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Professional Development for Geometry Teachers Under Common Core State Standards in Mathematics

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PROFESSIONAL DEVELOPMENT FOR GEOMETRY TEACHERS UNDER COMMON CORE STATE STANDARDS IN MATHEMATICS

A Thesis

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TABLE OF CONTENTS

ABSTRACT

This thesis offers a model professional development workshop to high school geometry teachers, with a focus on the Common Core State Standards in geometry, including a description of the workshop, materials to assist in the presentation, and follow-up materials. This workshop, which is based upon curriculum written for EngageNY under the direction of Common Core, Inc., has been presented by the author on four separate occasions to a total of approximately two hundred teachers and other school and district personnel over the past few months. Feedback obtained from attendees has been uniformly positive, indicating that the information and understanding obtained as a direct result of the workshop experience will be useful both in teaching the curriculum and in assisting other geometry teachers in the attendees' schools and districts.

CHAPTER 1: INTRODUCTION

In this introduction, I explain the need for professional development on the Common Core State Standards in High School Geometry, briefly describe how this thesis meets this need, and provide an overall guide to the content of the professional development offered.

With the implementation of the more rigorous Common Core State Standards, teachers need more focused, in-depth knowledge of math content, especially since much material previously taught at one grade level has been moved down and will now be taught one or more grade levels earlier. Many teachers are unfamiliar with the more in-depth treatment of the math topics under the new standards. Geometry, in particular, has undergone a major shift in emphasis and teachers must be given the opportunity to engage in activities which require them to grapple with the concepts before they ask their students to do so.

In the past, much professional development for teachers has been “one-shot sit-and-get” workshops designed to impart to teachers the latest strategies “guaranteed” to improve their teaching. Seldom has any follow-up been offered, nor have the strategies proved to be of any lasting value in increasing students’ performance (Darling-Hammond, 2009). Mathematician Hung-Hsi Wu, a long-time, well-informed commentator on the state of mathematics instruction in the USA, insists that teacher professional development needs to focus on ridding teachers of their reliance on “textbook school mathematics” (TSM) and replacing the math they have learned with appropriate content knowledge. “Unless we can improve teachers’ content knowledge,

the Common Core State Standards will fail disastrously" (Wu 2014). He further states that "the need for content-based PD becomes urgent" (Wu 2011).

In the white paper *Professional Learning in the Learning Profession*, Darling-Hammond and her colleagues (2009) identify four key components required for successful professional development designed to improve teaching. "Professional development should (1) be intensive, ongoing, and connected to practice; (2) focus on student learning and address the teaching of specific curriculum content; (3) align with school improvement priorities and goals; and (4) build strong working relationships among teachers." The workshop described in this thesis addresses all four of these components. It is intensive and immediately connected to the practice of teaching geometry and there is a component that addresses ongoing professional development. The workshop requires teachers to focus directly on the specific content their students will be learning. Common Core State Standards are a key component of school improvement priorities and goals and the workshop directly supports these goals. Finally, the workshop attendees are encouraged to network with one another and with the presenter to share ideas and insights as they teach the curriculum.

In 2011, the state of New York contracted with Common Core, Inc. to write mathematics curricula for pre-kindergarten through 12th grade aligned with the Common Core State Standards in Mathematics. New York created a web site, EngageNY.org, to showcase the curricula written by Common Core, Inc. and to provide teachers with access to a variety of resources available for learning and teaching the new standards. The EngageNY High School Geometry Course consists of five modules: Congruence, Proof & Constructions; Similarity, Proof & Trigonometry; Extending to Three Dimensions;

Connecting Algebra and Geometry through Coordinates; and Circles with and without Coordinates. New York has provided its own teachers with professional development designed to give the teachers an overview of the geometry curriculum and a somewhat more detailed exploration of the modules throughout the course of this past school year. Professional development linked to the EngageNY geometry course was limited.

This thesis focuses on the first module – arguably the most novel module in the geometry curriculum as it represents the most sweeping changes under the Common Core State Standards in Mathematics. Module I (Congruence, Proof & Constructions, is a critical introduction to one of the major shifts in geometry content under the Common Core State Standards in Mathematics – the focus on proving congruence by using rigid motions. In the past, teachers have only taught transformations at the very end of the curriculum, and then only if time permitted. In addition, rigid motions were taught solely by using the coordinate plane with "nice" transformations (e.g. rotations of 90° clockwise around the origin or reflection across the y-axis). Now, teachers must not only become familiar with teaching transformations as functions (without regard to the coordinate plane), they must also be prepared to show students how to use rigid motions to demonstrate that two figures are congruent. Even the notation used in the Engage NY curriculum is foreign to most teachers, who often have education degrees as opposed to degrees in mathematics. Seeing $R_{A'',100^\circ}\left(T_{\overrightarrow{B'B''}}\left(r_{\overline{DE}}(\triangle ABC)\right)\right)$ and recognizing this as a rotation following a translation following a reflection of a triangle is new to a great many high school teachers. In short, there is currently a need for significant professional development for teachers of high school geometry. This thesis is a specific

example of a professional development model designed to address this high-priority need.

Using the EngageNY curriculum for Module I in Geometry, I present a workshop (Appendix I) wherein I guide teachers to work through the materials in the module, much as their students will be expected to do, but in an abbreviated time. The workshop begins by exploring Lesson 1 in full, both teacher materials and student materials. Teachers work the exercises under the guidance of the presenter, who explains the rationale for each section of the lesson. After fully exploring Lesson 1, we discuss the layout of the entire module, again with a rationale provided for each of the seven topics and the flow of lessons within each topic. The presenter uses sample problems from most lessons to demonstrate the flow of ideas and to explain how each topic builds upon the preceding topics. Teachers and other workshop attendees work through problems from the Mid-Module Assessment and End-of-Module Assessment; then score the work using the rubric in the teacher materials. Participants receive a packet of handouts with all problems discussed during the workshop, an overview of the entire high school math curriculum, a complete listing of the Common Core State Standards in High School Mathematics, and a description of the Standards for Mathematical Practice (Appendix II).

Appendix III includes follow-up materials that are designed to be used by teacher professional learning communities to engage in an in-depth exploration of geometry topics alluded to in Module I. These materials are intended to enable teachers to “see beyond” the immediate needs of their students to better guide them through current concepts in geometry.

CHAPTER 2: SURVEY OF THE CURRICULUM

This chapter describes how the content of Module 1 fits into the high school geometry curriculum and how the module addresses the major changes in geometry under the Common Core State Standards. It elaborates on the professional development currently available for teachers of high school geometry, explaining why this workshop approach meets research-based requirements for effective professional development.

UNIT 1 OVERVIEW: CONTENT AND PURPOSE

The EngageNY curriculum in high school geometry begins with a module entitled “Congruence, Proof, and Constructions.” The module begins with basic constructions and leads up to an introduction to axiomatic systems. The first few lessons (1-5) aim to solidify students’ understanding of basic terminology. In the second group of lessons (6-11), students solve problems involving unknown angles and provide justification for their solutions. Following this, the module moves on to the topic of rigid motions and transformations, requiring students to examine precise definitions for each of the three rigid motions and to interpret transformations as functions or compositions of functions. Only then, after a thorough grounding in basic geometry vocabulary and conceptual knowledge of rigid motions, do students turn to defining congruence in terms of rigid motions. The unit continues with an examination of various geometric figures and proofs of some of their properties. This allows students to explore some advanced constructions. Finally, students return to the topic of axiomatic systems, reviewing all facts, theorems, and assumptions encountered throughout the unit.

Module I of the EngageNY Geometry curriculum puts emphasis on the importance of defining congruence in terms of rigid motions and sets the stage for exploring a fully developed axiomatic system. The focal standards for Module I include all thirteen congruence standards (HSG-CO.1-13), which are organized under the following four headings:

1. Experiment with transformations in the plane.
2. Understand congruence in terms of rigid motions.
3. Prove geometric theorems.
4. Make geometric constructions.

See Appendix II for a complete copy of the standards associated with these headings.

The module itself has seven divisions, each of which includes two or more lessons. These divisions are as follows:

1. Basic constructions
2. Unknown angles
3. Transformations/Rigid motions
4. Congruence
5. Proving properties of geometric figures
6. Advanced constructions
7. Axiomatic systems

The module assumes that students have certain foundational knowledge from their work in geometry in 8th grade. Specifically, students are expected to be familiar with the properties of rotations, reflections, and translations through experimental verification, to

be able to demonstrate congruence by using a series of rigid motions, to be able to describe the effects of transformations on two-dimensional figures using coordinates, and to be able to use informal arguments to establish specific facts about angles and triangles. Throughout, the module mobilizes several of the standards for mathematical practice, including students' ability to construct viable arguments and critique the reasoning of others (MP.3), model with mathematics (MP.4), use appropriate tools strategically (MP.5), and attend to precision (MP.6).

WHAT PROFESSIONAL DEVELOPMENT AND SUPPORT FOR TEACHERS IS CURRENTLY AVAILABLE FOR UNIT I?

The EngageNY high-school geometry curriculum is radically different from the curriculum in typical American geometry textbooks, necessitating a major effort to provide teachers with appropriate professional development. EngageNY provides a website that includes the geometry modules and copies of PowerPoint presentations presented to teachers at New York state-sponsored trainings. Published copies of the both teacher materials and student materials are available for purchase through Eureka Math, a division of Common Core, Inc. (the company under contract to write the curriculum for New York). Common Core, Inc. also offers professional development workshops, tailored to schools' and districts' specific needs, for a contractual price. These workshops are an outgrowth of the materials that form the basis for this thesis. Finally, WestEd offers an online professional development workshop to "Geometric Transformations in the Common Core" – a workshop that partially overlaps the content of this thesis, but does not specifically tie in with the EngageNY curriculum. I have

located no other sources of professional development that deal with Unit 1 of the EngageNY geometry curriculum to date.

WHAT DEFINES EFFECTIVE PROFESSIONAL DEVELOPMENT?

“The professional development system for teachers is, by all accounts, broken,” claims Heather C. Hill in a 2009 article written for the Phi Delta Kappan (Hill, 2009). Every time a new initiative appears on the education horizon, from the New Math of the 1960s to differentiated instruction, to No Child Left Behind, there are those quick to offer massive “new” forms of professional development to familiarize teachers with the reform efforts.

The move to the Common State Standards requires a hard look at what defines truly effective professional development. As teachers, schools, and districts evaluate available models, it is useful to review what we know about effective professional development. Guskey (2009) discusses the requirement that “effective professional learning time must be well organized, carefully structured, clearly focused, and purposefully directed.” He states that there is no single list of “best practices” defining effective professional development, instead citing the need for effective school leaders to “begin all professional development endeavors by clearly focusing on learning and learners…and work to find the most appropriate adaptation of core elements (i.e. time, collaboration, school-based orientation, and leadership) to specific contexts.” There is no single model for professional development that will work for all schools or all school districts.

Darling-Hammond noted in her 2009 article, that “it is useful to put teachers in the position of studying the very material that they intend to teach to their own students.” Taber (1998), in describing Delaware’s program for teacher development notes that teachers value “the collaborative trying out and discussion of the “investigations” (they were to teach students) beforehand.” Mathematics Science Partnership programs (funded by the National Science Foundation) typically offer this type of “immersion” program to teachers and have proven highly effective in improving both teacher knowledge and student achievement (Gersten, 2014). St. John (2009), in his description of California’s COME ON program, states that “teachers need time to deepen their mathematical knowledge and understanding.” The COME ON program was developed to train teachers to use the Interactive Math materials so widely accepted in California and many other parts of the country in the last decade. A major emphasis of the training was in allowing the teachers to experience the work as their students would experience it, and to discuss potential difficulties anticipated on the part of students.

Clearly, for professional development to be useful to teachers and to have a chance at positively impacting student learning, teachers must be immersed in the material they will be expected to teach their students. Curriculum, particularly in math, has been “pushed down” to lower grade levels under Common Core State Standards, and is often taught two or more years earlier than previously encountered. The impact on students is great, but we must also consider that the teachers themselves may well be unfamiliar with the new material, and will certainly need to learn or relearn many of the concepts they will be expected to teach. For teachers to be able to respond

adequately to students' needs for greater depth of understanding, the teachers themselves must increase the depth of their own knowledge and understanding.

The Association of Public and Land-Grant Universities notes that Colleges of Education and math departments in universities are beginning to respond to this perceived need and are reviewing courses offered to pre-service teachers to ensure that these teachers are well grounded in the new standards in Mathematics. The APLU, in its discussion paper "Science and Mathematics Teacher Imperative" (2011), has suggested partnerships between universities and K-12 school districts to help pre-service and current teachers transition to teaching with greater depth of understanding. Some programs are being made available to teachers already in the profession, providing in-depth study of topics typically taught in middle and secondary schools to teachers at all grade levels. The Louisiana Math and Science Teacher Institute is one such program, offering (under a grant provided by the National Science Foundation) a tremendous benefit to a growing cadre of teacher leaders in Baton Rouge and surrounding parishes.

HOW DOES THE PROFESSIONAL DEVELOPMENT OFFERED HERE MEET THESE IDENTIFIED CRITERIA FOR EFFECTIVENESS?

Perhaps the most focused description of good PD is the Darling-Hammond white paper (2009) referred to above. The four features that she identifies are addressed to the extent appropriate in the workshop presented here.

(1) "Intensive, ongoing, and connected to practice." The workshop is intensive in the time-frame of six hours in which it is offered. New concepts are brought into

focus through activities that require understanding and application. For example, showing that two figures are congruent entails explicit demonstration of the transformations that account for the congruence. The workshop is not itself ongoing, but it comes with supporting follow-up-materials. It is closely connected to practice because it uses the exact curriculum that teachers will deliver.

(2) “Focus on student learning and address the teaching of specific curriculum content.” Teachers in the workshop play the role of students and experience the things that students experience. All activities are keyed explicitly to the curriculum.

(3) “Align with school improvement priorities and goals.” The move to Common Core is among the major goals for virtually every school improvement plan.

(4) “Build strong working relationships among teachers.” This is the case to the extent we have control of the teachers’ time. The workshop activities are collaborative; teachers are frequently asked to critique one another’s ideas and to discuss possible student misconceptions. In addition, follow-up materials are included to assist teachers who wish to continue collaborating throughout the school year. However, we obviously have no control over time spent collaboratively after the conclusion of the workshop.

The workshop offered in this thesis puts teachers in the position of studying material they are expected to teach their students, a critical component of effective professional development identified by Darling-Hammond and supported by Gersten and St. John (for in-service teachers) and by the APLU (for pre-service teachers). The

workshop is “well-organized, carefully structured, clearly focused, and purposefully directed” as required by Guskey. It focuses on a major shift in the geometry curriculum and is directed toward enabling high school teachers to thoroughly understand this shift and be able to teach it to their own students.

CHAPTER 3: DESIGN FOR PROFESSIONAL DEVELOPMENT

In the present chapter, I provide specifics about the intended audience for the materials, the anticipated outcomes of the professional development outlined in this thesis, and the structure of the workshop in terms of topics covered, materials needed, the work the participants will be doing, and the approximate timing of the presentation.

SPECIFICATIONS

For Whom Are the Materials Intended?

The professional development outlined in this thesis are intended to be used by (1) teachers who will teach the EngageNY High School Geometry curriculum, (2) teachers who want to better understand Common Core State Standards in Geometry, (3) professional development providers who will use the material to provide professional development, and (4) those who teach other levels of mathematics to identify what knowledge students are expected to gain in the geometry curriculum or what knowledge students need to have mastered prior to studying geometry.

What Will the Professional Development Achieve?

The professional development is intended to help teachers gain a more in-depth understanding of rigid motions in general and of the role they play in the new geometry standards in particular. The program is designed to immerse teachers in the concepts they will be teaching their students, allowing the teachers to experience the content first-hand and to grapple with unfamiliar tools and new depth of ideas before their students.

By the end of the workshop, teachers are expected to be able to provide reasoned responses to the following key questions:

1. What is a transformation?
2. What role do transformations play in Common Core State Standards in Mathematics, specifically in the geometry standards?
3. What is *congruence*?
4. How has the definition of congruence changed under Common Core State Standards in Mathematics?
5. What is an axiomatic system and how is such a system used in geometry?

In addition to the content knowledge outlined above, teachers who complete this workshop will be able to anticipate the specific areas in the geometry curriculum where students may experience difficulty and will identify strategies to address these difficulties. Teachers will experience the struggles themselves and collaborate with one another as they will expect their students to do.

The additional exploratory materials, condensed from course notes prepared by Dr. James Madden, will allow professional learning communities of geometry teachers to grapple with axiomatic systems and with rigid motions at a depth greater than that taught to high school students.

STRUCTURE

Topics

The professional development presentation (Appendices I and II) begins with an overview of the entire high school math curriculum as promulgated by EngageNY, and shows where Geometry Module I fits into this curriculum. The presenter then briefly identifies and describes the five modules in the entire Geometry curriculum.

Lesson 1 (Constructing an Equilateral Triangle) is presented in its entirety to give participants an in-context grounding in the structural components of lessons in the geometry curriculum. Participants are expected to complete all problems the students will complete, and will discuss each topic to be presented to students.

Next, presenters go through the instructional sequence in Module I, beginning with the module overview, giving a brief description of the seven topics within the module, pointing out the key changes in geometry under Common Core State Standards, and then providing at least one exemplar problem from each lesson in the module to highlight the concepts to be taught. The end-of-module assessment is provided in its entirety. Throughout the presentation, participants are encouraged to work collaboratively to solve the exemplar problems while the presenter walks around the room to provide assistance and encouragement as needed.

After a brief review of the module highlights, participants will work together to identify specific plans for implementing Module I at the beginning of the school year.

Finally, the presenter outlines follow-up professional development (Appendix III)

as a means for allowing geometry teachers to gain greater depth of understanding of the new geometry standards.

The specific work that participants do in the workshop is as follows. Participants are provided with a copy of full Module I (teacher and student materials), a booklet (Appendix II) containing the specific problems worked throughout the presentation, a complete listing of the high school geometry standards under Common Core State Standards, and a listing and description of the Standards for Mathematical Practices. Materials needed to complete the problems include rulers, compasses, pencils, markers, and paper. Participants are expected to work the problems throughout the presentation and discuss potential student misconceptions with one another.

Finally, as part of the in-depth follow-up study of axiomatic systems and rigid motion transformations, teachers are provided with guided problem solving activities to use during meetings of professional learning communities of geometry teachers. The follow-up meetings may take place in person or via internet meetings or discussion boards.

Timing. The core professional development is expected to take approximately six hours as presented. The follow-up study is expected to take approximately two to four hours per month of discussion (either in-person or via internet discussion boards) throughout the school year, not counting the preparatory reading and exploration.

EXPECTED OUTCOMES

By participating in the program, teachers will gain a greater, more in-depth understanding of the high school geometry congruence standards and how they may best be taught to high school students. They will also identify potential sources of student misunderstanding and identify scaffolds to ensure that students are brought to greater understanding.

CHAPTER 4: IMPLEMENTATION EXPERIENCE AND CONCLUSIONS

In this final chapter, I discuss my own experiences presenting this workshop, including dates, locations, numbers of participants, and specific ideas learned from each presentation.

IMPLEMENTATION EXPERIENCE

To date, I have presented this workshop on four separate locations in four parishes in Louisiana, to a total of approximately two hundred teachers and administrators. Table 1 below shows specific information about each of the four workshop presentations to date.

Table 1. Implementation Experience

Date	Location	Type of facility	Approximate number of participants	Comments
May 29, 2014	Raceland, LA	High school classroom	15	More exemplar problems were added. Time was available to move to an available computer lab to show participants how instruction might be enhance using Geogebra software. Participants welcomed the opportunity to exchange e-mail information and plan to collaborate via the internet over the course of the school year.
July 16, 2014	Lafayette, LA	University of Louisiana – Lafayette (moved to local community college)	60	Flexibility is important; a bomb scare at the university required us to change our venue to the local community college. Participants particularly value anecdotal information about classroom experiences with the curriculum.

(Table 1 continued)

Date	Location	Type of facility	Approximate number of participants	Comments
July 18, 2014	Alexandria, LA	High school library	50	I learned to involve the audience to a greater extent. Instead of working through the exemplar problems myself, I invited audience members to explain their approaches, which was very well-received.
July 25, 2014	Bossier City, LA	Large classroom in church building	60	Round tables were made available, facilitating better discussion among participants.

All presentations were very well-received, with participants expressing both gratitude and major relief at having the opportunity to experience these materials as students and to engage in conversations with a presenter who both assisted with the writing of the materials and who has actually used the materials in a classroom setting. The participating teachers stated that they now felt more secure in their understanding of the curriculum and in the requirements of the Common Core State Standards in Mathematics. They felt ready to teach the concepts, and expressed a hope that further presentations would be made available for the remaining modules.

CONCLUDING THOUGHT

Common Core State Standards are here to stay. Our students must be expected to understand mathematical concepts at a much greater depth than previously experienced to allow them to compete in an increasingly complex world market. If

students are to meet these rigorous expectations, teachers must be prepared to help them. Teachers must themselves be exposed to greater depth of knowledge of their content than they have previously been teaching. The professional development structure outlined in this thesis is easily adaptable to other modules in geometry and to any of the middle and high school mathematics modules as developed under the auspices of EngageNY.

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APPENDIX I

Starting the Year in Geometry

A Close Look at Congruence,
Proof, and Transformations

Participant Poll

- Classroom teacher
- Math trainer / coach
- Principal or school leader
- District representative / leader
- Other

Curriculum Map

	Grade 9 – Algebra I	Grade 10 – Geometry	Grade 11 – Algebra II	Grade 12 – Precalculus	
20 days	M1: Relationships Between Quantities and Reasoning with Equations and Their Graphs (40 days)	M1: Congruence, Proof, and Constructions (45 days)	M1: Polynomial, Rational, and Radical Relationships (45 days)	M1: Complex Numbers and Transformations (40 days)	20 days
20 days					20 days
20 days	M2: Descriptive Statistics (25 days)	M2: Similarity, Proof, and Trigonometry (45 days)	M2: Trigonometric Functions (20 days)	M2: Vectors and Matrices (40 days)	20 days
20 days	M3: Linear and Exponential Functions		M3: Functions (45 days)		20 days
	State Examinations	State Examinations	State Examinations	State Examinations	
20 days	(35 days)	M3: Extending to Three Dimensions (10 days)		M3: Rational and Exponential Functions (25 days)	20 days
20 days	M4: Polynomial and Quadratic Expressions, Equations and Functions (30 days)	M4: Connecting Algebra and Geometry through Coordinates (25 days)		M4: Trigonometry (20 days)	20 days
20 days			M4: Inferences and Conclusions from Data (40 days)		20 days
20 days	M5: A Synthesis of Modeling with Equations and Functions (20 days)	M5: Circles with and Without Coordinates (25 days)		M5: Probability and Statistics (25 days)	20 days
20 days	Review and Examinations	Review and Examinations	Review and Examinations	Review and Examinations	20 days

Key:	Number and Quantity and Modeling	Geometry and Modeling	Algebra and Modeling	Statistics and Probability and Modeling	Functions and Modeling

Session Objectives

- Examine the sequence of concepts across the module.
- Study mathematical models and instructional strategies from the high school curriculum.
- Review examples that highlight themes and changes according to the Common Core State Standards.
- Prepare to implement this and other high school modules.

