1-x)\gamma) \quad . \tag{9}$$

2.2 Analysis of the potential representation

Casting the original nonlinear dispersive evolution equation in the potential representation associates to the wave function $u(\xi)$ a space-time (i.e. $u-\xi$) trajectory of a particle moving in the potential $\mathcal{F}(u)$ with zero total energy. Different types of solutions of the original nonlinear evolution equation can be attributed to different kinds of trajectories in this phase space. For example, closed trajectories in bounded regions of the phase space correspond to periodic solutions, whereas solitary wave solutions are represented in phase space by separatrix trajectories [7]. In the following, the properties and conditions of solitary waves and of solitary waves with compact supportare discussed in more detail. However, the so-called *kinks*, solitary waves which are represented in phase space by separatrix trajectories with two cusps, are not considered.

Solitons

In the potential representation, necessary and sufficient conditions for solitary waves read:

$$\begin{aligned} \mathcal{F}(a) = \mathcal{F}'(a) = 0, \quad \mathcal{F}''(a) \leq 0, \\ \mathcal{F}(b) = 0, \quad \mathcal{F}'(b) \neq 0, \qquad a < b \quad . \end{aligned} \tag{10}$$

That is, the potential function must have at least a two-fold zero at a point $u = a$ with negative or vanishing curvature and must have a zero at $b > a$ with nonvanishing slope. In addition, $\mathcal{F}(u)$ must not have singularities in the interval $[a, b]$, and therefore $\mathcal{F}'(b)$ is greater than 0. The reasons underlying (10) are as follows: A particle at $u = a$ is at rest (the potential energy is equal to the total energy) and does not experience any force (the gradient of the potential is zero). It takes the particle infinitely long to leave this point. It moves in a positive u direction, is reflected at $u = b$, moves back in a negative u direction and reaches a again after another infinite time span, i.e. $\lim_{|\xi|\to\infty} u(\xi) \to a$, $u(0) = b$. The simple zero of the potential function, b, thus corresponds to the amplitude of the solitary wave.

Localized solitary waves (solitons) require $a = 0$. For dark solitary waves, i.e. solitary waves with negative amplitude, $\mathcal{F}(u)$ has to fulfill (10) with $b < a$ (the expression *dark soliton* has been introduced in the context of nonlinear optical pulse propagation [**?**]). The above conditions for solitons lead to $C_1 = C_2 = 0$ in (6). One also finds in general as a necessary premise for soliton solutions resulting from equations of the form (1) that $m = 1$ which is a consequence of the dynamical term u_t.

Focusing on $K(n, m)$-type equations with

$$\mathcal{V}(u) = nAu^{n-1} + v \tag{11}$$

one finds for the potential function

$$\mathcal{F}(u) = -\frac{2A}{m(m+n)}u^{n-m+2} - \frac{2v}{m(m+1)}u^{3-m} + C_1 u^{2-m} + C_2 u^{2-2m} \quad . \tag{12}$$

The application of the conditions dicussed above for solitons explicitely to this potential function yield besides $C_1 = C_2 = 0$

$$\begin{aligned} v &> 0,\ \ A < 0 \\ b &= \left(\frac{v(n+1)}{2|A|}\right)^{\frac{1}{n-1}} \end{aligned} \tag{13}$$

That is, the soliton moves in a positive x direction ($v > 0$) and the parameter A has to be smaller than zero. The amplitude of the soliton is proportional to the $(n-1)$-th root of the velocity. For dark solitons, i.e. antisolitons, one finds instead $A > 0$ for n even and $A < 0$.

Finally one can deduce basic properties of the width L of the soliton. The width is calculated at a certain height of the soliton, e.g. at half the maximum height

$$L = 2\int_{b/2}^{b} \frac{\mathrm{d}u}{\sqrt{-\mathcal{F}(u)}} = 2\int_{b/2}^{b} \frac{\mathrm{d}u}{\sqrt{\frac{2A}{n+1}u^{n+1} + vu^2}} = \frac{2}{\sqrt{v}}\int_{1/2}^{1} \frac{\mathrm{d}u}{\sqrt{u^{n+1} + u^2}} \tag{14}$$

The last step follows with $2A/(v(n+1)) = b^{1-n}$. One thus finds that

$$L \sim \frac{1}{\sqrt{v}} = \sqrt{\frac{n+1}{2|A|b^{n-1}}} \tag{15}$$

for any n.

