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Finite Deformations of Circular Arches.

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Finite Deformations of Circular Arches

A Dissertation

**Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy**

in

The Department of Engineering Science

by

**Ronald Steven Reagan
B.S., Louisiana State University, 1964
M.S., Rice University, 1966
August, 1970**

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NOTATION

Dimensional Quantities		Dimensionless Quantities	
Symbol	Meaning, Units	Symbol	Meaning
A	Cross-sectional area, L^2	a	$\frac{1}{2}$ angle subtended by arch
B	Bending, L^{-1}	b	Bending
		c	Extensibility = $\frac{I}{AL^2}$
		d_x	Horizontal deflection of arch midpoint, positive to the right
		d_y	Vertical deflection of arch midpoint, positive downward
E	Modulus of elasticity, FL^{-2}	e	Extension
<u>F</u>	Internal force, F	<u>f</u>	Internal force
G	Rotational spring constant, F	g	Rotational spring constant
H	Extensional spring constant, FL^{-1}	h	Extensional spring constant
I	Second moment of area, L^4	<u>i</u> , <u>j</u>	Unit vectors
K	Curvature, L^{-1}	k	Curvature
		<u>k</u>	Unit vector
L	$\frac{1}{2}$ length of undeformed arch, L		
M	Internal couple, FL	m	Internal couple
		<u>n</u>	Unit normal vector
<u>P</u>	Load, FL^{-1}	p	Load
		p	Load parameter for hydrostatic loading
		p_o	Load parameter for vertical loading
<u>Q</u>	Position vector, L	q	Position vector

R	Radius of curvature, L	r	Radius of curvature
S	Arc length, L	s	Arc length
		<u>t</u>	Unit tangent vector
U	Potential energy, FL	u	Potential energy
		v	Tangent angle
		w	Angle between chord and line of translation
X,Y	Cartesian coordinates locating centroidal axis of arch, L	x,y	Cartesian coordinates locating centroidal axis of arch
Z	Coordinate locating point in cross-section, L		
σ	Fiber stress, FL ⁻²	ϵ	Fiber strain

Subscripts:

- o denotes quantities associated with the undeformed configuration.
- n denotes a component of a vector in the direction of n.
- t denotes a component of a vector in the direction of t.
- x denotes a component of a vector in the direction of i.
- y denotes a component of a vector in the direction of j.

Superscripts:

- * denotes system properties which vary from 0 to 1.
- ' denotes differentiation with respect to s_0 .

Vector quantities are underlined.

ABSTRACT

A nonlinear two-point boundary-value problem was formulated whose solutions are the equilibrium configurations of initially circular arches. The formulation, which consists of a set of six nonlinear ordinary differential equations with nonlinear boundary conditions, is exact within the following assumptions:

1. The arch centroidal axis is constrained to remain in a plane.
2. The assumption of engineering beam theory that cross-sectional planes perpendicular to the centroidal axis of the arch remain plane as the arch deforms is used. Shearing deformations are neglected.
3. Linearly elastic material behavior is assumed.
4. The ratio of the thickness of the arch to the radius of the undeformed arch is assumed to be small.

Dimensionless variables were introduced such that both extensible and inextensible arches could be studied.

Two loading cases, hydrostatic loading and vertical loading, were considered. The hydrostatic loading consisted of a pressure which was uniformly distributed with respect to the arc length of the deformed centroidal axis and which remained normal to the centroidal axis as the arch deformed. The vertical loading was taken as a pressure, uniformly distributed with respect to the undeformed arc length of the centroidal axis, and which remained in a vertical direction. Both loadings can be completely specified by a loading parameter which was taken as the intensity of the applied pressure.

Two algorithms were developed for obtaining numerical solutions of the boundary-value problem for various values of the loading parameter.

One algorithm called SYM was designed to find geometrically symmetric configurations, and the other called ASYM was designed to find the asymmetric ones. Both used a shooting method to solve the boundary-value problem and an extrapolation technique for following plots of load parameter versus deflection. SYM included the computation of the load parameter corresponding to the first relative maximum in the load-deflection curve. ASYM was designed to find the load parameter corresponding to bifurcation of the arch from a symmetric configuration to an adjacent asymmetric one.

The algorithms SYM and ASYM were used to study the structural behavior of several different arches. Results were presented for the following cases:

1. Clamped, hydrostatically loaded, semicircular arches were considered with varying amounts of extensibility.
2. Several clamped, hydrostatically loaded, shallow arches which were studied previously by Kerr and El-Bayoumy using a more approximate shallow arch theory were considered.
3. Clamped and pinned, inextensible, vertically loaded arches were considered with total subtended angles of 60, 120, and 180 degrees.

These results showed that the effect of extensibility on the buckling load for clamped extensible arches is not very significant for semicircular (deep) arches and is significant for shallow arches. The shallow arch theory results of Kerr and El-Bayoumy provided an excellent approximation to the results for the shallow arches considered herein. The structural behavior of the inextensible, vertically loaded arches was quite similar to that of the corresponding hydrostatically loaded arches. However, the non-funicular vertical loading did produce significant prebuckling displacements for the higher arches.

CHAPTER 1. INTRODUCTION

The linear theory of curved or straight bars forms a rational basis for the structural analysis of framed structures such as buildings, bridges, transmission towers, etc. This linear theory is based on linearized versions of the equations governing the behavior of the bars. Unfortunately, the linear theory of bars with its associated mathematical model is unable to predict physically familiar behavior. One example of such behavior is the elastic buckling of thin axially-loaded rods which has been observed by most youngsters who play with sticks.

The equations governing the static equilibrium of straight or curved bars may be categorized as follows:

1. Equilibrium Equations. These differential equations follow from the static equilibrium of an arbitrarily small element of the bar.
2. Strain Measure Definitions. These equations define the quantities, such as change in curvature, angle of twist, etc., which are taken as measures of the internal strain in the bar, in terms of the quantities which are used to specify the geometry of the deformed bar.
3. Constitutive Equations. These are relations between the strain measures and the internal stress resultants.
4. Boundary Conditions. These are equations which express the constraints provided at the boundaries of the bar in terms of the stress resultants and the geometry of the bar at the boundaries.

In general all of these equations are nonlinear. To linearize them additional assumptions must be introduced which may be viewed as constraints on the admissible configurations of the mathematical model of the bar.

For many cases of practical interest the external loading can be specified by the value of a single parameter called the load parameter.

Only such loading systems will be considered herein. Examples of such loading systems are concentrated or distributed forces or couples, distributed temperature changes, and displacements.

The linear boundary-value problem which results from the linearized versions of the governing equations has a unique solution for many special cases of practical interest. By a unique solution it is meant that there is a unique equilibrium configuration for each different value of the load parameter.

The key to the phenomenon of buckling mentioned earlier is that certain structures may have more than one equilibrium configuration for the same value of the load parameter. In terms of the axially-loaded bar there exist two equilibrium configurations, a straight unbuckled one and a bent buckled one, for values of the axial load (which may be taken as the load parameter) greater than the Euler critical load. In order to predict such behavior analytically a more realistic mathematical model must be considered for which some of the assumptions (constraints) of the linear model have been relaxed. Such formulations result in nonlinear boundary-value problems.

This study is concerned with the equilibrium configurations of plane circular arches acted on by uniformly distributed forces. The intensity of the uniformly distributed force is taken as the load parameter. For such arches there can be more than one equilibrium configuration associated with a given value of the load parameter.

First a brief review of the work of other investigators will be presented. Following that the scope of the present study will be outlined.

Review of the Work of Other Investigators

There is an extensive literature dealing with the finite deformations and stability of circular arches and rings with much of the work being published during the decade of 1960-1969. Some twenty works will be referred to herein. Several older references may be found in the paper by Oran and Reagan^a(17). Most approaches to the problem require the solution of systems of nonlinear algebraic equations, a task which generally requires the use of a digital computer. The emergence of readily available digital computers in the early 1960's and the development of FORTRAN, a programming language for scientific and engineering computations, made most of these studies possible. The external loadings, constitutive equations and geometric constraints, and the methods of solution used by previous investigators will be discussed below.

Several different kinds of external loading have been considered. Most authors have considered either a single concentrated force which remains in the same direction throughout the deformation process (3, 4, 5, 7, 13, 18, 20, 24, 27) or a hydrostatic pressure (1, 2, 11, 12, 14, 17, 20, 21, 22, 26). Two other kinds of uniform loadings, one in which the loads are initially radial and remain in that direction throughout deformation (17, 21, 26) and the other called central loading in which the lines of action of the loads pass through the original center of curvature (20, 26), have been considered. Lo and Conway (16) studied the effect of equal and opposite couples applied at the ends of the arch. Since both the distribution and direction of the applied loading may be

^a Numbers enclosed in parentheses appearing in the text refer to items in the List of References

varied along the length of the arch, the number of possible loading cases is endless.

Most authors, with the exception of Wah (23), assumed that the arch or ring centroidal axis is constrained to remain in a plane. Except for Smith and Simitzes (21) who considered shearing deformations, previous authors have used the assumption of engineering beam theory that material planes which are normal to the arch centroidal axis in the undeformed configuration remain plane and stay normal to the centroidal axis during the deformation. Some authors (4, 17, 18, 22) restricted their studies to arches or rings with inextensible centroidal axes. The only authors who treated nonlinear constitutive equations were Antman (1) and Lee and Murphy (14), with the other authors using the linear one-dimensional constitutive relation between fiber stress and fiber strain. Most investigators have assumed that the thickness of the arch is thin with respect to its radius with the exception of Huddleston (7), Langhaar, Boresi, and Carver (13), Wempner and Kesti (26) who considered the Winkler-Bach theory for thick arches. Either clamped or pinned ends were assumed by all authors except Oran and Reagan (17) who considered the ends to be supported by rotational, horizontal, and tangential springs.

The exact (within the assumptions of engineering beam theory) formulation of the problem of finding the planar equilibrium configurations of statically loaded arches and rings leads to a sixth-order boundary-value problem of six first-order nonlinear ordinary differential equations with nonlinear boundary conditions. Three of the differential equations are the exact equations of equilibrium for a planar system and the other three follow from the exact differential geometry of the de-

formed plane curve. The six dependent variables in the differential equations may be taken as the three internal stress resultants (two components of the internal force and an internal couple), two coordinates (cartesian, polar, etc.) locating a point on the deformed centroidal axis, and some measure of the slope at that point, respectively. The independent variable in the equations is some measure of the arc distance measured along the deformed centroidal axis. In addition to the differential equations and boundary conditions, constitutive equations which may also be nonlinear are required to complete the formulation.

The solution of the exact nonlinear boundary-value problem is a formidable task. This is due to the fact that the nonlinear differential equations can have many solutions for the same value of the load parameter. In terms of the problem at hand this means that an arch can have many equilibrium configurations with branches laced through its infinite-dimensional configuration space. At present there is no general procedure for finding all these equilibrium configurations. For some cases an investigator is at least assured that there is one solution since Antman (2) has proven existence theorems for the nonlinear boundary-value problem governing the behavior of hydrostatically loaded, nonlinearly elastic rods of arbitrary initial shape. Solutions in terms of known functions can be found only for a very few special cases, an example being the inextensible arch loaded by a single concentrated load which was solved by DaDeppo and Schmidt (4) using elliptic functions. Antman (1) shows how solutions for hydrostatically loaded arches may be found in terms of definite integrals.

