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A simple entanglement measure for multipartite pure states

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Abstract

A simple entanglement measure for multipartite pure states is formulated based on the partial entropy of a series of reduced density matrices. Use of the proposed new measure to distinguish disentangled, partially entangled, and maximally entangled multipartite pure states is illustrated.
Keywords: Entanglement measure, multipartite, pure states.

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Entanglement plays an important role in the theory of quantum information and quantum computation.[1,2] A major challenge that remains is how to define good measures of entanglement since simple measures that classify and quantify entanglement of a given state should enhance our understanding of the phenomenon. Although, many measures of entanglement have been proposed,[3−26] most involve extremizations that are difficult to manage analytically.[8]

There has been a lot of work on multipartite entanglement. For example, Bennett *et al* in [17] introduced exact and asymptotic measures for multipartite pure state entanglement, in which a minimal reversible entanglement generating set (MREGS) was defined. In [18], reversibility of local transformations of multipartite entanglement was studied. Relations between tripartite pure state entanglement and additivity properties of the bipartite relative entropy of entanglement were established in [19]. Upper and lower bounds to the relative entropy of entanglement of multi-party systems in terms of the bi-partite entanglements of formation and distillation and entropies of various subsystems were discussed in [20]. Recently, the structure of a reversible entanglement generating set for three-particle states were investigated in [21]. In the connection with the logarithmic negativity discussed in [22], an operational interpretation of the logarithmic negativity has been found.[23] All these works help us to get better understandings of the multipartite entanglement.

A good definition of an entanglement measure can be used to distinguish entangled, partially entangled, and disentangled states, and this in turn should be useful in understanding the extent those particles are entangled and how many ways a multipartite system can be entangled. For a bipartite pure state, the problem has been solved.[13] In this case, an entanglement measure can be defined in terms of the von Neumann entropy. However, the problem still remains open for a system with more than three particles. The situation becomes more difficult and unclear for mixed states. In the following, we will concentrated on multipartite pure states, for which, as for the spin-$\frac{1}{2}$ case, there are two degrees of freedom with $\sigma = 0$ or 1 for each particle.

For a system of N such identical particles, any wavefunction $|\Psi\rangle$ can be expanded in terms of basis vectors $|\sigma_1, \sigma_2, \cdots, \sigma_N\rangle$ in the tensor product space $(V_2 \bigotimes)^N$ as

$$|\Psi\rangle = \sum_{\sigma_1 \cdots \sigma_N} C_{\sigma_1 \cdots \sigma_N} |\sigma_1, \cdots, \sigma_N\rangle, \tag{1}$$

where $\sigma_i = 0$ or 1 for $1 \le i \le N$ and $C_{\sigma_1 \cdots \sigma_N}$ is the normalized expansion coefficient. The corresponding density matrix is

$$\rho_\Psi = |\Psi\rangle\langle\Psi|. \tag{2}$$

Let $a^\dagger_{i\sigma}$ ($a_{i\sigma}$) with $i = 1, 2, \cdots, N$, be particle creation (annihilation) operators that satisfy

$$[a_{i\sigma}, a^\dagger_{j\sigma'}]_\pm \equiv a_{i\sigma} a^\dagger_{j\sigma'} \pm a^\dagger_{j\sigma'} a_{i\sigma} = \delta_{ij} \delta_{\sigma\sigma'}, \tag{3a}$$

$$[a_{i\sigma}, a_{j\sigma'}]_\pm = [a^\dagger_{i\sigma}, a^\dagger_{j\sigma'}]_\pm = 0, \tag{3b}$$

for spin-$\frac{1}{2}$ fermions or 2-component bosons. The wavefunction $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = \sum_{\sigma_1\cdots\sigma_N} C_{\sigma_1\cdots\sigma_N} a^\dagger_{1\sigma_1} a^\dagger_{2\sigma_2} \cdots a^\dagger_{N\sigma_N} |0\rangle, \tag{4}$$

where $|0\rangle$ is the vacuum state. Under the replacement $a^\dagger_{i\sigma_i} \to X_{i\sigma_i}$, where $X_{i\sigma_i}$ is simply a symbol, the operator form in front of the vacuum state on the right-hand-side of Eq. (4) becomes a homogeneous polynomial of degree N in terms of the $\{\mathbf{X}_i\}$,

$$F_C(\mathbf{X}_1, \cdots, \mathbf{X}_N) = \sum_{\sigma_1\cdots\sigma_N} C_{\sigma_1\cdots\sigma_N} X_{1\sigma_1} \cdots X_{N\sigma_N}. \tag{5}$$

