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Impulsive control systems

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IMPULSIVE CONTROL SYSTEMS

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Abstract

Impulsive control systems arose from classical control systems described by differential equations where the control functions could be unbounded. Passing to the limit of trajectories whose velocities are changing very rapidly leads to the state vector to “jump”, or exhibit impulsive behavior. The mathematical model in this thesis uses a differential inclusion and a measure-driven control, and it becomes possible to deal with the discontinuity of movements happening over a small interval. We adopt the formulism of impulsive systems in which the velocities are decomposed by the slow and fast ones. The fast time velocity is expressed as the multiplication of point-mass measure with a state-depended term.

Our methodology is deeply grounded in the concept of a graph completion, which is a technique to interpret and make rigorous the multiplication of a discontinuous function with a vector-valued measure. After reviewing how this concept is used to define the trajectory of impulsive system, the thesis works out a sampling method to estimate a solution and simultaneously construct a control measure, which is the first part of my research. The second part studies the stability of systems through invariance properties. Invariance of the system involves evolution properties on a given closed set with respect to the initial state belonging to that set. The third and last part of the thesis considers the Hamilton-Jacobi (HJ) theory of impulsive systems, which is related to the minimal time problem, an optimization topic of considerable interest. The minimal time function is uncovered to be the unique solution of HJ equation. Many discussions have earlier been offered in non-impulsive systems, especially in autonomous case, and we

attempt to extend these results to impulsive control system. Final thoughts and considerations are put in the last chapter of conclusions and future work.

Chapter 1
Introduction and Preliminaries

This thesis studies impulsive differential systems from the viewpoint of nonsmooth analysis. The term “differential systems” refers to differential equation-like dynamics but in which the righthand sides can take multiple values, and are often called differential inclusions. The term “impulsive” refers to the property that the state variable can exhibit “jumps”, whereas in the classical theory these arcs would be absolutely continuous.

The basic methodology throughout the thesis relies on concepts and techniques of nonsmooth and variational analysis. After a brief description of an impulsive system, we provide a brief overview of relevant topics in this preliminary chapter. The basic concepts are developed more completely in the standard references [8, 1, 16]. Also we review several useful background results from classical real analysis which can be found in either [10] or [11], and some measurable selection theory. Chapter Two begins with some background in differential inclusions in the presence of no impulses (that is, the measure μ in (1.3) does not appear), and then describes the relevant literature of impulsive systems. We provide a new approach to obtain a trajectory through graph completion and sampling method in Chapter Three, present our new results on the invariance of impulsive systems in Chapter Four, and sketch a new Hamilton-Jacobi Theory in Chapter Five.

1.1 The Impulsive Control System

The theory of impulsive systems in the context of control theory was initiated by Rishel [15], and was given a more full treatment by Warga [23]. The dynamics can be described by

$$dx = f\big(x(t), u(t)\big)dt + g\big(x(t), u(t)\big)\mu(dt), \tag{1.1}$$

where the state variable trajectory $x(t)$ and is of bounded variation defined on an interval $[0, T]$ and mapping into $\mathbb{R}^n$. The data f and g are given and defined on Euclidean space of a suitalbe dimension. The control variable $u(t)$ is measurable function on $[0, T]$ and maps into a compact subset $U \subset \mathbb{R}^k$, and μ is a vector-valued Borel measure defined on $[0, T]$. Whether μ is a priori given or can also be considered as a control input will be an important theme later in the thesis. In any case, dx is a Borel measure defined on the measurable subsets of $[0, T]$, and the equation (1.1) is to be interpreted in the sense of the measure dx equaling the measure of the righthand side. The cited works [15, 23] both emphasized a time-reparameterization in defining the nature of a solution, but importantly, the function g was independent of the state variable x. Indeed, it is a delicate matter to define precisely the meaning of the term

$$g\big(x(t), u(t)\big)\mu(dt), \tag{1.2}$$

since at an atom of μ, the multiplication is not well-defined. Indeed, in that case, the measure dx in (1.1) will be expected to also have an atom, and the multiplication in (1.2) becomes ambiguous and open to ambiguity and interpretation. These issues were first well appreciated by Dal Maso and Rampazzo [9] in the context of differential equations, and later by Bressan and Rampazzo [5, 4] with control functions added into the dynamics. There has been a lot of work since by a variety of authors, including [3, 12, 13, 14, 18, 19, 22, 26, 27].

The present work is largely based on that of Silva and Vinter [18], in which we adopt their problem formulation of a differential inclusion. This has the form

$$dx \in F\big(x(t)\big)dt + G\big(x(t)\big)\mu(dt), \tag{1.3}$$

where now the data F and G are so-called multifunctions (or set-valued maps). The same issues of interpretation of a solution are present, and will be further described in detail below. The inclusion (1.3) can be seen as a generalization of (1.1) by setting $F(x) = \{f(x,u) : u \in U\}$ and $G(x) = \{g(x,u) : u \in U\}$, but in fact the two formulations are equivalent in a sense to be described in Section 1.3. Silva and Vinter considered only scalar-valued measures, and incorporated the "vector" complication into G, but there are important reasons why this may not be desirable. We are motivated by the work of Wolenski and Zabic [26], where the measure μ retains its vector quality.

1.2 Proximal Normals

We give in this section the definition of proximal normal cone to a possibly nonsmooth set, which is a generalization of the classical normal vectors from differential geometry. This object plays an important role in characterization of invariance properties. We first illustrate the idea of a proximal normal somewhat informally.

Let S be a closed nonempty set in $\mathbb{R}^n$. The *Euclidean distance function* associated to S is defined by

$$d_S(x) := inf\{\|x - s\| : s \in S\},$$

for all $x \in \mathbb{R}^n$. The distance function associated with S can be viewed geometrically by considering a point $x \notin S$, and obtaining the set of all points $s \in S$ whose distance $d_S(x)$ to x is minimal. Such points exist since S is a closed set, and such a point$s$ is called a *projection* of x onto S. The set of all such closest points is denoted by $proj_S(x)$:

$$proj_S(x) = \{s \in S | d_S(x) = \|x - s\|\}.$$

The vector $x-s$ is an example of a *proximal normal* direction from S to x. More precisely, a proximal normal to S at s, is any vector of the form $\zeta = t(x-s)$ where $t \geq 0$ and $s \in proj_S(x)$. The set of all proximal normals is called the *proximal normal cone* to S, and is denoted by $N_S^P(s)$. Note that although every $x \notin S$ has $proj_S(x) \neq \varnothing$, it does not follow that every $s \in S$ belongs to some $proj_S(x)$, and hence may have its only proximal normal vector as the trivial one. The following formula precisely defines the proximal normal cone:

$$N_S^P(s) := \{\zeta | \exists t > 0 \text{ so that } d_S(s + t\zeta) = t\|\zeta\|\}.$$

A characterization of a proximal normal vector is that $\zeta \in N_S^P(s)$ if and only if there exists $\sigma \geq 0$ and $\eta > 0$ such that

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2 \text{ for all } s' \in S \cap (s + \eta B),$$

where B is a unit closed ball. A seemingly more particular but nonetheless equivalent characterization is that $\zeta \in N_S^P(s)$ if and only if there exists $\sigma \geq 0$ such that

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2 \text{ for all } s' \in S. \tag{1.4}$$

Equation (1.4) is called the *proximal normal inequality*, and can be found as Proposition 1.5 in [8].

A few other basic definitions follow. A set $K \subset \mathbb{R}^n$ is called a *cone* if it is closed with respect to the multiplication of any positive scalar. That is, for all $k \in K$ and for all $\lambda > 0$, $\lambda k \in K$ as well. A set S is *convex* if it contains the linear segment connecting any two of its points. This means that s_1, $s_2 \in S$ and $0 \leq \lambda \leq 1$ implies $\lambda s_1 + (1-\lambda)s_2 \in S$. The intersection of all convex sets containing a set

$S \in \mathbb{R}^n$, is called the *convex hull* of S and denoted by *co* S. Obviously a set S is convex if and only if it equals its convex hull. The notation $\overline{co}\ S$ denotes the closure of *co* S.

1.3 Measurable Properties of Multifunctions

An aspect of recent and current research in controlled dynamical system theory is carried out on the platform of *differential inclusion* theory, which can be viewed as a generalization of a differential equation that allows for multivalued righthand sides. There is a considerable literature in this subject, cf. [2, 20, 24, 25]. We take this section to review some of the basic concepts of multifunctions, and in particular, state and prove the measurable selection theorem, which has important and ubiquitous applications in control. We offer a complete proof since it is not so easily found elsewhere. A *multifunction* F is a set-valued map that is described by writing $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$. That is, F assigns to each $x \in \mathbb{R}^m$ a corresponding set $F(x) \subseteq \mathbb{R}^n$.

Recall that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is measurable if every inverse images $f^{-1}(C)$ is a measurable subset of $\mathbb{R}^m$ for each closed set $C \subseteq \mathbb{R}^n$. This motivates the following definition.

Definition 1.1. *A multifunction is called measurable if for any closed set $C \subseteq \mathbb{R}^n$, the set*

$$\{x \in \mathbb{R}^m : F(x) \cap C \neq \varnothing\}$$

is Lebesgue measurable in $\mathbb{R}^m$. The multifunction F is said to be closed-valued (or simply closed) if all the sets $F(x)$ are closed in the usual topological sense.

The following is a list of notations and conventions that will be adopted:

$domF = \{x \in \mathbb{R}^m \| F(x) \neq \varnothing\};$	(effective) domain of F,
$graph\,F = \{(x, v) \| v \in F(x)\};$	graph of F,
$imF = \{v \| \exists x \in \mathbb{R}^m \text{ so that } v \in F(x)\};$	image of F,
$F(X) = \cup_{x \in X} F(x)$	range of $X \subseteq \mathbb{R}^m$
$F^{-1}(C) = \bigcup_{v \in C} F^{-1}(v) = \{x \in \mathbb{R}^m \| F(x) \cap C \neq \varnothing\}$	inverse images of F

The inverse multifunction $F^{-1} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is most easily defined by specifying its graph $graphF^{-1}$ to be the set $\{(v, x) : (x, v) \in graphF\}$. We occasionally say that a multifunction F is defined only on subset $X \subset \mathbb{R}^n$, but it can equally be viewed as being defined on all of $\mathbb{R}^n$ by just setting $F(x) = \varnothing$ for $x \notin X$. In this way, every multifunction has an inverse F^{-1}.

The following are characterizations of measurability.

Proposition 1.2. *For a closed-valued multifunction $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the following properties are equivalent:*

(a) *F is measurable;*

(b) *$F^{-1}(C)$ is measurable for all open sets C;*

(c) *$F^{-1}(C)$ is measurable for all compact sets C;*

(d) *The function $x \mapsto d_{F(x)}(\zeta)$ is a measurable function of $x \in \mathbb{R}^m$ for each $\zeta \in \mathbb{R}^n$.*

Proof. The proofs of (a)-(c) are straightforward and are thus omitted. The proof of (d) follows by noting that for any ζ in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have

$$\{x \in X | d_{F(x)}(\zeta) \leq \alpha\} = \{x \in X | F(x) \cap (\zeta + \alpha B) \neq \varnothing\}.$$

The result immediately follows from this. $\square$

The relation between a given measurable multifunction and a measurable function living within it is reflected by the measurable selection theorem (Theorem 2.2 [20]), which asserts that such a measurable function exists. Its proof is offered by construction. This theorem is a crucial tool throughout the thesis, and since its proof is somewhat complicated and perhaps not readily available, we give complete details here for the sake of completeness.

Theorem 1.3 (Measurable Selection)**.** *Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be measurable, closed-valued, and with $X = domF \neq \varnothing$. Then there exists a measurable function $f : X \to \mathbb{R}^n$ such that $f(x) \in F(x)$ for all $x \in X$.*

Proof. Recall from Proposition 1.2(d) that the function $x \mapsto d_{F(x)}(\zeta)$ is measurable on X. We will prove the theorem by constructing a function $f(\cdot)$ as a pointwise limit of a sequence of measurable functions $\{f_i(\cdot)\}_i$.

Let $\{\zeta_i\}$ be a countable dense subset of $\mathbb{R}^n$, and construct the first function as follows,

$$f_0(x) = \text{the first } \zeta_i \text{ such that } d_{F(x)}(\zeta_i) \leq 1.$$

We claim the functions $f_0(\cdot)$ and $x \mapsto d_{F(x)}(f_0(x))$ are both measurable. To see this, observe that f_0 assumes countably many values, and that, for each i,

$$\{x|f_0(x) = \zeta_i\} = \bigcap_j \{x|d_{F(x)}(\zeta_j) > 1\} \cap \{x|d_{F(x)}(\zeta_i) \leq 1\},$$

where the intersection is over $j = 1, ..., i-1$. This implies that f_0 is measurable. To see function $x \mapsto d_{F(x)}(f_0(x))$ is also measurable, we need only note

$$\{x|d_{F(x)}(f_0(x)) > \alpha\} = \bigcup_{j \in \mathbb{N}} \Big[\{x|f_0(x) = \zeta_j\} \cap \{x|d_{F(x)}(\zeta_j) > \alpha\} \Big],$$

We pursue the process begun above by defining for each i a function f_{i+1} such that $f_{i+1}(x)$ is the first ζ_j for which both the following hold:

$$\|\zeta_j - f_i(x)\| \leq \frac{2}{3} d_{F(x)}(f_i(x)), \qquad d_{F(x)}(\zeta_j) \leq \frac{2}{3} d_{F(x)}(f_i(x)).$$

We will prove each $\{f_i\}_i$ is measurable by induction. Suppose f_k is measurable for all $k = 0, ..., i$, and for such k note that

$$\{x|d_{F(x)}(f_k(x)) > \alpha\} = \bigcup_{j\in\mathbb{N}}\Big[\{x|f_k(x) = \zeta_j\} \cap \{x|d_{F(x)}(\zeta_j) > \alpha\}\Big].$$

Hence we conclude that $x \mapsto d_{F(x)}(f_k(x))$ is measurable for all $k = 0, ..., i$. By the definition of $f_{i+1}(x)$ we have for each j

$$\begin{aligned}&\{x|f_{i+1}(x) = \zeta_j\} = \\ &\bigcap_{k=1}^{j-1}\Big[\Big\{x|d_{F(x)}(f_i(x)) < \frac{3}{2}\|\zeta_k - f_i(x)\|\Big\} \cup \Big\{x|d_{F(x)}(f_i(x)) < \frac{3}{2}d_{F(x)}(\zeta_k)\Big\}\Big] \\ &\quad \cap \Big\{x|d_{F(x)}(f_i(x)) \geq \frac{3}{2}\|\zeta_j - f_i(x)\|\Big\} \cap \Big\{x|d_{F(x)}(f_i(x)) \geq \frac{3}{2}d_{F(x)}(\zeta_j)\Big\}.\end{aligned}$$

Note that $\{x|f_{j+1}(x) = \zeta_j\}$ is measurable, since the sets of right hand side of equation are all measurable. Hence, $f_{i+1}(x)$ is measurable function since its range is at most countable.

Furthermore, we deduce the inequalities

$$d_{F(x)}(f_{i+1}(x)) \leq \left(\frac{2}{3}\right)^i d_{F(x)}(f_0(x)) \leq \left(\frac{2}{3}\right)^{i+1},$$

together with $\|f_{i+1}(x) - f_i(x)\| \leq \left(\frac{2}{3}\right)^{i+1}$. It follows that $\{f_i(x)\}$ is a Cauchy sequence converging to a value $f(x)$ in $F(x)$ for each x, and that f is a measurable selection for F. □

A specific control application of measurable selection theory is stated below, and is usually referred to as Filippov's theorem (Theorem 2.3 of [20]). We will use it later to describe the connection between a controlled differential equation and its related differential inclusion.

Corollary 1.4 (Filippov's Selection Theorem)**.** *Let* $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ *be a continuous function, and let* $v : \mathbb{R}^n \to \mathbb{R}^m$ *be a measurable function. Assume that*

$U \subset \mathbb{R}^k$ is a compact set such that $v(x) \in f(x, U) := \{f(x, u) : u \in U\}$ for almost all $x \in X$, where $\varnothing \neq X \subseteq \mathbb{R}^n$ is measurable. Then there exists a measurable function $u : X \to U$ satisfying $v(x) = f(x, u(x))$.

This statement actually resembles an implicit function theorem, but does not require any of regularity properties.

Proof. The multifunction $F(x) = \{u \in U : v(x) = f(x, u)\}$ is measurable whose domain contains X. By Theorem 1.3, there exists a measurable selection $u(x)$, and it fulfils the conclusion of the corollary. $\square$

Chapter 2
Differential Inclusions

In this chapter, we give an extensive review of the theory of differential inclusions. More details are contained in the references [2] and [20].

2.1 Assumptions and Growth Estimates

Given a multifunction $F : \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a time interval $[0, T]$, a *differential inclusion* has the form

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \quad a.e. \quad t \in [0, T) \\ x(0) = x_0. \end{cases} \tag{2.1}$$

A *solution* (also referred to *trajectory*) $x(\cdot)$ of the differential inclusion is an absolutely continuous function $x : [0, T) \to \mathbb{R}^n$ which satisfies (2.1). The concept of a differential inclusion coincide that of a differential equation when the multifunction F is singleton-valued; that is, when $F(t, x) = \{f(t, x)\}$. The assumptions invoked on F will closely mirror those usually invoked in differential equation theory, and when the data has only measurable t-dependence, the solution is sometimes referred to as a solution in *Carathéodory* sense.

The main development of differential inclusion theory subsumes the theory of a *standard control system*, where the latter has the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \quad a.e. \quad t \in [0, T) \\ u(t) \in U \quad a.e. \quad t \in [0, T) \\ x(0) = x_0. \end{cases} \tag{2.2}$$

The data maps $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and the (measurable) control function $u(\cdot)$ takes its values over a fixed subset U of $\mathbb{R}^m$. The relationship between (2.1) and

(2.2) was alluded to before, but can now be made precisely. The bridge between the problem data is the multifunction $F(t,x) = f(t,x,U)$. If $\big(x(\cdot), u(\cdot)\big)$ satisfies (2.2), then it is clear $x(\cdot)$ satisfies (2.1). Conversely, if $x(\cdot)$ satisfies (2.1), then Filippov's Selection Theorem implies (that under mild hypothesis on f), there is a measurable function $u(\cdot)$ with values in U such that $\big(x(\cdot), u(\cdot)\big)$ satisfies (2.2).

Definition 2.1 (Continuity)**.** *A multifunction $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is called upper semi-continuous at $x \in \mathbb{R}^m$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x' - x\| < \delta \Rightarrow F(x') \subset F(x) + \varepsilon B.$$

F is said to be upper semi-continuous on a set $X \subseteq \mathbb{R}^m$ if it is so at every point $x \in X$.

A multifunction F is called lower semi-continuous at x if for any $y \in F(x)$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x' - x\| < \delta \Rightarrow y \subset F(x') + \varepsilon B.$$

F is said to be lower semi-continuous if it is so at every point $x \in X$.

