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Generalized q-Deformed Symplectic sp(4) Algebra for Multi-shell Applications

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Abstract. A multi-shell generalization of a fermion representation of the q-deformed compact symplectic $sp_q(4)$ algebra is introduced. An analytic form for the action of two or more generators of the $Sp_q(4)$ symmetry on the basis states is determined and the result used to derive formulae for the overlap between number preserving states as well as for matrix elements of a model Hamiltonian. A second-order operator in the generators of $Sp_q(4)$ is identified that is diagonal in the basis set and that reduces to the Casimir invariant of the $sp(4)$ algebra in the non-deformed limit of the theory. The results can be used in nuclear structure applications to calculate β-decay transition probabilities and to provide for a description of pairing and higher-order interactions in systems with nucleons occupying more than a single-j orbital.

1. Introduction

The symplectic $sp(4)$ algebra, which is isomorphic to $so(5)$ [1, 2, 3], has been successfully used for a description of pairing correlations when two types of particles are taken into account [4, 5]. The algebra can be generalized to multiple levels [6, 2, 7] so the particles can occupy more than a single orbit. When applied to nuclear structure, this generalization of the $sp(4)$ algebra makes a study of nuclei with mass numbers $56 < A < 100$ possible.

An additional degree of freedom can be introduced through a q-deformation of the classical $sp(4)$ Lie algebra [8, 9]. While this preserves the underlying symmetry, it introduces non-linear terms into the theory. In contrast with the usual formulation of q-deformation for the symplectic $sp(4)$ algebra and its $su(2)$ subalgebras that is normally used in mathematical studies [10, 11, 12] and in nuclear physics applications [13, 14, 15], we have discovered a new formulation that depends upon the dimensionality of the underlying space [16]. Because of this dependence, a generalization of the q-deformed symplectic $sp_q(4)$ algebra to a multi-orbit case is an interesting exercise that introduces new elements into the theory.

To apply the q-deformed theory (single-level or multiple-orbit) to real physical systems, the action of the generators of $Sp_q(4)$ on the basis states must be known. We have derived an analytical form for accomplishing this. The results can be used to build a q-deformed analog of the second-order Casimir invariant of the $sp(4)$ algebra. The results also provide for analytical matrix elements of a model interaction and therefore for an exact solution of the corresponding Hamiltonian problem. The results are what is needed for nuclear structure applications and an investigation into the physical significance of q-deformation.

2. Generalized sp(4) algebra and its q-deformed extension

The multi-shell generalization of the fermion realization of $sp_{(q)}(4)$ follows the single-j construction of the algebra [16]. The $sp(4)$ algebra, which is isomorphic to $so(5)$, is realized in terms of creation and annihilation fermion operators $c^{\dagger}_{j,m,\sigma}$ and $c_{j,m,\sigma}$, which describe a particle of type σ ($= \pm 1$ for protons/neutrons) in a state of total angular momentum j (half-integer) with a third projection m ($-j \leq m \leq j$). For a given σ, the dimension of the fermion space is $2\Omega = \sum_j 2\Omega_j = \sum_j (2j+1)$, where the sum $\sum_j$ is over all orbits that are considered to be active.

The deformation of the $sp_q(4)$ algebra is introduced in terms of q-deformed creation and annihilation operators $\alpha^{\dagger}_{j,m,\sigma}$ and $\alpha_{j,m,\sigma}$, $(\alpha^{\dagger}_{j,m,\sigma})^* = \alpha_{j,m,\sigma}$, where $\alpha^{(\dagger)}_{j,m,\sigma} \rightarrow c^{(\dagger)}_{j,m,\sigma}$ in the limit $q \rightarrow 1$. The deformed single-particle operators are defined through their anticommutation relation for every j, σ, and m in a similar way as for the single-level problem [16]

$$\begin{array}{ll} \{\alpha_{j,\sigma,m}, \alpha^{\dagger}_{j,\sigma,m'}\}_{\pm 1} = q^{\pm \frac{N_\sigma}{2\Omega}} \delta_{m,m'}, & \{\alpha_{j,\sigma,m}, \alpha^{\dagger}_{j',\sigma',m'}\} = 0, \sigma \neq \sigma', j \neq j', \\ \{\alpha^{\dagger}_{j,\sigma,m}, \alpha^{\dagger}_{j',\sigma',m'}\} = 0, & \{\alpha_{j,\sigma,m}, \alpha_{j',\sigma',m'}\} = 0, \end{array} \quad (1)$$

where the two Cartan generators $N_\sigma=\sum_j \sum_{m=-j}^{j} c^\dagger_{j,m,\sigma} c_{\ j,m,\sigma}$ count the number of particles of each type σ and by definition the q-anticommutator is given as $\{A,B\}_k = AB + q^k BA$.