Linear classical buckling theory may be used to find the buckling loads for some arch geometries and loadings. This formulation leads to an eigenvalue problem of homogeneous differential equations. It gives correct results only if the arch is inextensible and the applied loading is a funicular loading for the undeformed configuration (17). A funicular loading is one that produces no internal moments or shears. The funicular loading for a circular arch is a uniformly distributed pressure acting in the radial direction. In this case the undeformed configuration is an equilibrium configuration (an exact solution of the nonlinear boundary-value problem) for all values of the load parameter and the eigenvalues (buckling loads) are values of the load parameter for which the arch has another equilibrium configuration adjacent to the undeformed (trivial) one. Associated with each eigenvalue is an eigenmode (buckling mode). A disadvantage to this approach is that it yields no information about the post-buckling behavior of the arch. It was used by Oran and Reagan (17).

Some investigators (3, 5, 10, 11, 13, 14, 16, 20, 21, 26) introduced approximate assumptions into the exact boundary-value problem which led to another boundary-value problem for which solutions could be found in terms of known functions. One set of assumptions, referred to herein as shallow arch theory, was introduced by Schreyer and Masur (20, p. 3) and subsequently used by others (11, 12, 14).

The finite-element method has been used by Walker (24) and Wempner and Patrick (27) to find equilibrium configurations of arches loaded with a single vertical concentrated load. Walker used shallow arch approximations, while Wempner and Patrick did not. In the finite-element method

the continuous arch is replaced by a set of discrete elements, each of which has a finite number of degrees of freedom. The equilibrium of each element and compatibility between elements leads to a set of nonlinear algebraic equations which must be solved numerically.

Other investigators have used numerical procedures to attack the exact nonlinear boundary-value problem directly. Tadjbakhsh and Odeh (10, 22) found some equilibrium configurations for inextensible rings loaded with a hydrostatic pressure using Newton-Raphson iteration in function space. Numerical integration techniques for nonlinear initial-value problems such as Runge-Kutta or predictor-corrector methods may be used to solve nonlinear boundary-value problems, provided that some scheme is devised to satisfy the boundary conditions at both ends. Such techniques are commonly referred to as shooting methods. Huddleston (7) used a shooting method to study extensible arches. He also used shooting methods for studying the behavior of other structures (6, 8).

Scope of the Present Study

The impetus for this study came from the previous investigations of Cenap Oran and the author (17). These investigators used linear classical buckling theory to find the buckling loads for inextensible circular arches under the action of funicular loadings, the only case for which that theory gives correct results. The present study was an attempt to devise a means of determining the buckling loads for extensible arches with non-funicular loadings which are not accessible to linear classical buckling theory.

In Chapter 2 a nonlinear boundary-value problem governing the static behavior of circular arches is derived. It is exact within the following set of assumptions:

1. In the undeformed (unloaded) configuration the centroidal axis of the arch is an arc of a circle. The undeformed geometry of the arch including the boundaries is assumed to be symmetric about a vertical line passing through the initial center of curvature. The deformed centroidal axis is constrained so that it remains in a plane.
2. Engineering beam theory is used. This assumes that material planes which are normal to the arch centroidal axis in the undeformed configuration remain plane and normal to the centroidal axis as the arch deforms. Deformations of the arch due to shear are neglected. These assumptions imply that the only important effect of the loading on the arch is the one-dimensional straining of fibers of the arch which are normal to the cross-sectional planes.
3. The arch is assumed to be thin, i.e. its thickness is much smaller than its radius.
4. The linear and homogeneous one-dimensional constitutive equation relating fiber stress to fiber strain is used.
5. Only conservative loadings are considered.

In the Appendix a shooting method is developed for solving the boundary-value problem. An algorithm called SYM, which uses the shooting method and is designed to find arch configurations that are geometrically symmetric, is described. The symmetric configurations are found for successive values of the load parameter. By a technique of following a plot of load parameter versus the vertical deflection of the midpoint of the arch, the load values are increased until a relative maximum in that curve is reached, and after that the load parameter is decreased. These computations are terminated after the first decrease in the load parameter since the configurations corresponding to the initial portion of the curve including the relative maximum are the ones of primary interest.

For some arches the symmetric configuration becomes unstable at a load parameter value less than the maximum one found by algorithm SYM, and the arch buckles (bifurcates) to an adjacent asymmetric configuration. The shooting method is used in another algorithm called ASYM which is also described in the Appendix. ASYM is used to find some of the adjacent asymmetric equilibrium configurations and to compute the load parameter value at which bifurcation occurs.

In Chapter 3 results are presented for several different undeformed geometries and for two different loading cases, hydrostatic loading and a vertical loading. The latter consists of a pressure which is uniformly distributed with respect to the undeformed arc length and remains in the vertical direction. This latter case was chosen because it had not been studied by any of the previous investigators.

No attempt was made to vary all the parameters in the problem. The author feels that it is impossible to pick a range of parameter values or a particular loading case which will be directly useful to practicing engineers. If a practicing engineer feels that nonlinear effects are important in a given structure, he can use a method such as the one presented here to analyze the behavior of that structure. In a sense, all that is presented here is a feasibility study illustrating the shooting method for solving the boundary-value problem.

In Chapter 4 some conclusions are presented. These are limited by the difficulty in generalizing from the results of the numerical solution of a small number of special problems.

CHAPTER 2. FORMULATION OF THE BOUNDARY-VALUE PROBLEM

In this chapter the problem of determining the equilibrium configurations of plane arches undergoing finite deformations is formulated as a boundary-value problem of six simultaneous first-order ordinary differential equations. First the differential equations, boundary conditions, and constitutive equations will be derived, and then these relations will be cast in a nondimensional form by introducing a convenient set of dimensionless quantities.

The following notational conventions will be used:

1. Symbols denoting vector quantities will be underlined, e.g. $\underline{Q}$.
2. All dimensional quantities will be denoted by majuscule letters. If a dimensionless quantity corresponding to a given dimensional quantity is needed, it will be denoted by the corresponding minuscule letter. All other dimensionless quantities will also be denoted by minuscule letters.
3. Components of vector quantities will be denoted by subscripts affixed to the symbol denoting the vector.

Differential Geometry of the Deformed Arch

Any geometrical configuration of a plane arch is completely specified by the position vector of the centroidal axis of the arch, $\underline{Q}(S)$, as shown in Fig. 2.1. The variable S is the arc length measured from some convenient reference point. The components of $\underline{Q}$ with respect to the right-handed cartesian coordinate system shown are denoted as X and Y , respectively.

The unit tangent vector $\underline{t}(S)$ is given in terms of $\underline{Q}$ as

$$\underline{t} = d\underline{Q}/dS \quad (2.1)$$

Fig. 2.1 Arch Geometry

Fig. 2.2 Sign Convention for Internal Stress Resultants

The tangent angle $v(S)$ is defined as the angle between the positive X -axis and $\underline{t}$, with the counterclockwise direction considered positive. The following relations exist between $\underline{t}$, v , and Q

$$t_x = \cos v = dX/dS \quad (2.2)$$

$$t_y = \sin v = dY/dS \quad (2.3)$$

The curvature K is defined as

$$K = -dv/dS \quad (2.4)$$

As shown in Fig. 2.1, $\underline{n}(S)$ is defined to be the unit vector which is perpendicular to $\underline{t}$ and points to the right as the centroidal axis is followed in the direction of increasing S .

Equilibrium Equations

The equilibrium equations are derived by requiring that the resultant of all the forces and moments acting on the free-body diagram of an element of the arch of length ΔS vanishes in the limit as $\Delta S \rightarrow 0$. This free-body diagram is shown in Fig. 2.3.

The resultant of the internal stresses is taken as a force $\underline{F}(S)$ and a couple $M(S)$. The components of $\underline{F}$ in the normal and tangential directions are denoted as F_n and F_t , respectively. The sign convention for these quantities is shown in Fig. 2.2, where the stress resultants are shown as vectors, and the unit vector $\underline{k} = \underline{i} \times \underline{j}$.

The externally applied loading considered is a pressure $\underline{P}(S)$ distributed along the centroidal axis of the arch. The normal and tangential components of $\underline{P}$ are denoted as P_n and P_t , respectively. No externally applied couples are considered in this study.

Fig. 2.3 Free-Body Diagram
of an Arch Element

Fig. 2.4 Undeformed Arch

Requiring the resultant force to vanish yields

$$\underline{F}(S+\Delta S) - \underline{F}(S) + \int_S^{S+\Delta S} \underline{P}(\bar{S}) d\bar{S} = \underline{0} \quad (2.5)$$

where $\bar{S}$ is a dummy variable of integration and $\underline{0}$ is the null vector.

After dividing both sides of Eq. 2.5 by ΔS and taking the limit as $\Delta S \rightarrow 0$, it becomes

$$d\underline{F}/dS + \underline{P} = \underline{0} \quad (2.6)$$

To evaluate $d\underline{F}/dS$ it is convenient to write $\underline{F}$ in the component form

$$\underline{F} = F_n \underline{n} + F_t \underline{t} \quad (2.7)$$

Differentiating Eq. 2.7 with respect to S yields

$$d\underline{F}/dS = dF_n/dS \underline{n} + F_n d\underline{n}/dS + dF_t/dS \underline{t} + F_t d\underline{t}/dS \quad (2.8)$$

The vectors $\underline{n}$ and $\underline{t}$ may be expressed in terms of v and the cartesian base vectors $\underline{i}$ and $\underline{j}$ as

$$\underline{n} = \sin v \underline{i} - \cos v \underline{j} \quad (2.9)$$

$$\underline{t} = \cos v \underline{i} + \sin v \underline{j} \quad (2.10)$$

Differentiating these with respect to S and using Eq. 2.4 yields

$$d\underline{n}/dS = -K \underline{t} \quad (2.11)$$

$$d\underline{t}/dS = K \underline{n} \quad (2.12)$$

Substituting Eqs. 2.11 and 2.12 into Eq. 2.8 and then substituting that result in Eq. 2.6 requires that

$$dF_n/dS + KF_t + P_n = 0 \quad (2.13)$$

$$dF_t/dS - KF_n + P_t = 0 \quad (2.14)$$

Requiring the sum of the moments about the left end to vanish yields

$$\begin{aligned} -\underline{k} [M(S+\Delta S) - M(S)] + [Q(S+\Delta S) - Q(S)] \times \underline{F}(S+\Delta S) \\ + \int_S^{S+\Delta S} \{ [Q(\bar{S}) - Q(S)] \times \underline{P}(\bar{S}) \} d\bar{S} = 0 \end{aligned} \quad (2.15)$$

Dividing both sides of Eq. 2.15 by ΔS and taking the limit as $\Delta S \rightarrow 0$ gives

$$-dM/dS \underline{k} + dQ/dS \times \underline{F} = 0 \quad (2.16)$$

Using Eqs. 2.1 and 2.7 and performing the cross product yields

$$dM/dS + F_n = 0 \quad (2.17)$$

Undeformed Geometry

This study is limited to arches with undeformed configurations which are circular and symmetric about a vertical line. The undeformed geometry of such an arch is completely specified by the quantities L and a , which are defined as one-half the arc length of the undeformed arch and one-half of the subtended angle, respectively. See Fig. 2.4.