Agenda

- **Lesson Structure**
- **Instructional Sequence**
- **Module Review**
- **Preparation for Implementation**

Lesson 1: Construct an Equilateral Triangle

Flow of the lesson:

- **Opening Exercise:** (1) A question that jogs students ideas on distance (2) key vocabulary review
- **Example 1:** Guide students to discovering how to construct an equilateral triangle (with the use of a compass and straightedge).
- **Example 2:** Read an excerpt from *Elements* that describes Euclid's approach to constructing an equilateral triangle; have students develop their own set of steps on how to construct an equilateral triangle.
- **Geometry Assumptions**: Introduce the notion that geometry is an axiomatic system- everything in the subject can be traced back to a short list of basic assumptions we take for granted.

Lesson 1: Construct an Equilateral Triangle

Student Outcomes

- Students learn to construct an equilateral triangle.
- Students communicate mathematical ideas effectively and efficiently.

Lesson 1: Lesson Notes

- Describe the purpose of the lesson and how the lesson fits into the overall story of the module.
- Notes prior knowledge expected of students.
- Explains the “flow” of the lessons.
- Provides any specific background knowledge useful for the teacher.

Lesson 1: Opening Exercise

Joe and Marty are in the park playing catch. Tony joins them, and the boys want to stand so that the distance between any two of them is the same.

- How do they figure this out precisely?
- What tool or tools could they use?

Lesson 1: Opening Exercise (cont.)

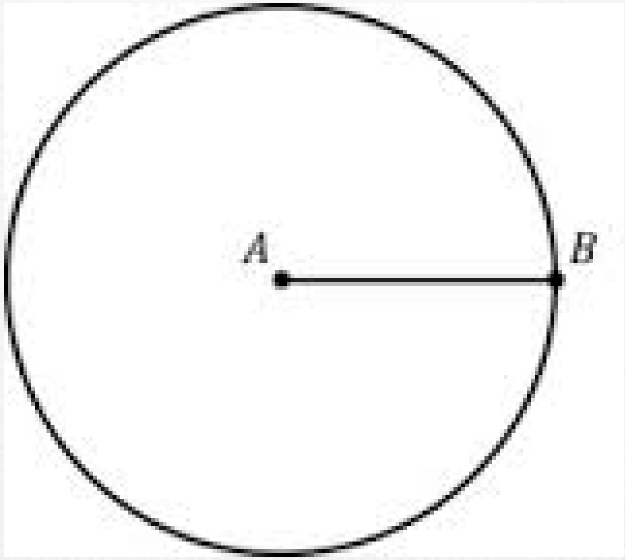
Fill in the blanks below as each term is discussed:

1. **segment** ______________ The _______ between points A and B is the set consisting of A, B, and all points on the line $\overleftrightarrow{AB}$ between A and B.

2. **radius** ______________ A segment from the center of a circle to a point on the circle.

3. **circle** ______________ Given a point C in the plane and a number $r > 0$, the _______ with center C and radius r is the set of all points in the plane that are distance r from the point C.

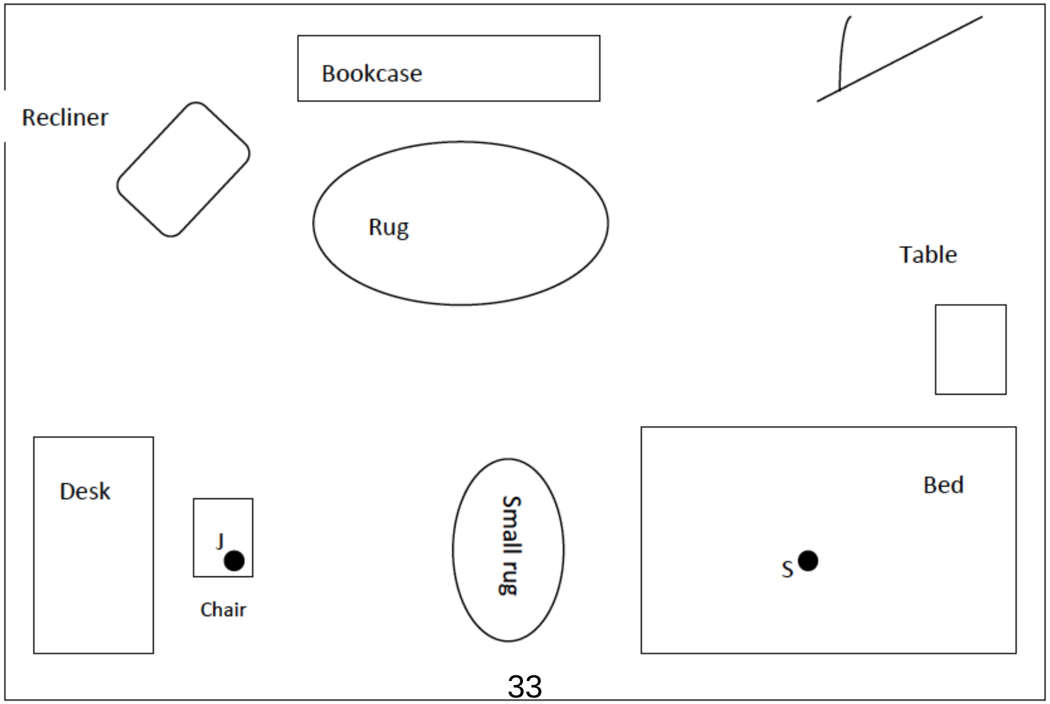
$Circle_A, radius\ AB$



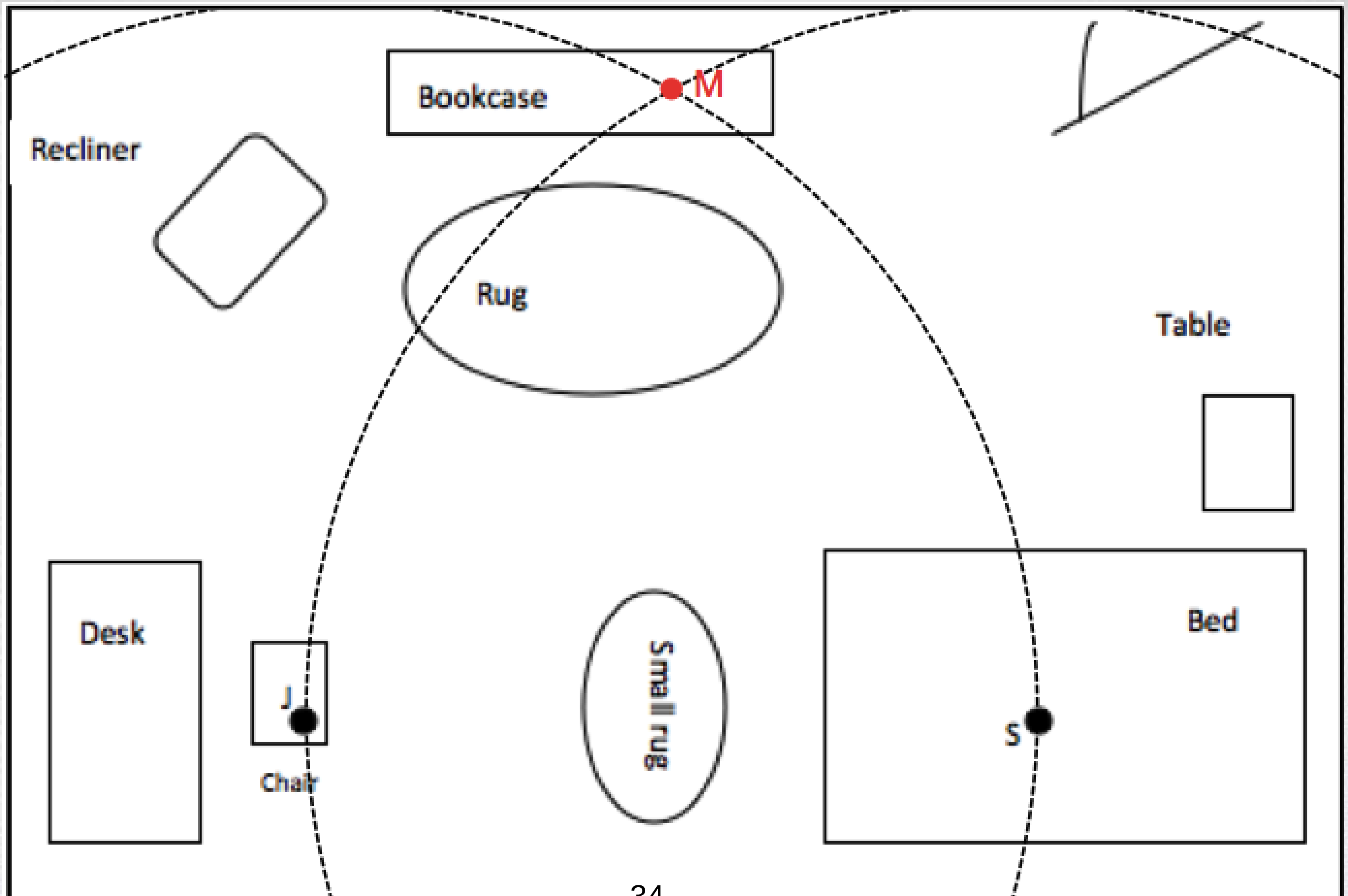
Notation is important!

Lesson 1, Example 1

Margie has three cats. She has heard that cats in a room position themselves at equal distances from one another and wants to test that theory. Margie notices that Simon, her tabby cat, is in the center of her bed (at **S**), while JoJo, her Siamese, is lying on her desk chair (at **J**). If the theory is true, where will she find Mack, her calico cat? Use the scale drawing of Margie's room shown below, together with (**only**) a compass and straightedge. Place an **M** where Mack will be if the theory is true.



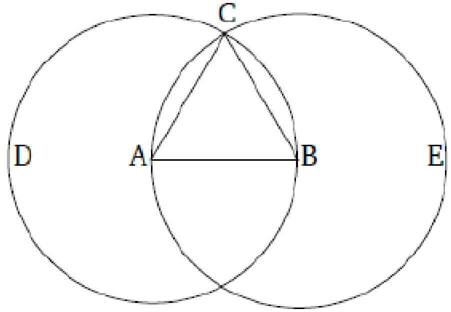
Lesson 1, Example 1



Lesson 1, Example 2

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

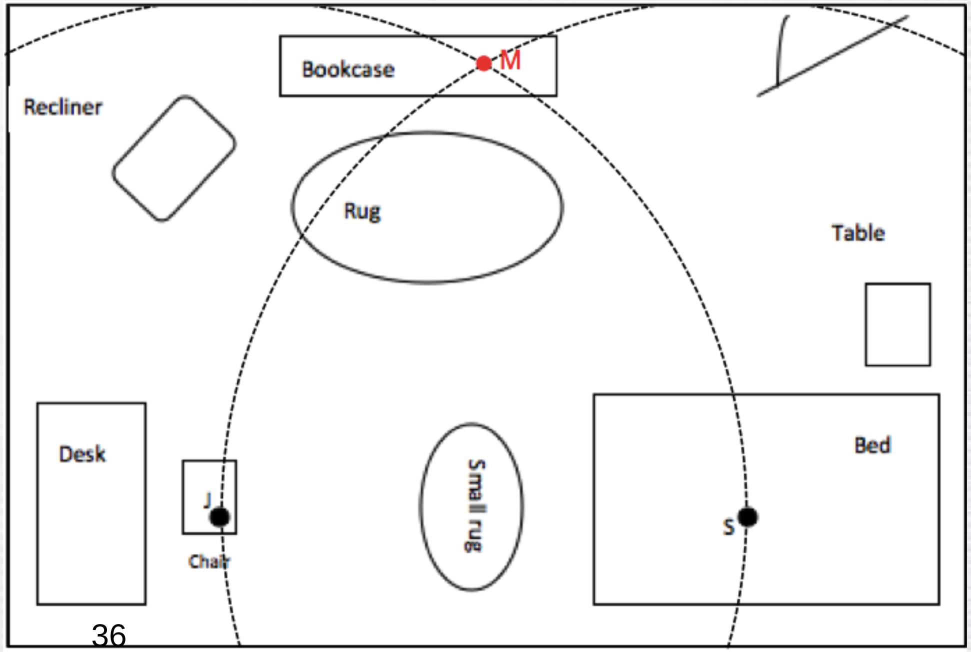
And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

In this margin, compare your steps with Euclid's.

Lesson 1, Example 2

1. Draw circle C_J: center J, radius JS.
2. Draw circle C_S: center S, radius SJ.
3. Label the intersection as M.
4. Join S, J, M.



Lesson 1: Geometry Assumptions

Discussion about points, lines, planes, and distance.

Lesson 1: Relevant Vocabulary

- Geometric construction
- Figure
- Equilateral triangle
- Collinear
- Length of a Segment
  - Notation in geometry
- Coordinate System on a Line

Lesson 1: Exit Ticket

We saw two different scenarios where we used the construction of an equilateral triangle to help determine a needed location (i.e., the friends playing catch in the park and the sitting cats). Can you think of another scenario where the construction of an equilateral triangle might be useful? Articulate how you would find the needed location using an equilateral triangle.

Lesson 1: Exit Ticket

Students might describe a need to determine the locations of fire hydrants, friends meeting at a restaurant, or parking lots for a stadium, etc.

Clear instructions for finding the needed location should be included in students' responses.

Lesson 1: Problem Set

1. Write a clear set of steps for the construction of an equilateral triangle. Use Euclid's Proposition 1 as a guide.

1. *Draw circle J: center J, radius JS.*
2. *Draw circle S: center S, radius SJ.*
3. *Label the intersection as M.*
4. *Join S, J, M.*

Lesson 1: Problem Set #2

Suppose two circles are constructed using the following instructions:

Draw circle: Center A, radius AB.

Draw circle: Center C, radius CD.

Under what conditions (in terms of distances AB, CD, AC) do the circles have

a. One point in common?

b. No points in common?

c. Two points in common?

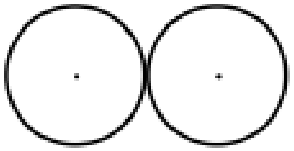
d. More than two points in common? Why?

Lesson 1: Problem Set #2

a. One point in common?

If $AB + CD = AC$ *or* $AC + AB = CD$ *or* $AC + CD = AB$.

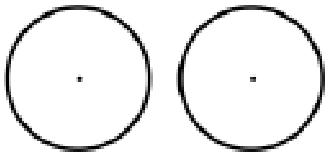
Ex.



b. No points in common?

If $AB + CD < AC$ *or* $AB + AC < CD$ *or* $CD + AC < AB$.

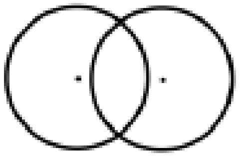
Ex.



Lesson 1: Problem Set #2

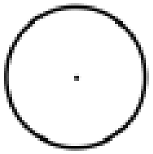
c. Two points in common?

If $AC < AB + CD$ *and* $CD < AB + AC$ *and* $AB < CD + AC$.

Ex. 

d. More than two points in common? Why?

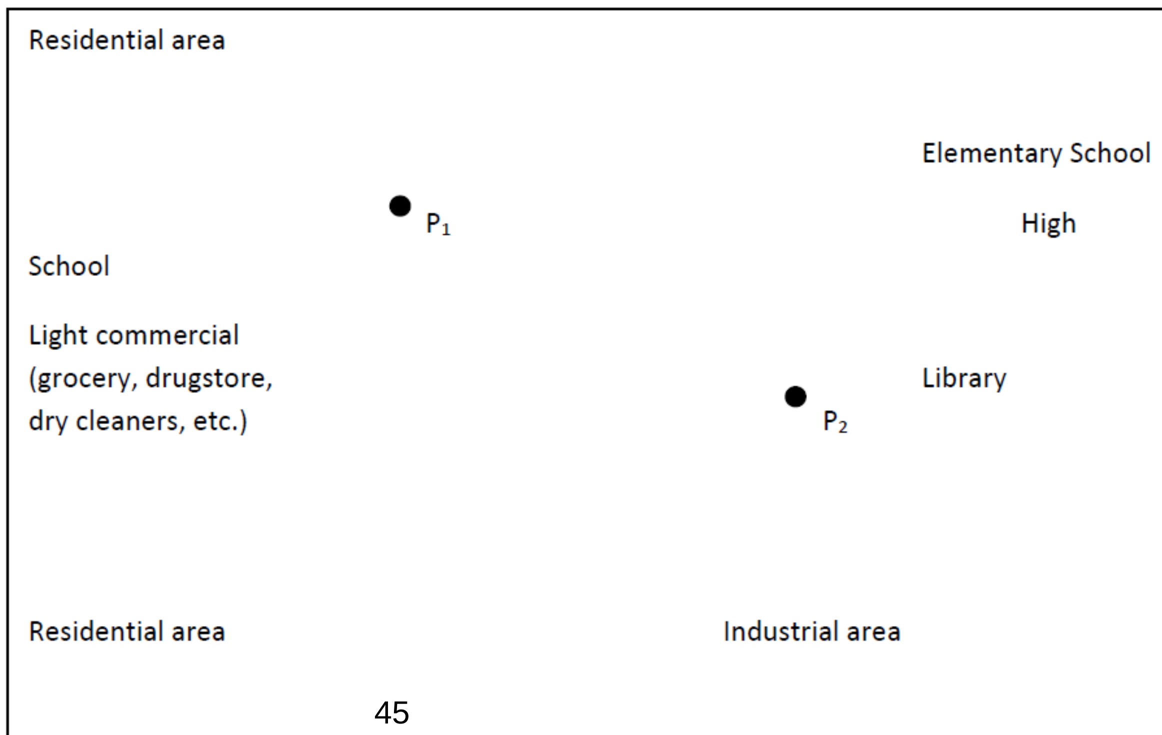
If $A = C$ *(same points) and* $AB = CD$.

Ex. 

Lesson 1 Problem Set #3

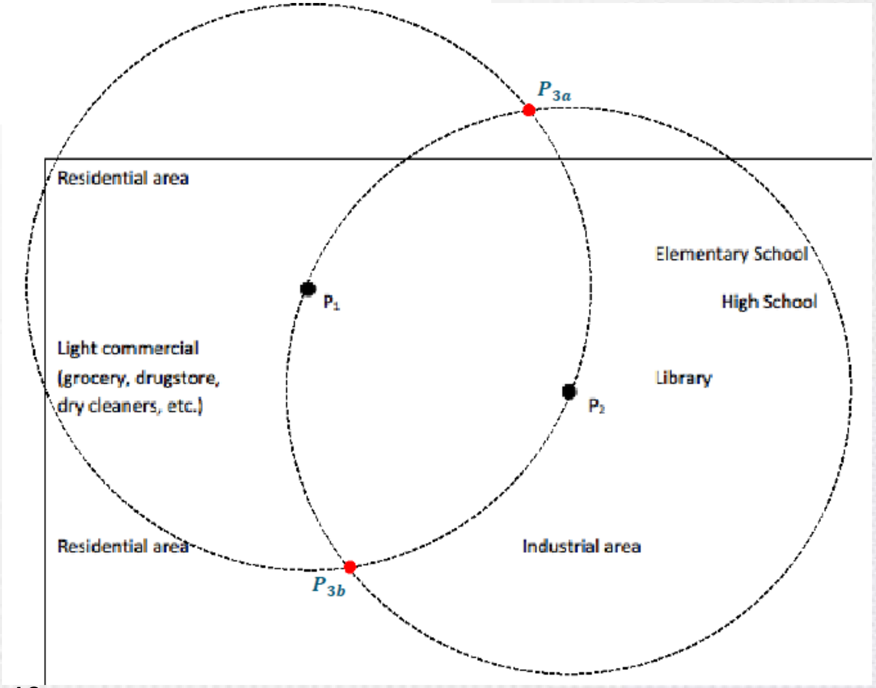
3. *You will need* a compass and straightedge.

   Cedar City boasts two city parks and is in the process of designing a third. The planning committee would like all three parks to be equidistant from one another to better serve the community. A sketch of the city appears below, with the centers of the existing parks labeled as P_1 and P_2. Identify two possible locations for the third park, and label them as P_{3a} and P_{3b} on the map. Clearly and precisely list the mathematical steps used to determine each of the two potential locations.



Lesson 1 Problem Set #3

1. *Draw a circle P_1: center P_1, radius $\overline{P_1P_2}$.*
2. *Draw a circle P_2: center P_2 radius $\overline{P_2P_1}$.*
3. *Label the two intersections of the circles as P_{3a} and P_{3b}.*
4. *Join P_1, P_2, P_{3a} and P_1, P_2, P_{3b}.*



What's In a Module?

Teacher Materials

- Module Overview
- Topic Overviews
- Daily Lessons
- Assessments

Student Materials

- Daily Lessons with Problem Sets

Copy Ready Materials

- Exit Tickets
- Assessments

What's In a Lesson?

Teacher Materials Lessons

- Student Outcomes and Lesson Notes (in select lessons)
- Classwork
  - General directions and guidance, including timing guidance
  - Bulleted discussion points with expected student responses
  - Student classwork with solutions (boxed)
- Exit Ticket with Solutions
- Problem Set with Solutions

Student Materials

- Classwork
- Problem Set

Types of Lessons

1. Problem Set

   Students and teachers work through examples and complete exercises to develop or reinforce a concept.

2. Socratic

   Teacher leads students in a conversation to develop a specific concept or proof.

3. Exploration

   Independent or small group work on a challenging problem followed by debrief to clarify, expand or develop math knowledge.

4. Modeling

   Students practice all or part of the modeling cycle with real-world or mathematical problems that are ill-defined.

Overview of the Year

- Module 1: Congruence, Proof, and Constructions
  - 45 days, 34 lessons
- Module 2: Similarity, Proof, and Trigonometry
  - 45 days, 33 lessons
- Module 3: Extending to Three Dimensions
  - 15 days, 13 lessons
- Module 4: Connecting Algebra to Geometry Through Coordinates
  - 20 days, 15 lessons
- Module 5: Circles with and without Coordinates
  - 25 days, 20 lessons

Agenda

- Lesson Structure
- Instructional Sequence
- Module Review
- Preparation for Implementation

Module Overview

- Overview
- Focus Standards
- Foundational Standards
- Mathematical Practices
- Terminology
- Suggested Tools
- Assessment Summary
- Topic Overviews

Module 1: Congruence, Proof, and Constructions

- **Topic A: Basic Constructions**
- **Topic B: Unknown Angles**
- **Topic C: Transformations/Rigid Motions**
- **Topic D: Congruence**
- Topic E: Proving Properties of Geometric Figures
- Topic F: Advanced Constructions
- Topic G: Axiomatic Systems

Key Changes in CCSS that Impact M1

"**The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation.** Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally)…

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries onto the other."

Key Changes in CCSS that Impact M1

- **Transformations**
  - How they are first introduced
  - The manner in which they are described and studied
  - Their use in the definition of congruence and similarity
  - Other uses to which transformations are put, e.g., reasoning and steps in proofs

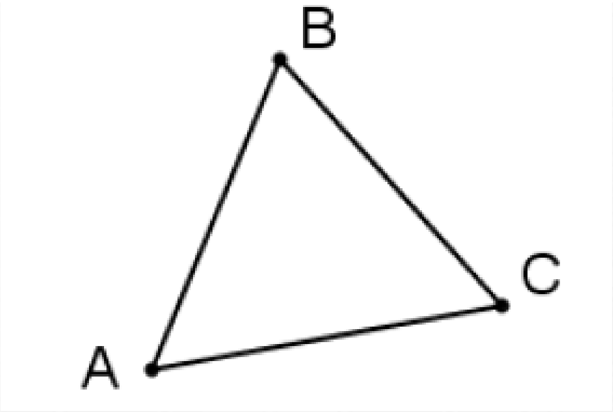
Topic A: Basic Constructions

- Standards
  - G-CO.1, G-CO.12, G-CO.13
- Constructions
  - How to:
    - construct an equilateral triangle
    - copy an angle
    - bisect an angle
    - construct a perpendicular bisector
    - determine the incenter and circumcenter of a triangle

Lesson 2: Construct an Equilateral Triangle

Exit Ticket

$\triangle ABC$ is shown below. Is it an equilateral triangle? Justify your response.