In the deformed case a pair of fermions can be created ($F_{0,\pm1} \stackrel{q\to1}{\to} A_{0,\pm1}$) or destroyed ($G_{0,\pm1} \stackrel{q\to1}{\to} B_{0,\pm1}$) by the operators:

$$F_{\frac{\sigma+\sigma'}{2}} = \frac{1}{\sqrt{2\Omega(1+\delta_{\sigma,\sigma'})}} \sum_j \sum_{m=-j}^{j} (-1)^{j-m} \alpha^\dagger_{j,m,\sigma} \alpha^\dagger_{j,-m,\sigma'}, \tag{2}$$

$$G_{\frac{\sigma+\sigma'}{2}} = \frac{1}{\sqrt{2\Omega(1+\delta_{\sigma,\sigma'})}} \sum_j \sum_{m=-j}^{j} (-1)^{j-m} \alpha_{j,-m,\sigma} \alpha_{j,m,\sigma'}, \tag{3}$$

where $F_{0,\pm1} = (G_{0,\pm1})^\dagger$. The number preserving Weyl generators are defined as:

$$T_+ = \frac{1}{\sqrt{2\Omega}} \sum_j \sum_{m=-j}^{j} \alpha^{\ \dagger}_{\ j,m,1} \alpha_{\ j,m,-1}, \quad T_- = \frac{1}{\sqrt{2\Omega}} \sum_j \sum_{m=-j}^{j} \alpha^\dagger_{j,m,-1} \alpha_{j,m,1}, \tag{4}$$

where ($T_\pm \stackrel{q\to1}{\to} \tau_\pm$). In addition to the Cartan generators $N_{\pm1}$ (or their linear combinations $N = N_{+1} + N_{-1}$ and $T_0 \equiv \tau_0 = (N_{+1} - N_{-1})/2$), the operators (2)-(4) close on the q-deformed $sp_q(4)$ algebra and their non-deformed counterparts close on the $sp(4)$ algebra. In physical applications, the number generators $N_{\pm1}$ along with the total number of particles N and the third projection T_0 (of the operator with two other components $T_\pm$), represent physical observables, which are always non-deformed.

In the deformed and non-deformed cases, the generators (2)-(4) are related to the corresponding single-level operators $X^{(j)}$ as $X = \sum_j \frac{\sqrt{\Omega_j}}{\sqrt{\Omega}} X^{(j)}$, where $X = \{F,\ G,\ T\}$ or $X \stackrel{q\to1}{=} \{A,\ B,\ \tau\}$. In the non-deformed limit, the ten operators $A^{(j)}_{0,\pm1}$, $B^{(j)}_{0,\pm1}$, $\tau^{(j)}_{0,\pm}$ and $N^{(j)}$ close on the $sp^{(j)}(4)$ algebra for each j-level and the direct sum holds, $sp(4) = \bigoplus_j sp^{(j)}(4)$.

A different situation occurs in the deformed case, where the single-level generators do not close within the $sp^{(j)}_q(4)$ algebra (e.g. $[T^{(j)}_+, T^{(j)}_-] \neq [2\frac{T^{(j)}_0}{2\Omega_j}]$) but rather within the generalized $sp_q(4)$ algebra, since

$$[T^{(j)}_+, T^{(j)}_-] = [2\frac{T_0}{2\Omega}], [F^{(j)}_0, G^{(j)}_0] = [\frac{N-2\Omega}{2\Omega}], \left[F^{(j)}_{\pm1}, G^{(j)}_{\pm1}\right] = \rho_\pm[\frac{N_{\pm1}-\Omega}{\Omega}],$$