A multifunciton F is called continuous at $x \in X$ if it is both upper and lower semi-continuous at x. F is said to be continuous if it is so at every point $x \in X$.

The following assumptions on F are assumed to hold throughout the rest of the thesis, and are referred to as the *standing hypotheses* (SH):

(a) For every (t,x), $F(t,x)$ is a nonempty, compact, and convex set.

(b) The multifunction $F(\cdot, x) : [0,t] \rightrightarrows \mathbb{R}^n$ $\mathcal{L} \times \mathcal{B}$ -measurable, and is upper semicontinuous in the x variable for almost every $t \in [0,T]$.

(c) There exist a positive constant γ and an integrable function $c(\cdot)$ such that the *linear growth condition* holds:

$$\|v\| \leq \gamma\|x\| + c(t) \quad \text{for all } v \in F(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^n.$$

The conditions on the values of F in (a) are so that solutions of (2.1) can be obtained through a limiting process. The role of (b) is so that measurability issues do not arise with the composition of F and an arc $x(\cdot)$. The role of linear growth condition (c) on a differential inclusion is same as in the classical theory of differential equations in that provides a priori bounds on all possible solutions to (2.1). These predicated bounds are obtained by using a version of *Gronwall's inequality.* We state this version and give its proof, since most references do not state it this way.

Lemma 2.2 (Gronwall's Lemma)**.** *Suppose $y(\cdot)$ is continuous and nonnegative on $[0,T]$, $\gamma \geq 0$, $r(\cdot)$ is nonnegative and nondecreasing on $[0,T]$, and the inequality*

$$y(t) \leq r(t) + \gamma \int_0^t y(s)\, ds \tag{2.3}$$

holds for all $t \in [0,T]$. Then

$$y(t) \leq r(t)e^{\gamma t} \tag{2.4}$$

holds for all $t \in [0,T]$.

Proof. Let $\phi(t) = \gamma \int_0^t y(s)\, ds$, and thus

$$\phi'(t) = \gamma y(t) \leq \gamma r(t) + \gamma\phi(t)$$

by (2.3). Multiplying this inequality by $e^{-\gamma t}$ and rearranging terms implies

$$\begin{aligned}\frac{d}{dt}\left[\phi(t)e^{-\gamma t}\right] &= \phi'(t)e^{-\gamma t} - \gamma\phi(t)e^{-\gamma t} \\ &= \left[\phi'(t) - \gamma\phi(t)\right]e^{-\gamma t} \\ &\leq \left[\gamma r(t) + \gamma\phi(t) - \gamma\phi(t)\right]e^{-\gamma t} \\ &= \gamma r(t)e^{-\gamma t}\end{aligned}$$

Now integrate from 0 to t, and the result is (observing that $\phi(0) = 0$)

$$\phi(t)e^{-\gamma t} \leq \gamma \int_0^t r(s)e^{-\gamma s}\, ds$$

Rearranging terms again, using the fact that $r(\cdot)$ is nondecreasing, and integrating, gives

$$\phi(t) \leq r(t) \int_0^t \gamma e^{-\gamma s}\, ds = r(t)\left[e^{\gamma t} - 1\right]$$

Finally, we have by (2.3) and the last estimate that

$$y(t) \leq r(t) + \phi(t) \leq r(t) + r(t)\left[e^{\gamma t} - 1\right] = r(t)e^{\gamma t},$$

which is the conclusion (2.4) □

We next see how this implies that solutions to the differential inclusion remain bounded.

Corollary 2.3. *Let $x(\cdot) : [0, T] \to \mathbb{R}^n$ be an absolutely continuous function defined on $[0, T]$ satisfying the differential inclusion (2.1). Then for all $t \in [0, T]$,*

$$\|x(t) - x_0\| \leq \left[t\gamma\|x_0\| + \int_0^t c(s)\, ds\right] e^{\gamma t}. \tag{2.5}$$

Proof. Note by (SH)(c) that

$$\|\dot{x}(t)\| \leq \gamma\|x(t)\| + c(t) \quad a.e. \quad t \in [0, T].$$

Let $y(t) := \|x(t) - x_0\|$, and so

$$
\begin{aligned}
y(t) &= \left\| \int_0^t \dot{x}(s)\, ds \right\| \le \int_0^t \|\dot{x}(s)\|\, ds \\
&\le \int_0^t \big[\gamma \|x(s)\| + c(s)\big]\, ds \\
&\le t\gamma\|x_0\| + \int_0^t c(s)\, ds + \int_0^t \gamma \|x(s) - x_0\|\, ds \\
&= r(t) + \gamma \int_0^t y(s)\, ds,
\end{aligned}
$$

where $r(t) = t\gamma\|x_0\| + \int_0^t c(s)\, ds$, a function that satisfies the criteria in Gronwall's Lemma. The conclusion (2.4) of Gronwall's Lemma says that (2.5) holds, as claimed. □

A discrete version of gronwall's Lemma is also useful in estimates that involve time discretizations.

Lemma 2.4 (Discrete Gronwall's Lemma)**.** *Let N be a positive integer, and suppose $x_0, x_1, \ldots, x_N$ are elements in $\mathbb{R}^n$ satisfying*

$$\|x_{j+1}\| \le \gamma \|x_j\| + c, \tag{2.6}$$

for each $j = 0, \ldots, N$, and where γ and c are nonegative numbers. Then

$$\|x_N\| \le c\frac{1-\gamma^N}{1-\gamma} + \gamma^N \|x_0\|. \tag{2.7}$$

Proof. The proof will be completed by mathematical induction. Note from (2.6) that for $N = 1$

$$\|x_1\| \le c + \gamma\|x_0\|,$$

and so inequality (2.7) obviously holds. Now suppose

$$\|x_{N-1}\| \le c\frac{1-\gamma^{N-1}}{1-\gamma} + \gamma^{N-1}\|x_0\|$$

holds. Combining this with (2.6), one obtains

$$\begin{aligned}
\|x_N\| &\leq \gamma\|x_{N-1}\| + c \\
&\leq c + c\gamma\frac{1-\gamma^{N-1}}{1-\gamma} + \gamma^N\|x_0\| \\
&= \frac{c}{1-\gamma}(1-\gamma+\gamma(1-\gamma^{N-1})) + \gamma^N\|x_0\| \\
&= c\frac{1-\gamma^N}{1-\gamma} + \gamma^N\|x_0\|,
\end{aligned}$$

which completes the proof. □

We record as a corollary the estimate that we will use from the Discrete Gronwall's Lemma.

Corollary 2.5. *With $x_1, \ldots, x_N$ and the constants $\gamma = 1 + \frac{\alpha}{N}$ and $c = \frac{\alpha}{N}$ as in Lemma 2.4, we have*

$$\|x_N\| \leq e^\alpha(1 + \|x_0\|) - 1.$$

Proof. Note that $c = \gamma - 1$ and

$$\gamma^N = \Big(1 + \frac{\alpha}{N}\Big)^N \leq e^\alpha.$$

The result immediately follows from Lemma 2.4. □

2.2 Time Discretization and Euler Solutions

Consider the differential inclusion (2.1). The most straightforward approach to constructing a trajectory is to first fix a measurable selection f of F. Recall this means that $f(t, x) \in F(t, x)$ for all (t, x). Now consider the differential equation

$$\text{(2.8)} \qquad \begin{cases} \dot{x}(t) = f\big(t, x(t)\big) \quad a.e. \quad t \in [a, b] \\ x(a) = x_0, \end{cases}$$

in which any solution will presumably satisfy (2.1). The main objection to this approach obviously lies in finding selections f with the regularity properties (e.g.

continuity) required in existence theory for a usual differential equation. Temporarily putting aside the selection issue, let us review the *Euler iterative scheme* for the so-called *Cauchy* or *initial-value problem* stated in (2.8), where f is simply any function from $[a, b] \times \mathbb{R}^n$ to $\mathbb{R}^n$. By discretizing in time, let

$$\pi = \{t_0, t_1, ..., t_{N-1}, t_N\}$$

be a uniform partition of $[a, b]$ where $t_0 = a$ and $t_N = b$. The mesh size of the partition is $h = (b - a)/N$, and generally $t_i = a + ih$, for $i = 1, 2, ..., N$. A piecewise affine arc $x^N(\cdot)$, called a *Euler polygonal arc*, and is defined by the nodes $x_0, x_1, ..., x_{N-1}$ obtained from the following iterative sampling scheme:

$$\begin{array}{rclrcl} v_0 & = & f(t_0, x_0) & x_1 & = & x_0 + hv_0 \\ v_1 & = & f(t_1, x_1) & x_2 & = & x_1 + hx_1 \\ & \vdots & & & \vdots & \\ v_i & = & f(t_i, x_i) & x_{i+1} & = & x_i + hv_i \\ & \vdots & & & \vdots & \\ v_{N-1} & = & f(t_{N-1}, x_{N-1}) & x_N & = & x_{N-1} + hv_{N-1} \end{array}$$

The Euler polygonal arc is given by

$$x^N(t) = x_i + (t - t_i)v_i \quad \text{whenever } t \in [t_i, t_{i+1}].$$

An *Euler solution* to the initial-value problem (2.8) is any uniform limit $x(\cdot)$ of Euler polygonal arcs $x^N(\cdot)$ as $N \to \infty$.

There are potential pathologies associated with these Euler solutions when f is discontinuous (e.g., 4.1.6 of [8]). On the other hand, with the linear growth restriction on f, we record here some basic results worth mentioning.

Theorem 2.6. *Suppose that for positive constants γ and c and for all $(t,x) \in [a,b] \times \mathbb{R}^n$, we have the linear growth condition*

$$\|f(t,x)\| \leq \gamma\|x\| + c,$$

(a) *At least one Euler solution x to the initial-value problem (2.8) exists, and any Euler solution is Lipschitz.*

(b) *Any Euler arc x for f on [a,b] satisfies*

$$\|x(t) - x(a)\| \leq (t-a)e^{\gamma(t-a)}(\gamma\|x(a)\| + c), \qquad a \leq t \leq b.$$

(c) *If f is continuous, then any Euler arc x of f on (a,b) is continuously differentiable on (a,b) and satisfies $\dot{x}(t) = f(t,x(t)), \forall t \in (a,b)$.*

Proof. Let $\pi := \{t_0, t_1, ..., t_N\}$ be a partition of $[a,b]$, and let x_π be the corresponding Euler polygonal arc, with the nodes of x_π being denoted $x_0, x_1, .., x_N$ as aforementioned. On the interval (t_i, t_{i+1}) we have

$$\|\dot{x}_\pi(t)\| = \|f(t_i, x_i)\| \leq \gamma\|x_i\| + c,$$

whence

$$\begin{aligned}
\|x_{i+1} - x_0\| &\leq \|x_{i+1} - x_i\| + \|x_i - x_0\| \\
&\leq (t_{i+1} - t_i)(\gamma\|x_i\| + c) + \|x_i - x_0\| \\
&\leq (t_{i+1} - t_i)(\gamma\|x_i\| - \gamma\|x_0\|) + (t_{i+1} - t_i)(\gamma\|x_0\| + c) + \|x_i - x_0\| \\
&\leq [(t_{i+1} - t_i)\gamma + 1]\,\|x_i - x_0\| + (t_{i+1} - t_i)(\gamma\|x_0\| + c) \\
&= (h\gamma + 1)\|x_i - x_0\| + h(\gamma\|x_0\| + c),
\end{aligned}$$

where h is the mesh size of time defined as before. Let $\delta := h\gamma + 1$, $\Delta = h(\gamma\|x_0\| + c)$ and $r_{i+1} := x_{i+1} - x_0$ with $r_0 = 0$. Then $\|r_{i+1}\| \leq \delta\|r_i\| + \Delta$, in which data meet the criteria of Discrete Gronwall's Lemma. Thus for $i = 1, 2, ..., N$, we have

$$\begin{aligned}
\|x_i - x_0\| = \|r_i\| &\leq \Delta \frac{1-\delta^i}{1-\delta} + \delta^i \|r_0\| \\
&= h(\gamma\|x_0\| + c)\frac{1-(h\gamma+1)^i}{-h\gamma} \\
&= (\gamma\|x_0\| + c)\left(\frac{1}{\gamma}\right)\left[\left(1 + \frac{(b-a)\gamma}{N}\right)^i - 1\right] \\
&= \left(\frac{e^{\gamma(b-a)} - 1}{\gamma}\right)(\gamma\|x_0\| + c) \\
&\leq M,
\end{aligned}$$

where

$$M := (b-a)e^{\gamma(b-a)}(\gamma\|x_0\| + c).$$

The last inequality above holds for

$$e^{\gamma(b-a)} - 1 \leq \gamma(b-a)e^{\gamma(b-a)},$$

which can be verified by expanding $e^{\gamma(b-a)}$ about $\gamma(b-a)$ in Taylor series. Therefore, all the nodes x_i lie in the closed ball $B(x_0; M)$; by convexity this is true for all $x_\pi(t)$ over $[a, b]$. Since the derivative along any linear portion of x_π is determined by the values of f at the nodes, we obtain as well the following bound on $[a, b]$:

$$\|\dot{x}_\pi\| \leq \max_i \|f(t_i, x_i)\| \leq k := \gamma(M + \|x_0\|) + c.$$

Therefore x_π is Lipschitz of rank k on $[a, b]$.

Now let π_j be a sequence of partitions such that their diameters $\mu_{\pi_j} \to 0$, and necessarily $N_j \to \infty$. Then the corresponding polygonal arc x_{π_j} on $[a, b]$ all satisfy

$$x_{\pi_j} = x_0, \qquad \|x_{\pi_j} - x_0\| \leq M, \qquad \|\dot{x}_{\pi_j}\| \leq k.$$

It follows that the family $\{x_{\pi_j}\}$ is equicontinuous and uniformly bounded; then, by the well-known theorem of Arzela and Ascoli, some subsequence of it converges uniformly to a continuous function x. The limit function inherits the Lipschitz rank k on $[a, b]$, and in consequence is absolutely continuous (i.e., x is an arc). Thus by definition x is an Euler solution of the initial-value problem (2.8) on $[a, b]$, and assertion (a) of the theorem is proved.

The inequality in part (b) of the theorem is inherited by x from the sequence of polygonal arcs generating it (we identify t with b). There remains to prove part (c) of the theorem.

Let x_{π_j} denote a sequence of polygonal arcs for problem (2.8) converging uniformly to an Euler solution x. As shown above, the arcs x_{π_j} all lie in a certain ball $B(x_0; M)$ and they all satisfy a Lipschitz condition of the same rank k. Since a continuous function on $\mathbb{R}^n$ is uniformly continuous on compact sets, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$t, \tilde{t} \in [a, b], \quad x, \tilde{x} \in B(x_0; M), \quad |t - \tilde{t}| < \delta,$$
$$\|x - \tilde{x}\| < \delta \quad \Longrightarrow \quad \|f(t, x) - f(\tilde{t}, \tilde{x})\| < \varepsilon.$$

Now let j be large enough so that the partition diameter μ_{π_j} satisfies $\mu_{\pi_j} < \delta$ and $k\mu_{\pi_j} < \delta$. For any point t at which $x_{\pi_j}(t)$ is not a node, we have $\dot{x}_{\pi_j}(t) = f(\tilde{t}, x_{\pi_j}(\tilde{t}))$ for some $\tilde{t}$ within $\mu_{\pi_j} < \delta$ of t. Thus, since

$$\|x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})\| \leq k\mu_{\pi_j} < \delta,$$

We deduce

$$\|\dot{x}_{\pi_j}(t) - f(t, x_{\pi_j}(t))\| = \|f(t, x_{\pi_j}(t)) - f(\tilde{t}, x_{\pi_j}(\tilde{t}))\| < \varepsilon.$$

It follows that for any t in $[a, b]$, we have

$$\begin{aligned}&\left\|x_{\pi_j}(t) - x_{\pi_j}(a) - \int_a^t f(t, x_{\pi_j}(\tau))\, d\tau\right\|\\ =&\left\|\int_a^t \{\dot{x}_{\pi_j}(\tau) - f(\tau, x_{\pi_j}(\tau))\}\, d\tau\right\|\\ <&\varepsilon(t-a) \leq \varepsilon(b-a).\end{aligned}$$

Letting $j \to \infty$, we obtain

$$\left\|x(t) - x_0 - \int_a^t f(\tau, x(\tau))\, d\tau\right\| \leq \varepsilon(b-a).$$

Since ε is arbitrary, it follows that

$$x(t) = x_0 + \int_a^t f(\tau, x(\tau))\, d\tau,$$

which implies (since the integrand is continuous) that the Euler arc x is C^1 and $\dot{x}(t) = f(t, x(t))$ for all $t \in (a, b)$. □

The following sequential compactness of trajectories property guarantees the existence of a solution of (2.1) under the standing hypotheses.

Theorem 2.7. *Let $\{x_i\}$ be a sequence of arcs on $[a, b]$ such that the set $x_i(a)$ is bounded, and satisfying*

$$\dot{x}_i(t) \in F(\tau_i(t), x_i(t) + y_i(t)) + r_i(t)B \quad a.e.,$$

where $\{y_i\}$, $\{r_i\}$ and $\{\tau_i\}$ are sequences of measurable functions on $[a, b]$ such that $y_i(\cdot)$ converges to 0 in L^2, $r_i \geq 0$ converges to 0 in L^2 and τ_i converges a.e. to t. Then there is a subsequence of $\{x_i\}$ which converges uniformly to an arc x which is trajectory of F, and whose derivatives converge weakly to $\dot{x}$.

Proof: From the differential inclusion and the linear condition of F we have the inequality,

$$\|\dot{x}_i(t)\| \leq \gamma\|x_i(t) + y_i(t)\| + |r_i(t)|.$$

Applying Gronwall's Lemma (Lemma 1.5), the inequality implies a uniform bound on $\|\dot{x}_i\|_\infty$ and hence on $\|\dot{x}_i\|_2$. Invoking weak compactness in $L_n^2[a,b]$ allows the extraction of a subsequence $\dot{x}_{i_j}$ converging weakly to a limit v_0; we may also suppose (by Arzela and Ascoli) that x_{i_j} converges uniformly to a continuous function x. Passing onto the limit in

$$x_{i_j} = x_{i_j}(a) + \int_a^t \dot{x}_{i_j}(s)ds$$

shows that $x(t) = x(a) + \int_a^t v_0(s)ds$, whence x is an arc and $\dot{x} = v_0$ $a.e.$ The fact that x is a trajectory for F is an immediate consequences of Theorem 3.5.24 of [8].