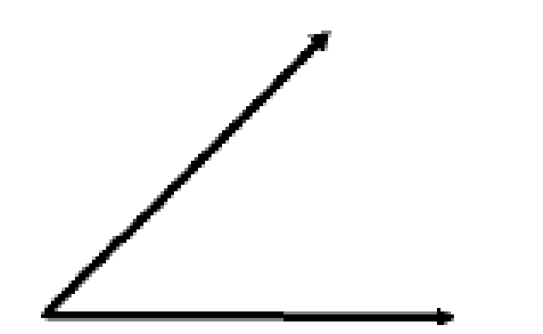
The triangle is not equilateral. Students may prove this by constructing two intersecting circles using any two vertices as the given starting segment. The third vertex will not be one of the two intersection points of the circles.

Lesson 3: Copy and Bisect and Angle

Exercise 2: Investigate How to Copy an Angle

You will need a compass and a straightedge.

You and your partner will be provided with a list of steps (in random order) needed to copy an angle using a compass and straightedge. Your task is to place the steps in the correct order, and then follow the steps to copy the angle below.



Lesson 3: Copy and Bisect an Angle

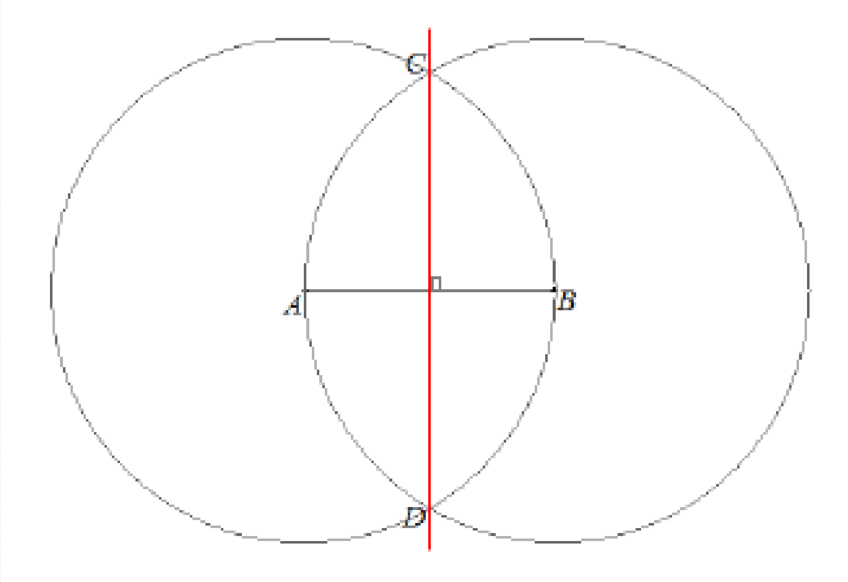
Steps needed (in correct order):

Steps to copy an angle:

1. *Label the vertex of the original angle as* B.
2. *Draw* $\overrightarrow{EG}$ *as one side of the angle to be drawn.*
3. *Draw circle* B*: center* B*, any radius.*
4. *Label the intersections of circle B with the sides of the angle as* A *and* C.
5. *Draw circle* E*: center* E*, radius* BA.
6. *Label intersection of circle E with* $\overrightarrow{EG}$ *as* F.
7. *Draw circle* F*: center* F*, radius* CA.
8. *Label either intersection of circle* E *and circle* F *as* D.
9. *Draw* $\overrightarrow{ED}$.

Lesson 4: Construct a Perpendicular Bisector

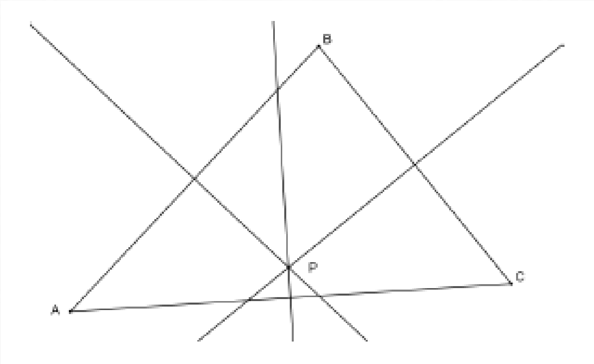
Students construct the perpendicular bisector of a line segment and examine the symmetries of the construction.



They also construct the perpendicular to a line from a point not on the line.

Lesson 5: Points of Concurrencies

Students construct the perpendicular bisectors of all three sides of a triangle and note that the three bisectors have a point of concurrency.

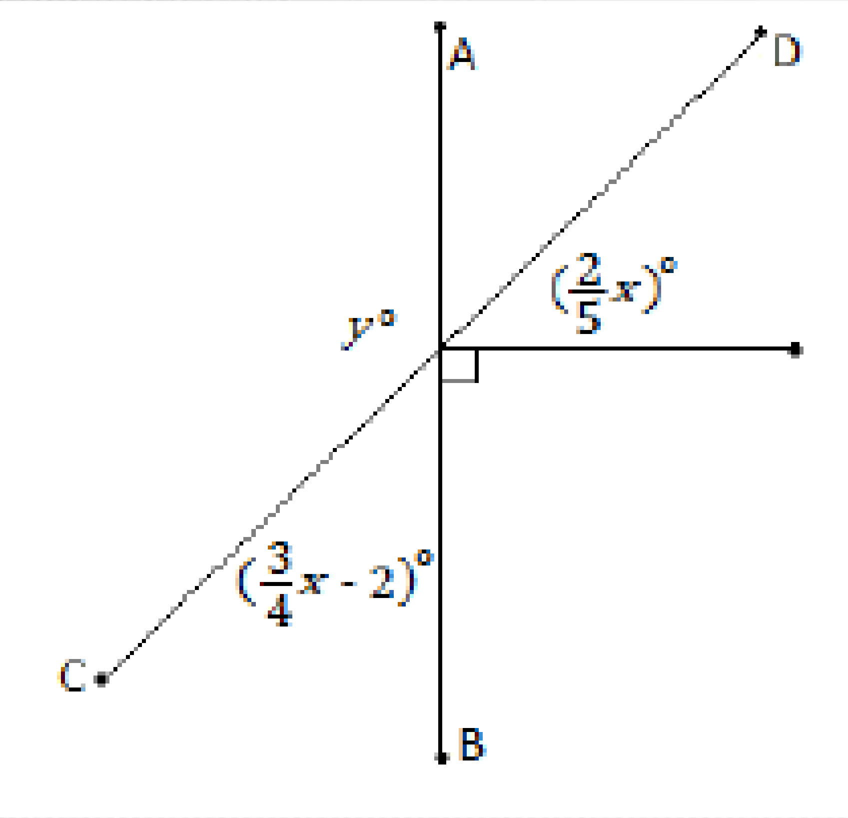


They are guided to explain WHY the bisectors are concurrent, drawing on their understanding of the properties of perpendicular bisectors and their relationship to endpoints of the segments they are bisecting.

Topic B: Unknown Angles

- Standards
  - G-CO.9
- Breakdown of Lessons 6-11:
  - Lessons 6-8: Unknown angle problems involving angles and lines at a point, transversals, and angles in a triangle. Students use simple justification in non-algebraic steps.
  - Lessons 9-11: Unknown angle proofs, non-numeric problems emphasizing justification of a relationship, sometimes a known existing relationship.

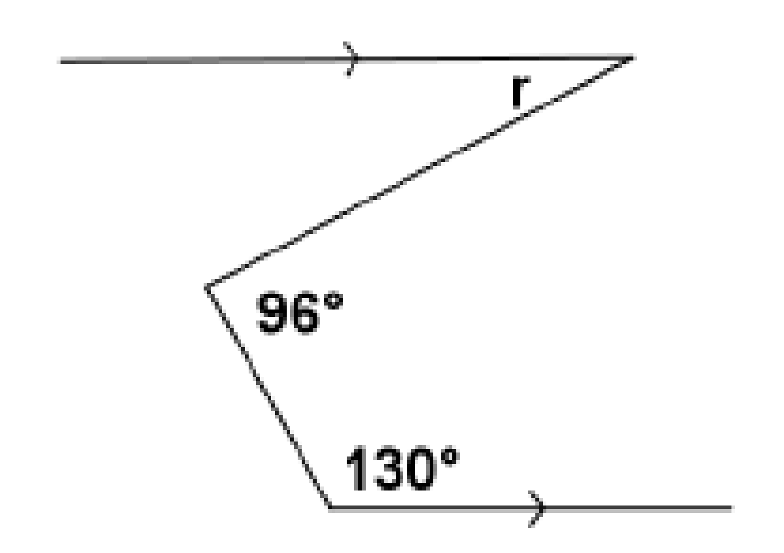
Lesson 6: Solve for Unknown Angles – Angles and Lines at a Point



$x = 80$; $y = 122$

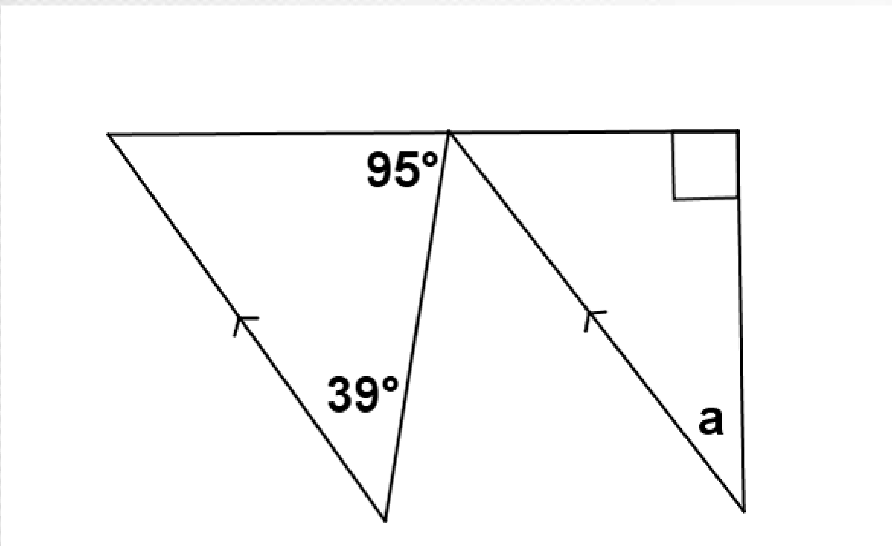
Consecutive adjacent angles on a line sum to 180°.

Lesson 7: Solve for Unknown Angles -- Transversals



$m\angle r = 46°$, *If parallel lines are cut by a transversal, then interior angles on the same side are supplementary; If parallel lines are cut by a transversal, then alternate interior angles are equal in measure*

Lesson 8: Solve for Unknown Angles – Angles in a Triangle



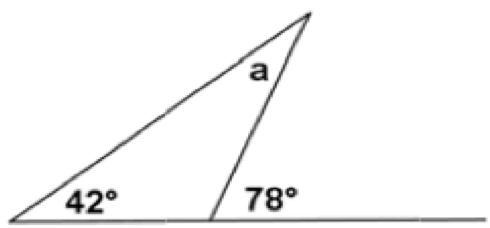
$m\angle a =$ 44°

Topic B: Unknown Angles

Lesson 8, Exercise 2:

In each figure, determine the measures of the unknown (labeled) angles. Give reasons for your calculations.

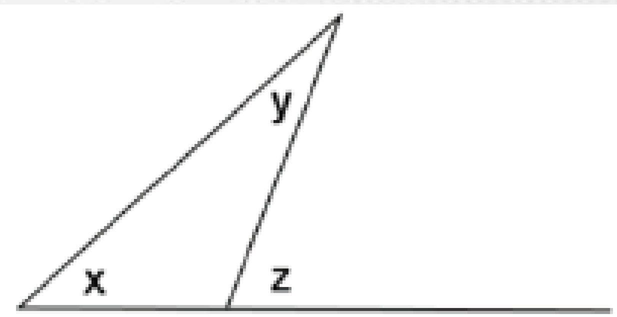
2.



$m\angle a = 36°$, *In a triangle, the exterior angle equals the sum of the two non-adjacent angles.*

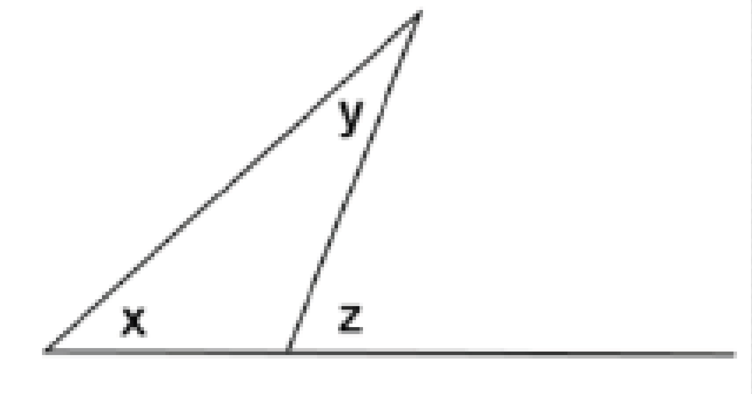
Lesson 9, Discussion:

What can be established about x, y, and z?



Topic B: Unknown Angles

What can be established about x, y, and z?

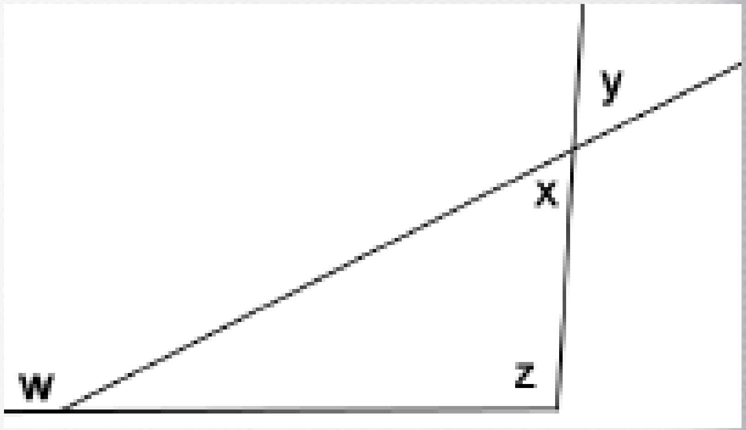


Label $\angle w$, as shown in the diagram.

$m\angle x + m\angle y + m\angle w = 180°$	Sum of the angles in a triangle is 180˚.
$m\angle w + m\angle z = 180°$	Linear pairs form supplementary angles.
$m\angle x + m\angle y + m\angle w = m\angle w + m\angle z$	Substitution Property of Equality
$\therefore m\angle x + m\angle y = m\angle z$	Subtraction Property of Equality

Lesson 9: Unknown Angle Proofs – Writing Proofs

Given the diagram to the right, prove that $m\angle w = m\angle y + m\angle z$.



$m\angle w = m\angle x + m\angle z$	*Exterior angle of a triangle equals the sum of the two interior opposite angles*
$m\angle x = m\angle y$	*Vertical angles are equal in measure*
$\therefore\ m\angle w = m\angle y + m\angle z$	*Substitution property of equality*

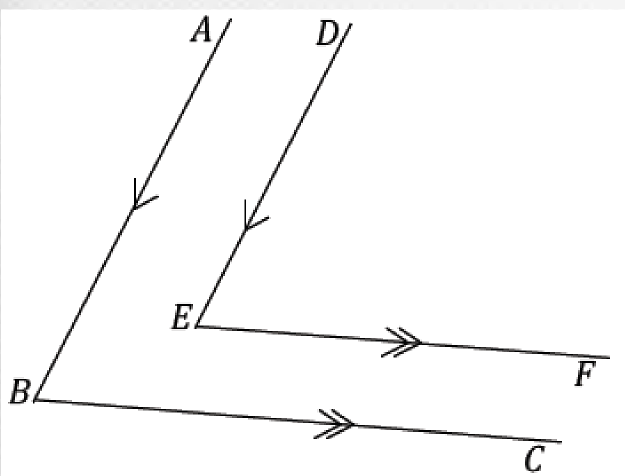
Lesson 10, Unknown Angle Proofs - Proofs with Constructions

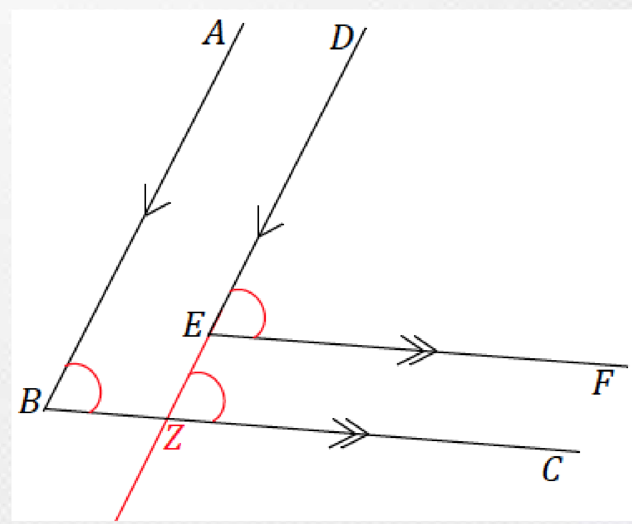
Problem Set #1

AB DE and $BC \| EF$. Prove that
m $ABC = m$ DEF.

Extend DE through BC, and mark the intersection with BC as Z.

$m\angle ABC = m\angle EZC$	*If parallel lines are cut by a transversal, the corresponding angles are equal.*
$m\angle EZC = m\angle DEF$	*If parallel lines are cut by a transversal, the corresponding angles are equal.*
$m\angle ABC = m\angle DEF$	*Transitive Property*





Lesson 11: Unknown Angle Proofs – Proofs of Known Facts

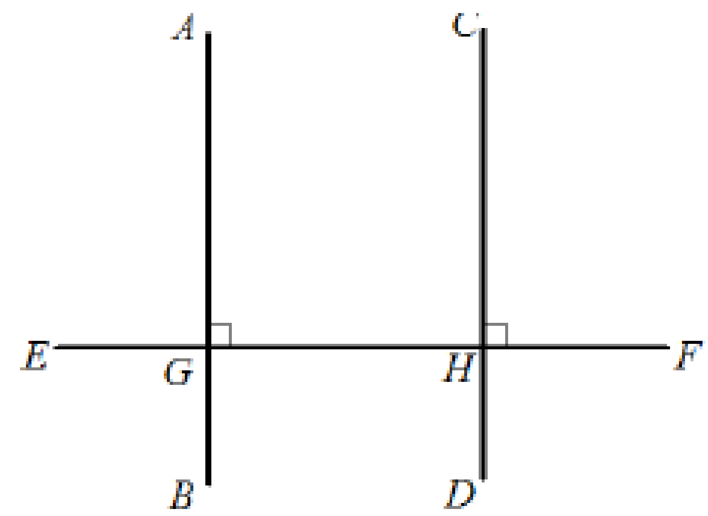
A theorem states that in a plane, if a line is perpendicular to one of two parallel lines and intersects the other, then it is perpendicular to the other of the two parallel lines.

Prove this theorem. (a) Construct and label an appropriate figure, (b) state the given information and the theorem to be proved, and (c) list the necessary steps to demonstrate the proof.

Lesson 11: Unknown Angle Proofs – Proofs of Known Facts

Given: $\overline{AB} \parallel \overline{CD}$, $\overline{EF} \perp \overline{AB}$, $\overline{EF}$ *intersects* $\overline{CD}$

Prove: $\overline{EF} \perp \overline{CD}$



$\overline{AB} \parallel \overline{CD}$, $\overline{EF} \perp \overline{AB}$	*Given*
$m\angle BGH = 90°$	*Definition of perpendicular lines*
$m\angle BGH = m\angle DHF$	*If parallel lines are cut by a transversal, then corresponding angles are equal in measure*
$\overline{EF} \perp \overline{CD}$	*If two lines intersect to form a right angle, then the two lines are perpendicular*

Topic C: Transformations/Rigid Motions

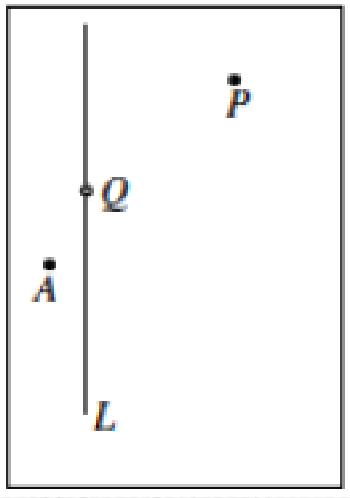
- Standards
  - G-CO.2, G-CO.3, G-CO.4, G-CO.5, G-CO.6, G-CO.7 G-CO.12
- Key concepts in Lessons 12-21:
  - Formal definition of each rigid motion (rotation, reflection, translation) and how each rigid motion is related to constructions
  - Compositions of rigid motions
  - Applications of rigid motions: discussion regarding correspondence, how congruence is defined

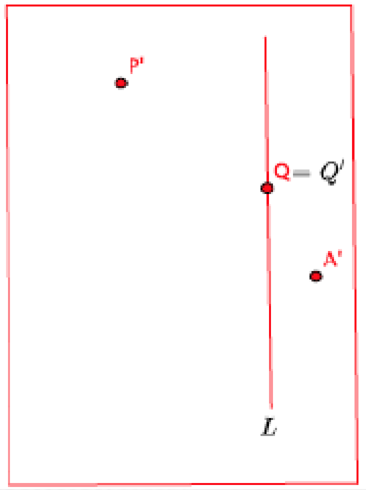
Rigid Motions in Grade 8

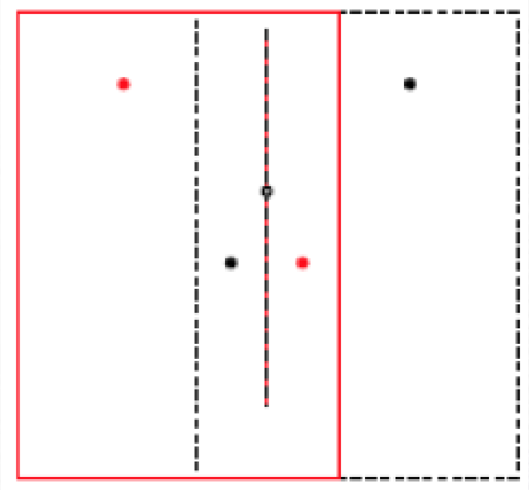
- What knowledge do students arrive with?
  - Rigid motions first introduced in Grade 8, Module 2
  - G8 focus:
    - Understanding of what a transformation is
    - Intuitive understanding and experimental verification of the properties of the rigid motions (translation, reflection, and rotation).
    - Descriptive definitions
    - Hands-on exploration with the use of transparencies

Grade 8, Module 2, Lesson 4

Grade 8 introduction to reflections:







Topic C: Transformations/Rigid Motions

Instructional Days:	10
Lesson 12:	Transformations—The Next Level (M)[1]
Lesson 13:	Rotations (E)
Lesson 14:	Reflections (E)
Lesson 15:	Rotations, Reflections, and Symmetry (E)
Lesson 16:	Translations (E)
Lesson 17:	Characterize Points on a Perpendicular Bisector (S)
Lesson 18:	Looking More Carefully at Parallel Lines (S)
Lesson 19:	Construct and Apply a Sequence of Rigid Motions (S)
Lesson 20:	Applications of Congruence in Terms of Rigid Motions (S)
Lesson 21:	Correspondence and Transformations (P)

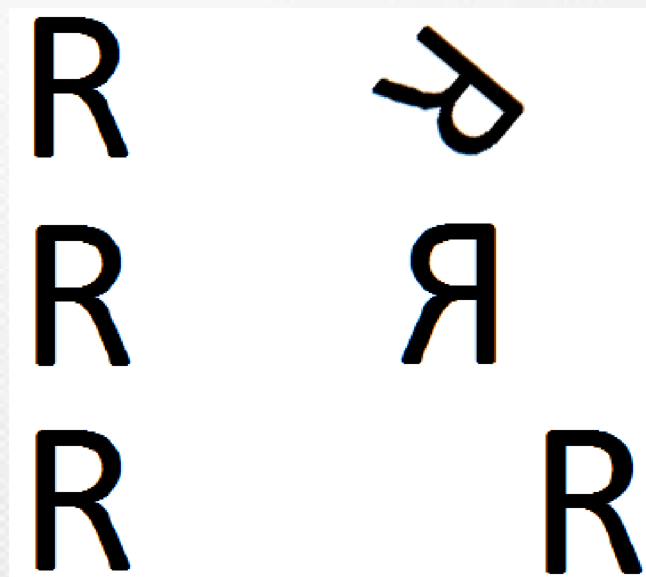
Lesson 12: Transformations – The Next Level

A transformation *F* of the plane is a function that assigns to each point *P* of the plane a unique point *F(P)* in the plane.