It should be understood that $\mathbf{X}_i$ is a two-value symbol with $\mathbf{X}_i = X_{i1}$ and X_{i0}. An alternative definition of entangled states can be stated as follows: The state $|\Psi\rangle$ is an N-particle entangled state if the corresponding polynomials $F_C(\mathbf{X}_1, \cdots, \mathbf{X}_N)$ on complex field $\mathcal{C}$ cannot be factorized into the following form

$$F_C(\mathbf{X}_1, \cdots, \mathbf{X}_N) = F_A(\mathbf{X}_{i_1}, \cdots, \mathbf{X}_{i_m}) F_B(\mathbf{X}_{i_{m+1}}, \cdots, \mathbf{X}_{i_N}) \tag{6}$$

for $1 \le m \le N-1$, where $\{i_1 \ne i_2 \ne \cdots \ne i_N\}$ can be in any ordering of $\{1, 2, \cdots, N\}$. Otherwise the state $|\Psi\rangle$ is not an N-particle entangled state. The state $|\Psi\rangle$ given in (4) is disentangled (separable) if the polynomials F_C can be factorized into a product of monomial of $\mathbf{X}_i$ as $\prod_{i=1}^N F_{A_i}(\mathbf{X}_i)$. In other cases, the state is partially entangled.

For $N = 2$, a criterion for distinguishing whether a homogeneous polynomial is factorizable can be established by using the von Neumann entropy of the reduced density matrix. Furthermore, the degree of entanglement can be quantified by the von Neumann entropy with

$$S_\Psi = -\mathrm{Tr}((\rho_\Psi)_i \,\mathrm{Log}_2\,(\rho_\Psi)_i) = -\frac{1}{2}\left(\mathrm{Tr}((\rho_\Psi)_1 \,\mathrm{Log}_2\,(\rho_\Psi)_1) + \mathrm{Tr}((\rho_\Psi)_2 \,\mathrm{Log}_2\,(\rho_\Psi)_2)\right), \tag{7}$$

where $i = 1$ or 2, and $(\rho_\Psi)_i$ ($i = 1$ or 2) is the reduced density matrix with particle 2 or 1, respectively, traced out. This definition and the correspondence between the factorizable (non-factorizable) case of (5) and a disentangled (entangled) state given in (4) is well-known, which provides with a clear quantification of entanglement for bipartite pure states. A state is separable if $S_\Psi = 0$, entangled if $S_\Psi \ne 0$, and maximally entangled if $S_\Psi = 1$. In (7), we have used the fact that $(\rho_\Psi)_1 = (\rho_\Psi)_2$.

However, there will be many new features for $N \ge 3$. Let $(\rho_\Psi)_{(12\cdots N-1)}$ be the reduced density matrix with the N-th particle traced out. There is a series of reduced density matrices with $N-1$ particles,

$$\{Q^{N-1}_\omega\,(\rho_\Psi)_{(12\cdots N-1)}\}, \tag{8}$$

where Q^{N-1}_ω is the left coset representative of the factor group $S_N/(S_{N-1} \otimes S_1)$, in which S_k is the permutation group, and ω is the normal ordered sequences.[24] Let g_i ($i = 1, 2, \cdots, N-1$) be generators of S_N, which are adjacent permutation of the i-th and $(i+1)$-th particles. When $N = 3$, one has $\{Q^2_1 = e, Q^2_2 = g_2, Q^2_3 = g_1 g_2\}$. Thus, one gets three two-particle reduced density matrices $(\rho_\Psi)_{(12)}$, $(\rho_\Psi)_{(13)}$, and $(\rho_\Psi)_{(23)}$ according to (8). Consequently, there will be a series of reduced density matrices with $N-2$, $N-$

3,⋯, 1 particle(s), $\{Q^{N-2}_{\omega_{N-2}}(\rho_\Psi)_{(12\cdots N-2)}\}$, $\{Q^{N-3}_{\omega_{N-3}}(\rho_\Psi)_{(12\cdots N-3)}\}$,⋯, $\{Q^{1}_{\omega_1}(\rho_\Psi)_{(1)}\}$, where $Q^{N-k}_{\omega_{N-k}}$ is the left coset representative of the factor group $S_N/(S_{N-k}\otimes S_k)$. For $N=3$, a complete set of reduced density matrices is $\{(\rho_\Psi)_{(12)},(\rho_\Psi)_{(13)},(\rho_\Psi)_{(23)},(\rho_\Psi)_{(1)},(\rho_\Psi)_{(2)},(\rho_\Psi)_{(3)}\}$. It can be shown that the state $|\Psi\rangle$ is not a genuine N-particle entangled state if the von Neumann entropy defined in terms of one of the reduced density matrices in the series $\{Q^{N-k}_{\omega_{N-k}}(\rho_\Psi)_{(12\cdots N-k)}\}$ $(k=1,2,\cdots,N-1)$ is zero because the corresponding homogeneous polynomial (5) is, at least, partially factorizable. Furthermore, unlike the $N=2$ case, it can be verified that values of the von Neumann entropy of reduced matrices for $N-k$ particles with $k=1,2,\cdots,N-1$, are not the same for fixed k. For example, generally, $(\rho_\Psi)_{(12)}\neq(\rho_\Psi)_{(23)}\neq(\rho_\Psi)_{(13)}$, and $(\rho_\Psi)_{(1)}\neq(\rho_\Psi)_{(2)}\neq(\rho_\Psi)_{(3)}$. In addition, it will be shown later that the maximal entropy calculated from $\{Q^{1}_{\omega}(\rho_\Psi)_{(1)}\}$ may be less than 1 when $N\geq 3$.