Corollary 2.8. *Let f be any selection of F, and let x be an Euler solution on $[a,b]$ of $\dot{x} = f(t,x), x(a) = x_0$. Then x is a trajectory of F on $[a,b]$.*

Proof. Let x_{π_j} denote a sequence of polygonal arcs whose uniform limit is x, as in the proof of Theorem 2.6. Let $t \in (a,b)$ be a non-partition point, and let $\tau_j(t)$ designate the partition point t_i immediately before t. Then

$$\dot{x}_{\pi_j}(t) = f(t_i, x_i) \in F(t_i, x_i) = F(\tau_j(t), x_{\pi_j}(t) + y_j(t)),$$

where $y_j(t) := x_i - x_{\pi_j}(t) = x_{\pi_j}(\tau_j(t)) - x_{\pi_j}(t)$. Since the functions x_{π_j} are Lipschitz with a common rank k, we have

$$\|y_j(t)\| \leq k \sup_{t\in[a,b]} |\tau_j(t) - t| \leq k\mu_{\pi_j}.$$

It follows that τ and y_j are measurable functions converging uniformly to t and 0, respectively. Consequently, the theorem asserts that x, the uniform limit of x_{π_j}, is a trajectory of F. □

When impulsive systems are introduced in the next chapter, we will see that a similar idea of an Euler approximation can be employed to a graph completion that "slows down" the impulsive atoms of the system.

2.3 Invariance Conditions

This section recalls invariance theory associated to the differential inclusion (2.1). The proofs can be found in [8]. In short, invariance properties deal with conditions under which a solution starts and remains in a given closed set C.

For simplicity, we assume in this section that F is *autonomous*; that is, $F(t,x) = F(x)$ does not explicitly depend on the time variable t. The next chapter will contain the application of this theory to an impulsive system.

Definition 2.9. *Consider the system (2.1) and a closed set $C \subseteq \mathbb{R}^n$.*

(a) (F,C) *is said to be* weakly invariant *provided for all $x_0 \in C$, there exists a trajectory $x(\cdot)$ of (2.1) defined on* $[0,\infty)$ *such that*

$$x(t) \in C \quad \forall t \geq 0.$$

(b) (F,C) *is* strongly invariant *provided that for all $x_0 \in C$ and all trajectories $x(\cdot)$ defined on* $[0,T]$ *(for any $T > 0$) with* $x(0) = x_0$ *satisfy*

$$x(t) \in C \qquad \forall t \in [0,T].$$

The following theorem specifies a weak invariance condition for the system.

Theorem 2.10. (F,C) *is weakly invariant if and only if for all $x \in C$ and $\zeta \in N_C^P(x)$, there exists a $v \in F(x)$ such that*

$$\langle v, \zeta \rangle \leq 0.$$

A characterization of strong invariance requires additional hypotheses. One common assumption that can be added to the standing hypotheses is that the multifunction $F(\cdot)$ is also *locally Lipschitz*. We define this property next.

Definition 2.11. *A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said locally Lipschitz if for every point $x \in domF$, there is a neighborhood $U = U(x)$ and a positive constant*

$L = L(x)$ such that

$$x_1, x_2 \in U \Rightarrow F(x_2) \subseteq F(x_1) + L\|x_2 - x_1\|B, \tag{2.9}$$

where the constant L is called the Lipschitz rank of $F(\cdot)$ on the set U. F is called globally Lipschitz if (2.9) holds with $U = \mathbb{R}^n$.

Theorem 2.12. *Suppose in addition to the standing hypotheses, F is locally Lipschitz. The system is strongly invariant if and only if for all $x \in C$, all $\zeta \in N_C^P(x)$ and all $v \in F(x)$ have that*

$$\langle v, \zeta \rangle \leq 0.$$

The above characterizations can be described succinctly in terms of Hamiltonian quanta. The *lower Hamiltonian h* and *upper Hamiltonian H* corresponding to a multifunction F mapping $\mathbb{R} \times \mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ are defined generally as follows,

$$h_F(t, x, p) := \min_{v \in F(t,x)} \langle v, p \rangle, \qquad H_F(t, x, p) := \max_{v \in F(t,x)} \langle v, p \rangle.$$

In the autonomous case, the multifunction F is independent of time t, so we notate simply,

$$h_F(x, p) := \min_{v \in F(x)} \langle v, p \rangle, \qquad H_F(x, p) := \max_{v \in F(x)} \langle v, p \rangle.$$

The conclusions of Theorems 2.10 and 2.12 can be respectively stated as

$$h_F\left(x, N_C^P(x)\right) \leq 0 \quad \text{and} \quad H_F\left(x, N_C^P(x)\right) \leq 0$$

for all $x \in C$.

There are other characterizations of invariance, but before we state those, we need to define other related concepts.

Definition 2.13. *The attainable set $\mathcal{A}(x_0;t)$ for $t \geq 0$ is the set of the all points of the form $x(t)$, where $x(\cdot)$ is any trajectory on $[0,t]$ satisfying $x(0) = x_0$.*

Definition 2.14. *The Bouligand (or contingent) tangent cone to a closed C at x, denoted $T_C^B(x)$, is defined as follows:*

$$T_C^B(x) := \left\{ \lim_{i\to\infty} \frac{x_i - x}{t_i} : x_i \xrightarrow{C} x, t_i \downarrow 0 \right\},$$

where $x_i \xrightarrow{C} x$ means that $x_i \in C$ for $i \in \mathbb{N}$, and that $\lim_{i\to\infty} x_i = x$.

This intuitive concept of tangency can be characterized by means of distance function that, $v \in T_C^B(x)$ if and only if

$$\liminf_{t\downarrow 0} \frac{d_C(x+tv)}{t} = 0.$$

For autonomous case, precisely the attainable set and these proximal concepts have relationship with variance conditions in the following equivalence theorems.

Theorem 2.15. *(Weak Invariance Case) The following are equivalent:*

(a) *$F(x) \cap T_C^B(x) \neq \varnothing$ for any $x \in C$.*

(b) *$F(x) \cap co\ T_C^B(x) \neq \varnothing$ for any $x \in C$.*

(c) *$h_F(x, N_C^P(x)) \leq 0$ for any $x \in C$.*

(d) *(F,C) is weakly invariant.*

(e) *For any $x \in C$, $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$ such that $\mathcal{A}(x_0;\delta) \cap C \neq \varnothing$.*

Theorem 2.16. *(Strong Invariance Case) Let F be locally Lipschitz. The following are equivalent:*

(a) *$F(x) \subseteq T_C^B(x)$ for any $x \in C$.*

(b) *$F(x) \subseteq co\ T_C^B(x)$ for any $x \in C$.*

(c) *$H_F(x, N_C^P(x)) \leq 0$ for any $x \in C$.*

(d) *(F,C) is strongly invariant.*

(e) *For any $x_0 \in C$, there exists $\varepsilon > 0$ such that $\mathcal{A}(x_0;\delta) \subseteq C$ for $t \in [0,\varepsilon]$.*

The proofs are contained in [8]. We will see the extension on impulsive system later. Actually after applying graph completion, impulsive system case can be handled as a non-impulsive case, and the invariance properties are obtained through the similar methods.

Chapter 3
Impulsive Systems and Their Solutions

Dynamical systems in which states may change rapidly with respect to different time scales leads to the mathematical model of an *impulsive system*. Examples of impulsive control could be generated, for example, from unavoidable noises, feedback adjustments, or externally required enforcements. Solutions (also called trajectories) can be defined in a variety of ways, and we choose the one that involves the technique of a graph completion. This is an additional piece of data which is used to deal with "jump" time-intervals to connect the discontinuous ends of the solution arc of bounded variation.

We review and modify in this chapter the solution concepts that were earlier developed in [26, 27]. Our new major contribution is the last section, where we introduce a new sampling method.

3.1 Introduction

The motivation to consider impulsive systems comes from the need to treat states that move at different scales. This idea integrates the effect of the "fast" movement occurring over a small time interval that on an infinitesimal scale is triggered by a point-mass measure. Meanwhile, traditional methods are also being applied on the "slow" movement as the usual time progression infinitesimally incremented by dt. We adopt the mathematical formalism introduced in [18, 19, 22], in which the control dynamic inclusion is the sum of a slow time velocity coming from a set $F(x)$ and a fast time velocity contributed by another set $G(x)d\mu$, where μ is a vector-valued measure. Technically, we formulate an impulsive system (called a *measure-driven* dynamical system in [19]) as follows,

$$
\begin{cases} dx \in F(x(t))dt + G(x(t))\mu(dt) \\ x(0-) = x_0, \end{cases} \tag{3.1}
$$

where $F(\cdot)$ and $G(\cdot)$ are multifunctions (set-valued maps) whose values, respectively, are subsets of $\mathbb{R}^n$ and $\mathcal{M}_{n\times m}$ (= the $n \times m$ matrices), and μ is a vector-valued measure with values in a close convex cone $K \subseteq \mathbb{R}^m$. The measure may have atoms (i.e. impulses) that may force the state trajectory $x(\cdot)$ to be discontinuous; $x(0-)$ refers to the initial state in a situation where 0 is an atom of the measure μ, and so that one may have the right hand limit $x(0+)$ different from the original starting point $x(0-)$. The definition in [19] of robust solution to (3.1) is an arc of bounded variation that satisfies the integral inclusion,

$$
\begin{cases} x(t) \in x_0 + \int_0^t F(x(\tau))d\tau + \int_{[0,t)} G(x(\tau))\mu(d\tau) \\ x(0-) = x_0. \end{cases} \tag{3.2}
$$

One may naturally expect the solution sets of (3.1) and (3.2) to coincide, however for both cases, the precise notion of solution needs further explanation. To see this, let us recall briefly the developing history in this field.

The pioneering study of impulsive system dates back to Rishel's work [15], in which the idea of handling impulses through graph completion (or called reparameterization) was raised. Warga [23] immediately extended this technique to a more general case. Then the key insight of *graph completion* was observed by Dal Maso and Rampazzo [9] to be necessary and effective on defining the multiplication of a point-mass measure with a state-dependent term. In the subtle view of graph completion, the graph of the distribution function $u(\cdot)$ of a measure μ was extended to a relation in graph space, which prescribes an arc to be influenced by the left and right hand limits of $u(\cdot)$ at the discontinuous points. The additional information in the extended time sector of graph is crucial since different graph completions can give rise to different solutions. The notation of a solution to a measure-driven *differential equation* (i.e. $F(\cdot)$ and $G(\cdot)$ are singleton-valued) in [9] is the first endeavor to define a solution of an auxiliary system that is reparameterized in time and depends on a given graph completion. The measure adopted in [9] is vector-valued, and is interpreted as the derivative of a control function of bounded variation, which is also the early exploration from this viewpoint.

As an application of vector-valued measure on impulsive systems, the idea of reparameterization solution is further constructed through setting control by Bressan, Rampazzo and others [3, 5, 4, 12, 14]. We also adopt some of these ideas in this thesis. The major improvement from a scalar-valued to a vector-valued measure is reflected on the prominent feature of graph completion to the latter. In the formulation of [19], the measure is scalar-valued and so the graph completion is a straight-line scalar completion of the distribution of μ, and the behavior of the trajectory during a jump is driven by a differential inclusion involving only G during a time interval with length equal to the magnitude of the measure's atom. In [3, 5, 26, 27], μ is vector-valued and the choice of a graph completion is incorporated into the definition of solution. This means the behavior during a jump depends on the particular graph completion.

We also need to mention related and independent work by Murray [13] who studied a proper extension of integral functionals from arcs of absolute continuity to ones of bounded variation. This approach can handle the dynamics system (3.1) by encoding them through the technique of infinite penalization. The extension requires the same type of graph completion of a vector-valued measure and the arcs to be bounded in variation, which are also employed by us here.

In [26], a related concept of solution, called *direct solution*, can be formulated directly from the differential form (3.1) of system by sorting the components of the decomposition of each measure into its absolutely continuous, continuous singular, and discrete parts (see [10], page 102). The direct solution is shown to be equivalent to the other solution concepts, and it enormously enriches tools to handle the impulsive part in systems. And it also provides insight into formulating invariance concepts, developing a Hamilton-Jacobi theory, and providing stability results. Some of these topics will be addressed in this thesis. The recent paper [27] tackles the two major issues (1) time discretization of, and (2) absolutely continuous approximation to (3.1), but here the measure is μ is fixed. These results inspire the approach of this thesis to construct solutions and impulsive measures that have desired properties.

Impulsive systems and graph completion are introduced precisely in Section 2, and more types of graph completion and their connections are built in section 3. Section 4 develops a sampling method to construct measure and approximate solutions of system (3.1).

3.2 Trajectories of Impulsive Systems

Consider the differential form (3.1) of an impulsive system. The following assumptions are in effect throughout the sequel, and are standard in this context.

(H1) A closed convex *pointed* cone $K \subset \mathbb{R}^m$, (where "pointed" is defined as $K \cap -K = \{0\}$);

(H2) A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with closed graph and convex values, and satisfying

$$f \in F(x) \qquad \Rightarrow \qquad \|f\| \leq c(1 + \|x\|) \qquad \forall x \in \mathbb{R}^n,$$

(where $c > 0$ is a given constant);

(H3) A multifunction $G : \mathbb{R}^n \rightrightarrows \mathcal{M}_{n\times m}$ (where $\mathcal{M}_{n\times m}$ denotes the $n \times m$ dimensional matrices with real entries) with closed graph and convex values, and satisfying

$$g \in G(x) \qquad \Rightarrow \qquad \|g\| \leq c(1 + \|x\|) \qquad \forall x \in \mathbb{R}^n.$$

The set of vector-valued Borel measures defined on the interval $[0, T] \subset \mathbb{R}$ with values in K is denoted by $\mathcal{B}_K([0, T])$. Suppose $\mu \in \mathcal{B}_K([0, T])$ is given.
The impulsive system (3.1) is a differential inclusion driven by the measure μ. If μ is absolutely continuous (with respect to Lebesgue Measure), then (3.1) is a non-impulsive system described by an ordinary inclusion, whose basic theory can be found, for example, in [2, 7, 20, 8].

A trajectory $x(\cdot)$ of (3.1) is a function of bounded variation, however, as previously mentioned, a further framework is required to define solutions so that the concpt is well-posed. The following notion of a graph completion was introduced in [3, 5].

Definition 3.1. *A graph completion of the distribution function* $u(\cdot) : [0, T] \to \mathbb{R}^m$ *of* μ*, given by* $u(t) = \mu([0, t])$*, is a Lipschitz continuous map* $(\psi_0, \psi) : [0, S] \to [0, T] \times \mathbb{R}^m$ *so that*

(GC1) $\psi_0(\cdot)$ *is non-decreasing;*

(GC2) *for every* $t \in [0, T]$*, there exists* $s \in [0, S]$ *so that* $(\psi_0(s), \psi(s)) = (t, u(t))$*; and*

(GC3) *for almost all* $s \in [0, S]$*,* $\dot{\psi}(s) \in K$*.*

A graph completion plays a fundamental role in the definition of a solution to an impulsive systems, for it pins down the behavior of the trajectory $x(\cdot)$ during the "jumps" of $u(\cdot)$ so that multiplication by $G(x)$ during this fast movement is unambiguous. The function ψ_0 is a remparameterized time variable. Since $\psi(\cdot)$ is defined to be Lipschitz, there exists a positive number r as the rank, such that

$$\|\dot{\psi}(s)\| \leq r. \tag{3.3}$$

Together with condition (GC3), the above implies

$$\dot{\psi}(s) \in K \cap B_r, \tag{3.4}$$

where B_r is a ball with radius r in $\mathbb{R}^m$. This restriction on $\psi(\cdot)$ is called *cone adherence*.

Suppose the measure $\mu \in \mathcal{B}_K[0, T]$ is given. We let $\mathcal{I}$ be the (at most countable) index set of atoms $\mathcal{T} = \{t_i\}_{i \in \mathcal{I}}$. Now consider a three-tuple

$$X_\mu := (x(\cdot), (\psi_0(\cdot), \psi(\cdot)), \{y_i(\cdot)\}_{i \in \mathcal{I}}) \tag{3.5}$$

with the following constituents: $x(\cdot) : [0, T] \to \mathbb{R}^n$ is of bounded variation with its points of discontinuity constrained in the set $\mathcal{T}$ of μ's atoms, $(\psi_0(\cdot), \psi(\cdot)) : [0, S] \to [0, T] \times \mathbb{R}^m$ is a graph completion of μ's distribution function $u(\cdot)$, and $\{y_i(\cdot)\}_{i \in \mathcal{I}}$ is a collection of Lipschitz functions with each defined on the nondegenerate interval $I_i := [s_i^-, s_i^+] := \psi_0^{-1}(t_i)$ and satisfying $y_i(s_i^\pm) = x(t_i\pm)$.

We see now the role a graph completion plays in defining the following *reparameterized solution* concept (or called *Bressan-Rampazzo (B-R) Solution*) of (3.1) with a slight modification of [5, 3].

Definition 3.2 (Reparameterized Solution)**.** *Consider a three-tuple X_μ as in* (3.5)*, and let*

$$y(s) = \begin{cases} x(t) & \text{if } s \notin \cup_{i\in\mathcal{I}} I_i, \quad t = \psi_0(s), \\ y_i(s) & \text{if } s \in I_i. \end{cases} \tag{3.6}$$

Then X_μ is a reparameterized solution of (3.1) *provided $y(\cdot)$ is Lipschitz on* $[0, S]$ *and satisfies*

$$\begin{cases} \dot{y}(s) & \in F(y(s))\dot{\psi}_0(s) + G(y(s))\dot{\psi}(s), \qquad a.e. \quad s \in [0, S], \\ y(0) & = x_0. \end{cases} \tag{3.7}$$

The second solution concept introduced in [26] requires trajectories stated more directly in the original time frame. Recall that an arc $x(\cdot)$ of bounded variation induces a measure dx that have decomposition of absolutely continuous, continuous singular, and discrete (i.e. purely atomic) parts, and so is written as

$$dx = \dot{x}(t)dt + dx_\sigma + dx_D,$$

where dx_σ is a singular continuous measure and $dx_D := \sum_{i\in\mathcal{I}} \delta^x_{t_i}$ is the discrete part with $\delta^x_{t_i}$ denoting the point mass jump of x at t_i, $x(t_i+) - x(t_i-)$. If $t = 0$ is an atom, then the initial condition is denoted by $x(0-)$. Correspondingly, the measure μ is decomposed into

$$\mu = \dot{u}dt + \mu_\sigma + \mu_D, \qquad \text{where } \mu_D = \sum_{i\in\mathcal{I}} \delta^u_{t_i}.$$

Definition 3.3 (Direct Solution). *The three-tuple* X_μ *in* (2.3) *is a direct solution of* (3.1) *provided*

(a) *for almost all* $t \in [0, T]$,

$$\begin{cases} \dot{x}(t) & \in F(x(t)) + G(x(t))\dot{u}(t), \\ x(0-) & = x_0; \end{cases}$$

(b) *there exists a bounded* μ_σ*-measurable selection* $\gamma \in G(x(t))$ *with*

$$dx_\sigma = \gamma(t)\mu_\sigma \qquad (\text{as a measure on } [0, T]); \text{ and}$$

(c) *the set of atoms of* dx *is* $\mathcal{T} = \{t_i\}_{i\in\mathcal{I}}$, *and for each* $i \in \mathcal{I}$, $y_i(s_i^-) = x(t_i-)$, $y_i(s_i^+) = x(t_i+)$, *and*

$$\dot{y}_i(s) \in G(y_i(s))\dot{\psi}(s) \qquad a.e. \quad s \in I_i.$$

The graph completion still play a fundamental role in the discrete part (c) of direct solution, and in effect circumscribes the fast velocities so that become available during jumps in t time. The direct solution concept gives us a clear view on the construction of measure and its relevant contribution to trajectory for each part. In [26], the two type of solutions are proved to be exactly equivalent, which inspires switches between graph completion and measure in many cases for convenience of illustration. We state the main theorem prove in [26].