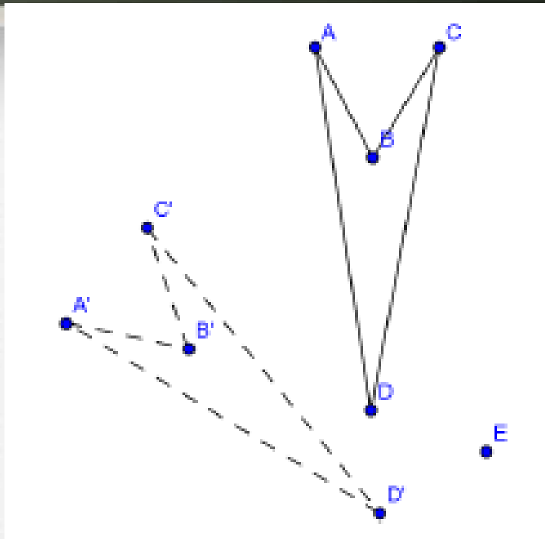
rotation

reflection

translation



Lesson 13: Rotations



For $0° < \theta < 180°$, the rotation of θ degrees around the center C is the transformation $R_{C,\theta}$ of the plane defined as follows:

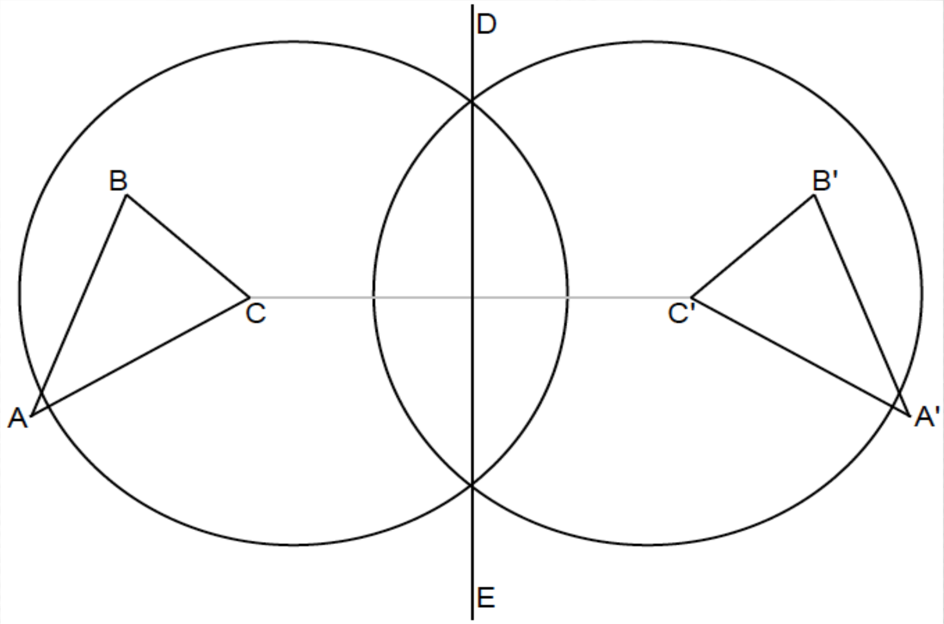
1. For the center point C, $R_{C,\theta}(C) = C$, and
2. For any other point P, $R_{C,\theta}(P)$ is the point Q that lies in the counterclockwise

Lesson 14: Reflections

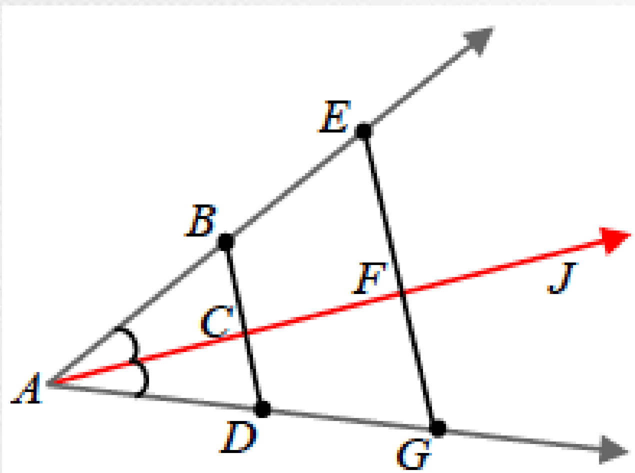
Exploratory Challenge

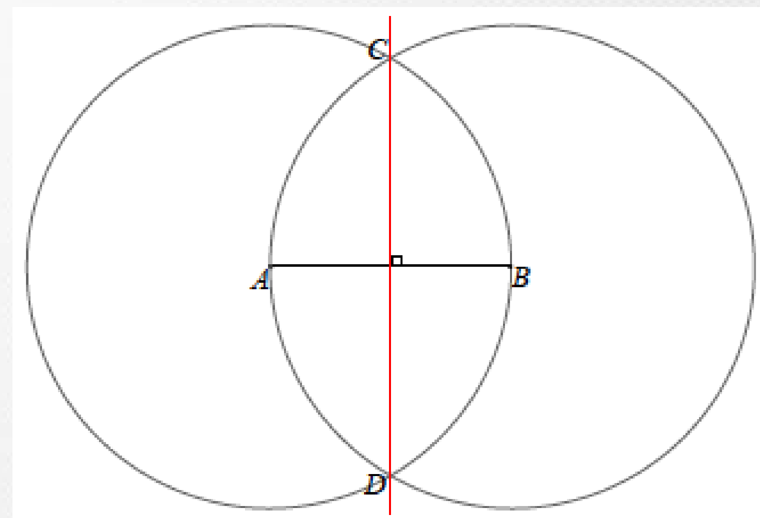
$\triangle ABC$ is reflected across DE and maps onto $\triangle A'B'C'$.

Use your compass and straightedge to construct the perpendicular bisector of each of the segments connecting A to A', B to B', and C to C'. What do you notice about these perpendicular bisectors?



Reflections and the Perpendicular Bisector Construction



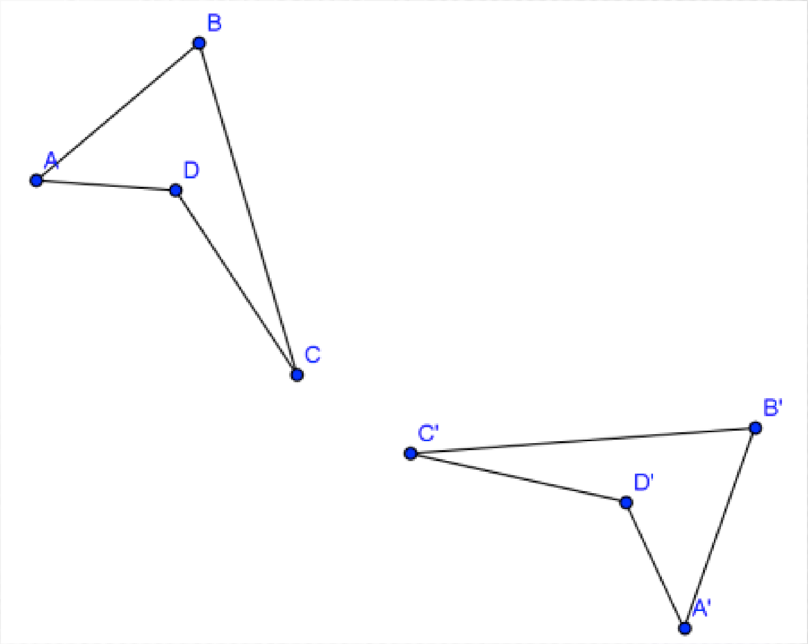


Lesson 14: Reflections

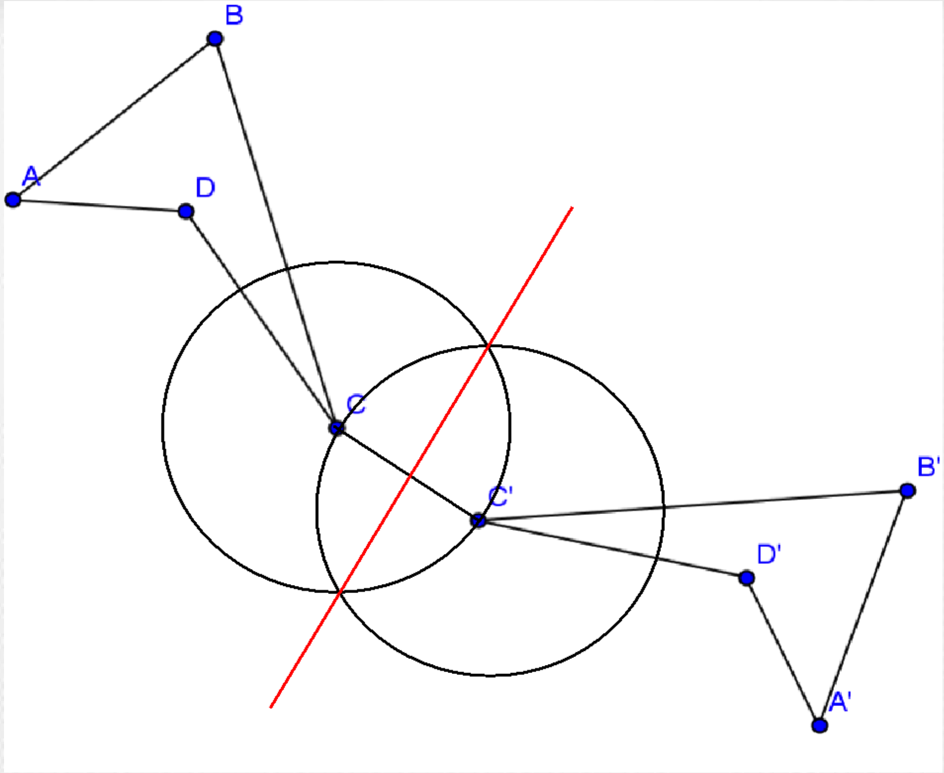
Example 1

Construct the segment that represents the line of reflection for quadrilateral $ABCD$ and its image $A'B'C'D'$.

What is true about each point on $ABCD$ and its corresponding point on $A'B'C'D'$ with respect to the line of reflection?



Lesson 14: Reflections



Lesson 14: Reflections

Definition of Reflection:

For a line l in the plane, a *reflection across* l is the transformation r_l of the plane defined as follows:

1. **For any point P on the line l, $r_l(P) = P$, and**

2. **For any point P not on l, $r_l(P)$ is the point Q so that l is the perpendicular bisector of the segment PQ.**

Lesson 14: Reflections

Definition of Reflection:

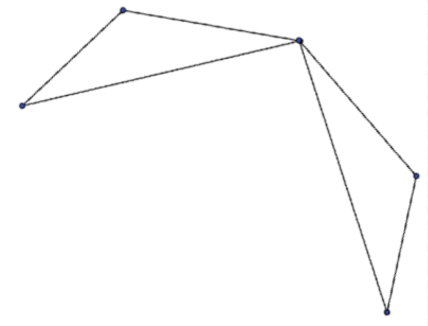
If the line is specified using two points, as in $\overleftrightarrow{AB}$, then the reflection is often denoted by $r_{\overleftrightarrow{AB}}$. Just as we did in the last lesson, let's examine this definition more closely:

- *A transformation of the plane*—the entire plane is transformed; what was once on one side of the line of reflection is now on the opposite side;
- $r_l(P) = P$ *means that the points on line l are left fixed*— the only part of the entire plane that is left fixed is the line of reflection itself;
- $r_l(P)$ is the point Q— the transformation r_l maps the point P to the point Q;
- so that l is the perpendicular bisector of the segment PQ— to find Q, first construct the perpendicular line m to the line l that passes through the point P. Label the intersection of l and m as N. Then locate the point Q on m on the other side of l such that $PN = NQ$.

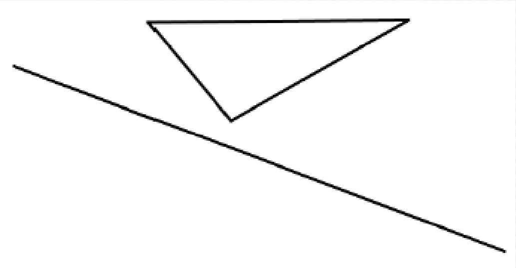
Lesson 14: Reflections

Application of Reflection:

1. Construct the line of reflection for the figures.

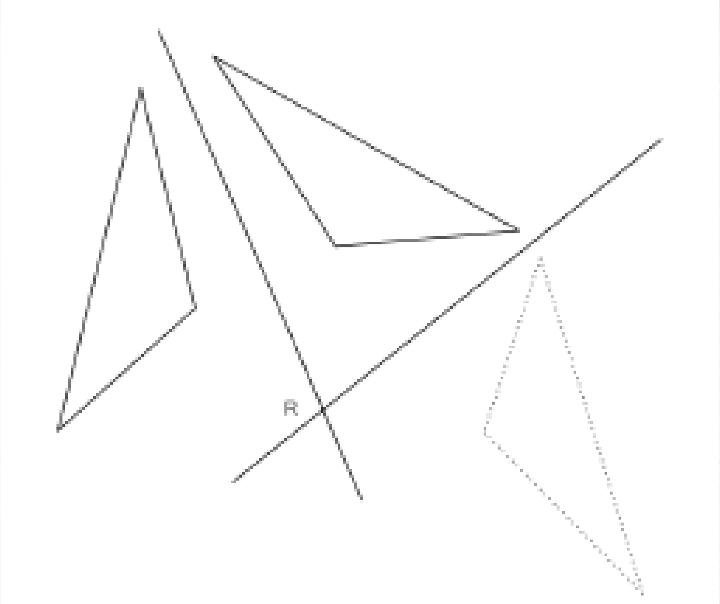


2. Reflect the given figure across the line of reflection provided.



Lesson 15: Rotations, Reflections, and Symmetry

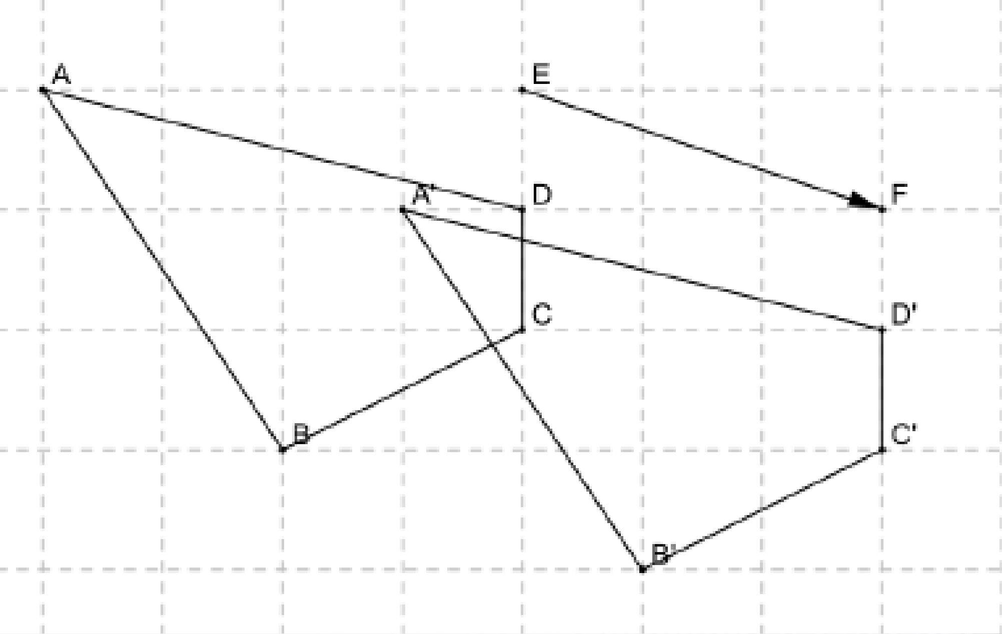
What is the relationship between a rotation and a reflection? Sketch a diagram that supports your explanation.



Reflecting a figure twice over intersecting lines yields the same results as a rotation.

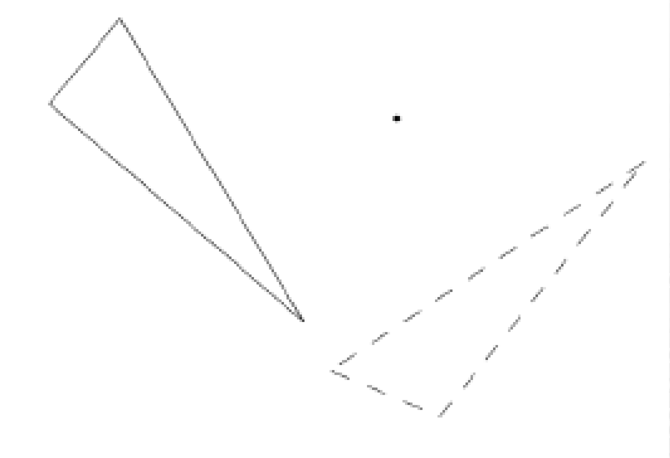
Lesson 16: Translations

Translate the figure one unit down and three units right. Draw the vector that defines the translation.



Lesson 17: Characterize Points on a Perpendicular Bisector

Example 3: Find the center of rotation for the transformation below. How are perpendicular bisectors a major part of finding the center of rotation? Why are they essential?



The center of rotation is the intersection of two perpendicular bisectors, each to a segment that joins a pair of corresponding parts (between the figure and its image).

Lesson 18: Looking More Carefully at Parallel Lines

Parallel Postulate: Through a given external point there is at most one line parallel to a given line.

In other words, we assume that for any point P in the plane not lying on a line l, every line in the plane that contains P intersects l except at most one line – the one we call *parallel* to l.

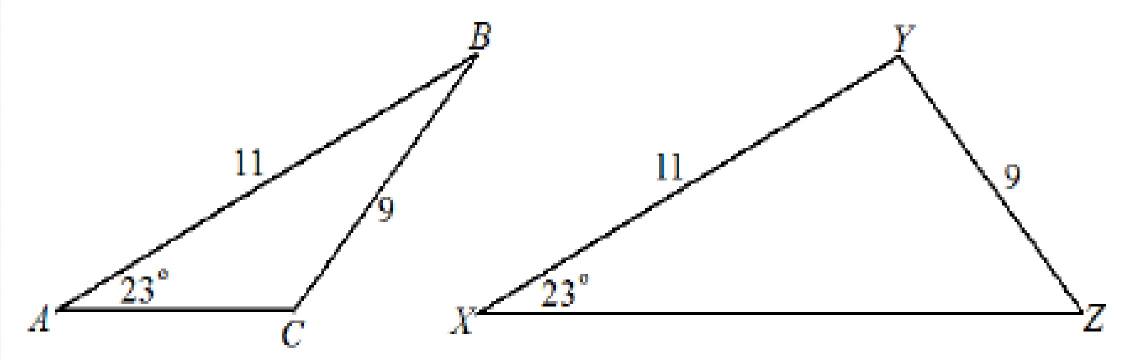
Lesson 19: Construct and Apply a Sequence of Rigid Motions

- We define two figures in the plane as congruent *if there exists a finite composition of basic rigid motions that maps one figure onto the other.*
- The idea of “Same size, same shape” only paints a mental picture; it is not specific enough.
- It is also not enough to say that two figures are alike in all respects except position in the plane.
- A congruence gives rise to a correspondence.

Lesson 20: Applications of Congruence in Terms of Rigid Motions

Problem Set #3:

Give an example of two triangles and a correspondence between their vertices such that (a) one angle in the first is congruent to the corresponding angle in the second and (b) two sides of the first are congruent to the corresponding sides of the second, but (c) the triangles themselves are not congruent.



$A \rightarrow X$

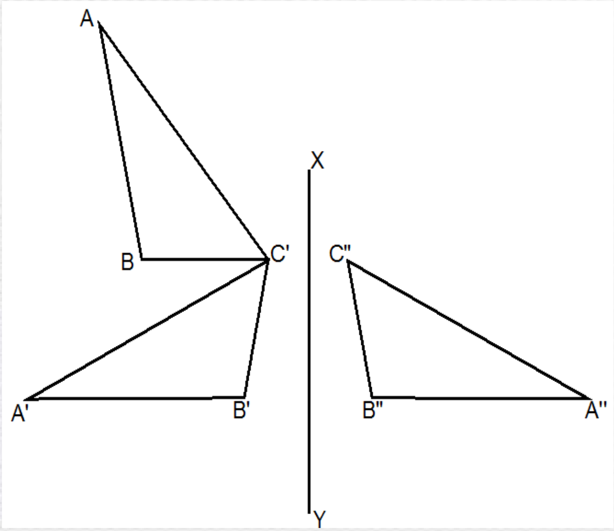
$B \rightarrow Y$

$C \rightarrow Z$

Lesson 21: Correspondence and Transformations

Exit Ticket

Complete the table based on the series of rigid motions performed on $\triangle\ ABC$ below.