Based on the above observations, we can defined a measure of genuine N-particle entanglement $\eta^{(N)}_\Psi$ as follows:

$$\eta^{(N)}_\Psi=\begin{cases}\frac{1}{N}\sum_{i=1}^{N}S_{(i)} & \text{if}\quad Q^{N-k}_{\omega_{N-k}}S_{(12\cdots N-k)}\neq 0\ \ \forall\ \ \omega_{N-k}\ \ \text{with}\ \ 1\leq k\leq N-1,\\ \\ 0 & \text{otherwise},\end{cases}\tag{9}$$

where

$$S_{(12\cdots N-k)}=-\mathrm{Tr}\left((\rho_\Psi)_{(12\cdots N-k)}\mathrm{Log}_2(\rho_\Psi)_{(12\cdots N-k)}\right)\tag{10}$$

is the partial von Neumann entropy with the k particles traced out. The state $|\Psi\rangle$ is, at least, partially separable when one of the values of partial entropy $\{Q^{N-k}_{\omega_{N-k}}S_{(12\cdots N-k)}\}$ is zero. In such case, the corresponding state is not a genuine N-particle entangled state. Otherwise, we can quantify the measure using the average one-particle reduced entropy defined in (9).

It is clear that (9) is zero for separable states. Furthermore, the entanglement measure should be invariant under local unitary transformations, and its expectation should not increase under local operations and classical communication (LOCC). In order to prove (9) satisfying the above requirements, we use the conclusions made in [17]. As has been noted in [17], partial entropies have the nice property that for pure states their average does not increase under LOCC. The entanglement measure (9) is defined in terms of the average one-particle reduced entropy. Therefore, its value should also not increase under LOCC. In addition, since the measure (9) is defined in terms of the average one-particle reduced entropy, its value should also be invariant under any local unitary transformation.

However, Eq. (9) does not tell us how many particles are disentangled from one another and how many of them are still entangled, when some of the values of the partial entropy $\{Q^{N-k}_{\omega_{N-k}}S_{(12\cdots N-k)}\}$ are zero. Actually, one need to calculate all values of the partial entropy $\{Q^{N-k}_{\omega_{N-k}}S_{(12\cdots N-k)}\}$ to get the full picture. In the following, as an example, we will apply the above definition of the entanglement measure given by (9) for the 3-particle case. For $N=3$, the complete basis space is eight dimensional $(2^N=8)$, of which the basis vectors are denoted as $\{|1\rangle=|000\rangle,|2\rangle=|110\rangle,|3\rangle=|101\rangle,|4\rangle=|011\rangle,|5\rangle=|111\rangle,|6\rangle=|001\rangle,|7\rangle=|010\rangle,|8\rangle=|100\rangle\}$. A general 3-particle state $|\Psi\rangle$ can be expanded in terms of these basis vectors with at most 8 terms. As a simple example, we assume a 3-particle state $|\Psi\rangle$ has three non-zero terms. There are 56 possible three-term linear combinations ($\binom{8}{3} = 56$) of these 8 basis vectors as listed in Table 1, with 24 combinations being states with 2-particle entangled and disentangled with another one. Therefore, those 24 states are partially entangled, and not genuine 3-particle entangled states. The remaining 32 such combinations are genuine 3-particle entangled states. In order to verify the effectiveness of the definition (9), we calculate a series of reduced density matrices for two cases with $|\Psi\rangle = |127\rangle$ and $|123\rangle$ given in the Table 1. In the case of $|127\rangle$, the wavefunction can be written as

$$|\Psi\rangle = \alpha|000\rangle + \beta|110\rangle + \gamma|010\rangle, \tag{11}$$

where α, β, and γ are nonzero complex numbers satisfying the normalization condition. The corresponding diagonalized reduced density matrices are

$$(\rho)_{(12)} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, \quad (\rho)_{(13)} = (\rho)_{(23)} = \begin{pmatrix} \frac{1}{2}(1 - \sqrt{1 - 4|\alpha|^2|\beta|^2}) & \\ & \frac{1}{2}(1 + \sqrt{1 - 4|\alpha|^2|\beta|^2}) \end{pmatrix},$$

$$(\rho)_{(1)} = (\rho)_{(2)} = (\rho)_{(13)}, \quad (\rho)_{(3)} = (\rho)_{(12)}. \tag{12}$$

