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Extremal Entanglement For Triqubit Pure States

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Abstract: A complete analysis of entangled triqubit pure states is carried out based on a new simple entanglement measure. An analysis of all possible extremally entangled pure triqubit states with up to eight terms is shown to reduce, with the help of SLOCC transformations, to three distinct types. The analysis presented are most helpful for finding different entanglement types in multipartite pure state systems.

Keywords: Extremal entanglement, triqubit pure states; entanglement measure

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Entanglement is a fundamental concept that underpins quantum information and computation.[1−3] As a consequence, the quantification of entanglement emerges as a central challenge. Many authors have contributed to this topic,[4−23] among which the basic requirements for entanglement measures proposed in [4] provide with guidelines for its definition. In [6], Bennett *et al* defined stochastic local operations and classical communication (SLOCC) based on the concept of local operations assisted with classical communication (LOCC). Dür *et al* applied such an operation to a triqubit pure state system and found that triqubit states can be entangled *at least* in two inequivalent ways,[7] namely, in the GHZ form[24] or the W form.[7] In this Letter, we will use a recently proposed entanglement measure[23] for N-qubit pure states to analyze all extremally entangled triqubit pure states with the constrained maximization, and to see whether there are other inequivalent types of entanglement.

In the following, based on the method used in [7], only entanglement properties of a single copy of a state will be considered. Therefore, asymptotic properties will not be discussed. At single copy level, it is well known that two pure states can always be transformed with certainty from each other by means of LOCC if and only if they are related by Local Unitary transformation (LU).[6] However, even in the simplest bipartite cases, entangled states are not always related by LU, and continuous parameters are needed to label all equivalence classes. Hence, it seems that one needs to deal with infinitely many kinds of entanglement. Fortunately, such arbitrariness has been overcome with the help of SLOCC.[7]

According to [23], for a genuine entangled N-qubit pure state Ψ, the measure can be defined by

$$E(\Psi)=\begin{cases}\frac{1}{N}\sum_{i=1}^{N}S_i & \text{if } S_i\neq 0\ \forall\ i,\\ \\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

where $S_i=-\mathrm{Tr}[(\rho_\Psi)_i\log_2(\rho_\Psi)_i]$ is the reduced von Neumann entropy for the i-th particle only with the other $N-1$ particles traced out, and $(\rho_\Psi)_i$ is the corresponding reduced density matrix. It can be verified that the state Ψ is partially separable when one of the reduced von Neumann entropies S_i is zero. In such cases the state Ψ is not a genuine entangled N-qubit state. Furthermore, definition (1) is invariant under LU, which is equivalent to LOCC for pure state system. Eq. (1) will be our unique benchmark for the degree of entanglement of N-qubit pure states. Using (1), we have successfully verified that there is only one type of extremal (maximal) entanglement for biqubit system, which is equivalent to the Bell type.[25]

The basis vectors of a triqubit system are denoted by

$$\begin{array}{l}\{W_1=|000\rangle, W_2=|110\rangle, W_3=|101\rangle, W_4=|011\rangle,\\ \bar{W}_1=|111\rangle, \bar{W}_2=|001\rangle, \bar{W}_3=|010\rangle, \bar{W}_4=|100\rangle\}.\end{array} \tag{2}$$

In order to study all possible forms of entangled triqubit states, entangled states are classified according to the number of terms involved in their expressions in terms of a linear combination of basis vectors given in (2). This means there will be up to eight terms in their expressions. Then, maximization constrained by the normalization condition for the entanglement measure (1) is performed to find the corresponding parameters and phase factors. In this way, all extremally entangled triqubit pure states can be obtained. It will be shown that values of the measure for different extremally entangled states are all different, and these extremally entangled forms can thus be recognized as the different types of entanglement for triqubit system since they can not be transformed into other forms by SLOCC.