All quantities associated with the undeformed configuration are denoted by the subscript o . Thus S_o , $\underline{Q}_o = X_o \underline{i} + Y_o \underline{j}$, and v_o are the arc length, position vector, and tangent angle, respectively, for the undeformed configuration of the arch. S_o is always measured from the center point of the arch centroidal axis, and the origin of the cartesian coordinate system is always taken as the center of curvature of the undeformed arch. From the geometry of Fig. 2.4 it follows that

$$X_0(S_0) = R_0 \sin(S_0/R_0) \quad (2.18)$$

$$Y_0(S_0) = R_0 \cos(S_0/R_0) \quad (2.19)$$

$$v_0(S_0) = -S_0/R_0 \quad (2.20)$$

where $R_0 = L/a =$ radius of undeformed arch.

Boundary Conditions

The arches are considered to be supported at each of their ends ($S_0 = \pm L$) by two linearly elastic springs, one of which resists translation along a given line and the other which resists rotation of the end, as shown in Fig. 2.4. Only symmetric springs are considered, that is the spring constants are the same for both ends of the arch and the line of translation at one end is the mirror image of the corresponding line for the other end. The spring constants for the rotational and translational springs are denoted by G and H , respectively, and the angle between the line of translation and the chord connecting the two boundary points in their undeformed positions is denoted by w , as shown in Fig. 2.4.

At each end of the arch three boundary conditions are prescribed. These conditions are relations between the values of the six functions - F_n , F_t , M , X , Y , and v - at each end of the arch. The boundary conditions follow from these three requirements:

1. The internal moment at the end of the arch must be equal in magnitude and opposite in direction to the resisting moment provided by the linearly elastic rotational spring.
2. The ends of the arch are constrained to remain on the line of translation.
3. The component of the internal force at each end of the arch in the direction of the line of translation must be equal in magnitude and opposite in direction to the resisting force provided by the linearly elastic translational spring.

Applying these requirements at the left end in the given order yields

$$M = G(a-v) \quad (2.20)$$

$$\frac{Y - R_o \cos a}{X + R_o \sin a} = \tan w \quad (2.21)$$

$$F_n \sin(v-w) + F_t \cos(v-w) = H \frac{X + R_o \sin a}{\cos w} \quad (2.22)$$

Similarly at the right end the boundary conditions are

$$M = G(a+v) \quad (2.23)$$

$$\frac{Y - R_o \cos a}{R_o \sin a - X} = \tan w \quad (2.24)$$

$$F_n \sin(v+w) + F_t \cos(v+w) = H \frac{X - R_o \sin a}{\cos w} \quad (2.25)$$

Strain Measures and Constitutive Equations

To complete the formulation, constitutive equations, which are relationships between the internal stress resultants and suitably defined measures of the internal strains, are needed. Engineering beam theory with shearing strains neglected is used to derive these equations. The strain measures arise from a comparison of a deformed element of the arch with the corresponding element of the undeformed arch. Both of these elements are shown in Fig. 2.5. First the strain measure e , which is called the extension, is defined as

$$e = \lim_{\Delta S_o \rightarrow 0} \frac{\Delta S - \Delta S_o}{\Delta S_o} = dS/dS_o - 1 \quad (e > -1) \quad (2.26)$$

The necessity of the restriction $e > -1$ may be seen by solving Eq. 2.26 for dS which yields

$$dS = (1+e)dS_o \quad (2.27)$$

Since both the quantities dS and dS_o are inherently greater than zero (they are lengths), it follows from Eq. 2.27 that $1+e$ must be greater than zero which requires that $e > -1$.

The length of a "fiber" of the undeformed element is denoted by ΔC_o and the length of the corresponding fiber in the deformed element by ΔC . The location of the fiber being considered is specified by the distance Z which is measured as shown in Fig. 2.5. Following engineering beam theory it is assumed that each cross-sectional plane remains perpendicular to the centroidal axis and suffers no deformation.

Then the engineering strain ϵ of the fiber at Z is defined as

$$\epsilon = \lim_{\Delta C_o \rightarrow 0} \frac{\Delta C - \Delta C_o}{\Delta C_o} = dC/dC_o - 1 \quad (2.28)$$

In the limit as $\Delta S_o \rightarrow 0$ the undeformed and deformed elements of the arch may be considered as circular ring segments of radii R_o and R , respectively. From the geometry of the elements it follows that

$$\frac{\Delta C_o}{\Delta S_o} = \frac{R_o + Z}{R_o} \quad (2.29)$$

and

Fig. 2.5 Undeformed and Deformed Arch Elements

Fig. 2.6 Aid in Physical Interpretation of B

Fig. 2.7 Area Element

$$\frac{\Delta C}{\Delta S} = \frac{R + Z}{R} \quad (2.30)$$

Substituting these into Eq. 2.28 yields

$$\epsilon = \lim_{\Delta S_0 \rightarrow 0} \frac{\frac{(R+Z)\Delta S}{R}}{\frac{(R_0+Z)\Delta S_0}{R_0}} - 1 = \frac{1 + KZ}{1 + K_0 Z} dS/dS_0 - 1 \quad (2.31)$$

where $K_0 = 1/R_0$ = the curvature of the undeformed arch.

Substituting Eq. 2.26 into Eq. 2.31 and combining terms leads to

$$\epsilon = \frac{e + [(1+e)K - K_0]Z}{1 + K_0 Z} \quad (2.32)$$

It is also assumed that the thickness of the arch is much smaller than the initial radius (which implies that $K_0 Z \ll 1$). Using this and introducing the strain measure B , called the bending, defined as

$$B = (1+e)K - K_0 \quad (2.33)$$

Eq. 2.32 becomes

$$\epsilon = e + BZ \quad (2.34)$$

While many authors have used the extensional strain measure e , the bending strain measure B was introduced by Antman (1). To gain a physical understanding of the strain measure B , consider Fig. 2.6 which shows an infinitesimal element of the undeformed arch and the corresponding infinitesimal element of the deformed arch. Both of these elements may

be considered as segments of circular rings with radii R_0 and R and lengths dS_0 and $dS = (1+e)dS_0$, respectively. In the figure each of these elements has been displaced as a rigid body until the cross-section of its left end is vertical. Such rigid body displacements have no effect on the strain measures, B and e . The angle between the right hand cross-section of the undeformed element and a vertical line is $K_0 dS_0$, while the corresponding quantity for the deformed element is $K dS = K(1+e)dS_0$. The difference of these two angles, $K dS - K_0 dS_0 = [K(1+e) - K_0] dS_0 = B dS_0$, is the clockwise rotation of the deformed right cross-sectional plane from the undeformed right cross-sectional plane. In terms of the geometry of these elements in their actual positions (without the artificially introduced rigid body displacements), $B dS_0$ is the sum of the clockwise rotation of the left end of the element and the counterclockwise rotation of the right end of the element, both measured from the undeformed configuration.

According to engineering beam theory it is assumed that the one-dimensional constitutive equation

$$\sigma = E \epsilon \quad (2.35)$$

is valid, where σ is the fiber stress and E is the modulus of elasticity.

In terms of σ the internal stress resultants, F_t and M , are given as

$$F_t = \int_A \sigma dA \quad (2.36)$$

and

$$M = \int_A \sigma Z dA \quad (2.37)$$

where the integration is over the cross-sectional area. From Eqs. 2.36, 2.35, and 2.34 the following constitutive equation for F_t

$$F_t = EAe \quad (2.38)$$

is obtained, where A is the cross-sectional area. From Eqs. 2.37, 2.35, and 2.34 the constitutive equation for M

$$M = EIB \quad (2.39)$$

follows, where I is the second moment of the area of the cross-section. Since shearing strains are neglected there is no constitutive equation for the shear force F_n . The constitutive Eqs. 2.38 and 2.39, which are linear and homogeneous, are special cases of the more general constitutive equations given by Antman (1) for hyperelastic bars. For the important special case of an inextensible arch ($e = 0$), the only applicable constitutive equation is Eq. 2.39, which becomes the familiar

$$M = EI(K - K_0) \quad (2.40)$$

Formulation as a Boundary-Value Problem

It is convenient to change the independent variable from S to S_0 . To this end the chain rule

$$d/dS = d/dS_0 \quad dS_0/dS = \frac{1}{1+e} \quad d/dS_0 \quad (2.41)$$

is used to convert the derivatives. Applying Eq. 2.41 to Eqs. 2.13, 2.14, 2.17, 2.2, 2.3, and 2.4, and multiplying both sides of each equation by $1+e$, it follows that

$$dF_n/dS_0 = -(1+e)KF_t - P_{on} \quad (2.42)$$

$$dF_t/dS_0 = (1+e)KF_n - P_{ot} \quad (2.43)$$

where $\underline{P}_0(S_0) = (1+e)\underline{P}(S(S_0))$ = the applied loading per unit undeformed length, and

$$dM/dS_0 = -(1+e)F_n \quad (2.44)$$

$$dX/dS_0 = (1+e)\cos v \quad (2.45)$$

$$dY/dS_0 = (1+e)\sin v \quad (2.46)$$

$$dv/dS_0 = -(1+e)K \quad (2.47)$$

The six first-order differential equations above, the strain measure definitions (Eqs. 2.26 and 2.33), the constitutive equations (Eqs. 2.38 and 2.39), and the boundary conditions (Eqs. 2.20-2.25), form a boundary-value problem for the determination of the dependent variables - $\underline{F}$, M , $\underline{Q}$, v , e , dS/dS_0 , K and B - with S_0 as the independent variable.

Formulation in Terms of Dimensionless Quantities

In this study the applied loadings to be considered later can be specified by a single parameter P . With the restriction that EI and EA are not functions of S_0 , the following constant quantities (system properties) - EI , EA , a , L , G , H , and w - appear in the boundary-value problem. Thus to find all the equilibrium configurations (specified by the dependent variables $\underline{F}$, M , $\underline{Q}$, v , etc.) of all the arches considered in this study it is necessary to vary each of the eight quantities - P , EI , EA , a , L , G , H , and w - independently. This in itself is a formidable task, but the problem is further complicated by the nonlinearity of the differential equations and the boundary conditions. The complication is that for a given arch (specified by the system properties) there may be more than one equilibrium configuration for a specified value of P , i.e. the arch may buckle. Therefore it is desirable, as in most problems, to reduce the

number of system properties by recasting the formulation in terms of dimensionless quantities.