The soliton solutions of $K(n,1)$ equations have the form (see section 3.2)

$$u(\xi) = (\pm)^n \left(\frac{(n+1)v}{2A}\right)^{\frac{1}{n-1}} \frac{1}{\cosh^{\frac{2}{n-1}}\left(\frac{(n-1)\sqrt{v}}{2}\xi\right)} \tag{16}$$

Compact Support Solutions

In this section we will discuss localized solitary waves with compact support, i.e.

$$u(\xi) \begin{cases} \neq & 0 \text{ if } \xi \in [\xi_1, \xi_2] \\ = & 0 \text{ else} \end{cases} \tag{17}$$

which here shall be called *compact support solutions* (CSS). These solutions were introduced as *compactons* in [1]. $u(\xi)$ being a solution of (1) requires $u^m(\xi)$ to be of class C_3 and $u(\xi)$ to be at least of class C_1 at the boundary of the compact ξ interval. Since unlike the soliton the CSS does not approach zero asymptotically but reaches it in a finite time ξ, the gradient of the potential function must never vanish in the interval $0 \leq u \leq b$ (b denotes the amplitude of the CSS), or, equivalently, $u_{\xi\xi} \neq 0$ for all $\xi \in [\xi_1, \xi_2]$. This leads to

$$\begin{aligned} \mathcal{F}(0) &= 0,\ \mathcal{F}'(0) < 0 \\ \mathcal{F}(b) &= 0,\ \mathcal{F}'(b) > 0 \end{aligned} \tag{18}$$

From the requirements for the wavefunction and its derivatives at the boundary of the finite ξ interval one obtains that either $m = 2$ or $m \geq 3$. However, with (18) one finds $m = 2$ and $C_1 = C_2 = 0$. The conditions for compactons can finally be stated as follows

$$\begin{aligned} &m = 2 \\ &v > 0,\ A < 0 \\ &b = \left(\frac{v(n+2)}{3|A|}\right)^{\frac{1}{n-1}} \end{aligned} \tag{19}$$

Similar to the soliton case, the amplitude is proportional to the $(n-1)$-th root of the velocity. CSS with negative amplitude require both $v < 0$, i.e. they are moving in negative x-direction, and either $A > 0$ for n odd or $A < 0$.

In the same way as for solitons one finds for the widths of a CSS the following relation

$$L \sim \sqrt{\frac{b}{v}} = \sqrt{\frac{1}{3}}\left(\frac{v}{3}\right)^{\frac{2-n}{2(n-1)}}\left(\frac{|A|}{(n+2)}\right)^{\frac{-1}{2(n-1)}} \tag{20}$$

For $n = 2$ one thus finds the known result, that the width of the CSS is independent of its speed or its height [1].

As examples for CSS we give the solutions of the $K(2,2)$ equation

$$u(\xi) = \begin{cases} \dfrac{4v}{3A}\cos^2\left(\dfrac{\sqrt{A}}{4}\xi\right) & \text{for} \quad |\sqrt{A}\xi/4| \leq \pi \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

and of the $K(3,2)$ equation

$$u(\xi) = \begin{cases} \sqrt{\dfrac{5v}{3|A|}}\mathrm{sn}^2\left(\left(\dfrac{v|A|}{240}\right)^{\frac{1}{4}}\xi\middle| -1\right) & \text{for} \quad 0 \leq \xi \leq \left(\dfrac{240}{v|A|}\right)^{\frac{1}{4}}\dfrac{2}{\sqrt{2}}K\left(\dfrac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

both clearly showing properties (19) and (20) found from the general analysis of the potential function. In the latter solution, $\mathrm{sn}(x|m)$ denotes the Jacobian elliptic function $\mathrm{sn}(x|m) = \sin(\mathrm{am}\, u)$ with the Jacobi amplitude $\mathrm{am}\, u$ and $u = u(x,m) = \int_0^x \mathrm{d}\theta(1 - m\sin^2\theta)^{-1/2}$. $K(m)$ denotes the quarter period $u(\pi/2, m)$ [9].