When a state Ψ is a linear combination of two terms W_i and W_j or $\bar{W}_i$ and $\bar{W}_j$, where $i,\ j \in P$ $(i \neq j)$, and $P = \{1, 2, 3, 4\}$, Ψ is a partially separable state. There are 12 such combinations. It can easily be verified that $E(\Psi) = 0$ in such cases according to the measure defined in (1). In such cases, one of reduced von Neumann entropy S_i $(i = 1, 2, 3)$ is zero. As an example, When $\Psi = aW_1 + be^{i\alpha}W_2$ with constraint $|a|^2 + |b|^2 = 1$, one obtains

$$(\rho_\Psi)_1 = (\rho_\Psi)_2 = \begin{pmatrix} |a|^2 & \\ & |b|^2 \end{pmatrix}, \quad (\rho_\Psi)_3 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}. \tag{3}$$

Therefore, in this case, generally $S_1 = S_2 \neq 0$, while $S_3 = 0$. According to (1), the measure $E(\Psi) = 0$ since one of the reduced von Neumann entropy, S_3 is zero. Analysis on the other 11 combinations are similar, which lead to the same conclusion. Similarly, when Ψ is a linear combination of two terms W_i and $\bar{W}_j$ with $i \neq j$, Ψ is a fully separable state, of which $E(\Psi) = 0$ is obvious. There are also 12 of such linear combinations. Ψ will be an entangled state only when it is a linear combination of W_i and $\bar{W}_i$. In such cases, one may write $\Psi = a\ W_j + be^{i\alpha}\ \bar{W}_j$, where $a,\ b$ are real, and α is a relative phase. Then, to maximize its measure (1) with the constraint $a^2 + b^2 = 1$, one can find parameters for the corresponding extremal cases. It can be verified easily that $E = E_{\max} = 1$ when $|a| = |b| = 1/\sqrt{2}$ in such cases, and there is no restriction on relative phase. Such exremally (maximally) entangled two-term states are nothing but the GHZ states.

When a state Ψ is a linear combination of three terms shown in (1), there are $\binom{8}{3} = 56$ forms of Ψ in total, among which 24 of them are partially separable, and the remaining 32 of them are genuine entangled states. When $\Psi = a\ W_i + be^{i\alpha}\ \bar{W}_i + ce^{i\beta}\ W_j$ or $\Psi = a\ W_i + be^{i\alpha}\ \bar{W}_i + ce^{i\beta}\ \bar{W}_j$, where $i,\ j \in P$ $(i \neq j)$, a, b, c, α and β are real, there are 24 such combinations. In such cases, one can verify that no extremal value of $E(\Psi)$ exists if coefficients a, b, c are all non-zero. When a state Ψ is a linear combination of (W_i, W_j, W_q) or that of $(\bar{W}_i, \bar{W}_j, \bar{W}_q)$, where i, j, $q \in P$ with $i \neq j \neq q$, the state is a genuine entangled state. Similar to the GHZ states, one can prove that extremal value of $E(\Psi)$ is 0.918296 for such states when the absolute value of all the expansion coefficients equal to $1/\sqrt{3}$. The result is also independent of the relative phases. These entangled states are nothing but the W states.

When a state Ψ is a linear combination of four terms shown in (1), there are 70 forms in total, among which 6 of them are partially separable, while the remaining 64 of them are genuine entangled states. They are classified into 5 types shown in Table 1, in which Yes (No) listed in the last column refers to the fact that there exists (does not exist) an extremal value of $E(\Psi)$ for the corresponding state with all expansion coefficients nonzero.

Table 1. Classification of the entangled states with four terms

Type	Terms involved	Total Number of Forms	Extremal entanglement
Type I_4	$W_i, W_j, \bar{W}_i, \bar{W}_j$ $(i, j \in P,\ i \neq j)$	6	Yes
Type II_4	W_i, W_j, W_t, W_q or $\bar{W}_i, \bar{W}_j, \bar{W}_t, \bar{W}_q$ $(i,\ j,\ t,\ q \in P,\ i \neq j \neq t \neq q)$	2	Yes
Type III_4	$W_i, W_j, W_t, \bar{W}_i$ or $\bar{W}_i, \bar{W}_j, \bar{W}_t, W_i$ $(i, j, t \in P,\ i \neq j \neq t)$	24	Yes
Type IV_4	$W_i, W_j, \bar{W}_i, \bar{W}_q$ $(i, j, q \in P,\ i \neq j \neq q)$	24	No
Type V_4	$W_i, W_j, W_t, \bar{W}_q$ or $\bar{W}_i, \bar{W}_j, \bar{W}_t, W_q$ $(i, j, t, q \in P, i \neq j \neq t \neq q)$	8	No