A statics problem such as this one may be nondimensionalized by scaling all the quantities using a characteristic force and a characteristic length. The characteristic quantities should be chosen carefully so that they remain finite over their ranges. To this end L is taken as the characteristic length. For finite L the arch degenerates to a straight beam as the angle α approaches zero, whereas if R_0 is chosen, the degenerate case corresponds to a point, which is not desirable. In this study EIL^{-2} , which remains finite, is chosen as the characteristic force. This is done so that both extensible and inextensible arches may be studied with the same formulation. If EA were taken as the characteristic force, the scaled quantities would vanish as EA became infinite (the inextensible case) or become infinite as EA approaches zero, which would not be desirable. On the other hand, the special cases of $EI = \infty$ or $EI = 0$ corresponding to complete bending rigidity or flexibility, respectively, cannot be handled within the present formulation. However, this is not a serious limitation since those special cases lie outside the realm of practical problems.

The dimensionless quantities are defined as follows:

$$s = S/L, \quad s_0 = S_0/L, \quad r_0 = R_0/L = 1/a, \quad q = Q/L, \quad q_0 = Q_0/L$$

$$k = LK, \quad k_0 = LK_0 = a, \quad b = BL + a$$

(2.48)

$$\underline{f} = \frac{FL^2}{EI}, \quad m = \frac{ML}{EI}, \quad p = \frac{PL^3}{EI}, \quad p_0 = \frac{P_0L^3}{EI}$$

$$c = \frac{EI}{EAL^2} = \frac{I}{AL^2}, \quad g = \frac{GL}{EI}, \quad h = \frac{HL^3}{EI}$$

Note that the system properties a and w , which were previously defined as dimensionless angles, do not appear above. Introducing Eqs. 2.48 into the boundary-value problem yields the following dimensionless formulation, where primes denote differentiation with respect to s_0 :

$$e = cf_t \quad (2.49)$$

$$s' = 1 + e \quad \text{Constitutive Equations and} \quad (2.50)$$

$$b = m + a \quad \text{Strain Measure} \quad (2.51)$$

$$k = b/s' \quad \text{Definitions} \quad (2.52)$$

$$f_n' = -s'kf_t - p_{on} \quad (2.53)$$

$$f_t' = s'kf_n - p_{ot} \quad (2.54)$$

$$m' = -s'f_n \quad \text{Differential} \quad (2.55)$$

$$x' = s'\cos v \quad \text{Equations} \quad (2.56)$$

$$y' = s'\sin v \quad (2.57)$$

$$v' = -b \quad (2.58)$$

$$\begin{aligned} \text{or} \quad v &= a - m/g & (g > 0) \\ m &= 0 & (g=0) \end{aligned} \quad (2.59)$$

$$\text{or} \quad x = \frac{\cos w}{h} [f_n \sin(v-w) + f_t \cos(v-w)] - r_0 \sin a \quad (h > 0) \quad \text{Boundary} \quad (2.60)$$

$$f_n = -f_t \cot(v-w) \quad (h=0) \quad \text{Condi-} \quad (2.61)$$

$$y = \tan w (x + r_0 \sin a) + r_0 \cos a \quad (2.61)$$

$$\begin{aligned} \text{or} \quad v &= m/g - a & (g > 0) \\ m &= 0 & (g=0) \end{aligned} \quad (2.62)$$

$$\text{or} \quad x = \frac{\cos w}{h} [f_n \sin(v+w) + f_t \cos(v+w)] + r_0 \sin a \quad (h > 0) \quad \text{Boundary} \quad (2.63)$$

$$f_n = -f_t \cot(v+w) \quad (h=0) \quad \text{Condi-} \quad (2.64)$$

$$y = \tan w (r_0 \sin a - x) + r_0 \cos a \quad (2.64)$$

The process of casting the formulation into dimensionless form reduced the seven dimensionless system properties to the five dimensionless system properties - c , a , g , h , and w . Hence to consider all arches in the restricted class of this study it is necessary to vary p_0 , c , a , g , h , and w independently, where p_0 is the parameter specifying the dimensionless loading p_0 .

Each of the system properties c , g , and h may vary from 0 to ∞ . For convenience in the numerical computations they are replaced by the dimensionless quantities c^* , g^* , and h^* which are defined as

$$c^* = \frac{c}{1+c} \quad (2.65)$$

$$g^* = \frac{g}{1+g} \quad (2.66)$$

$$h^* = \frac{h}{1+h} \quad (2.67)$$

and which vary from 0 to 1. The system property c which appears in Eq. 2.49 is called the extensibility. System property values corresponding to various special cases are listed below.

$$c = c^* = 0 \longleftrightarrow e = 0 \quad (\text{inextensible arch})$$

$$c = \infty (c^* = 1) \longleftrightarrow f_t = 0$$

$$g = g^* = 0 \longleftrightarrow m = 0 \quad (\text{boundaries free to rotate})$$

$$g = \infty (g^* = 1) \longleftrightarrow v = v_0 \quad (\text{boundary rotation prevented})$$

$$h = h^* = 0 \longleftrightarrow (\text{boundaries free to translate})$$

$$h = \infty (h^* = 1) \longleftrightarrow (\text{boundary translation prevented})$$

Loading Cases Considered

Two kinds of loading referred to herein as hydrostatic and vertical loadings, respectively, are considered. Hydrostatic loading consists of a pressure which is uniformly distributed with respect to the length of the deformed centroidal axis and which remains normal to it. Vertical loading is a pressure which is uniformly distributed with respect to the arc length of the undeformed centroidal axis and which remains in the vertical direction.

For hydrostatic loading p_0 is specified as

$$p_0 = s'p = ps'n \quad (2.68)$$

where p is the dimensionless load parameter for the hydrostatic loading. Note that p , which is not equal to the magnitude of the vector $\underline{p}$ which is non-negative, may be positive or negative depending on whether the loading acts in the same or opposite direction as the unit normal vector $\underline{n}$.

For vertical loading p_0 is specified as

$$p_0 = -p_0j = p_0(\cos v \underline{n} - \sin v \underline{t}) \quad (2.69)$$

where Eqs. 2.9 and 2.10 have been used and p_0 is the dimensionless load parameter for the vertical loading.

Potential Energy of the System

The mechanical system composed of the arch, its elastic supports, and either of the loadings considered herein is conservative. That is, the work done by all of the forces (external and internal) acting on the system, when it is displaced (without any additional forces added), from one configuration to any other configuration, is independent of the path followed by the system. Choosing the undeformed configuration as a datum,

the potential energy U is defined as the work that would be done by all the forces acting on the system during an imagined displacement from the deformed configuration to the datum configuration. During this displacement the loading parameter, p or p_0 , is held constant. The potential energy may be considered as the sum of two parts, the potential energy of the internal forces, U_i , called strain energy and the potential energy of the external forces, U_e .

$$U = U_i + U_e \quad (2.70)$$

The dimensionless potential energy is defined as

$$u = \frac{UL}{EI} \quad (2.71)$$

The strain energy of the system is the sum of contributions from the arch itself and from the elastic springs, which are denoted as U_a and U_s , respectively. The strain energy of the arch is found by summing all the strain energies of the fibers of the arch. The strain energy stored in a given fiber during the displacement is the average force, $\frac{1}{2}\sigma dA$, multiplied by the total elongation of the fiber which is $\epsilon dC_0 = \epsilon(1+K_0Z)dS_0 \doteq \epsilon dS_0$, by Eqs. 2.28 and 2.29 and the assumption that the arch is thin. Thus by integration over the volume of the arch it follows that

$$U_a = \frac{1}{2} \int_{-L}^L \int_A \sigma \epsilon dA dS_0 = \frac{1}{2} \int_{-L}^L \int_A E(e + BZ)^2 dA dS_0 = \frac{1}{2} \int_{-L}^L (EAe^2 + EIB^2) dS_0 \quad (2.72)$$

using Eqs. 2.35 and 2.34. The strain energy of each rotational spring is the product of $G/2$ and the square of the angle of rotation, while that of each extensional spring is the product of $H/2$ and the square of the

displacement of the end. Summing the strain energies of all the springs yields

$$U_s = \frac{G}{2} \left\{ \left[a-v(-L) \right]^2 + \left[a+v(L) \right]^2 \right\} + \frac{H}{2\cos^2 w} \left\{ \left[X(-L)+R_o \sin a \right]^2 + \left[X(L)-R_o \sin a \right]^2 \right\} \quad (2.73)$$

The dimensionless strain energy is determined using Eqs. 2.71-2.73 as

$$u_i = \frac{g}{2} \left\{ \left[a-v(-1) \right]^2 + \left[a+v(1) \right]^2 \right\} + \frac{h}{2\cos^2 w} \left\{ \left[x(-1)+r_o \sin a \right]^2 + \left[x(1)-r_o \sin a \right]^2 \right\} + \frac{1}{2} \int_{-1}^1 \left[\frac{e^2}{c} + (b-a)^2 \right] ds_o \quad (2.74)$$

For hydrostatic loading the potential energy of the external forces is equal to the product of $-P$ with the signed area enclosed between the deformed and undeformed centroidal axes. This may be seen by considering the arch to be submerged in a tank of incompressible weightless fluid and forming a part of the boundary of the tank. Then if a hydrostatic pressure is applied at the open surface of the tank, a hydrostatic loading, as defined herein, will be applied to the arch. If the arch is displaced from its deformed configuration to its undeformed configuration while holding the pressure applied to the fluid constant, the work done by the loading on the arch will equal the work done by the externally applied pressure. This work which is the potential energy of the external forces is equal to the product of $-P$ and the area of fluid displaced. The area

of the fluid displaced is equal to the signed area between the deformed and undeformed centroidal axes so that the assertion made above is correct. In computing the signed area, a formula for the area of a quadrilateral in terms of the cartesian coordinates of its vertices is used. This formula is

$$\text{Area} = \frac{1}{2} [(X_1 - X_3)(Y_2 - Y_4) + (X_2 - X_4)(Y_3 - Y_1)] \quad (2.75)$$

where X_i , Y_i ($i = 1, 4$) are the X and Y coordinates, respectively, of the i th vertex with the vertices numbered in counterclockwise order. Applying Eq. 2.75 to the area element shown in Fig. 2.7 and integrating leads to the following formula for U_e

$$U_e = - \frac{P}{2} \int_{-L}^L \left[(X - X_o) \left(\frac{dY}{dS_o} + \frac{dY_o}{dS_o} \right) - (Y - Y_o) \left(\frac{dX}{dS_o} + \frac{dX_o}{dS_o} \right) \right] dS_o \quad (2.76)$$

which becomes

$$u_e = - \frac{P}{2} \int_{-1}^1 [(x - x_o)(y' + y'_o) - (y - y_o)(x' + x'_o)] ds_o \quad (2.77)$$

when expressed in terms of dimensionless variables.