Theorem 3.4. *Suppose* $\mu \in \mathcal{B}_K([0, T])$ *and* X_μ *is as in* (2.3). *Then* X_μ *is a reparameterized solution of* (3.1) *if and only if* X_μ *is a direct solution of* (3.1).

3.3 Graph Completions

It is convenient to standardize the graph completion in order to develop a sampling method of approximate solutions. The most natural choice of a graph completion is the following one called the *canonical* graph completion. With this choice replacing an original graph completion, we will also see that the property of being a solution of (3.1) does not change either. This will mean that the solution property does not depend on the choice of graph completion.

Definition 3.5. *A* canonical graph completion *of a distribution function* $u(\cdot) : [0,T] \to \mathbb{R}^m$ *of* μ, *given by* $u(t) = \mu([0,t])$, *is a Lipschitz continuous pair* $(\phi_0, \phi) : [0,S] \to [0,T] \times \mathbb{R}^m$ *so that*

(CG1) $\phi_0(\cdot)$ *is the filled-in inverse of* $\eta(t) := t + \|\mu\|([0,t])$, *which means that*

$$\phi_0(s) = t \qquad \textit{for} \quad \eta(t-) \leq s \leq \eta(t+); \tag{3.8}$$

(CG2) *For every* $t \in [0,T]$, *there exists* $s \in [0,S]$ *so that* $(\phi_0(s), \phi(s)) = (t, u(t))$; *and*

(CG3) *For almost all* $s \in [0,S]$, $\dot{\phi}(s) \in K$.

The canonical graph completion essentially fixes the choice of the temporal component, as it is given by (3.8); recall that $\eta(t-)$ and $\eta(t+)$ respectively denote the left-hand limit $\lim_{t \uparrow t_0} \eta(t)$ and the right-hand limit $\lim_{t \downarrow t_0} \eta(t)$. Observe that $\eta(t-) = \eta(t+)$ if and only if t is not an atom of μ. If 0 is an atom of μ, then $\eta(0-) = 0$ by convention.

Obviously, the canonical graph completion is one type of graph completion, since $\phi_0(\cdot)$ is a non-decreasing Lipschitz continuous function. We shall adopt the canonical graph completion to represent an impulsive solution, but there is no loss in generality, as we will see. The relationship between a general graph completion and the canonical one is described in the following lemma, which uncovers the rescaling between them without changing the solution.

Lemma 3.6. *Suppose measure* $\mu \in \mathcal{B}_K([0,T])$ *is given and a three tuple*

$$X_\mu := (x(\cdot), (\psi_0(\cdot), \psi(\cdot)), \{y_i(\cdot)\}_{i \in \mathcal{I}})$$

is a Bressan-Rampazzo solution of (3.1)*, where the pair* $(\psi_0, \psi) : [0, S] \to [0, T] \times \mathbb{R}^m$ *is a graph completion. Then, there exists* $\overline{S} > 0$ *and an absolutely continuous function* $\Psi : [0, \overline{S}] \to [0, S]$ *so that the pair* $(\phi_0, \phi) : [0, \overline{S}] \to [0, T] \times \mathbb{R}^m$,

$$(\phi_0, \phi)(s) := (\psi_0, \psi)(\Psi(s)) \tag{3.9}$$

is the canonical graph completion and there exists a Bressan-Rampazzo solution

$$\overline{X}_\mu := (\bar{x}(\cdot), (\phi_0(\cdot), \phi(\cdot)), \{\bar{y}_i(\cdot)\}_{i \in \mathcal{I}})$$

such that $\bar{x}(t) = x(t)$ *and* $\bar{y}_i(s) = y_i(\Psi(s))$.

Proof. Firstly, define an "inverse" $\Upsilon : [0, T] \to [0, S]$ of $\psi_0(\cdot)$ as $\Upsilon(t) = \psi_0^{-1}(t+)$. Let $\phi_0(\cdot)$ be given by (3.8), and let $\bar{I}_i := [\bar{s}_i^-, \bar{s}_i^+] := \phi_0^{-1}(t_i)$ for all $t_i \in \mathcal{T}$. The function $\Psi : [0, \overline{S}] \to [0, S]$ is constructed in the following way.

$$\Psi(s) := \begin{cases} \Phi_i(s), & \text{for } s \in \bar{I}_i, i \in \mathcal{I}, \\ (\Upsilon \circ \phi_0)(s), & \text{for } s \notin \cup_i \bar{I}_i, \end{cases}$$

where for each $i \in \mathcal{I}$, $\Phi_i(\cdot)$ is the linear function mapping $[\bar{s}_i^-, \bar{s}_i^+]$ to $[s_i^-, s_i^+]$ given by

$$\Phi_i(s) = s_i^+ \frac{s - \bar{s}_i^-}{\bar{s}_i^+ - \bar{s}_i^-} + s_i^- \frac{\bar{s}_i^+ - s}{\bar{s}_i^+ - \bar{s}_i^-}.$$

We see that $\Psi\{\cdot\}$ gives $\phi_0 = \psi_0 \circ \Psi$, since $\phi_0(s) = t_i = \psi_0(\Psi(s))$ when $s \in \bar{I}_i$, and $\phi_0(s) = \psi_0(\psi_0^{-1}(\phi_0(s))) = \psi_0(\Psi(s)) = \psi_0(\Upsilon(\phi_0(s)))$ when $s \notin \cup_i \bar{I}_i$. The spatial component is obtained by defining

$$\phi(s) := \psi(\Psi(s)).$$

With this definition of $\big(\phi_0(\cdot), \phi(\cdot)\big)$, we only need to show (CG3) holds to assert that the pair $(\phi_0(\cdot), \phi(\cdot))$ is a canonical graph completion. Actually condition (CG3) holds almost everywhere on [0,S] because

$$\dot{\phi}(s) = \dot{\psi}(\Psi(s))\dot{\Psi}(s) \in K.$$

Next, we need to prove that there exists a solution tuple $\overline{X}_\mu$ of (3.1) under the choice of graph completion $(\phi_0, \phi)(\cdot)$. In fact, from inclusion (2.5), there exist measurable selections $f(s) \in F(s)$ and $g(s) \in G(y(s))$ for any $s \in [0, S]$ so that

$$\dot{y}(s) = f(s)\dot{\psi}_0(s) + g(s)\dot{\psi}(s). \tag{3.10}$$

For almost $s \in [0, \overline{S}]$, let $\bar{y}(s) := y(\Psi(s))$, $\bar{x}(t) := \bar{y}(\eta(t))$ on $[0, T]$ and for all $i \in \mathcal{I}$, and $\bar{y}_i(s) = \bar{y}(s)$ on $\bar{I}_i$. Also let $\bar{f}(s) = f(\Psi(s))$ and $\bar{g}(s) = g(\Psi(s))$. Therefore,

$$\begin{aligned}
\dot{\bar{y}}(s) &= \dot{y}(\Psi(s))\dot{\Psi}(s) && \text{(the definition of } \bar{y}\text{)} \\
&= f(\Psi(s))\dot{\psi}_0(\Psi(s))\dot{\Psi}(s) + g(\Psi(s))\dot{\psi}(\Psi(s))\dot{\Psi}(s) && (3.10) \\
&= \bar{f}(s)\dot{\phi}_0 + \bar{g}(s)\dot{\phi}(\Psi(s)) && \text{(the definition of } \phi_0\text{)} \\
&\in F(\bar{y}(s))\dot{\phi}_0(s) + G(\bar{y}(s))\dot{\phi}(s).
\end{aligned}$$

We finish the proof by noting for all $t \in [0, T]$ and $s = \eta(t)\pm$, that

$$\bar{x}(t\pm) = y(\Psi(\eta(t\pm))) = y(\Psi(s)) = x(t\pm).$$

When $s \in \bar{I}_i$, we have

$$\bar{y}_i(s) = y_i(\Psi(s)) = y_i(\Psi(s)).$$

This shows that $\overline{X}_\mu$ is a B-R solution. □

This lemma suggests that the canonical graph completion can rescale the differential inclusion of system and still reserves its reparameterized solution that is originally defined by a general graph completion. Without loss of generality, usually we take $(\psi_0, \psi) = (\phi_0, \phi)$ and the three tuple

$$X_\mu := (x(\cdot), (\phi_0(\cdot), \phi(\cdot)), \{y_i(\cdot)\}_{i \in \mathcal{I}}) \tag{3.11}$$

which is a Bressan-Rampazzo solution of (3.1) with $(\phi_0, \phi) : [0, S] \to [0, T] \times \mathbb{R}^m$. And with no special indication, we always mean canonical graph completion as referring to graph completion throughout the remainder of thesis.

To inspire other concepts of graph completion, we use the following notation. Let $supp\ \mu_\sigma \subseteq [0, T]$ denote the closed support of μ_σ. Define

$$\Gamma := \eta(supp\ \mu_\sigma) \subseteq [0, S],$$

which has Lebesgue measure $\|\mu_\sigma\|$, and set

$$\widetilde{\Gamma} := \Gamma \bigcup (\cup_{i \in \mathcal{I}} I_i).$$

Then $\dot{\psi}_0(s) = 0$ *a.e.* $s \in \widetilde{\Gamma}$ and $\dot{\phi}_0 > 0$ *a.e.* $s \in [0, S] \backslash \widetilde{\Gamma}$. $\eta(\cdot)$ and $\phi_0(\cdot)$ are as in Definition 3.5.

Note that the discussion in (3.3) and (3.4) indicates a restrict constrain to the second component of graph completion $(\phi_0, \phi)(\cdot)$ that $|\dot{\phi}| \leq r$, where r is a positive number. Actually the graph completion is not uniquely determined for the diversity of spatial component. And in general, various $\phi(\cdot)$ produce various solution of (3.1) even when $F(\cdot)$ and $G(\cdot)$ are singleton-valued [3]. Therefore the current concepts of graph completions are not well-defined to determine a unique graph completion pair $(\phi_0, \phi)(\cdot)$ from (3.1) and then a solution. But we find that $\dot{\phi}(s) = r(s)k(s)$ for all $s \in [0, S]$, where $0 \leq r(s) \leq r$ and $k(s) \in K_1 := K \cap S_1$. The jump from $u(t_i-)$ to $u(t_i+)$ is discomposed into two parts with one controlling the direction of jump and the other one determining jump size. This idea points out a way to normalize the spatial term $\dot{\phi}(s)$ without changing system (3.1). And then a equivalent transformation of (2.7) is obtained,

$$\dot{y} \in F(y(s))\dot{\phi}_0(s) + G(y(s))r(s)k(s).$$

Furthermore, we can squeeze or stretch $r(s)$ so that $0 \leq r(s) \leq 1$ by rescaling $\phi_0(s)$ on $s \in [0, S]$.

Basing on the these thoughts, we offer another graph completion which will be used to construct sampling approximation to solutions.

Definition 3.7. *Normalized graph completion of distribution function* $u(\cdot) : [0, T] \to \mathbb{R}^m$ *of* μ*, given by* $u(t) = \mu([0, t])$*, is a Lipschitz continuous pair map* $(\varphi_0, \varphi) : [0, \overline{S}] \to [0, T] \times \mathbb{R}^m$ *so that*

(NG1) $0 \leq \dot{\varphi}_0 \leq 1$ *almost everywhere on* $[0, \overline{S}]$,

(NG2) *for every* $t \in [0, T]$ *there exists* $s \in [0, \overline{S}]$ *so that* $(\varphi_0(s), \varphi(s)) = (t, u(t))$ *and*

(NG3) $\dot{\varphi}(s) = (1 - \dot{\varphi}_0(s))k(s)$, *for almost all* $s \in [0, \overline{S}]$, *where* $k(s) \in K_1 = K \cap S_1$.

Clearly, normalized graph completion is one kind of graph completion. We can use it to reparameterize solution of (3.1). Naturely, using Lemma 3.6, a given normalized graph completion can be rescaled to a canonical graph completion. Inversely, the conclusion is summarized in the following lemma.

Lemma 3.8. *Suppose measure* $\mu \in \mathcal{B}_K([0, T])$ *is given and a three tuple*

$$X_\mu := (x(\cdot), (\phi_0(\cdot), \phi(\cdot)), \{y_i(\cdot)\}_{i \in \mathcal{I}})$$

is a Bressan-Rampazzo solution of (3.1)*, where the pair* $(\phi_0, \phi) : [0, S] \to [0, T] \times \mathbb{R}^m$ *is a canonical graph completion. Then, there exists* $\overline{S} > 0$ *and an absolutely continuous non-decreasing function* $\Phi : [0, S] \to [0, \overline{S}]$ *so that the pair* $(\varphi_0, \varphi) : [0, \overline{S}] \to [0, T] \times \mathbb{R}^m$,

$$(\varphi_0, \varphi)(s) := (\phi_0, \phi)(\Phi^{-1}(s)) \tag{3.12}$$

is the normalized graph completion and there exists a Bressan-Rampazzo solution

$$\overline{X}_\mu := (\bar{x}(\cdot), (\varphi_0(\cdot), \varphi(\cdot)), \{\bar{y}_i(\cdot)\}_{i\in\mathcal{I}})$$

such that $\bar{x}(t) = x(t)$ *and* $\bar{y}_i(s) = y_i(\Phi^{-1}(s))$.

Proof. Suppose that $(\phi_0, \phi)(\cdot)$ is a canonical graph completion. Define

$$\bar{k}(s) := \begin{cases} \frac{\dot{\phi}(s)}{1-\dot{\phi}_0(s)}, & \text{when } \dot{\phi}_0(s) \neq 1, \\ \frac{\dot{\phi}(s)}{\|\dot{\phi}(s)\|}, & \text{when } \dot{\phi}_0(s) = 1. \end{cases} \tag{3.13}$$

Note that if $\dot{\phi}_0(s) = 1$ for $s \in I$, an interval in $[0, S]$, then on interval $\phi_0(I)$, measure μ is inactive and its distribution is constant, i.e. $\dot{\phi}(s) = 0$ on I. Thus, for $s \in I$, $\dot{\phi}(s) = \bar{k}(s)(1 - \dot{\phi}_0)(s)$ trivially holds for both sides equal to 0. From (3.13), we conclude

$$\dot{\phi}(s) = \bar{k}(s)(1 - \dot{\phi}_0(s)) \tag{3.14}$$

almost everywhere on $[0, S]$.

Note $\dot{u}(t) = \dot{\phi}(\eta(t))$ for $t \notin \mathcal{T}$. For almost $s \in [0, S]\backslash\widetilde{\Gamma}$, $\dot{\phi}_0(s) > 0$ and

$$s - \phi_0(s) = \|\mu\|([0, \phi_0(s)]) = \int_0^s \|\dot{\phi}(s')\|ds',$$

which means $1 - \dot{\phi}_0(s) = \|\dot{\phi}(s)\|$ after taking derivative. This implies

$$\|\bar{k}(s)\| = \frac{\|\dot{\phi}(s)\|}{1 - \dot{\phi}_0(s)} = 1. \tag{3.15}$$

For almost all $s \in [0, S]\backslash\widetilde{\Gamma}$, $\dot{\phi}(s)$ belongs to the coK and

$$\bar{k}(s) = \frac{\dot{\phi}(s)}{1 - \dot{\phi}_0(s)} \in K, \tag{3.16}$$

whence $\bar{k}(s) \in K_1 = K \cap S_1$. For almost all $s \in \widetilde{\Gamma}$, $\dot{\phi}_0(s) = 0$ and

$$\bar{k}(s) = \dot{\phi}(s) \in \overline{B_r}. \tag{3.17}$$

Define function $\lambda(\cdot)$ on [0,S]

$$\lambda(s) = \begin{cases} 1 & \text{for } s \notin \cup_i I_i, \\ \|\dot{\phi}(s)\| & \text{for } s \in \cup_i I_i, \end{cases}$$

and let $\Phi : [0, S] \to [0, \overline{S}]$ be

$$\Phi(s) := \int_0^s \lambda(s')ds'.$$

Define

$$k(s) = \frac{\bar{k}(\Phi^{-1}(s))}{\|\bar{k}(\Phi^{-1}(s))\|},$$

$$\varphi_0(s) := \phi_0(\Phi^{-1}(s))$$

and

$$\varphi(s) := \phi(\Phi^{-1}(s)).$$

Almost everywhere on $[0, \overline{S}]$,

$$\dot{\varphi}_0(s) = \dot{\phi}_0(\Phi^{-1}(s))\frac{d\Phi^{-1}(s)}{ds} = \frac{\dot{\phi}_0(\Phi^{-1}(s))}{\|\dot{\phi}(\Phi^{-1}(s))\|}.$$

Then for almost $s \notin \cup_i \bar{I}_i$, where $\bar{I}_i := \varphi_0^{-1}(t_i)$, $t_i \in \mathcal{T}$,

$$\dot{\varphi}_0(s) = \dot{\phi}_0(\Phi^{-1}(s)),$$

and

$$\dot{\varphi}(s) = \dot{\phi}(\Phi^{-1}(s))\frac{d\Phi^{-1}(s)}{ds} = \frac{\dot{\phi}(\Phi^{-1}(s))}{\|\dot{\phi}(\Phi^{-1}(s))\|}.$$

Hence, for almost $s \notin \cup_i \bar{I}_i$, applying (3.15) and (3.16),

$$\dot{\varphi}(s) = \dot{\phi}(\Phi^{-1}(s)) = (1 - \dot{\phi}_0(\Phi^{-1}(s)))k(\Phi^{-1}(s)) = (1 - \dot{\varphi}_0(s))k(s).$$

For almost $s \in \cup_i \bar{I}_i$, using $\dot{\phi}_0(s) = 0$ together with (3.17), we also have

$$\dot{\varphi}(s) = \frac{\dot{\phi}(\Phi^{-1}(s))}{\|\dot{\phi}(\Phi^{-1}(s))\|} = \frac{\bar{k}(\Phi^{-1}(s))}{\|\bar{k}(\Phi^{-1}(s))\|} = (1 - \dot{\phi}_0(s))k(s).$$

So far, we show that the pair $(\varphi_0, \varphi)(\cdot)$ is the normalized graph completion to seek.