For the Type I_4 case, $\Psi = a\ \bar{W}_i + be^{i\alpha}W_i + ce^{i\beta}W_j + de^{i\gamma}\bar{W}_j$, where $i \neq j$, and $a,\ b,\ c,\ d,\ \alpha,\ \beta,\ \gamma$ are real with constraint $a^2+b^2+c^2+d^2 = 1$. One can verify that $E(\Psi) = 1$ when $|a| = |b|,\ |c| = |d|,\ \theta = \alpha - \beta - \gamma = (2n+1)\pi$ with integer n. Such states are equivalent to the GHZ type states after LU transformations. For Type II_4 case, $\Psi = aW_i + be^{i\alpha}W_j + ce^{i\beta}W_t + de^{i\gamma}W_q$, where $i \neq j \neq t \neq q$. When $|a| = |b| = |c| = |d| = \frac{1}{2}$ for arbitrary $\alpha,\ \beta,\ \gamma$, the corresponding states are also equivalent to the GHZ type states. For Type III_4 case, $\Psi = aW_j + be^{i\alpha}W_t + ce^{i\beta}W_i + de^{i\gamma}\bar{W}_i$, where $i \neq j \neq t$. In this case, the reduced density matrices are

$$\rho_A = \begin{pmatrix} c^2 & 0 \\ 0 & a^2+b^2+d^2 \end{pmatrix},\ \rho_B = \begin{pmatrix} b^2+d^2 & ade^{i\gamma} \\ ade^{-i\gamma} & a^2+c^2 \end{pmatrix},$$

$$\rho_C = \begin{pmatrix} a^2+d^2 & bde^{i\gamma - i\alpha} \\ bde^{-i\gamma+i\alpha} & b^2+c^2 \end{pmatrix}. \tag{4}$$

The corresponding entanglement measure is

$$\begin{aligned} E(\Psi) = \tfrac{1}{3}\{&-c^2\log_2 c^2 - (a^2+b^2+d^2)\log_2(a^2+b^2+d^2) - \tfrac{1}{2}(1-u)\log_2[\tfrac{1}{2}(1-u)] \\ &-\tfrac{1}{2}(1+u)\log_2[\tfrac{1}{2}(1+u)] - \tfrac{1}{2}(1-v)\log_2[\tfrac{1}{2}(1-v)] - \tfrac{1}{2}(1+v)\log_2[\tfrac{1}{2}(1+v)]\}, \end{aligned} \tag{5}$$

where

$$u = \sqrt{1-4(a^2b^2+a^2c^2+c^2d^2)},\quad v = \sqrt{1-4(a^2b^2+b^2c^2+c^2d^2)}. \tag{6}$$

We found that there are three extremal values for the measure with $E_1(\Psi) = 1$ when $|a| = |b| = 0$, $|c| = |d| = 1/\sqrt{2}$, and $E_2(\Psi) = 0.918296$ when $|a| = |b| = |c| = 1/\sqrt{3}$, $|d| = 0$, and $E_3(\Psi) = 0.893295$ when $|a| = |b| = 0.462175$, $|c| = 0.653614$, $|d| = 0.381546$, respectively. There is also no restriction on relative phases. The first and the second cases clearly correspond to the GHZ and W types, respectively. While the third case with $E_3(\Psi) = 0.893295$ has not been reported previously. In order to study whether this type of entangled states is equivalent to GHZ or W types, in the following, we will briefly review the SLOCC used in [7] : States Ψ and Φ are equivalent under SLOCC if an invertible local operator (ILO) relating them exists, which is denoted as $Q_A \otimes Q_B \otimes Q_C$. Typically, these ILOs are elements of the complex general linear group $GL_A(2,c) \otimes GL_B(2,c) \otimes GL_C(2,c)$ with each copy operates on the corresponding local basis.