For vertical loading the computation of the potential energy is much simpler. In this case the potential energy is found by considering the work done by the resultant of the pressure acting on a typical element of the arch during a displacement of the arch from the deformed configuration to the undeformed configuration. The work done by the vertical load acting on such an element is the product of the resultant force, $P_o dS_o$, and the vertical displacement of the element, $Y - Y_o$. Integrating

over the length of the arch yields

$$U_e = \int_{-L}^L p_o (Y - Y_o) dS_o \quad (2.78)$$

which becomes

$$u_e = p_o \int_{-1}^1 (y - y_o) ds_o \quad (2.79)$$

when expressed in terms of dimensionless quantities.

CHAPTER 3. NUMERICAL RESULTS

The nonlinear behavior of fifteen different arches was analyzed using the algorithms SYM and ASYM which are described in the Appendix. The term arch is taken in this chapter to mean a special case of the more general mechanical system considered in Chapter 2. Such a special case is specified by prescribing an external loading and assigning values for the dimensionless system properties - a , c^* , g^* , h^* , and w . Each arch is designated by a label which indicates the loading case (H for hydrostatic loading or V for vertical loading) followed by a number.

Selected graphical and tabular results are presented for each arch in a uniform manner. The graphical results are presented using two figures per arch. At the top of the first figure a plot of the load parameter versus d_y , the vertical deflection of the midpoint of the arch, appears. Below it the deflected shape of the arch is shown for selected values of the load parameter. See, for example, Fig. 3.1. Critical load parameter values, the value corresponding to bifurcation and the maximum value, are indicated on the load-deflection plot. The second figure consists of curves showing the distribution of the internal stress resultants - f_n , f_t , and m - with respect to the undeformed arc length s_0 , for selected load parameter values. See, for example, Fig. 3.2. For geometrically symmetric configurations only the left half ($-1 \leq s_0 \leq 0$) of the plots of the deflected shapes and distributions of the internal stress resultants is shown. Consideration of Eq. 2.55 which states that $m' = -s'f_n$ shows that the extrema of the m - s_0 curve should correspond to the

zeroes of the f_n-s_0 curve. Also for inextensible arches, $s' = 1$, and the slope of the $m-s_0$ curve is equal to the negative of the ordinate of the f_n-s_0 curve. Also a table entry is provided for each arch which gives the maximum load parameter and the bifurcation load parameter and the corresponding values of d_y .

As discussed in the Appendix, no asymmetric equilibrium configurations for arches with the angle α less than 90 degrees could be found using algorithm ASYM. Therefore the bifurcation load parameter could not be found for these arches. However, the computations in the first part of ASYM (in which a small constant horizontal force is added to the given hydrostatic or vertical loading) did show that the horizontal (asymmetric) deflections of the shallow arch increased rapidly as the load parameter approached a certain value. This is quite similar to what happens when an arch is tested in the laboratory, since for any real arch there will not be perfect symmetry of the undeformed shape and the loading cannot be applied in a perfectly symmetric manner. For these shallower arches an approximate value of the bifurcation load parameter is presented. This value is the ordinate of the horizontal asymptote which the points on the load parameter versus d_x curve, where d_x is the horizontal deflection of the midpoint of the arch, seem to approach when the computations described above are performed.

In Table 3.1 symmetric and asymmetric buckling loads (critical values of the load parameter) for inextensible arches which were obtained by Oran and Reagan (17) are given. Results are shown for several values of α and for two different loading cases, the hydrostatic loading discussed

Table 3.1 Buckling Loads for Inextensible
Arches Taken from Oran and Reagan (17)

a. Clamped Arches

a	Hydrostatic Loading		Radial Loading	
	Asymmetric Buckling	Symmetric Buckling	Asymmetric Buckling	Symmetric Buckling
.1 radian	2.02	3.32	2.02	3.32
30 deg.	10.5	17.3	10.7	17.5
60 deg.	20.8	34.2	21.8	35.7
90 deg.	31.0	50.0	34.9	55.2

b. Pinned Arches

a	Hydrostatic Loading		Radial Loading	
	Asymmetric Buckling	Symmetric Buckling	Asymmetric Buckling	Symmetric Buckling
30 deg.	4.99	11.4	5.17	11.5
60 deg.	9.18	22.0	10.0	23.1
90 deg.	11.6	31.0	12.7	34.8

in Chapter 2 and a case referred to as radial loading. The radial loading is a uniformly distributed pressure such that the resultant force acting on each small element of the arch acts parallel to the original radial direction for the small element as the arch deforms. These results will be compared with the results for hydrostatically and vertically loaded arches which follow.

Results for Hydrostatic Loading

Nine hydrostatically loaded arches were studied. Only clamped ($g^* = h^* = 1$, $w = \text{arbitrary}$) arches were considered.

For five of the arches, which were designated by the labels H1-H5, the angle α was taken as 90 degrees (a semicircular arch) and the value of the extensibility c^* was taken as 0, .00001, .0001, .001, and .01, respectively. The tabular results for these arches appear in Table 3.2, and the graphical results are shown in Figs. 3.1-3.8.

Table 3.2 Results for Clamped, Semicircular
($a = 90$ deg.) Arches With Hydrostatic Loading

Label	c^*	Maximum p	d_y at Maximum p	Bifurcation p	d_y at Bifurcation p
H1	0	52.025 ^a	0 ^a	31.006	0
H2	.00001	49.205	.00699	31.010	.00030
H3	.0001	47.631	.02496	31.058	.00297
H4	.001	43.653	.10409	31.553	.03082
H5	.01	b	b	b	b

Table 3.3 Results for Clamped Shallow ($a = .1$ radian =
5.729577 deg.) Arches With Hydrostatic Loading and
Rectangular Cross-Sections

Label	c^*	Maximum p	d_y at Maximum p	Bifurcation p	d_y at Bifurcation p
H6	$.23148 \times 10^{-4}$	1.6607	.01537	1.65 ^c	.014 ^c
H7	$.13021 \times 10^{-4}$	2.0220	.01114	1.84 ^c	.0069 ^c
H8	$.83333 \times 10^{-5}$	2.2693	.00853	1.91 ^c	.0042 ^c
H9	$.37037 \times 10^{-5}$	2.6196	.00518	1.97 ^c	.0018 ^c

NOTES:

- ^a This result was taken from Oran and Reagan (17) and corresponds to bifurcation from the undeformed configuration to an adjacent deformed symmetric configuration.
- ^b For arch H5 the restriction $e > -1$ required by Eq. 2.26 was violated before a relative maximum in the p - d_y curve was attained.
- ^c For these shallower arches no asymmetric equilibrium configurations and bifurcation points could be found using algorithm ASYM. However it was observed that the slope of the p - d_x curve became very small as the applied load approached the value shown, with the arch subjected to a small horizontal force in addition to the applied hydrostatic or vertical loading.

Fig. 3.1 Load-Deflection Curve and Deformed Shapes for Arch H2

Fig. 3.2 Distributions of the Internal Stress Resultants for Arch H2

Fig. 3.3 Load-Deflection Curve and Deformed Shapes for Arch H3

Fig. 3.4 Distributions of the Internal Stress Resultants for Arch H3

Fig. 3.5 Load-Deflection Curve and Deformed Shapes for Arch H4

Fig. 3.6 Distributions of the Internal Stress Resultants for Arch H4

Fig. 3.7 Load-Deflection Curve and Deformed Shapes for Arch H5

Fig. 3.8 Distributions of the Internal Stress Resultants for Arch H5

These arches were considered so that the effect of extensibility on the behavior could be studied. For a clamped inextensible arch the trivial solution

$$f_n(s_0) = m(s_0) = 0, \quad f_t(s_0) = -pr_0 \quad (3.1)$$

$$x(s_0) = x_0(s_0), \quad y(s_0) = y_0(s_0), \quad v(s_0) = v_0(s_0)$$

is an exact solution to the nonlinear boundary-value problem formulated in Eqs. 2.49-2.64. The results of Oran and Reagan (17), which are based on linear classical buckling theory, predict that for an inextensible arch bifurcation from the undeformed (trivial) solution to an adjacent asymmetric deformed configuration occurs when $p = 31.006$. This result was reproduced in the analysis of arch H1 by algorithm ASYM. The results of Oran and Reagan also predict that bifurcation from the undeformed configuration to an adjacent symmetric configuration occurs when $p = 52.025$. This latter result could not be obtained by either of the algorithms SYM or ASYM since no provision was made in them to force (possibly by adding a small concentrated vertical load at the center) the arch to go from one symmetric configuration to an adjacent symmetric configuration.

Considering the results for arches H2-H4 in Table 3.2 it appears that the maximum and bifurcation load parameter values for the extensible arches approach the symmetric and asymmetric buckling loads, respectively, for the inextensible arch as the extensibility c^* is decreased toward zero. Changing the value of c^* from .00001 to .001 affects the bifurcation p value only slightly (from 31.010 to 31.553), but it greatly changes the associated value of d_y (from .00030 to .03082). For the

arch H5 ($c^* = .01$) the restriction of Eq. 2.26 that $e > -1$ was violated before a relative maximum in the $p-d_y$ curve was attained. The effect of extensibility may be graphically seen by comparing the deflected shapes for arches H2-H5. For arch H2 the deflected shape shown in Fig. 3.1 lies on both sides of the undeformed shape. As the value of c^* increases less of the arch length is displaced to the outside of the undeformed configuration. This may be seen in Figs. 3.3, 3.5, and 3.7 for arches H3-H5, respectively. One-half of the deformed lengths of the arches for the configurations corresponding to the end points on the appropriate $p-d_y$ curves were .99968, .99691, .97984, and .80651 for the arches H2-H5, respectively.

Comparison of the distributions of the internal stress resultants for arches H2-H5 is difficult since for each arch the results are shown for different values of p . In addition to the general relation mentioned above between the f_n-s_0 and $m-s_0$ curves, several relations follow from the equilibrium equations for the free-body of the whole arch. For the symmetric configurations it may be shown that

$$f_t(-1) = -pr_0 \quad (3.2)$$

and for the asymmetric configurations it may be shown that

$$f_t(-1) + f_t(1) = -2pr_0 \quad (3.3)$$

and

$$f_n(-1) = -f_n(1) \quad (3.4)$$

Note that Eqs. 3.2-3.4 are only valid for hydrostatically loaded arches.

All these requirements were satisfied by the numerical results obtained for arches H1-H5.