where by definition $[X]_k = \frac{q^{kX}-q^{-kX}}{q^k-q^{-k}}$ and $\rho_\pm = \frac{q^{\pm1}+q^{\pm\frac{1}{2\Omega}}}{2}$. The rest of the commutation relations remain within the single-j q-deformed algebra, for example $[T^{(j)}_0, T^{(j)}_\pm] = \pm T^{(j)}_\pm$. However, several of these relations, like $[T^{(j)}_l, F^{(j)}_0]_{\frac{[2]}{2}} = \frac{1}{2\sqrt{\Omega_j}} F^{(j)}_l (q^{\frac{N_{-l}}{2\Omega}} + q^{-\frac{N_{-l}}{2\Omega}})$, $l = \pm1$, include a multiplicative q-factor with a dependence on the averaged multi-level number, $N_{\pm1}/(2\Omega)$. This behavior of $sp^{(j)}_q(4)$ can be traced back to the generalized q-deformation (1), where the anticommutation relations of two fermions on a single-j level depend on the total number of particles of one kind averaged over the multi-shell space. Another interesting consequence of (1) is the single-j q-deformed quantity

$$\sum_m \alpha^\dagger_{j,\sigma,m} \alpha_{j,\sigma,m} = 2\Omega_j[\frac{N_\sigma}{2\Omega}],\ \sigma = \pm1. \tag{5}$$

In the non-deformed limit, the left-hand side of (5) represents the single-level number operator $N_\sigma^{(j)}$, while in the deformed extension the zeroth approximation of (5) gives an even distribution of the particles over the entire multi-level space weighed by the single-j dimension. In this way, the q-deformation for the generalized $sp_q(4)$ algebra introduces probability features at the constituent single-j levels of the theory.

In the deformed case (as in the "classical" case), each finite representation is spanned by completely paired states, which are constructed as pairs of fermions coupled to a total angular momentum and parity $J^\pi = 0^+$ [17],

$$|n_1, n_0, n_{-1})_q = (F_1)^{n_1} (F_0)^{n_0} (F_{-1})^{n_{-1}} |0\rangle \,, \tag{6}$$

where $F_{0,\pm1}$ are defined in (2) and n_1, n_0, n_{-1} are the total number of pairs of each kind, $(\sigma, \sigma') = (++), (+-), (--)$, respectively. The basis states (6) in a multiple-orbit space of dimension 2Ω is a linear combination of the single-level basis states which depend on what pairs occupy which levels [18, 19].

The states (6) are eigenvectors of the total number operators $N_{\pm1}$ with eigenvalues $N_\pm$, where $N_\pm = 2n_{\pm1} + n_0$. Both N and τ_0 (T_0) are diagonal in the basis (6) with eigenvalues $n = 2(n_1 + n_{-1} + n_0)$ and $i = n_1 - n_{-1}$, respectively. While the single-j fermion number operators $N_{\pm1}^{(j)}$ project onto the single-level basis, the q-deformed analog (5) is diagonal in the basis (6) with eigenvalue $2\Omega_j[\frac{2n_\sigma+n_0}{2\Omega}]$, $\sigma = \pm1$.

The generalized model has the same symmetry properties as the single-level realization of the theory [16]. All formulae that are constructed in terms of commutation relations of the single-level generators (like action of a group generator on the basis states, Casimir invariants, eigenvalues, normalization coefficients of the basis vectors) coincide at the algebraic level and have the same form under the substitution $\Omega_j \to \Omega$. The three important reduction limits of the $sp_q(4)$ algebra to $u_q(2)$ are summarized in Table 1 with the eigenvalues of the first and second-order Casimir invariants and the basis states given for each limit. Also, the q-deformed symplectic algebra reverts back to the "classical" limit when $q \to 1$.

For nuclear structure applications we use the set of the commutation relations that is symmetric with respect to the exchange $q \leftrightarrow q^{-1}$ [16]. The q-coefficients, $\Psi_{lk}(N_p)$,

Table 1. Reduction limits of the $sp_q(4)$ algebra, $\mu = \{T, 0, \pm\}$.