Particularly for the $K(2,2)$ case, we want to mention that a solution (21) added to a constant δ is again a solution of a $K(2,2)$ equation. If $u(\xi)$ solves the $K(2,2)$ equation $(u^2)_{\xi\xi\xi} = A(u^2)_\xi + vu_\xi$ with the potential function (see eq. (12))

$$\mathcal{F}(u) = -\frac{A}{4}u^2 - \frac{v}{3}u$$

then $U(\xi) = u(\xi) + \delta$ obeys, according to (9), the following potential representation

$$(U_\xi)^2 = \frac{A}{4}U^2 + (\frac{v}{3} + \frac{A}{2}\delta)U - \frac{A}{4}\delta^2 - \frac{v}{3}\delta \tag{23}$$

for the transformed equation $((U-\delta)^2)_{\xi\xi\xi} = A(U^2)_\xi + (v - 2A\delta)U_\xi$ with $C_1 = C_2 = 0$. Eq. (23) may, however, as well be interpreted as the potential representation of a $K(2,2)$ equation with $C_1 = -A\delta^2/4 - v\delta/3$. The solution $U(\xi)$ moves with the velocity $V = v + 3/2|A|\delta = 3/4|A|(b + 2\delta)$, v being the velocity of $u(\xi)$ [10].

The above results show that in nonlinear dispersive, non-dissipative dynamical systems soliton solutions may only occur if the dispersion is linear ($m = 1$) whereas compact support solutions require a quadratic dispersion ($m = 2$). This is summarized in Fig. 1 showing a chart of $K(n,m)$ equations.

3 Classification of $K(n,m)$-equations by a point transformation

3.1 General considerations

In this section it will be shown that $K(n,m)$-type equations can be transformed into other $K(N,M)$-type equations with different arguments N and M. The applied (point) transformation uniquely connects the elements of certain sets of $K(n,m)$ equations. It constitutes an equivalence relation between those connected equations and thus serves as a tool to classify nonlinear dispersive evolution equations of the $K(n,m)$ type. The benefit of this classification is clearly the fact that solutions to any element (equation) of an equivalence class can be traced back to the solution of the representative of this class. However, this transformation requires to restrict the considerations to a certain subset of traveling solutions which is defined by fixing the initially arbitrary integration constants in the potential representation $C_1 = C_2 = 0$. In the case of $K(n,1)$ or $K(n,2)$ equations this is the necessary condition for solitons and compact support solutions.

Transformations of the potential representation, being a first order ODE, are fully covered by the theory of point transformations [11]. Here we choose the transformation $u(\xi) = \omega^{\pm q}(\xi)$ which belongs to a certain class of point transformations that transport the potential representation of a nonlinear dispersive differential equation of type $K(n,m)$ into the potential representation of another $K(N,M)$ equation. The general derivation of admissible transformations is carried out in Appendix B. The transformation presented above has been chosen for further considerations because it allows for an entirely analytical treatment.

This simple point transformation suggests diagram (Fig. **??**) in which the $K(n,m)$ and the $K(N,M)$ equation are connected via the respective potential representations. The subscript 0 of the potential function indicates the special choice $C_1 = C_2 = 0$. Thus, if $u(\xi)$ is a solution of the $K(n,m)$ equation, $\omega(\xi)$ is a solution of the $K(N,M)$ equation.

The integers M and N fulfill the relations

$$\left.\begin{array}{rcl} M &=& q(m-1)+1 \\ N &=& q(n-1)+1 \end{array}\right\} \qquad \text{for } \; u \to \omega^{+q} \tag{24}$$

$$\left.\begin{array}{rcl} M &=& q(n-m)+1 \\ N &=& q(n-1)+1 \end{array}\right\} \qquad \text{for } \; u \to \omega^{-q} \tag{25}$$

For M, N, m and n being integers, q can take on any rational number $q = (N-1)/(n-1)$ but in fact, the diagram remains valid for any value of q. Note, however, that although in this way the $K(n,m)$ equations may be transformed into equations with arbitrary, infinitesimal nonlinearities $n,\ m = 1+\varepsilon$, the resulting equations do not represent analytic continuations of the linear cases and the respective solutions are not smoothly connected.

The $K(n,m)$ equations with $m = (n+1)/2$ are, as a peculiarity of the transformation, again transformed into equations with $M = (N+1)/2$ by both (**??**) and (24).