From inclusion (3.7), there exists a measurable selection $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ so that on $[0, S]$

$$\dot{y}(s) = f(s)\dot{\phi}_0(s) + g(s)\dot{\phi}(s).$$

For all $s \in [0, \overline{S}]$, let $\bar{y}(s) := y(\Phi^{-1}(s))$, let $\bar{x}(t) = \bar{y}(\eta(t))$ on $[0, T]$, and for all $i \in \mathcal{I}$ let $\bar{y}_i(s) = \bar{y}(s)$ on $\bar{I}_i$. Moreover, let $\bar{f}(s) = f(\Phi^{-1}(s))$ and $\bar{g}(s) = g(\Phi^{-1}(s))$. The proof completes for

$$\begin{aligned}
\dot{\bar{y}} &= \dot{y}(\Phi^{-1}(s))\frac{d}{ds}\Phi^{-1}(s) \\
&= [f(\Phi^{-1}(s))\dot{\phi}(\Phi^{-1}(s)) + g(\Phi^{-1}(s))\dot{\phi}(\Phi^{-1}(s))]\frac{d}{ds}\Phi^{-1}(s) \\
&= \bar{f}(s)\dot{\varphi}_0(s) + \bar{g}(s)\dot{\varphi}(s),
\end{aligned}$$

where $\bar{f}(s) \in F(\bar{y}(s))$ and $\bar{g}(s) \in G(\bar{y}(s))$. □

This lemma shows that a given graph completion associated with a solution of (3.1) can be normalized and still preserve the property of being a solution. In particular, the reparameterized time interval $[0, S]$ can be chosen to have the form

$$\dot{\varphi}(s) = (1 - \dot{\varphi}_0(s))k(s) \quad \text{with} \quad k(s) \in K_1,$$

and thus is decomposed into two factors, one labeling the direction of the jump and the other one determining its size. This will be convenient later in the sampling method that limits to a solution of (3.1).

3.4 A Sampling Method by Constructing Measure and Graph Completion

An Euler-type discretization procedure is introduced to approximate discrete solutions (also called *sample trajectories*) in [27], when the measure μ and a graph completion are given. A sequence of approximation is shown to graph-converge in the Hausdorff metric to some solution X_μ of (3.1) by the means of limit of a subsequence. The objective in this thesis is to offer a sampling method that construct the measure and graph completion, which also give us hint to design feedback controls for some purposes, such as invariance, controllability, and optimization.

Before that, we need to recall and extend Euler method in differential inclusion of impulsive system in which the control value set K is given supposedly.

With $S > 0$ and $x_0 \in C$ fixed, let $N > 0$ be an integer and let $h := \frac{S}{N}$ be the mesh size. Let $s_0^N = 0$ and $s_j^N = jh$ for $j = 1, 2, ..., N$. The sampled nodes $\{x_j^N\}_{j=0}^N$ are defined as follows, for the initial pieces, $x_0^N := x_0$ and

$$x_1^N := x_0^N + \lambda_0^N h f_0^N + (1 - \lambda_0^N) h g_0^N k_0^N$$

with

$$\lambda_0^N \in [0,1] \qquad f_0^N \in F(x_0^N) \qquad k_0^N \in K_1 \qquad g_0^N \in G(x_0^N).$$

Generally for $j = 1, 2, ..., N$, $x_j^N := x_{j-1}^N + \lambda_{j-1}^N h f_{j-1}^N + (1 - \lambda_{j-1}^N) h g_{j-1}^N k_{j-1}^N$ with

$$\lambda_{j-1}^N \in [0,1] \qquad f_{j-1}^N \in F(x_{j-1}^N) \qquad k_{j-1}^N \in K_1 \qquad g_{j-1}^N \in G(x_{j-1}^N).$$

We denote the graph of a sampled trajectory by Ω^Nas

$$\Omega^N := \{(s_j^N, x_j^N) : j = 0, 1, ..., N\}. \tag{3.18}$$

One trick needed to mention is that there exists a constant c_1 independent of N and j so that

$$\max_j \{\|x_j\|, \|f_j\|, \|g_j\|\} \le c_1 \tag{3.19}$$

for all j and $N \in \mathbb{N}$. Indeed, with c as in (H2) and (H3), we have

$$\begin{aligned} \|x_{j+1}\| &\leq \|x_j\| + h\|f_j\| + \|g_j\|h \\ &\leq \|x_j\| + 2c(1 + \|x_j\|)h \\ &= h\alpha + [1 + h\alpha]\|x_j\|, \end{aligned}$$

where $\alpha := 2c$. It follows from the discrete Gronwall inequality that

$$\|x_j\| \leq e^{\alpha S}(1 + \|x_0\|) - 1,$$

and then (3.19) holds by (H2) and (H3) with $c_1 := c[e^{\alpha S}(1 + \|x_0\|)]$.

$dist_{\mathcal{H}}(\cdot, \cdot)$ denotes Hausdorff distance between two compact sets, and the graph of function $y(\cdot) : [0, S] \to \mathbb{R}^N$ is denoted as

$$gr\ y = \{(s, y(s)) : s \in [0, S]\}.$$

The sampled trajectories approach to the solution of impulsive system by constructing graph completion and measure, which is realized by the following theorem.

Theorem 3.9. *Suppose that $S > 0$ and $x_0 \in C$are given. For every sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories, there exist a time length T, a solution X_μ as in* (3.11) *with some measure $\mu \in \mathcal{B}_K([0, T])$ and a sequence $\{\widetilde{\Omega}^{N_k}\}_{N_k}$ constructed from $\{\Omega^N\}_N$ so that*

$$dist_{\mathcal{H}}(\widetilde{\Omega}^{N_k}, gr\ y) \to 0 \quad as \quad k \to \infty,$$

where $y(\cdot)$ is defined as in three-tuple solution X_μ of (3.1).

Proof. Let $S > 0$ and $x_0 \in C$. For any integer $N \geq 0$, let $\Omega^N = \{(s_j^N, x_j^N) : j = 1, 2, ..., N\}$,

$$\lambda_j^N \in [0, 1], \qquad f_j^N \in F(x_j^N), \qquad k_j^N \in K_1, \text{ and } g_j^N \in G(x_j^N), \text{ for } j = 1, 2, ..., N,$$

which are chosen as aforementioned.

Define $\lambda^N(\cdot)$ on $[0,S]$ so that for every $N \in \mathbb{N}$

$$\lambda^N(s) := \lambda_j^N \text{ on } [s_{j-1}, s_j],\ j = 1,2,...,N.$$

Define T as

$$T := \limsup_{N\to\infty} \sum_{j=1}^{N} \lambda_j^N \frac{S}{N}.$$

Without loss of generality, there exists a selection of integer sequence $\{N_k\}_k$ such that

$$\sum_{j=1}^{N_k} \lambda_j^{N_k} \frac{S}{N_k} \to T \text{ as } N_k \to \infty.$$

And then

$$\sum_{j=1}^{N_k} (1-\lambda_j^{N_k})h \to S - T \text{ as } N_k \to \infty,$$

where $h^{N_k} = \frac{S}{N_k}$. Notice we also can rewrite the summation expression in integral form for convenience of discussing later:

$$T = \lim_{N_k\to\infty} \int_0^S \lambda^{N_k}(s')ds',$$

by the definition of $\lambda^N(\cdot)$.

Let $\mathcal{D}$ be denoted as $\mathcal{D} = \{Sq : q \text{ is a rational in } [0,1]\}$, which is a dense subset of $[0,S]$. We index all rational number in $[0,1]$ as $\{q_i\}_{i\in\mathbb{N}}$, where specially let $q_1 = 1$, (i.e. $q_1 S = S$). We construct $\varphi_0 : [0,S] \to [0,T]$ by the sampled trajectories as follows.

For $s_1 = q_1 S = S$, we use the subsequence $\{N_k\}_k$ to set $\varphi_0(s_1) = T$. For $s_2 = q_2 S$, similarly we have a sequence as follows,

$$\left\{ \int_0^{s_2} \lambda^{N_k}(s')ds' \right\}_k. \tag{3.20}$$

Without loss of generality, by passing to a subsequence of $\{N_k\}_k$, we suppose the sequence in (3.20) converges to the supremum value. Then we set

$$\varphi_0(s_2) = \lim_{N_k\to\infty} \int_0^{s_2} \lambda^{N_k}(s')ds'.$$

Repeating this process, we have a sequence $\{N_k\}_k$ without loss of generality so that

$$\varphi_0(s_i) = \lim_{N_k\to\infty} \int_0^{s_i} \lambda^{N_k}(s')ds', \quad \text{for } i \in \mathbb{N}.$$

So far, we have defined the $\varphi_0(s)$ for $s \in \mathcal{D}$. For any $s \in [0, S]$, coincidentally define

$$t = \varphi_0(s) := \lim_{s'\to s, s'\in\mathcal{D}} \varphi_0(s'). \tag{3.21}$$

We notice $\varphi_0(\cdot) : [0, S] \to [0, T]$ is Lipschitz of rank 1 and increasing, which means $0 \le \dot{\varphi}_0(s) \le 1$ for $s \in [0, S]$.

In fact, for any $s_1, s_2 \in \mathcal{D}$ with $s_1 \le s_2$,

$$\varphi_0(s_2) - \varphi_0(s_1) = \lim_{N_k\to\infty} \int_{s_1}^{s_2} \lambda^{N_k}(s')ds' \ge 0,$$

and

$$\lim_{N_k\to\infty} \int_{s_1}^{s_2} \lambda^{N_k}(s')ds' \le \int_{s_1}^{s_2} 1ds' = s_2 - s_1.$$

Generally for any $0 \le s_1 \le s_2 \le 1$, we have the same results by taking limits as (3.21), which indicates such defined $\varphi_0(\cdot)$ satisfying the condition of normalized complete graph.

Then we obtain the function $\lambda : [0, S] \to [0, 1]$ defined as $\lambda(s) := \dot{\varphi}_0(s)$ for $s \in [0, S]$. Let us focus on the following variational inclusion of impulsive system by using Euler method,

$$\dot{y}(s) \in \lambda(s)F(y(s)) + (1 - \lambda(s))G(y(s))K_1, \tag{3.22}$$

Given an integer $N > 0$, define the mesh size of each piece over $[0, T]$ by

$$h_j := \int_{(j-1)h}^{jh} \dot{\varphi}_0(s) ds,$$

where $h = S/N$. Then for all j, $h_j \leq h$. Let $s_0^N = 0$ and $s_j^N = jh$ for $j = 1, 2, ..., N$. The sampled nodes $\{x_j^N\}_{j=0}^N$ are defined as follows, for the initial pieces, $x_0^N := x_0$ and

$$x_1^N := x_0^N + \lambda_0^N h f_0^N + (1 - \lambda_0^N) h g_0^N k_0^N, \text{ where } \lambda_0^N = h_0/h \in [0, 1]$$

$$f_0^N \in F(x_0^N) \qquad k_0^N \in K_1 \qquad g_0^N \in G(x_0^N).$$

Generally for $j = 1, 2, ..., N$, $x_j^N := x_{j-1}^N + \lambda_{j-1}^N h f_{j-1}^N + (1 - \lambda_{j-1}^N) h g_{j-1}^N k_{j-1}^N$, where $\lambda_{j-1}^N = h_{j-1}/h \in [0, 1]$

$$f_{j-1}^N \in F(x_{j-1}^N) \qquad k_{j-1}^N \in K_1 \qquad g_{j-1}^N \in G(x_{j-1}^N).$$

We can construct Euler polygonal arc by these sampling points as follows,

$$y^N(s) := x_{j-1}^N + \frac{s - s_{j-1}}{h}(x_j^N - x_{j-1}^N), \text{ for } s \in [s_{j-1}^N, s_j^N],$$

$j = 1, 2, ..., N$.

Then a sequence of Euler polygonal $\{y^N(\cdot)\}_N$ is obtained. For each $N \in \mathbb{N}$, $\widetilde{\Omega}^N$ is the sampled trajectory in s-time:

$$\widetilde{\Omega}^N := \{(s_j, x_j) : j = 0, 1, ..., N\}.$$

Easily we have the following inequality, which is noted to be used later.

$$dist_{\mathcal{H}}(\widetilde{\Omega}^N, gr\ y^N(\cdot)) \leq \sqrt{2} max\{h, 2c_1 h\}. \tag{3.23}$$

Define $k^N(\cdot)$ on $[0, S]$ so that

$$k^N(s) := k_j^N \text{ on } [s_{j-1}, s_j], \quad j = 1, 2, ..., N.$$

Define the multifunction $M : [0, S] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$M(s, y) := \lambda(s)F(y) + (1 - \lambda(s))G(y)\overline{co}K_1, \tag{3.24}$$

where $0 \notin \overline{co}K_1$ for K is pointed, $M(\cdot, \cdot)$ is $\mathcal{L} \times \mathcal{B}$ measurable, has nonempty convex values, and has linear growth. Moreover, $M(s, \cdot)$ has closed graph for almost all $s \in [0, S]$. We claim there exist the sequences of positive numbers δ_N and r_N so that $\delta_N \to 0$ and $r_N \to 0$, where the limits take as $N \to \infty$, and that satisfy

$$inf\{\|\dot{y}^N(s) - v\| : v \in M(s, y^N(s) + \delta_N \overline{\mathbb{B}})\} \leq r_N \quad a.e. \quad s \in [0, S]. \tag{3.25}$$

To see this, let $\delta_N = 2c_1 \frac{S}{N}$ where c_1 is as in (3.19). Note for each $j = 0, 1, ..., N-1$ and $s \in [s_{j-1}^N, s_j^N]$ that

$$\begin{aligned} \|y^N(s) - x_j^N\| &\leq \|x_{j+1}^N - x_j^N\| \\ &= \|\lambda_j^N h f_j^N + (1 - \lambda_j^N) h g_j^N k_j^N\| \\ &\leq h[\|f_j^N\| + \|g_j^N\|] \\ &\leq \delta_N. \end{aligned}$$

Now let

$$v^N(s) := \lambda(s) f_j^N + (1 - \lambda(s)) g_j^N k^N(s), \tag{3.26}$$

and note that $v^N(s) \in M(s, x_j^N)$ for almost $s \in [s_j^N, s_{j+1}^N]$. Recall that for all $s \in [s_j^N, s_{j+1}^N]$,

$$\dot{y}^N(s) = \lambda_j^N f_j^N + (1 - \lambda_j^N) g_j^N k_j^N = \lambda^N(s) f_j^N + (1 - \lambda^N(s)) g_j^N k^N(s).$$

Thus

$$\begin{aligned} &\max_{s \in [0,S]} \|\dot{y}^N(s) - v^N(s)\| \\ \leq\ &\max_{s \in [s_j^N, s_{j+1}^N]} \|(\lambda^N(s) - \lambda(s)) f_j^N + (\lambda(s) - \lambda^N(s)) g_j^N k^N(s)\| \\ \leq\ &r_N, \end{aligned}$$

where r_N is the quantity defined as

$$r_N := 2c_1 \max_{s\in[0,S]} \{|\lambda^N(s) - \lambda(s)|\}$$

such that $r_N \to 0$, as $N \to \infty$. In fact, $\lambda^N(s) = \lambda_j^N = \frac{h_j}{h}$ over $[s_j, s_{j+1}]$, then by the definition of h_j and Lipschitz property of φ_0, easily we have $|\lambda(s) - \lambda^N(s)| < h = S/N$ on $[0, S]$. So far, we show that (3.25) holds.

From the compactness of trajectories theorem (Theorem 4.1.11 of [8]), there exists a trajectory $y(\cdot)$ of M and a subsequence $\{y^{N_k}(\cdot)\}_k$ of $\{y^N(\cdot)\}_N$ so that $y^{N_k}(\cdot) \to y(\cdot)$ uniformly on $[0, S]$. One sees easily this means

$$dist_{\mathcal{H}}(gr\ y^{N_k}(\cdot), gr\ y(\cdot)) \to 0, \quad as \quad k \to \infty. \tag{3.27}$$

And the compactness of trajectories theorem also implies multifunction selections f and g, and a function $\bar{k}(\cdot) : [0, S] \to \overline{co}K_1$ along as $\dot{y}^{N_k}$ converges to $\dot{y}$ weakly so that

$$\dot{y}(s) = \lambda(s)f(y(s)) + (1 - \lambda(s))g(y(s))\bar{k}(s).$$

Bundling the definition of $\varphi_0(\cdot)$, we get the pair $(\varphi_0, \varphi)(\cdot) : [0, S] \to [0, T] \times \mathbb{R}^m$, where $\varphi(s)$ is defined as

$$\varphi(s) := \int_0^s (1 - \lambda(s'))\bar{k}(s')ds',$$

and functions $\eta : [0, T] \to [0, S]$ and $u : [0, T] \to \mathbb{R}^m$ as

$$\eta(t) := \varphi_0^{-1}(t+), \qquad u(t) := \varphi(\eta(t)).$$

And let $\mu \in \mathcal{B}_K[0, T]$ such that $u(\cdot)$ is its distribution. The pair $(\varphi_0(\cdot), \varphi(\cdot))$ is a graph completion of measure μ according to Definition 2.1, since $\dot{\varphi}(s) = (1 - \dot{\varphi}_0(s))\bar{k}(s) \in K$. We define other components of a solution X_μ to (3.1) as follows. Let $x(\cdot) : [0,T] \to \mathbb{R}^n$ be given by $x(t) = y(\eta(t))$, and the functions $y_i(\cdot)$ for $i \in \mathcal{I}$ be as the restriction of $y(\cdot)$ to I_i.

By Lemma 3.6 and 3.8, we also can reparameterize the solution of (3.1) by a normalized graph completion. Without loss generality, we say, the normalized graph completion pair is $(\varphi_0(\cdot), \varphi(\cdot))$ defined on [0,S], in which S is also rescaled. The procedure to gain all components is similar as above, and the key point is to define $k(\cdot) : [0, S] \to K_1$ so that for any $s \in [0, S]$,

$$k(s) = \begin{cases} 0 & \text{when } \lambda(s) = 1, \\ \frac{\bar{k}(s)}{\|\bar{k}(s)\|} & \text{when } \lambda(s) \neq 1. \end{cases}$$

In the end, by the triangle inequality, one has

$$dist_{\mathcal{H}}(\widetilde{\Omega}^N, gr\ y(\cdot)) \leq dist_{\mathcal{H}}(\widetilde{\Omega}^N, gr\ y^N(\cdot)) + dist_{\mathcal{H}}(gr\ y^N(\cdot), gr\ y(\cdot)).$$

By passing to the subsequence $\{N_k\}$ and combining (3.23) and (3.27), we obtain the statement of theorem

$$dist_{\mathcal{H}}(\widetilde{\Omega}^{N_k}, gr\ y) \to 0 \quad as \quad k \to \infty,$$

where $\widetilde{\Omega}^{N_k}$ is a related sequence to Ω^N in some extension. Finally using the equivalence for solutions defined as in Definition 2.2 and Definition 2.3, such defined X_μ is exactly what we seek, which completes the proof. □

Usually, if measure is given, the trajectory obtained from (3.1) will not meet the dynamic need for some optimization purpose. The lemma introduced above suggests a way to construct a trajectory and simultaneously get a measure control. This method will be used widely and developed in invariance conditions and other classic optimization problems for impulsive case.