For the remaining four hydrostatically loaded arches system properties were chosen to correspond to some of the cases presented by Kerr and El-Bayoumy (11) so that the results could be directly compared with their results. Their study was restricted to shallow, clamped arches with a rectangular cross-section that is one unit wide and h units deep. The shallow arch assumptions of Schreyer and Masur (20) were used in their analysis. In their formulation, since the shape of the cross-section was specified, only one system property which they took as $\frac{R_0 a^2}{h}$ was needed in place of the two, a and c^* , used in the present study. Values of a and c^* were chosen for arches H6-H9 to correspond to the cases $\frac{R_0 a^2}{h} = 6, 8, 10, \text{ and } 15$, respectively. To this end the value of a was kept constant at $.1 \text{ radian} = 5.729577 \text{ degrees}$ for all four arches and values of $c^* = 1/43201, 1/76801, 1/120001, \text{ and } 1/270001$ were taken for arches H6-H9, respectively. The decimal approximations to these c^* values appear with the tabular results in Table 3.3

The linear classical buckling results (17) for clamped arches with $a = .1 \text{ radian}$ are $p = 2.02$ for asymmetric buckling and $p = 3.32$ for symmetric buckling as shown in Table 3.1. Again, as was observed for the semicircular arches, the critical load parameter values for the extensible arches approach those for the inextensible arches as the extensibility c^* is decreased toward zero. But contrary to the results for the semicircular arches, the linear classical buckling result for the bifurcation load parameter for the shallow arches is not a very good

approximation to the corresponding result for the extensible arches. The load parameter results presented in Table 3.3 agree to three significant figures with the results presented by Kerr and El-Bayoumy.

Results for Vertical Loading

Six arches were considered with vertical loading. This loading case was considered because no other results for it could be found in the literature. For this loading case linear classical buckling theory cannot be used to find the buckling loads since the loading is not a funicular loading for the circular arch. Only inextensible arches with either clamped ($g^* = h^* = 1$, $w = \text{arbitrary}$) or pinned ($g^* = 0$, $h^* = 1$, $w = \text{arbitrary}$) arches were considered. The clamped arches were designated as V3, V6, and V9 corresponding to the values 30, 60, and 90 degrees for α , respectively. Similarly, the pinned arches were labeled as V13, V16, and V19. The tabular results appear in Table 3.4a for the clamped arches and in Table 3.4b for the pinned ones. The graphical results for the clamped arches appear in Figs. 3.9-3.14 and those for the pinned arches are found in Figs. 3.15-3.20.

Table 3.4 Results For Inextensible
Arches With Vertical Loading

a. Clamped Arches

Label	a, deg.	Maximum p_o	d_y at Maximum p_o	Bifurcation p_o	d_y at Bifurcation p_o
V3	30	14.572	.01945	10.7 ^c	.0032 ^c
V6	60	25.904	.08465	22.1 ^c	.030 ^c
V9	90	37.444	.21768	31.969	.09307

b. Pinned Arches

Label	a, deg.	Maximum p_o	d_y at Maximum p_o	Bifurcation p_o	d_y at Bifurcation p_o
V13	30	10.203	.03876	5.13 ^c	.0023 ^c
V16	60	19.170	.17082	9.44 ^c	.015 ^c
V19	90	31.272	.43103	9.6709	.031470

NOTE:

^c See note appearing below Table 3.3.

Fig. 3.9 Load-Deflection Curve and Deformed Shapes for Arch V3

Fig. 3.10 Distributions of the Internal Stress Resultants for Arch V3

Fig. 3.11 Load-Deflection Curve and Deformed Shapes for Arch V6

Fig. 3.12 Distributions of the Internal Stress Resultants for Arch V6

Fig. 3.13 Load-Deflection Curve and Deformed Shapes, for Arch V9

Fig. 3.14 Distributions of the Internal Stress Resultants for Arch V9

Fig. 3.15 Load-Deflection Curve and Deformed Shapes for Arch V13

Fig. 3.16 Distributions of the Internal Stress Resultants for Arch V13

Fig. 3.17 Load-Deflection Curve and Deformed Shapes for Arch V16

Fig. 3.18 Distributions of the Internal Stress Resultants for Arch V16

Fig. 3.19 Load-Deflection Curve and Deformed Shapes for Arch V19

Fig. 3.20 Distributions of the Internal Stress Resultants for Arch V19

A comparison of these results shows that as the arch becomes shallower (α is decreased) that the magnitudes of both the deformations and critical loads decrease. For the arches with $\alpha = 30$ degrees the deflections are quite small (the order of 2-4% of the length of the arch), but for the arches with $\alpha = 90$ degrees, they are quite large (the order of 20-40% of the length of the arch).

Comparing the results given in Table 3.1 for the hydrostatically and radially loaded arches with those for the vertically loaded arches given in Table 3.4, it may be seen that the asymmetric buckling loads do not differ much for all three loading cases. The agreement is best for the arches with $\alpha = 30$ degrees, which is as it should be, since in the limit as α goes to zero the arch approaches a straight bar and all three loading cases become identical.

CHAPTER 4. CONCLUSIONS

Method of Analysis

The formulation which appears in Eqs. 2.49-2.64 is a convenient one for studying the finite deformations of circular arches. It includes the effects of extensibility and non-funicular loading which were not considered in the previous study by Oran and Reagan (17).

The algorithm SYM which uses the shooting method described in the Appendix was successfully used to determine symmetric configurations for all the arches considered in this study. It was used to follow the load parameter versus vertical deflection curve through a relative maximum. Then by interpolation an approximation for the maximum load parameter value was computed.

The algorithm ASYM which also uses the shooting method and is designed to find asymmetric (postbuckling) equilibrium configurations worked best for the higher arches considered. For the shallower, stiffer arches no asymmetric configurations could be found. This failure of ASYM is not too serious because: 1) ASYM did yield an estimate of the buckling load, and 2) shallow arch theories provide acceptable results for the shallower arches. The latter was verified by the agreement between the results presented herein and those of Kerr and El-Bayoumy (11).

Structural Behavior of Circular Arches

The following conclusions concerning the structural behavior of circular arches were drawn from the numerical results presented in Chapter 3:

1. The effect of extensibility on the asymmetric buckling load for clamped, semicircular arches is not very significant. However the addition of a seemingly small amount of extensibility (say $c^* = .001$) does introduce significant prebuckling displacements.
2. The assumptions on which this formulation was based must be applied with care. In particular the engineering beam theory and the linear constitutive equation on which it is based are valid only for small strains. For some of the geometries considered, especially the semicircular arches, the prebuckling deformations are so large that yielding of the material would probably occur before elastic buckling. The limitations of the engineering beam theory were made quite clear to the author when the restriction $e > -1$ appearing in Eq. 2.26 was violated by algorithm SYM in the analysis of arch H5. This was a case when a physical impossibility was possible in the mathematical model, i.e. the computer "sees" nothing wrong in "squeezing" a segment of the arch into a zero or even a negative length.
3. The shallow arch theory assumptions as presented by Schreyer and Masur (20) provided an excellent approximation to the more exact formulation used herein. This was observed in the agreement of the results presented here and those of Kerr and El-Bayoumy (11) for the four shallow arches H6-H9 with hydrostatic loading.
4. Inextensible, clamped or pinned, circular arches with vertical

loading buckle asymmetrically in much the same way as the corresponding hydrostatically or radially loaded arches. However the prebuckling deformations for vertically loaded, high arches were quite large (20-40% of the length of the arch).

5. The response of the arches considered herein was quite nonlinear as may be seen in the load-deflection curves presented in Chapter 3. The shape of the load-deflection curves is similar to that obtained when nonlinear material properties are introduced into small deformation theories for bars. See, for example, Reagan and Krah1 (19). This similarity would make it difficult to separate the geometric and material nonlinearities in an actual laboratory test of an arch.

Suggestions for Further Research

A general procedure for finding all of the significant equilibrium configurations of structures is needed. It should be less intuitive and more direct than the techniques used in algorithms SYM and ASYM.

The formulation presented here could be extended to allow three-dimensional displacements of the arch. This would increase the order of the boundary-value problem to twelve and introduce additional system properties. Such a formulation could be used to study the out-of-plane buckling of arches.

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NOTATION FOR THE APPENDIX

$A_1, A_2, A_3,$ B_1, B_2, B_3	Constants appearing in Eq. A.3 and Eq. A.4
B	Extrapolation distance regulator appearing in Eqs. A.13, A.16, and A.17
D	3x3 matrix defined in Eq. A.23
E	Upper bound on the absolute values of the residuals
$F(X)$	Column vector iteration function defined by Eq. A.22
L	Index on the loading parameter values, p_L , with the maximum number of load values allowed denoted by L_a
R	Column vector of residuals
S_U, S_V	Scale factors used in the extrapolation procedure
T	Quantity defined following Eq. A.12
U, V	Scaled coordinates used to describe the curve in the extrapolation procedure, with the unscaled coordinates denoted by $\bar{U}$ and $\bar{V}$
X	Column vector of missing boundary values at the left end of the arch
X^n	n th iterate of X in the shooting method for solving the boundary-value problem
a	$\frac{1}{2}$ of the angle subtended by the undeformed arch
c^*	Extensibility defined in Eq. 2.65
g^*	Rotational spring constant defined in Eq. 2.66
h^*	Extensional spring constant defined in Eq. 2.67
d_x	Horizontal deflection of the center point of the arch centroidal axis, positive to the right
d_y	Vertical deflection of the center point of the arch centroidal axis, positive downward

f_n, f_t	Normal and tangential components, respectively, of the internal force vector
i	Integer index used in various places
k	Integer index used in various places
m	Internal couple
n	Index used to count the iterations in the shooting method, with n_a equal to the maximum number allowed
p	Load parameter for hydrostatic loading
p_o	Load parameter for vertical loading
q, q_o	Position vectors of the deformed and undeformed centroidal axes, respectively
r_o	Radius of the undeformed arch
s	Deformed arc length
s_o	Undeformed arc length
t	Approximate arc length used in the extrapolation procedure
u	Potential energy
v	Tangent angle
w	Angle defining the line of translation of the boundaries
x, y	Cartesian components of q
x_o, y_o	Cartesian components of q_o

APPENDIX. ALGORITHMS

This appendix contains a discussion of the two algorithms, SYM and ASYM, which were used to obtain the results appearing in Chapter 3. SYM was used to determine the geometrically symmetric equilibrium configurations, while ASYM was used to find the geometrically asymmetric ones. FORTRAN IV programs using double-precision arithmetic and based on each of the algorithms were written for the IBM 360/65 computer at the Louisiana State University Computer Research Center. Before describing SYM and ASYM, two important elements of the algorithms, and extrapolation procedure and a shooting method for solving the boundary-value problem, will be discussed.

The notation used in the Appendix will be somewhat different from that used in the main text. Some of the symbols used in the main text will retain their old definitions, while others used in the main text will have new definitions in the Appendix. Since only dimensionless quantities are needed in the Appendix, the notational convention used in the main text concerning minuscule and majuscule letters will not be used. See the separate listing of notation which applies only to the Appendix which appears at the beginning of this Appendix.

Extrapolation Procedure

The extrapolation procedure described here is used extensively in algorithms SYM and ASYM. Consider a two-dimensional curve which is described approximately by a discrete set of points with cartesian coordinates $(\bar{U}_i, \bar{V}_i)$, $i = 1, L-1$, which are ordered with respect to distance along the curve. See Fig. A.1. It is desired to determine approximations

Fig. A.1 Illustration Showing
Extrapolation Procedure

Fig. A.2 Typical Load-
Deflection Curve

for the coordinates $(\bar{U}_L, \bar{V}_L)$ of an additional point, the L th point, by extrapolating from the given $L-1$ points. Only the last three of the points - $L-3$, $L-2$, and $L-1$ - are used in the procedure.