Therefore, the corresponding values of partial entropy are $S_{(12)} = 0$, $S_{(13)} = S_{(23)} \neq 0$, $S_{(1)} = S_{(2)} \neq 0$, and $S_{(3)} = 0$. The values of entropy $S_{(3)} = S_{(12)} = 0$ indicate that in this case particle 3 is disentangled from particle 1 and 2, while the values of entropy $S_{(1)} = S_{(2)} \neq 0$ indicate that particle 1 and 2 are still entangled. According to definition (9), therefore, the 3-particle state $|127\rangle$ is not a genuine 3-particle entangled state. When $|\Psi\rangle = |123\rangle$, the corresponding diagonalized reduced density matrices are

$$(\rho)_{(3)} = (\rho)_{(12)} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \\ & |\gamma|^2 \end{pmatrix}, \quad (\rho)_{(2)} = (\rho)_{(13)} = \begin{pmatrix} |\beta|^2 & \\ & ||\alpha|^2 + \gamma|^2 \end{pmatrix},$$

$$(\rho)_{(1)} = (\rho)_{(23)} = \begin{pmatrix} |\alpha|^2 & \\ & |\beta|^2 + |\gamma|^2 \end{pmatrix}. \tag{13}$$

Hence, the corresponding values of partial entropy are all non-zero, which indicate that the state $|123\rangle$ is a genuine 3-particle entangled state. In this case, the values of reduced entropy $(S)_{(i)}$ for $i = 1, 2, 3$ are not the same in general. To maximize (9) with the results given in (13) and the constraint $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$, one finds that $S_{(i)} = 0.918296$ for $i = 1, 2, 3$. Up to a phase factor, the corresponding coefficients are $|\alpha| = |\beta| = |\gamma| = \frac{1}{\sqrt{3}}$, which gives the maximally entangled 3-particle state with 3 terms. Actually, this state belongs to the W-state family.[25] The reduced entropy $S_{(i)}$ is different from that of two-term GHZ-state case,[26] in which $S_{(i)} = 1$. It has been verified that the definition (9) is indeed invariant under any local unitary transformation. This procedure enabled us to analyze all possible 3-particle entangled pure states with at most 8 terms. The detailed results will be reported elsewhere. The generalization to multipartite entangled pure states is straightforward.

In summary, we have formulated a simple entanglement measure for multipartite pure states based on partial entropy of a series of reduced density matrices. The new definition seems suitable to distinguish from disentangled, partially entangled, and maximally entangled multipartite pure states. However, entanglement measure of a multipartite mixed state is much more difficult to be defined than that of a multipartite pure state studied in this paper. Much work remain to be done for multipartite mixed states.

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Table 1. All possible 3-particle entangled states with 3 nonzero terms. There are 24 one-particle separable states (Case I) and 32 genuine 3-particle entangled states (Case II). The symbol $|ijk\rangle$ means that the state is a linear combination of the i-th, j-th, and k-th basis vectors defined in the text.

Case I							
$\|127\rangle$	$\|128\rangle$	$\|136\rangle$	$\|138\rangle$	$\|146\rangle$	$\|147\rangle$	$\|167\rangle$	$\|168\rangle$
$\|178\rangle$	$\|235\rangle$	$\|238\rangle$	$\|245\rangle$	$\|247\rangle$	$\|257\rangle$	$\|258\rangle$	$\|278\rangle$
$\|345\rangle$	$\|346\rangle$	$\|356\rangle$	$\|358\rangle$	$\|368\rangle$	$\|456\rangle$	$\|457\rangle$	$\|467\rangle$
Case II							
$\|123\rangle$	$\|124\rangle$	$\|125\rangle$	$\|126\rangle$	$\|134\rangle$	$\|135\rangle$	$\|137\rangle$	$\|145\rangle$
$\|148\rangle$	$\|156\rangle$	$\|157\rangle$	$\|158\rangle$	$\|234\rangle$	$\|236\rangle$	$\|237\rangle$	$\|246\rangle$
$\|248\rangle$	$\|256\rangle$	$\|267\rangle$	$\|268\rangle$	$\|347\rangle$	$\|348\rangle$	$\|357\rangle$	$\|367\rangle$
$\|378\rangle$	$\|458\rangle$	$\|468\rangle$	$\|478\rangle$	$\|567\rangle$	$\|568\rangle$	$\|578\rangle$	$\|678\rangle$