The equation $K(n,m)$ can be directly transformed into $K(N,M)$ by using the potential representation of the $K(N,M)$ equation as a consistency relation. The diagram (Fig. 2) can thus be closed (see Fig. 3). To see this one calculates explicitely for the case (**??**) with $u = \omega^{+q}$

$$\begin{array}{rcl} (u^m)_{\xi\xi\xi} &=& A\,(u^n)_\xi + v u_\xi \\ (\omega^{mq})_{\xi\xi\xi} &=& \left(\omega^{q-1+M}\right)_{\xi\xi\xi} = A\,(\omega^{nq})_\xi + v\,(\omega^q)_\xi \end{array} \tag{26}$$

Expanding $(\omega^{q-1+M})_{\xi\xi\xi}$ eq. (26) can be rearranged

$$\begin{array}{rcl} \left(\omega^M\right)_{\xi\xi\xi} &=& -M(q-1)\,(q+2M-4)\,\omega^{M-3}\,(\omega_\xi)^2\,\omega_\xi - \dfrac{3}{2}M(q-1)\omega^{M-2}\left[(\omega_\xi)^2\right]_\xi \\ && +\dfrac{AM}{\omega^{q-1}mq}\,(\omega^{nq})_\xi + \dfrac{vM}{\omega^{q-1}mq}\,(\omega^q)_\xi \end{array} \tag{27}$$

$$\stackrel{!}{=} \; A'\left(\omega^N\right)_\xi + v'\omega_\xi \quad . \tag{28}$$

>From the requirement that (27) equals a $K(N,M)$ equation with coefficients A' and v' (**??**) we infer also the existence of a potential representation

$$(\omega_\xi)^2 = -\mathcal{F}(\omega) = \frac{2A'}{M(M+N)}\omega^{N-M+2} + \frac{2v'}{M(+1)}\omega^{3-M} \;, \quad \left[(\omega_\xi)^2\right]_\xi = -\mathcal{F}'(\omega)\omega_\xi \;. \tag{29}$$

Here we have made the special choice $C_1 = C_2 = 0$ which is a necessary condition for the general case (see also appendix B). Inserting this condition for the squared derivative of the transformed wavefunction ω into (27) one gets

$$\begin{aligned}\left(\omega^M\right)_{\xi\xi\xi} &= \frac{MA\,n}{m}\omega^{N-1}\omega_\xi + \frac{Mv}{m}\omega_\xi \\ &-(q-1)\,(q+2M-4)\left(\frac{2A'}{(M+N)}\omega^{N-1} + \frac{2v'}{(M+1)}\right)\omega_\xi \\ &-3(q-1)\left(\frac{A'(N-M+2)}{(M+N)}\omega^{N-1} + \frac{v'(3-M)}{(M+1)}\right)\omega_\xi \quad . \end{aligned} \tag{30}$$

Comparing the last equation with (27) yields the conditions for the new coefficients

$$\frac{Mn}{m}A - \frac{2(q-1)\,(q+2M-4)}{(M+N)}A' - \frac{3(q-1)(N-M+2)}{(M+N)}A' = NA' \tag{31}$$

$$\frac{M}{m}v - \frac{2(q-1)\,(q+2M-4)}{(M+1)}v' - \frac{3(q-1)(3-M)}{(M+1)}v' = v' \tag{32}$$

leading to

$$A' = \frac{AM(M+N)}{q^2m(m+n)} \tag{33}$$

$$v' = \frac{vM(M+1)}{q^2m(m+1)} \tag{34}$$

Similar one obtains for the case (24) a $K(N,M)$ equation with coefficients

$$A' = \frac{vM(M+N)}{q^2m(m+1)} \tag{35}$$

$$v' = \frac{AM(M+1)}{q^2m(m+n)} \tag{36}$$

One finds here that the transformation (24) interchanges the nonlinear convection term and the dynamical term. The transformation (24) is thus a purely mathematical relation between the ODEs considered with any physical implication removed.

To illustrate this result, a chart of the $K(n,m)$ equations shown in Figure **??** gives three sets of $K(n,m)$ equations connetcted by the point transformation discussed, i.e. three different equivalent classes. In fact, under the assumption of certain kinds of traveling solutions, the latter being specified by a particular choice of integration constants in the potential representation, any $K(n,m)$ equation belongs to a certain equivalence class whose representant is characterized by m and n with $(m-1)$ and $(n-1)$ having no common divisors and $n \geq 2m-1$. The latter restriction is based on the fact that any $K(n,m)$ equation with $n < 2m-1$ is connected to a $K(n,m)$ equation with $n > 2m-1$ through a transformation (24).