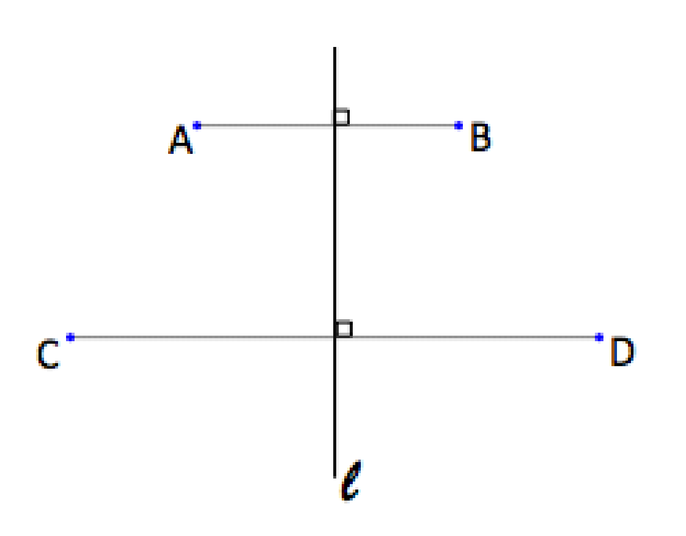


Sequence of rigid motions (2)	
Composition in function notation	
Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

Example from Mid-Module Assessment

Given in the figure below, line l is the perpendicular bisector of $\overline{AB}$ and of $\overline{CD}$.

a. Show $\overline{AC} \cong \overline{BD}$ using rigid motions.

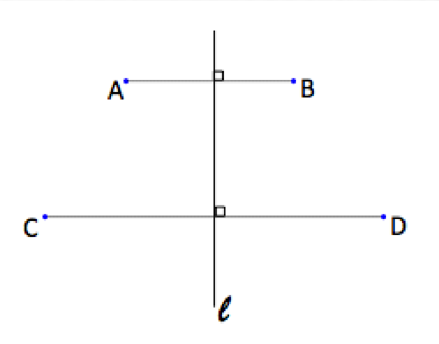


b. Show $\angle ACD \cong \angle BDC$.

c. Show $\overline{AB} \parallel \overline{CD}$.

Example from Mid-Module Assessment

Given in the figure below, line l is the perpendicular bisector of $\overline{AB}$ and of $\overline{CD}$.



a. Show $\overline{AC} \cong \overline{BD}$ using rigid motions.

Since l is the perpendicular bisector of $\overline{AB}$ and $\overline{CD}$, the reflection through line l brings A to B and C to D. Because reflections take line segments to congruent line segments, $\overline{AC}$ is congruent to $\overline{BD}$.

b. Show $\angle ACD \cong \angle BDC$.

The reflection through line l brings A to B and C to D and D to C. Therefore ray $\overrightarrow{CA}$ goes to ray $\overrightarrow{DB}$. Ray $\overrightarrow{CD}$ goes to ray $\overrightarrow{DC}$. The image of $\angle ACD$ is therefore congruent to $\angle BDC$.

c. Show $\overline{AB} \parallel \overline{CD}$.

$\overline{AB} \parallel \overline{CD}$ because the perpendicular bisector intersects the two lines creating congruent corresponding angles.

Topic D: Congruence

- Standards

  G-CO. 7, G-CO.8

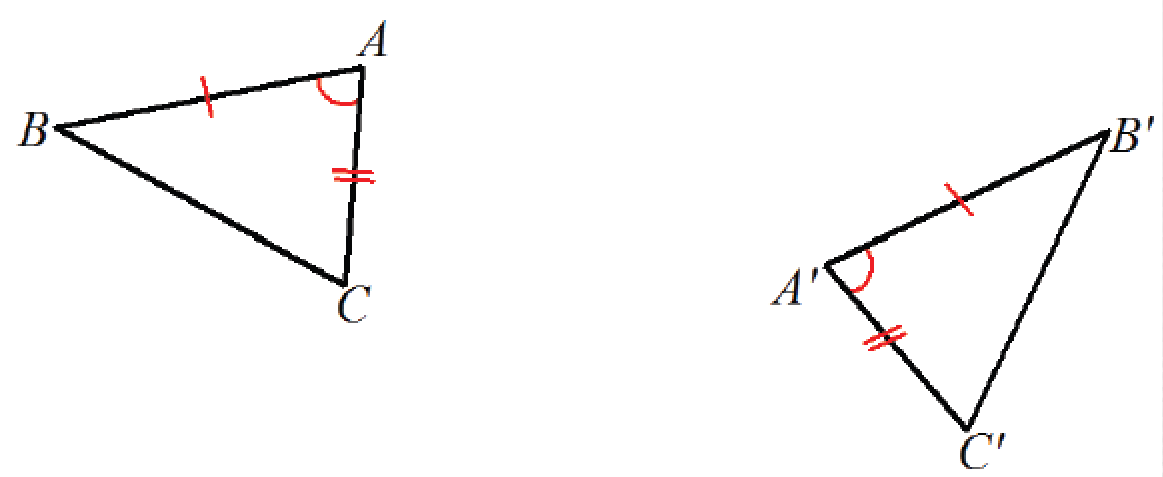
- Key theme in Topic D:

Your students might say, “Well, if ALL the corresponding sides are equal in measure, don’t we OBVIOUSLY have congruent triangles?” The answer is, “NO! We have agreed to use the word *congruent* to mean *there exists a composition of basic rigid motions of the plane that maps one figure to the other.* We will see that SAS, ASA, and SSS imply the existence of the rigid motion needed, but precision demands that we explain **how** and **why**.”

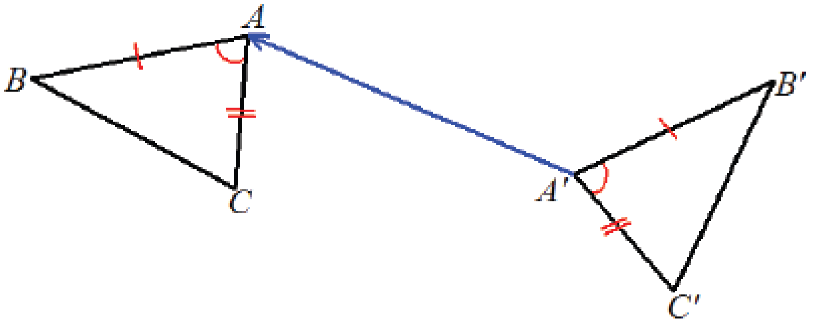
Topic D: Congruence

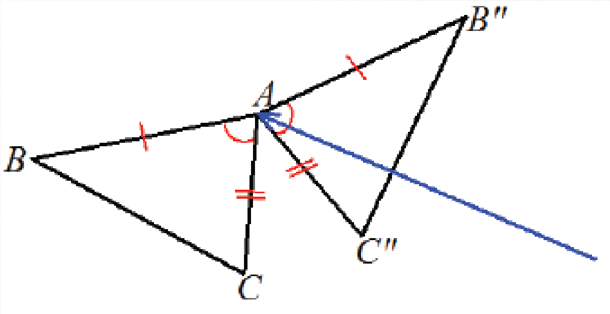
Instructional Days:	6
Lesson 22:	Congruence Criteria for Triangles—SAS (P)[1]
Lesson 23:	Base Angles of Isosceles Triangles (E)
Lesson 24:	Congruence Criteria for Triangles—ASA and SSS (P)
Lesson 25:	Congruence Criteria for Triangles—AAS and HL (E)
Lessons 26–27:	Triangle Congruency Proofs (P, P)

Lesson 22: Congruence Criteria for Triangles- SAS

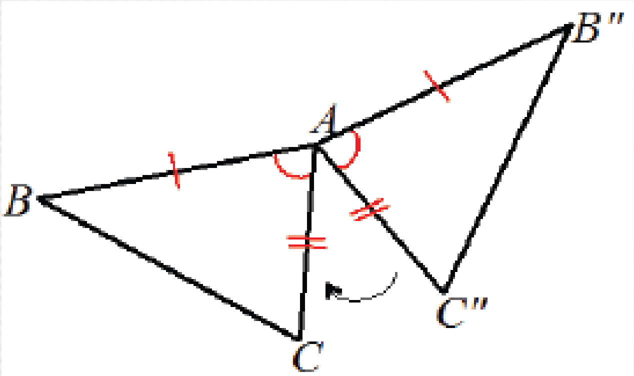


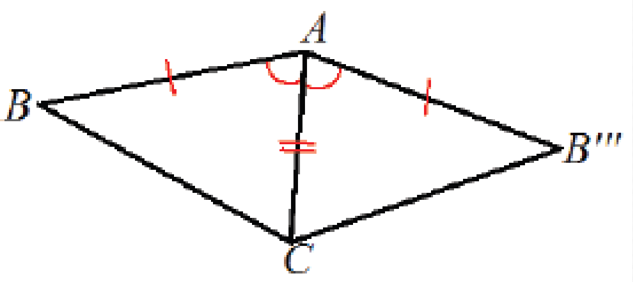
Lesson 22: Congruence Criteria for Triangles-SAS



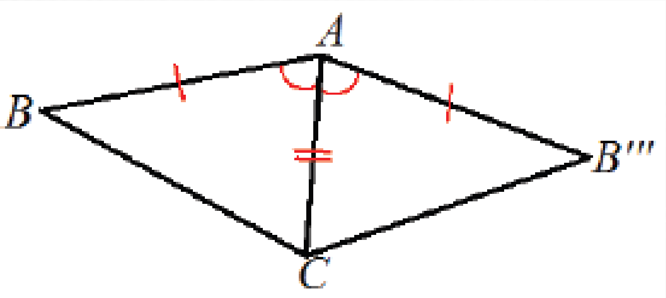


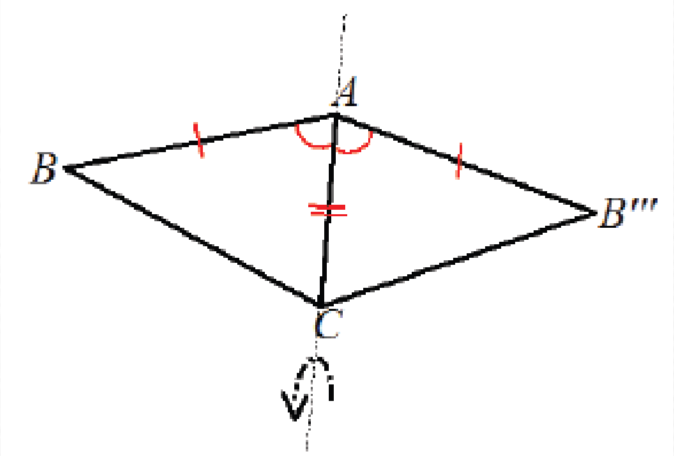
Lesson 22: Congruence Criteria for Triangles-SAS





Lesson 22: Congruence Criteria for Triangles-SAS



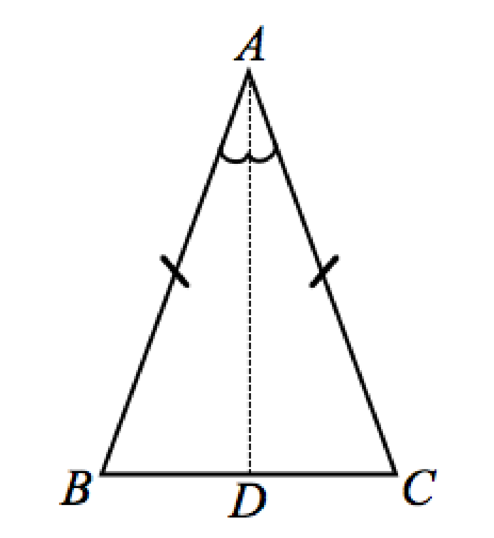


Lesson 23: Bases Angles of Isosceles Triangles

Given: Isosceles $\triangle ABC$, with $AB = BC$

Prove: $\angle B \cong \angle C$

Construction: **Draw the angle bisector $\overrightarrow{AD}$ of $\angle A$, where D is the intersection of the bisector and $\overline{BC}$. We are going to use this auxiliary line towards our SAS criteria.**



$AB = AC$	*Given*
$AD = AD$	*Reflexive property*
$\mathrm{m}\angle BAD = \mathrm{m}\angle CAD$	*Definition of an angle bisector*
$\triangle ABD \cong \triangle ACD$	*Side Angle Side criteria*
$\angle B \cong \angle C$	*Corresponding angles of congruent triangles are congruent*

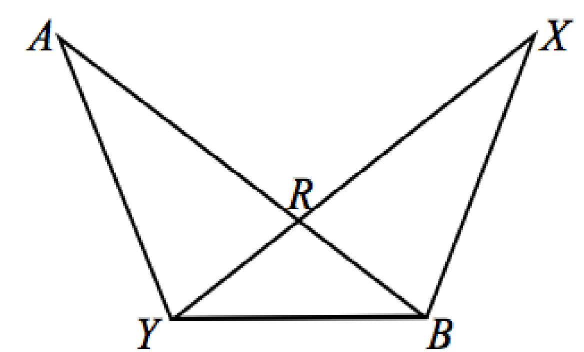
Lesson 24: Congruence Criteria for Triangles – ASA and SSS

Based on the information provided, determine whether a congruence exists between triangles. If a congruence exists between triangles or if multiple congruencies exist, state the congruencies and the criteria used to determine them.

Given: $RY = RB, AR = XR.$

$\triangle ARY \cong \triangle XRB$, SAS

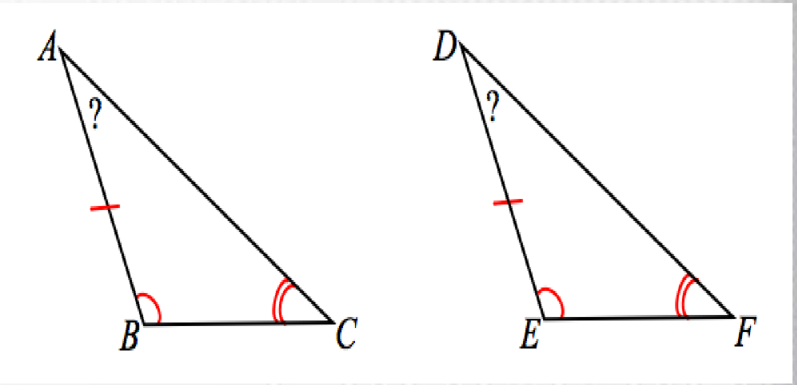
$\triangle ABY \cong \triangle XBY$, SAS



Lesson 25: Congruence Criteria for Triangles – AAS and HL

Given two triangles ABC and A'B'C'. If $AB = A'B'$ (Side), $m\angle B = m\angle B'$ (Angle), and $m\angle C = m\angle C'$ (Angle), then the triangles are congruent.

Proof: Consider a pair of triangles that meet the AAS criteria. If you knew that two angles of one triangle corresponded to and were equal in measure to two angles of the other triangle, what conclusions can you draw about the third angles of each triangle?



Since the first two angles are equal in measure, the third angles must also be equal in measure.

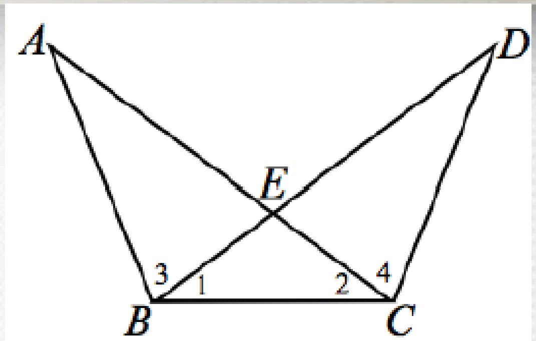
Given this conclusion, which formerly learned triangle congruence criteria can we use to determine if the pair of triangles are congruent?

ASA

Lesson 26, Exercise 5

5. **Given:** $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4.$

Prove: $AC = BD.$



$\angle 1 \cong \angle 2$	*Given*
$BE = CE$	*When two angles of a triangle are congruent, it is an isosceles triangle.*
$\angle 3 \cong \angle 4$	*Given*
$\angle AEB \cong \angle DEC$	*Vertical angles are congruent*
$\triangle ABC \cong \triangle DCB$	*ASA*
$\angle A \cong \angle D$	*Corresponding angles of congruent triangles are congruent.*
$BC = BC$	*Reflexive property*
$\triangle ABC \cong \triangle DCB$	*AAS*
$AC = BD$	*Corresponding sides of congruent triangles are congruent.*

Topic E: Proving Properties of Geometric Figures

G-CO.C.9, G-CO.C.10, G-CO.C.11

Prove theorems about lines and angles.

Prove theorems about triangles.

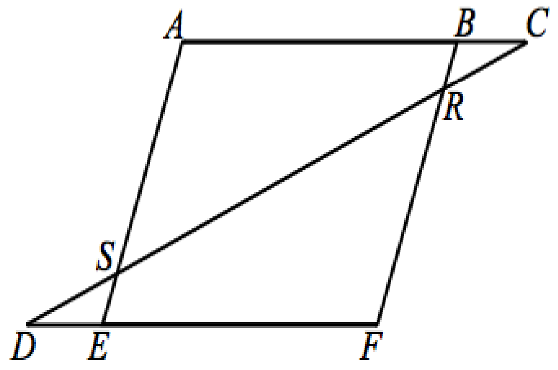
Prove theorems about parallelograms.

Instructional Days:	3
Lesson 28:	Properties of Parallelograms $(P)^2$
Lessons 29–30:	Special Lines in Triangles (P, P)

Lesson 28: Properties of Parallelograms

Given: Parallelogram $ABFE$, $CR = DS$.

Prove: $BR = SE$.

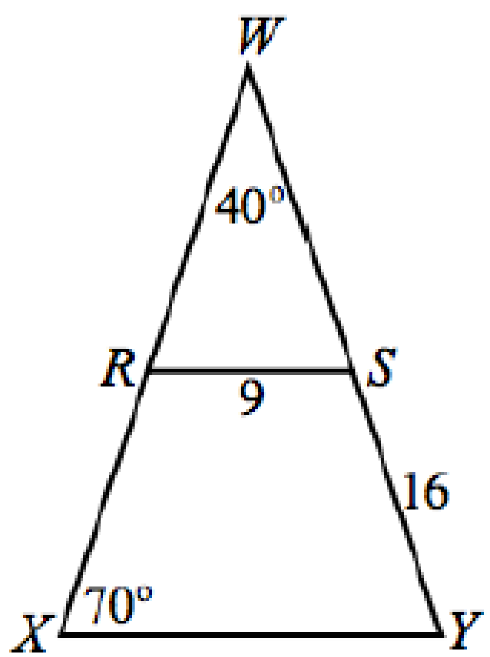


$m\angle BCR = m\angle EDS$	*If parallel lines cut by a transversal, then alternate interior angles are equal in measure*
$\angle ABF \cong \angle FEA$	*Opposite angles of a parallelogram are congruent*
$\angle CBR \cong \angle DES$	*Supplements of congruent angles are congruent*
$CR = DS$	*Given*
$\triangle CBR \cong \triangle DES$	*AAS*
$BR = SE$	*Corresponding sides of congruent triangles are equal in length*

Lesson 29: Special Lines in Triangles -- Midsegments

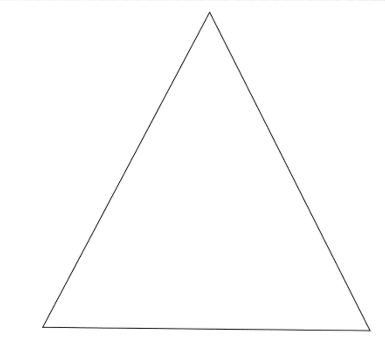
R and S are the midpoints of $\overline{XW}$ and $\overline{WY}$ respectively.

What is the perimeter of $\triangle WXY$? ____82____



Lesson 30: Special Lines in Triangles -- Medians

Ty is building a model of a hang glider using the template below. To place his supports accurately, Ty needs to locate the center of gravity on his model.



1. Use your compass and straightedge to locate the center of gravity on Ty's model.
2. Explain what the center of gravity represents on Ty's model.
3. Describe the relationship between the longer and shorter sections of the line segments you drew as you located the center of gravity.

The centroid divides the length of each median in a ratio of 2:1.

Topic F: Advanced Constructions

Focus Standard:

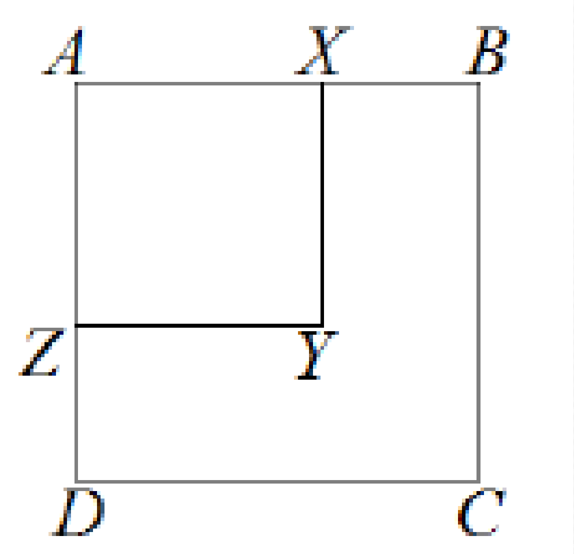
G-CO.D.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Instructional Days:	2
Lesson 31:	Construct a Square and a Nine-Point Circle (E)[1]
Lesson 32:	Construct a Nine-Point Circle (E)

Lesson 31: Construct a Square and a Nine-Point Circle

Exit Ticket

Construct a square $ABCD$ and a square $AXYZ$ so that $\overline{AB}$ contains X and $\overline{AD}$ contains Z.

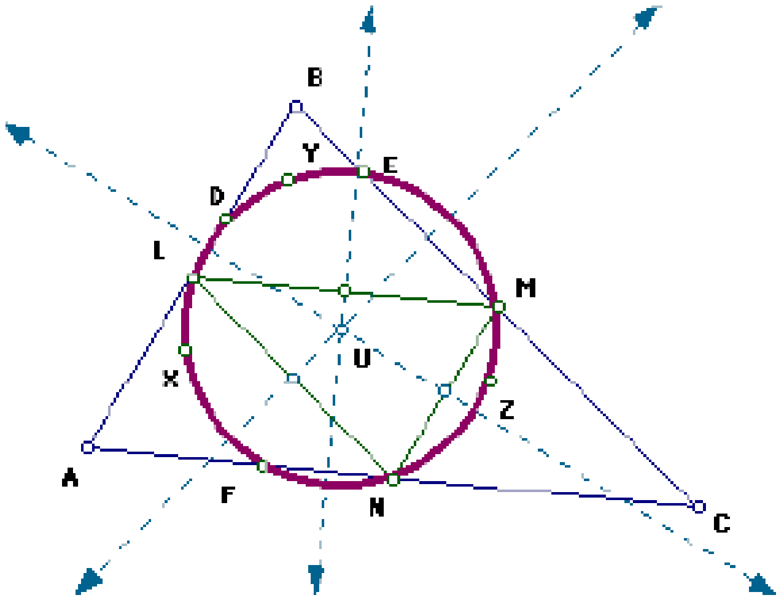


Lesson 32: Construct a Nine-Point Circle

There are two constructions for finding the center of the nine-point circle. With a partner, work through both constructions.

Construction 1

1. To find the center of the circle, draw inscribed $\triangle LMN$.
2. Find the circumcenter of $\triangle LMN$, and label it as U.



Recall that the circumcenter of a triangle is the center of the circle that circumscribes the triangle, which in this case, is the nine-point circle.