$u_q^\mu(2)$	Generators	Eigenvalues		Basis states
		$\mathbf{C}_1(u_q^\mu(2))$	$\mathbf{C}_2(su_q^\mu(2))$	
$u_q^T(2)$	$T_\pm,\ T_0;\ N$	$n = 2n_1 + 2n_{-1} + 2n_0$	$2\Omega\left[\frac{1}{2\Omega}\right][T]_{\frac{1}{2\Omega}}[T+1]_{\frac{1}{2\Omega}}$	$\|n, T, i\rangle$
$u_q^0(2)$	$F_0,\ G_0,\ \frac{N-2\Omega}{2};\ T_0$	$i = n_1 - n_{-1}$	$2\Omega\left[\frac{1}{2\Omega}\right]\left[\frac{2\Omega-2(n_1+n_{-1})}{2}\right]_{\frac{1}{2\Omega}} \times\left[\frac{2\Omega-2(n_1+n_{-1})}{2}+1\right]_{\frac{1}{2\Omega}}$	$\|n_1, n_0, 0)$ $\|0, n_0, n_{-1})$
$u_q^\pm(2)$	$F_{\pm1},\ G_{\pm1},\ \frac{N_{\pm1}-\Omega}{2};\ N_{\mp1}$	$2n_{\mp1} + n_0$	$\rho_\pm\ \Omega\left[\frac{1}{\Omega}\right]\left[\frac{\Omega-n_0}{2}\right]_{\frac{1}{\Omega}}\left[\frac{\Omega-n_0}{2}+1\right]_{\frac{1}{\Omega}}$	$\|n_1, 0, n_{-1})$ $\|n_1, 1, n_{-1})$

obtained in [16] for the following commutation relations

$$[T_l, Y_{\pm k}] = \pm\frac{Y_{\pm l \pm k}\Psi_{\pm 1}(N_{\pm k})}{2[2]\sqrt{\Omega}},\ l,k \neq 0; \quad [F_l, G_{-k}] = \frac{T_{l+k}\Psi_{|l|-|k|}(N_{l-k})}{2[2]\sqrt{\Omega}},\ l+k \neq 0,$$

(where $Y = F$ (G) for the '+' ('−') case) can be written in a compact way as

$$\Psi_{\pm 1}(N_p) = 2\sqrt{\rho_+\rho_-}\left[2_{N_p \pm 1/2-\Omega}\right]_{\frac{1}{2\Omega}} = \begin{cases} \Psi_{l0}(N_p) \\ \Psi_{0k}(N_p) \end{cases}, \tag{7}$$

where we define $[2_X]_{\frac{1}{2\Omega}} \equiv \frac{[2X]_{\frac{1}{2\Omega}}}{[X]_{\frac{1}{2\Omega}}} = q^{\frac{X}{2\Omega}} + q^{-\frac{X}{2\Omega}} \stackrel{q\to 1}{\to} 2.$

3. Action of first and second-order operators on the basis vectors

In addition to a generalization of the $sp_q(4)$ algebra to multi-j shells, an algebraic form for the action of the product of two or more generators of the symplectic symmetry can be given. This allows one to calculate the overlaps between states in a number preserving sequence, to build a q-deformed second-order diagonal operator in terms of all the ten generators of $Sp_q(4)$, and to obtain the matrix elements of a model Hamiltonian.

3.1. The $su_q^T(2)$ limit

In the q-deformed case, the commutators of the raising (lowering) $T_\pm$ operator with the pair creation operators, $(F_\mp)^{n_{\mp 1}}$ and $(F_0)^{n_0}$, which enter into the construction of the basis states (6), are

$$\begin{aligned} [T_\pm, (F_\mp)^{n_{\mp 1}}] &= F_0 (F_\mp)^{n_{\mp 1}-1}\frac{\sqrt{\rho_+\rho_-}}{\sqrt{\Omega}[2]}[n_{\mp 1}]_{\frac{1}{2\Omega}}\left[2_{N_{\mp 1}+n_{\mp 1}-1/2-\Omega}\right]_{\frac{1}{2\Omega}} \\ [T_\pm, (F_0)^{n_0}]_{\left(\frac{[2]}{2}\right)^{n_0}} &= F_\pm (F_0)^{n_0-1}\frac{1}{2\sqrt{\Omega}}\sum_{p=0}^{n_0-1}\frac{[2]^p}{2^p}[2_{N_{\mp 1}+n_0-1-p}]_{\frac{1}{2\Omega}} \end{aligned} \tag{8}$$

With the use of (8), the general formula for the action of the k-th order product of $T_\pm$ on the lowest (highest) weight basis state can be determined,

$$\begin{aligned} T_\pm^k (F_\mp)^{n_{\mp 1}}|0\rangle \ = \ & \sum_{i=0}^{\lfloor k/2 \rfloor}\left(\frac{\sqrt{\rho_+\rho_-}}{\sqrt{\Omega}[2]}\left[2_{n_{\mp 1}-\frac{1}{2}-\Omega}\right]_{\frac{1}{2\Omega}}\right)^{k-i}\frac{[n_{\mp 1}]_{\frac{1}{2\Omega}}!\ \theta(k,i)}{[n_{\mp 1}-k+i]_{\frac{1}{2\Omega}}!} \\ & \times (F_\pm)^i (F_0)^{k-2i}(F_\mp)^{n_{\mp 1}-k+i}|0\rangle, \end{aligned} \tag{9}$$