In the following, we prove that the Type III_4 entangled states is inequivalent to the GHZ and W types under the SLOCC. To do so, we take

$$\Psi = x_1W_2 + x_2W_3 + x_3W_4 + x_4\bar{W}_4 \tag{7}$$

as a representative of Type III$_4$ entangled states since the proof for other cases is similar, where x_i ($i = 1, \cdots, 4$) are arbitrary nonzero complex numbers. Let Q_A, Q_B, and Q_C be ILOs for particles A, B, and C, respectively. First, we prove that (7) is inequivalent to a GHZ type state. Notice that Eq. (7) can be rewritten as

$$\Psi = |1\rangle_A \left(x_1|10\rangle_{BC} + x_2|01\rangle_{BC}\right) + \left(x_3|011\rangle_{ABC} + x_4|100\rangle_{ABC}\right). \tag{8}$$

The first term in (8) is a product of a Bell type state and a single particle state, while the second term is a GHZ type state. After $Q_A \otimes Q_B \otimes Q_C$ transformation, the first term will remain a product of a single particle state and a Bell type state, while the second term will always remain a GHZ type state. Therefore, (7) is inequivalent to GHZ type state if x_1 and x_2 are all non-zero. It is clear that there is a phase transition from the Type III$_4$ to the GHZ type when the parameters x_1 and x_2 tend to zero. Such a transition belongs to the first class phase transition. Similarly, the Type III$_4$ entangled state becomes separable when x_3 and x_4 tend to zero. Second, we prove that (7) is also inequivalent to W type state. Eq. (7) can also be written as

$$\Psi = \left(x_1|110\rangle_{ABC} + x_2|101\rangle_{ABC} + x_3|011\rangle_{ABC}\right) + x_4|100\rangle_{ABC}. \tag{9}$$

The first term in (9) belongs to a W type state, while the second term is a product of three single-particle states. After arbitrary $Q_A \otimes Q_B \otimes Q_C$ transformation, the first term will remain a W type state, while the second term will always a product of three single-particle states. Hence, the Type III$_4$ entangled state is also inequivalent to a W Type if x_4 is nonzero. Similarly, the Type III$_4$ entangled state will become a W type state only when x_4 tends to zero. Furthermore, it can be shown by a simple analysis that invertible transformation from (7) with all parameters nonzero to either GHZ or W state does not exist. Therefore, for nonzero x_i ($i = 1, \cdots, 4$), the Type III$_4$ is a new type of entanglement in triqubit pure system, which is inequivalent to the GHZ and W types. This result is not so surprising since the Type III$_4$ states were not studied in [7]. There is no extremal value of the measure for types IV$_4$ and V$_4$ with all expansion coefficients nonzero.

When a state Ψ is a linear combination of five terms, there are $\binom{8}{5} = 56$ of them, which can be classified into three types. If Ψ is a linear combination of $(W_1, W_2, W_3, W_4, \bar{W}_i)$ or $(\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4, W_i)$, where $i \in P$, it is called Type I$_5$. If Ψ is a linear combination of $(W_i, W_j, W_q, \bar{W}_i, \bar{W}_j)$ or $(\bar{W}_i, \bar{W}_j, \bar{W}_q, W_i, W_j)$, where $i,\ j,\ q \in P$, it is called Type II$_5$. If Ψ is a linear combination of $(W_i, W_j, W_q, \bar{W}_i, \bar{W}_t)$ or $(\bar{W}_i, \bar{W}_j, \bar{W}_q, W_i, W_t)$, where $i,\ j,\ q,\ t \in P,\ i \neq j \neq t \neq q$, it is called Type III$_5$. For the Type I$_5$ cases, $\Psi = aW_1 + be^{i\alpha}W_2 + ce^{i\beta}W_3 + de^{i\gamma}W_4 + fe^{i\sigma}\bar{W}_i$, where $a,\ b,\ c,\ d,\ f,\ \alpha,\ \beta,\ \gamma,\ \sigma$ are real. When $|a| = \frac{2}{3},\ |b| = |c| = |d| = \frac{1}{3}$, and $|f| = \frac{\sqrt{2}}{3}$ with arbitrary phases, there exists a extremal value of the measure with $E(\Psi) = 0.918296$. It can be verified that such states are equivalent to W type states with respect to SLOCC. For the Type II$_5$ and Type III$_5$ cases, we found that there is no extremal value of the measure exists with all expansion coefficients nonzero. Similar analysis for states with six, seven, and eight terms were also carried out. After tedious computations, we did not find any other new type of extremally entangled states. The results are consistent with the conclusion made in [10] and [22] that entangled triqubit pure states can always be expressed as a linear combination of five appropriate terms chosen from (2).