Finally we want to mention a peculiarity arising for $m=1,\ q=2$. In this case the first term on the r.h.s. in eq. (27) does not contribute and only the derivative of the potential function $\mathcal{F}(\omega)$ enters the transformation of the $K(n,1)$ equation. This allows the potential function $\mathcal{F}(\omega)$ to contain an arbitrary constant, corresponding to the term $C_2\omega^{2-2m} = C_2$ in (12), and according, the transformation $u = \omega^2$, $\mathcal{F}(u)$ may additionally contain the term $4C_2u^{2-m} = 4C_2u$. Here $\mathcal{F}(u)$ belongs to a $K(n,1)$ equation with the parameters A and v and, accordingly, $\mathcal{F}(\omega)$ is the potential function of a $K(2n-1,1)$ equation with the parameters $A' = n/(2(n+1))A$ and $v' = v/4$. In this way, pairs of $K(n,1)$ equations become connected. In [6] this property has been adresses especially for the pair $K(3,1) - K(5,1)$, see Fig. 4

3.2 Example: The equivalence class of the KdV equation

The KdV equation reads

$$u_{\xi\xi\xi} = A\left(u^2\right)_\xi + vu_\xi \quad .$$

According to section 2.2, we choose $A < 0$ to have soliton solutions. A soliton solution has the form

$$u(\xi) = \frac{3v}{2|A|}\frac{1}{\cosh^2\left(\frac{\sqrt{v}}{2}\xi\right)} \quad .$$

Using the results from the precious section one can immediately state solutions to any $K(N,1)$ and $K(N,N)$ equation. On the one hand one finds that with the transformation $u = w^{+q}$ solutions to $K(N,1) = K(q+1,1)$ equations read (the parameters v and A have been expressed by the transformed parameters v' and A')

$$w(\xi) = (\pm)^N \left(\frac{(N+1)v'}{2A'}\right)^{\frac{1}{N-1}} \frac{1}{\cosh^{\frac{2}{N-1}}\left(\frac{(N-1)\sqrt{v'}}{2}\xi\right)} \quad . \tag{37}$$

As indicated by the resulting factor $(\pm)^N$ on the right hand side, one finds immediately that the $K(N,1)$ equations with a symmetric potential function, i.e. with odd N, have both soliton and anti-soliton solutions. On the other hand the solutions for the resulting $K(N,N) = K(q+1,q+1)$ equations of the transformation $u = w^{-q}$ read

$$w(\xi) = (\pm)^N \left(\frac{2v'N}{(N+1)A'}\right)^{\frac{1}{N-1}} \cosh^{\frac{2}{N-1}}\left(\frac{\sqrt{A'}(N-1)}{2N}\xi\right) \quad . \tag{38}$$

To compare this result with the literature, e.g. [12], one needs to recall that the potential function belonging to the transformed $K(N,N)$ equation is just the negative of the usual potential function of the $K(n,n)$ equations used in the literature. Changing the signs accordingly results in the change $\xi \to \mathrm{i}\xi$, i.e. the hyperbolic cosine is changed to a trigonometric cosine. Eq. (38) then coincides with the result of [12]. For $m = n = 2$

the solution (38) can be compactified. With the appropriate changes it represents the CSS (compare section 2.2)

$$w(\xi) = \begin{cases} \dfrac{4v'}{3A'} \cos^2 \left(\dfrac{\sqrt{A'}}{4} \xi \right) & \text{for} \quad |\sqrt{A'}\xi/4| \leq \pi \\ 0 & \text{else} \end{cases} \quad . \tag{39}$$

Finally, a certain nonlinear dispersive evolution equation of non-$K(n,m)$ type will be investigated that nevertheless can be considered as an element of the KdV equivalence class. The equation under consideration reads

$$\beta \left(u^2\right)_{\xi\xi\xi} + \varepsilon u_{\xi\xi\xi} = -\alpha \left(u^2\right)_{\xi} - u_t \quad . \tag{40}$$

With $\beta(u^2)_{\xi\xi\xi} + \varepsilon u_{\xi\xi\xi} = \beta((u+\varepsilon/(2\beta))^2)_{\xi\xi\xi}$ and $u_t = -vu_\xi$, the corresponding potential representation is found using (9). Basically we have dealt with this problem already in section 2.2. Thus we may immediately assume the solution to have the form of a CSS (39) added to a constant. In particular, the potential representation reads

$$\left(u_\xi\right)^2 = -\frac{\alpha}{4\beta} u^2 + \left(-\frac{\alpha\varepsilon}{4\beta^2} + \frac{(\alpha\varepsilon + v\beta)}{3\beta^2} \right) u - C_1 - \frac{\alpha\varepsilon^2}{16\beta^3} + \frac{\varepsilon(\alpha\varepsilon + v\beta)}{6\beta^3} \quad . \tag{41}$$