where $k \leq n_{\pm 1}$ and the functions in the sum are defined as

$\theta(k,0) = 1, \forall\ k$

$$\theta(k,i) = \begin{cases} \dfrac{\theta(k-1,i)\left[2_{n_{\mp 1}-i-\frac{1}{2}-\Omega}\right]_{\frac{1}{2\Omega}}}{\left[2_{n_{\mp 1}-\frac{1}{2}-\Omega}\right]_{\frac{1}{2\Omega}}} + \frac{\theta(k-1,i-1)}{2\sqrt{\Omega}}\sum_{p=0}^{k-2i}\frac{[2]^p}{2^p}[2_{k-2i-p}]_{\frac{1}{2\Omega}},\ i \leq \lfloor k/2 \rfloor \\ 0,\ i > \lfloor k/2 \rfloor \end{cases}. \tag{10}$$

This implies that $\theta(k, \frac{k}{2}) = \frac{\theta(k-1,\frac{k}{2}-1)}{\sqrt{\Omega}}$ when k is even. Starting from the lowest (highest) weight basis state the action of the $T_\pm$ operator (9) gives all the number preserving vectors with a definite maximum value of the T quantum number. (Recall that for given n and i, T takes the values, $T = \frac{\tilde{n}}{2},\ \frac{\tilde{n}}{2}-2, ...,\ 2\lceil\frac{i}{2}\rceil$, where $\tilde{n} = \min\{n, 4\Omega - n\}$.) The rest of the vectors with lower T values and the same (n, i) quantum numbers can be found as independent and orthogonal vectors to those constructed in (9).

In nuclear systems, the $T_{\pm}$ generators represent the raising and lowering isospin operators and as such they generate $\beta^{\mp}$-decay transitions in an isobaric sequence. It follows that formula (9) derived above is used extensively in the calculation of the strength of these transitions. Also, the construction of the isospin states (9) allows one to compute overlaps with the pair states (6) and with the eigenvectors of a model Hamiltonian.

3.2. Action of products of two generators on the basis states

We are also able to give an analytical form of the action of the anticommutator $\{T_+,\ T_-\} = T_+T_- + T_-T_+$ on the basis states

$$\begin{aligned}\{T_+, T_-\}\,|n_1, n_0, n_{-1}) &= \frac{M^T_{-1,+2,-1}}{\Omega}\,|n_1-1, n_0+2, n_{-1}-1) \\ &+ \frac{M^T_{0,0,0}}{\Omega}\,|n_1, n_0, n_{-1}) + \frac{M^T_{+1,-2,+1}}{2\Omega}\,|n_1+1, n_0-2, n_{-1}+1)\,, \quad (11)\end{aligned}$$

where the coefficients $M^T_{n'_1,n'_0,n'_{-1}}$ (with n'_1, n'_0, n'_{-1} indicating the number of pairs of each kind added (+)/removed (−)) are q-deformed functions of the pair numbers given in terms of the n_{-1}, n_0, n_1 by

$$\begin{aligned}M^T_{-1,+2,-1} &= \frac{1}{4[2]^2}\{\Psi(n_0, n_1-1)\Psi(n_0+1, n_{-1}-1) \\ &+ \Psi(n_0, n_{-1}-1)\Psi(n_0+1, n_1-1)\} \\ M^T_{0,0,0} &= \frac{1}{4[2]}\{\Phi(n_0-1)\left(\Psi(n_0-1, n_1) + \Psi(n_0-1, n_{-1})\right) \\ &+ \Phi(n_0)\left(\Psi(n_0, n_1-1) + \Psi(n_0, n_{-1}-1)\right)\} \\ M^T_{+1,-2,+1} &= \Phi(n_0-1)\Phi(n_0-2), \quad (12)\end{aligned}$$

where we define

$$\Phi(n_0) = \sum_{k=0}^{n_0} \frac{[2]^k}{2^k}\,[2_{n_0-k}]_{\frac{1}{2\Omega}}\,,$$

$$\begin{aligned}\Psi(n_0, n_{\pm 1}) &= [n_0 + 2n_{\pm 1} + 1]_{\frac{1}{2\Omega}} - [n_0 - 1]_{\frac{1}{2\Omega}} + [n_0 + 2n_{\pm 1} + 2 - 2\Omega]_{\frac{1}{2\Omega}} - [n_0 - 2\Omega]_{\frac{1}{2\Omega}} \\ &= 2\sqrt{\rho_+\rho_-}\,[n_{\pm} + 1]_{\frac{1}{2\Omega}}\left[2_{n_0+n_{\pm}+1/2-\Omega}\right]_{\frac{1}{2\Omega}}\,. \quad (13)\end{aligned}$$