Topic G: Axiomatic Systems

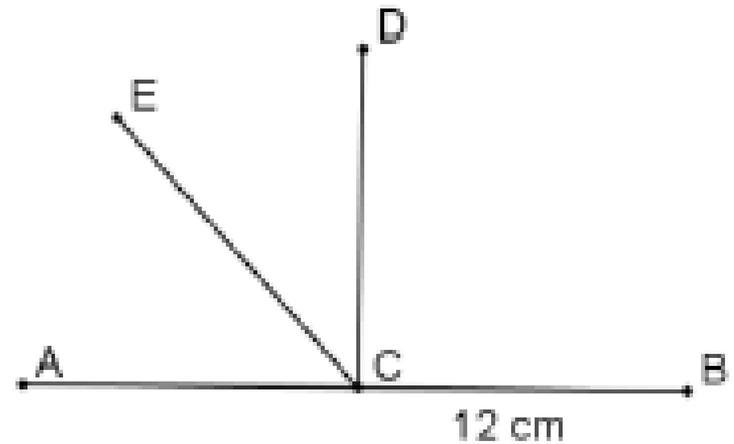
- Standards
  - All
- A mathematician's story:
- Many years ago I taught calculus for business majors. I started my class with an offer: "We have to cover several chapters from the textbook and there are approximately forty formulas. I may offer you a deal: you will learn just four formulas and I will teach you how to get the rest out of these formulas." The students gladly agreed.

Instructional Days:	2
Lessons 33–34:	Review of the Assumptions (P, P)[1]

Lesson 33: Review of the Assumptions

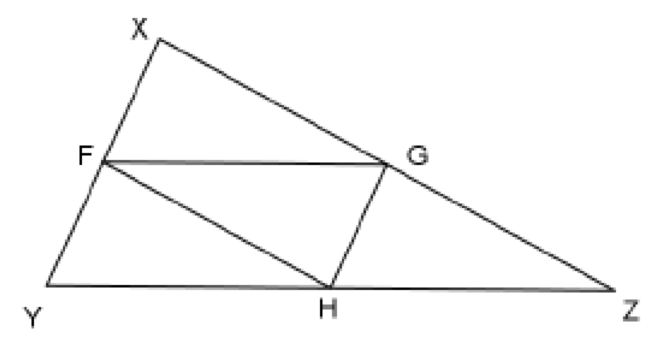
The perpendicular bisector of a segment is the line that passes through the midpoint of a line segment and is perpendicular to the line segment.

In the diagram below, $\overline{DC}$ is the $\perp$ bisector of $\overline{AB}$, and $\overline{CE}$ is the angle bisector of $\angle ACD$. Find the measures of $\overline{AC}$ and $\angle ECD$.



Lesson 34: Review of the Assumptions

Problem Set #3



$XY = 12$

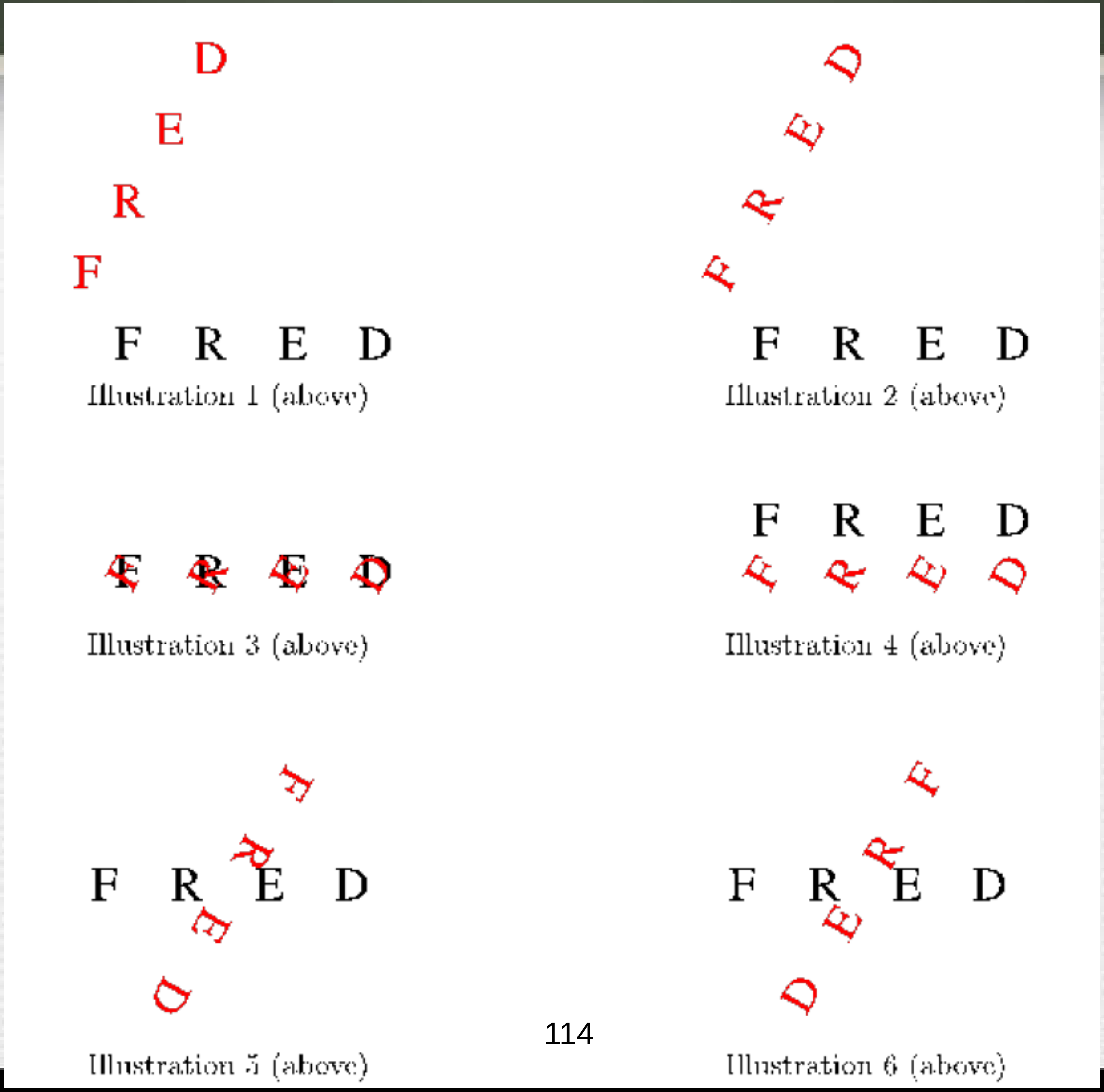
$XZ = 20$

$ZY = 24$

F, G, and H are midpoints of the sides on which they are located. Find the perimeter of $\triangle FGH$. Justify your solution.

28. *The midsegment is half the length of the side of the triangle it is parallel to.*

End-of-Module Assessment



End-of-Module Assessment

1. Each of the illustrations on the next page shows in black a plane figure consisting of the letters F, R, E, and D evenly spaced and arranged in a row. In each illustration, an alteration of the black figure is shown in red. In some of the illustrations, the red figure is obtained from the black figure by a geometric transformation consisting of a single rotation. In others, this is not the case.

   a. Which illustrations show a single rotation?

   Illustrations 2 and 5

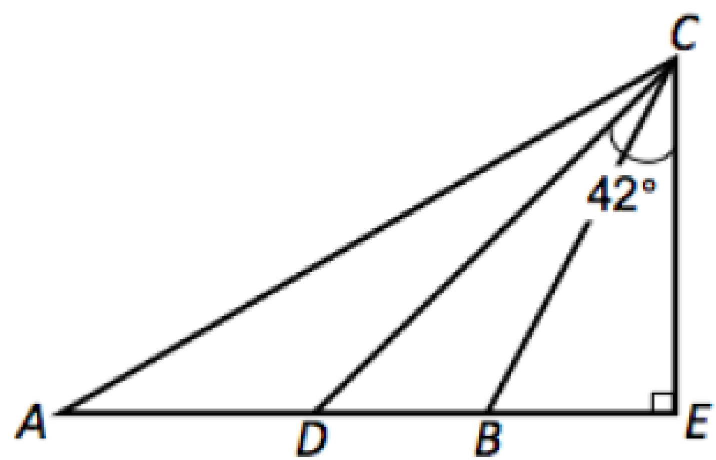
   b. Some of the illustrations are not rotations or even a sequence of rigid transformations. Pick one such illustration and use it to explain why it is not a sequence of rigid transformations.

   Illustration 1 shows translations of individual letters F, R, E, and D; but each letter is translated a different distance. Since translation requires a shift of the entire plane by the same distance, Illustration 1 does not qualify.

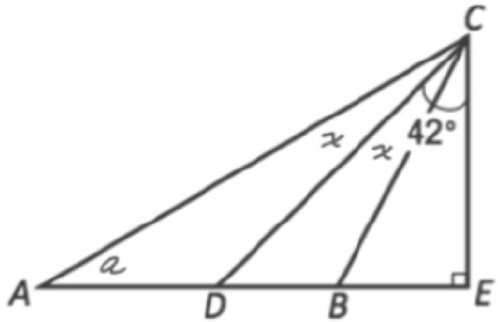
End-of-Module Assessment

2. In the figure below, $\overline{CD}$ bisects $\angle ACB$, $AB = BC$, $\angle BEC = 90°$, and $\angle DCE = 42°$.

Find the measure of angle $\angle A$.



End-of-Module Assessment



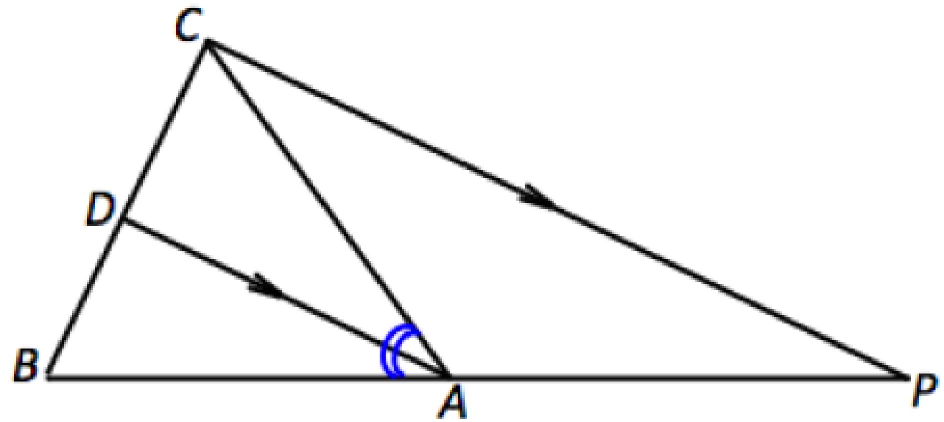
Label the angles as shown.
($\angle ACD \cong \angle DCB$ since CD bisects $\angle ACB$)
Since $AB = BC$, $\triangle ABC$ is isosceles, therefore $2x = a$

$\angle A + \angle ACE + \angle E = 180°$
$a + (x + 42) + 90 = 180$
$2x + x + 132 = 180$
$x = 16$
Since $a = 2x$, $\angle A = 32°$

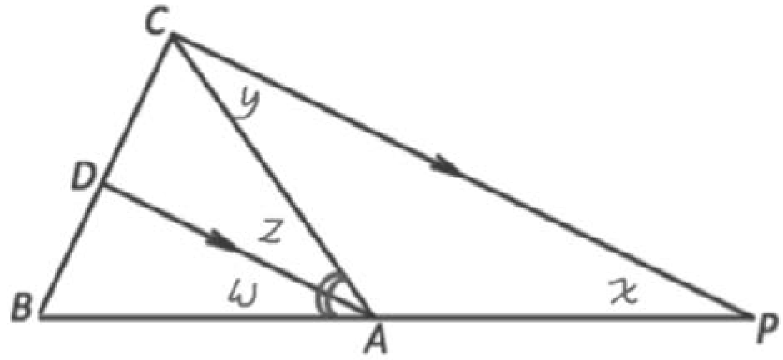
End-of-Module Assessment

3. In the figure below, $\overline{AD}$ is the angle bisector of $\angle BAC$. $\overline{BAP}$ and $\overline{BDC}$ are straight lines and $\overline{AD} \parallel \overline{PC}$.

   Prove that $AP = AC$.



End-of-Module Assessment

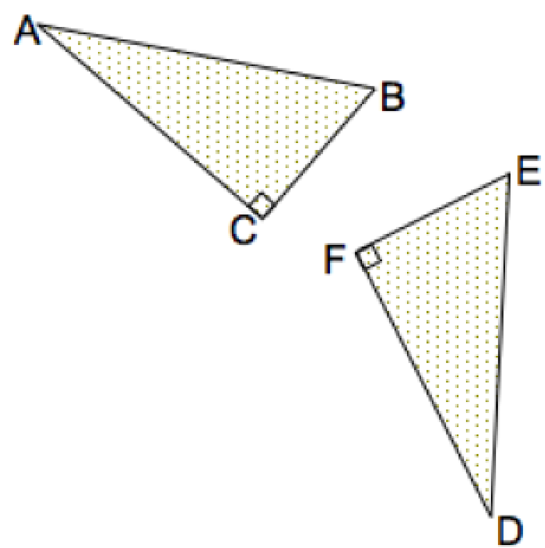


Label w, x, y, and z as shown.

1. $z = w$	AD is the angle bisector of $\angle BAC$
2. $z = y$	Alt. int. $\angle$s; $DA \parallel CP$
3. $w = x$	Corr. $\angle$s; $DA \parallel CP$
4. $x = y$	
5. $\triangle ACP$ is isosceles	Base $\angle$s converse
6. $AC = AP$	Defn. of isos. $\triangle$

End-of-Module Assessment

4. The triangles ΔABC and ΔDEF in the figure below such that $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$.



a. What criteria for triangle congruence (ASA, SAS, SSS) implies that $\Delta ABC \cong \Delta DEF$? SAS

b. Describe a sequence of rigid transformations that shows $\Delta ABC \cong \Delta DEF$.

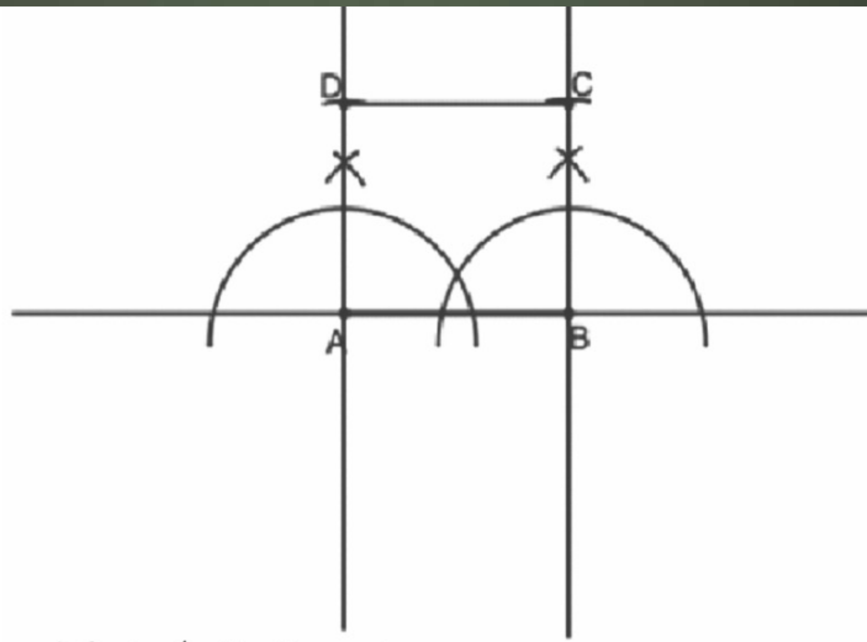
translation, rotation, reflection

End-of-Module Assessment

5.

a. Construct a square $ABCD$ with side $\overline{AB}$. List the steps of the construction.

End-of-Module Assessment

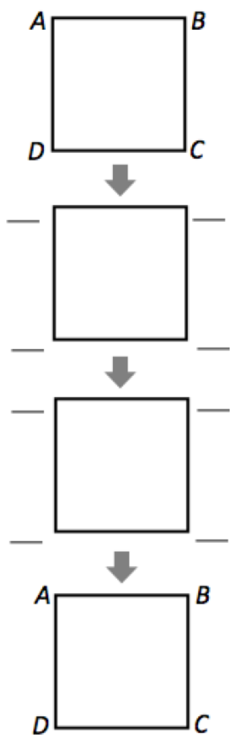


1. Extend $\overline{AB}$ in both directions.
2. Construct a perpendicular bisector to AB through A; construct a perpendicular bisector to AB through B.
3. Construct a circle with center A and radius AB; construct a circle with center B and radius AB.
4. Select a point where circle A meets the perpendicular through A and call that point D. On the same side of AB as D, select the point where circle B meets the perpendicular through B and call that point C.
5. Draw segment CD.

End-of-Module Assessment

b. Three rigid motions are to be performed on square $ABCD$. The first rigid motion is the reflection through line $\overline{BD}$. The second rigid motion is a 90° clockwise rotation around the center of the square.

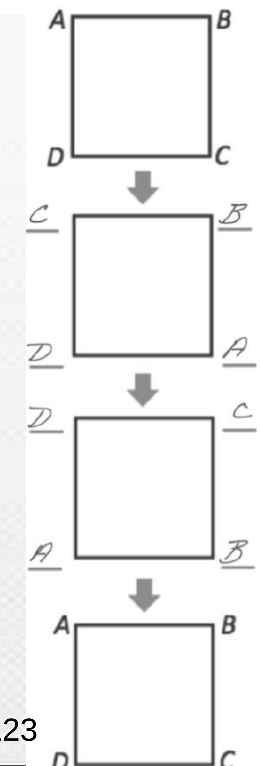
Describe the third rigid motion that will ultimately map $ABCD$ back to its original position. Label the image of each rigid motion A, B, C, D in the provided blanks.



Rigid Motion 1 Description: Reflection through line $\overline{BD}$

Rigid Motion 2 Description: 90° clockwise rotation around the center of the square.

Rigid Motion 3 Description: ____________________



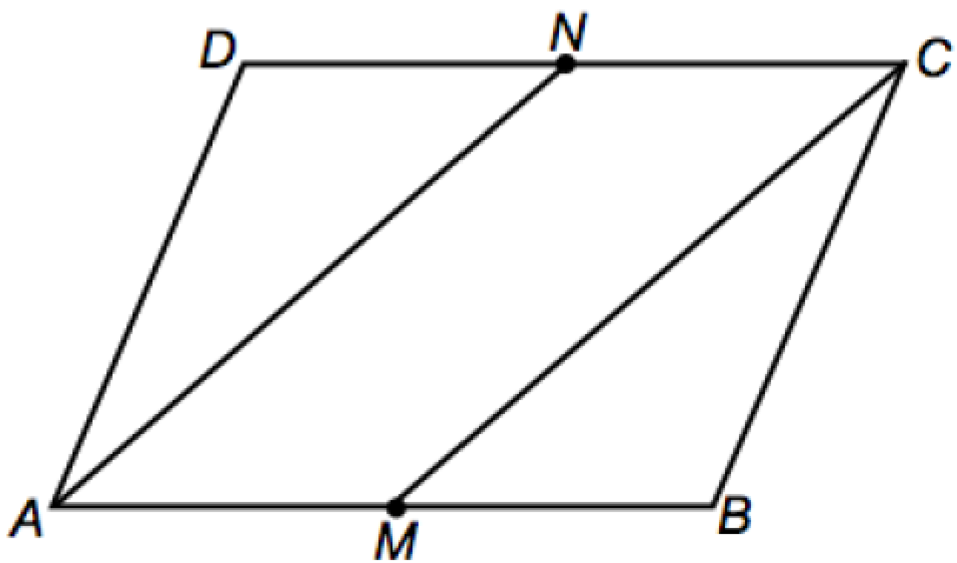
Rigid Motion 1 Description: Reflection through line $\overline{BD}$

Rigid Motion 2 Description: 90° clockwise rotation around the center of the square.

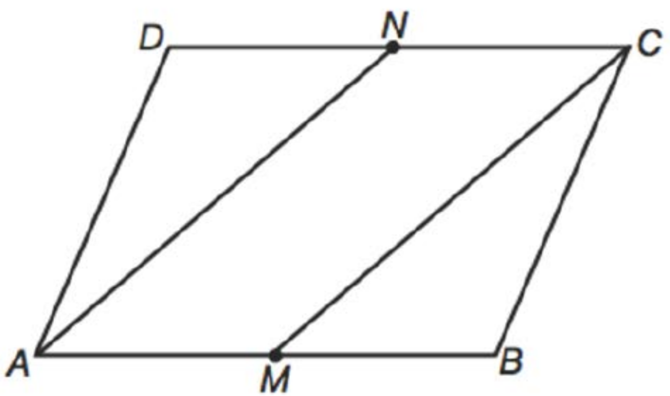
Rigid Motion 3 Description: Reflection through the line connecting the midpoint of AD and the midpoint of BC.

End-of-Module Assessment

6. Suppose that $ABCD$ is a parallelogram and that M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Prove that $AMCN$ is a parallelogram.



End-of-Module Assessment



Given: ABCD is a ||-ogram; M is the midpoint of AB
N is the midpoint of DC.

Prove: AMCN is a ||-ogram.

1. $AB = DC$	opp. sides of ||-ogram
2. $NC = \frac{1}{2} DC$	N is the midpoint of DC.
$= \frac{1}{2} AB$	
$= AM$	M is the midpoint of AB
3. $AB \parallel DC$	Defn. of ||-ogram; ABCD
4. $AM \parallel NC$	M is on line AB, N is on line DC
5. ∴ AMCN is a ||-ogram	opp. sides are equal and ||.

Agenda

Lesson Structure

Instructional Sequence

Module Review

Preparation for Implementation

Key Themes of Module 1: Congruence, Proof, and Constructions

- Module is anchored by the definition of congruence
- Emphasis is placed on extending the meaning and use of vocabulary in constructions
- There is an explicit recall and application of facts learned over the last few years in unknown angle problems and proofs
- Triangle congruence criteria are indicators that a rigid motion exists that maps one triangle to another; each criterion can be proven to be true with the use of rigid motions.

Agenda

Lesson Structure

Instructional Sequence

Module Review

Preparation for Implementation

Practice a Planning Protocol

- With any topic from *A Story of Functions*, read the module overview and the topic opener.
- Study the module assessment, paying particular attention to the sample responses provided.

Practice a Planning Protocol

- Read through the first lesson of the topic.
- Then, take note of the lesson objective and re-examine the exit ticket with the objective in mind. What major concept is necessary to successfully complete the exit ticket?
- Study the concept development and problem set. How do the CD/PS develop the major concept that is required in the exit ticket? What parts of the CD/PS go beyond this major concept?

Practice a Planning Protocol

- How will this knowledge enable teachers to support specific groups of learners?
- Turn to the subsequent lesson, and examine the exit ticket. How does this exit ticket build on the last? How are the two exit tickets similar and how are they different?
- Will students have an opportunity in the second lesson to continue development of the first lesson's objective? What level of mastery of the first lesson's objective is necessary in preparation for the second lesson?

Practice a Planning Protocol

- How does the new plan for implementation impact the student debrief?
- Are any adjustments needed to the fluency and/or application components of the lesson?

Practice a Planning Protocol

- Repeat this process for each lesson.
  - Read lesson.
  - Study exit ticket. Identify critical portions of concept development and problem set.
  - Consider needs of specific students.
  - Refer to subsequent exit ticket. Revise implementation plan as needed.
  - Make adjustments to the student debrief as needed.
  - Consider the other lesson components, ensuring a balance of rigor.

Biggest Takeaway

- What is your biggest takeaway with respect to Module 1?
- How can you support successful implementation at your school/s given your role?