In summary, by using the entanglement measure (1), a complete analysis for entangled triqubit pure states has been carried out. Three types of extremally entangled triqubit pure states have been identified by using the constrained maximization. The extremal values of these three types of entanglement are 1, 0.918296, 0.893295, respectively, which show that the GHZ type state with

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{10a}$$

and the W type with

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{10b}$$

are all the special cases of the Type III_4 states. Similarly, the type III_4 states can always be transformed by SLOCC to the following form

$$|\mathrm{III}_4\rangle = a|110\rangle + b|101\rangle + c|011\rangle + d|100\rangle \tag{10c}$$

with $|a| = |b| = 0.462175,\ |c| = 0.653614,\ |d| = 0.381546$, respectively. These three types of entangled states cannot be transformed from one type into another type by SLOCC, which, therefore, are recognized to be all inequivalent types of entanglement. Graphical descriptions of these three types of tripartite entanglement are shown in Fig. 1. The relations among these three types of entanglement are shown in Fig. 2.

We also found that the entanglement measure defined by (1) is effective in classifying different genuinely entangled tripartite pure states. The most important conclusion is that the number of basic ways of entanglement equals to the number of extremally entangled types. Therefore, extremal entanglement is a necessary condition in finding different ways of entanglement in multipartite pure state systems under the SLOCC. The analysis can be extended to other multipartite pure state systems in a straightforward manner.

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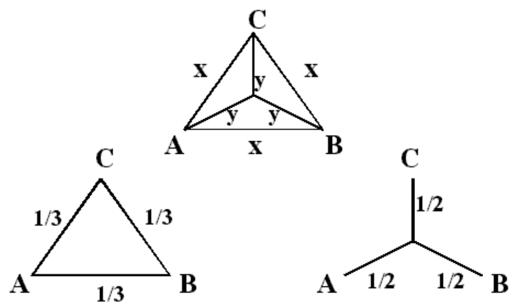

Figure 1: Graphical descriptions of three types of entanglement. The one in the upper middle represents the type III_4 with x $= |ab|/3 = \frac{|bc|}{3\sqrt{2}} = \frac{|ac|}{3\sqrt{2}}$, and y $= \frac{|cd|}{2\sqrt{3}}$, where $|a|$, $|b|$, $|c$, and $|d|$ are given after Eq. (10c). The lower left one represents the W type, in which the value besides each edge is $x_1x_2 = x_2x_3 = x_1x_3 = \frac{1}{3}$ with $x_1 = x_2 = x_3 = 1/\sqrt{3}$, and the Y type connection within the triangle disappears since $x_4 = 0$ in this case. The lower right one represents the GHZ type, in which the value besides each line is $x_3x_4 = 1/2$ with $x_3 = x_4 = 1/\sqrt{2}$, and the triangle disappears since $x_1 = x_2 = 0$ in this case.

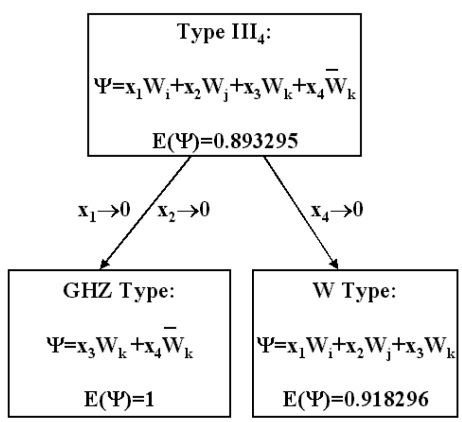

Figure 2: Relations among three inequivalent types of entanglement, where $i,\ j,\ k \in P$ with $i \neq j \neq k$, and the measures listed are the corresponding extremal values according to (2). These three types of entangled states can always be transformed under SLOCC into the corresponding forms shown by Eq. (10).