First the coordinates of the last three points are scaled by dividing by the scale factors S_U and S_V ,

$$U_i = \bar{U}_i / S_U \quad (i = L-3, L-1) \quad (A.1)$$

$$V_i = \bar{V}_i / S_V \quad (i = L-3, L-1) \quad (A.2)$$

with U_i and V_i denoting the scaled coordinates. The scale factors S_U and S_V are taken as $|\bar{U}_k - \bar{U}_{k-1}|$ and $|\bar{V}_k - \bar{V}_{k-1}|$, respectively, where k is the largest value of i such that $\bar{U}_i \neq \bar{U}_{i-1}$ and $\bar{V}_i \neq \bar{V}_{i-1}$. In the event no such i exists (which is unlikely) the scale factors $S_U = S_V = 1$ are used.

Then it is assumed that the curve can be approximated locally by the expressions

$$U(t) = A_1 + A_2 t + A_3 t^2 \quad (A.3)$$

and

$$V(t) = B_1 + B_2 t + B_3 t^2 \quad (A.4)$$

where the parameter t = an approximate arc distance measured from point $L-3$ and A_1, A_2, A_3, B_1, B_2 , and B_3 are constants to be determined. The arc distance t is found by summing the chord lengths between successive points. Thus

$$t_{L-3} = 0 \quad (A.5)$$

$$t_{L-2} = \sqrt{(U_{L-2} - U_{L-3})^2 + (V_{L-2} - V_{L-3})^2} \quad (A.6)$$

$$t_{L-1} = t_{L-2} + \sqrt{(U_{L-1} - U_{L-2})^2 + (V_{L-1} - V_{L-2})^2} \quad (A.7)$$

with t_{L-1} = approximate arc distance to point L-1, etc. The constants A_1 - B_3 are determined by forcing Eqs. A.3 and A.4 to give correct coordinates for the three points L-3, L-2, and L-1. Doing this yields

$$A_1 = U_{L-3} , \quad B_1 = V_{L-3} \quad (A.8)$$

$$A_2 = \frac{1}{T} [t_{L-1}^2 (U_{L-2} - U_{L-3}) - t_{L-2}^2 (U_{L-1} - U_{L-3})] \quad (A.9)$$

$$B_2 = \frac{1}{T} [t_{L-1}^2 (V_{L-2} - V_{L-3}) - t_{L-2}^2 (V_{L-1} - V_{L-3})] \quad (A.10)$$

$$A_3 = \frac{1}{T} [-t_{L-1} (U_{L-2} - U_{L-3}) + t_{L-2} (U_{L-1} - U_{L-3})] \quad (A.11)$$

$$B_3 = \frac{1}{T} [-t_{L-1} (V_{L-2} - V_{L-3}) + t_{L-2} (V_{L-1} - V_{L-3})] \quad (A.12)$$

where $T = t_{L-2}t_{L-1}^2 - t_{L-1}t_{L-2}^2$.

The arc distance to the sought point, t_L , is taken as

$$t_L = t_{L-1} + 2^{-B}(t_{L-1} - t_{L-2}) \quad (A.13)$$

where the exponent B is an integer called the extrapolation distance regulator. For positive, zero, or negative values of B the arc distance from point L to point L-1 is less than, equal to, or greater than, respectively, the arc distance from point L-1 to point L-2. B is useful in varying how "far" the extrapolated point is from the given points.

The desired result then may be expressed as

$$\bar{U}_L = S_U(A_1 + A_2 t_L + A_3 t_L^2) \quad (A.14)$$

$$\bar{V}_L = S_V(B_1 + B_2 t_L + B_3 t_L^2) \quad (A.15)$$

by applying Eqs. A.3 and A.4 at point L. This extrapolation procedure is more convenient for the algorithms used in this study than conventional extrapolation treating $\bar{V}$ as a function of $\bar{U}$. Since $\bar{U}$ and $\bar{V}$ play symmetric roles in this procedure, it can handle situations such as zero or infinite slopes or double-valued functions ($\bar{V}(\bar{U})$ or $\bar{U}(\bar{V})$) which cannot be treated by conventional extrapolation. On the other hand, Eqs. A.14 and A.15 are less accurate (in the sense that they do not give exact results for a parabola) than conventional quadratic extrapolation, since they approximate the arc length. For the case $L = 3$ the linear extrapolation formulas corresponding to Eqs. A.14 and A.15 are

$$\bar{U}_3 = \bar{U}_2 + 2^{-B}(\bar{U}_2 - \bar{U}_1) \quad (\text{A.16})$$

and

$$\bar{V}_3 = \bar{V}_2 + 2^{-B}(\bar{V}_2 - \bar{V}_1) \quad (\text{A.17})$$

Shooting Method

A shooting method is used to solve the boundary-value problem, which may be viewed as finding the zero of a set of three simultaneous nonlinear functions of three variables. The shooting method consists of the following steps:

1. Guesses are made for the three missing (to be discussed later) boundary values, denoted by X_1 , X_2 , and X_3 , respectively, at the left end of the arch.
2. The remaining three boundary values at the left end of the arch are computed from the boundary conditions, Eqs. 2.59-2.61.

3. The differential equations are integrated numerically beginning at the left end and continuing to the right end. This yields six boundary values at the right end of the arch.

4. In general the six boundary values at the right end computed in step 3 will not satisfy the boundary conditions at the right end. Three residuals (amounts by which the three right boundary conditions are not satisfied) are denoted as R_1 , R_2 , and R_3 . These residuals may be viewed as functions of the missing boundary conditions at the left end - X_1 , X_2 , and X_3 . For the correct values of X_1 , X_2 , and X_3 , then $R_1 = R_2 = R_3 = 0$.

5. The guesses for X_1 , X_2 , and X_3 are refined and steps 2-4 are repeated. If the absolute values of the residuals are all less than a specified tolerance, the procedure is terminated, and if they are not, step 5 is repeated until they are.

Since the partial derivatives of the functions R_1 , R_2 , and R_3 with respect to X_1 , X_2 , and X_3 are not available, a modified Newton-Raphson method which does not require derivatives of the functions, is used in step 5 above to refine the guesses for X_1 , X_2 , and X_3 . Matrix notation is convenient for the description of it with the shooting method stated as

$$R(X) = 0 \quad (A.18)$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad (A.19)$$

and

$$R(X) = \begin{bmatrix} R_1(X) \\ R_2(X) \\ R_3(X) \end{bmatrix} = \begin{bmatrix} R_1(X_1, X_2, X_3) \\ R_2(X_1, X_2, X_3) \\ R_3(X_1, X_2, X_3) \end{bmatrix} \quad (A.20)$$

The modified Newton-Raphson iterative procedure for improving an initial guess for a zero of Eq. A.18 is given by the recursion formula

$$X^{n+1} = F(X^n) \quad (A.21)$$

where X^n and X^{n+1} are the n th and $n+1$ th iterates, respectively, and $F(X)$ is the iteration function

$$F(X) = X - D^{-1}R(X) \quad (A.22)$$

and D is the 3×3 matrix

$$D = \begin{bmatrix} \frac{R_1(X_1+R_1, X_2, X_3)-R_1}{R_1} & \frac{R_1(X_1, X_2+R_2, X_3)-R_1}{R_2} & \frac{R_1(X_1, X_2, X_3+R_3)-R_1}{R_3} \\ \frac{R_2(X_1+R_1, X_2, X_3)-R_2}{R_1} & \frac{R_2(X_1, X_2+R_2, X_3)-R_2}{R_2} & \frac{R_2(X_1, X_2, X_3+R_3)-R_2}{R_3} \\ \frac{R_3(X_1+R_1, X_2, X_3)-R_3}{R_1} & \frac{R_3(X_1, X_2+R_2, X_3)-R_3}{R_2} & \frac{R_3(X_1, X_2, X_3+R_3)-R_3}{R_3} \end{bmatrix} \quad (A.23)$$

where $R_1 = R_1(X_1, X_2, X_3)$, etc. This procedure is very similar to one presented by Wegge (25). The only difference is that in Wegge's procedure the argument $X_i + R_i$ appearing in the numerator of each element and the divisor R_i appearing in each element of the matrix D are replaced by $X_i - R_i$ and $-R_i$, respectively. In either case the matrix D is an approximation to the Jacobian matrix which appears in the usual Newton-Raphson

method. Notice that each step in the iteration requires four numerical integrations of the differential equations to determine each of the vectors $R(X)$, $R(X_1+R_1, X_2, X_3)$, $R(X_1, X_2+R_2, X_3)$, and $R(X_1, X_2, X_3+R_3)$ which appear in Eq. A.23.

The numerical integration of the differential equations was done with the subroutine DRKGS in the IBM System/360 Scientific Subroutine Package (SSP) (9). DRKGS (which is the double-precision version of the SSP subroutine RKGS) uses a fourth-order Runge-Kutta integration scheme. It is very convenient to use since it automatically adjusts the step size used in the integration (by halving or doubling) so as to make a measure of the truncation error less than a user-supplied tolerance. If more than ten bisections of the original (user-supplied) step size are required, the subroutine terminates and returns to the main program.

Algorithm SYM

Algorithm SYM is a procedure for finding geometrically symmetric configurations of an arch, beginning from the unloaded configuration (p or $p_0 = 0$) and continuing with varying values of p or p_0 until a local maximum in the p or p_0 versus d_y curve is attained. See Fig. A.2. From here on whenever p is mentioned it is to be understood that either p or p_0 is to be used for hydrostatic or vertical loading, respectively. Using the extrapolation procedure described above, the p - d_y curve is followed to obtain the successive values of p . Thus what is incremented during the procedure is scaled distance along the p - d_y curve, and not either p or d_y . Load values are numbered by an integer index L , with $L = 1$ corresponding to the unloaded configuration. For each load value, the corresponding values of f_n , f_t , m , x , y , and v at the left end and

d_y are stored for later use in extrapolation to pick the next load value and initial guesses for the missing boundary values corresponding to the next load value. A brief block diagram of SYM appears in Fig. A.3.

Each block is numbered and will be further described below.

Block 1 The system properties a , c^* , g^* , h^* , and w are read. Also the following computational parameters are read; the first nonzero load value p_2 , an allowable upper bound for the boundary condition residuals (denoted by E), an integer denoting the loading case (hydrostatic or vertical), the number of load values allowed (denoted by L_a), the number of Newton-Raphson iterations allowed for each load value (denoted by n_a), and the number of equally spaced points (called net points) along the length of the undeformed arch for which results are desired.

Block 2 The quantities q_0 and q'_0 at each net point are computed for later use in Block 8.