The second expression of $\Psi(n_0, n_{\pm 1})$ (13) allows the non-diagonal term that scatters two identical particle pairs of opposite kinds into two non-identical particle pairs (12) to be rewritten as

$$\begin{aligned}M^T_{-1,+2,-1} &= \frac{\rho_+\rho_-}{[2]^2}\,[n_1]_{\frac{1}{2\Omega}}\,[n_{-1}]_{\frac{1}{2\Omega}} \times \\ &\{\left[2_{n_0+n_1-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_0+n_{-1}-\Omega+\frac{1}{2}}\right]_{\frac{1}{2\Omega}} + \left[2_{n_0+n_1-\Omega+\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_0+n_{-1}-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\}. \quad (14)\end{aligned}$$

In a similar way, the action of the second-order product, F_0G_0, on the basis states yields the following non-diagonal term

$$\begin{aligned} -\frac{1}{\Omega}M^P_{-1,+2,-1} &= -\frac{1}{\Omega}\tilde{n}_1\tilde{n}_{-1} \\ &= -\frac{\rho_+\rho_-}{[2]^2\Omega}[n_1]_{\frac{1}{2\Omega}}[n_{-1}]_{\frac{1}{2\Omega}}\left[2_{n_1-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_{-1}-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}, \end{aligned} \quad (15)$$

and for $F_{+1}G_{+1}+F_{-1}G_{-1}$ it is

$$-\frac{1}{\Omega}M^P_{+1,-2,+1} = -\frac{1}{\Omega}\frac{\sqrt{\rho_+\rho_-}}{[2]}\sum_{k=1}^{n_0-1}S_q(k), \quad (16)$$

where we define $\tilde{n}_{\pm1} \equiv \frac{1}{2[2]}([2n_{\pm1}-1]_{\frac{1}{2\Omega}} + [2n_{\pm1}-2\Omega]_{\frac{1}{2\Omega}} + [2\Omega]_{\frac{1}{2\Omega}} + 1) = \frac{1}{[2]}\sqrt{\rho_+\rho_-}\,[n_{\pm1}]_{\frac{1}{2\Omega}}\left[2_{n_{\pm1}-\Omega-1/2}\right]_{\frac{1}{2\Omega}} \stackrel{q\to1}{\to} n_{\pm1}$, and $S_q(k) \equiv [2_{k-\Omega-1/2}]_{\frac{1}{2\Omega}}\sum_{i=0}^{k-1}\frac{[2]^i}{2^i}[2_{k-1-i}]_{\frac{1}{2\Omega}} \stackrel{q\to1}{\to} 4k$ [20]. The diagonal elements, $M^P_{0,0,0}$, of F_0G_0 and $F_{+1}G_{+1}+F_{-1}G_{-1}$ are discussed in detail in [20].

3.3. Second-order operators

The analytical relations (12)-(16) allow us to find a q-deformed second-order operator, $O_2(sp_q(4))$, that is diagonal in the q-deformed basis and that in the limit when q goes to one reverts to the second-order Casimir invariant of the $sp(4)$ algebra [3, 16],

$$O_2(sp_q(4)) = \frac{\gamma_1}{2}(\{F_{+1},G_{+1}\}+\{F_{-1},G_{-1}\}) + \gamma_0\frac{C_2(su^0_q(2))}{\Omega} + \frac{C_2(su^T_q(2))}{\Omega}, \quad (17)$$