Chapter 4
Invariance Properties

Variance properties of trajectories satisfying the system represent its stability somehow by testing if trajectories remain in a given set. We have classical theory of dynamics system mostly focusing on the differential equation model. The goal of this chapter is to generalize the theory in differential inclusions and impulsive systems.

4.1 Weak Invariance

We now consider the invariance property of system (3.1) by tracking its trajectories along a close set $C \subseteq \mathbb{R}^n$. Weak invariance claims the existence of a characterized trajectory lying in C over all slow and fast times.

Definition 4.1. *Given $C \subseteq \mathbb{R}^n$ a closed set, the system is weak invariant on C if and only if for any $S > 0$ and $x_0 \in C$, there exist a time $T \in [0, S]$, a measure $\mu \in \mathcal{B}_K[0, T]$ and a three-tuple solution X_μ of the system such that $x(t) \in C$ for all $t \in [0, T]$ and for each fast time arc $\{y_i(\cdot)\}$, $y_i(s) \in C$ for all $s \in I_i$.*

Weak invariance actually represents the proximal character of impulsive system that pulls the trajectory back towards the inner of given set. Precisely the theorem is narrated as following with notation $K_1 := K \cap S_1$, where S_1 is the surface of unit ball. Utilizing the techniques developed in the sampled trajectories and approximation in Section 3.4, we are to prove the Weak Invariance Theorem.

Theorem 4.2. *The system* (3.1) *is weak invariant on a closed set C if and only if for each $x_0 \in C$ and $\zeta \in N_C^p(x_0)$, there exist $\lambda \in [0, 1]$ and $v \in F(x_0)\lambda + (1 -$*

$\lambda)G(x_0)K_1$ *so that*

$$\langle \zeta, v\rangle \leq 0.$$

Proof. ($\Longleftarrow$) Supposing for each $x_0 \in C$ and $\zeta \in N_C^p(x_0)$, there exist $\lambda \in [0,1]$ and $v \in F(x_0)\lambda + (1-\lambda)G(x_0)K_1$ so that

$$\langle \zeta, v\rangle \leq 0,$$

we need to show it implies the weak invariance of system (3.1) on closed set C. Let $S > 0, x_0 \in C$ and $N \in \mathbb{N}$. As the same sampling scheme introduced previously, we set h and $\{s_j\}$ and get the sequence $\{s_j, x_j\} : j = 0, 1, ..., N$, which satisfies the given condition such that

$$\text{for a } c(x_j) \in proj_C(x_j), \quad \langle \lambda_j h f_j + (1-\lambda_j) h g_j k_j, x_j - c(x_j)\rangle \leq 0, \tag{4.1}$$

where $\{\lambda_j\}$,$\{k_j\}$,$\{f_j\}$, and $\{g_j\}$ are also from the sampling scheme. By Theorem 3.9, there exists a measure $\mu \in \mathcal{B}_K([0,T])$ and solution X_μ of system (3.1) so that the graphes of sampled trajectories converge to the graph of $y(\cdot)$ in Hausdorff metric. We claim that $y(s) \in C$ for all $s \in [0, S]$, where $y(\cdot)$ is defined as in three-tuple solution of (3.1).

In fact, $x_0 \in C$,

$$d_C(x_1) \leq \|x_1 - x_0\| \leq \lambda_0 h\|f_0\| + (1-\lambda_0)\|g_0\|\|k_0\| \leq 2hc_1,$$

where c_1 is from (3.5). Moreover,

$$\begin{aligned}
d_C^2(x_2) &\leq \|x_2 - c(x_1)\|^2 \\
&= \|x_2 - x_1\|^2 + \|x_1 - c(x_1)\|^2 + 2\langle x_2 - x_1, x_1 - c(x_1)\rangle \\
&\leq 4h^2c_1^2 + d_C^2(x_1) + 2\int_{s_1}^{s_2} \langle \dot{y}^N(s), x_1 - c(x_1)\rangle ds \\
&\leq 8h^2c_1^2 + 2\int_{s_1}^{s_2} \langle \lambda_1 h f_1 + (1-\lambda_1)hg_1, x_1 - c(x_1)\rangle ds \\
&\leq 8h^2c_1^2,
\end{aligned}$$

where the last inequality is justified by (4.1). Generally,

$$d_C^2(x_j) \leq 4jh^2c_1^2 \leq 4Shc_1^2.$$

As $N \to \infty$, $h = S/N \to 0$. Meanwhile the nodes $\{x_j\}$ of sampling scheme converge to C, which implies $y(s) \in C$ on $[0, S]$ by applying Theorem 3.9. This direction of proof is completed.

($\Longrightarrow$) Suppose the system (3.1) is weak invariance on C. Let $x_0 \in C$, and let the three-tuple X_μ as in (3.5) be a solution that lies in C with $x(0) = x_0$ and where measure μ is constructed by the normalized graph completion $(\varphi_0, \varphi)(\cdot)$. For $\zeta \in N_C^P(x_0)$ there is a $\sigma > 0$ satisfying

$$\langle \zeta, x - x_0 \rangle \leq \sigma \|x - x_0\|^2, \tag{4.2}$$

for all $x \in C$ by Proposition 1.1.5 in [8].

Since $0 \leq \dot{\varphi}_0(s) \leq 1$, we have $\varphi_0(s) \leq s$. So there exists a sequence $\{s_j\}$ decreasing and converging to 0 and such that the following limit exists:

$$\lambda := \lim_{j \to +\infty} \frac{\varphi_0(s_j)}{s_j} = \dot{\varphi}_0(0).$$

If time $t = 0$ is an atom with $\eta(0+) = a > 0$ then $\lambda = 0$ and for a large j, $s_j \in \varphi_0^{-1} = [0, a]$. By the definition of weak variance, any trajectory $y(\cdot)$ corresponding to the solution X_μ satisfies $\dot{y}(s) \in G(y(s))\dot{\varphi}(s)$ and $\dot{\varphi}(s) \in K_1$ almost everywhere on $[0, a]$. Moreover,

$$\frac{y(s_j) - x_0}{s_j} = \frac{1}{s_j} \int_0^{s_j} \dot{y}(s) ds \in \frac{1}{s_j} \int_0^{s_j} G(y(s))\dot{\varphi}(s) ds \in G(x_0)K_1 + o(j),$$

where $o(j) \to 0$ as $j \to \infty$. Thus, $\frac{y(s_j) - x_0}{s_j}$ has at least one cluster point and by passing to a subsequence we can assume it is the only cluster point denoted by

v, which belongs to $G(x_0)K_1$. Since $y(s) \in C$ on $[0, \eta(T)]$, by using (4.2),

$$\begin{aligned} \langle \zeta, v \rangle &= \lim_{j\to\infty} \langle \zeta, \frac{y(s_j) - x_0}{s_j} \rangle \\ &\le \lim_{j\to\infty} \frac{\sigma}{s_j} \|y(s_j) - x_0\|^2 = 0, \end{aligned}$$

in which the last equality is obtained for $y(\cdot)$ is Lipschitz with rank 1 and $y(s_j) - x_0 \le s_j \to 0$ as $j \to \infty$.

Otherwise we suppose that time $t = 0$ is not an atom and let $t_j := \varphi_0(s_j)$. For all j, any trajectory $y(\cdot)$ corresponding to solution X_μ satisfies

$$\begin{aligned} \frac{y(s_j) - x_0}{s_j} &= \frac{1}{s_j}\int_0^{s_j} f(s)\dot{\varphi}_0(s)ds + \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\varphi}(s)ds \\ &= \frac{\varphi_0(s_j)}{s_j}\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt + \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\varphi}(s)ds, \end{aligned}$$

where $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ are selections and $\bar{f}(t) = f(\eta(t+))$ on $[0, t_j]$. And notice a property of normalized graph completion, $\dot{\varphi}(s) = k(s)(1 - \dot{\varphi}_0(s))$ with $k(s) \in K_1$. Thus

$$\begin{aligned} \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\varphi}(s)ds &= \frac{s_j - \varphi_0(s_j)}{s_j}\frac{1}{s_j - \varphi_0(s_j)}\int_0^{s_j} g(s)\dot{\varphi}(s)ds \\ &= \frac{s_j - \varphi_0(s_j)}{s_j}\frac{1}{s_j - \varphi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \varphi_0(s)). \end{aligned}$$

Then we get

$$\begin{aligned} \frac{y(s_j) - x_0}{s_j} &= \frac{\varphi_0(s_j)}{s_j}\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt \\ &+ \frac{s_j - \varphi_0(s_j)}{s_j}\frac{1}{s_j - \varphi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \varphi_0(s)). \end{aligned}$$

By the definition of λ, $\lim_{j\to+\infty} \frac{\varphi_0(s_j)}{s_j} = \lambda$ and $\lim_{j\to+\infty} \frac{s_j - \varphi_0(s_j)}{s_j} = (1 - \lambda)$. And since

$$\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt \in F(x_0) + o(j),$$

$$\frac{1}{s_j - \varphi_0(s_j)} \int_0^{s_j} g(s)k(s)d(s - \varphi_0(s)) \in G(x_0)K_1 + o(j),$$

we can find clustering points for the two parts,

$$\left\{ \frac{1}{t_j} \int_0^{t_j} \bar{f}(t)dt \right\} \qquad \text{and} \qquad \left\{ \frac{1}{s_j - \varphi_0(s_j)} \int_0^{s_j} g(s)k(s)d(s - \varphi_0(s)) \right\},$$

in $F(x_0)$ and $G(x_0)K_1$ respectively. By passing to subsequence, we can assume the convergence as

$$v := \lim_{j \to \infty} \frac{y(s_j) - x_0}{s_j} \in \lambda F(x_0) + (1 - \lambda)G(x_0)K_1.$$

Similarly, since $y(s) \in C$ on $[0, \eta(T)]$ and $y(\cdot)$ is Lipschitz with rank 1,

$$\begin{aligned} \langle \zeta, v \rangle &= \lim_{j \to \infty} \langle \zeta, \frac{y(s_j) - x_0}{s_j} \rangle \\ &\leq \lim_{j \to \infty} \frac{\sigma}{s_j} \|y(s_j) - x_0\|^2 = 0. \end{aligned}$$

So far, the proof of inverse direction is done. □

4.2 Strong Invariance

The system *(3.1)* is call *strongly invariant* if every trajectory remains in a given closed set C for all fast and slow times. Comparing with weak invariance property, both F and G are required to satisfy the additional assumption, *locally Lipschitz* (see Definition 2.11) with respect to the Hausdorff distance.

Definition 4.3. *The system* (3.1) *is called strong invariance on a closed set* $C \subseteq \mathbb{R}^n$ *if for every* $x_0 \in C$ *and any* $T > 0$*, all measure* $\mu \in \mathcal{B}_K[0, T]$ *and all corresponding three-tuple solution* X_μ *of system with* $x(0) = x_0$ *satisfy that* $x(t\pm) \in C$ *for all* $t \in [0, T]$ *and* $y_i(s) \in C$ *as* $s \in I_i$ *for each fast time arc* $\{y_i(\cdot)\}_i$.

We also have the similar result on the proximal characterization of strong invariance as follows.

Theorem 4.4. *The system (3.1) is strong invariant on a closed set C if and only if for each $x \in C$ and $\zeta \in N_C^P(x)$ we have*

$$\langle v, \zeta \rangle \leq 0 \tag{4.3}$$

for all $\lambda \in [0,1]$ and every $v \in \lambda F(x) + (1-\lambda)G(x)K_1$.

Proof. ($\Longrightarrow$) Suppose that system (3.1) is strongly invariant on a closed set C. Any arc $y(\cdot)$ (see definition 3.2) corresponding to solution X_μ that satisfies

$$\dot{y}(s) \in F(y(s))\dot{\varphi}_0(s) + G(y(s))\dot{\varphi}(s), \tag{4.4}$$

remains within the set C, where the pair $(\varphi_0, \varphi)(\cdot)$ is a normalized graph completion of an arbitrary measure in $\mathcal{B}_K([0,T])$. For any fixed $x \in C$, let λ be any number in $[0,1]$ and let v be an arbitrary element in $\lambda F(x) + (1-\lambda)G(x)K_1$, or namely say $v := \lambda f + (1-\lambda)gk$ for some $f \in F(x)$, $g \in G(x)$ and $k \in K_1$ respectively. We need to show the inequality (4.3) holds.

For any y, we define $\bar{v}(y)$ to be the closest point to $v = \lambda f + (1-\lambda)gk$ in $\lambda F(y) + (1-\lambda)G(y)k$. Easily see that $\bar{v}(x) = v$ and the multifunction $\lambda F + (1-\lambda)Gk$ is locally Lipschitz for both F and G are locally Lipschitz. It is also implied that the single-valued multifunction $\mathcal{V}(y) = \{\bar{v}(y)\}$ holds the properties of close graph, convex values and linear growth inherited from F and G. For $S = 1$, consider the measure $\mu \in \mathcal{B}_K([0,\lambda])$ so that the pair $\varphi_0(s) := \lambda s$ and $\varphi(s) := (1-\lambda)ks$ represents a normalized graph completion with this measure on $[0,1]$. With such choice of measure μ, the strongly invariant system (4.4) can be transformed as follows,

$$\dot{y} \in \lambda F + (1-\lambda)Gk. \tag{4.5}$$

Obviously, strongly invariant is weakly invariant, therefore the system (4.5) is weakly invariant also. We can prove function $\bar{v}(y)$ is a continuous selection of $\lambda F(y)+(1-\lambda)G(y)k$. (In fact, easily we can see locally Lipschitz multifunctions F, G are Lipschitz on bounded set $y+B\subset\mathbb{R}^n$. Then for $y_1, y_2\in y+B$, $\|\bar{v}(y_1)-\bar{v}(y_2)\|\leq\lambda L_1\|y_1-y_2\|+(1-\lambda)L_2\|y_1-y_2\|\leq\max\{L_1,L_2\}\|y_1-y_2\|$, where L_1, L_2 supposedly are Lipschitz ranks of F and G on $y+B$ respectively.) Therefore, the system $\dot{y}\in\mathcal{V}(y)$ is also weakly invariant, which means for the point $x\in C$ picked early, and $v=\bar{v}\in\mathcal{V}(x)$ we get

$$\langle v,\zeta\rangle\leq 0,\qquad\text{for all }\zeta\in N_C^P(x). \tag{4.6}$$

($\Longleftarrow$) Now take $T\geq 0$ and $\mu\in\mathcal{B}_K([0,T])$ arbitrarily. Supposedly, for each $x\in C$ and $\zeta\in N_C^P(x)$, we have $\langle v,\zeta\rangle\leq 0$ for all $\lambda\in[0,1]$ and for every $v\in\lambda F(x)+(1-\lambda)G(x)K_1$. We need to show the system (3.1) is strongly invariant on C. For any $x_0\in C$, let the three tuple X_μ as (3.5) be a solution of (3.1) with $x(0-)=x_0$. The given condition implies that for all $y\in C$,

$$\max\langle v,\zeta\rangle\leq 0,\qquad\forall\zeta\in N_C^P(y). \tag{4.7}$$

Consider the differential inclusion of system by the normalized graph completion:

$$\dot{y}\in F(y)\dot{\varphi}_0(s)+(1-\dot{\varphi}_0(s))G(y)K_1, \tag{4.8}$$

where $y(\cdot)$ is associated to X_μ and defined as (3.6). Define multifunction $M:[0,S]\times\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ by

$$M(s,y):=F(y)\dot{\varphi}_0(s)+(1-\dot{\varphi}_0(s))G(y)K_1, \tag{4.9}$$

where $S:=\varphi_0^{-1}(T+)$. Given an arbitrary arc $y(\cdot)$ of system (3.1), there exists a selection $\bar{f}$ of M such that $\dot{y}=\bar{f}(s,y)$, $y(s)=x_0$.

Let $m > 0$ be such that any $y(\cdot)$ given above satisfies $\|y(t) - x_0\| < m$, for $s \in [0, S]$. If y lies in $x_0 + mB$ and $c \in proj_C(y)$, then

$$\|c - x_0\| \leq \|c - y\| + \|y - x_0\| \leq 2\|y - x_0\|,$$

so that $c \in x_0 + 2mB$.

Since F and G are local Lipschitz, easily we see M also be local Lipschitz. Now let L be a Lipschitz constant for M on $x_0 + 2mB$, and consider any $y \in x_0 + mB$ and $c \in proj_C(y)$. Then $y - c \in N_C^P(c)$. Since $\bar{f}(s, y) \in M(s, y)$, there exists $v \in M(s, c)$ such that $\|v - \bar{f}(s, y)\| \leq L\|c - y\| = Ld_C(y)$. By the given condition, we have $\langle v, y - c\rangle \leq 0$. We deduce

$$\langle \bar{f}(s, y), y - c\rangle \leq Ld_C(y)^2.$$

Then for any $0 \leq \tau < s \leq S$,

$$\begin{aligned}
d_C^2(y(s)) &\leq \|y(s) - c(\tau)\|^2 \\
&= \|y(\tau) - c(\tau)\|^2 + \|y(s) - y(\tau)\|^2 + 2\langle y(s) - y(\tau), y(\tau) - c(\tau)\rangle \\
&\leq d_C^2(y(\tau)) + 2\int_\tau^s \langle \dot{y}(r), y(\tau) - c(\tau)\rangle dr \\
&\leq d_C^2(y(\tau)) + 2\int_\tau^s \langle \bar{f}(r, y(r)), y(\tau) - c(\tau)\rangle dr.
\end{aligned}$$

We conclude that

$$d_C^2(y(s)) - d_C^2(y(\tau)) \leq 2\int_\tau^s \langle \bar{f}(r, y(r)), y(\tau) - c(\tau)\rangle dr.$$

With both sides above divided by $s - \tau$, and taking limit $s - \tau \to 0$, we get

$$\begin{aligned}
\frac{d}{dt} d_C^2(y(s)) &\leq 2\langle \bar{f}(s, y(s)), y(s) - c(s)\rangle \\
&\leq 2Ld_C^2(y(s)).
\end{aligned}$$

Thus we obtain

$$\frac{d}{ds} d_C(y(s)) \leq Ld_C(y(s)), \qquad s \in [0, S], \qquad d_C(y(0)) = 0,$$

which implies that $d_C(y(s)) = 0$ by Gronwall inequality. Thus

$$y(s) \in C \quad \text{for all } s \in [0, S].$$

In other words, $x(t\pm) \in C$ for all $t \in [0, T]$ and each fast time arc $\{y_i(\cdot)\}$ satisfies $y_i(s) \in C$ for all $s \in I_i$. By the arbitrary selection of T, μ and trajectory X_μ of system (3.1), the strong invariance of system (3.1) is proved. $\square$

Remark. This theorem is completely consistent with non-impulsive case. In fact, for $G(x) = 0$, we see the condition (4.3) holds for every $\lambda \in [0, 1]$ and every $v \in \lambda F(x)$. That means for all x a selection $f(x) \in F(x)$ and $0 \leq \lambda \leq 1$, so that

$$\langle v, \zeta \rangle = \langle \lambda f, \zeta \rangle \leq \langle f, \zeta \rangle \leq 0,$$

which is a well-known invariance condition for the non-impulsive case. Inversely, if we have $\langle f, \zeta \rangle \leq 0$ for any $f \in F$, then multiplying $\lambda \leq [0, 1]$ on both sides of this inequality, we get the condition (4.3).