Block 3 Approximations for the missing boundary values at the left end (denoted by the column vector X^1) for $L = 2$ corresponding to the p_2 specified in Block 1 are computed. The crude assumption (which is correct only for unbuckled, hydrostatically loaded, inextensible rings) that

$$f_n(-1) = m(-1) = 0 \quad (A.24)$$

$$f_t(-1) = -p/r_0 = -pa \quad (A.25)$$

$$q(-1) = q_0(-1) \quad (A.26)$$

$$v(-1) = v_0(-1) \quad (A.27)$$

Fig. A.3 Block Diagram of SYM

is used.

Block 4 See Fig. A.4 for an expansion of Block 4. The blocks in Fig. A.4 are further described below.

Block 4.1 The load index is incremented by 1.

Block 4.2 For $L=2$, control is transferred directly to Block 5, since X^1 for $L=2$ was already computed in Block 3.

Block 4.3 For $L>2$, the extrapolation process, Eqs. A.14-A.15 (for $L>3$) or Eqs. A.16-A.17 (for $L=3$), is used to pick a load parameter value p_L and an approximate d_y , with $\bar{U} = d_y$ and $\bar{V} = p$.

Block 4.4 If $\bar{U}_{L-3} < \bar{U}_{L-2} < \bar{U}_{L-1}$, the value of $p_L = \bar{V}_L$ is improved by using interpolatory quadratic extrapolation treating $\bar{V}$ as a function of $\bar{U}$ to find a better approximate value of $\bar{V}_L = p_L$ corresponding to the value of $\bar{U}_L = d_y$ found in Block 4.3.

Blocks 4.5-4.11 Initial guesses for the three missing boundary values X^1 are computed using extrapolation by Eqs. A.14-A.15 or A.16-A.17 with $\bar{U} = d_y$ and $\bar{V} = X_1^1, X_2^1$, and X_3^1 , successively.

Different choices are made for the missing boundary values X_1 depending on the values of g^* and h^* as may be seen in the block diagram Fig. A.4.

Block 5 An attempt is made to solve the boundary-value problem using the shooting method described above. Sometimes the shooting method fails due to the following circumstances:

1. The method fails to converge after the allowable number of iterations n_a specified in Block 1.
2. The determinant of the matrix D becomes too small so that D^{-1} cannot be found.

Fig. A.4 Expansion of Block 4 in Fig. A.3

3. More than ten bisections of the specified step size (the arc distance between two adjacent net points) are required by subroutine DRKGS.

Failure of the shooting method usually occurs because a value of p is chosen by extrapolation for which there is no equilibrium configuration with missing boundary values which are sufficiently close to the initial guesses x^1 .

An expansion of Block 5 appears in Fig. A.5. The configurations are forced to be geometrically symmetric by considering only the left half of the arch ($-1 \leq s_0 \leq 0$) with the boundary conditions Eqs. 2.59-2.61 at the left end, $s_0 = -1$, and the boundary conditions

$$f_n(0) = x(0) = v(0) = 0 \quad (\text{A.28})$$

at the center, $s_0 = 0$.

Block 7 If the shooting method fails as discussed above under Block 5, the extrapolation regulator B is incremented by 1 and the load index L is decremented by 1. This enables the shooting method to be tried again in Blocks 4 and 5 with the extrapolation distance reduced to half of its value for the previous unsuccessful attempt.

Block 8 The deformed arc length, $s(s_0)$, for each of the net points and the potential energy u are calculated by numerical integration of Eqs. 2.27 (in dimensionless form), 2.74, and 2.77 or 2.79 using Simpson's rule.

Block 9 In this block the results (s , f_n , f_t , m , x , y , and v) at each of the net points for the L th load value are printed.

Fig. A.5 Expansion of Block
5 in Fig. A.3

Block 10 The current load value is compared with the last one. A decrease in load indicates that a local maximum in the $p-d_y$ curve has been found.

Block 11 If L equals the number of load values allowed, L_a (specified in Block 1), the computations are terminated. Otherwise control is transferred to Block 4.

Block 12 An approximation to the maximum load and the corresponding d_y value are found by passing an interpolating parabola through the last three points on the $p-d_y$ curve and then finding the maximum of that parabola. The potential energy u corresponding to the maximum load is also found by interpolation. Then the computations are terminated.

Block 13 The extrapolation regulator B is set equal to zero in preparation for the first attempt to solve the boundary-value problem for the next load value p_{L+1} .

Algorithm ASYM

Algorithm ASYM was designed to find asymmetric configurations near a point of bifurcation and then to find an approximation for the load parameter value at which bifurcation occurs. The main difficulty in this procedure lies in obtaining two equilibrium configurations so that the extrapolation procedure may be used to follow a load-deflection curve. The following physically motivated process, which is similar to one used by Lee, Manuel, and Rossow (15) to find a bifurcation point for a portal frame, was used with some success:

1. Equilibrium configurations were found for successive values of the load parameter using a procedure similar to that outlined for SYM. One difference from SYM was that the given hydrostatic or vertical loading system was augmented by a small horizontal force. This horizontal force was applied with constant intensity over a center portion of the arch length, $-.1 \leq s_0 \leq .1$, and zero intensity over the rest of the arch length. The full arch length was considered and the boundary conditions Eqs. 2.62-2.64 were used at the right end. Another difference from SYM was that the plot of p versus d_x , where d_x is the horizontal deflection of the midpoint, was followed by extrapolation instead of the p - d_y curve. Corresponding to that, the variable d_x was used in ASYM in the extrapolation process to obtain initial guesses for the shooting method and the successive load values, instead of d_y which was used in SYM. A semi-log plot of p_0 versus d_x for the arch V19 described in Chapter 3, which was obtained by the above described procedure, appears in Fig. A.6(a). The points so obtained are indicated by small, circles with selected points numbered in the order that they were obtained. It may be considered typical for all the arches considered in Chapter 3. The loading process was stopped after a fixed number of load values (this number was 20 for Fig. A.6(a)).

2. The equilibrium configurations associated with the last two points obtained in step 1 were used as approximations to asymmetric configurations for the arch with the artificially added horizontal load removed. Then a procedure similar to SYM was used to follow

Fig. A.6

P_o - d_x Plots
for Arch V19

the $p-d_x$ curve (in the opposite direction from step 1) until d_x becomes nearly zero. In this procedure the approximate boundary values found in step 1 are used as initial guesses for the shooting method. Following that an approximation for the bifurcation load was found using parabolic extrapolation. The points thus obtained for arch V19 are shown on an expanded $p-d_x$ plot in Fig. A.6(b) as plus signs (+) and are numbered 1'-4'. The last points obtained in step 1 are also shown on this plot and are indicated by circles.

The physical reasoning behind the procedure outlined above was the following: For load parameter values sufficiently less than the bifurcation load parameter, the symmetric configurations are stable, which implies that the application of a small disturbance, e.g. a small horizontal force, to the arch should produce only a small change in the configuration. For load parameter values near the bifurcation load, 1) the symmetric configuration of the arch is unstable and small disturbances should produce large changes in the configuration, forcing the arch to assume an asymmetric configuration, and 2) the asymmetric configuration of the arch is stable and small disturbances should produce only small changes in the configuration.

One difficulty in applying the algorithm ASYM was that the magnitude of the disturbing force had to be chosen. Several trials were usually required to find a suitable value. If the disturbing force is taken too small, the resulting d_x values will be insignificant (less than the roundoff errors) and the arch will not assume an asymmetric configuration as the bifurcation point is approached. If it is taken too large,

the asymmetric configurations which are found in step 1 will not be adequate approximations to provide starting values for the shooting method in step 2. The intensity of the horizontal load applied over the center portion of arch V19 was .00230.

The algorithm ASYM worked very well for the semicircular arches considered in this study, but when it was tried for the shallower arches, no asymmetric configurations could be found. However, the $p-d_x$ curves obtained for these arches were similar in shape to Fig. A.6(a), but they were much flatter, indicating that the behavior of the arch was very sensitive to small changes in the load parameter. The extrapolation technique had difficulty in following such a flat curve. It would predict load values for which no equilibrium configuration existed, so that the shooting method would not converge. When this happened the extrapolation distance was reduced as described in SYM and the shooting method was tried again. Progress along the $p-d_x$ curve was quite slow because of this. Then when the disturbing horizontal force was removed and step 2 was tried, the shooting method converged to the symmetric configuration instead of an asymmetric one. Several modifications of the algorithm ASYM were tried in an effort to correct these problems, but none of them were successful. They were as follows:

1. Since the value of d_x may range over several factors of ten in the loading process of step 1, it was thought that it might be an improvement to follow the $p-d_y$ curve and d_y was used instead of d_x in the extrapolation procedure.

2. The distribution of the disturbing horizontal pressure was changed from the one described above (which was a discontinuous function at the points $s_0 = \pm 1$) to the continuous function $e^{-20s_0^2}$. This function approximates a concentrated load with a "spike" centered about the origin, $s_0 = 0$.

3. The disturbing horizontal force was replaced by a vertical loading distributed as a dipole, $s_0 e^{-20s_0^2}$. This loading produces a counterclockwise couple.

Even though the bifurcation point could not be obtained for the shallower arches, the value of the load parameter for which the p - d_x curve became flat should be a good approximation to the bifurcation load. Such approximate values are presented in Chapter 3.

Remarks About SYM and ASYM

One drawback to the use of the algorithms SYM and ASYM was that they required large amounts of computer time. The Newton-Raphson method converged quite rapidly, usually in three or four iterations, but the variable-step Runge-Kutta integration routine had to use a very small step size to obtain sufficiently accurate results. Since each Newton-Raphson iteration required four integrations of the differential equations, the computer time required was greatly dependent on the step size. The values 10^{-7} and 10^{-8} were commonly used for the allowable upper bounds on the boundary condition residuals and the truncation error measure required by DRKGS, respectively. For these values the computer time required by the programs SYM and ASYM for each arch was 5 and 10 minutes, respectively, on the IBM 360-65.

VITA

Ronald Steven Reagan was born in Meridian, Mississippi, on March 2, 1943. As the son of an enlisted man in the U.S. Air Force he attended public schools in New York, Texas, Oklahoma, Japan, and Mississippi, and graduated from Biloxi High School in Biloxi, Mississippi, in June, 1960. In September, 1960, he entered Louisiana State University in Baton Rouge, Louisiana, and received the degree of Bachelor of Science in Civil Engineering in June, 1964. In September, 1964, he entered Graduate School at Rice University, in Houston, Texas, and received the Master of Science Degree in Civil Engineering in June, 1966. He was married in December, 1966, to Llewellyn Jordan Morrow. After an additional year of graduate study at Rice, he transferred to the Engineering Science Department at Louisiana State University in Baton Rouge, Louisiana. He has held summer jobs with the Mississippi State Highway Department in 1961 and 1962, Chevron Oil Company in 1963, General Dynamics Corporation in 1964, Brown and Root, Inc. in 1965, and Bernard Johnson Engineers in 1966. He is presently a candidate for the degree of Doctor of Philosophy in the Department of Engineering Science. After graduation he will work as a structural engineer for Meyer Manufacturing, Inc. in Hazleton, Pennsylvania.

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Date of Examination:

July 13, 1970