where the γ-coefficients are q-functions of the pair numbers, $\gamma_1 = \frac{M^T_{+1,-2,+1}}{2M^P_{+1,-2,+1}} \stackrel{q\to1}{\to} 2$ and $\gamma_0 = \frac{\left[2_{n_0+n_1-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_0+n_{-1}-\Omega+\frac{1}{2}}\right]_{\frac{1}{2\Omega}}+\left[2_{n_0+n_1-\Omega+\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_0+n_{-1}-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}}{2\left[2_{n_1-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}\left[2_{n_{-1}-\Omega-\frac{1}{2}}\right]_{\frac{1}{2\Omega}}} \stackrel{q\to1}{\to} 1$. The Casimir invariants in (17) are $C_2(su^T_q(2)) = \Omega(\{T_+,T_-\}+\left[\frac{1}{\Omega}\right][T_0]^2_{\frac{1}{2\Omega}})$ and $C_2(su^0_q(2)) = \Omega(\{F_0,G_0\}+\left[\frac{1}{\Omega}\right]\left[\frac{N}{2}-\Omega\right]^2_{\frac{1}{2\Omega}})$ [16].

The second-order operator can be written in terms of the Casimir operators of all four limits, $\{+,-,0,T\}$, as

$$O_2(sp_q(4)) = \sum_{k=+,-,0,T}\gamma_k\frac{C_2(su^k_q(2))}{\Omega} - \frac{\gamma_1}{2}\left[\frac{2}{\Omega}\right]\{\rho_+\left[\frac{N_1-\Omega}{2}\right]^2_{\frac{1}{\Omega}} + \rho_-\left[\frac{N_{-1}-\Omega}{2}\right]^2_{\frac{1}{\Omega}}\} \quad (18)$$

where $\gamma_\pm \equiv \gamma_1$, $\gamma_T \equiv 1$ and $C_2(su^\pm_q(2)) = \frac{\Omega}{2}(\{F_{\pm1},G_{\pm1}\}+\rho_\pm\left[\frac{2}{\Omega}\right]\left[\frac{N_{\pm1}-\Omega}{2}\right]^2_{\frac{1}{\Omega}})$ [16]. Its eigenvalue in the basis set (6) (see Table 1 and (12)-(16)) is

$$\begin{aligned} \langle O_2(sp_q(4))\rangle &= \gamma_1(\rho_++\rho_-)\left[\frac{1}{\Omega}\right]\left[\frac{\Omega-n_0}{2}\right]_{\frac{1}{\Omega}}\left[\frac{\Omega-n_0}{2}+1\right]_{\frac{1}{\Omega}} \\ &\quad -\frac{\gamma_1}{2}\left[\frac{2}{\Omega}\right]\{\rho_+\left[\frac{2n_++n_0-\Omega}{2}\right]^2_{\frac{1}{\Omega}} + \rho_-\left[\frac{2n_-+n_0-\Omega}{2}\right]^2_{\frac{1}{\Omega}}\} \end{aligned}$$

$$+\,2\gamma_0\left[\frac{1}{2\Omega}\right]\left[\frac{2\Omega-2(n_1+n_{-1})}{2}\right]_{\frac{1}{2\Omega}}\left[\frac{2\Omega-2(n_1+n_{-1})}{2}+1\right]_{\frac{1}{2\Omega}}$$
$$+\,\frac{M^T_{0,0,0}}{\Omega}+\left[\frac{1}{\Omega}\right][n_1-n_{-1}]^2_{\frac{1}{2\Omega}}\quad\stackrel{q\to 1}{\to}\Omega+3. \tag{19}$$

The second-order operator (17) is a Casimir invariant only in the non-deformed limit of the theory. In that limit its eigenvalue $(\Omega+3)$ labels the $Sp(4)$ representations. While an explicit form for the second-order Casimir operator of $sp_q(4)$ for other q-deformed schemes can be given [12], this is not the case here because the scheme includes, by construction [16], a dependence on the shell structure which is suitable for nuclear physics applications. Nevertheless, the importance of the second-order operator (17) in the q-deformed case is obvious. It is an operator that consists of number preserving products of all ten q-deformed group generators, and the pair basis states (6), which span the entire space for a given $Sp_q(4)$ representation, are its eigenvectors. Its zeroth-order approximation commutes with the generators of the q-deformed symplectic symmetry, which means that only the higher-order terms introduce a dependence on the quantum numbers that label the states. It also gives a direct relation between the expectation values of the second-order products of the generators that build $O_2(sp_q(4))$.