I now know…

I need to figure out…

THANK YOU!!!!!

APPENDIX II

Participants' Handouts

Lesson 1: Construct an Equilateral Triangle

Classwork

Opening Exercise

Joe and Marty are in the park playing catch. Tony joins them, and the boys want to stand so that the distance between any two of them is the same. Where do they stand?

How do they figure this out precisely? What tool or tools could they use?

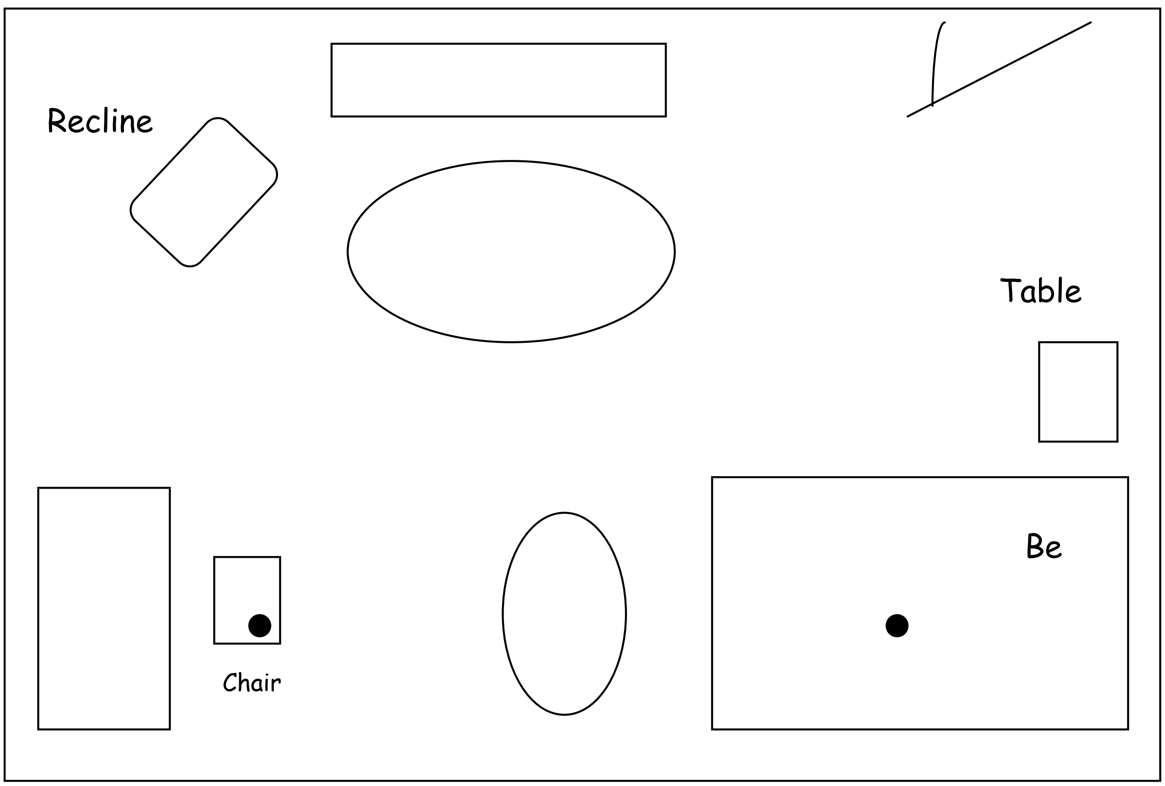
Fill in the blanks below as each term is discussed:

1.	____________________	The _______ between points A and B is the set consisting of A, B, and all points on the line AB between A and B.
2.	____________________	A segment from the center of a circle to a point on the circle.
3.	____________________	Given a point C in the plane and a number $r > 0$, the _______ with center C and radius r is the set of all points in the plane that are distance r from point C.

Example 1: Sitting Cats

You will need a compass and a straightedge.

Margie has three cats. She has heard that cats in a room position themselves at equal distances from one another and wants to test that theory. Margie notices that Simon, her tabby cat, is in the center of her bed (at S), while JoJo, her Siamese, is lying on her desk chair (at J). If the theory is true, where will she find Mack, her calico cat? Use the scale drawing of Margie's room shown below, together with (only) a compass and straightedge. Place an M where Mack will be if the theory is true.



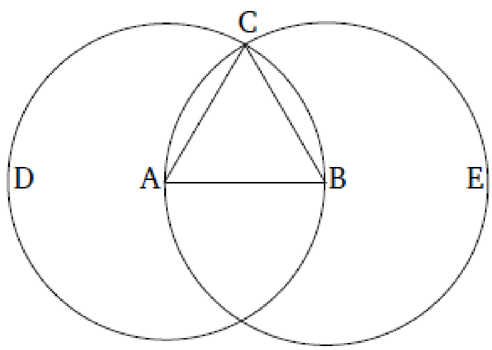
Mathematical Modeling Exercise: Euclid, Proposition 1

Let's see how Euclid approached this problem. Look at his first proposition, and compare his steps with yours.

In this margin, compare your steps with Euclid's.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Problem Set

1. Write a clear set of steps for the construction of an equilateral triangle. Use Euclid's Proposition 1 as a guide.

2. Suppose two circles are constructed using the following instructions:

   Draw circle: Center A, radius AB.

Draw circle: Center C, radius CD.

Under what conditions (in terms of distances AB, CD, AC) do the circles have

a. One point in common?

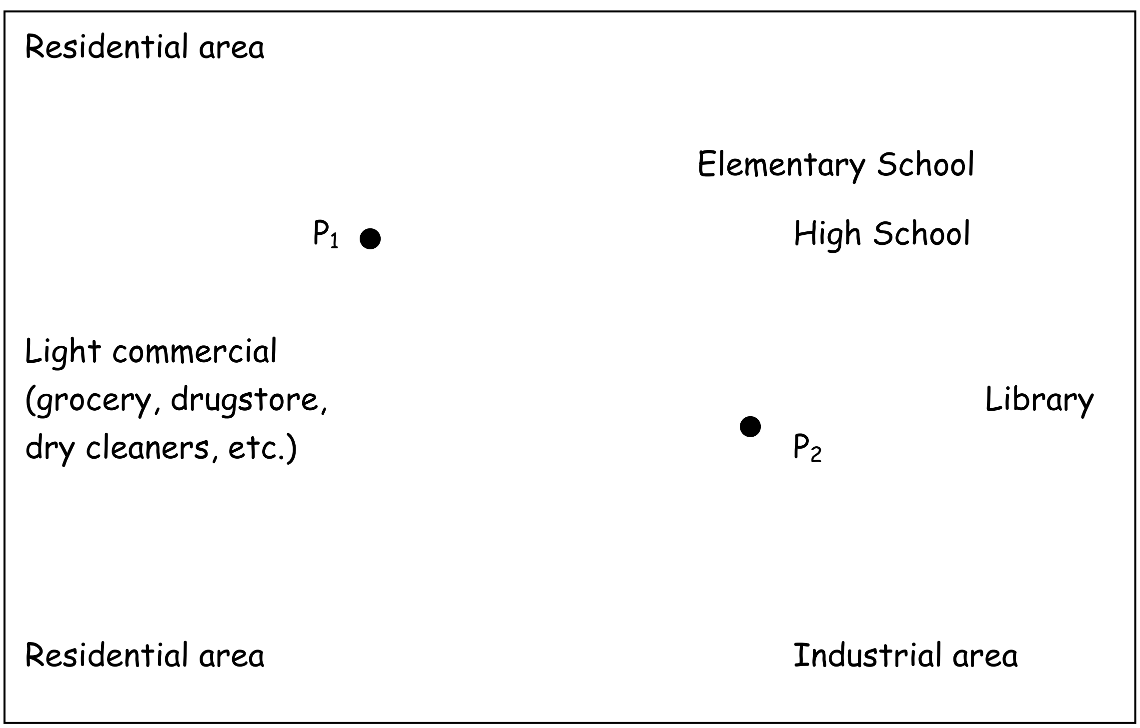
b. No points in common?

c. Two points in common?

d. More than two points in common? Why?

3. *You will need* a compass and straightedge.

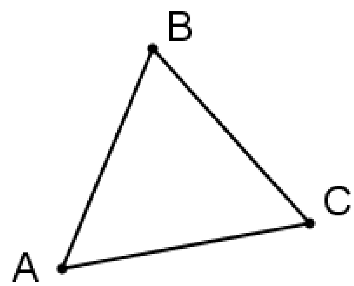
Cedar City boasts two city parks and is in the process of designing a third. The planning committee would like all three parks to be equidistant from one another to better serve the community. A sketch of the city appears below, with the centers of the existing parks labeled as P_1 and P_2. Identify two possible locations for the third park, and label them as P_{3a} and P_{3b} on the map. Clearly and precisely list the mathematical steps used to determine each of the two potential locations.



Lesson 2: Construct an Equilateral Triangle

Exit Ticket

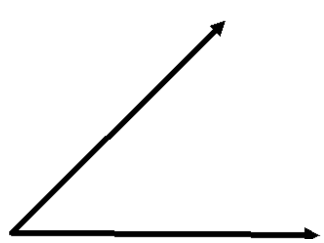
$\triangle ABC$ is shown below. Is it an equilateral triangle? Justify your response.



Mathematical Modeling Exercise 2: Investigate How to Copy an Angle

You will need a compass and a straightedge.

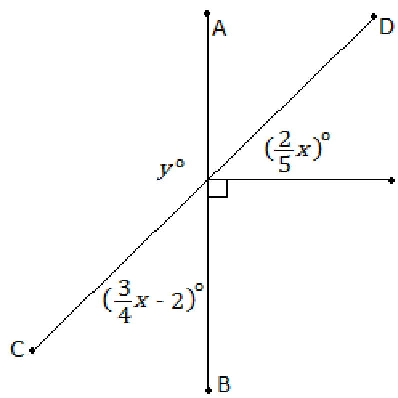
You and your partner will be provided with a list of steps (in random order) needed to copy an angle using a compass and straightedge. Your task is to place the steps in the correct order, then follow the steps to copy the angle below.



Steps needed (in correct order):

Lesson 6: Solve for Unknown Angles—Angles and Lines at a Point

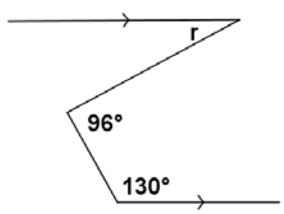
Exercise 10



x = __________ y = __________

Lesson 7: Solve for Unknown Angles—Transversals

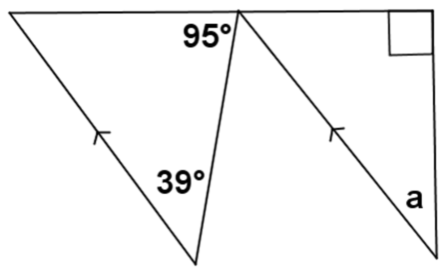
Exercise 10



m∠r = ________________________

Lesson 8: Solve for Unknown Angles—Angles in a Triangle

Problem Set #1

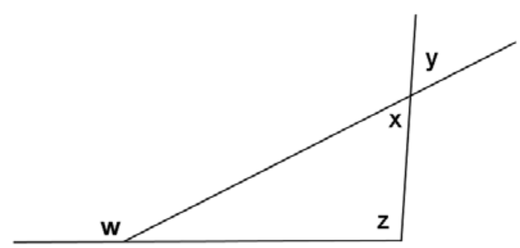


$m\angle a$ = __________

Lesson 9: Unknown Angle Proofs—Writing Proofs

Exercise 2b

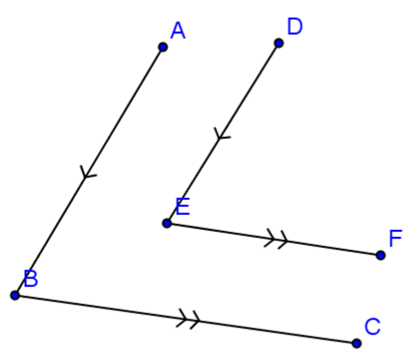
Given the diagram to the right, prove that $\mathrm{m}\angle w = \mathrm{m}\angle y + \mathrm{m}\angle z$.



Lesson 10: Unknown Angle Proofs—Proofs with Constructions

Problem Set

1. In the figure to the right, $\overline{AB} \parallel \overline{DE}$ and $\overline{BC} \parallel \overline{EF}$. Prove that $\mathrm{m}\angle ABC = \mathrm{m}\angle DEF$.



Lesson 11: Unknown Angle Proofs – Proofs of Known Facts

Problem Set

2. A theorem states that *in a plane, if a line is perpendicular to one of two parallel lines and intersects the other, then it is perpendicular to the other of the two parallel lines.*

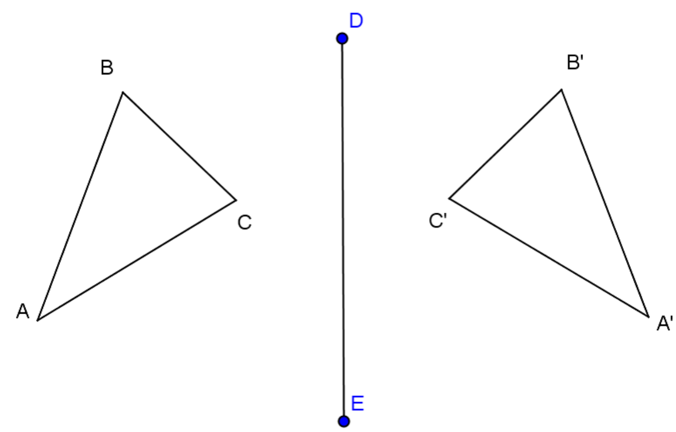
   Prove this theorem. (a) Construct and label an appropriate figure, (b) state the given information and the theorem to be proven, and (c) list the necessary steps to demonstrate the proof.

Lesson 14: Reflections

Exploratory Challenge

$\triangle ABC$ is reflected across $\overline{DE}$ and maps onto $\triangle A'B'C'$.

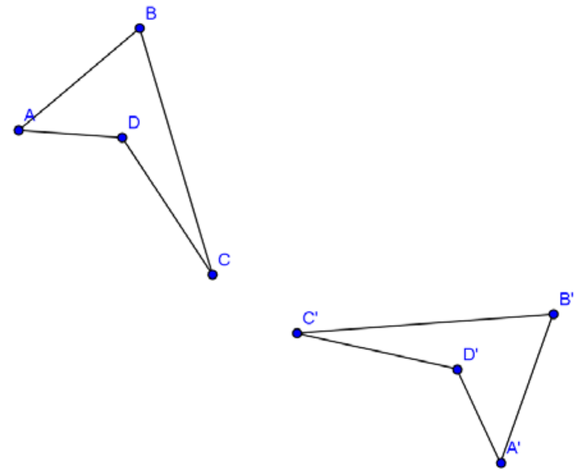
Use your compass and straightedge to construct the perpendicular bisector of each of the segments connecting A to A', B to B', and C to C'. What do you notice about these perpendicular bisectors?



Example 1

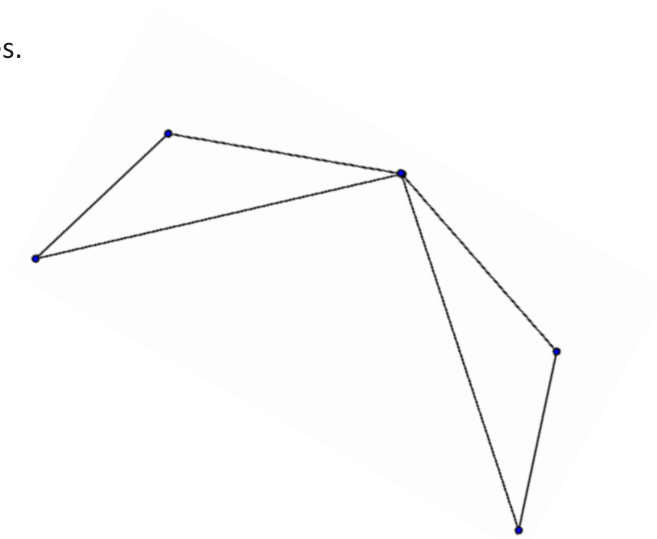
Construct the segment that represents the line of reflection for quadrilateral $ABCD$ and its image $A'B'C'D'$.

What is true about each point on $ABCD$ and its corresponding point on $A'B'C'D'$?

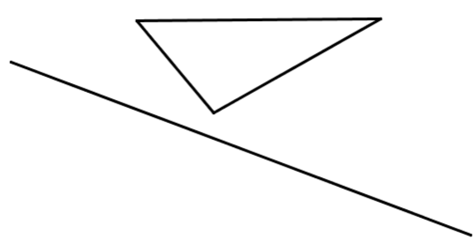


Exit Ticket

1. Construct the line of reflection for the figures.



2. Reflect the given figure across the line of reflection provided.



Lesson 15: Rotations, Reflections, and Symmetry

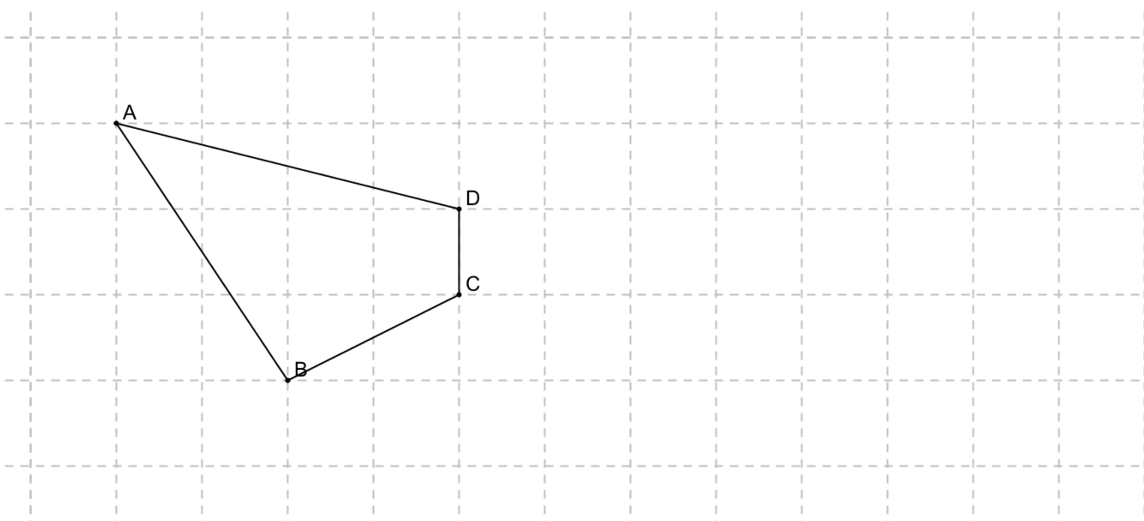
Exit Ticket

What is the relationship between a rotation and reflection? Sketch a diagram that supports your explanation.

Lesson 16: Translations

Exit Ticket

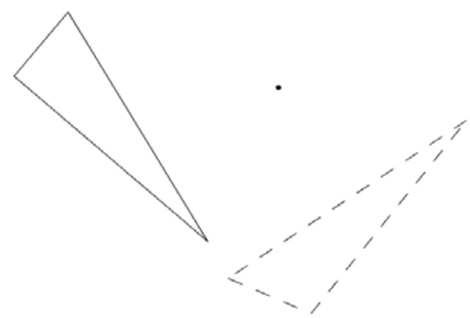
Translate the image one unit down and three units right. Draw the vector that defines the translation.



Lesson 17: Characterize Points on a Perpendicular Bisector

Example 3

Find the center of rotation for the transformation below. How are perpendicular bisectors a major part of finding the center of rotation? Why are they essential?



Lesson 20: Applications of Congruence in Terms of Rigid Motions

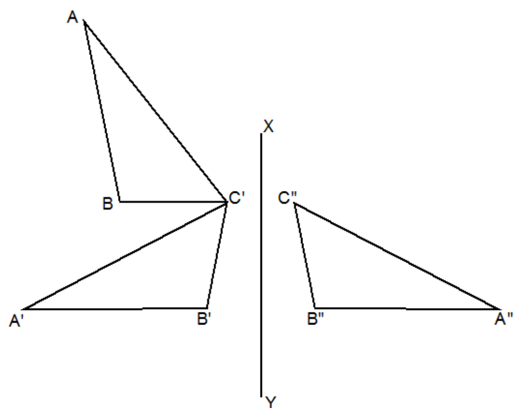
Problem Set #3

Give an example of two triangles and a correspondence between their vertices such that (a) one angle in the first is congruent to the corresponding angle in the second and (b) two sides of the first are congruent to the corresponding sides of the second, but (c) the triangles themselves are not congruent.

Lesson 21: Correspondence and Transformations

Exit Ticket

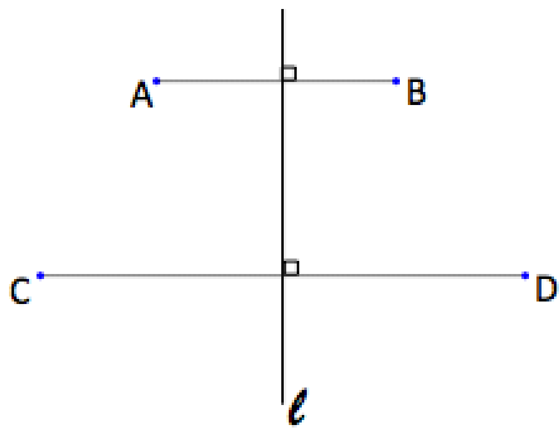
Complete the table based on the series of rigid motions performed on $\triangle ABC$ below.



Sequence of rigid motions (2)	
Composition in function notation	
Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

Example from Mid-Module Assessment

6. Given in the figure below, line l is the perpendicular bisector of $\overline{AB}$ and of $\overline{CD}$.



a. Show $\overline{AC} \cong \overline{BD}$ using rigid motions.

b. Show $\angle ACD \cong \angle BDC$.

c. Show $\overline{AB} \parallel \overline{CD}$.

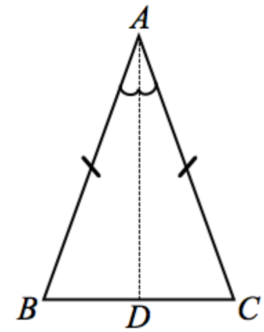
Lesson 23: Base Angles of Isosceles Triangles

Discussion

Prove Base Angles of an Isosceles are Congruent: Transformations

Given: Isosceles $\triangle\ \boldsymbol{ABC}$, with $\boldsymbol{AB = AC}$.

Prove: $\angle B \cong \angle C$.



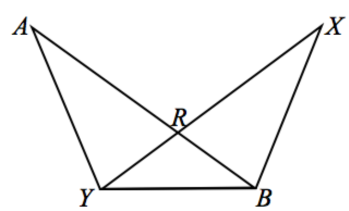
Construction: Draw the angle bisector $\overrightarrow{\mathrm{AD}}$ of $\angle\mathrm{A}$, where D is the intersection of the bisector and $\overline{\mathrm{BC}}$. We need to show that rigid motions will map point B to point C and point C to point B.

Lesson 24: Congruence Criteria for Triangles—ASA and SSS

Exercises

Based on the information provided, determine whether a congruence exists between triangles. If a congruence exists between triangles or if multiple congruencies exist, state the congruencies and the criteria used to determine them.