C_2 is set equal to zero to avoid singularities of the potential function. Setting

$$U(\xi) = u(\xi) + \frac{\varepsilon}{2\beta} - \frac{2\left(\alpha\varepsilon + v\beta\right)}{3\alpha\beta} + \sqrt{\frac{4\left(\alpha\varepsilon + v\beta\right)^2}{9\alpha^2\beta^2} - C_1} \quad ,$$

the potential representation can be transformed into the form

$$\left(U_\xi\right)^2 = -\frac{\alpha}{4\beta} U^2 + \sqrt{\frac{\left(\alpha\varepsilon + v\beta\right)^2}{9\beta^4} - \frac{\alpha^2 C_1}{4\beta^2}}\, U \quad . \tag{42}$$

With the preceding analysis the solution for $U(\xi)$ and thus for $u(\xi)$ can be given immediately

$$\begin{aligned} u(\xi) \quad &= \quad \frac{4}{3\alpha\beta}\sqrt{(\alpha\varepsilon + v\beta)^2 - 9/4\alpha^2\beta^2 C_1} \cos^2 \left(\frac{\sqrt{\alpha/\beta}}{4} \xi \right) \\ &\quad -\frac{\varepsilon}{2\beta} + \frac{2\left(\alpha\varepsilon + v\beta\right)}{3\alpha\beta} - \sqrt{\frac{4\left(\alpha\varepsilon + v\beta\right)^2}{9\alpha^2\beta^2} - C_1} \quad . \end{aligned} \tag{43}$$

The solution $u(\xi)$ thus has the amplitude

$$b = \frac{4}{3\alpha\beta}\sqrt{(\alpha\varepsilon + v\beta)^2 - 9/4\alpha^2\beta^2 C_1} \tag{44}$$

and the width

$$L = \frac{4}{\sqrt{\alpha/\beta}} \quad . \tag{45}$$

It moves with the speed

$$v' = 3\left(-\frac{\alpha\varepsilon}{4\beta^2} + \frac{(\alpha\varepsilon + v\beta)}{3\beta^2}\right) \tag{46}$$

and is shifted by

$$\delta = -\frac{\varepsilon}{2\beta} + \frac{2\,(\alpha\varepsilon + v\beta)}{3\alpha\beta} - \sqrt{\frac{4\,(\alpha\varepsilon + v\beta)^2}{9\alpha^2\beta^2} - C_1} \quad . \tag{47}$$

For $C_1 = 0$ these equations simplify enomously. One finds the simple relation for the speed

$$v' = \frac{3\alpha}{4\beta}\left(\frac{\varepsilon}{\beta} - b\right) \quad . \tag{48}$$

Here one finds that $b_{crit} = \varepsilon/\beta$ constitutes a critical amplitude. Solutions with $b > b_{crit}$ move to the left whereas solutions with $b < b_{crit}$ move to the right. Solutions with $b = b_{crit}$ are at rest. This property of traveling modes in systems having both quadratic and linear dispersion is documented in [10, 13].

4 Summary

In this article we have presented a potential picture for nonlinear dispersive wave equations. The potential picture provides a simplified representation of the original wave equation in terms of a nonlinear first order ordinary differential equation which is valid for the set of traveling solutions. The potential representation allows for an easy and intuitive way to identify different types of possible solutions. It proves to be extremely useful for the examination of nonlinear dynamical systems, since it provides direct access to various properties of the solutions without the actual need to solve the underlying nonlinear dispersive wave equation. A general investigation of the potential picture reveals that solitons may exist only in systems with linear dispersion whereas compact support solutions may arise as possible modes in systems with quadratic dispersion. Moreover, the specification of this concept to so-called $K(n, m)$ equations - a certain kind of nonlinear dispersive wave equations - directly gives the relations between the amplitude of the solitary wave and its speed or the width of the wave and its speed and its height, respectively. Only for the compact support solution of the $K(2, 2)$ equation is the width of independent of its speed or its height.