The analytical formulae, which were derived above, are also used for finding the matrix elements of the interaction in a system with symplectic dynamical symmetry. The model Hamiltonian [20] is another second-order operator that is expressed in terms of the generators of the $Sp_q(4)$ group [17, 16]:

$$H_q=\,-\bar{\epsilon}^q N-G_qF_0G_0-F_q(F_{+1}G_{+1}+F_{-1}G_{-1})-\frac{E_q}{2\Omega}(C_2(su^T_q(2))-\Omega\left[\frac{N}{2\Omega}\right])$$
$$-\,C_q2\Omega\left[\frac{1}{\Omega}\right](\left[\frac{N}{2}-\Omega\right]^2_{\frac{1}{2\Omega}}-[\Omega]^2_{\frac{1}{2\Omega}})-(D_q-\frac{E_q}{2\Omega})\Omega\left[\frac{1}{\Omega}\right][T_0]^2_{\frac{1}{2\Omega}}\,, \tag{20}$$

where $\epsilon^q=\bar{\epsilon}^q+(\frac{1}{2}-2\Omega)C_q+\frac{D_q}{4}>0$ is the Fermi level of the system, $G_q,\ F_q,\ E_q,\ C_q$ and D_q are constant interaction strength parameters. The classical Hamiltonian, H_{cl}, is obtained in the limit $q\to 1$. For a nuclear system with N_+ valence protons and N_- valence neutrons, the interaction represents proton-neutron and identical-particle isovector pairing (with $G_q\geq 0$ and $F_q\geq 0$ strength parameters), a symmetry term (E_q), a diagonal proton-neutron isoscalar force (E_q and C_q) [21, 22] and an isospin breaking term ($D_q\neq\frac{E_q}{2\Omega}$). The quantum extension of the $sp(4)$ algebra introduces higher-order interactions and accounts for non-linear effects. The q-deformed model can be applied to multi-j major shells, like $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$, and is exactly solvable. The second-order diagonal operator $O_2(sp_q(4))$ (17) sets a linear dependence between the q-deformed pairing and symmetry energies, which allows for a reduction of the number of the phenomenological parameters in the Hamiltonian (20).

3.4. Boson approximation

Although the fermion generalization allows many j-orbitals to be considered, the dimension of the space should not be allowed to grow too large because the effect of the deformation diminishes as the size of the model space grows. For example, in the case of very large Ω the anticommutation relation of the fermions (1) reduces to the simpler form $\{\alpha_{j,\sigma,m}, \alpha^{\dagger}_{j,\sigma,m'}\}_{q^{\pm 1}} = \delta_{m,m'}$ and all q-brackets $[X]$ or $[X]_{\frac{1}{(2)\Omega}}$ go to X when X is not a function of Ω. In this limit the pair-operators obey boson commutations and a boson approximation is achieved. The model Hamiltonian (20) is diagonal in the pair states and does not include scattering between different kinds of bosons. In the limit of large $\Omega \gg \{\frac{N}{2}, N_{\pm 1}\}$, the action of the operators $A \cdot B$ $(F \cdot G)$ on the basis states counts the number of different kinds of bosons: $\langle F_0 G_0 \rangle$ counts the number of non-identical particle pairs and remains non-deformed; $\langle F_{\pm} G_{\pm} \rangle$ counts the number of identical particle pairs and only scales its non-deformed analog by a factor of $\frac{1+q^{\pm 1}}{2}$. As another direct consequence of the dependence of the deformation on the space dimension is that in this large Ω limit the direct product of the single-j quantum symplectic algebras holds, $sp_q(4) = \bigoplus_j sp_q^{(j)}(4)$, as for the non-deformed case for all Ω.

4. Conclusion

In this article we introduced a multi-shell extension of the quantum $sp_q(4)$ algebra. While in the non-deformed case this is a direct sum of the single-j symplectic $sp_q^{(j)}(4)$ algebras, in the deformed case a direct sum result is only achieved in a boson approximation of the theory that is applicable in the large space limit, $\Omega \gg \{\frac{N}{2}, N_{\pm 1}\}$. The dependence of the deformation on the dimensionality of the space makes the generalization unique and non-trivial.

We also derived an analytical solution for the action of the q-deformed raising and lowering operators on the basis states. This, in turn, allows one to calculate the overlap between number preserving states. It also makes the construction of q-deformed basis vectors with a definite isospin value possible and allows one to calculate β-decay transition probabilities between these states.

We were also able to obtained formulae for computing the action of the product of two generators of the $Sp_q(4)$ group on the basis states. From this we found a q-deformed second-order operator in the group generators that is diagonal in the basis set with its zeroth-order approximation commuting with all the $Sp_q(4)$ generators. The results can also be used to provide for an exact solutions of a q-deformed model Hamiltonian.

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