3. Given: $RY = RB,\ AR = XR.$

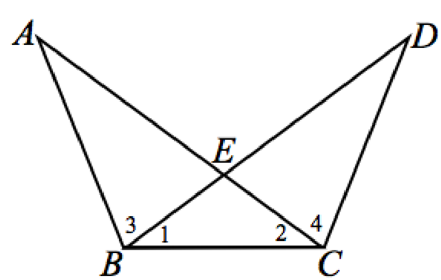


Lesson 26: Triangle Congruency Proofs

Exercises

5. Given: $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4.$

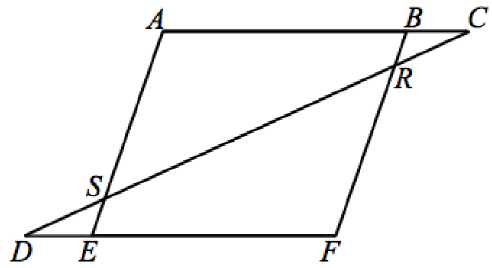
   Prove: $\overline{AC} \cong \overline{BD}.$



Lesson 28: Properties of Parallelograms

Problem Set

5. Given: Parallelogram $ABFE$, $CR = DS$.
   Prove: $BR = SE$.



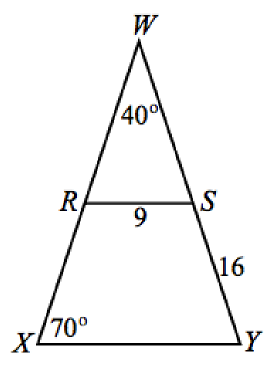
Lesson 29: Special Lines in Triangles

Exit Ticket

Use the properties of midsegments to solve for the unknown value in each question.

R and S are the midpoints of $\overline{XW}$ and $\overline{WY}$, respectively.

What is the perimeter of $\triangle WXY$? ___________________



Lesson 30: Special Lines in Triangles

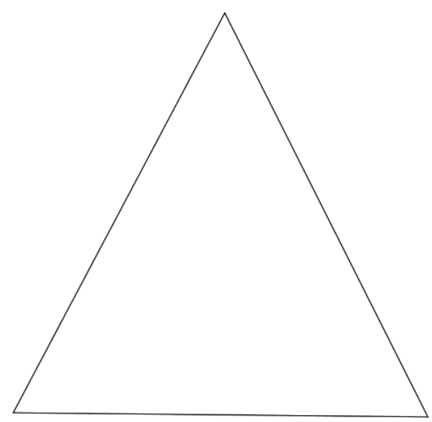
Problem Set

Ty is building a model of a hang glider using the template below. To place his supports accurately, Ty needs to locate the center of gravity on his model.

6. Use your compass and straightedge to locate the center of gravity on Ty's model.

7. Explain what the center of gravity represents on Ty's model.

8. Describe the relationship between the longer and shorter sections of the line segments you drew as you located the center of gravity.

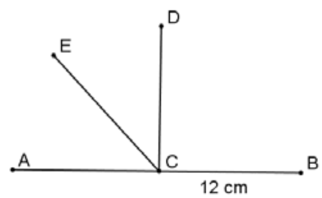


Lesson 31: Construct a Square and a Nine-Point Circle

Exit Ticket

Construct a square $ABCD$ and a square $AXYZ$ so that $\overline{AB}$ contains X and $\overline{AD}$ contains Z.

Lesson 33: Review of the Assumptions

The perpendicular bisector of a segment is the line that passes through the midpoint of a line segment and is perpendicular to the line segment.	In the diagram below, $\overline{DC}$ is the $\perp$ bisector of $\overline{AB}$, and $\overline{CE}$ is the angle bisector of $\angle ACD$. Find the measures of $\overline{AC}$ and $\angle ECD$. 	

Lesson 34: Review of the Assumptions

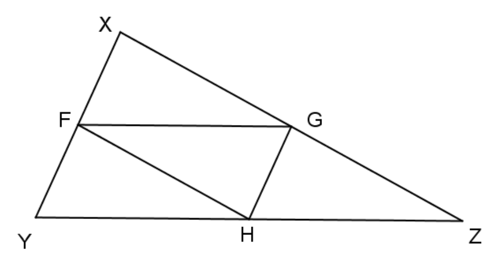
Problem Set

3. $XY = 12$

   $XZ = 20$

   $ZY = 24$

   F, G, and H are midpoints of the sides on which they are located. Find the perimeter of $\triangle FGH$. Justify your solution.



1. Each of the illustrations on the next page shows in black a plane figure consisting of the letters F, R, E, and D evenly spaced and arranged in a row. In each illustration, an alteration of the black figure is shown in gray. In some of the illustrations, the gray figure is obtained from the black figure by a geometric transformation consisting of a single rotation. In others, this is not the case.

   a. Which illustrations show a single rotation?

   b. Some of the illustrations are not rotations or even a sequence of rigid transformations. Select one such illustration and use it to explain why it is not a sequence of rigid transformations.

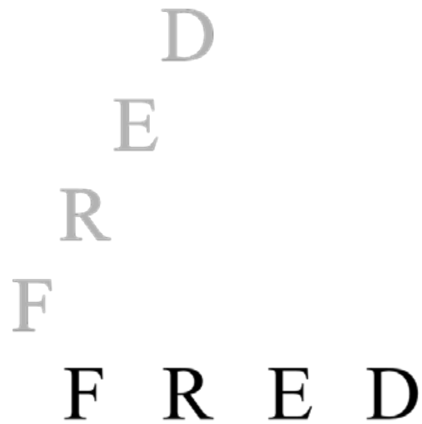


Illustration 1

Illustration 2

Illustration 3

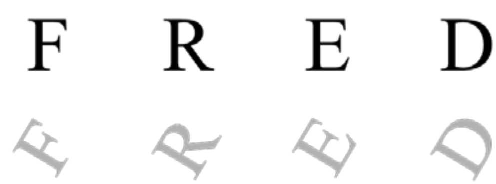


Illustration 4

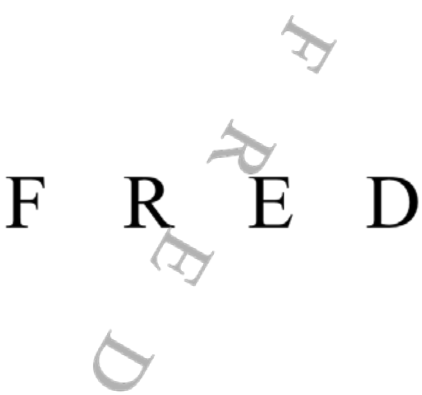


Illustration 5

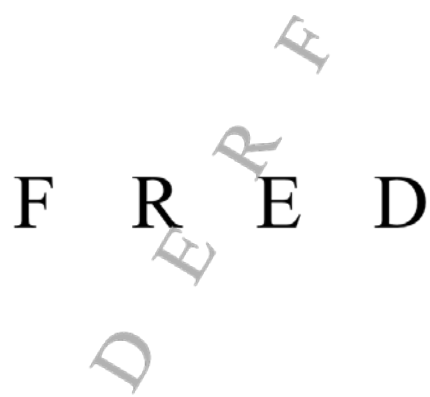
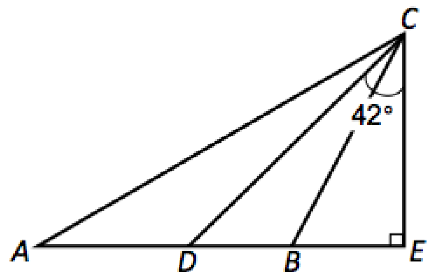


Illustration 6

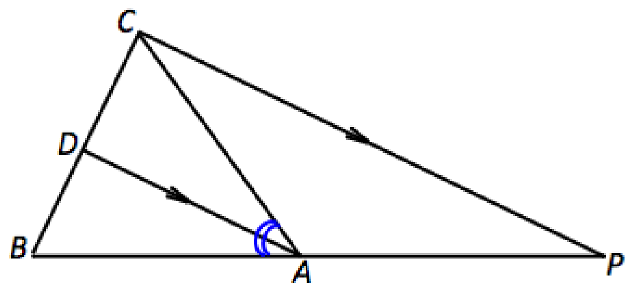
2. In the figure below, $\overline{CD}$ bisects $\angle ACB$, $AB = BC$,$\angle BEC = 90°$, and $\angle DCE = 42°$.

Find the measure of angle $\angle A$.

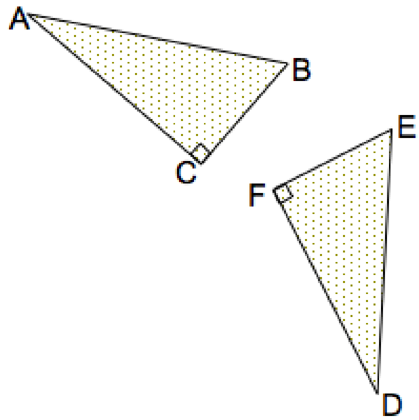


3. In the figure below, $\overline{AD}$ is the angle bisector of $\angle BAC$. $\overline{BAP}$ and $\overline{BDC}$ are straight lines, and $\overline{AD} \parallel \overline{PC}$.

Prove that $AP = AC$.



4. The triangles ΔABC and ΔDEF in the figure below are such that $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$.



a. Which criteria for triangle congruence (ASA, SAS, SSS) implies that $\Delta ABC \cong \Delta DEF$?

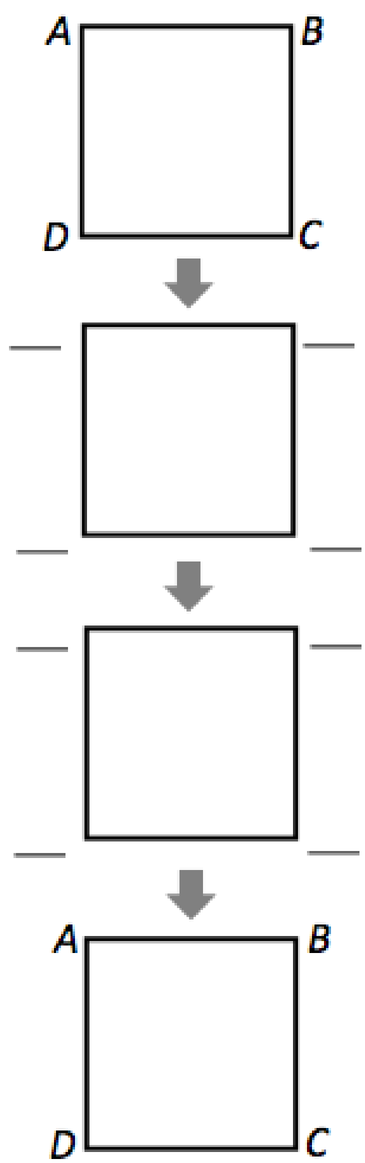
b. Describe a sequence of rigid transformations that shows $\Delta ABC \cong \Delta DEF$.

5.

a. Construct a square $ABCD$ with side $\overline{AB}$. List the steps of the construction.

b. Three rigid motions are to be performed on square $ABCD$. The first rigid motion is the reflection through line $\overline{BD}$. The second rigid motion is a $90°$ clockwise rotation around the center of the square.

Describe the third rigid motion that will ultimately map $ABCD$ back to its original position. Label the image of each rigid motion A, B, C, and D in the blanks provided.



Rigid Motion 1 Description: Reflection through line $\overline{BD}$

Rigid Motion 2 Description: $90°$ clockwise rotation around the center of the square.

Rigid Motion 3

Description: ______________________

6. Suppose that $ABCD$ is a parallelogram and that M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Prove that $AMCN$ is a parallelogram.

Common Core Mathematics High School Acronyms

- Number & Quantity
  - N-RN – The Real Number System
  - N-Q – Quantities
  - N-CN – The Complex Number System
  - N-VM – Vector and Matrix Quantities
- Algebra
  - A-SSE – Seeing Structure in Equations
  - A-APR – Arithmetic with Polynomials and Rational Expressions
  - A-CED – Creating Equations
  - A-REI – Reasoning with Equations and Inequalities
- Functions
  - F-IF – Interpreting Functions
  - F-BF – Building Functions
  - F-LE – Linear, Quadratic and Exponential Models
  - F- TF – Trigonometric Functions
- Geometry
  - G-CO - Congruence
  - G-SRT – Similarity, Right Triangles, & Trigonometry
  - G-C – Circles
  - G-GPE – Expressing Geometric Properties with Equations
  - G-GMD – Geometric Measurement & Dimension
  - G-MG – Modeling with Geometry
- Statistics and Probability
  - S-ID – Interpreting Categorical & Quantitative Data
  - S-IC – Making Inferences & Justifying Conclusions
  - S-CP – Conditional Probability and Rules of Probability
  - S-MD – Using Probability to Make Decisions

Mathematics - High School Geometry: Introduction

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations.

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Mathematical Practices

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.

5. Use appropriate tools strategically.

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

Geometry Overview

Congruence

• Experiment with transformations in the plane

• Understand congruence in terms of rigid motions

• Prove geometric theorems

• Make geometric constructions

Similarity, Right Triangles, and Trigonometry

• Understand similarity in terms of similarity transformations

• Prove theorems involving similarity

• Define trigonometric ratios and solve problems involving right triangles

• Apply trigonometry to general triangles

Circles

• Understand and apply theorems about circles

• Find arc lengths and areas of sectors of circles

Expressing Geometric Properties with Equations

• Translate between the geometric description and the equation for a conic section

• Use coordinates to prove simple geometric theorems algebraically

Geometric Measurement and Dimension

• Explain volume formulas and use them to solve problems

• Visualize relationships between twodimensional and three-dimensional objects

Modeling with Geometry

• Apply geometric concepts in modeling situations

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, & Trigonometry G-SRT

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Apply trigonometry to general triangles

9. (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

2. Derive the equation of a parabola given a focus and directrix.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Use coordinates to prove simple geometric theorems algebraically

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★

Geometric Measurement & Dimension G-GMD

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

Visualize relationships between two-dimensional and three-dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Mathematics: Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities

in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient

students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Appendix III

Exploration of Axiomatic Systems and Rigid Motion Transformations

2300 years ago in ancient Greece, Euclid codified an axiomatic system to explore the measurement of the earth, a system that has been assiduously studied by students in secondary school and beyond ever since. Not only aspiring mathematicians are impressed by Euclid's elegance; he has been known to influence the logical practices of such legendary figures as Abraham Lincoln, who determined that a thorough study of Euclid's methods was critical to the successful practice of law:

> Lincoln, you never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.
>
> ~Ketcham, Henry (The Life of Abraham Lincoln)
>
> http://www.authorama.com/life-of-abraham-lincoln-10.html

High school geometry is the first immersion of students into the study of an axiomatic system, and their understanding of its rationale is vital to their ability to reason critically. Students are expected, in tenth grade, to begin with a small number of assumptions and, from those few assumptions, build an entire system of logical statements. Let's explore that process.

The structure of a Euclidean Proposition

Euclid used a six-part structure in his propositions and constructions, as follows:

- The **enunciation**, in which the proposition is stated, generally in the form of a conditional (e.g. *If two sides of a triangle have the same length, then the angles opposite those sides have the same measure.*) Conditionals consist of a hypothesis (the part after the *If*) and the conclusion (the part after the *then*).
- The **setting out**, which identifies and explains the notation needed to work with the enunciation (*Let A, B, and C be the vertices of a triangle.)* and formulates the hypothesis in terms of this notation (*Given that sides AB and AC have the same length.).*
- The **goal**, which restates the conclusion from the enunciation with the required notation (*Show that ∠ACB is congruent to ∠ABC.).*

- The **construction**, in which any additional lines or circles are added to the given figure, if necessary.
- The **proof**, which delineates a series of statements and justifications, leading from the given assumptions to the goal. Statements must be offered in a logical progression, and justifications must represent either (1) given facts or assumptions or (2) facts or definitions that have been previously established, either in earlier proofs or in the current proof. Note that proofs need not be in the traditional "two-column" format, with statements in the left-hand column and corresponding reasons to the right; they may be presented either in paragraph form (as was typical for Euclid) or in flow-chart form, which many students are more easily able to follow and understand.
- The **conclusion**, which is simply a restatement of what has been demonstrated. This is generally the last line or sentence in a formal proof.

Try This #1

Euclid's Proposition I.16 is reproduced at the right. Identify each part of the proposition below.

Proposition 16.

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let **ABC** be a triangle, and let one side of it **BC** be produced to **D**;

I say that the exterior angle **ACD** is greater than either of the interior and opposite angles **CBA**, **BAC**.

Let **AC** be bisected at **E** [I. 10], and let **BE** be joined and produced
in a straight line to **F**;

let **EF** be made equal to **BE**[I. 3], let **FC** be joined [Post. 1], and let **AC** be drawn through to **G** [Post. 2].

ECF; [C. N. 5]

therefore the angle ACD is greater than the angle BAE.

Similarly also, if **BC** be bisected, the angle **BCG**, that is, the angle **ACD** [I. 15], can be proved greater than the angle **ABC** as well.

Therefore etc.

Euclid's "Bricks and Mortar"

Euclid built his logical system using a few very basic assumptions that lay the foundation for his entire logical structure. He first describes several geometric ideas which defy formal definition, including *point, line*, and *plane*. He then establishes a few specific "postulates" – that we can draw a segment between any two points; that we can extend segments infinitely to create lines; that, given any two points, we can create a circle centered around one of the two points and passing through the other; that all right angles are congruent; and, that, "if a straight line falling on two lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles" – the often-debated parallel postulate.

Try This #2

Euclid defined *right angles* as follows:

> When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands. (Euclid, *Elements.* Book I, Definitions)

How does Euclid's definition of *right angle* compare with the common definition used today? Why was Euclid's definition stated in this manner?

Try This #3

Examine the parallel postulate below in detail. Draw a diagram that illustrates Euclid's reasoning.

> If a straight line falling on two lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
>
> Euclid, *Elements*. Book I. Postulates.

Bricks and Mortar – Beyond the Foundation

The parallel postulate provides the beginning of a train of logical thought leading up to Euclid's Propositions 27 – 30, the transversal propositions. Let's examine the propositions that lead up to those which so many high school students must master to continue with their studies.

- The parallel postulate itself, as noted above.

- The first 26 propositions:

- Proposition 1.
  On a given finite straight line to construct an equilateral triangle.
- Proposition 2.
  To place at a given point (as an extremity) a straight line equal to a given straight line.
- Proposition 3.
  Given two unequal straight lines, to cut off from the greater a straight line equal to the less.
- Proposition 4.
  If two triangles have the two sides equal to two sides respectively, and have angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.
- Proposition 5. *
  In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.
- Proposition 6.
  If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.
- Proposition 7.
  Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.
- Proposition 8.
  If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.
- Proposition 9.
  To bisect a given rectilineal angle.
- Proposition 10.
  To bisect a given finite straight line.

- Proposition 11.
  To draw a straight line at right angles to a given straight line from a given point on it.
- Proposition 12.
  To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.
- Proposition 13. *
  If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.
- Proposition 14. *

If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

- Proposition 15. *
  If two straight lines cut one another, they make the vertical angles equal to one another.
- Proposition 16. *
  In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.
- Proposition 17. *
  In a triangle two angles taken together in any manner are less than two right angles.
- Proposition 18.
  In any triangle the greater side subtends the greater angle.
- Proposition 19.
  In any triangle the greater angle is subtended by the greater side.
- Proposition 20.
  In any triangle two sides taken together in any manner are greater than the remaining one.
- Proposition 21.
  If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.
- Proposition 22.
  Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I.20]
- Proposition 23.
  On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.
- Proposition 24.
  If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

- Proposition 25.
  If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater that the other.
- Proposition 26.
  If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, of that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.

Try This #4
Six of the first 26 propositions noted above are marked with an asterisk (*). Are each of the six related in any way to the transversal propositions noted below (Propositions 27-30)? Do each of the six depend on the Parallel Postulate? Do any other propositions from the first 26 lead up to the transversal propositions?

Proposition 27.

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

Proposition 28.

If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

Proposition 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

Proposition 30.

Straight lines parallel to the same straight line are also parallel to one another.

Try This #5
Demonstrate (prove) the transversal propositions noted above using the six-part structure delineated at the beginning of this study. You may wish to consult the wording in *Elements*, but using more modern wording is suggested.

Transformations of the Euclidean Plane – Rigid Motions
Properties of the Euclidean plane are described in the postulates and in the propositions deduced from the postulates. One idea that does NOT appear in Euclid's *Elements* is the use of number to describe any measurements – of segments or of angles. Instead, Euclid simply compared segments and angles to one another without reference to any unit of measurement. In working with transformations, we will instead agree to use the more modern "ruler postulate" which allows us to refer to specific measurements according to a standard unit of measure, and will denote the *distance* between two points as $d(A, B)$.

The notion of a *perpendicular bisector* of a line segment is critical to understanding the three rigid motion transformations – reflections, rotations, and translations. Another important concept involved in transformations is the realization that all transformations are *functions* – rules that map each point in the plane to its image as defined under the particular transformation defined. An *isometry* is a mapping of points in the plane that preserves both distance and angle measure.

Try This #6
Construct a line segment $\overline{AB}$ and, using only a compass and straightedge, construct the perpendicular bisector of $\overline{AB}$.

Prove that the perpendicular bisector you constructed is the set of all points equidistant from A and B. Which of Euclid's propositions will you use?

Rigid Motion Transformations Defined
A basic rigid motion is a mapping of the plane onto itself and is one of the following:

- The *reflection in line ℓ*- denoted r_ℓ - is the transformation such that (a) if P is a point on ℓ, then $r_\ell(P) = P$; or, if P is not on ℓ, then $r_\ell(P)$ is the point on the opposite half-plane of ℓ such that ℓ is the perpendicular bisector of $\overline{Pr_\ell(P)}$.
- The *rotation by the oriented angle θ about C* – denoted $R_{C,\theta}$ – is the transformation that maps point C to point C and maps each point other than C to the point $R_{C,\theta}(P)$ on circle CP (the circle with center C through point P) such that the oriented angle $\angle PCR_{C,\theta}(P) = \theta$.

- The translation that maps point A to point B – denoted T_{AB} - is the transformation that maps each point P to the point $T_{AB}(P)$ such that segment $\overline{PT_{AB}(P)}$ is parallel to $\overline{AB}$ (or is on $\overleftrightarrow{AB}$ if P is on $\overleftrightarrow{AB}$) and is of the same length and direction.

Try This #7

Using a compass and straightedge, draw a scalene triangle ΔABC and line ℓ such that vertex C lies on line ℓ. Construct $\Delta A'B'C'$ such that ℓ is the line of reflection between the two triangles.

How did perpendicular bisectors play a role in your construction? Explain how your reflection complied with the definition above.

Try This #8

Use the same reflection from Try This #7. Draw a second line (ℓ_2) that intersects ℓ. Construct $\Delta A''B''C''$ such that ℓ_2 is the perpendicular bisector of $\Delta A'B'C'$ and $\Delta A''B''C''$.

You just constructed a reflection of the triangle $\Delta A'B'C'$. How is its image ($\Delta A''B''C''$) related to the original triangle in Try This #7 (ΔABC)? Compare your work with the definition of *rotation* above.

Try This #9

Return to your construction in Try This #7. Draw another line (ℓ_3) parallel to ℓ. Now reflect $\Delta A'B'C'$ across ℓ_3. What do you observe? Does your new construction meet the definition of *translation* noted above? Why or why not?

You have now worked extensively with perpendicular bisectors. Explain the role played by this construction in all three rigid motion transformations.

VITA

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