Furthermore, it has been shown that the potential representations of a certain $K(n, m)$ equation can be transformed into the respective potential representation of another $K(n,m)$-type equation by a simple point transformation. Using the potential representation as a consistency relation for the derivative of the wave function, $K(n,m)$ equations can be transformed directly into another. In this way the $K(n,m)$ equations are divided into equivalence classes each containing the set of equations that are connected via the point transformation considered. This transformation requires a further restriction of the admissible solutions. In addition to the requirement of focusing on traveling solutions, the solutions are specified by fixing the initially arbitrary integration constants to zero. An important property of point transformations is its invertibility. All elements of an equivalent class are uniquely connected with one another.

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A Derivation of the integrating factor

We start from the general nonlinear dispersive evolution equation (2) for traveling solutions,

$$(u^m)_{\xi\xi\xi} = \mathcal{V}(u)u_\xi \quad , \tag{49}$$

which can be integrated immediately with respect to ξ, giving

$$(u^m)_{\xi\xi} = \int_0^u \mathrm{d}t\mathcal{V}(t) + C_1 \tag{50}$$

where C_1 is an arbitrary constant. We introduce an integrating factor $f(u,u_\xi)$ that has to fulfill

$$\begin{aligned}
&(u^m)_{\xi\xi} f(u,u_\xi) - \int_0^u \mathrm{d}t\mathcal{V}(t)f(u,u_\xi) - C_1 f(u,u_\xi) \\
&= \left[(u^m)_\xi f(u,u_\xi) - \int_0^u \mathrm{d}t\mathcal{V}(t)F - C_1 F\right]_\xi - (u^m)_\xi f_\xi(u,u_\xi) + \left[\frac{\partial}{\partial\xi}\int_0^u \mathrm{d}t\mathcal{V}(t)\right] F \quad (51) \\
&= \left[a(u,u_\xi)\,(u^m)_\xi - b(u)\int_0^u \mathrm{d}t\mathcal{V}(t) - C_1 F\right]_\xi \quad (52)
\end{aligned}$$

with $F = \int \mathrm{d}\xi f(u,u_\xi)$. The function $a(u,u_\xi)$ can be choosen to be $af(u,u_\xi)$ with a a constant. Now, $b(u)$ and $f(u,u_\xi)$ can be determined via the differential equations

$$af_\xi(u,u_\xi)\,(u^m)_\xi = (1-a)f(u,u_\xi)\,(u^m)_{\xi\xi} \tag{53}$$

$$\int_0^u \mathrm{d}t\mathcal{V}(t)f(u,u_\xi) = b'(u)u_\xi \int_0^u \mathrm{d}t\mathcal{V}(t) + b(u)\left[\frac{\partial}{\partial\xi}\int_0^u \mathrm{d}t\mathcal{V}(t)\right]u_\xi \quad . \tag{54}$$

The first equation yields

$$f(u,u_\xi) = \left[(u^m)_\xi\right]^{\frac{1-a}{a}} \quad . \tag{55}$$

The second equation can be easily solved with $f(u,u_\xi) = F'u_\xi$ which is a consequence of the ansatz that $b(u)$ is a function of u only. This assumption gives $a = 1/2$ and one gets for $b(u)$ and for $f(u,u_\xi)$

$$b(u) = \frac{1}{\int_0^u \mathrm{d}t\mathcal{V}(t)} \int_0^u \mathrm{d}tF' \int_0^t \mathrm{d}x\mathcal{V}(x) \quad , \tag{56}$$

$$f(u,u_\xi) = (u^m)_\xi \quad . \tag{57}$$

B Derivation of the point transformation

We start from the general potential representation of the $K(n,m)$ equations,

$$(u_\xi)^2 = \frac{2A}{m(m+n)}u^{n-m+2} + \frac{2v}{m(m=1)}u^{3-m} + C_1u^{2-m} + C_2u^{2-2m} \quad . \tag{58}$$

Assuming the point transformation $u = \theta(\omega)$ one is led to

$$(\omega_\xi)^2 = \frac{2A}{m(m+n)}\frac{\theta^{n-m+2}}{\theta'^2} + \frac{2v}{m(m+1)}\frac{\theta^{3-m}}{\theta'^2} + C_1\frac{\theta^{2-m}}{\theta'^2} + C_2\frac{\theta^{2-2m}}{\theta'^2} \quad . \tag{59}$$

Eq. (59) is again be the potential function of a $K(N,M)$ equation (with different arguments N and M), i.e.