Chapter 5
Hamilton-Jacobi Theory

In this chapter, the goal is to prove that minimal time function is the unique proximal solution to the Hamilton-Jacobi (HJ) equation, which is a direct development of autonomous differential inclusion system [24].

5.1 Minimal Time Function

In this section, we extend the concept of minimal time function from autonomous non-impulsive systems to impulsive ones. The minimal time problem consists of a given closed set C (the "target set") and a control system in which the goal is to steer an initial state x to the target set along a trajectory in minimal time.

Given a measure $\mu \in \mathcal{B}_K([0,T])$, $(\phi_0, \phi)(\cdot)$ is a canonical graph completion of system (3.1). $y(\cdot)$ associated to a solution tuple X_μ is called generalized trajectory of (3.1), satisfying

$$\dot{y} \in F(y)\dot{\phi}_0(s) + G(y)\dot{\phi}(s), \tag{5.1}$$

Correspondingly, variable $s \in [0, S]$ is named as generalized time variable, and the multifunction $M(s,y)$ defined as the following is the generalized differential multifunction,

$$M(s,y) := F(y)\dot{\phi}_0(s) + G(y)\dot{\phi}(s), \tag{5.2}$$

We will discuss the minimal time function on means of generalization unless otherwise stated.

Suppose closed set $C \subset \mathbb{R}^n$ is the target. The minimal time function $\widetilde{T}_C : \mathbb{R}^n \to [0, \infty]$ is defined as follows. If $x \notin C$, then

$$(5.3) \qquad \begin{aligned} \widetilde{T}_C(x) := inf\{S : & \text{ there exists } y(\cdot) \text{ satisfying (5.2)} \\ & \text{with } y(0) = x \text{ and } y(S) \in C\}. \end{aligned}$$

If no trajectory of M originating from x can reach C in finite time, then the above infimum is taken over empty set, and write $\widetilde{T}_C(x) = \infty$ as convention in this case. If $x \in C$, then $\widetilde{T}_C(x) = 0$ by definition, which is consistent with the definition above if the trajectories are allowed to be defined on the degenerate interval $[0, 0]$.

Suppose $U \subseteq \mathbb{R}^n$ is open and $x \in U$. As S is called *escape time* from U, it is defined and written as $S := Esc(y(\cdot); U)$. The set of all trajectories of M originating from x that remain in U over a maximal interval is denoted by $\Upsilon_{(M,U)}(x)$. That is, $\Upsilon_{(M,U)}(x)$ is composed by those trajectories $y(\cdot)$ of M defined on a half-open interval $[0, S)$ with $y(0) = x$ and $S = Esc(y(\cdot); U)$. The set of endpoints of all trajectories of M is denoted by $R_M^{(S)}(x)$ and is called the *reachable set* (from x and at time S). That is, $R_M^{(S)}(x) := \{y(S) : y(\cdot) \text{ is a trajectory of } M \text{ satisfying } y(0) = x\}$. The notation $R_M^{(\leq S)}$ denotes the set all points reachable from x at a time less than or equal to S.

It turns out that (H1)-(H3) are not sufficient to give many of desired properties of $\widetilde{T}_C(\cdot)$. Actually the following useful assumption is needed to exclude some weird trajectories into our discussion. This assumption is not set merely on F, but both F and C.

(H4) For all $x \in C$ and $y(\cdot) \in \Upsilon_{(M,\mathbb{R}^n)}(x)$, if

$$Esc(y(\cdot); \mathbb{R}^n) < \infty$$

then

$$Esc(\bar{y}(\cdot); C^c) < Esc(y(\cdot); \mathbb{R}^n),$$

where $\bar{y}(\cdot)$ is a restriction of $y(\cdot)$ on C^c.

Roughly speaking, if (H4) holds, any trajectory of M escaping to infinity in finite time must pass through C.

The following two facts of $\widetilde{T}_C(x)$ are immediate consequences of the definition (5.3).

$$\widetilde{T}_C(x) = inf\{S \geq 0 : R_M^{(S)}(x) \cap C \neq \varnothing\},$$

and if assumptions (H1)-(H3) hold and $x \notin C$,

$$\widetilde{T}_C(x) = inf\{Esc(y(\cdot); C^c) : y(\cdot) \in \Upsilon_{(F,C^c)}(x)\}.$$

To get more characterizations of the minimal time function, we need to recall the following basic definitions and do some preparation in theory. The *epigraph* of function $f : X \to \mathbb{R}$ is given by

$$epi\ f := \{(x, y) \in dom\ f \times \mathbb{R} : y \geq f(x)\}.$$

A function $\theta : X \to \mathbb{R}$ is *lower semicontinuous* at x provided that

$$\theta(x) \leq \liminf_{x' \to x} \theta(x').$$

Suppose $\theta : \mathbb{R}^n \to (-\infty, \infty]$ is lower semicontinuous and $x \in dom\ \theta := \{x' : \theta(x') < \infty\}$. A vector $\xi \in \mathbb{R}^n$ is a *proximal subgradient* of θ at x provided $(\xi, -1) \in N^P_{epi\ \theta}(x, \theta(x))$. The set(which could be empty) of all proximal subgradients of $\theta(\cdot)$ at x is denoted by $\partial_P \theta(x)$. If $x \notin dom\ \theta$, then $\partial_P \theta(x) = \varnothing$ by definition.

Proposition 5.1. *Suppose multifunctions F and G and set K satisfy* (H1)-(H3).

(a) *For each compact set $U \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exists $0 < \tau \leq S$ such that*

$$R_M^{(\leq\tau)}(U) := \bigcup_{x \in U} R_M^{(\leq\tau)}(x) \subset U + \varepsilon B.$$

(b) *If $y(\cdot)$ is a trajectory of M on $[0, S)$ originating from x with $S < \infty$ and satisfies*

$$\liminf_{s \uparrow S} \|y(s)\| < \infty,$$

then the limits of $y(s)$ exists as $s \uparrow S$.

(c) *There exists $S > 0$ such that* (5.1) *admits at least one solution.*

Proof. For an arbitrary number $\varepsilon > 0$, define $k := sup\{\|v\| : v \in M(s, U + \varepsilon B)$, where $s \in \mathbb{R}\}$. Indeed, the supremum value can be obtained, since U is compact, $|\dot{\phi}_0(\cdot)| \leq 1$ and $\|\dot{\phi}(\cdot)\| \leq r$ (where r is as in (3.3)). Let $\tau = \varepsilon/k$. For any $x \in U$, let $y(\cdot)$ be a trajectory of M such that $y(0) = x$, and define $s_0 := sup\{s : 0 \leq s \leq \tau$ with $y(s') \in x + \varepsilon B$ over $[0, s]\}$. Note that $s_0 > 0$ since $y(\cdot)$ is continuous. If $0 < s \leq s_0$, then $y(s) \in x + \varepsilon B \subseteq U + \varepsilon B$, and hence $\|\dot{y}(s)\| \leq k$ a.e. $s \in [0, s_0]$. Therefore, for $s \in [0, s_0]$,

$$\|y(s) - x\| = \left\| \int_0^{s_0} \dot{y}(s) \right\| ds \leq \int_0^{s_0} \|\dot{y}(s)\| \, ds \leq k s_0 \leq k\tau \leq \varepsilon. \tag{5.4}$$

Actually $s_0 = \tau$ by the above inequalities for the following reasoning: if one of inequalities above is strict, then for $s \in [0, s_0]$, $y(s) \in x + \varepsilon B$. By the definition of s_0 and the continuity of $y(\cdot)$, we have $s_0 = \tau$. Otherwise, all equalities also show $s_0 = \tau$.

Since the trajectory $y(\cdot)$ is arbitrary originating from x, it follows that

$$R_M^{(\leq\tau)}(x) \subset U + \varepsilon B.$$

With x taken arbitrarily over U, the statement of (a) holds naturally.

(b) Let $y(\cdot)$ be a trajectory satisfying the condition given. Obviously, there is a compact set U such that $y(s) \in U$ as $s \uparrow S$ always. By the discussion of (a), k defined above can be obtained so that $k = sup\{\|v\| : v \in M(s, U + B)$, where $s \in \mathbb{R}\}$ (with $\varepsilon = 1$). Accordingly, choose τ as (a). Choose s_0 such that $\max\{S - \tau, 0\} \leq s_0 < S$ and $y(s_0) \in U$, and by part (a) it follows that $y(s) \in U + B$ for all $s \in [s_0, S)$. By the choice of k, we have for all $s_0 \leq s < s' \leq S$ that

$$\|y(s') - y(s)\| \leq \int_s^{s'} \|\dot{y}(\tau)\| \, d\tau \leq k(s' - s). \tag{5.5}$$

Since S is assumed to be finite, it follows from (5.5) that the limit of $y(s)$ as $s \uparrow S$ is bounded, and so exists for the continuity of trajectory y.

(c) This is a direct conclusion from Theorem 2.6(c) with a selection $f \in F$ and $g \in G$ such that $f(y)\dot{\phi}_0(s) + g(y)\dot{\phi}(s)$ is continuous. □

Proposition 5.2. *Suppose set K and multifunctions F and G satisfy* (H1)-(H3) *and* (H4). *If $x \in C^c \cap dom\widetilde{T}_C$, then there exists $y(\cdot) \in \Upsilon_{(F,C^c)}(x)$ with $Esc(y(\cdot), C^c) = \widetilde{T}_C(x)$ and $y(\widetilde{T}_C(x)) \in C$. That is, the infimum in* (5.3) *is attainable for some $y(\cdot)$. Furthermore, $\widetilde{T}_C(\cdot)$ is lower semi-continuous on $\mathbb{R}^n$.*

Proof. Suppose $x \in C^c \cap dom\widetilde{T}_C$, which also means $x \notin C$ and $\widetilde{T}_C(x) < \infty$. Let $\{y_i(\cdot)\}$ be a minimizing sequence of (5.3), so we have $y_i(\cdot) \in \Upsilon_{(M,C^c)}(x)$ and $S_i := Esc(y_i(\cdot); C^c) \to \widetilde{T}_C(x)$ as $i \to \infty$ and $y_i(S_i) \in C$ for all i. Let

$$S := \inf\{s \in [0, \widetilde{T}_C(x)] : \limsup_{i \to \infty} \|y_i(s)\| = \infty\},$$

If the limsup is always finite, then we take $S = \widetilde{T}_C(x)$ by convention. By Proposition 5.1(a), we note that $S > 0$, and for any $s < S$, the sequence $y_i(\cdot)$ is uniformly bounded on the interval $[0,s]$ with $y_i(0) = x$ for all i. By using a diagonal process, there are a subsequence $\{y_{i_j}(\cdot)\}$ and its pointwise convergent function $\bar{y}(\cdot)$ so that $y_{i_j}(s') = \bar{y}(s') + \varepsilon_j(s')$ on $[0,s]$, where $\varepsilon_j(\cdot)$ converges to 0 in $L^2([0,s])$. Then by the compactness of trajectories theorem (Theorem 2.7), without loss of generality, we may say that $y_i(\cdot)$ converges uniformly to a trajectory $y(\cdot)$ on each compact interval of $[0,T)$. Since $S \le \widetilde{T}_C(x)$, we have

$$\liminf_{s\uparrow S} \|y(s)\| < \infty,$$

since otherwise $Esc(y(\cdot);\mathbb{R}^n) = S < \infty$ and (H4) would be violated for

$$Esc(y(\cdot);C^c) = Esc(y(\cdot);\mathbb{R}^n) = S.$$

Hence, by Proposition 5.1(b), the limit $\lim_{s\uparrow S} y(s) =: y(S)$ exists.

To prove that the infimum in (5.3) is attainable, we need to show $y(S) \in C$. We first claim $S = \widetilde{T}_C(x)$. Indeed, let $U := \{y(s) : s \in [0,S]\}$ and choose τ as in Proposition 5.1(a) associated to the compact set $U + B$ and $\varepsilon = 1$. If $S < \widetilde{T}_C(x)$, then there exist s_0 and $0 < \tau_0 \le \tau$ so that

$$0 < s_0 < S < s_0 + \tau_0 \le \widetilde{T}_C(x).$$

Since $y_i(s_0) \to x(s_0)$ as $i \to \infty$, we have $y_i(s_0) \in U + B$ for all large i, and by Proposition 5.1(a), it follows that $y_i(s_0 + \tau_0) \in U + 2B$ for all large i. However, the definition of T as an infimum promises that $\limsup_{i\to\infty} \|x_i(s_0 + \tau_0)\| = \infty$, a contradiction. Hence, $S = \widetilde{T}_C(x)$ as claimed.

Next to show that $y(S) \in C$, let k be as in proof of Proposition 5.1(a) associated to the compact set $U + 2B$ and $\varepsilon = 2$. Now let $\eta > 0$ be small enough and choose s_1 such that

$$S - \min\{\tau, \eta/k\} < s_1 < S \quad \text{and} \tag{5.6}$$

$$\|y(S) - y(s_1)\| < \eta. \tag{5.7}$$

We now choose i large enough such that

$$\|y(s_1) - y_i(s_1)\| < \eta \quad \text{and} \tag{5.8}$$

$$(S_i - s_i) < \max\{\tau, \eta/k\}. \tag{5.9}$$

Note that (5.9) is reasonable by (5.6) and the fact $S_i \to \widetilde{T}_C(x) = S$. Moreover, $y_i(s_1) \in U + B$ naturally means $y_i(s_1) \in U + 2B$ for $s \in [s_1, T_i]$, and consequently

$$\|\dot{y}_i(s)\| \leq k \quad a.e. \quad s \in [s_1, T_i]. \tag{5.10}$$

Since $y_i(S_i) \in C$, we have

$$\begin{aligned}
d_C(y(S)) &\leq \|y(S) - y_i(S_i)\| \\
&\leq \|y(S) - y(s_1)\| + \|y(s_1) - y_i(s_1)\| + \|y_i(s_1) - y_i(S_i)\| \\
&\leq 2\eta + \int_{s_1}^{S_i} \|\dot{y}_i(s)\| \, ds \\
&\leq 2\eta + k(S_i - s_1) \\
&\leq 3\eta,
\end{aligned}$$

where the third inequality holds by (5.7) and (5.8), the forth one by (5.10), and the last one by (5.9). Letting $\eta \to 0$, we see $y(S) \in C$.

To prove lower semicontinuity, suppose $x_i \to x$, and we may assume without loss of generality that $\widetilde{T}_C(x_i)$ converges, namely to $S < \infty$. For each $i \in \mathbb{N}$, let $y_i(\cdot) \in \Upsilon_{(M,C^c)}(x_i)$ satisfy $Esc(y_i(\cdot); C^c) = \widetilde{T}_C(x_i)$ and $y_i(\widetilde{T}_C(x_i)) \in C$, which are promised by conclusion above. We also can produce a trajectory $y(\cdot) \in \Upsilon_{(M,C^c)}(x)$ so that $Esc(y(\cdot); C^c) \leq S$ and $y(S) \in C$ along the process completely analogous to the proof above. (The only difference between here and the above is the initial value $y_i(0) = x_i$ of the trajectory $y_i(\cdot)$, but the estimates still work.) Since $\widetilde{T}_C(x)$ is defined as an infimum, we have $\widetilde{T}_C(x) \leq Esc(y(\cdot); C^c) \leq S = \lim_{i\to\infty} \widetilde{T}_C(x_i)$, which proves that $\widetilde{T}_C(\cdot)$ is lower semicontinuous. $\square$

5.2 Invariance

In this section, the connection between minimal time function $\widetilde{T}_C$ and invariance is obtained through certain lower semi-continuous function θ. The invariance defined anew reflects another feature, which is connected to previous concept of target set, but more emphasized in the means of modifying the given data.

Definition 5.3. *Suppose $E \subseteq \mathbb{R}^n$ is nonempty, $U \subseteq \mathbb{R}^n$ is open, and $M : \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction.*

(a) *(M, E) is called weakly invariant in U provided that for all $y \in E \cap U$, there exists a trajectory $y(\cdot) \in \Upsilon_{(M,U)}(y)$ that satisfies $y(s) \in E$ for all $s \in [0, Esc(y(\cdot); U))$.*

(b) *(M, E) is strongly invariant in U provided that for every $y \in E \cap U$, every trajectory $y(\cdot) \in \Upsilon_{(M,U)}(y)$ satisfies $y(s) \in E$ for all $s \in [0, Esc(y(\cdot); U))$.*

We write $-M \times \{1\}$ for the multifunction defined at $(s, y, r) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ as

$$(-M \times \{1\})(s, y, r) := \{-v : v \in M(s, y)\} \times \{1\} \subset \mathbb{R}^{n+1}.$$

The multifunction $M \times \{-1\}$ is defined similarly. We will show the connection between minimal time function and the new defined invariance as the below proposition.

Proposition 5.4. *Suppose multifuncitons F, G and a closed set K satisfies* (H1)-(H3)*, and let $E := \text{epi } \widetilde{T}_C$.*

(a) *If* (H4) *holds, then $(M \times \{-1\}, E)$ is weakly invariant in $U := C^c \times \mathbb{R}$.*

(b) *$(-M \times \{1\}, E)$ is strongly invariant in $\mathbb{R}^{n+1}$.*

Proof. (a) Let $(y, r) \in E \cap U$, and hence $y \in C$ and $\widetilde{T}_C(y) \leq r < \infty$. By Lemma 5.2, there exists $y(\cdot) \in \Upsilon_{(M,C^c)}(y)$ satisfying $Esc(y(\cdot); C^c) = \widetilde{T}_C(y)$. By the definition of minimal time function, we have

$$\widetilde{T}_C(y(s)) = \widetilde{T}_C(y) - s \leq r - s \quad \text{for all} \quad s \in [0, \widetilde{T}_C(y)]. \tag{5.11}$$

Define $z(s) := (y(s), r-s)$ for $s \in [y(s), r-s)$ for $s \in [0, \widetilde{T}_S(y))$. Then $Esc(z(\cdot); U) = \widetilde{T}_C(y)$, and clearly $z(\cdot) \in \Upsilon_{(M\times\{-1\})}(y, r)$. Moreover, it follows immediately from (5.11) that $z(s) \in E$ for all $s \in [0, Esc(z(\cdot); U))$, which yields (a). (b) let $(y, r) \in E$ and suppose $z(\cdot) \in \Upsilon_{(-M\times 1, \mathbb{R}^{n+1})}(y, r)$. Then $z(\cdot)$ has the representation $z(s) = (\bar{y}(s), r + s)$ for $s \in [0, S)$, where $\bar{y}(\cdot) \in \Upsilon_{(-M,\mathbb{R}^n)}(y)$ and

$$S := Esc(z(\cdot); \mathbb{R}^{n+1}) = Esc(\bar{y}(\cdot); \mathbb{R}^n).$$

Fix $s \in [0, S)$. We need to show that $z(s) \in E$. For $s' \in [0, s]$, define $y(s') = \bar{y}(s - s')$. It is clear that $y(\cdot)$ is a trajectory for M since $\bar{y}(\cdot)$ is a trajectory of $-M$. Hence, by the definition of minimal time function, we have

$$\widetilde{T}_C(y(s)) + s \geq \widetilde{T}_C(y(0)).$$

Together with the fact $(y, r) \in E$, the above inequality induces that

$$r + s \geq \widetilde{T}_C(y) + s = \widetilde{T}_C(\bar{y}(0)) + s = \widetilde{T}_C(y(s)) + s \geq \widetilde{T}_C(y(0)) = \widetilde{T}_C(\bar{y}(s)),$$

which means that $z(s) = (\bar{y}(s), r + s) \in E$, so (b) holds. □

The notations of invariance and the discussion in Lemma 5.2 inspire us to compare the minimal time function and certain lower semicontinuous functions θ, as the following result.

Proposition 5.5. *Suppose multifunctions F and G and set K satisfy* (H1) – (H3) *and $\theta : \mathbb{R}^n \to (-\infty, \infty]$ is lower semi-continuous and satisfies $\theta(s) = 0$ for all $s \in C$. Let $E :=$ epi θ and $U := C^c \times \mathbb{R}$.*

(a) *If* (H4) *also holds,* $(M \times \{-1\}, E)$ *is weakly invariant in* U *and* $\theta(\cdot)$ *is bounded below on* $\mathbb{R}^n$*, then* $\theta(x) \geq \widetilde{T}_C(y)$ *for all* $y \in \mathbb{R}^n$*.*

(b) *If* $(-M \times \{1\}, E)$ *is strongly invariant in* $\mathbb{R}^{n+1}$*, then* $\theta(y) \leq \widetilde{T}_C(y)$ *for all* $y \in \mathbb{R}^n$*.*

Proof. Suppose $y \in \mathbb{R}^n$. The statement is trivial if $y \in C$ or if $\theta(y) = \infty$, so we assume

$$y \in C^c \cap dom\theta.$$

By the weak invariance of $(M\times\{-1\}, E)$, there exists a $z(\cdot) \in \Upsilon_{(M\times\{-1\},U)}(y, \theta(y))$ that remains in E. Note that $z(\cdot)$ has the form $z(s) = (y(s), \theta(y) - s)$, where $y(\cdot) \in \Upsilon_{(M,C^c)}(x)$. By the nature of U, we have

$$S := Esc(z(\cdot), U) = Esc(y(\cdot), C^c). \tag{5.12}$$

Specially observe that the statement "$z(\cdot)$ remains in C" equivalently means that

$$\theta(y(s)) \leq \theta(x) - s \quad \text{for all} \quad s \in [0, S). \tag{5.13}$$

Since the semi-continuous function $\theta(\cdot)$ is bounded below, (a) follows from (5.13) in which $S < \infty$. Assumption (H4) and (5.12) implies that $\inf_{s\in[0,S)} \|y(s)\| < \infty$, and so it follows that by Proposition 5.1(b) that $y(s) \to y \in C$ as $s \uparrow S$. We simply set $y(S) := y$. The lower semicontinuity of θ implies that (5.13) holds for $s = S$ as well, and the boundary condition on θ means that $\theta(y(S)) = 0$. Hence we have $\theta(y) \geq S$. Finally, the definition of $\widetilde{T}_C$ yields that $S \geq \widetilde{T}_C(y)$, and we conclude that $\theta(y) \geq S \geq \widetilde{T}_S(y)$, which finish the proof of (a). (b) Suppose $y \in \mathbb{R}^n$. If $\widetilde{T}_C(y) = \infty$ or $y \in C$, the conclusion is trivial, so we assume $y \in C^c \cap dom\, \widetilde{T}_C$. Let $\eta > 0$. There exists $y(\cdot) \in \Upsilon_{(M,C^c)}(y)$ with $Esc(y(\cdot); C^c) =: S < \widetilde{T}_C(y) + \eta$ and $y(S) \in C$. Let $z(s) := (y(S - s), s)$, which is a trajectory of $-M \times \{1\}$ originating from $(y(S), 0) \in E$. by the strong invariance condition, the trajectory

$z(\cdot)$ remains in E, and hence

$$s \geq \theta(y(S-s)) \quad \text{for all} \quad s \in [0, S].$$

Letting $s = S$, we see

$$\widetilde{T}_C(y) + \eta > S \geq \theta(y(0)) = \theta(y),$$

which proves (b) by letting $\eta \to 0$.

$\square$

5.3 HJ Inequalities and HJ Equation

In Chapter 4, we have explored the weak and strong invariance properties of impulsive system. Comparing with the new definition of invariance, we can easy see the equivalence of different versions.

Theorem 5.6. *Suppose multifunctions F and G and set K satisfy* (H1) – (H3), *$E \subseteq \mathbb{R}^n$ is closed, and U is open.*

(a) *Then (M, E) is weakly invariant in U if and only if $h_M((s, y), \zeta) \leq 0$ for all $y \in E \cap U$ and $\zeta \in N_E^P(y)$.*

(b) *(M, E) is strongly invariant in U if and only if $h_M((s, y), -\zeta) \geq 0$ for all $y \in E \cap U$ and $\zeta \in N_E^P(y)$.*

Remark. In this theorem, part (a) and (b) are equivalent to Theorem 4.2 and 4.4 respectively. The key point to bridge the two kinds of formulation is that for $y \in E \cap U$, $N_E^P(y) \subseteq N_{E \cap \overline{U}}^P(y)$. Here, the significance of $E \cap \overline{U}$ is analogous to the former target set C. And also note that $h_M(s, y, -\zeta) \geq 0$ is equivalent to say $H_M(s, y, \zeta) \leq 0$

We next interpret the above results in terms of epigraphs of lower semicontinuous functions.

Proposition 5.7. *Suppose multifunctions F and G and set K satisfies* (H1)-(H3), $\theta : \mathbb{R}^n \to (-\infty, \infty]$ *is lower semicontinuous, and* $E = epi\ \theta$.

(a) *Then* $(M \times \{-1\}, E)$ *is weakly invariant in* $C^c \times \mathbb{R}$ *if and only if*

$$1 + h_M(s, y, \xi) \leq 0 \quad \textit{for all} \quad y \notin C \textit{ and } \xi \in \partial_P \theta(y).$$

(b) *Suppose* (H4) *is satisfied. Then* $(M \times \{-1\}, E)$ *is strongly invariant in* $C^c \times \mathbb{R}$ *if and only if*

$$1 + h_M(s, y, \xi) \geq 0 \quad \textit{for all} \quad y \in \mathbb{R}^n \textit{ and } \xi \in \partial_P \theta(y).$$

Proof. (a) Let $(s, y, \xi) \in \mathbb{R}^{2n+1}$, $r \in \mathbb{R}$, and $\rho < 0$. Then we note that

$$\begin{aligned} h_{(M\times\{-1\})}\big((s,y,r),(\xi,\rho)\big) &= \inf_{v\in M(s,y)} \{\langle v, \xi\rangle - \rho\} \\ &= -\rho\left(1 + h_M\left(s, y, -\frac{\xi}{\rho}\right)\right). \end{aligned} \tag{5.14}$$

($\Longleftarrow$) Suppose $y \notin C$ and $\xi \in \partial_P \theta(y)$. By Theorem 5.6(a), we have

$$h_{(M\times -1)}\big((s, y', r), -\zeta\big) \leq 0 \tag{5.15}$$

for all $(y', r) \in E$, $y' \notin C$, and $\zeta \in N_E^P(y, r)$. Using the values $(y', r) = (y, \theta(y)) \in epi\theta = E$ and $\zeta = (\xi, -1)$, where $\xi \in \partial_P \theta(y)$, we see from (5.14) and (5.15) that

$$1 + h_M(s, y, \xi) \leq 0$$

($\Longrightarrow$) Let $(y, r) \in E \cap C^c \times \mathbb{R}$ and $\zeta = (\xi, \rho) \in N_E^P(y, r)$. By the property of epigraph, we have $\rho < 0$. Let us assume first that $\rho < 0$, from which it follows that $r = \theta(y)$. Since $N_E^P(y, \theta(y))$ is a cone, we have $(-\xi/\rho, -1) \in N_E^P(y, \theta(y))$, and consequently, $-\xi/\rho \in \partial_P \theta(y)$. By (5.14), we have

$$h_{(M\times\{-1\})}\big((s, y, \theta(y)), (\xi, \rho)\big) = -\rho\big(1 + h_M(s, y, -\xi/\rho)\big) \geq 0. \tag{5.16}$$

From the inequality of assumption and $\rho < 0$. Now suppose $\rho = 0$. It is easily checked that $(\xi, 0) \in N_E^P(y, \theta(y))$ as well, and so by Rockafellar's horizontality theorem [17], there exist sequences $\{y_i\}$, $\{\xi_i\}$, and $\{\rho_i\}$ such that $y_i \to x$, $\theta(y_i) \to \theta(y)$, $\xi_i \to \xi$, $\rho_i < 0$, and $\rho_i \uparrow 0$ and $-\xi_i/\rho_i \in \partial_P\theta(x_i)$. by (5.16) we have

$$-\rho_i(1 + h_M\big(s, y_i, -\xi_i/\rho_i)\big) \geq 0.$$

for all i, and letting $i \to \infty$ yields $h_F(s, y, \xi) \geq 0$, and hence

$$h_{(M\times\{-1\})}\big((s, y, r), (\xi, 0)\big) = h_M(s, y, \xi) \geq 0. \tag{5.17}$$

In view of (5.16) and (5.17), it follows from theorem 5.6 (a) that $(M \times \{-1\}, E)$ is weakly invariant on $C^c \times \mathbb{R}$. (b) Comparing to (5.14), the transformation needed here is

$$h_{(-M\times\{1\})}\big((s, y, r), -(\xi, \rho)\big) = -\rho\big(1 + h_M(s, y, -\xi/\rho)\big). \tag{5.18}$$

The proof of the equivalence in (b) is virtually identical to the one of (a), where Theorem 5.6(b) is cited accordingly. □

Finally, we now characterize $\widetilde{T}_C(\cdot)$ as the solution of the HJ equation on C^c that satisfies certain boundary conditions.

Theorem 5.8. *Suppose multifunctions F and G and set K satisfies* (H1)-(H3)*, and $C \subset \mathbb{R}$ is closed set that* (H4) *holds. Then there exists a unique lower semi-continuous function $\theta : \mathbb{R}^n \to (-\infty, \infty]$ bounded below on $\mathbb{R}^n$ and satisfying the following conditions. And the unique such function is $\theta(\cdot) = \widetilde{T}_C(\cdot)$.*

(HJ) *For each $y \notin C$ and $\xi \in \partial_P\theta(y)$, we have*

$$1 + h_M(s, y, \xi) = 0.$$

(ABC) *Each* $y \in C$ *satisfies* $\theta(y) = 0$ *and*

$$1 + h_M(s, y, \xi) \geq 0,$$

for all $\xi \in \partial_P \theta(y)$.

Proof. It is obvious that $\widetilde{T}_C(\cdot)$ is bounded below by zero and it is lower semicontinuous by Proposition 5.2. It equals zero on C by definition. Proposition 5.4(a) and 5.7(a) together imply that

$$1 + h_M(s, y, \xi) \geq 0 \quad \text{for all} \quad y \notin C \text{ and } \xi \in \partial_P \widetilde{T}_C(y). \tag{5.19}$$

Similarly, Proposition 5.4(b) and 5.7(b) together imply that

$$1 + h_M(s, y, \xi) \geq 0 \quad \text{for all} \quad y \in \mathbb{R}^n \text{ and } \xi \in \partial_P \widetilde{T}_C(y). \tag{5.20}$$

Combining (5.19) and (5.20), we show (HJ) and (ABC) hold for $\theta(\cdot) := \widetilde{T}_C(\cdot)$. To obtain the uniqueness, suppose $\theta(\cdot)$ is lower semicontinuous, bounded below, and satisfies (HJ) and (ABC). By Proposition 5.5(a) and 5.7(a), we conclude that $\theta(y) \geq \widetilde{T}_C(y)$ for all $y \in \mathbb{R}^n$. Similarly, by Proposition 5.5(b) and 5.7(b), we conclude that $\theta(y) \leq \widetilde{T}_C(y)$ for all $y \in \mathbb{R}^n$. Obviously, $\theta(\cdot) = \widetilde{T}_C(\cdot)$. □

Remark. (HJ) is the Hamilton-Jacobi equation as Hamiltonian applies to minimal time problem. (ABC) is an analytic boundary condition.

Chapter 6
Conclusions and Future Work

The contribution of this thesis is a new sampling approach through graph completion to estimate a solution of impulsive system. During this process, we can construct the driven measures and the related control function. We also show that the impulsive systems have the weak and strong invariance properties with multifunctions F and G satisfying linear growth condition and the range of measure restricted to a given cone.

The next challenge is to apply impulsive control systems on Hamilton-Jocobi theory. The work in Chapter 5 is not as complete as the heretofore progress of existed achievements in autonomous control system. The fundamental shortcoming is that the minimal function is not well defined on original time, although, the parameterized time S can figured out a bound of the real time length by

$$T := \int_0^S \dot{\varphi}_0 \, ds \leq S, \quad \text{where } 0 \leq \dot{\varphi}_0 \leq 1.$$

However, we can view the generalized differential inclusion (5.2) as one kind of simplified nonautonomous systems, since the righthand side of inclusion is a linear combination. The effect of temporal term need pay more attention to breakthrough some details. After completely understanding this simplified version, our goal and interest next is to precisely develop the HJ theory in the theme of original time. The following minimal time function may be defined:

$$
\begin{aligned}
T_C(x) := inf\{T = \phi_0(S) : \ & (\dot{\phi}_0(\cdot), \dot{\phi}(\cdot)) \text{ is defined as in } X_\mu \text{ of } (3.1) \\
& \text{and } y(\cdot) \text{ satisfies } (5.2) \\
& \text{with } y(0) = x \text{ and } y(S) \in C\}.
\end{aligned}
$$

With the new defined minimal time function, JH theory worth further investigation.

By JH theory, a specific way to seek the minimal time is suggested, if a lower semi-continuous function satisfies the JH equation and analytic boundary condition. Hinted by this idea, maybe we can design the numerical methods through discretizing time, to estimate trajectories of systems and achieve the optimization objective. Simultaneously, we could obtain a control function along this process. Besides, the general optimal control problem on impulsive system emerges to our horizon.

Finally, with the development of study on hybrid systems in recent years, its links to the impulsive systems need come into our notice.

References

[1] J. P. Aubin, *Viability Theorey*, Birkhaäuser, Boston, 1991.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, Germany, 1984.

[3] A. Bressan, Impulsive control systems, in Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control, B. Mordukhovich and H. H. Sussmann, eds., Springer, New York, pp. 1-22, 1996.

[4] A. Bressan and F. Rampazzo, Impulsive systems with commutative vector fields, J. of Optimization Theory and Applications **71**, pp. 67-83, 1991.

[5] A. Bressan and F. Rampazzo, Impulsive control systems without commutativivty assumptions, J. of Optimization Theory and Applications **81**, pp. 435-457, 1994.

[6] G. Buttazzo, M. Giaquinta and S. Hildebrandt, *One-dimensional Variational Problems: An Introduction*, Oxford University Press, Oxford, 1999.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society, Wiley-Interscience, Toronto, 1983.

[8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

[9] G. Dal Maso and F. Rampazzo, *On systems of ordinary differential equations with measures as controls*, Differential Integral Equations, v. 4, pp. 739-765, 1991.

[10] G. B. Folland, *Real Analysis: Modern Techniques and Their applications*, Wiled-Interscience, New York, 1984.

[11] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, New York, 1965.

[12] M. Motta and F. Rampazzo, Dynamic programming for nonlinear systems driven by ordinary and impulsive control, SIA J. Control and Optimization **34**, pp. 199-225, 1996.

[13] J. M. Murray, *Existence theorems for optimal control and calculus of variation problems where the states can jump*, SIAM J. Control, v. 8, pp. 1-34, 1998.

[14] F. Rampazzo and C. Sartori, The minimum time function with unbounded controls, J. of Math. Systems, Estimation, and Control, **8**, pp. 1-34, 1998.

[15] R. Rishel, *An Extended Pontryagin Principle for Control Systems Whose Control Laws Contain Measures*, SIAM J. Control, pp. 191-205, v. 3, 1965.

[16] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[17] R. T. Rockafellar, *Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization optimization*, Math. Oper. Res., 6, pp.424-436, 1970.

[18] G. N. Silva and R. B. Vinter, *Measure driven differential inclusions*, J. of Mathematical Analysis and Applications, pp. 767-746, v. 202, 1996.

[19] G. N. Silva and R. B. Vinter, *Necessary conditions for optimal impulsive control system*, SIAM J. of Control and Optimization, pp. 1829-1846, v. 35, 1998.

[20] G. V. Smirnov, *Introduction to the theory of Differential Inclusions*, American Mathematical Society, Providence, RI, 2002.

[21] R. B. Vinter, *Optimal Control*, Birkhäuser, Boston, 2000.

[22] R. B. Vinter and F.M.F.L. Pereira, *A maximum principle for optimal processes with discontinuous trajectories*, SIAM J. of Control and Optimization, pp. 205-229, v. 26, 1988.

[23] J. Warga, *Optimal Control of Differntial and Functional Equations*, Academic Press, 1972.

[24] Peter R. Wolenski, *A Uniqueness Theorem for Differntial Inclusions*, J. Differential Equations **84**, no.1, pp. 165-182, 1990.

[25] Peter R. Wolenski, *The Exponential Formula for the Reachable Set of a Lipschitz Differential Inclusion*, SIAM J. on Control and Optimization **28**, no.5, 1990.

[26] Peter R. Wolenski and Stanislav Žabić, *A differential solution concept for impulsive systems*, Dynamics of Continuous, Discrete and Impulsive Systems, Proceedings **2**, pp. 199-210, 2006.

[27] Peter R. Wolenski and Stanislav Žabić, *A Sampling Method and Approximation Results for Impulsive Systems*, SIAM Journal on Control and Optimization, Volume 46, Issue 3, June 2007.

[28] Peter R. Wolenski and Yu Zhuang, *Proximal Analysis and the Minimal Time Function*, SIAM J. Control Optimization, Vol. 36, No. 3, pp. 1048-1072, 1998.

Vita

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