$$(\omega_\xi)^2 = \frac{2A'}{M(M+N)}\omega^{N-M+2} + \frac{2v'}{M(M+1)}\omega^{3-M} + C_1'\omega^{2-M} + C_2'\omega^{2-2M} \quad . \tag{60}$$

The four terms on the r.h.s. constitute (in combination with (59)) four differential equations that determine the transformation $\theta(\omega)$. With only two free parameters M and N, they can not be fulfilled simultaneously so that we have to assume $C_1 = C_2 = 0$. One is left with

$$\begin{aligned} \frac{\theta^{n-m+2}}{\theta'^2} &= \mathcal{A}\omega^{N-M+2} \\ \frac{\theta^{3-m}}{\theta'^2} &= \mathcal{B}\omega^{3-M} \end{aligned} \tag{61}$$

with

$$\begin{aligned} A' &= \frac{\mathcal{A}M(M+N)}{n(m+n)}A \\ v' &= \frac{\mathcal{B}M(M+1)}{m(m+1)}v \end{aligned} \tag{62}$$

or alternatively

$$\begin{aligned} \frac{\theta^{n-m+2}}{\theta'^2} &= \mathcal{B}\omega^{3-M} \\ \frac{\theta^{3-m}}{\theta'^2} &= \mathcal{A}\omega^{N-M+2} \end{aligned} \tag{63}$$

with

$$\begin{aligned} A' &= \frac{\mathcal{B}M(M+1)}{n(m+n)}A \\ v' &= \frac{\mathcal{A}M(M+N)}{m(m+1)}v \quad . \end{aligned} \tag{64}$$

In the following we will only deal with the transformations (61). One finds, in general,

$$\theta = \left(\frac{\mathcal{A}}{\mathcal{B}}\omega^{N-1}\right)^{\frac{1}{n-1}} \quad . \tag{65}$$

Further we first consider the cases covered by $m \neq 1$, $n \neq m$, $M \neq 1$, $N \neq M$. Straightforward integration of the equations (61) leads to

$$\begin{aligned} \theta &= \left(\frac{1}{\mathcal{A}}\left(\frac{m-n}{M-N}\right)^2 \omega^{M-N}\right)^{\frac{1}{m-n}} \\ &= \left(\frac{1}{\mathcal{B}}\left(\frac{m-1}{M-1}\right)^2 \omega^{M-1}\right)^{\frac{1}{m-1}} \end{aligned}$$

These equations together with (65) yield the conditions

$$\begin{aligned} &\frac{N-1}{n-1} = \frac{M-n}{m-n} = \frac{M-1}{m-1} \\ &\frac{1}{\mathcal{A}}\left(\frac{m-n}{M-N}\right)^2 = \frac{1}{\mathcal{B}}\left(\frac{m-1}{M-1}\right)^2 = 1 \\ &\mathcal{A} = \mathcal{B} \end{aligned}$$

that are fulfilled by

$$M - N = q(m-n) \tag{66}$$

$$M - 1 = q(m-1) \tag{67}$$

$$\mathcal{A} = \mathcal{B} = \frac{1}{q^2} \quad . \tag{68}$$

Therefore the parameters M and N of the $K(N, M)$ equation belonging to the transformed potential representation are

$$\begin{aligned} M &= q(m-1)+1 \\ N &= q(n-1)+1 \quad . \end{aligned} \tag{69}$$

The transformation (61) finally reads

$$\theta = \omega^{+q} \tag{70}$$

The sign of the exponent is fixed by the requirement $M,\ N > 0$.

Let us now turn to the cases in which either $m = 1$ or $m = n$ or $M = 1$ or $M = N$. One finds that the combinations $m = M = 1$ or $m - n = M - N = 0$ in the case (69) are likewise covered by (70). The remaining four combinations, i.e. $m = 1$ and $N \neq M$, $m = 1$ and $N = M$, $m = n$ and $N \neq M$, or $m = n$ and $M = 1$, respectively, do not lead to solutions of (61) that are in agreement with (65) and can thus be discarded. Equivalently one finds for the alternatively possible transformation (63)

$$\begin{aligned} M &= q(n-m)+1 \\ N &= q(n-1)+1 \quad . \end{aligned} \tag{71}$$

Here the transformation reads

$$\theta = \omega^{-q} \quad . \tag{72}$$

In additionally to the requirement $M,\ N > 0$, one has to assume in this case that $n \geq m$.

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